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Action accessibility via centralizers

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ABSTRACT

In this paper we first introduce a non-symmetric notion of centralization between a relation S and an equivalence relation R, which coincides with Smith centralization in the case S is an equivalence relation too. We then prove that in any action accessible category in the sense of Bourn and Janelidze (2009) [11], the centralizer of an equivalence relation R, defined as in [11], actually has a stronger property, namely it is an equivalence relation, which is the largest among all the relations S centralizing R in the non-symmetric sense mentioned above. As a main result, we show that the existence of centralizers for any equivalence relation with this stronger property actually characterizes action accessibility for exact protomodular categories.

0. Introduction

The notion of *action accessible category* was recently introduced by Bourn and Janelidze in [11], and developed in a more general concept by Bourn in [6], in the context of pointed protomodular categories (categories where split short five lemma holds, see [10]). Via the equivalence between internal actions and split extensions in the category of groups as in any other semi-abelian category (in the sense of [17]), it is possible to represent any action on a fixed object X by a split extension with kernel X. The crucial notion of faithful split extension introduced in [11] is an interpretation in terms of split extensions of the classical notion of faithful action. A pointed protomodular category $\mathbb{C}$ is said to be *action accessible* if any split extension in $\mathbb{C}$ admits a morphism in a faithful one. This assignment is not unique, but we will show that it can be given in a canonical way (see Corollary 2.8), as it happens in groups, where this procedure is obtained by zeroing the elements acting trivially.

The relevance of action accessibility was pointed out in the very recent paper [12] by Bourn and Montoli, where action accessible categories which are Barr-exact turn out to be a good context for an internal version of Schreier–Mac Lane theorem on obstructions to extensions. The reason relies on a very important property of action accessible categories: the existence of centers, and more generally of centralizers. In fact, it was proved in [11] that in an action accessible category, for any normal subobject X of A, there exists a normal subobject $Z(X, A)$ of A, cooperating with X in A in the sense of Definition 2.1, which is larger than any other subobject of A having the same property. Moreover, Bourn and Janelidze showed that a similar property holds for any equivalence relation R on A: namely, there exists a largest equivalence relation $E_A(R)$, such that $E_A(R)$ and R centralize each other in the sense of Smith (see Definition 3.2). Furthermore, they also showed that in a homological action accessible category two normal subobjects cooperate if and only if the corresponding equivalence relations centralize each other.

Note that the centralizer of an equivalence relation R is the largest among equivalence relations that centralize R, while the centralizer of a normal subobject X is the largest among (not necessarily *normal*) subobjects that cooperate with X. This is a stronger property, which has no counterpart in terms of equivalence relations.

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In order to face this lack of analogy, in this paper we introduce a non-symmetric notion of centralization between a relation S and an equivalence relation R, which coincides with Smith centralization in the case S is an equivalence relation too (see Proposition 3.6). It turns out (see Proposition 4.1) that in any action accessible category the centralizer of an equivalence relation R is an equivalence relation which is the largest among all the relations S that centralize R in the non-symmetric sense, so that it can be viewed as a *non-symmetric centralizer*, according to Definition 4.3. This result shows the analogy existing in the action accessible case between centralizers of equivalence relations and centralizers of normal subobjects. This stronger property actually allows a characterization of action accessible categories via the existence of centralizers. We show that, in any homological category with (non-symmetric) centralizers for equivalence relations, faithful split extensions are exactly those with trivial centralizer, in the sense of Proposition 4.4. This is a property of action accessible categories (as shown in [6] for the more general case of faithful groupoids). Actually, we prove that, for an exact pointed protomodular category, the existence of (non-symmetric) centralizers is equivalent to action accessibility.

1. Action accessible categories

Let $\mathbb{C}$ be a pointed protomodular category, essentially a category where split short five lemma holds (see [1] for the definition and several characterizations). Given an object $X \in \mathbb{C}$, a split extension with kernel X is a diagram:

$$X \xrightarrow{x} A \underset{s}{\overset{p}{\rightleftarrows}} B$$

such that $ps = 1_B$ and $x = \ker p$. We will denote such a split extension by (B, A, p, s, x). Given another split extension (D, C, q, t, k) with the same kernel X, a morphism $(g, f) : (B, A, p, s, x) \to (D, C, q, t, k)$ is a pair (g, f) of arrows such that $k = fx$, $qf = gp$ and $fs = tg$ in

$$\begin{array}{ccccc} X & \xrightarrow{x} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B \\ \Vert & & {\scriptstyle f}\downarrow & (*) & \downarrow{\scriptstyle g} \\ X & \xrightarrow{k} & C & \underset{t}{\overset{q}{\rightleftarrows}} & D \end{array} \qquad (1)$$

Split extensions with fixed kernel X and morphisms between them form a category, which we will denote by $\mathrm{SplExt}_{\mathbb{C}}(X)$, or simply by $\mathrm{SplExt}(X)$.

Remark 1.1. Note that, whenever it exists, the morphism f in diagram (1) is uniquely determined by the rest of the data, and this follows from the fact that the pair (x, s) is jointly strongly epimorphic by protomodularity. The same property also makes condition $qf = gp$ redundant, since it can be deduced by the others. Moreover, in every diagram of the form (1) the square $(*)$ is a pullback by protomodularity, since p and q have the same kernel X.

It is well known that any split extension (B, A, p, s, x) in the category of groups gives rise to an action of the group B on X, namely the action induced by the conjugation action of $s(B)$ on $x(X)$ in A. This correspondence holds, more generally, in every pointed category $\mathbb{C}$ with finite limits and finite coproducts, as explained in [3], where the notion of *internal object action* was introduced. We recall here the definition: for $\mathrm{Pt}_B(\mathbb{C})$ being the category of points over B in $\mathbb{C}$, we have an adjunction:

$$\mathbb{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Pt}_B(\mathbb{C}) \quad (F \dashv G) \qquad (2)$$

where, on objects:

$$X \overset{F}{\longmapsto} \left(B+X \underset{\iota_B}{\overset{[1,0]}{\rightleftarrows}} B \right) \quad \text{and} \quad \left(A \underset{s}{\overset{p}{\rightleftarrows}} B \right) \overset{G}{\longmapsto} \ker p$$

And the corresponding monad $GF(-)$ on $\mathbb{C}$ is denoted by $B\flat(-)$ (so, as an object, we denote $B\flat X = \ker[1, 0]$).

Definition 1.2. The algebras for the monad $B\flat(-)$ induced by the adjunction (2) above are called *internal B-actions* in $\mathbb{C}$. We denote by $\mathbb{C}^B$ the category of these algebras.

The comparison functor $\mathrm{Pt}_B(\mathbb{C}) \to \mathbb{C}^B$ associates with every point (A, p, s) a B-action ξ as described in the following diagram (where X is the kernel of p and ξ is induced by the universal property of X):

$$\begin{array}{ccccc} B\flat X & \xrightarrow{\ker[1,0]} & B+X & \underset{\iota_B}{\overset{[1,0]}{\rightleftarrows}} & B \\ {\scriptstyle \xi}\,\vdots & (*) & \downarrow{\scriptstyle [s,k]} & & \| \\ X & \xrightarrow{k} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B \end{array}$$

When $\mathbb{C}$ is the category of groups, given a group action ξ of B over K, we can always associate with it a semidirect product $K \rtimes_\xi B$ and then a point $X \rtimes_\xi B \underset{\langle 0,1\rangle}{\overset{\pi_B}{\rightleftarrows}} B$. It turns out that the corresponding B-action is exactly the starting ξ. However, unlike the case of groups, or, more generally, the case of a semi-abelian category, the correspondence between actions and split extensions is not a category equivalence in general. And when it is not an equivalence, the notion of split extension of an object B with kernel X can be considered as an alternative notion of B-action on X. In particular, the classical notion of faithful action suggests:

Definition 1.3 (*[11], Definition 1.2*)**.** An object in $\mathrm{SplExt}(X)$ is said to be *faithful* if any object in $\mathrm{SplExt}(X)$ admits at most one morphism into it.

The term *faithful* is justified indeed by the fact that, if $\mathbb{C}$ is the category of groups, the notion of faithful split extension corresponds, via the canonical equivalence between split extensions and actions given by the semidirect product construction, to the classical notion of faithful action.

Definition 1.4 (*[11], Definition 2.1*)**.** Let $\mathbb{C}$ be a pointed protomodular category. An object in $\mathrm{SplExt}_{\mathbb{C}}(X)$ is said to be *accessible* if it admits a morphism into a faithful one. If, for any $X \in \mathbb{C}$, every object in $\mathrm{SplExt}_{\mathbb{C}}(X)$ is accessible, we say that $\mathbb{C}$ is an *action accessible* category.

In particular, if $\mathbb{C}$ is *action representative* in the sense of [2], then it is action accessible. The converse is not true: indeed, for example, the category of rings is action accessible ([11], Proposition 2.2) but not action representative, as shown in [4].

Other examples of action accessible categories can be obtained from the following result:

Proposition 1.5 (*[11], Proposition 2.3*)**.** *If $\mathbb{C}$ is an action accessible homological category and $\mathbb{B}$ is a Birkhoff subcategory of $\mathbb{C}$, then $\mathbb{B}$ is also action accessible.*

Moreover Montoli proved in [18] that every *category of interest* in the sense of Orzech [19] is action accessible. Categories of interest are: the categories of groups, rings, Lie and Leibniz algebras, Poisson algebras and others, but not, for example, Jordan algebras. Finally, Bourn proved in [6] that all topological models of action accessible varieties are action accessible.

2. Properties of action accessible categories: centralizers

The notion of cooperating morphisms we refer to was introduced by Huq [16] under the name of commuting morphisms and later developed by Bourn [5] in the context of unital categories. For a complete treatment we also refer to [1].

Definition 2.1. Let $\mathbb{C}$ be a pointed protomodular category. Two morphisms f and g with the same codomain *cooperate* if there exists a morphism φ making the following diagram commutative:

$$\begin{array}{ccccc} X & \xrightarrow{\langle 1,0\rangle} & X\times Y & \xleftarrow{\langle 0,1\rangle} & Y \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \varphi} & \swarrow{\scriptstyle g} & \\ & & Z & & \end{array}$$

When it exists, the morphism φ is unique and it is called the *cooperator* of f and g.

In the case X and Y are subobjects of Z, we say that X and Y *cooperate* in Z (and we write $[X, Y]_Z = 0$) when the two inclusions cooperate.

Lemma 2.2. *Given a split extension (B, A, p, s, x) of X in a pointed protomodular category and a subobject $J \xrightarrow{j} B$, then the following are equivalent:*

1. $[X, J]_A = 0$*, that is, there exists a cooperator $X \times J \xrightarrow{\varphi} A$ for the maps x and sj;*

2. *j gives rise to a morphism* (j, φ) *of split extensions:*

$$\begin{array}{ccccc} X & \xrightarrow{\langle 1,0\rangle} & X \times J & \underset{\langle 0,1\rangle}{\overset{\pi_J}{\rightleftarrows}} & J \\ \| & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{x} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B \end{array} \tag{3}$$

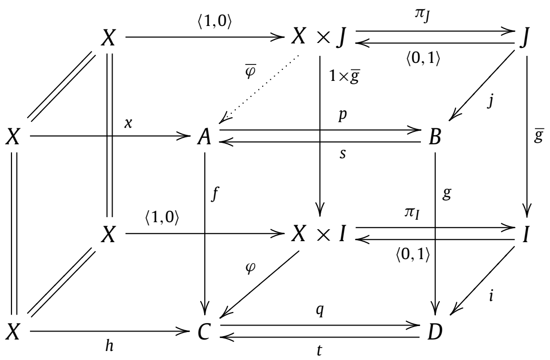
Proof. Trivial by Remark 1.1. □

Lemma 2.3. *For every morphism of split extensions in a pointed protomodular category:*

$$\begin{array}{ccccc} X & \xrightarrow{x} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B \\ \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{k} & C & \underset{t}{\overset{q}{\rightleftarrows}} & D \end{array}$$

if $I \xrightarrow{i} D$ *is a subobject of D with* $[X, I]_C = 0$*, then the pullback* $J \xrightarrow{j} B$ *of i along g is such that* $[X, J]_A = 0$.

Proof. In the diagram



since $[X, I]_C = 0$, by Lemma 2.2, the bottom right hand square is a pullback, as well as the rear one. By the universal property of pullbacks, there exists a morphism $\overline{\varphi} : X \times J \to A$ making the top right hand square a pullback. Again by Lemma 2.2, we can conclude that $[X, J]_A = 0$. □

Lemma 2.4. *For every morphism of split extensions in a pointed protomodular category:*

$$\begin{array}{ccccc} X & \xrightarrow{x} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B \\ \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{k} & C & \underset{t}{\overset{q}{\rightleftarrows}} & D \end{array}$$

the kernel of g is a normal subobject of A that cooperates with X in A.

Proof. Let us call $Z \xrightarrow{z} B$ the kernel of g: it is the pullback of 0 in D, and since trivially $[X, 0]_C = 0$, by Lemma 2.3, $[X, Z]_A = 0$. Furthermore the map sz is the kernel of f and then Z is normal in A. □

From now on in this section, assume that the category $\mathbb{C}$ is action accessible. The following proposition is inspired by the results in [11] and gives a definition of centralizer relative to a split extension.

Proposition 2.5. *Given any morphism from a split extension in* $\mathbb{C}$ *to a faithful one:*

$$\begin{array}{ccccc} X & \xrightarrow{x} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B \\ \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{k} & C & \underset{t}{\overset{q}{\rightleftarrows}} & D \end{array}$$

the kernel of g is the largest subobject of B cooperating with X in A. We will call this object $Z(X, B)$*, the* centralizer *of X in B.*

Proof. Such a morphism always exists in action accessible categories by definition and, by Lemma 2.4, $Z(X, B)$ is a normal subobject of A cooperating with X. Conversely, let $J \xrightarrow{j} B$ be a subobject of B such that $[X, J]_A = 0$, that is, there exists a cooperator $X \times J \xrightarrow{\varphi} A$ for the maps x and sj. Consider the following diagram:

$$\begin{array}{ccccc}
X & \xrightarrow{\langle 1,0\rangle} & X \times J & \underset{\langle 0,1\rangle}{\overset{\pi_J}{\rightleftarrows}} & J \\
\| & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle j} \\
X & \xrightarrow{x} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B \\
\| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{k} & C & \underset{t}{\overset{q}{\rightleftarrows}} & D
\end{array}$$

By Lemma 2.2, the pair (j, φ) is a morphism of split extensions, and so is (g, f) by our assumption; therefore $(gj, f\varphi)$ is a morphism of split extensions by composition. On the other hand, there is another morphism between the same two extensions, namely:

$$\begin{array}{ccccc}
X & \xrightarrow{\langle 1,0\rangle} & X \times J & \underset{\langle 0,1\rangle}{\overset{\pi_J}{\rightleftarrows}} & J \\
\| & & \downarrow{\scriptstyle k\pi_X} & & \downarrow{\scriptstyle 0} \\
X & \xrightarrow{k} & C & \underset{t}{\overset{q}{\rightleftarrows}} & D
\end{array}$$

and since the lower split extension is faithful, $(f\varphi, gj) = (k\pi_X, 0)$ and in particular $gj = 0$, so J is contained in the kernel of g. $\square$

As a corollary, we recover the result of Proposition 5.2 in [11], which gives a construction of the classical centralizer of a normal subobject:

Corollary 2.6. *For any normal subobject $X \xrightarrow{x} A$ of an object A, let (R, r_0, r_1, s_0) be the equivalence relation on A associated with X, and consider a morphism*

$$\begin{array}{ccccc}
X & \xrightarrow{\langle 0,x\rangle} & R & \underset{s_0}{\overset{r_0}{\rightleftarrows}} & A \\
\| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{k} & C & \underset{t}{\overset{q}{\rightleftarrows}} & D
\end{array}$$

of split extensions, where the lower split extension is faithful.
Then the kernel of g is the centralizer $Z(X, A)$ of X in A, that is, the largest subobject of A cooperating with X in A.

Proof. We already know, by Proposition 2.5, that the kernel of g is the largest subobject of A cooperating with X in R. But, for any $Y \xrightarrow{y} A$, $[X, Y]_R = 0$ if and only if $[X, Y]_A = 0$. Indeed, if η is the cooperator of $\langle 0, x\rangle$ and $s_0 y$, then $r_1\eta$ is the cooperator of x and y. Vice versa, if φ is the cooperator of x and y, then $\langle 0, x\rangle$ and $s_0 y$ cooperate by means of the arrow $\langle y\pi_Y, \varphi\rangle$. $\square$

Remark 2.7. Given any $Y \xrightarrow{y} A$, thanks to the following morphism:

$$\begin{array}{ccccc}
X & \xrightarrow{x'} & P & \underset{s'}{\overset{p'}{\rightleftarrows}} & Y \\
\| & & \downarrow{\scriptstyle y'} & (*) & \downarrow{\scriptstyle y} \\
X & \xrightarrow{\langle 0,x\rangle} & R & \underset{s_0}{\overset{r_0}{\rightleftarrows}} & A
\end{array}$$

where $(*)$ is constructed as a pullback and $x' = \ker(p')$, we can write $Z(X, Y) = Z(X, A) \wedge Y$. As a special case we recover the centralizer of Proposition 2.5 as $Z(X, B) = Z(X, A) \wedge B$.

The result in Proposition 2.5 is independent from the chosen faithful extension and, if moreover the category $\mathbb{C}$ is regular, it leads to the following construction (see Proposition 2.1 of [12] for an analogous result for the case of equivalence relations).

Corollary 2.8. *Every split extension in a homological action accessible category* $\mathbb{C}$ *admits a morphism onto a* canonical faithful extension*:*

$$\begin{array}{ccccc} X & \xrightarrow{x} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B \\ \| & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_0} \\ X & \xrightarrow{\bar{x}} & T_1 & \underset{\bar{s}}{\overset{\bar{p}}{\rightleftarrows}} & T_0 \end{array}$$

with the property that any other morphism (f, g) *from* $X \xrightarrow{x} A \underset{s}{\overset{p}{\rightleftarrows}} B$ *to a faithful extension factors through* (τ_1, τ_0)*, which is its regular epi part.*

In other words, the category of faithful split extensions with fixed kernel X *is a (regular-epi-)reflective subcategory of* $\mathrm{SplExt}_{\mathbb{C}}(X)$.

Proof. Since $\mathbb{C}$ is action accessible, the above extension always admits a morphism to a faithful one:

$$\begin{array}{ccccc} X & \xrightarrow{x} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B \\ \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{k} & C & \underset{t}{\overset{q}{\rightleftarrows}} & D \end{array}$$

As in Lemma 2.4, let us call z the kernel of g. Now take the (regular epi, mono) factorizations (τ_0, m_0) and (τ_1, m_1) of g and f respectively. Since sz is a normal subobject of A by Lemma 2.4, then $(\tau_1, \tau_0) = (\mathrm{coker}(sz), \mathrm{coker}(z))$ and, by the property of cokernels, a split extension $(\overline{p}, \overline{s})$ is induced:

$$\begin{array}{ccccc} X & \xrightarrow{x} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B \\ \| & & \downarrow{\scriptstyle \tau_1} & (*) & \downarrow{\scriptstyle \tau_0} \\ X & \xrightarrow{\bar{x}} & T_1 & \underset{\bar{s}}{\overset{\bar{p}}{\rightleftarrows}} & T_0 \end{array}$$

Since $\ker(\tau_1) = \ker(\tau_0) = Z(X, B)$, by protomodularity $(*)$ is a pullback and $\ker(\overline{p}) = X$.

The split extension $(T_0, T_1, \bar{p}, \bar{s}, \bar{x})$ is faithful, since it has a monomorphism (m_1, m_0) into a faithful one. By Proposition 2.5, the kernel $z : Z(X, B) \to B$ of g is independent from the chosen faithful extension and then any other morphism (f', g') from (B, A, p, s, x) to a faithful extension factors through (τ_1, τ_0), which is its regular epi part. □

The failure of the property of having normal centralizers provide us a criterion to prove that a given category is not action accessible. For instance, the category of Jordan algebras is not action accessible, as the following example shows.

Example 2.9. Let A be the Jordan algebra given by the vector space $A = \mathbb{R}x \oplus \mathbb{R}y \oplus \mathbb{R}t$, endowed with a distributive and commutative product whose multiplication table for generators is the following:

$*$	x	y	t
x	x	0	y
y	0	0	x
t	y	x	t

The subobject $K = \mathbb{R}x \oplus \mathbb{R}y$ generated by x and y is a normal subobject, i.e. an ideal of A in the sense of Higgins [15]. On the other hand, $C(K, A) = \mathbb{R}y$, which is the largest subobject cooperating with K, is not normal (since $y * t = x$), while it should be so if the category of Jordan algebras were action accessible.

Example 2.10. The second example is given by the category $\mathbb{C}$ whose objects are groups with an additional binary associative and distributive operation $*$, and morphisms are group homomorphisms preserving $*$. As before, normal subobjects are ideals in the sense of Higgins [15]. Consider in $\mathbb{C}$ the object given by the additive group:

$$B = \langle x, y, t \rangle \qquad \text{with } x + y = y + x,\ y + t = t + y$$

(as a group, B is isomorphic to the product of the free group generated by $\{x, t\}$ with the free group generated by $\{y\}$) endowed with an associative and distributive operation $*$, whose multiplication table for generators is the following:

$*$	x	y	t
x	x	0	0
y	0	y	t
t	0	t	t

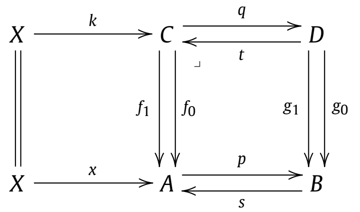
If K is the ideal of B generated by x, then the largest subobject cooperating with K is the subobject $C(K, B) = \langle y \rangle$ generated by y, which is not normal, since $y * t = t \notin C(K, B)$. And this proves that $\mathbb{C}$ is not action accessible.

It is worth observing that the category of Example 2.9 satisfies all the axioms of a category of interest in the sense of [19] except for (7), while the category of Example 2.10 satisfies all the axioms but (8). In fact, these examples were presented in [14] in order to show that axioms (7) and (8) are strictly necessary to have normal centralizers and, at the same time, to recover Huq commutator as Higgins commutator (see Theorem 5.3.6 in [14]).

3. A non-symmetric version of centralization of relations

Throughout this section let $\mathbb{C}$ be a homological category. Since in this section we deal with relations, we state here the following lemma, which will be useful later:

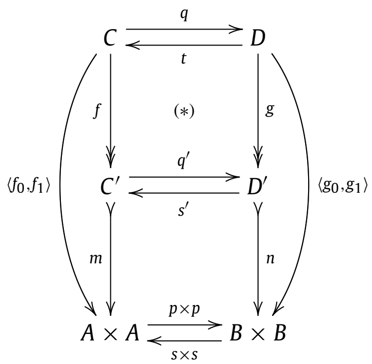
Lemma 3.1. *Every pair of parallel morphisms in* SplExt(X)*:*



(4)

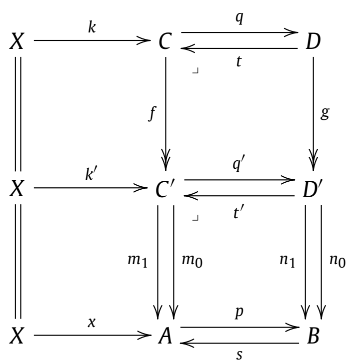
factors through a pair of jointly monic pairs (i.e. relations on A and B respectively).

Proof. First of all recall that, by protomodularity, the right hand squares with parallel arrows of the same index are pullbacks (q and p having the same kernel X). Now take the (regular epi, mono) factorizations of the pair $(\langle g_0, g_1\rangle, \langle f_0, f_1\rangle)$:



We are going to prove that the commutative square $(*)$, formed by $q'f = gq$, above is a pullback. As already observed, the right hand squares in diagram (4) are pullbacks, so that f_0 and g_0 have the same kernel, and the same holds for f_1 and g_1. Moreover, for any two morphisms u and v, $\ker\langle u, v\rangle \cong \ker u \wedge \ker v$, thus $\ker\langle f_0, f_1\rangle \cong \ker f_0 \wedge \ker f_1 \cong \ker g_0 \wedge \ker g_1 \cong \ker\langle g_0, g_1\rangle$. But by construction $f = \mathrm{coker}(\ker\langle f_0, f_1\rangle)$ and $g = \mathrm{coker}(\ker\langle g_0, g_1\rangle)$, so they are two regular epimorphisms with isomorphic kernels, and this implies, by protomodularity, that the square $(*)$ is a pullback.

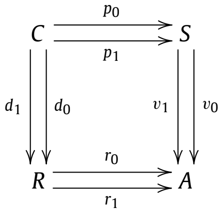
As a consequence, $\ker q' = X$ and diagram (4) factorizes as follows:



where $m = \langle m_0, m_1\rangle$, $n = \langle n_0, n_1\rangle$, and this completes the proof, since by construction $((n_0, n_1), (m_0, m_1))$ is a pair of jointly monic pairs. □

We recall here the definition of equivalence relations centralizing each other in the sense of Smith (see for example [7] or [13]):

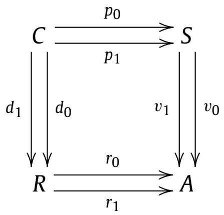
Definition 3.2. Two equivalence relations (R, r_0, r_1, s_0) and (S, v_0, v_1, u) on A centralize each other, and we will write $[R, S]_A = 0$, if there exists a centralizing double relation between them, that is an equivalence relation C on both R and S such that, in the diagram below, the four squares where parallel arrows have the same index are pullbacks:



We can extend this definition to the non-symmetric case, where only one of the relations is requested to be an equivalence.

Definition 3.3. Let S be a relation on A and R an equivalence relation on A. We will say that S centralizes R, and we will write $[R, S[_A = 0$, if there exists an equivalence relation (C, p_0, p_1, t_0) on S with (C, d_0, d_1) a relation on R such that

1. In the diagram below, the four squares where parallel arrows have the same index are pullbacks:

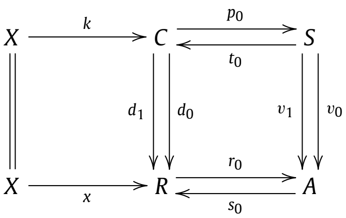


2. If $k : X \to C$ is a kernel of p_0 (or p_1 equivalently), then $d_0 k = d_1 k$.

The fact that C is a relation on R comes for free because d_0 and d_1 are jointly monic, since they are pullbacks of v_0 and v_1 respectively.

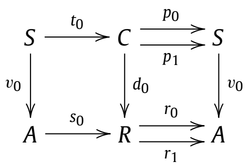
As easy consequences of Definition 3.3, we obtain:

Lemma 3.4. *If $[R, S[_A = 0$, in the following diagram (where t_0 is the common section of p_0 and p_1 and s_0 is the common section of r_0 and r_1):*



$x = d_0 k = d_1 k$ is a kernel of r_0 and both (v_0, d_0) and (v_1, d_1) are morphisms of split extensions.

Proof. The left hand squares commute by definition. Moreover, in the diagram below:



the whole rectangle and the right hand squares commute, so the left hand square also commute since r_0 and r_1 are jointly monic. The same argument holds by replacing (v_0, d_0) with (v_1, d_1). □

Lemma 3.5. *If $[R, S[_A = 0$ and $h : Y \to C$ is a kernel of d_0, then $p_0 h = p_1 h$.*

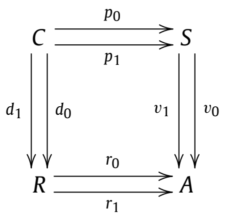
Proof. We claim that $t_0 p_0 h = h$. Indeed, since p_0 and d_0 are jointly monic, it follows from $p_0 t_0 p_0 h = p_0 h$, and $d_0 t_0 p_0 h = s_0 v_0 p_0 h = s_0 r_0 d_0 h = s_0 r_0 0 = 0 = d_0 h$. After that we have $p_0 h = p_0 t_0 p_0 h = p_1 t_0 p_0 h = p_1 h$. □

The following proposition explains the link between the classical case and this non-symmetric version when we have two equivalence relations.

Proposition 3.6. *Let R and S be equivalence relations on A, then:*

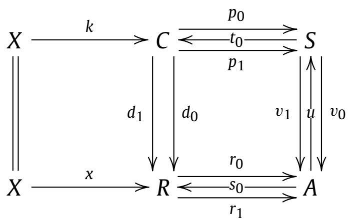
$$[R, S]_A = 0 \iff [R, S[_A = 0$$

Proof. Suppose that R and S centralize each other in the classical sense, then we have an equivalence relation C on both R and S:



where the four commutative squares are pullbacks. Using the reflexivity of C as equivalence relation on R, as in Lemma 3.5, Condition 2 in Definition 3.3 comes for free. So the implication $[R, S]_A = 0 \Rightarrow [R, S[_A = 0$ is proved.

Vice versa, suppose S centralizes R in the sense of Definition 3.3:



To conclude the proof we have to show that C is an equivalence relation on R, and in order to do this, since the category is Mal'tsev, it suffices to exhibit a common section for d_0 and d_1.

Consider the pullback given by the four arrows of index 0. Since $v_0 u r_0 = r_0$, then there exists a unique arrow $c : R \to C$ such that:

$$\begin{cases} p_0 c = u r_0 \\ d_0 c = 1_R \end{cases}$$

Moreover,

$$\begin{cases} p_0 c s_0 = u r_0 s_0 = u = p_0 t_0 u \\ d_0 c s_0 = s_0 = s_0 v_0 u = d_0 t_0 u \end{cases} \qquad \text{gives } c s_0 = t_0 u$$

because p_0 and d_0 are jointly monic. Furthermore,

$$\begin{cases} p_0 c x = u r_0 x = 0 = p_0 k \\ d_0 c x = x = d_0 k \end{cases} \qquad \text{gives } c x = k$$

Now recall that the category is protomodular, so x and s_0 are jointly epic and

$$\begin{cases} d_1 c s_0 = d_1 t_0 u = s_0 v_1 u = s_0 \\ d_1 c x = d_1 k = d_0 k = d_0 c x = x \end{cases} \qquad \text{gives } d_1 c = 1_R$$

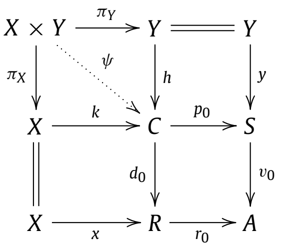
Therefore c is a common section for d_0 and d_1, so that C is an equivalence relation on R and this proves the implication $[R, S[_A = 0 \Rightarrow [R, S]_A = 0$. □

Proposition 3.7.

$$[R, S[_A = 0 \Rightarrow [X, Y]_A = 0$$

That is, in the notation of Definition 3.3, with $x = d_0 k : X \to R$ and $y = p_0 h : Y \to S$ being the kernels of r_0 and v_0 respectively, the morphisms $r_1 x : X \to A$ and $v_1 y : Y \to A$ cooperate.

Proof. Consider the following commutative diagram:



By hypothesis the right lower square is a pullback, and since $v_0 y \pi_Y = 0 = r_0 x \pi_X$ there exists a unique $\psi : X \times Y \to C$ such that:

$$\begin{cases} p_0\psi = y\pi_Y = p_0 h \pi_Y \\ d_0\psi = x\pi_X = d_0 k \pi_X \end{cases}$$

Now observe that ψ is the cooperator of h and k in C, since $\psi\langle 1, 0\rangle = k$ (and similarly $\psi\langle 0, 1\rangle = h$) because:

$$\begin{cases} p_0\psi\langle 1, 0\rangle = p_0 h \pi_Y \langle 1, 0\rangle = 0 = p_0 k \\ d_0\psi\langle 1, 0\rangle = d_0 k \pi_X \langle 1, 0\rangle = d_0 k \end{cases}$$

and p_0, d_0 are jointly monic. Now define $\eta = r_1 d_1 = v_1 p_1$, then we have:

$$\begin{cases} \eta\psi\langle 1, 0\rangle = r_1 d_1 \psi\langle 1, 0\rangle = r_1 d_1 k = r_1 x \\ \eta\psi\langle 0, 1\rangle = v_1 p_1 \psi\langle 0, 1\rangle = v_1 p_1 h = v_1 y \end{cases}$$

that is, $[X, Y]_A = 0$ with $\eta\psi$ being the needed cooperator. □

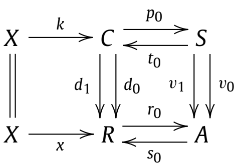
4. A characterization of action accessibility

Now we are going to find a characterization of faithful split extensions by means of centralizers fulfilling a stronger property, based on the previous definition of non-symmetric centralization.

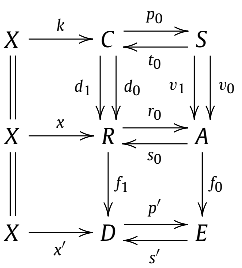
In [11] the authors showed that in homological action accessible categories any equivalence relation R admits a *centralizer* $E_A(R)$, defined as the largest equivalence relation centralizing R in the sense of Definition 3.2. Actually, a stronger version of this property holds:

Proposition 4.1. *Let $\mathbb{C}$ be a homological action accessible category. Then, for any equivalence relation R, its centralizer $E_A(R)$ contains any relation S on A with $[R, S[_A = 0$.*

Proof. Let (R, r_0, r_1, s_0) be an equivalence relation on A in $\mathbb{C}$ and consider the associated split epimorphism (R, r_0, s_0). Given any relation S on A with $[R, S[_A = 0$, by Lemma 3.4, we know that the diagram



gives rise to a pair of morphisms of split extensions on X. Now, if (f_0, f_1) is a morphism into a faithful split extension of X, then $f_0 v_0 = f_0 v_1$:



This means that S must be contained in the kernel pair $R[f_0]$ of f_0. But in [11] it is shown that this kernel pair coincides with the centralizer $E_A(R)$. □

Observe that the normal subobject of A associated with $E_A(R)$ actually coincide with $Z(X, A)$, as proved in [11], Proposition 5.2.

We will see that this stronger property of centralizers of equivalence relations is characteristic of action accessible exact categories. In the following example, we exhibit an equivalence relation R in a semi-abelian category with a centralizer (in the sense of Bourn and Janelidze) which does not contain a relation S with $[R, S[_A = 0$.

Example 4.2. Consider the Jordan algebra $A = \mathbb{R}x \oplus \mathbb{R}y \oplus \mathbb{R}t$ of Example 2.9. We take the equivalence relation R associated with the normal subobject $K = \mathbb{R}x \oplus \mathbb{R}y$. Since we know that the largest subobject of A that cooperates with K is $\mathbb{R}y$, then the only normal subobject of A cooperating with K is $\{0\}$, so that $E_A(R)$ is given by the discrete relation Δ_A.

Consider now the equivalence relation S associated with $\mathbb{R}y$ in K, that is $S = \{(k, k') \in K \times K \mid k - k' \in \mathbb{R}y\}$. This is also a relation on A, but not an equivalence relation (since it is not reflexive). Since Δ_K is normal in S, in the following commutative diagram (where v_i, for $i = 0, 1$, is any of the two projections of S on A):

$$\begin{array}{ccccc}
\Delta_K & \longrightarrow & S & \xrightarrow{q} & S/\Delta_K \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle v_i} & & \downarrow{\scriptstyle 0} \\
K & \longrightarrow & A & \xrightarrow{p} & A/K
\end{array}$$

the right hand square is a pullback by protomodularity since p and q are regular epimorphisms with isomorphic kernels. Hence $[R, S[= 0$ via the kernel pair of q.

Consequently, we introduce the following definition:

Definition 4.3. A *non-symmetric centralizer* for an equivalence relation R on A is an equivalence relation $E_A(R)$ on A such that:

1. $[R, E_A(R)]_A = 0$
2. $E_A(R)$ contains any relation S on A with $[R, S[_A = 0$

From now on, with the term *centralizer* we will refer to the *non-symmetric centralizer* defined above.

Given a normal subobject X of A, we denote by $Z_A(X)$ the normal subobject of A associated with $E_A(R)$, where R is the equivalence relation on A associated with X. Obviously $Z_A(X)$ cooperates with X in A (since $[R, E_A]_A = 0$ implies that the corresponding kernels cooperate).

Now we are ready to give our characterization of faithful split extensions:

Proposition 4.4. *Let $\mathbb{C}$ be a homological category with centralizers for equivalence relations. Given a split extension (B, A, p, s, x), we can define E_B as the pullback of $E_A(R[p])$ along $s \times s$:*

$$\begin{array}{ccc}
E_B & \longrightarrow & E_A \\
{\scriptstyle\langle w_0, w_1\rangle}\downarrow & & \downarrow{\scriptstyle\langle z_0, z_1\rangle} \\
B \times B & \xrightarrow{s \times s} & A \times A
\end{array}$$

and its associated normal subobject Z_B as the pullback of $Z_A(X)$ along s.

The following conditions are equivalent:

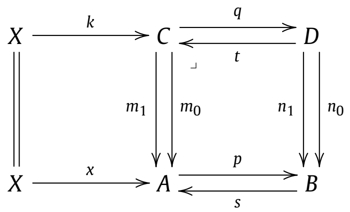
1. $X \xrightarrow{x} A \underset{s}{\overset{p}{\rightleftarrows}} B$ *is faithful;*
2. $E_B = \Delta_B$ *(or, equivalently, $Z_B = 0$).*

Proof. Suppose $Z_B \neq 0$. Since Z_B considered as a subobject of A is less or equal to Z_A, we have $[X, Z_B]_A = 0$. Consequently, thanks to Lemma 2.2, Z_B gives rise to two (different) morphisms of split extension into (B, A, p, s, x):

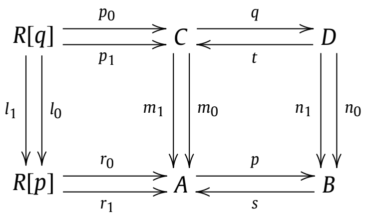
$$\begin{array}{ccccc}
X & \xrightarrow{\langle 1,0\rangle} & X \times Z_B & \underset{\langle 0,1\rangle}{\overset{\pi_{Z_B}}{\rightleftarrows}} & Z_B \\
\| & & {\scriptstyle\varphi}\downarrow\downarrow{\scriptstyle x\pi_X} & & {\scriptstyle j}\downarrow\downarrow{\scriptstyle 0} \\
X & \xrightarrow{x} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B
\end{array}$$

This means that (B, A, p, s, x) is not faithful.

Vice versa, let $E_B = \Delta_B$ and suppose there exist two morphisms of split extension into (B, A, p, s, x):



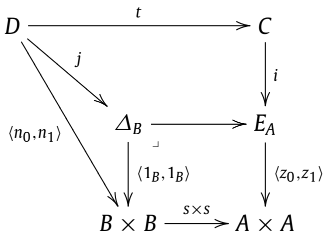
By Lemma 3.1, we can assume that (m_0, m_1) and (n_0, n_1) are jointly monic pairs. This means that C is a relation on A. Since both the right hand squares involving q and p are pullbacks, the corresponding morphisms between the kernel pairs $R[q]$ and $R[p]$ give rise to four pullbacks:



Since $R[q]$ is an equivalence relation on C, in order to show that $[R[p], C[_A = 0$, we need to verify Condition 2 of Definition 3.3. Consider $\langle 0, k\rangle : X \to R[q]$ as a kernel of p_0. $l_0\langle 0, k\rangle = \langle 0, x\rangle$, because

$$\begin{cases} r_0 l_0\langle 0, k\rangle = m_0 p_0\langle 0, k\rangle = 0 = r_0\langle 0, x\rangle \\ r_1 l_0\langle 0, k\rangle = m_0 p_1\langle 0, k\rangle = m_0 k = x = r_1\langle 0, x\rangle \end{cases}$$

Since also $m_1 k = x$, the same argument shows that $l_1\langle 0, k\rangle = \langle 0, x\rangle$, so that also Condition 2 is fulfilled and we can conclude that $[R[p], C[_A = 0$ and then $C \leq E_A(R[p])$. So there exists a monomorphism $i : C \to E_A$ such that $z_0 i = m_0$ and $z_1 i = m_1$ and this induces a monomorphism $j : D \to E_B = \Delta_B$:



with $n_0 = 1_B j = n_1$. This means that (B, A, p, s, x) is faithful. □

Now we are ready to state the main result of the present paper. Before doing this, we recall a useful characterization of centralization of equivalence relations (Theorem 5.2 in [8], adapted to the case where normal subobjects coincide with kernels):

Proposition 4.5. *In a pointed exact protomodular category $\mathbb{C}$, let R and S be two equivalence relations on an object A and $y : Y \to A$ the normal subobject associated with S. Then $[R, S] = 0$ if and only if $s_0 y : Y \to R$ is normal.*

Theorem 4.6. *Let $\mathbb{C}$ be a pointed exact protomodular category, so that kernel pairs coincide with equivalence relations. The following are equivalent:*

1. *$\mathbb{C}$ is action accessible;*
2. *$\mathbb{C}$ has centralizers for equivalence relations (in the sense of Definition 4.3).*

Proof. We have already seen in Proposition 4.1 that homological action accessible categories have centralizers. Now we are going to prove that if in $\mathbb{C}$ any equivalence relation has a centralizer, then $\mathbb{C}$ is action accessible.

We have to show that any split extension (B, A, p, s, x) admits a morphism into a faithful one. If the given one is itself faithful, simply take the identity. If not, by Proposition 4.4, $E_B \neq \Delta_B$, hence $E_A \neq \Delta_A$. Thanks to the following inclusion:

$$\begin{array}{ccccc}
X & \xrightarrow{x} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B \\
\Vert & & \Big\downarrow{\scriptstyle \langle sp, 1_A\rangle} & & \Big\downarrow{\scriptstyle s} \\
X & \xrightarrow[\langle 0,x\rangle]{} & R[p] & \underset{s_0}{\overset{r_0}{\rightleftarrows}} & A
\end{array}$$

it suffices to find a morphism into a faithful split extension from the lower canonical split extension.

By hypothesis, $[R[p], E_A]_A = 0$, so there exists a double centralizing equivalence relation C:

$$\begin{array}{ccc}
C & \underset{p_1}{\overset{p_0}{\rightrightarrows}} & E_A \\
{\scriptstyle d_1}\Big\downarrow\Big\downarrow{\scriptstyle d_0} & & {\scriptstyle z_1}\Big\downarrow\Big\downarrow{\scriptstyle z_0} \\
R[p] & \underset{r_1}{\overset{r_0}{\rightrightarrows}} & A
\end{array}$$

Let us consider then the coequalizers $q : A \to \overline{B}$, $\overline{q} : R[p] \to \overline{A}$ of E_A and C respectively. Since $\mathbb{C}$ is exact, we can apply Barr–Kock Theorem (see [9]) to the following diagram:

$$\begin{array}{ccc}
C & \underset{t_0}{\overset{p_0}{\rightleftarrows}} & E_A \\
{\scriptstyle d_1}\Big\downarrow\Big\downarrow{\scriptstyle d_0} & & {\scriptstyle z_1}\Big\downarrow\Big\downarrow{\scriptstyle z_0} \\
R[p] & \underset{s_0}{\overset{r_0}{\rightleftarrows}} & A \\
\Big\downarrow{\scriptstyle \overline{q}} & & \Big\downarrow{\scriptstyle q} \\
\overline{A} & \underset{\overline{s}}{\overset{\overline{p}}{\rightleftarrows}} & \overline{B}
\end{array}$$

and conclude that the lower square is a pullback. This means that $\ker \overline{p} = X$ and $(\overline{B}, \overline{A}, \overline{p}, \overline{s}, \overline{x})$ is a split extension on X. We want to show that it is faithful, applying Proposition 4.4.

Consider the following composite of morphisms in $\mathrm{SplExt}(X)$:

$$\begin{array}{ccccc}
X & \xrightarrow{\langle 0,x\rangle} & R[p] & \underset{s_0}{\overset{r_0}{\rightleftarrows}} & A \\
\Vert & & \Big\downarrow{\scriptstyle \overline{q}} & & \Big\downarrow{\scriptstyle q} \\
X & \xrightarrow{\overline{x}} & \overline{A} & \underset{\overline{s}}{\overset{\overline{p}}{\rightleftarrows}} & \overline{B} \\
\Vert & & \Big\downarrow{\scriptstyle \langle \overline{s}\overline{p}, 1_{\overline{A}}\rangle} & & \Big\downarrow{\scriptstyle \overline{s}} \\
X & \xrightarrow[\langle 0,\overline{x}\rangle]{} & R[\overline{p}] & \underset{\overline{s}_0}{\overset{\overline{r}_0}{\rightleftarrows}} & \overline{A}
\end{array}$$

Let $E_{\overline{A}}$ be the centralizer of $R[\overline{p}]$, and $z_{\overline{A}} : Z_{\overline{A}} \to \overline{A}$ the associated normal subobject. By Proposition 4.5, $\overline{s}_0 z_{\overline{A}}$ is normal. Now take the following diagram where the two squares are constructed as pullbacks:

$$\begin{array}{ccc}
K & \xrightarrow{k} & A \\
\Big\downarrow{\scriptstyle q'} & & \Big\downarrow{\scriptstyle q} \\
Z_{\overline{B}} & \xrightarrow{z_{\overline{B}}} & \overline{B} \\
\Big\downarrow{\scriptstyle s'} & & \Big\downarrow{\scriptstyle \overline{s}} \\
Z_{\overline{A}} & \xrightarrow{z_{\overline{A}}} & \overline{A}
\end{array}$$

then, by composition, s_0k results to be the pullback of $\bar{s}_0 z_{\bar{A}}$ along $\langle \overline{sp}, 1_{\bar{A}} \rangle \bar{q}$, and then it is normal in $R[p]$. So, again by Proposition 4.5, for the equivalence relation E_K associated to $k : K \to A$, we have $[E_K, R[p]]_A = 0$ and consequently $E_K \leq E_A$, since E_A is the centralizer. But this means that $qk = 0$ and then $Z_{\bar{B}} = 0$ because $(z_{\bar{B}}, q')$ is a (mono, regular epi)-factorization. Finally, by Proposition 4.4, the split extension

$$X \xrightarrow{\bar{x}} \bar{A} \underset{\bar{s}}{\overset{\bar{p}}{\rightleftarrows}} \bar{B}$$

is faithful and the proof is completed. $\square$

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