Production Theory: Accounting for Firm Heterogeneity and Technical Change

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(Article begins on next page)

Production theory: accounting for firm heterogeneity and technical change*

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Abstract

The paper presents a new framework to assess firm level heterogeneity and to study the rate and direction of technical change. Building on the analysis of revealed short-run production functions by Hildenbrand (1981), we propose the (normalized) volume of the zonotope composed by vectors-firms in a narrowly defined industry as an indicator of inter-firm heterogeneity. Moreover, the angles that the main diagonal of the zonotope form with the axes provides a measure of the rates and directions of technical change over time. The proposed framework can easily account for n-inputs and m-outputs and, crucially, the measures of heterogeneity and technical change do not require many of the standard assumptions from production theory.

JEL codes: D24; D61; C67; C81; O30

Keywords: Production theory, Heterogeneous firms, Activity Analysis, Technical change, Zonotopes, Production functions

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1 Introduction

In recent years an extremely robust evidence regarding firm– and plant– level longitudinal microdata has highlighted striking and persistent heterogeneity across firms operating in the same industry. A large body of research from different sectors in different countries (cf. Baily et al.; 1992; Baldwin and Rafiquzzaman; 1995; Bartelsman and Doms; 2000; Disney et al.; 2003; Dosi; 2007; Syverson; 2011, among many others) documents the emergence of the following "stylized facts": first, wide asymmetries in productivity across firms; second, significant heterogeneity in relative input intensities even in presence of the same relative input prices; third, high intertemporal persistence in the above properties. Fourth, such heterogeneity is maintained also when increasing the level of disaggregation, thus plausibly reducing the diversity across firms' output.

The latter property has been vividly summarized by Griliches and Mairesse (1999): "*We [...] thought that one could reduce heterogeneity by going down from general mixtures as "total manufacturing" to something more coherent, such as "petroleum refining" or "the manufacture of cement." But something like Mandelbrot's fractal phenomenon seems to be at work here also: the observed variability-heterogeneity does not really decline as we cut our data finer and finer. There is a sense in which different bakeries are just as much different from each others as the steel industry is from the machinery industry.*"

The bottom line is that firms operating in the same industry display a large and persistent degree of technological heterogeneity while there does not seem to be any clear sign that either the diffusion of information on different technologies, or the working of the competitive mechanism bring about any substantial reduction of such an heterogeneity, even when involving massive differences in efficiencies, as most incumbent theories would predict.

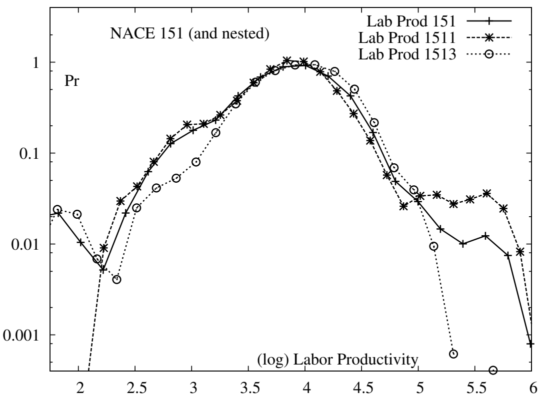
This evidence poses serious challenges not only to theory of competition and market selection, but also to any theoretical or empirical analysis which relies upon some notion of industry or sector defined as a set of production units producing under rather similar input prices with equally similar technologies, and the related notion of "the technology" of an industry represented by means of a sectoral production function. Indeed, the aggregation conditions needed to yield the canonic production functions building from the technologies of micro entities are extremely demanding, basically involving the identity of the latter up to a constant multiplier (cf. Fisher 1965 and Hulten 2001).

Note that these problems do not only concern the neoclassical production function, whose well known properties may either not fit empirical data or fit only spuriously,[1] but also non neoclassical representations of production at the industry level. If input-output coefficients *à la* Leontief (1986) are averages over a distribution with high standard deviation and high skewness, average input coefficients may not provide a meaningful representation of the technology of that industry. Moreover, one cannot take for granted that changes of such coefficients can be interpreted as indicators of technical change as they may be just caused by some changes in the distribution of production among heterogeneous units, characterized by unchanged technologies.

How does one then account for the actual technology - or, better, the different *techniques* - in such industry? Hildenbrand (1981) suggests a direct and agnostic approach which instead of estimating some aggregate production function, offers a representation of the empirical production possibility set of an industry in the short run based on actual microdata. Each production unit is represented as a point in the input-output space whose coordinates are input requirements and output levels at full capacity. Under the assumptions of divisibility and additivity of production processes,[2] the production possibility set is represented geometrically by the space formed by the finite sum of all the line segments linking the origin and the points representing

[1] Shaikh (1974), for instance, shows that Cobb-Douglas production functions with constant returns to scale, neutral technological change and marginal products equal to factor rewards in presence of constant distributional shares of labour and capital (wages and profits) tend to yield a good fit to the data for purely algebraic reasons.

[2] Already not entirely innocent assumptions: for a discussion cf. Dosi and Grazzi (2006).



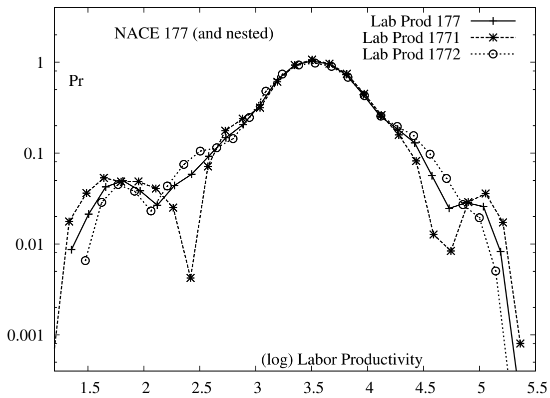


Figure 1: Empirical distribution of labor productivity in three and (nested) four-digit NACE sectors. Densities estimates are obtained using the Epanenchnikov kernel with the bandwidth set using the optimal routine described in Silverman (1986).

each production unit, called a *zonotope* (see also below). Hildenbrand then derives the actual "production function" (one should more accurately say "feasible" functions) and shows that "*short-run efficient production functions do not enjoy the well-known properties which are frequently assumed in production theory. For example, constant returns to scale never prevail, the production functions are never homothetic, and the elasticities of substitution are never constant. On the other hand, the competitive factor demand and product supply functions [...] will always have definite comparative static properties which cannot be derived from the standard theory of production*" (Hildenbrand; 1981, p. 1095).

In this paper we move a step forward and show that by further exploiting the properties of zonotopes it is possible to obtain rigorous measures of heterogeneity and technical change without imposing on data a model like that implied by standard production functions. In particular, we develop measures of technical change that take into consideration the entire *observed* production possibility set derived from *observed* heterogeneous production units, instead of considering only an *efficient frontier*. In that, our representation of industry-level dynamics bear some complementarities to non-parametric estimates of (moving) efficiency frontiers (cf. Farrell 1957, Färe et al. 1994).

The promise of the methodology is illustrated in this work with reference to the evidence on micro data of Italian industries and the dynamics of their distributions.

The rest of the work is organized as follows. We start with an empirical illustration of the general point (Section 2). Next, Section 3 builds on the contribution of Hildenbrand (1981) and introduces the (normalized) volume of the zonotope as a measure of industry heterogeneity. We then proposes a measure of technical change based on the zonotope's main diagonal and we assess the role of firm entry and exit on industry level productivity growth. Section 4 presents an empirical application on manufacturing firms in narrowly defined industries. Section 5 discusses the implications of this work and further applications of the proposed methodology.

2 Persistent micro heterogeneity: an illustration

In order to vividly illustrate the ubiquitous, wide and persistent heterogeneity across firms within the same lines of business and in presence of roughly identical relative prices, consider two sectors of the Italian industry which one could expect not too different in terms of output, namely meat products, NACE 151, and knitted and crocheted articles, NACE 177, see Figure 1. Each of the two plot reports the distribution of labor productivity in a three-digit NACE sector

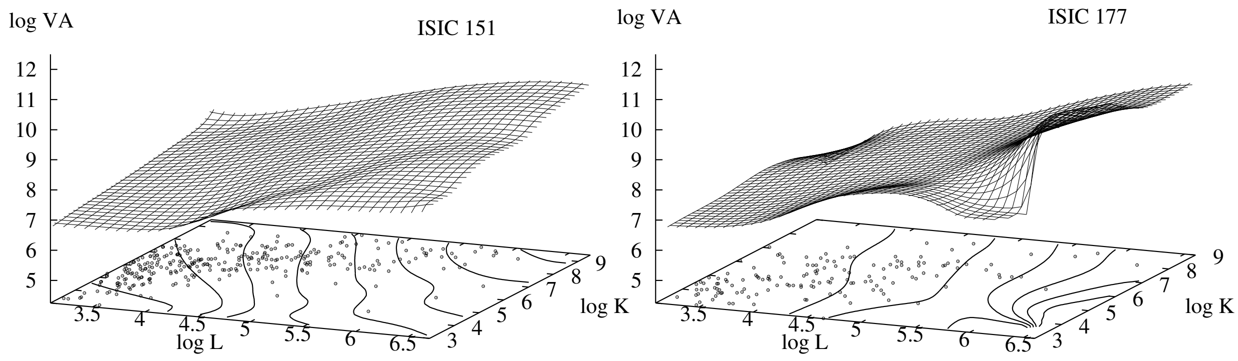


Figure 2: Adopted techniques and output level in two different three-digit NACE sectors, in 2006.

and it shows the coexistence of firms with much different levels of productivity across firms highlighting a ratio ‘top to bottom’ greater than 5 to 1 (in logs!). Disaggregation well reveals the ‘scale freeness’ of such distribution: the width of their support does not shrink if one considers the four-digit NACE sectors nested therein. The observed heterogeneity is not the result of the chosen level of industry aggregation.

Further evidence that firm-level techniques do not belong to the same ‘production function’ - at least of any canonic form - stem from the lack of correlation between labour productivies and ‘capital productivities’ (i.e. value added/ capital stock). In our two foregoing sectors is -.02 and .2, and over all the 3-digit sectors of the Italian manufacturing industry it ranges between -0.07 and .425 with a median of .13.

Figure 2 graphically illustrates the point in our two sectors, with on the axes inputs (labour and capital) and output (value added). Using the kernel estimation techniques, smooth surfaces have been obtained from the discrete sets of observations. As a reference, the location of the observed amount of inputs (l, k) has been reported on the bottom of plots, each dot represents the input mix of a firm. The “isoquants” report on the l-k plane the correspondent levels of output. Note, first, that dots are quite dispersed over the plane and do not seem to display any regularity resembling conventional isoquant (a feature already emphasized by Hildenbrand 1981). Second, output does increase - as it should be expected - in both inputs. However, this happens in quite non-monotonic manners: given a quantity of one input, different firms attain the same level of output with very different levels of the other input. In other words, overall degrees of efficiency seemingly widely differ.

Further, over time heterogeneity is very persistent. In our two sectoral illustrations the autocorrelation coefficients in firm-level labour productivities over a two years period rests around .8, as it does in most of the comparable 3-digit industrial sectors. And, such an evidence is quite in tune with both the parametric and non-parametric estimates discussed in Bartelsman and Dhrymes (1998); Haltiwanger et al. (1999); Dosi and Grazzi (2006) among others.

All together, the evidence is robustly ‘Schumpeterian’ consistent with idiosyncratic firm-level capabilities, quite inertial over time and rather hard to imitate (much more on that in Winter 2005, 2006; Nelson 2008; Dosi et al. 2008).

Granted that, how does one concisely represent the corresponding distributions of micro coefficients and their dynamics over time?

3 Accounting for heterogeneous micro-techniques

Without loss of generality it is possible to represent the *actual technique* of a production unit by means of a *production activity* represented by a vector (Koopmans; 1977; Hildenbrand; 1981)

$$a = (\alpha_1, \ldots, \alpha_l, \alpha_{l+1}) \in \mathbb{R}_+^{l+1}.$$

A production unit, which is described by the vector a, produces during the current period α_{l+1} units of output by means of $(\alpha_1, \ldots, \alpha_l)$ units of input.[3] Also notice that in this framework it is possible to refer to the *size* of the firm as to the length of vector a, which can be regarded as a multi-dimensional extension of the usual measure of firm size, often proxied either by the number of employees, sales or value added. In fact, this measure allows to employ both measures of input and output in the definition of firm size.

In this framework, as noted by Hildenbrand (1981), the assumption of constant returns to scale (with respect to variable inputs) for individual production units is not necessary: indeed it is redundant if there are "many" firms in the industry. Anyhow, the short run production possibilities of an industry with N units at a given time are described by a finite family of vectors $\{a_n\}_{1\le n\le N}$ of production activities. In order to analyze such a structure Hildenbrand introduces a novel *short-run feasible industry production function* defined by means of a *Zonotope* generated by the family $\{a_n\}_{1\le n\le N}$ of production activities. More precisely let $\{a_n\}_{1\le n\le N}$ be a collection of vectors in $\mathbb{R}^{l+1}$, $N \ge l+1$. To any vector a_n we may associate a line segment

$$[0, a_n] = \{x_n a_n \mid x_n \in \mathbb{R}, 0 \le x_n \le 1\}.$$

Hildenbrand defines the *short run total production set* associated to the family $\{a_n\}_{1\le n\le N}$ as the Minkowski sum

$$Y = \sum_{n=1}^{N} [0, a_n]$$

of line segments generated by production activities $\{a_n\}_{1\le n\le N}$. More explicitly, it is the Zonotope

$$Y = \{y \in \mathbb{R}_+^{l+1} \mid y = \sum_{n=1}^{N} \phi_n a_n, 0 \le \phi_n \le 1\}.$$

Remark 3.1 *Geometrically a Zonotope is the generalization to any dimension of a Zonohedron that is a convex polyhedron where every face is a polygon with point symmetry or, equivalently, symmetry under rotations through* 180°. *Any Zonohedron may equivalently be described as the Minkowski sum of a set of line segments in three-dimensional space, or as the three-dimensional projection of an hypercube. Hence a Zonotope is either the Minkowski sum of line segments in an l-dimensional space or the projection of an $(l+1)$-dimensional hypercube. The vectors from which the Zonotope is formed are called its generators.*[4]

Analogously to parallelotopes and hypercubes, Zonotopes admit diagonals. We define the main diagonal of a Zonotope Y as the diagonal joining the origin $O = (0, \ldots, 0) \in Y \subset \mathbb{R}^{l+1}$ with its opposite vertex in Y. Algebraically it is simply the sum $\sum_{n=1}^{N} a_n$ of all generators, that is, in our framework, the sum of all production activities in the industry. In the following, we will denote by d_Y such diagonal and we will call it production activity of the industry.

Denote by D the projection of Y on the firsts l coordinates, i.e.

[3]Our considerations hold also for the multi-output case.

[4]The interested reader can refer to Ziegler (1995) for a survey on Zonotopes.

$$D = \{v \in R^l_+ \mid \exists x \in \mathbb{R}_+ \text{ s.t. } (v, x) \in Y\}$$

and the production function $F : D \longrightarrow \mathbb{R}_+$ associated with Y as

$$F(v) = \max\{x \in \mathbb{R}_+ \mid (v, x) \in Y\}.$$

In the definition above the aggregation of the various production units implies a "frontier" associating to the level v_1, ..., v_l of inputs for the industry the maximum total output which is obtainable by allocating, without restrictions, the amounts v_1, ... v_l of inputs in a most efficient way over the individual production units. However, as argued by Hildenbrand (1981) it might well be that the distribution of technological capabilities and/or the market structure and organization of the industry is such that the efficient production function couldn't be the focal reference either from a positive or from a normative point of view in so far as the "frontier", first, does not offer any information on the actual technological set-up of the whole industry, and, second, does not offer any guidance to what the industry would look like under an (unconstrained) optimal allocation of resources. This notwithstanding, estimates of the "frontier" offer important clues on the moving best-practice opportunities and the distance of individual firms from them. Here is also the notional complementarity between this approach and the contributions in the Data Envelopment Analysis (DEA) tradition, see Farrell (1957); Charnes et al. (1978); Daraio and Simar (2007); Simar and Zelenyuk (2011) for major contributions in the field and Murillo-Zamorano (2004) for a review. In the DEA approach one focuses on a measure of firm's efficiency which is provided by the distance between any single firm and the efficient frontier. Conversely, in our approach, the way in which a firm contributes to industry heterogeneity depends on how such firms *combines and compares* with *all* other firms. A similar point applies to how technical change is measured, see below.

The representation of any industry at any one time by means of the Zonotope provides a way to assess and measure the overall degree of heterogeneity of an industry. As we shall show below, it allows also to account for its variation of production techniques adopted by firms in any industry and, at least as important, it allows to ascertain the rate and direction of technical change.

3.1 Volume of Zonotopes and heterogeneity

Start by noting that if all firms in an industry with N enterprises were to use the same technique in a given year, all the vectors of the associated family $\{a_n\}_{1\leq n\leq N}$ of production activities would be multiples of the same vector. Hence they would lie on the same line and the generated zonotope would coincide with the diagonal $\sum_{n=1}^{N} a_n$, that is a degenerate zonotope of null volume. This is the case of one technology only and perfect homogeneity among firms. At the opposite extreme the maximal heterogeneity would feature an industry wherein there are zero inputs and other firms producing little output with a large quantity of inputs. This case of maximal heterogeneity is geometrically described by vectors that generate a zonotope which is almost a parallelotope.

In the following we provide the formula to compute the volume of the zonotope.

Let $A_{i_1,\ldots,i_{l+1}}$ be the matrix whose rows are vectors $\{a_{i_1}, \ldots, a_{i_{l+1}}\}$ and $\Delta_{i_1,\ldots,i_{l+1}}$ its determinant. In our framework, the first l entries of each vector stand for the amount of the inputs used in the production process by each firm, while the last entry of the vector is the output. It is well known that the volume of the zonotope Y in $\mathbb{R}^{l+1}$ is given by:

$$Vol(Y) = \sum_{1\leq i_1<\ldots<i_{l+1}\leq N} \mid \Delta_{i_1,\ldots,i_{l+1}} \mid$$

where $\mid \Delta_{i_1,\ldots,i_{l+1}} \mid$ is the module of the determinant $\Delta_{i_1,\ldots,i_{l+1}}$.

Our main interest lies in getting a *pure* measure of the heterogeneity in techniques employed by firms within any given industry that allows for comparability across firms and time; that is, a measure which is independent both from the unit in which inputs and output are measured and from the number of firms making up the sector. The volume of the zonotope itself depends both from the units of measure involved and from the number of firms. In order to address these issues we need a way to normalize the zonotope's volume yielding a new index which is dimensionless and independent from the number of firms.

The normalization we introduce is a generalization of the well known Gini index, which we call *Gini volume* of the zonotope. Analogously to the original index, we will consider the *ratio* of the volume of the zonotope Y generated by the production activities $\{a_n\}_{1\leq n\leq N}$ over a *total* volume of an industry with production activity $d_Y = \sum_{n=1}^{N} a_n$. It is an easy remark that the parallelotope is the zonotope with largest volume if the main diagonal is fixed. If P_Y is the parallelotope of diagonal d_Y, its volume $Vol(P_Y)$, i.e. the product of the entries of d_Y, is obviously the maximal volume that can be obtained once we fix the industry production activity $\sum_{n=1}^{N} a_n$, that is the *total* volume of an industry with production activity $d_Y = \sum_{n=1}^{N} a_n$.

Note that alike the complete inequality case in the Gini index, i.e. the case in which the index is 1, also in our framework the complete heterogeneity case is not feasible, since in addition to firms with large values of inputs and zero output it would imply the existence of firms with zero inputs and non zero output. It has to be regarded as a *limit*, conceptually alike and opposite to the 0 volume case in which all techniques are equal, i.e. the vectors $\{a_n\}_{1\leq n\leq N}$ are proportional and hence lie on the same line.

In what follows we consider the Gini volume defined above for the short run total production set Y:

$$G(Y) = \frac{Vol(Y)}{Vol(P_Y)} \quad . \tag{1}$$

3.2 Unitary production activities

An interesting information is provided by comparison of the Gini volume $G(Y)$ of the short run total production set Y and the same index computed for the zonotope $\overline{Y}$ generated by the normalized vectors $\{\frac{a_n}{\|a_n\|}\}_{1\leq n\leq N}$, i.e. the unitary production activities. The Gini volume $G(\overline{Y})$ evaluates the heterogeneity of the industry in a setting in which all firms have the same size (norm is equal to one). Hence the only source of heterogeneity is the difference in adopted techniques, since differences in firm size do not contribute to the volume.

Comparing the Gini volume of the zonotope Y with that of the unitary zonotope $\overline{Y}$ is informative about the relative contribution of large and small firms to the overall heterogeneity in techniques within the given industry. Intuitively, if the Gini volume $G(Y)$ of Y is bigger than $G(\overline{Y})$ this means that big firms contribute to heterogeneity more than the small ones, and viceversa, if the volume $G(Y)$ is smaller than $G(\overline{Y})$.

3.3 Solid Angle and external production activities

Let us move further and introduce the *external* zonotope Y_e. In order to define it we need the notion of solid angle. Let us start with the solid angle in a 3-dimensional space, but the idea can be easily generalized to the $(l+1)$ dimension.

In geometry, a solid angle (symbol: Ω) is the two-dimensional angle in three-dimensional space that an object subtends at a point. It is a measure of *how large* the object appears to an observer looking from that point. In the International System of Units, a solid angle is a dimensionless unit of measurement called a *steradian* (symbol: sr). The measure of a solid angle Ω varies between 0 and 4π steradian.

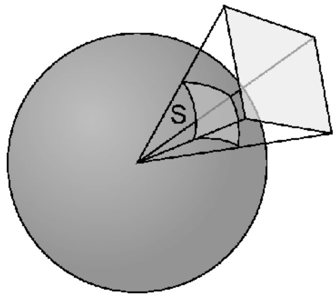


Figure 3: The solid angle of a pyramid generated by 4 vectors.

More precisely, an object's solid angle is equal to the area of the segment of a unit sphere, centered at the angle's vertex, that the object covers, as shown in figure 3.

In our framework the production activities are represented by a family $\{a_n\}_{1\leq n\leq N}$ of vectors. Their normalization $\{\frac{a_n}{\|a_n\|}\}_{1\leq n\leq N}$ will generate an arbitrary pyramid with apex in the origin. Note that in general, not all vectors a_i, $i=1,\dots,N$ will be edges of this pyramid. Indeed it can happen that one vector is *inside* the pyramid generated by others. We will call *external* vectors those vectors $\{e_i\}_{1\leq i\leq R}$ of the family $\{a_n\}_{1\leq n\leq N}$ such that their normalizations $\{\frac{e_i}{\|e_i\|}\}_{1\leq i\leq R}$ are edges of the pyramid generated by the vectors $\{\frac{a_n}{\|a_n\|}\}_{1\leq n\leq N}$. All the others will be called *internal.*

This pyramid will subtend a solid angle Ω, smaller or equal than $\frac{\pi}{2}$ as the entries of our vectors are positive. We will say that the *external* vectors of the family $\{a_n\}_{1\leq n\leq N}$ subtend the solid angle Ω if it is the angle subtended by the generated pyramid.

We define the *external zonotope* Y_e as the one generated by vectors $\{e_i\}_{1\leq i\leq R}$. A pairwise comparison between $G(Y_e)$ and $G(Y)$ shows the relative importance of the *density* of internal activities in affecting our proposed measure of heterogeneity.

Solid angle of an arbitrary pyramid. In $\mathbb{R}^3$ the solid angle of an arbitrary pyramid defined by the sequence of unit vectors representing edges $\{s_1, s_2, \dots, s_n\}$ can be computed as

$$\Omega = 2\pi - \arg \prod_{j=1}^{n} (< s_j, s_{j-1} >< s_j, s_{j+1} > - < s_{j-1}, s_{j+1} > +i \mid s_{j-1}s_js_{j+1} \mid) \qquad (2)$$

where parentheses $< s_j, s_{j-1} >$ are scalar products, brackets $\mid s_{j-1}s_js_{j+1} \mid$ are scalar triple products, i.e. determinants of the 3×3 matrices whose rows are vectors s_{j-1}, s_j, s_{j+1}, and i is the imaginary unit. Indices are cycled: $s_0 = s_n$ and $s_{n+1} = s_1$ and arg is simply the argument of a complex number.

The generalization of the definition of solid angle to higher dimensions simply needs to account for the l-sphere in an $l+1$-dimensional space.

3.4 Technical Change

Let us consider a non-zero vector $v = (x_1, x_2, \dots, x_{l+1}) \in \mathbb{R}^{l+1}$ and, for any $i \in 1, \dots, l+1$ the projection map

$$\begin{aligned} pr_{-i} : \mathbb{R}^{l+1} &\longrightarrow \mathbb{R}^l \\ (x_1, \dots, x_{l+1}) &\mapsto (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{l+1}) \quad . \end{aligned}$$

Using the trigonometric formulation of the Pythagoras' theorem we get that if ψ_i is the angle that v forms with the x_i axis, $\theta_i = \frac{\pi}{2} - \psi_i$ is its complement and $\|v_i\|$ is the norm of the projection vector $v_i = pr_{-i}(v)$, i.e. the length of the vector v_i, then the tangent of θ_i is:

$$\mathrm{tg}\theta_i = \frac{x_i}{\|v_i\|}.$$

In our framework we are primarily interested in the angle θ_{l+1} that the diagonal of the zonotope, i.e. the vector d_Y, forms with the space generated by all inputs. This can easily be generalized to the case of multiple outputs, so that if we have m different outputs we will consider the angles θ_i for $l < i \leq l + m$.[5]

In order to assess if and to what extent productivity is growing in a given industry, it is possible to analyze how the angle θ_{l+1} varies over the years. For example if the angle θ_{l+1} increases then productivity increases. This is indeed equivalent to state that the industry is able to produce more output, given the quantity of inputs, than it was able to before. Conversely, a decrease in θ_{l+1} stands for a productivity reduction.

Also notice that it is possible to study how the relative inputs use changes over the years. To do this, it is enough to consider the angles that the input vector, i.e. the vector with entries given by only the inputs of d_Y, forms with different input axis. More precisely, if there are l inputs and m outputs and the vectors of production activities are ordered such that the first l entries are inputs, then we can consider the projection function on the first l coordinates:

$$\begin{aligned} pr_l : \mathbb{R}^{l+m} &\longrightarrow \mathbb{R}^l \\ (x_1, \ldots, x_{l+m}) &\mapsto (x_1, \ldots, x_l) \quad . \end{aligned}$$

The change over time of the angle φ_i between the projection vector $pr(d_Y)$ and the x_i axis, $1 \leq i \leq l$, captures the changes in the relative intensity of input i over time with respect to all the other inputs.

It is also informative to measure the changes in the *normalized* angles $\overline{\theta}_i$. Indeed, as we have done for volumes, we can consider the normalized production activities $\{\frac{a_n}{\|a_n\|}\}_{1\leq n\leq N}$. Call $d_{\overline{Y}}$ the resulting industry production activity. Of course, one can study how it varies over time and this is equivalent to study how the productivity of an industry changes independently from the size of the firms. In particular the comparison of the changes of two different angles, θ_i and $\overline{\theta}_i$, reveals the relative contribution of bigger and smaller firms to productivity changes and hence, on the possible existence of economies/diseconomies of scale.

For the sake of simplicity and for coherence with the one output case we consider here the variation of the tangent of the angles instead of the angles themselves, noting that if an angle increases then its tangent increases too.

3.5 Entry and exit

Under what circumstances does the entry of a new firm increase or decrease the heterogeneity of a given industry? In order to compute how entries and exits impact on industry heterogeneity it is enough to remark that, by the definition of volume, given a zonotope Z in the space $\mathbb{R}^{l+1}$ generated by vectors $\{a_n\}_{1\leq n\leq N}$ and a vector $b = (x_1, \ldots, x_{l+1}) \in \mathbb{R}^{l+1}$, the volume of the new zonotope X generated by $\{a_n\}_{1\leq n\leq N} \cup \{b\}$ can be computed as follow:

$$Vol(X) = Vol(Z) + V(x_1, \ldots, x_{l+1})$$

where $V(x_1, \ldots, x_{l+1})$ is a real continuous function on $\mathbb{R}^{l+1}$ defined as:

$$V(x_1, \ldots, x_{l+1}) = \sum_{1\leq i_1<\ldots<i_l\leq N} \mid \Lambda_{i_1,\ldots,i_l} \mid,$$

[5] This is the generalization to the multi-dimensional case of the 1-input and 1-output setting. Indeed, in this simpler case, industry productivity is the quotient o/i between output and input, i.e. the tangent of the angle θ_2 that the vector (i, o) forms with the input axis.

$\Lambda_{i_1,\ldots,i_l}$ being the determinant of the matrix $B_{i_1,\ldots,i_l}$ whose rows are vectors $\{b, a_{i_1}, \ldots, a_{i_l}\}$.

If $d_Z = (d_1, \ldots, d_{l+1})$ is the diagonal of the zonotope Z, then the diagonal of X will be $d_X = d_Z + b = (d_1 + x_1, \ldots, d_{l+1} + x_{l+1})$. The heterogeneity for the *new* industrial set-up will be the continuous real function

$$G(X) = \frac{Vol(Z) + V(x_1, \ldots, x_{l+1})}{Vol(P_X)} = \frac{Vol(Z) + V(x_1, \ldots, x_{l+1})}{\Pi_{i=1}^{l+1}(d_i + x_i)}$$

and the tangent of the angle with the input space will be the continuous real function

$$\mathrm{tg}\theta_{l+1}(x_1, \ldots, x_{l+1}) = \frac{d_{l+1} + x_{l+1}}{\|pr_{-(l+1)}(d_X)\|}$$

Studying the variation (i.e. gradient, hessian etc...) of these real continuous functions is equivalent to analyze the impact of a new firm on the industry. So, for example, when these functions increase then the new firm positively contributes both to industry heterogeneity and productivity. We consider as an example the entry of a firm in the 3-dimensional case. If Z is the zonotope generated by vectors $\{a_n\}_{1\le n\le N}$ in $\mathbb{R}^3$ with entries $a_n = (a_n^1, a_n^2, a_n^3)$, the function $V(x_1, x_2, x_3)$ for a generic vector $b = (x_1, x_2, x_3)$ is

$$V(x_1, x_2, x_3) = \sum_{1\le i<j\le N} \mid x_1(a_i^2 a_j^3 - a_i^3 a_j^2) - x_2(a_i^1 a_j^3 - a_i^3 a_j^1) + x_3(a_i^1 a_j^2 - a_i^2 a_j^1) \mid .$$

The diagonal of the new zonotope X is

$$d_X = (\sum_{i=1}^N a_i^1 + x_1, \sum_{i=1}^N a_i^2 + x_2, \sum_{i=1}^N a_i^3 + x_3).$$

We get the Gini volume for X as:

$$G(X) = \frac{Vol(Z) + \sum_{1\le i<j\le N} \mid x_1(a_i^2 a_j^3 - a_i^3 a_j^2) - x_2(a_i^1 a_j^3 - a_i^3 a_j^1) + x_3(a_i^1 a_j^2 - a_i^2 a_j^1) \mid}{\sum_{i,j,k=1}^N (a_i^1 + x_1)(a_j^2 + x_2)(a_k^3 + x_3)} \quad , \quad (3)$$

where $Vol(Z)$ and $\{a_n^1, a_n^2, a_n^3\}_{1\le n\le N}$ are constants and the tangent of the angle with the input space is:

$$\mathrm{tg}\theta_3(x_1, x_2, x_3) = \frac{\sum_{i=1}^N a_i^3 + x_3}{\sqrt{(\sum_{i=1}^N a_i^1 + x_1)^2 + (\sum_{i=1}^N a_i^2 + x_2)^2}} \quad .$$

If we set the output x_3 constant or we fix the norm of b, i.e. the size of the firm, setting $x_3 = \sqrt{\|b\| - x_1^2 - x_2^2}$ then $G(X)$ and $\mathrm{tg}\theta_3(x_1, x_2, x_3)$ become two variables functions, $G(X) = G(X)(x_1, x_2)$ and $\mathrm{tg}\theta_3(x_1, x_2)$, which can be easily studied from a differential point of view.

It is important to notice that all the foregoing measures not only can be easily applied to any $l+1$-dimensional case with multi-dimensional outputs (i.e., for example, l inputs and m outputs in the space $\mathbb{R}^{l+m}$), but also to the more general case of a vector space V over a field $\mathbb{K}$. Indeed all the tools we introduced hold for any finite dimensional vector space. In that respect recall that the set $\mathcal{H}om(V, W)$ of all linear maps between two vector spaces V and W over the same field $\mathbb{K}$ is a vector space itself. Hence we can consider the vector space $\mathcal{H}om(\mathbb{R}^l, \mathbb{R}^m)$ in which a vector is a linear function from $\mathbb{R}^l$ to $\mathbb{R}^m$. More in general, our model applies to all finite dimensional topological vector spaces such as, for example, the space of degree $l+1$ polynomials over a field $\mathbb{K}$, the finite dimensional subspaces of smooth functions on $\mathbb{R}$ and so on.

	Year 1			Year 2			Year 3			Year 4		
#	L	K	VA	L	K	VA	L	K	VA	L	K	VA
1	7.0	4.0	9.0	7.0	4.0	9.0	**7.0**	**4.0**	**9.0**	**7.0**	**4.0**	**9.0**
2	1.0	4.0	5.0	1.0	4.0	5.0	**1.0**	**4.0**	**5.0**	**1.0**	**4.0**	**5.0**
3	**6.0**	**2.0**	**9.0**	**6.0**	**2.0**	**9.0**	**6.0**	**2.0**	**9.0**	**6.0**	**2.0**	**9.0**
4	**1.5**	**8.0**	**10.0**	**1.5**	**8.0**	**10.0**	**1.5**	**8.0**	**10.0**	**1.5**	**8.0**	**10.0**
5	5.0	2.0	8.0	5.0	2.0	8.0	5.0	2.0	8.0	5.0	2.0	8.0
6	**1.0**	**3.0**	**8.0**	**1.0**	**3.0**	**8.0**	**1.0**	**3.0**	**8.0**	**1.0**	**3.0**	**8.0**
7	**2.0**	**2.0**	**7.0**	**2.0**	**2.0**	**7.0**	**2.0**	**2.0**	**7.0**	**2.0**	**2.0**	**7.0**
8	3.0	5.0	7.0	3.0	5.0	7.0	3.0	5.0	7.0			
9	**2.5**	**2**	**2**	**2.5**	**2**	**2**						
10	**5.0**	**6.0**	**4.0**	4.0	4.0	6.0	**4.0**	**4.0**	**6.0**	**4.0**	**4.0**	**6.0**

Table 1: Production schedules in year 1 to 4, Number of employees (L), Capital (K) and Output (VA). External production activities in bold.

3.6 A toy illustration

Consider the production schedules of 10 hypothetical firms composing an industry as reported in Table 1, with two inputs, labor, on the x axis, and capital, on the y axis, and one output, on the z axis, measured in terms of value added; "external" production activities are in bold. Figure 4 show the initial zonotope of the industry and Figure 5 reports the solid angles in year 1 and 2, respectively.[6]

In order to better evaluate the proposed measure of heterogeneity and technical change, and, even more relevant, their dynamics over time, we allow for a change in only one of the firm (vector) going from one period to the other, as reported in Table 1. In particular, from period 1 to 2 only the production schedule of firm 10 changes with unequivocal productivity increases as both inputs decrease while output increases. Then, from period 2 to period 3 the ninth firm exits the industry. The property of the vector representing the ninth firm is that it is an "external" vector: hence removing it significantly affects the shape of the zonotope. Finally, from period 3 to 4 firm 8 leaves the industry. However this time it is a firm represented by an "internal" vector.

How do these changes, i.e. a firm increasing productivity and two different firms exiting, affect industry heterogeneity and the extent and direction of technical change?

Let us introduce a few notations in order to study this easy example. Denote by $a_j^t \in \mathbb{R}^3$ the 3-dimensional vector representing the production activity of the firm j in the year t, $1 \leq j \leq 10$ and $1 \leq t \leq 4$ (e.g. $a_1^1 = (7.0, 4.0, 9.0)$ and $a_2^2 = (1.0, 4.0, 5.0)$). The zonotope at year t will be denoted by Y^t and the industry production activity will be $d_{Y^t} = \sum_{j=1}^{10} a_j^t$, $1 \leq t \leq 4$.

Then the matrices described in section 3.1 will be 3×3 matrices $A_{i,j,k}^t$ with vectors a_i^t, a_j^t, a_k^t as columns and determinants $\Delta_{i,j,k}^t$.

The volumes of zonotopes Y^t are given by

$$Vol(Y^t) = \sum_{1 \leq i < j < k \leq 10} \mid \Delta_{i,j,k}^t \mid \quad , \qquad 1 \leq t \leq 4$$

and yielding the following values:

$$Vol(Y^1) = 8265 \quad Vol(Y^2) = 6070 \quad Vol(Y^3) = 4664.5 \quad Vol(Y^4) = 3402.$$

[6] Numerical calculations for this toy illustration as well as for the empirical analysis that follows have been performed using the software zonohedron, written by Federico Ponchio. The code and instructions are available at: `http://vcg.isti.cnr.it/~ponchio/zonohedron.php`.

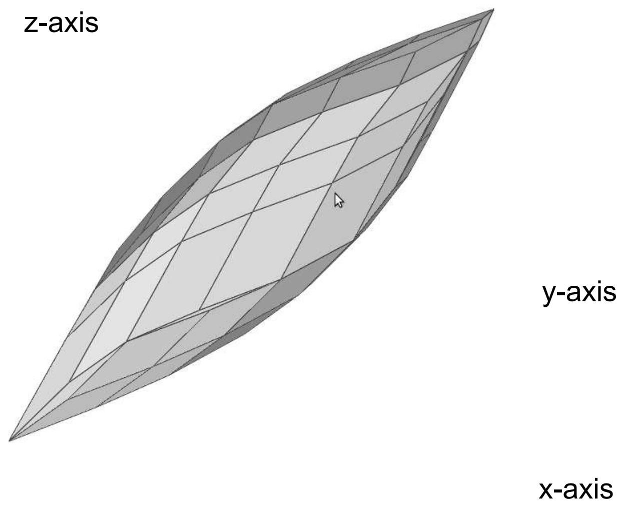


Figure 4: 3D representation of the zonotope of the toy illustration.

The norm of the projection on the space of inputs of the 3-dimensional diagonal vector $d_{Y^t} = (d_1^t, d_2^t, d_3^t)$, $1 \leq t \leq 4$, is $\|pr_{-3}(d_{Y^t})\| = \sqrt{(d_1^t)^2 + (d_2^t)^2}$.
and we get following numerical values:

$$\|pr_{-3}(d_{Y^1})\| = 50.997 \quad \|pr_{-3}(d_{Y^2})\| = 48.84 \quad \|pr_{-3}(d_{Y^3})\| = 45.67 \quad \|pr_{-3}(d_{Y^4})\| = 39.96$$

The Gini volume will be:

$$G(Y^t) = \frac{Vol(Y^t)}{d_1^t d_2^t d_3^t}$$

The numerical results for $1 \leq t \leq 4$ are shown in Table 2.

As illustrated in Section 3.4 the variation over time of the angle θ_3 that the diagonal of the zonotope Y^t forms with the plane x, y of inputs can be used to assess if and to what extent productivity is growing in a given industry. Similarly if φ_1^t is the angle that the diagonal of Y^t forms with the x axis, then $\mathrm{tg}\varphi_1^t$ allows to study how the relative inputs use changes over the years. Using the notation introduced above, they are, respectively,

$$\mathrm{tg}\theta_3^t = \frac{d_3^t}{\|pr_{-3}(d_{Y^t})\|} \quad \text{and} \quad \mathrm{tg}\varphi_1^t = \frac{d_2^t}{\|d_1^t\|},$$

where the first is the index of the technical efficiency in the production of output, i.e. a measure of improvement in "total factor productivity", and the second index captures the relative "intensity" of the first input (the second one can be obtained as $\mathrm{tg}\varphi_2^t = \frac{1}{\mathrm{tg}\varphi_1^t}$). Table 2 displays the values of Gini volumes for the zonotopes Y^t, the zonotopes $\overline{Y}^t$ generated by the normalized production activities $\{\frac{a_j^t}{\|a_j^t\|}\}_{1\leq j\leq 10}$ and the zonotopes Y_e^t generated by the external production activities which are in bold in Table 1. Moreover it also reports the solid angle, the ratio of the Gini volumes of Y^t over the Gini volumes of Y_e^t and the angles that account for the rate and direction of technical change.

Going from year 1 to 2, firm 10 displays an unequivocal increase in productivity. As shown in Figure 5 the normalized vector accounting for the production activity of firm 10 rotates inward: in period 1 $\overline{a}_{10}^1$ is a boundary (normalized) vector, whether in period 2, $\overline{a}_{10}^2$ is an "internal" vector. Since a boundary vector (firm) shifts inward, production techniques are more similar

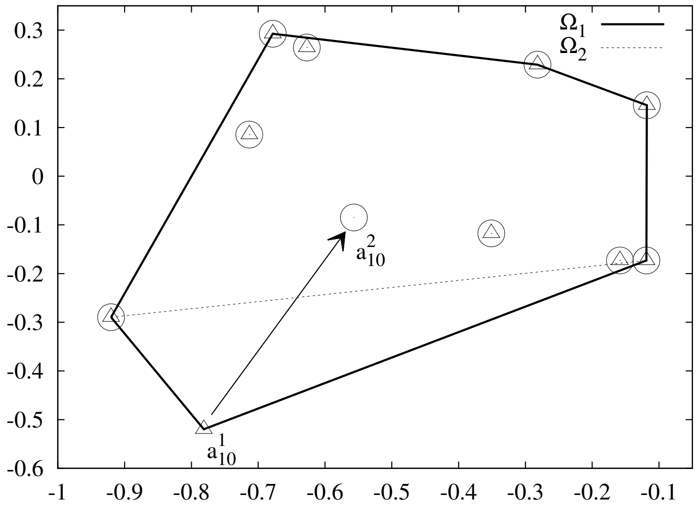


Figure 5: The plot depicts a planar section of the solid angle generated by all vectors in year 1 and 2. The section plane is the one perpendicular to the vector sum of generators in year 1. The vector of firm 10 (a_{10}) moves inward from year 1 to year 2.

	Year 1	Year 2	Year 3	Year 4
$G(Y^t)$	0.09271	0.07196	0.06518	0.06880
$G(\overline{Y}^t)$	0.09742	0.07905	0.06795	0.07244
$G(Y_e^t)$	0.12089	0.09627	0.07297	0.07297
Solid Angle	0.28195	0.22539	0.15471	0.15471
$G(Y^t)$ / $G(Y_e^t)$	0.70593	0.74748	0.89324	0.94285
$\text{tg}\theta_3^t$	1.3532	1.4538	1.51066	1.55133
$\text{tg}\varphi_1^t$	1.11765	1.09091	1.11475	1.05455

Table 2: Volumes and angles accounting for heterogeneity and technical change, respectively, in the four years of the toy illustration.

in period 2, hence heterogeneity within the industry reduces. This is captured by our proposed measures which all vary in the expected direction. The Gini index, $G(Y)$, the Gini index on normalized, $G(\overline{Y})$, and "external" vectors reduce from year 1 to year 2. As apparent from Figure 5 also the solid angle reduces. The ratio $G(Y^t)$ / $G(Y_e^t)$ increases suggesting that internal vectors now contribute more to the volume as compared to external vectors. The variation of the tangent of the angle θ_3 that the diagonal of the zonotope forms with the plane of inputs is our measure of technical change. From year 1 to 2, firm 10, the least efficient, becomes more productive, and this positively contributes to productivity growth at the industry level as captured by the increase in the tangent of the angle θ_3. The last indicator of Table 2 is informative of the direction of technical change. The decrease in $\text{tg}\varphi_1^t$ suggest that technical change is biased in the capital saving direction.

From year $t = 2$ to year $t = 3$ firm 9, an external vector, leaves the industry.[7] The outcomes are smaller Gini volumes for all our zonotopes. The solid angle reduces, too, while the tangent of the angle θ_3 increases, suggesting that the exit of firm 9 resulted in a a further efficiency gain

[7]Note that, intuitively, external vectors are the analogous to the support of an empirical distribution.

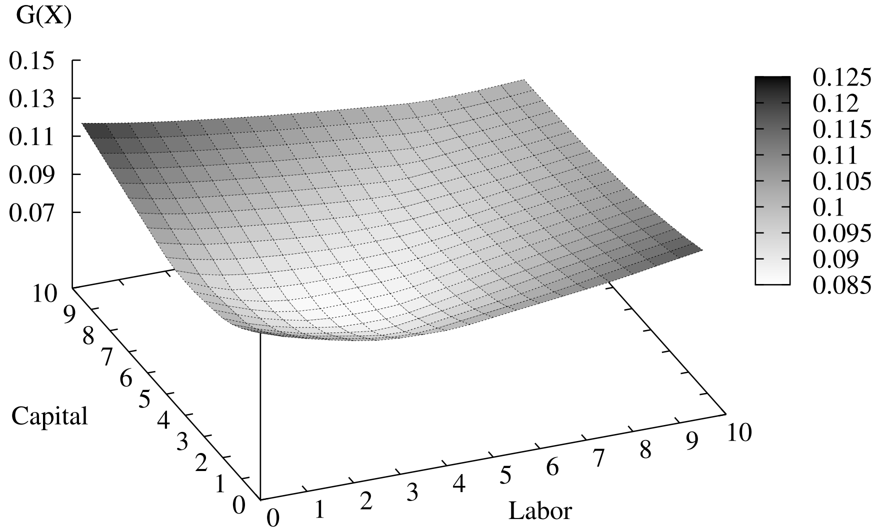


Figure 6: Variation of heterogeneity (on the z axis) when a firm of labor x, capital y and fixed value added enters the industry.

for the industry. Technical change is now in the labor saving direction as $\text{tg}\varphi_1^t$ increases.

From period $t = 3$ to $t = 4$ an "internal" vector, firm 8, drops out of the industry. In this case all our measures of Gini volumes point to an increase in heterogeneity, except, obviously, $G(Y_e^t)$ since the boundary vectors do not change. Again the exit of firm 8 positively contributes to productivity growth in the industry, as shown by the increase in $\text{tg}\theta_3^t$ with capital saving bias, as $\text{tg}\varphi_1^t$ decreases.

Section 3.5 above discussed how the present framework can account for firm entry and exit. In this respect, graph in Figure 6 represents how the heterogeneity changes when a generic firm of fixed value added ($VA = 5$) enters the industry in year 1 of the example. The function plotted in Fig. 6 is the function G(X) in equation (3) with $Z = Y^1$, $N = 10$ and vectors $a_n = a_n^1$. The value on the z-axis is the degree of heterogeneity in terms of the Gini index for the different values that labour and capital might take. Analogous graphs can be plotted in order to visualize technical change.

4 An empirical application

In the following we put the model at work on longitudinal firm-level data of an ensemble of Italian 4-digit industries (chosen on the grounds of the numerosity of observations) over the period 1998-2006. Values have been deflated with the industry-specific production price index. Output is measured by (constant price) valued added (thousands of euro), capital is proxied by (deflated) gross tangible assets (thousands of euro) and labour is simply the number of employees (full time equivalent). More details on the databank are in Appendix A. The list of sectors and the number of observations is reported in Table 3 together with the number of "external vectors", as defined above, in brackets.

Figure 7 reports the actual analog of Figure 5 and it shows the coordinates of the normalized vectors on the unit sphere for firms making up the industry in 2002 and 2006. Both plots show that the solid angle provides a snapshot of the extreme techniques at use in a given industry. For

NACE Code	Description			
1513	Meat and poultrymeat products	162 (7)	162 (10)	190 (9)
1721	Cotton-type weaving	139 (9)	119 (11)	113 (7)
1772	Knitted & crocheted pullovers, cardigans	137 (8)	117 (10)	100 (7)
1930	Footwear	616 (9)	498 (6)	474 (9)
2121	Corrugated paper and paperboard	186 (7)	176 (9)	199 (11)
2222	Printing n.e.c.	297 (11)	285 (10)	368 (8)
2522	Plastic packing goods	204 (7)	217 (10)	253 (11)
2524	Other plastic products	596 (9)	558 (9)	638 (10)
2661	Concrete products for construction	208 (8)	231 (11)	272 (7)
2663	Ready-mixed concrete	103 (8)	114 (8)	147 (10)
2751	Casting of iron	94 (7)	77 (9)	88 (9)
2811	Metal structures and parts of structures	402 (9)	378 (8)	565 (10)
2852	General mechanical engineering	473 (11)	511 (8)	825 (11)
2953	Machinery for food & beverage processing	131 (6)	134 (7)	159 (6)
2954	Machinery for textile, apparel & leather	191 (10)	170 (10)	154 (12)
3611	Chairs and seats	205 (8)	201 (10)	229 (7)

Table 3: NACE sectors for the empirical analysis. Number of observations in 1998, 2002 and 2002. In brackets the number of external vectors in each year.

the same reason, this measure can change a lot following a variation in the adopted technique by one firm only. Hence we will not refer to the solid angle as our measure of heterogeneity, but we'd rather focus on some normalized measures of the zonotope's volume.

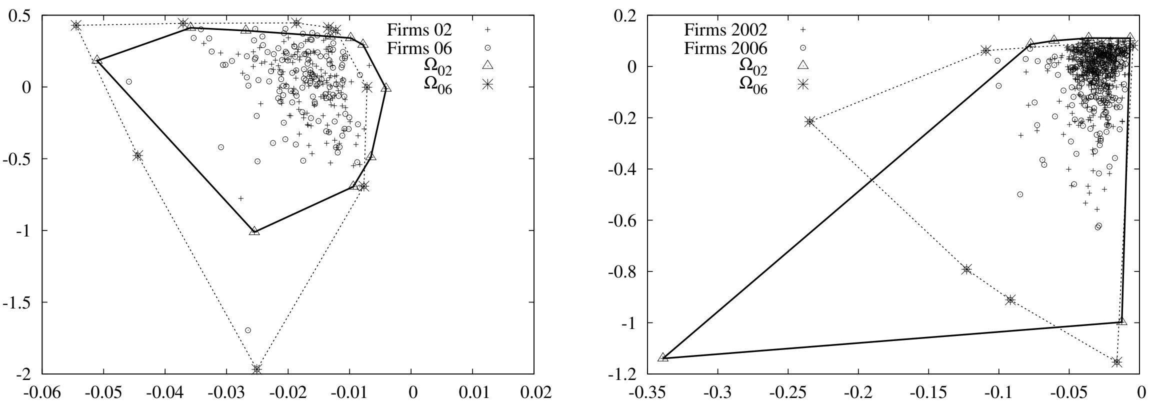


Figure 7: The plot depicts a planar section of the solid angle generated by all vectors in year 2002 and 2006. The section plane is the one perpendicular to the vector sum of generators. Meat and poultrymeat (sector 1513) on the left and Footware (sector 1930) on the right.

4.1 Within Industry Heterogeneity and its dynamics

Table 4 reports the normalized volumes for the sectors under analysis. The first set of columns report $G(Y)$ which, to repeat, is the ratio between the zonotope's volume and the volume of the parallelotope build on the zonotope's main diagonal, for 1998, 2002 and 2006. Notice that the volume of the cuboid (denominator) is much bigger than that of the zonotope (nominator) because, intuitively, the parellelotope is the notional production set formed by production activities that produce no output with positive amounts of inputs, and conversely, others which produce a high quantity of output with no input. That is why the ratio, $G(Y)$, although small

NACE Code	I $G(Y)$ '98	'02	'06	II $G(\overline{Y})$ '98	'02	'06	III $G(Y_e)$ '98	'02	'06
1513	0.059	0.051	0.062	0.082	0.062	0.096	0.391	0.201	0.301
1721	0.075	0.068	0.103	0.075	0.078	0.124	0.135	0.120	0.133
1772	0.160	0.122	0.136	0.154	0.126	0.130	0.142	0.273	0.172
1930	0.108	0.139	0.150	0.110	0.115	0.123	0.361	0.562	0.249
2121	0.108	0.043	0.062	0.081	0.064	0.081	0.257	0.105	0.178
2222	0.062	0.077	0.087	0.077	0.086	0.115	0.239	0.328	0.356
2522	0.065	0.061	0.070	0.071	0.064	0.074	0.197	0.261	0.266
2524	0.089	0.083	0.094	0.097	0.088	0.096	0.458	0.269	0.307
2661	0.079	0.088	0.099	0.100	0.094	0.110	0.376	0.234	0.352
2663	0.066	0.067	0.088	0.111	0.106	0.111	0.306	0.192	0.277
2751	0.035	0.037	0.070	0.064	0.055	0.073	0.174	0.107	0.184
2811	0.105	0.109	0.109	0.117	0.113	0.122	0.327	0.480	0.416
2852	0.088	0.102	0.110	0.100	0.103	0.111	0.227	0.395	0.391
2953	0.072	0.095	0.096	0.098	0.104	0.111	0.233	0.155	0.248
2954	0.078	0.074	0.093	0.086	0.130	0.113	0.170	0.141	0.352
3611	0.078	0.099	0.118	0.107	0.096	0.121	0.288	0.233	0.281

Table 4: Normalized volumes in 1998, 2002 and 2006 for selected 4 digit sectors.

in absolute value, points nonetheless at big differences in the production techniques employed by firms in the industry. The dynamics over time of the ratio within any one industry allows to investigate how heterogeneity in the adopted techniques evolves. $G(Y)$ displays indeed an increase over time in most sectors, suggesting that heterogeneity has not been shrinking, but if anything it has increased.[8] Since $G(Y)$ is a ratio, we can also compare this measure of heterogeneity across industries. As it might be expected, there are relevant differences in such degrees of heterogeneity, with $G(Y)$ varying in the range .03-.16. Interestingly, also sectors that are supposed to produce rather homogeneous output, such as 2661, Concrete, and 2663, Ready-mixed concrete, display a degree of heterogeneity comparable, if not higher, to that of many other sectors.

The second set of columns reports the value of $G(\overline{Y})$, that is the Gini volume of the *unitary* zonotope. As discussed in Section 3, in this case the zonotope is formed by vectors having the same (unitary) length; hence size plays no role. For most of sectors, $G(\overline{Y})$ is bigger than $G(Y)$ suggesting that, within any industry, smaller firms contribute relatively more to heterogeneity than bigger ones. In particular, in some industries, such as the Ready-mixed concrete (NACE 2663), industry heterogeneity almost doubles when all firms are rescaled to have the same size. Finally, also $G(\overline{Y})$ display an increasing trend over time, from 2002 to 2006, pointing to growing differences in the techniques in use.

Finally, $G(Y_e)$ (column III) reports the Gini volume for the zonotope built on the external vectors only. As it could be expected, for all sectors $G(Y_e)$ is bigger than $G(Y)$ as the former maps a sort of "overall frontier" made of better and worse techniques.

4.2 Assessing industry level technical change

Next, let us take to the data the analysis of technical change by means of the angle that the main diagonal of the zonotope forms with the input plane.

As shown in the toy illustration above, an increase in the tangent of the angle with the plane of inputs is evidence of an increase of efficiency of the industry. The first two columns of Table 5

[8]This result is coherent with the evidence shown in Dosi et al. (2012) on Italian firms, although employing a different methodology to explore heterogeneity.

NACE Code	(a) rates of growth of tg θ_3 1998-2002	2002-2006	(b) rates of growth of tg $\overline{\theta}_3$ 1998-2002	2002-2006
1513	-11.9073	-11.4541	-3.5540	-3.8569
1721	10.5652	4.3723	3.6084	1.4179
1772	-2.5717	32.9763	-0.3235	3.1497
1822	30.2406	42.1259	2.0202	1.9519
1930	3.1152	25.2797	0.3005	1.9487
2121	-6.8362	-8.8206	-3.1977	-4.4401
2222	-23.8199	2.8973	-8.2509	1.0216
2522	-18.0316	-16.8038	-8.2018	-8.9011
2524	-15.2821	0.4118	-4.6589	0.1282
2661	6.7277	-18.5953	1.4119	-4.6636
2663	25.4457	-19.6427	6.6499	-5.9972
2751	-33.6675	12.9994	-15.6436	5.9044
2811	6.4256	-7.9102	0.9619	-1.2723
2852	-12.0712	2.1536	-2.4139	0.4289
2953	19.3637	-4.7927	1.1784	-0.3032
2954	-0.3020	-21.2919	-0.0209	-1.8708
3611	-17.9141	0.0892	-2.0263	0.0102

Table 5: Angles of the zonotope's main diagonal, rates of growth. (a) original zonotope; (b) *unitary* zonotope.

reports the rates of growth of tgθ_3 respectively for the period 1998-2002 and 2002-2006.[9]

Overall, not many sectors display a constant increase of productivity (i.e. increase in tgθ_3^t) in all periods. Reassuringly, the results from the method are broadly in line with the rougher evidence stemming from sheer sector-level average productivities and the simple observation of their micro distribution, highlighting a widespread stagnation in Italian manufacturing over the first decade of the new millennium (cf. Dosi et al. 2012). Notice that, in tune with it, also the values of the *unitary* zonotope point to the same pattern.

The change over time of the angle φ_i between the projection vector $pr(d_Y)$ and the x_i axis captures the changes of the quantity of input i over time with respect to all the other inputs. Results are reported in Table 6. For some sectors the value of tgφ_1 decreases over time, suggesting that industries have moved towards relatively more labor intensive techniques (indeed a result which might reveal the peculiarities of the most recent patterns of growth, or more precisely, lack of it, of the Italian economy).

[9]Note that changing the unit of measurement, i.e. considering value added in millions (rather than thousands) of euro of course changes the value of the angle, but the variation over time - our proxy of technical change - is not affected by the unit of measure.

Nace Code	(a) tg φ_1 1998	2002	2006	(b) tg $\overline{\varphi}_1$ 1998	2002	2006
1513	23.0224	25.5508	27.0043	21.8256	22.6442	20.1068
1721	21.0047	21.3726	18.4777	15.1804	16.0332	12.8776
1772	6.8281	6.7041	5.5909	5.0274	6.2307	5.6227
1930	4.5795	5.3113	5.0533	3.8295	4.5798	4.4675
2121	39.0274	39.3436	40.0129	23.9678	26.6887	24.8705
2222	19.1097	26.1785	24.3621	13.2630	17.2095	14.6962
2522	30.2555	37.3270	43.8918	23.9336	27.9305	27.1886
2524	17.9118	21.4862	19.9956	13.4808	15.7137	14.6993
2661	14.1626	16.5427	17.9402	10.5512	12.3908	12.0912
2663	26.9417	26.3218	26.9437	15.1537	16.5539	15.0585
2751	21.8179	36.6899	31.9027	19.4979	30.2839	24.7586
2811	9.1053	9.7865	9.8170	6.9113	7.8659	6.6393
2852	10.0784	13.1988	13.2519	7.9099	10.4204	10.0850
2953	5.4316	5.3541	5.9111	4.1020	5.0619	4.7180
2954	5.0435	5.1530	5.8891	4.0276	3.7809	4.6053
3611	5.7162	6.3274	6.2222	4.5100	5.3190	5.0401

Table 6: Angles of the zonotope's main diagonal. (a) Angles on production inputs plane, original zonotope; (b) *unitary* zonotope.

5 Conclusions

How does one synthetically accounts for the actual "state of the technology" of any industry when firm-level techniques are widely and persistently heterogeneous? Hildenbrand (1981) suggested a seminal methodology focusing on the geometric properties of the *actual* activities - that is the actual input-output relations - displayed by the firms composing the industry. And he analyzed the features of such constructs in terms of the standard properties normally postulated by production functions but not born by the actual data. Here we pushed the investigation some steps further. First, we used different measures of volume of the geometrical objects defined by firms' activities as measures of inter-firm technological heterogeneity. Second, we investigated the properties of the dynamics of such objects over time as meaningful proxies for industry-level technological change quite independent of any behavioral assumptions on allocative strategies of individual firms, and on the relationships between input prices and input intensities.

A straightforward step ahead involves the disentangling between movement of the "frontier" however defined and movements of the weighted an unweighted distributions of techniques across firms. And another one entails indeed the study of the relationships of the foregoing dynamics with relative input prices, if any, and, probably more important, with the patterns of learning and competition characteristic of each industry.

Appendix

The database employed for the analyses, Micro.3, has been built through to the collaboration between the Italian statistical office, ISTAT, and a group of LEM researchers from the Scuola Superiore Sant'Anna, Pisa.[10]

Micro.3 is largely based on the census of Italian firms yearly conducted by ISTAT and contains information on firms above 20 employees in all sectors[11] of the economy for the period 1989-2006. Starting in 1998 the census of the whole population of firms only concerns companies with more than 100 employees, while in the range of employment 20-99, ISTAT directly monitors only a "rotating sample" which varies every five years. In order to complete the coverage of firms in the range of employment 20-99 Micro.3 resorts, from 1998 onward, to data from the financial statement that limited liability firms have to disclose, in accordance to Italian law.[12]

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[10] The database has been made available for work after careful censorship of individual information. More detailed information concerning the development of the database Micro.3 are in Grazzi et al. (2009).

[11] In the paper we use the Statistical Classification of Economic Activities known as NACE, Revision 1.1.

[12] Limited liability companies (*società di capitali*) have to hand in a copy of their financial statement to the Register of Firms at the local Chamber of Commerce

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