On Fano manifolds of large pseudoindex

ON FANO MANIFOLDS OF LARGE PSEUDOINDEX

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Abstract. We describe Fano manifolds of large pseudoindex that are rationally connected with respect to some numerically independent families of rational curves.

1. Introduction

Let X be a Fano manifold, *i.e.* smooth complex projective variety whose anticanonical bundle $-K_X$ is ample. A Fano manifold is associated with two invariants, namely the *index*, r_X, defined as the largest integer dividing $-K_X$ in the Picard group of X, and the *pseudoindex*, i_X, defined as the minimum anticanonical degree of rational curves on X. It is known that these invariants satisfy the relations $1 \leq r_X \leq i_X \leq \dim X + 1$ ([17] and [11]). Moreover, the index of a Fano manifold X is related with the dimension of X and the Picard number, ρ_X, of X by the following conjecture of Mukai ([22]):

$$\rho_X(r_X - 1) \leq \dim X, \text{ with equality if and only if } X = (\mathbb{P}^{r_X-1})^{\rho_X}.$$

The first step towards this conjecture was made in [32], where the notion of pseudoindex was introduced. In general, when dealing with Fano manifolds of large Picard number, it can happen that the index is equal to one even for simple varieties such as $\mathbb{P}^s \times \mathbb{P}^{s+1}$, so it seems that in studying these varieties the pseudoindex could be a more useful invariant than the index. In particular, the above conjecture has been restated ([7]) by replacing the index with the pseudoindex, so the conjecture has the following generalized form:

$$\rho_X(i_X - 1) \leq \dim X, \text{ with equality if and only if } X = (\mathbb{P}^{i_X-1})^{\rho_X}.$$

We consider Fano manifolds of large pseudoindex, more precisely we are interested in Fano manifolds of pseudoindex $i_X > \frac{\dim X}{3}$, since under this assumption the generalization of the conjecture of Mukai has been proved ([26, Theorem 3], [23, Theorem 5.1]; see also [32, Theorem A] and [29, Corollary 4.3] for $i_X \geq \frac{\dim X+2}{2}$). This paper is intended as a first step to the actual classification of Fano manifolds with $i_X \geq \frac{\dim X+1}{3}$, and it deals with Fano manifolds with Picard number $\rho_X \geq 3$. In general when the Picard number of the variety is large, namely $\rho_X \geq 4$, the setting is quite easy to be understood; as to next case, namely $\rho_X = 3$, these varieties are more difficult to classify. However, by looking at the proof of the generalized Mukai conjecture, one can see that X is rationally connected with respect to some families of rational curves and that these families have "good" properties. So we can make use of such families of rational curves to study the

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manifolds we are interested in. This allows us to give the complete classification of Fano manifolds of pseudoindex $i_X \geq \frac{\dim X+2}{3}$ and Picard number $\rho_X \geq 3$:

Theorem. *Let X be a Fano manifold of pseudoindex $i_X \geq \frac{\dim X+2}{3}$ and Picard number $\rho_X \geq 3$. Then one of the following holds:*

$i_X = \frac{\dim X+3}{3}$	$\rho_X = 3$	$X = \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X-1}$
$i_X = \frac{\dim X+2}{3}$	$\rho_X = 3$	$X = \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X}$
		$X = \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X-1} \times \mathbb{Q}^{i_X}$, *with* $i_X \geq 3$
		$X = \mathbb{P}^{i_X-1} \times \mathbb{P}_{\mathbb{P}^{i_X}}(T_{\mathbb{P}^{i_X}})$
		$X = \mathbb{P}^{i_X-1} \times \mathrm{Bl}_{\mathbb{P}^{i_X-2}} \mathbb{P}^{2i_X-1}$
	$\rho_X = 4$	$X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

When $i_X = \frac{\dim X+1}{3}$ and $\rho_X \geq 3$ things are much more complicated. However we can give the complete classification both in case $\rho_X \geq 4$ (Proposition 5.1), and in case $\rho_X = 3$ if X is rationally connected with respect to three unsplit families of rational curves, one of them having anticanonical degree greater than i_X (Theorem 5.7).

The paper is organized as follows: in Section 2 we collect basic material concerning definitions and results on extremal contractions, on families of rational curves and on chains of rational curves on projective manifolds; in Section 3 we consider families of rational curves on Fano manifolds and we give results on the extremality of some of these families; in Section 4 we study Fano manifolds of Picard number $\rho_X \geq 3$ and pseudoindex $i_X \geq \frac{\dim X+2}{3}$ and we give the complete classification of such manifolds; in the last section we address our investigation to Fano manifolds of Picard number $\rho_X \geq 3$ and pseudoindex $i_X = \frac{\dim X+1}{3}$.

2. Background material

Let X be a smooth complex projective variety.

Definition 2.1. A *contraction* $\varphi\colon X \to Y$ is a proper surjective map with connected fibers onto a normal variety Y. If the canonical bundle K_X is not nef, then the negative part of the closure $\overline{\mathrm{NE}}(X)$ of the cone of effective 1-cycles into the $\mathbb{R}$-vector space of 1-cycles modulo numerical equivalence is polyhedral, by the Cone Theorem. By the Contraction Theorem, every face in this part of the cone, called *extremal face*, is associated with a contraction, called *extremal contraction* or *Fano–Mori contraction*.
An extremal contraction associated with an extremal face of dimension one, *i.e.* with an *extremal ray*, is called an *elementary contraction*; if $\dim Z < \dim Y$ then it is called *of fiber type*, otherwise it is called *birational*. If the codimension of the exceptional locus of an elementary birational contraction is equal to one, the contraction is called *divisorial*, otherwise it is called *small*. The *length* of an extremal ray is defined as the minimum anticanonical degree of rational curves whose numerical equivalence class belongs to the ray; a rational curve attaining the length of the ray is called *minimal* curve of the ray. A Cartier divisor which is the pull-back of an ample divisor A on Y is called a *supporting divisor* of the contraction φ.

Remark 2.2. Fibers of contractions associated with different extremal rays can meet at most at points.

Definition 2.3. We call $\mathbb{P}^r$*-bundle* a morphism whose general fibers are $\mathbb{P}^r$.

Definition 2.4. A *family of rational curves* V on X is an irreducible component of the scheme $\mathrm{RatCurves}^n(X)$ (see [18, Definition II.2.11]).
Given a rational curve we will call a *family of deformations* of that curve any irreducible component of $\mathrm{RatCurves}^n(X)$ containing the point parameterizing that curve.
We define $\mathrm{Locus}(V)$ to be the set of points of X through which there is a curve among those parameterized by V; we say that V is a *covering family* if $\mathrm{Locus}(V) = X$ and that V is a *dominating family* if $\overline{\mathrm{Locus}(V)} = X$.
By abuse of notation, given a line bundle $L \in \mathrm{Pic}(X)$, we will denote by $L \cdot V$ the intersection number $L \cdot C$, with C any curve among those parameterized by V.
We will say that V is *unsplit* if it is proper; clearly, an unsplit dominating family is covering.
We denote by V_x the subscheme of V parameterizing rational curves passing through a point x and by $\mathrm{Locus}(V_x)$ the set of points of X through which there is a curve among those parameterized by V_x. If, for a general point $x \in \mathrm{Locus}(V)$, V_x is proper, then we will say that the family is *locally unsplit*; by Mori's Bend and Break arguments, if V is a locally unsplit family, then $-K_X \cdot V \leq \dim X + 1$.
If X admits dominating families, we can choose among them one with minimal degree with respect to a fixed ample line bundle A, and we call it a *minimal dominating family.* Such a family is locally unsplit.

Definition 2.5. Let U be an open dense subset of X and $\pi\colon U \to Z$ a proper surjective morphism to a quasi-projective variety; we say that a family of rational curves V is a *horizontal dominating family with respect to* π if $\mathrm{Locus}(V)$ dominates Z and curves parameterized by V are not contracted by π. If such families exist, we can choose among them one with minimal degree with respect to a fixed ample line bundle and we call it a *minimal horizontal dominating family* with respect to π; such a family is locally unsplit.

Remark 2.6. By fundamental results in [21], a Fano manifold admits dominating families of rational curves; also horizontal dominating families with respect to proper morphisms defined on an open set exist, as proved in [19]. In the case of Fano manifolds with "minimal" we will mean minimal with respect to $-K_X$, unless otherwise stated.

Definition 2.7. We define a *Chow family of rational 1-cycles* $\mathcal{W}$ to be an irreducible component of $\mathrm{Chow}(X)$ parameterizing rational and connected 1-cycles.
We define $\mathrm{Locus}(\mathcal{W})$ to be the set of points of X through which there is a cycle among those parameterized by $\mathcal{W}$; notice that $\mathrm{Locus}(\mathcal{W})$ is a closed subset of X ([18, II.2.3]). We say that $\mathcal{W}$ is a *covering family* if $\mathrm{Locus}(\mathcal{W}) = X$.
If V is a family of rational curves, the closure of the image of V in $\mathrm{Chow}(X)$, denoted by $\mathcal{V}$, is called the *Chow family associated with* V.

Remark 2.8. If V is proper, *i.e.* if the family is unsplit, then V corresponds to the normalization of the associated Chow family $\mathcal{V}$.

Definition 2.9. Let V be a family of rational curves and let $\mathcal{V}$ be the associated Chow family. We say that V (and also $\mathcal{V}$) is *quasi-unsplit* if every component of any reducible cycle parameterized by $\mathcal{V}$ has numerical class proportional to the numerical class of a curve parameterized by V.

Definition 2.10. Let $V^1, \dots, V^k$ be families of rational curves on X and $Y \subset X$. We define $\mathrm{Locus}(V^1)_Y$ to be the set of points $x \in X$ such that there exists a curve C among those parameterized by V^1 with $C \cap Y \neq \emptyset$ and $x \in C$. We inductively define $\mathrm{Locus}(V^1, \dots, V^k)_Y := \mathrm{Locus}(V^2, \dots, V^k)_{\mathrm{Locus}(V^1)_Y}$. Notice that, by this definition, we have $\mathrm{Locus}(V)_x = \mathrm{Locus}(V_x)$. Analogously we define $\mathrm{Locus}(\mathcal{W}^1, \dots, \mathcal{W}^k)_Y$ for Chow families $\mathcal{W}^1, \dots, \mathcal{W}^k$ of rational 1-cycles.

Notation. We denote by ρ_X the Picard number of X, *i.e.* the dimension of the $\mathbb{R}$-vector space $\mathrm{N}_1(X)$ of 1-cycles modulo numerical equivalence. If Γ is a 1-cycle, then we will denote by $[\Gamma]$ its numerical equivalence class in $\mathrm{N}_1(X)$; if V is a family of rational curves, we will denote by $[V]$ the numerical equivalence class of any curve among those parameterized by V.
If $Y \subset X$, we will denote by $\mathrm{N}_1(Y, X) \subseteq \mathrm{N}_1(X)$ the vector subspace generated by numerical classes of curves of X contained in Y; moreover, we will denote by $\mathrm{NE}\,(Y, X) \subseteq \mathrm{NE}(X)$ the subcone generated by numerical classes of curves of X contained in Y.

We will make frequent use of the following dimensional estimates:

Proposition 2.11. ([18, IV.2.6]) *Let V be a family of rational curves on X and $x \in \mathrm{Locus}(V)$ a point such that every component of V_x is proper. Then*

(a) $\dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) \geq \dim X - K_X \cdot V - 1$;
(b) $\dim \mathrm{Locus}(V_x) \geq -K_X \cdot V - 1$.

Definition 2.12. We say that k quasi-unsplit families $V^1, \dots, V^k$ of rational curves are numerically independent if, in $\mathrm{N}_1(X)$, we have $\dim\langle [V^1], \dots, [V^k] \rangle = k$.

Lemma 2.13. (*Cf.* [1, Lemma 5.4]) *Let $Y \subset X$ be a closed subset and $V^1, \dots, V^k$ numerically independent unsplit families of rational curves such that $\langle [V^1], \dots, [V^k] \rangle \cap NE(Y, X) = \underline{0}$. Then either $\mathrm{Locus}(V^1, \dots, V^k)_Y = \emptyset$ or*

$$\dim \mathrm{Locus}(V^1, \dots, V^k)_Y \geq \dim Y + \sum -K_X \cdot V^i - k.$$

Definition 2.14. Let $Y \subset X$ be a closed subset, let V be a dominating family of rational curves on X and denote by $\mathcal{V}$ be the associated Chow family; define $\mathrm{ChLocus}(\mathcal{V})_Y$ to be the set of points $x \in X$ such that there exist cycles $\Gamma_1, \dots, \Gamma_m$ with the following properties:

- Γ_i belongs to the family $\mathcal{V}$;
- $\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$;
- $\Gamma_1 \cap Y \neq \emptyset$ and $x \in \Gamma_m$,

i.e. $\mathrm{ChLocus}(\mathcal{V})_Y$ is the set of points that can be joined to Y by a connected chain of at most m cycles belonging to the family $\mathcal{V}$.

We will use the description of the numerical expression of curves in $\mathrm{ChLocus}(\mathcal{V})_Z$, with $Z \subset X$ a closed subset and V a quasi-unsplit family of rational curves, as stated in [27, Lemma 1.10].

Lemma 2.15. (Cf. [6, Proof of Lemma 1.4.5], [29, Lemma 3.2 and Remark 3.3]) *Let $Z \subset X$ be a closed subset and let V be a quasi-unsplit family of rational curves. Then every curve contained in $\mathrm{ChLocus}(\mathcal{V})_Z$ is numerically equivalent to a linear combination with rational coefficients*

$$\lambda_V C_V + \lambda_Z C_Z,$$

with C_V a curve among those parameterized by V, C_Z a curve in Z and $\lambda_Z \geq 0$.

Define a relation of *rational connectedness with respect to* $\mathcal{V}$ on X in the following way: two points x and y of X are in $\mathrm{rc}(\mathcal{V})$-relation if there exists a chain of cycles in $\mathcal{V}$ which joins x and y, *i.e.* if $y \in \mathrm{ChLocus}(\mathcal{V})_x$. In particular, X is *rc*($\mathcal{V}$)*-connected* ifwe have $X = \mathrm{ChLocus}(\mathcal{V})_x$.

The family $\mathcal{V}$ defines a proper prerelation in the sense of [18, Definition IV.4.6]. This prerelation is associated with a fibration, which we will call the *rc*($\mathcal{V}$)*-fibration*:

Theorem 2.16. ([18, IV.4.16], *Cf.* [9]) *Let X be a normal and proper variety and $\mathcal{V}$ a proper prerelation; then there exists an open subvariety $X^0 \subset X$ and a proper morphism with connected fibers $\pi \colon X^0 \to Z^0$ such that*

- $\langle \mathcal{U} \rangle$ *restricts to an equivalence relation on* X^0*;*
- $\pi^{-1}(z)$ *is a* $\langle \mathcal{U} \rangle$*-equivalence class for every* $z \in Z^0$*;*
- $\forall\, z \in Z^0$ *and* $\forall\, x, y \in \pi^{-1}(z)$*,* $x \in \mathrm{ChLocus}(\mathcal{V})_y$ *with* $m \leq 2^{\dim X - \dim Z^0} - 1$.

Clearly X is $\mathrm{rc}(\mathcal{V})$-connected if and only if $\dim Z^0 = 0$.

Given $\mathcal{V}^1, \ldots, \mathcal{V}^k$ Chow families of rational 1-cycles, it is possible to define a relation of $\mathrm{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-connectedness, which is associated with a fibration, that we will call $\mathrm{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-fibration. The variety X will be called *rc*($\mathcal{V}^1, \ldots, \mathcal{V}^k$)*-connected* if the target of the fibration is a point.

Notation. In the next sections for simplicity we will write $\mathrm{Locus}(V)_x$ to mean $\mathrm{Locus}(V)_x$ for a general point $x \in \mathrm{Locus}(V)$, and $\mathrm{Locus}(V^\alpha, \ldots, V^\beta)_{x_\alpha}$ to mean $\mathrm{Locus}(V^\alpha, \ldots, V^\beta)_{x_\alpha}$ for a general point $x_\alpha \in \mathrm{Locus}(V^\alpha)$, unless otherwise stated.

3. Families of rational curves and extremal rays

We start this section by recalling the following general construction.

Construction 3.1. ([26, Construction 1]) Let X be a Fano manifold; let V^1 be a minimal dominating family of rational curves on X and consider the associated Chow family $\mathcal{V}^1$. If X is not $\mathrm{rc}(\mathcal{V}^1)$-connected, let V^2 be a minimal horizontal dominating family with respect to the $\mathrm{rc}(\mathcal{V}^1)$-fibration, $\pi_1 \colon X \dashrightarrow Z^1$. If X is not $\mathrm{rc}(\mathcal{V}^1, \mathcal{V}^2)$-connected, we denote by V^3 a minimal horizontal dominating family with respect to the $\mathrm{rc}(\mathcal{V}^1, \mathcal{V}^2)$-fibration, $\pi_2 \colon X \dashrightarrow Z^2$, and so on. Since $\dim Z^{i+1} < \dim Z^i$, for some integer k we have that X is $\mathrm{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-connected.

By abuse of notation, we will write V^i instead of $\mathcal{V}^i$ if the family is unsplit.

Remark 3.2. Examples of the above construction are given in [23, Examples 4.2]. Note that at each step the dimension drops at least by $\dim \mathrm{Locus}(V^i)_{x_i}$; moreover, if a family V^i is dominating, the minimality assumption implies $-K_X \cdot V^1 \leq -K_X \cdot V^i$.

Remark 3.3. Let X be a Fano manifold of dimension $\dim X \geq 3$, pseudoindex $i_X \geq \frac{\dim X + 1}{3}$ and Picard number $\rho_X \geq 3$. By looking at the proofs of [23, Theorem 5.1] and [26, Theorem 5], we see that, if one of the families V^j as in Construction 3.1 is not unsplit, then $\dim X = 5$, $i_X = 2$ and X is $\mathrm{rc}(V^1, V^2, V^3)$-connected.

Lemma 3.4. *Let X be a Fano manifold of dimension $\dim X \geq 3$ and pseudoindex $i_X \geq \frac{\dim X + 1}{3}$ such that X is rc(V^1, V^2, V^3)-connected with respect to three families of rational curves as in Construction 3.1. Then $-K_X \cdot V^1 = i_X$.*

Proof. Assume to get a contradiction that $-K_X \cdot V^1 \geq i_X+1$. Since at the i-th step in Construction 3.1 the dimension drops at least by $\dim \operatorname{Locus}(V^i)_{x_i}$, by Proposition 2.11 we obtain that $i_X = \frac{\dim X+1}{3}$, $-K_X \cdot V^1 = i_X+1$, $-K_X \cdot V^2 = -K_X \cdot V^3 = i_X$ and that the family V^2 and V^3 are dominating; we thus have a contradiction with Remark 3.2. $\square$

The following result is an immediate consequence of Lemma 2.15.

Lemma 3.5. *Let X be a Fano manifold such that X is rc(V^1,V^2,V^3)-connected with respect to three unsplit families of rational curves as in Construction 3.1. If $X = \operatorname{ChLocus}(V^\beta, V^\alpha)_G$ with $G \subset X$ a closed subset such that $\mathrm{N}_1(G,X) = \langle [V^\gamma] \rangle$ and $\{\alpha,\beta,\gamma\} = \{1,2,3\}$, then $\langle [V^\alpha],[V^\beta] \rangle$ is extremal.*

Proof. By repeated applications of Lemma 2.15, we can write the numerical class of any curve in X as $\lambda_\alpha[V^\alpha] + \lambda_\beta[V^\beta] + \lambda_\gamma[V^\gamma]$, with $\lambda_\gamma \geq 0$. Therefore, if C_a and C_b are curves of X whose numerical classes satisfy $[C_a]+[C_b] \in \langle [V^\alpha],[V^\beta] \rangle$, it is clear that $[C_a],[C_b] \in \langle [V^\alpha],[V^\beta] \rangle$. $\square$

Lemma 3.6. *Let X be a Fano manifold of pseudoindex $i_X \geq \frac{\dim X+2}{3}$ such that X is rc(V^1,V^2,V^3)-connected with respect to three dominating families of rational curves as in Construction 3.1. Then $NE(X) = \langle [V^1],[V^2],[V^3] \rangle$.*

Proof. From Remark 3.3, we know that V^1, V^2 and V^3 are unsplit. We prove first that at least two of these families span extremal rays.

Suppose to get a contradiction that the numerical classes of two of these families, say V^α and V^β, do not span an extremal ray.

Since $[V^\alpha]$ does not span an extremal ray, by [8, Proposition 1] there exists an irreducible component G of a rc(V^α)-equivalence class of dimension at least $-K_X \cdot V^\alpha$. By computing the dimension of $\operatorname{Locus}(V^\beta,V^\gamma)_G$ with Lemma 2.13, we derive $X = \operatorname{Locus}(V^\beta,V^\gamma)_G$ (and $-K_X \cdot V^i = i_X = \frac{\dim X+2}{3}$ for $i=1,2,3$), so $\langle [V^\beta],[V^\gamma] \rangle$ is extremal by Lemma 3.5. By exchanging the role of V^α and V^β, we get that also $\langle [V^\alpha],[V^\gamma] \rangle$ is extremal, hence $[V^\gamma]$ spans an extremal ray, say $\mathbb{R}_+[V^\gamma]$. Denote by π_γ the contraction associated with $\mathbb{R}_+[V^\gamma]$ and put $G_\gamma := (\pi_\gamma)^{-1}(\pi_\gamma(G))$. Since, by Lemma 2.13, $X = \operatorname{Locus}(V^\beta)_{G_\gamma}$, it follows that G_γ intersects all the rc(V^β)-equivalence classes; we derive that these classes are equidimensional, so $[V^\beta]$ spans an extremal ray, which is a contradiction.

Therefore at least two families among V^1,V^2,V^3 span extremal rays.

Now, by computing the dimension of the general fibers of the contractions associated with these extremal rays with [33, Theorem 1.1] and recalling Remark 2.2, we see that X does not admit any extremal ray associated with a small contraction. So the last assertion follows by repeated applications of [10, Lemma 2.4]. $\square$

Lemma 3.7. *Let X be a Fano manifold of pseudoindex $i_X = \frac{\dim X+1}{3}$ such that X is rc(V^1,V^2,V^3)-connected with respect to three unsplit dominating families of rational curves as in Construction 3.1. Then the numerical class of at least one of these families spans an extremal ray.*

Proof. Suppose that the numerical classes of two of these families, say V^α and V^β, do not span an extremal ray. We claim that the numerical class of the third family, say V^γ, spans an extremal ray.

Since $[V^\alpha]$ does not span an extremal ray, by [8, Proposition 1] there exists an irreducible component G of a rc(V^α)-equivalence class of dimension at least $-K_X \cdot$

V^α. Then, by Lemma 2.13, $\dim \mathrm{Locus}(V^\beta, V^\gamma)_G \geq \dim X - 1$.
If equality holds, denote by D an irreducible component of maximal dimension of $\mathrm{Locus}(V^\beta, V^\gamma)_G$. If D is positive on V^β or V^γ, then $X = \mathrm{ChLocus}(V^\beta, V^\gamma)_G$ and so $\langle [V^\beta], [V^\gamma] \rangle$ is extremal by Lemma 3.5. If $D \cdot V^\beta = 0$ and $D \cdot V^\gamma = 0$, then $D|_D$ is nef since curves in D can be written as $\lambda_\alpha [V^\alpha] + \lambda_\beta [V^\beta] + \lambda_\gamma [V^\gamma]$, with $\lambda_\alpha \geq 0$; so $\langle [V^\beta], [V^\gamma] \rangle$ is extremal.
If else, $\langle [V^\beta], [V^\gamma] \rangle$ is extremal by Lemma 3.5.
Therefore we get the claim by exchanging the role of V^α and V^β. □

Lemma 3.8. *Let X be a Fano manifold of pseudoindex $i_X \geq \frac{\dim X + 2}{3}$ such that X is rc(V^1, V^2, V^3)-connected with respect to three families of rational curves as in Construction 3.1.*

(a) *If V^2 is not dominating, then $[V^1]$ and $[V^3]$ span two extremal rays.*
(b) *If V^3 is not dominating, then $[V^1]$ and $[V^2]$ span two extremal rays.*

Proof. From Remark 3.3, we know that V^1, V^2 and V^3 are unsplit. Assume that at least one family between V^2 and V^3 is not dominating.

By construction we have $-K_X \cdot V^i = i_X = \frac{\dim X + 2}{3}$ for $i = 1, 2, 3$ and exactly one family, say V^α, between V^2 and V^3 is covering. Note that the non-covering family, say V^β, is horizontal and dominating with respect to the rc(V^1, V^α)-fibration. Since, by Lemma 2.13, $X = \mathrm{Locus}(V^\alpha, V^1)_{\mathrm{Locus}(V^\beta)_{x_\beta}}$, $\langle [V^1], [V^\alpha] \rangle$ is extremal by Lemma 3.5.

Since $X = \mathrm{Locus}(V^1)_{\mathrm{Locus}(V^\beta, V^\alpha)_{x_\beta}}$, by Lemma 2.15 every curve in X is numerically equivalent to $\lambda_1 C_1 + \lambda_\Gamma \Gamma$, with $[C_1] \in [V^1]$, Γ an effective curve in $\mathrm{Locus}(V^\beta, V^\alpha)_{x_\beta}$ and $\lambda_\Gamma \geq 0$. Moreover, for a curve C such that $[C] \in \langle [V^1], [V^\alpha] \rangle$, it must be $[\Gamma] \in \langle [V^1], [V^\alpha] \rangle \cap \langle [V^\alpha], [V^\beta] \rangle$, hence $[\Gamma] = \mu_\alpha [V^\alpha]$. So $[C] = \lambda_1 [V^1] + \lambda_\alpha [V^\alpha]$, with $\lambda_\alpha \geq 0$. Therefore, if C_a and C_b are curves such that $[C_a] + [C_b] \in \mathbb{R}_+ [V^1]$, being $\mathbb{R}_+[V^1] \subset \langle [V^1], [V^\alpha] \rangle$ we easily derive that $[C_a], [C_b] \in \mathbb{R}_+[V^1]$, so $[V^1]$ spans an extremal ray.

If V^2 is not dominating, by construction $X = \mathrm{Locus}(V^3, V^1)_{\mathrm{Locus}(V^2)_{x_2}} = \mathrm{Locus}(V^1, V^3)_{\mathrm{Locus}(V^2)_{x_2}}$, so we argue as before by exchanging the role of V^1 and V^3. We thus obtain part (a) of the statement.

If V^3 is not dominating, we can argue as in the previous case by exchanghing V^2 and V^3. We thus obtain part (b) of the statement. □

Corollary 3.9. *Let X be a Fano manifold of pseudoindex $i_X = \frac{\dim X + 2}{3}$ such that X is rc(V^1, V^2, V^3)-connected with respect to three families of rational curves as in Construction 3.1. Then X admits (at least) two extremal rays associated with contractions of fiber type and no extremal rays associated with small contractions.*

Proof. By construction, at most one family between V^2 and V^3 is not dominating. It thus follows from Lemma 3.6 and Lemma 3.8 that X admits two extremal rays associated with contractions of fiber type.

Now, by computing the dimension of the general fibers of the contractions associated with these extremal rays with [33, Theorem 1.1] and recalling Remark 2.2, if X admits an extremal ray associated with a birational contraction, then by using the same theorem and remark we see that all the non-trivial fibers of this contraction have dimension equal to i_X; therefore this contraction cannot be small, again by [33, Theorem 1.1]. □

By arguing as in the proof of Lemma 3.8, we have the following

Lemma 3.10. *Let X be a Fano manifold of pseudoindex $i_X = \frac{\dim X+1}{3} \geq 2$ such that X is rc(V^1, V^2, V^3)-connected with respect to three unsplit families of rational curves as in Construction 3.1 with $-K_X \cdot V^2 = i_X + 1$.*

(a) *If V^2 is not dominating, then $[V^1]$ and $[V^3]$ span two extremal rays.*
(b) *If V^3 is not dominating, then $[V^1]$ and $[V^2]$ span two extremal rays.*

Lemma 3.11. *Let X be a Fano manifold of pseudoindex $i_X = \frac{\dim X+1}{3}$ such that X is rc(V^1, V^2, V^3)-connected with respect to three unsplit families of rational curves as in Construction 3.1 with $-K_X \cdot V^3 = i_X + 1$. Then the numerical classes of at least two of these families span extremal rays:*

(a) *if V^3 is not dominating, then $[V^1]$ and $[V^2]$ span two extremal rays.*
(b) *if V^3 is dominating and $[V^3]$ does not span an extremal ray, then $[V^1]$ and $[V^2]$ span two extremal rays.*
(c) *if V^3 is dominating and $[V^3]$ spans an extremal ray, then at least one between $[V^1]$ and $[V^2]$ spans an extremal ray.*

Proof. If V^3 is not dominating, then $-K_X \cdot V^2 = i_X$ and V^2 is dominating. Therefore we get case (a) of the statement by arguing as in the proof of Lemma 3.8.

If V^3 is dominating and $[V^3]$ does not span an extremal ray, by [8, Proposition 3] there exists an irreducible component G of a rc(V^3)-equivalence class of dimension at least $-K_X \cdot V^3$. It follows that V^2 is dominating with $-K_X \cdot V^2 = i_X$. By Lemma 3.7 at least one between $[V^1]$ and $[V^2]$ spans an extremal ray, say $\mathbb{R}_+[V^\alpha]$; so let π_α be the contraction associated with $\mathbb{R}_+[V^\alpha]$ and put $G_\alpha := (\pi_\alpha)^{-1}(\pi_\alpha(G))$. Denoted by V^β the third family, $X = \text{Locus}(V^\beta)_{G_\alpha}$, so G_α intersects all the rc(V^β)-equivalence classes; we derive that these classes are equidimensional, so $[V^\beta]$ spans an extremal ray.

Assume now that V^3 is dominating and $[V^3]$ spans an extremal ray.
If V^2 is not dominating, then it is horizontal and dominating with respect to the rc(V^1, V^3)-fibration. Therefore we can argue as in the proof of part (a) of Lemma 3.8 and we obtain that $[V^1]$ spans an extremal ray.
If V^2 is dominating, then we can argue as in the proof of Lemma 3.6 and we obtain that at least one between $[V^1]$ and $[V^2]$ spans an extremal ray. □

4. Fano manifolds with $i_X \geq \frac{\dim X+2}{3}$ and $\rho_X \geq 3$

In this section we deal with Fano manifolds of Picard number $\rho_X \geq 3$ and pseudoindex $i_X \geq \frac{\dim X+2}{3}$ and we give the complete list of these varieties.

We start by considering manifolds with Picard number $\rho_X \geq 4$:

Proposition 4.1. *Let X be a Fano manifold of pseudoindex $i_X \geq \frac{\dim X+2}{3}$ and Picard number $\rho_X \geq 4$. Then $\dim X = 4$, $i_X = \frac{\dim X+2}{3}$ and $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Clearly $\dim X > 1$, and then the statement follows by [23, Theorem 5.1]. □

As to next case, *i.e.* $\rho_X = 3$, we recall that by [26, Theorem 3] we have

Proposition 4.2. *Let X be a Fano manifold of pseudoindex $i_X \geq \frac{\dim X+3}{3}$ and Picard number $\rho_X \geq 3$. Then $i_X = \frac{\dim X+3}{3}$ and $X = \mathbb{P}^{\frac{\dim X}{3}} \times \mathbb{P}^{\frac{\dim X}{3}} \times \mathbb{P}^{\frac{\dim X}{3}}$.*

Therefore in the rest of the section we have to deal with Fano manifolds with pseudoindex $i_X = \frac{\dim X+2}{3}$ and Picard number $\rho_X = 3$.

Remark 4.3. Let X be a Fano manifold. For some integer k, X is $\mathrm{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$-connected with respect to the Chow families $\mathcal{V}^1, \dots, \mathcal{V}^k$ associated with k numerically independent families of rational curves $V^1, \dots, V^k$ as in Construction 3.1. Assume now that X has pseudoindex $i_X = \frac{\dim X+2}{3}$ and Picard number $\rho_X = 3$. By looking at the proof of [23, Theorem 5.1], we see that $k = 3$ and each family V^i is unsplit, so we know that X is $\mathrm{rc}(V^1, V^2, V^3)$-connected.

We consider first Fano manifolds which are $\mathrm{rc}(V^1, V^2, V^3)$-connected and admit an extremal ray associated with a birational contraction.

Proposition 4.4. *Let X be a Fano manifold of pseudoindex $i_X = \frac{\dim X+2}{3}$ such that X is $rc(V^1, V^2, V^3)$-connected with respect to three families of rational curves as in Construction 3.1. If X admits a birational elementary contraction, then $X = \mathrm{Bl}_{\mathbb{P}^{i_X-2}\times\mathbb{P}^{i_X-1}} \mathbb{P}^{2i_X-1} \times \mathbb{P}^{i_X-1}$.*

Proof. In view of Lemma 3.6, Lemma 3.8 and Corollary 3.9 we know that X admits two different extremal rays, say $R_1 := \mathbb{R}_+[V^1]$ and $R_a := \mathbb{R}_+[V^a]$, $a \in \{2, 3\}$, associated with contractions of fiber type and it does not admit any extremal ray associated with a small contraction. Denote by $\sigma\colon X \to X'$ a birational elementary contraction, which is thus divisorial, and by R_σ the extremal ray associated with σ.

Now we can compute the dimension of the non-trivial fibers of σ and the dimension of the general fibers of the contractions associated with R_1 and R_a by combining [33, Theorem 1.1] with Remark 2.2; we obtain that all the non-trivial fibers of σ have dimension equal to i_X. Then, by [3, Theorem 5.1], σ gives X as the blow-up of a smooth variety X' along a smooth center of dimension $2i_X - 3$. Moreover, recalling Remark 2.2 we know that X cannot have any other extremal ray whose exceptional locus is contained in the exceptional locus of σ, so X' is a Fano manifold by [33, Proposition 3.4].

We claim that $\mathrm{NE}(X) = \langle R_1, R_a, R_\sigma\rangle$.
Note that it is enough to prove that each extremal ray of X lies on an extremal face with $[V^1]$. So, assume to get a contradiction that there exists an extremal ray R that is not contained in an extremal face with $[V^1]$. If either R is associated with a contraction of fiber type, or R is associated with a divisorial contraction whose exceptional locus E_R satisfies $E_R \cdot V^1 > 0$, then a family of deformation V^R of a minimal curve in R is horizontal and dominating with respect to the $\mathrm{rc}(V^1)$-fibration, so we have a contradiction by [10, Lemma 2.4]. Therefore R is divisorial with $E_R \cdot V^1 = 0$; then there exists an extremal ray R' on an extremal face with $[V^1]$ such that $E_R \cdot R' < 0$, hence R' is divisorial; so we have a contradiction.

Denoted by E_σ the exceptional locus of σ and by $\overline{R}$ the extremal ray on which E_σ is positive, let $\Sigma\colon X \to Y$ be the contraction associated with the extremal face $\langle \overline{R}, R_\sigma\rangle$. Then we have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[dr, "\Sigma"'] & X' \arrow[d, "\psi"] \\ & Y. \end{tikzcd}$$

A general fiber F_Σ of Σ contains $\mathrm{Locus}(\overline{R})_{F_\sigma}$, with F_σ a general non-trivial fiber of σ, and has dimension $\leq \dim X - \dim F$, where F is a general fiber of the contraction associated with the extremal ray different from R_σ and $\overline{R}$. In view of Lemma 2.13 we get $\dim F_\Sigma = 2i_X - 1$. Moreover, F_Σ is a Fano manifold of pseudoindex i_X

and it admits an extremal ray, of length equal to i_X, associated with a divisorial contraction, so $F_\Sigma = \mathrm{Bl}_{\mathbb{P}^{i_X-2}}\mathbb{P}^{2i_X-1}$ by [4, Theorem 1.1]. Therefore the general fiber of ψ is $\mathbb{P}^{2i_X-1}$ and ψ is a contraction of fiber type associated with an extremal ray of the Fano manifold X'. Then, by [4, Theorem 1.1], $X' = \mathbb{P}^{2i_X-1} \times \mathbb{P}^{i_X-1}$, and the statement follows. □

Next we will consider Fano manifolds of pseudoindex $i_X = \frac{\dim X+2}{3}$ without extremal rays associated with birational contractions. We will need the following

Lemma 4.5. *Let Y be a Fano manifold of dimension $\dim Y > 3$, Picard number $\rho_Y = 2$ and pseudoindex $i_Y = \frac{\dim Y+1}{2}$. Assume that the extremal rays of Y are associated with contractions of fiber type, one of them, say $\varphi\colon Y \to T$, being a $\mathbb{P}^{i_Y-1}$-bundle. Then φ is equidimensional and one of the following holds:*

(1) $Y = \mathbb{P}^{i_Y-1} \times \mathbb{Q}^{i_Y}$.
(2) $Y = \mathbb{P}^{i_Y-1} \times \mathbb{P}^{i_Y}$.
(3) $Y = \mathbb{P}_{\mathbb{P}^{i_Y}}(T_{\mathbb{P}^{i_Y}})$, *where $T_{\mathbb{P}^{i_Y}}$ is the tangent bundle on $\mathbb{P}^{i_Y}$.*

Proof. First we show that φ is equidimensional.
We can argue as in Step 2 of the proof of [28, Proposition 6]; so assume by contradiction that φ has a jumping fiber J. By computing the dimension of the general fiber of the elementary contractions of Y with [33, Theorem 1.1] and recalling Remark 2.2, we derive that $\dim J = i_Y$. Moreover, being φ an elementary contraction, the image of the jumping fibers in T has codimension $m \geq 3$. By taking $\dim T - m$ hyperplane sections A_j of T, we have a contraction $\varphi|_{\varphi^{-1}(\cap A_j)}\colon \varphi^{-1}(\cap A_j) \to \cap A_j$, with general fiber $\mathbb{P}^{i_Y-1}$ and some isolated jumping fibers of dimension i_Y. Moreover, we are in the assumptions of [2, Lemma 3.3], so we derive that this contraction is supported by a divisor of the form $K_{\varphi^{-1}(\cap A_j)} + i_Y L$, where L is a $\varphi|_{\varphi^{-1}(\cap A_j)}$-ample line bundle on $\varphi^{-1}(\cap A_j)$ such that L restricts as $\mathcal{O}(1)$ on each non-jumping fiber of $\varphi|_{\varphi^{-1}(\cap A_j)}$. We now get a contradiction with [5, Theorem 4.1].

Therefore T has dimension i_Y and Picard number 1, it is smooth by [14, Lemma 2.12], it is a Fano manifold by [19, Corollary 2.9] and it has pseudoindex $\geq i_Y$ by [7, Lemme 2.5(a)]. So either $i_T = i_Y$ or $i_T = i_Y + 1$, which give $T = \mathbb{Q}^{i_Y}$ by [20, Theorem 0.1] and [12, Theorem C], and $T = \mathbb{P}^{i_Y}$ by [11, Corollary 0.3], respectively.

Now, the fibrations which are not projectivization of vector bundles come from torsion elements in $H^2(T, \mathcal{O}_T^*)$ (see for instance [13, pg. 223]). So, since T is a rational variety and the Brauer group is a birational invariant by [15, III Corollary 7.3], $Y = \mathbb{P}_T(\mathcal{F})$ with $\mathcal{F}$ a vector bundle of rank i_Y on T. Moreover, up to a twist, we can assume that $0 < c_1(\mathcal{F}) \leq i_Y$, and we can argue as in [28, Section 4]; in particular, the tautological line bundle $\xi_{\mathcal{F}}$ is nef.

If $T = \mathbb{Q}^{i_Y}$, the restriction of $\mathcal{F}$ to any line is $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus i_Y}$. So we get case (1) of the statement. If else $T = \mathbb{P}^{i_Y}$, the restriction of $\mathcal{F}$ to any line is either $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus i_Y}$ (if $c_1(\mathcal{F}) = i_Y$), or $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}{}^{\oplus(i_Y-1)}$ (if $0 < c_1(\mathcal{F}) < i_Y$); so $\mathcal{F}$ is uniform and by [34, Proposition 1.9], recalling that Y does not admit extremal rays associated with birational contractions, we get cases (2) and (3) of the statement. □

Remark 4.6. We remark here for later use that if a Fano threefold Y, with $i_Y \geq 2$ and $\rho_Y = 2$, has only extremal rays associated with contractions of fiber type, then either $Y = \mathbb{P}^1 \times \mathbb{P}^2$, or $Y = \mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$ (e.g. see [25, Proposition 5.1 (b)]).

Remark 4.7. Notice that for each variety Y classified in Lemma 4.5 and Remark 4.6, if Y is the target of an equidimensional $\mathbb{P}^{r-1}$-bundle $\varphi\colon X \to Y$ of a Fano manifold X, we can argue in the same way as in the above proof to show that there exists a vector bundle $\mathcal{E}$ on Y of rank r such that $X = \mathbb{P}_Y(\mathcal{E})$.

In dealing with Fano manifolds of pseudoindex $i_X = \frac{\dim X+2}{3}$ without extremal rays associated with birational contractions, we will make use of the following

Lemma 4.8. *Let X be a Fano manifold of pseudoindex i_X admitting an extremal ray associated with a $\mathbb{P}^{i_X-1}$-bundle $\varphi\colon X \to Y$ giving $X = \mathbb{P}_Y(\mathcal{E})$.*

(1) *If $Y = \mathbb{P}^{i_X-1} \times \mathbb{Q}^{i_X}$, then $X = \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X-1} \times \mathbb{Q}^{i_X}$.*
(2) *If $Y = \mathbb{P}_{\mathbb{P}^{i_X}}(T_{\mathbb{P}^{i_X}})$, then $X = \mathbb{P}^{i_X-1} \times \mathbb{P}_{\mathbb{P}^{i_X}}(T_{\mathbb{P}^{i_X}})$, where $T_{\mathbb{P}^{i_Y}}$ is the tangent bundle on $\mathbb{P}^{i_Y}$.*

Proof. Denote by R_j, $j = 1, 2$, the extremal rays of Y. Let Γ_j be minimal curve in R_j, and let $\phi_j\colon \mathbb{P}^1 \to \Gamma_j$ be the normalization. Since $i_Y = i_X$ and both the extremal rays of Y have length $i_Y = \operatorname{rk}\mathcal{E}$, by [7, Lemme 2.5] $\phi_j^*\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus i_X}$. Up to a twist we can assume $a = 1$, and $K_Y + \det\mathcal{E}$ is trivial on both the extremal rays of Y, so $K_Y + \det\mathcal{E} = \mathcal{O}_Y$. Therefore $X = \mathbb{P}^{i_X-1} \times Y$ by [25, Proposition 4.4]. $\square$

Theorem 4.9. *Let X be a Fano manifold of pseudoindex $i_X = \frac{\dim X+2}{3}$ such that X is rc(V^1, V^2, V^3)-connected with respect to three families of rational curves as in Construction 3.1. Then one of the following holds:*

(1) $X = \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X}$.
(2) $X = \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X-1} \times \mathbb{Q}^{i_X}$, *with* $i_X \geq 3$.
(3) $X = \mathbb{P}^{i_X-1} \times \mathbb{P}_{\mathbb{P}^{i_X}}(T_{\mathbb{P}^{i_X}})$.
(4) $X = \mathbb{P}^{i_X-1} \times \mathrm{Bl}_{\mathbb{P}^{i_X-2}}\mathbb{P}^{2i_X-1}$.

Proof. Recall that V^1, V^2 and V^3 are unsplit. Moreover, by construction we have that V^1 is covering, $-K_X \cdot V^1 = -K_X \cdot V^2 = i_X$ and one of the following occurs:

(i) the families V^2 and V^3 are covering and $-K_X \cdot V^3 = i_X + 1$;
(ii) at least one family between V^2 and V^3 is covering and $-K_X \cdot V^3 = i_X$.

Note that case (i) leads to case (1) of the statement by [29, Theorem 1.1].

We can thus assume that X is not rc(V^1, V^2, V^3)-connected with respect to families of rational curves as in (i) and that we are in case (ii). Denote by V^a a covering family between V^2 and V^3. By Lemma 3.6 and Lemma 3.8 we know that X admits two extremal rays, $\mathbb{R}_+[V^1]$ and $\mathbb{R}_+[V^a]$, associated with contractions of fiber type. Note that they both have length equal to i_X. Let F_b be a general non-trivial fiber of the contraction φ_b associated with the other extremal ray, say R_b. Then $\dim F_b \leq \dim X - 2(i_X - 1) = i_X$.

If φ_b is birational, we obtain case (4) of the statement by Proposition 4.4.

Assume now that φ_b is of fiber type.

If the length of R_b were greater than i_X, then, by a direct computation, this length would be equal to $i_X + 1$. So we could construct a covering family of rational curves V^b from φ_b. Then V^1, V^a, V^b would be as in case (i). Hence we can reduce to the case that R_b has length equal to i_X. Now, the general fiber F_b has dimension equal to either i_X or $i_X - 1$. Moreover, in view of Remark 2.2, X has no other extremal rays.

Notice that if $\dim X = 4$, then we have case (3) of the statement by [25, Proposition 5.1(b)], so we can assume that $\dim X > 4$ (and so $i_X > 2$).

If the contraction, say $\varphi\colon X \to Y$, associated with one of the rays has general fiber G of dimension i_X, then all the contractions are equidimensional. Notice that $\rho_G = 1$ by [32, Theorem A]. In view of [16, Theorem 1.3] applied to the general fiber of the elementary contractions different from φ, and by [20, Theorem 0.1] and [12, Theorem C] applied to the general fiber of φ, X has two equidimensional $\mathbb{P}^{i_X-1}$-bundles and one equidimensional $\mathbb{Q}^{i_X}$-fibration. Denote by $\varphi_i\colon X \to Y_i$, $i = 1, 2$, the $\mathbb{P}^{i_X-1}$-bundles. Then each Y_i is smooth, it is a Fano manifold by [19, Corollary 2.9] and, since $\rho_{Y_i} = 2$, it has pseudoindex $i_{Y_i} = i_X$ by combining parts (a) and (b) of [7, Lemme 2.5]. Moreover, by [33, Proof of Lemma 3.1], the cone $\mathrm{NE}(Y_i)$ is generated by the classes of images of extremal rational curves from X, so Y_i has two extremal rays of fiber type, one of them being a $\mathbb{P}^{i_X-1}$-bundle. Therefore Y_i satisfies the assumptions of Lemma 4.5, and by Remark 4.7 $X = \mathbb{P}_{Y_i}(\mathcal{E}_i)$ with $\mathcal{E}_i$ a vector bundle of rank i_X on Y_i. Since the contractions associated with the two extremal rays of Y_i are equidimensional with fibers of dimension $i_X - 1$ and i_X, respectively, $Y_i = \mathbb{P}^{i_Y-1} \times Z$, with either $Z = \mathbb{Q}^{i_Y}$ or $Z = \mathbb{P}^{i_Y}$. In the former case we get case (2) of the statement by Lemma 4.8. In the latter, denote by Ψ the contraction associated with the extremal face of $\mathrm{NE}(X)$ generated by the two extremal rays different from the one associated with φ_i. Then Ψ does not contracts curves contracted by φ_i and its target is a variety of dimension $i_X - 1$; therefore, by [25, Lemma 4.1], $X = \mathbb{P}^{i_X-1} \times Y$, which gives a contradiction with the elementary contractions of X.

Otherwise, each contraction has general fiber of dimension $i_X - 1$. If all the contractions are equidimensional, by [16, Theorem 1.3] X has three equidimensional $\mathbb{P}^{i_X-1}$-bundle structures, $\varphi_i\colon X \to Y_i$, with $i = 1, a, b$, associated with the three extremal rays. Arguing as before, we have that, for each i, Y_i is a Fano manifold of dimension $2i_X - 1$, Picard number 2 and pseudoindex i_X satisfying the assumptions of Lemma 4.5, so $X = \mathbb{P}_{Y_i}(\mathcal{E}_i)$ by Remark 4.7 and one of the following holds:

(a) $Y_i = \mathbb{P}^{i_Y-1} \times Z$, with either $Z = \mathbb{Q}^{i_Y}$ or $Z = \mathbb{P}^{i_Y}$;
(b) $Y_i = \mathbb{P}_{\mathbb{P}^{i_Y}}(T_{\mathbb{P}^{i_Y}})$, where $T_{\mathbb{P}^{i_Y}}$ is the tangent bundle on $\mathbb{P}^{i_Y}$.

If one of the Y_i is as in case (b), we get case (3) of the statement by Lemma 4.8. We can thus assume that no one of the Y_i is as in case (b), and so each Y_i is as in case (a). By taking into account Lemma 4.8, we see that it cannot be $Z = \mathbb{Q}^{i_Y}$, since otherwise we would reach a contradiction with the type of elementary contractions of X. Therefore it can only be $Y_i = \mathbb{P}^{i_Y-1} \times \mathbb{P}^{i_Y}$ for each $i = 1, 2, 3$, which leads to a contradiction by considering the three extremal faces spanned by any pairs of extremal rays of X.

We are left to show that all the contractions are equidimensional. So, assume to get a contradiction, that one of the elementary contractions of X, say φ_J has a jumping fiber J, clearly $\dim J = i_X$. Denote by V^α and V^β the families of deformations of minimal curves of the other extremal rays, R_α and R_β, and by $\varphi_\alpha\colon X \to Y_\alpha$ and $\varphi_\beta\colon X \to Y_\beta$ the associated contractions. By counting the dimension with Lemma 2.13, we have $X = \mathrm{Locus}(V^\alpha, V^\beta)_J$. Moreover, since $\mathrm{Locus}(V^\alpha)_J$ intersects each fiber of φ_β, this contraction is equidimensional. By exchanging the role of V^α and V^β we obtain that also φ_α is equidimensional. So, in view of [16, Theorem 1.3], φ_α and φ_β are $\mathbb{P}^{i_X-1}$-bundles, while φ_J is a non-equidimensional $\mathbb{P}^{i_X-1}$-bundle. Arguing as before we get that Y_α and Y_β are Fano manifolds of dimension $2i_X - 1$, pseudoindex equal to i_X, $X = \mathbb{P}_{Y_\alpha}(\mathcal{E}_\alpha) = \mathbb{P}_{Y_\beta}(\mathcal{E}_\beta)$. By combining Lemma 4.5 and Lemma 4.8 and taking into account the descriptions of the elementary contractions of X, we see that the only possibility is that $Y_\alpha = Y_\beta = \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X}$. Then $\bar{\varphi_\alpha} \circ \varphi_\alpha$, where $\bar{\varphi_\alpha}$ is the projection $Y_\alpha \to \mathbb{P}^{i_X-1}$, does not contracts curves of R_β, therefore $X = \mathbb{P}^{i_X-1} \times Y_\beta$ by [25, Lemma 4.1] and we have a contradiction since Y_β does not have contractions with jumping fibers. □

Remark 4.10. In particular, from Theorem 4.9 it follows that, if X is a Fano manifold of pseudoindex $i_X = \frac{\dim X+2}{3}$ such that X is $\mathrm{rc}(V^1, V^2, V^3)$-connected with respect to three families of rational curves as in Construction 3.1, then $-K_X \cdot V^i = i_X$ for any $i = 1, 2, 3$, unless $X = \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X}$. Moreover, any X is a product with $\mathbb{P}^{i_X-1}$ as a factor.

5. Fano manifolds with $i_X = \frac{\dim X+1}{3}$ and $\rho_X \geq 3$

In this section we start considering Fano manifolds of pseudoindex $i_X = \frac{\dim X+1}{3} \geq 2$ and Picard number $\rho_X \geq 3$.

The following result concerns manifolds with Picard number $\rho_X \geq 4$.

Proposition 5.1. *Let X be a Fano manifold of pseudoindex $i_X = \frac{\dim X+1}{3} \geq 2$ and Picard number $\rho_X \geq 4$. Then one of the following holds:*

(1) $\dim X = 8$, *and* $X = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.
(2) $\dim X = 5$, *and one of the following holds:*
  (2a) $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.
  (2b) $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$.
  (2c) $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$.
  (2d) $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{Bl}_p(\mathbb{P}^3)$.

Proof. By [23, Theorem 5.1], we immediately get that the dimension of X can only be equal to either 8, or 5. Moreover, by the same theorem, if $\dim X = 8$ then we are in case (1) of the statement, while if $\dim X = 5$ and $\rho_X = 5$ then we are in case (2a). So we are left to deal with $\dim X = 5$ and $\rho_X = 4$. In this last case, by the classification in [10, Theorem 1.1], we see that the Kleiman–Mori cone of X is generated by four extremal rays and at most one of them is not associated with a contraction of fiber type. If all these contractions are of fiber type, by [25, Proposition 5.1] we get cases (2b) and (2c) of the statement; if else, by [25, Proposition 5.2] we get cases (2d) of the statement. □

Even by comparing this first result with the corresponding one in the previous section, *i.e.* with Proposition 4.1, it is clear that the classification of Fano manifolds of pseudoindex $\frac{\dim X+1}{3}$ is much more complicated than the classification of Fano manifolds of pseudoindex $\geq \frac{\dim X+2}{3}$.

However we can still consider Fano manifolds X that are $\mathrm{rc}(V^1, V^2, V^3)$-connected with respect to three families of rational curves as in Construction 3.1 and have Picard number $\rho_X = 3$. By following the same ideas as in the previous section, we obtain the complete classification of such varieties when $-K_X \cdot V^i \neq i_X$ for at least one of these families (Theorem 5.7).

If X is $\mathrm{rc}(V^1, V^2, V^3)$-connected with respect to three families of rational curves as in Construction 3.1, we know by Lemma 3.4 that $-K_X \cdot V^1 = i_X$. In the next proposition we see when $-K_X \cdot V^2 \neq i_X$.

Proposition 5.2. *Let X be a Fano manifold of pseudoindex $i_X = \frac{\dim X+1}{3} \geq 2$ such that X is rc(V^1, V^2, V^3)-connected with respect to three families of rational curves as in Construction 3.1. Then $-K_X \cdot V^2 = i_X$, unless one of the following holds:*

(1) $X = \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X} \times \mathbb{P}^{i_X}$;
(2) $X = \mathrm{Bl}_{\mathbb{P}^{i_X-1}\times\mathbb{P}^{i_X-1}} \mathbb{P}^{2i_X} \times \mathbb{P}^{i_X-1}$;
(3) $X = \mathrm{Bl}_{\mathbb{P}^{i_X-2}\times\mathbb{P}^{i_X-1}} \mathbb{P}^{2i_X} \times \mathbb{P}^{i_X-1}$;
(4) $X = \mathrm{Bl}_{\mathbb{P}^{i_X-2}\times\mathbb{P}^{i_X}} \mathbb{P}^{2i_X-1} \times \mathbb{P}^{i_X}$.

Proof. Assume that $-K_X \cdot V^2 \geq i_X + 1$. By Lemma 3.4, the covering family V^1 has anticanonical degree $-K_X \cdot V^1 = i_X$; moreover, by Construction 3.1 it is immediate to derive that $-K_X \cdot V^2 = i_X + 1$. Then one of the following holds:

(i) V^2 and V^3 are covering and $-K_X \cdot V^3 = i_X + 1$;
(ii) V^2 is covering, V^3 is not covering and $-K_X \cdot V^3 = i_X$;
(iii) V^2 is not covering, V^3 is covering and $-K_X \cdot V^3 = i_X$.

Case (i) leads to case (1) of the statement by [29, Theorem 1.1], so we can now assume that X does not admit three families of rational curves as in this case.

Assume that we are in case (ii). By Lemma 3.10 we know that $[V^1]$ and $[V^2]$ span two extremal rays, whose associated contractions are of fiber type.

Let F be a non-trivial fiber of the contraction associated with an extremal ray R of X different from the previous ones. By combining Remark 2.2 with [33, Theorem 1.1] we get that $\dim F = i_X - 1$ or i_X. However it cannot be $\dim F = i_X - 1$, since in this case by the same theorem the contraction would be of fiber type and a family of deformation of a minimal curve would be covering with anticanonical degree i_X, contradicting the minimality of V^2. Moreover, if $\dim F = i_X$ and the contraction were of fiber type, we would reach the setting of case (i). Therefore by [33, Theorem 1.1] we get that the contraction associated with R is divisorial, R has length i_X and all the non-trivial fibers have dimension i_X. Then by [3, Theorem 5.1] X is the blow-up of a smooth variety along a smooth subvariety of dimension $2(i_X - 1)$. Denoted by $\sigma\colon X \to X'$ this contraction, X' is a Fano manifold by [33, Proposition 3.4].

Denote by F_σ a non-trivial fiber of σ. Since $X = \mathrm{Locus}(V^1, V^2)_{F_\sigma} = \mathrm{Locus}(V^2, V^1)_{F_\sigma}$, by repeated applications of Lemma 2.15 we see that the numerical class of every curve in X can be written as a linear combination with nonnegative coefficients of $[V^1]$, $[V^2]$ and R, hence $\mathrm{NE}(X) = \langle [V^1], [V^2], R\rangle$.

Denote by $\Sigma\colon X \to Y$ the contraction associated with the extremal face $\langle \overline{R}, R\rangle$, where $\overline{R}$ is an extremal ray that is positive on the exceptional locus of σ. Then we have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[dr, "\Sigma"'] & X' \arrow[d, "\psi"] \\ & Y. \end{tikzcd}$$

Suppose first that $\overline{R}$ is the ray spanned by $[V^2]$. A general fiber F_Σ of Σ has dimension equal to $2i_X$. Moreover, F_Σ is a Fano manifold of pseudoindex i_X and it admits an extremal ray, of length i_X, associated with a divisorial contraction whose non-trivial fibers have dimension i_X. It follows by [3, Theorem 5.1] that F_Σ is the blow-up of a smooth variety along a smooth subvariety of dimension $i_X - 1$; hence $F_\Sigma = \mathrm{Bl}_{\mathbb{P}^{i_X-1}} \mathbb{P}^{2i_X}$ by the proof of [4, Theorem 1.3]. Therefore, the general fiber of

ψ is $\mathbb{P}^{2i_X}$ and ψ is a contraction of fiber type associated with an extremal ray of the Fano manifold X'. Then $X' = \mathbb{P}^{2i_X} \times \mathbb{P}^{i_X-1}$ by [4, Theorem 1.1], so we obtain case (2) of the statement.

If otherwise $\overline{R}$ is the ray spanned by $[V^1]$, by arguing as above we get case (4) of the statement.

Therefore we are left to consider Fano manifolds that are rationally connected as in case (iii) and that are not rationally connected as in the previous cases. By Lemma 3.10 we know that $[V^1]$ and $[V^3]$ span two extremal rays, whose associated contractions are of fiber type. We see that X has an elementary contractions which turns out to be a blow-up and, by arguing as before, we get case (3) of the statement. $\square$

Next, we want to describe X such that $-K_X \cdot V^2 = i_X$ and $-K_X \cdot V^3 \neq i_X$. We start with the following

Lemma 5.3. *Let X be a Fano manifold of pseudoindex $i_X = \frac{\dim X+1}{3} \geq 2$ such that X is rc(V^1, V^2, V^3)-connected with respect to three unsplit families of rational curves as in Construction 3.1. Then $-K_X \cdot V^3 \leq i_X + 1$, unless $X = \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X+1}$.*

Proof. If $-K_X \cdot V^3 \geq i_X + 2$, then $\sum_{i=1}^{3} \dim \operatorname{Locus}(V^i)_{x_i} = \dim X$ by Proposition 2.11. It follows that V^1, V^2 and V^3 are dominating with $-K_X \cdot V^1 = -K_X \cdot V^2 = i_X$ and $-K_X \cdot V^3 = i_X + 2$, so we conclude by [29, Theorem 1.1]. $\square$

Proposition 5.4. *Let X be a Fano manifold of pseudoindex $i_X = \frac{\dim X+1}{3} \geq 2$ such that X is rc(V^1, V^2, V^3)-connected with respect to three unsplit families of rational curves as in Construction 3.1. If $-K_X \cdot V^2 = i_X$ and X admits a birational elementary contraction, then $-K_X \cdot V^3 = i_X$ unless one of the following holds:*

1. $X = \operatorname{Bl}_{\mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X-1}} \mathbb{P}^{2i_X} \times \mathbb{P}^{i_X-1}$;
2. $X = \operatorname{Bl}_{\mathbb{P}^{i_X-2} \times \mathbb{P}^{i_X-1}} \mathbb{P}^{2i_X} \times \mathbb{P}^{i_X-1}$;
3. $X = \operatorname{Bl}_{\mathbb{P}^{i_X-2} \times \mathbb{P}^{i_X}} \mathbb{P}^{2i_X-1} \times \mathbb{P}^{i_X}$.

Proof. Assume that $-K_X \cdot V^3 > i_X$.

We have $-K_X \cdot V^1 = i_X$ by Lemma 3.4 and $-K_X \cdot V^3 = i_X + 1$ by Lemma 5.3.

If V^3 is dominating and $[V^3]$ spans an extremal ray, by Lemma 3.11 at least one between $[V^1]$ and $[V^2]$, say $[V^\alpha]$, spans an extremal ray. By combining [33, Theorem 1.1] with Remark 2.2 we see that every non-trivial fiber of the birational contraction $\sigma\colon X \to X'$ has dimension equal to i_X, and so σ is divisorial and associated with an extremal ray R of length equal to i_X, by [33, Theorem 1.1]. Then σ gives X as the blow-up of a smooth variety X' along a smooth center of dimension $2(i_X - 1)$ by [3, Theorem 5.1] and X' is a Fano manifold by [33, Proposition 3.4].

Denote by F_σ a non-trivial fiber of σ. Since $X = \operatorname{Locus}(V^\alpha, V^3)_{F_\sigma} = \operatorname{Locus}(V^3, V^\alpha)_{F_\sigma}$, by repeated applications of Lemma 2.15 we see that the numerical class of every curve in X can be written as a linear combination with nonnegative coefficients of $[V^\alpha]$, $[V^3]$ and R, hence $\operatorname{NE}(X) = \langle [V^\alpha], [V^3], R \rangle$.

Therefore we can consider the contraction $\Sigma\colon X \to Y$ associated with the extremal face $\langle \overline{R}, R \rangle$, where $\overline{R}$ is an extremal ray that is positive on the extremal locus of σ.

If $\overline{R}$ is the ray spanned by $[V^3]$, then, by arguing as in the proof of Proposition 5.2, we obtain case (1) in the statement. If otherwise $\overline{R}$ is the ray spanned by $[V^\alpha]$, by arguing as above we get case (3) of the statement.

We can thus assume that X is not $\mathrm{rc}(V^1, V^2, V^3)$-connected as above, so we confine to consider manifolds $\mathrm{rc}(V^1, V^2, V^3)$-connected with respect to three unsplit families of rational curves as in Construction 3.1 such that $-K_X \cdot V^1 = -K_X \cdot V^2 = i_X$, $-K_X \cdot V^3 = i_X + 1$ and V^3 is not dominating, or V^3 is dominating but $[V^3]$ does not span an extremal ray. First of all, by Lemma 3.11 we know that V^2 is dominating and that $[V^1]$ and $[V^2]$ span two extremal rays (associated with contractions of fiber type). Moreover, in view of Remark 2.2, we derive that X does not admit any other extremal ray associated with a contraction of fiber type and that the dimension of the general non-trivial fiber of the birational contraction $\sigma \colon X \to X'$ is equal to either i_X or $i_X + 1$.

If σ has a fiber F_σ such that $\dim F_\sigma = i_X + 1$, then $X = \mathrm{Locus}(V^1, V^2)_{F_\sigma} = \mathrm{Locus}(V^2, V^1)_{F_\sigma}$, by repeated applications of Lemma 2.15 we see that the numerical class of every curve in X can be written as a linear combination with nonnegative coefficients of $[V^1]$, $[V^2]$ and R, hence $\mathrm{NE}(X) = \langle [V^1], [V^2], R \rangle$. Moreover, a family of deformations of a minimal rational curve in R is horizontal and dominating with respect to the $\mathrm{rc}(V^1, V^2)$-fibration; so R has length equal to $i_X + 1$. Therefore the contraction associated with R is divisorial by [33, Theorem 1.1], and so it is a blow-up by [3, Theorem 5.1]. Now we can consider the contraction $\Sigma \colon X \to Y$ associated with the extremal face $\langle \overline{R}, R \rangle$, where $\overline{R}$ is an extremal ray that is positive on the exceptional locus of σ. By arguing as in the proof of Proposition 5.2, we obtain case (2) in the statement.

Therefore we are left to show that the general non-trivial fiber of σ cannot have dimension equal to i_X. Notice that the argument above actually shows that X has no small contractions. In particular this implies that V^3 cannot be dominating, since otherwise, by [10, Lemma 2.4], $[V^3]$ would lie in an extremal face both with $[V^1]$ and $[V^2]$, so it would span an extremal ray associated with a contraction of fiber type, which is a contradiction. Then $X = \mathrm{Locus}(V^2, V^1)_{\mathrm{Locus}(V^3)_{x_3}} = \mathrm{Locus}(V^1, V^2)_{\mathrm{Locus}(V^3)_{x_3}}$; hence, by repeated applications of Lemma 2.15, we see that the numerical class of every curve in X can be written as a linear combination with nonnegative coefficients of $[V^1]$, $[V^2]$ and $[V^3]$, so $\mathrm{NE}(X) = \langle [V^1], [V^2], [V^3] \rangle$. The birational contraction is then associated with $\mathbb{R}_+[V^3]$, so we reach a contradiction by [33, Theorem 1.1]. □

Proposition 5.5. *Let X be a Fano manifold of pseudoindex $i_X = \frac{\dim X + 1}{3} \geq 2$ such that X is $\mathrm{rc}(V^1, V^2, V^3)$-connected with respect to three unsplit families of rational curves as in Construction 3.1. If $-K_X \cdot V^2 = i_X$ and X does not admit any birational elementary contraction, then $-K_X \cdot V^3 = i_X$ unless one of the following holds:*

(1) $X = \mathbb{P}^{i_X - 1} \times \mathbb{P}^{i_X - 1} \times \mathbb{P}^{i_X + 1}$;
(2) $X = \mathbb{P}^{i_X - 1} \times \mathbb{P}^{i_X - 1} \times \mathbb{Q}^{i_X + 1}$;
(3) $X = \mathbb{P}^{i_X - 1} \times \mathbb{P}^{i_X} \times \mathbb{Q}^{i_X}$, $i_X \geq 3$;
(4) $X = \mathbb{P}^{i_X - 1} \times \mathbb{P}_{\mathbb{P}^{i_X}}(\mathcal{O}_{\mathbb{P}^{i_X}}^{\oplus i_X} \oplus \mathcal{O}_{\mathbb{P}^{i_X}}(1))$;
(5) $X = \mathbb{P}^{i_X - 1} \times \mathbb{P}_{\mathbb{P}^{i_X}}(T_{\mathbb{P}^{i_X}} \oplus \mathcal{O}_{\mathbb{P}^{i_X}}(1))$;
(6) $X = \mathbb{P}^{i_X} \times \mathbb{P}_{\mathbb{P}^{i_X}}(T_{\mathbb{P}^{i_X}})$.

Proof. Assume that $-K_X \cdot V^3 \geq i_X + 1$. In view of Lemma 5.3, either we are in case (1) of the statement, or $-K_X \cdot V^3 = i_X + 1$. So we have to deal with this last case.

We claim that V^3 is dominating. Assume to get a contradiction that any minimal horizontal dominating family for the $\mathrm{rc}(V^1, V^2)$-fibration is not dominating. Then, by Lemma 3.11, $[V^1]$ and $[V^2]$ span two extremal rays. A family of deformation of a minimal curve of the third extremal ray of X is a dominating family of curves which is horizontal and dominating with respect to the $\mathrm{rc}(V^1, V^2)$-fibration, so it has anticanonical degree $\geq i_X + 2$ (and so equal to $i_X + 2$). Therefore, by [29, Theorem 1.1], X is as in case (1) of the statement, which is a contradiction.

The same argument applies to show that $[V^3]$ spans an extremal ray. Moreover, by Lemma 3.11, at least one between $[V^1]$ and $[V^2]$ spans an extremal ray. Since X has exactly three extremal rays associated with contraction of fiber type, by [10, Lemma 2.4] we derive that both $[V^1]$ and $[V^2]$ span extremal rays.

For each $i = 1, 2, 3$, denote by $R_i := \mathbb{R}_+[V^i]$ the extremal rays of X and by F_i the general fiber of the contraction $\varphi_i \colon X \to Y_i$ associated with R_i. By [33, Theorem 1.1] combined with Remark 2.2, we get that F_i, for $i = 1, 2$, has dimension equal to either $i_X - 1$ or i_X.

If, for $i = 1$ or 2, $\dim F_i = i_X$, then the three contractions are equidimensional. In particular, the contraction with $(i_X - 1)$-dimensional fibers is a $\mathbb{P}^{i_X-1}$-bundle, while $\varphi_3 \colon X \to Y_3$ is a $\mathbb{P}^{i_X}$-bundle, both by [16, Theorem 1.3]. Then Y_3 is smooth, it is a Fano manifold by [19, Corollary 2.9] and, since $\rho_{Y_3} = 2$, it has pseudoindex $i_{Y_3} = i_X$ by combining cases (a) and (b) of [7, Lemme 5.2]. Moreover, by [33, Proof of Lemma 3.1], the cone $\mathrm{NE}(Y_3)$ is generated by the classes of images of extremal rational curves from X, so $\mathrm{NE}(Y_3) = \langle \varphi_3(R_1), \varphi_3(R_2) \rangle$. Notice that the extremal ray of Y_3 which is the image of the extremal ray of X associated with the $\mathbb{P}^{i_X-1}$-bundle is a $\mathbb{P}^{i_X-1}$-bundle of Y_3. So Y_3 is one of the varieties classified in Lemma 4.5 and Remark 4.6, and so, by Remark 4.7, $X = \mathbb{P}_{Y_3}(\mathcal{E})$ with $\mathcal{E}$ a vector bundle of rank $i_X + 1$ on Y_3. Moreover the extremal face spanned by $\langle R_1, R_2 \rangle$ gives a contraction onto a i_X-dimensional variety and this contraction does not contracts curves of R_3, so $X = \mathbb{P}^{i_X} \times Y_3$ by [25, Lemma 4.1], and we get case (3) of the statement by Lemma 4.5 and Remark 4.6.

Therefore we can assume $\dim F_1 = \dim F_2 = i_X - 1$. Clearly $\dim F_3 = i_X$ or $\dim F_3 = i_X + 1$.

In the last case the contractions φ_i, $i = 1, 2, 3$ are equidimensional; moreover, $\rho_{F_3} = 1$ by [32, Theorem A], so, by [16, Theorem 1.3], [20, Theorem 0.1] and [12, Theorem C], φ_1 and φ_2 are $\mathbb{P}^{i_X-1}$-bundles, φ_3 is a $\mathbb{Q}^{i_X+1}$-fibration and the Y_i are smooth for each $i = 1, 2, 3$. Moreover, for both $i = 1, 2$, Y_i is a Fano manifold by [19, Corollary 2.9] and $\mathrm{NE}(Y_i) = \langle \varphi_i(R_j), \varphi_i(R_3) \rangle$, where $j \neq i, 3$, by [33, Proof of Lemma 3.1], so Y_i has two elementary contractions of fiber type, which are equidimensional and have fibers of dimension $i_X + 1$ and $i_X - 1$, respectively. Then $\bar{\varphi}_i \circ \varphi_i$, where $\bar{\varphi}_i$ is the contraction associated with $\varphi_i(R_3)$ is a proper morphism which does not contracts curves of R_j and has $(i_X - 1)$-dimensional target.

The fibers of the contraction $Y_j \to Z_i$ associated with $\varphi_j(R_i)$ are dominated by $\mathbb{P}^{i_X-1}$ while the fibers of the contraction $Y_j \to Z_3$ associated with $\varphi_j(R_3)$ are dominated by $\mathbb{Q}^{i_X+1}$, so the elementary contractions of Y_j are equidimensional and they are a $\mathbb{P}^{i_X-1}$-bundle and either a $\mathbb{P}^{i_X+1}$-bundle, or a $\mathbb{Q}^{i_X+1}$-fibration. Clearly

$\dim Z_3 = i_X - 1$, therefore, being dominated by $\mathbb{P}^{i_X-1}$, it is $Z_3 = \mathbb{P}^{i_X-1}$. By arguing as in Remark 4.7 we get $Y_j = \mathbb{P}_{\mathbb{P}^{i_X-1}}(\mathcal{F}_j)$ for a vector bundle $\mathcal{F}_j$ on $\mathbb{P}^{i_X-1}$; so $Y_j = \mathbb{P}^{i_X-1} \times Z_i$ by [25, Lemma 4.1]. Then, as Remark 4.7, we get $X = \mathbb{P}_{Y_j}(\mathcal{E}_j)$ for a vector bundle $\mathcal{E}_j$ on Y_j, hence $X = \mathbb{P}^{i_X-1} \times Y_j$ by [25, Lemma 4.1]. So, if Y_j were as in the former case, by [31, Theorem A] it would be $Y_j = \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X+1}$, which leads to a contradiction with the elementary contractions of X. It follows that Y_j has a $\mathbb{Q}^{i_X+1}$-fibration. Therefore Z_i is dominated by $\mathbb{Q}^{i_X+1}$, so it can be either $\mathbb{Q}^{i_X+1}$ or $\mathbb{P}^{i_X+1}$. Since the last case leads to a contradiction with the elementary contractions of X, we have $Z_i = \mathbb{Q}^{i_X+1}$, so we get case (2) of the statement.

We can now assume that $\dim F_1 = \dim F_2 = i_X - 1$ and $\dim F_3 = i_X$.

Notice that at most one among these morphisms can be non equidimensional. In fact, if φ_α is such a morphism, then, by computing the dimensions with Lemma 2.13, we see that $X = \mathrm{Locus}(V^\beta)_{\mathrm{Locus}(V^\gamma)_{G_\alpha}} = \mathrm{Locus}(V^\gamma)_{\mathrm{Locus}(V^\beta)_{G_\alpha}}$, where G_α is a jumping fiber of φ_α. So both the $\mathrm{rc}(V^\beta)$-fibration and the $\mathrm{rc}(V^\gamma)$-fibration are equidimensional, and we are done.

In particular, this implies that at least one of the $\mathbb{P}^{i_X-1}$-bundles is equidimensional, so, up to exchange R_1 with R_2, we can assume that $\varphi_2\colon X \to Y_2$ is equidimensional. Then Y_2 is smooth of dimension $2i_X$, it is a Fano manifold by [19, Corollary 2.9] and, since $\rho_{Y_2} = 2$, it has pseudoindex equal to either i_X or $i_X + 1$ by combining cases (a) and (b) of [7, Lemme 5.2].

If $i_{Y_2} = i_X + 1$, then $Y_2 = \mathbb{P}^{i_X} \times \mathbb{P}^{i_X}$ by [29, Corollary 4.3], so we have the following diagram:

$$\begin{tikzcd}
Y_1 \arrow[d] & X \arrow[l, "\varphi_1"'] \arrow[r, "\varphi_3"] \arrow[d, "\varphi_2"] & Y_3 \arrow[d, "\phi_3"] \\
\mathbb{P}^{i_X} & \mathbb{P}^{i_X} \times \mathbb{P}^{i_X} \arrow[l, "\bar\varphi_1"'] \arrow[r, "\bar\varphi_3"] & \mathbb{P}^{i_X}.
\end{tikzcd}$$

In particular, $\bar\varphi_3$ is equidimensional with i_X-dimensional fibers, so the same holds for φ_3. Then φ_3 is an equidimensional $\mathbb{P}^{i_X}$-bundle onto the smooth variety Y_3 of dimension $2i_X - 1$, which is a Fano manifold by [19, Corollary 2.9]. In particular, since the general fiber of ϕ_3 is dominated by a general fiber of the $\mathbb{P}^{i_X-1}$-bundle φ_2, it follows that ϕ_3 is a $\mathbb{P}^{i_X-1}$-bundle, so Y_3 is one of the varieties described in Lemma 4.5 and Remark 4.6. Moreover, in view of Remark 4.7, there exists a vector bundle $\mathcal{E}$ on Y_3 such that $X = \mathbb{P}_{Y_3}(\mathcal{E})$. Now, since $\bar\varphi_1 \circ \varphi_2\colon X \to \mathbb{P}^{i_X}$ does not contracts curves in R_3, $X = \mathbb{P}^{i_X} \times Y_3$ by [25, Lemma 4.1]. Therefore the only possibility is case (6) of the statement.

From now on, we can thus assume $i_{Y_2} = i_X$.

We claim that the contraction $\bar\varphi_3$ which is associated with the extremal ray $\varphi_2(R_3)$ is equidimensional. This is obvious if its general fiber has dimension $i_X + 1$, since fibers of $\bar\varphi_1$ have dimension $\geq i_X - 1$ (and so $= i_X - 1$), in view of Remark 2.2. The only case to deal with is the one in which the general fiber of $\bar\varphi_3$ has dimension i_X. Assume to get a contradiction that $\bar\varphi_3$ has a jumping fiber, say J, whose dimension is thus equal to $i_X + 1$. Moreover, being $\bar\varphi_3$ associated with an extremal ray, the image of the jumping fibers in T has codimension $m \geq 3$. By taking $\dim T - m$ hyperplane sections A_k of T, we have a contraction $\bar\varphi_3|_{\bar\varphi_3^{-1}(\cap A_k)}\colon \bar\varphi_3^{-1}(\cap A_k) \to \cap A_k$, with general fiber $\mathbb{P}^{i_X}$ and some isolated jumping fibers of dimension $i_X + 1$. Moreover, we are in the assumptions of [2, Lemma 3.3], so we derive that this contraction is supported by a divisor of the form $K_{\varphi^{-1}(\cap A_k)} + (i_X+1)L$; we now get a contradiction with [5, Theorem 4.1].

We assume first that $\bar{\varphi}_3$ is equidimensional with i_X-dimensional fibers. This implies that also φ_3 is equidimensional. We recall the diagram

$$\begin{tikzcd} Y_1 \arrow[d] & X \arrow[l, "\varphi_1"'] \arrow[r, "\varphi_3"] \arrow[d, "\varphi_2"] & Y_3 \arrow[d, "\phi_3"] \\ Z_1 & Y_2 \arrow[l, "\bar{\varphi}_1"'] \arrow[r, "\bar{\varphi}_3"] & Z_3, \end{tikzcd}$$

in which we know that φ_3 is a $\mathbb{P}^{i_X}$-bundle; then also $\bar{\varphi}_3$ is a $\mathbb{P}^{i_X}$-bundle. It follows that Z_3 is smooth of dimension i_X, it is a Fano manifold by [19, Corollary 2.9] and it has pseudoindex equal to either i_X or i_X+1 by part (a) of [7, Lemme 5.2], hence Z_3 is either $\mathbb{Q}^{i_X}$ or $\mathbb{P}^{i_X}$. On the other hand, also Y_3 is smooth, Fano with $i_{Y_3} = i_X$ and it has dimension equal to $2i_X - 1$; since ϕ_3 is a $\mathbb{P}^{i_X-1}$-bundle (being its general fiber of dimension $i_X - 1$ and dominated by $\mathbb{P}^{i_X-1}$), Y_3 is one of the varieties classified in Lemma 4.5 and Remark 4.6.
We claim that $Z_3 = \mathbb{P}^{i_X}$. In fact if Z_3 were $\mathbb{Q}^{i_X}$, then $Y_3 = \mathbb{P}^{i_X-1} \times \mathbb{Q}^{i_X}$; by arguing as in Remark 4.7 we would get $Y_2 = \mathbb{P}^{i_X} \times \mathbb{Q}^{i_X}$ and so that $X = \mathbb{P}_{Y_2}(\mathcal{E})$ for some vector bundle $\mathcal{E}$ of rank i_X on the Fano manifold Y_2. Since $(\psi_3 \circ \varphi_3)$, where ψ_3 is the contraction associated to the other extremal ray of Y_3, is a morphism onto a $(i_X - 1)$-dimensional variety which does not contract curves contracted by φ_2, by [25, Lemma 4.1] we would have $X = \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X} \times \mathbb{Q}^{i_X}$, which is not possible in view of the contractions of X. It follows that $Z_3 = \mathbb{P}^{i_X}$. Moreover, one of the following holds:

(i) $Y_3 = \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X}$;
(ii) $Y_3 = \mathbb{P}_{\mathbb{P}^{i_X}}(T_{\mathbb{P}^{i_X}})$.

In case (i), by arguing as in Remark 4.7 we get $Y_2 = \mathbb{P}_{\mathbb{P}^{i_X}}(\mathcal{O}_{\mathbb{P}^{i_X}}^{\oplus i_X} \oplus \mathcal{O}_{\mathbb{P}^{i_X}}(1))$, or $Y_2 = \mathbb{P}_{\mathbb{P}^{i_X}}(T_{\mathbb{P}^{i_X}} \oplus \mathcal{O}_{\mathbb{P}^{i_X}}(1))$. Since $(\psi_3 \circ \varphi_3)$, where ψ_3 is the contraction associated to the other extremal ray of Y_3, is a morphism onto a (i_X-1)-dimensional variety which does not contract curves contracted by φ_2, by [25, Lemma 4.1] we have $X = \mathbb{P}^{i_X-1} \times Y_2$. So we have cases (4) and (5) of the statement.
In case (ii), if $\dim Z_1 = i_X$, then $X = \mathbb{P}^{i_X} \times Y_3$ by [25, Lemma 4.1]. If else, $\dim Z_1 = i_X + 1$, then $\bar{\varphi}_1$ is a $\mathbb{P}^{i_X-1}$-bundle which is not equidimensional, since otherwise we would reach a contradiction with [30, Theorem 2]; so, being φ_1 an equidimensional $\mathbb{P}^{i_X-1}$-bundle, we thus get a contradiction since $\varphi_2^{-1}(J)$, with J a jumping fiber of $\bar{\varphi}_1$ is a jumping fiber of φ_1, as in the proof of [25, Proposition 3.10].

So now we can assume that $\bar{\varphi}_3$ is equidimensional with (i_X+1)-dimensional fibers. Since $\bar{\varphi}_1$ is equidimensional with (i_X-1)-dimensional fibers, the same holds for φ_1. Moreover, $\bar{\varphi}_3$ is either a $\mathbb{P}^{i_X+1}$-bundle, or a $\mathbb{Q}^{i_X+1}$-fibration. Arguing as before, in both the cases we derive that $X = \mathbb{P}_{Y_1}(\mathcal{G})$, for some vector bundle $\mathcal{G}$ on the smooth Fano $(2i_X)$-fold Y_1, the morphism $\bar{\varphi}_3 \circ \varphi_2 \colon X \to Z_3$ is onto a (i_X-1)-dimensional variety and does not contracts curves in R_1; then by [25, Lemma 4.1] $X = \mathbb{P}^{i_X-1} \times Y_1$, and Y_1 is one of the variety we discussed in the above part of the proof. $\square$

Remark 5.6. In Lemma 5.3, Proposition 5.4 and Proposition 5.5, if X has dimension greater than five, we know by Remark 3.3 that the families V^1, V^2 and V^3 are always unsplit.

We can summarize the previous results as follows:

Theorem 5.7. *Let X be a Fano manifold of pseudoindex $i_X = \frac{\dim X+1}{3} \geq 2$ such that X is rc(V^1, V^2, V^3)-connected with respect to three unsplit families of rational curves as in Construction 3.1. Then $-K_X \cdot V^i = i_X$ for any $i = 1, 2, 3$, unless one of the following holds:*

(1) $X = \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X+1}$;
(2) $X = \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X-1} \times \mathbb{Q}^{i_X+1}$;
(3) $X = \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X} \times \mathbb{P}^{i_X}$;
(4) $X = \mathbb{P}^{i_X-1} \times \mathbb{P}^{i_X} \times \mathbb{Q}^{i_X}$, $i_X \geq 3$;
(5) $X = \mathbb{P}^{i_X-1} \times \mathbb{P}_{\mathbb{P}^{i_X}}(\mathcal{O}_{\mathbb{P}^{i_X}}^{\oplus i_X} \oplus \mathcal{O}_{\mathbb{P}^{i_X}}(1))$;
(6) $X = \mathbb{P}^{i_X-1} \times \mathbb{P}_{\mathbb{P}^{i_X}}(T_{\mathbb{P}^{i_X}} \oplus \mathcal{O}_{\mathbb{P}^{i_X}}(1))$;
(7) $X = \mathbb{P}^{i_X} \times \mathbb{P}_{\mathbb{P}^{i_X}}(T_{\mathbb{P}^{i_X}})$;
(8) $X = \mathbb{P}^{i_X-1} \times \mathrm{Bl}_{\mathbb{P}^{i_X-1}} \mathbb{P}^{2i_X}$;
(9) $X = \mathbb{P}^{i_X-1} \times \mathrm{Bl}_{\mathbb{P}^{i_X-2}} \mathbb{P}^{2i_X}$;
(10) $X = \mathbb{P}^{i_X} \times \mathrm{Bl}_{\mathbb{P}^{i_X-2}} \mathbb{P}^{2i_X-1}$.

Remark 5.8. Note that in Lemma 5.3, Proposition 5.4 and Theorem 5.7, the families V^1, V^2 and V^3 are always unsplit as soon as $\dim X > 5$ (cf. Remark 3.3).

Remark 5.9. Note that when we consider Fano manifolds which are rc(V^1, V^2, V^3)-connected with respect to three unsplit families of rational curves as in Construction 3.1, if we assume that the pseudoindex $i_X = \frac{\dim X+2}{3}$ then there is exactly one exception to the condition $-K_X \cdot V^i = i_X$ for any $i = 1, 2, 3$ (cf. Remark 4.10), while if $i_X = \frac{\dim X+1}{3}$ then there are more exceptions (cf. Theorem 5.7). Moreover, if $i_X = \frac{\dim X+2}{3}$ then X is a product with $\mathbb{P}^{i_X-1}$ as a factor (cf. Remark 4.10), while this is no longer true if $i_X = \frac{\dim X+1}{3}$ (cf. Theorem 5.7).

Remark 5.10. Some of the above results can be seen as special cases in the characterization of Fano manifolds which are rationalliy connected with respect to unsplit families of rational curves $V^1, \ldots, V^k$ whose anticanonical degrees satisfy the condition $\sum_{i=1}^k -K_X \cdot V^i = \dim X + k - 1$. However, the complete proof of the characterization of these varieties is quite long and complicated, so it will appear somewhere else [24].

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