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RULED FANO FIVEFOLDS OF INDEX TWO

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Abstract. We classify Fano fivefolds of index two which are projectivization of rank two vector bundles over four dimensional manifolds.

1. Introduction

A smooth complex projective variety X is called Fano if its anticanonical bundle $-K_X$ is ample; the index of X, r_X, is the largest natural number m such that $-K_X = mH$ for some (ample) divisor H on X, while the pseudoindex, i_X, is the minimum anticanonical degree of rational curves on X. Since X is smooth, $\mathrm{Pic}(X)$ is torsion free, and therefore the divisor L satisfying $-K_X = r_X L$ is uniquely determined and called the fundamental divisor of X.

By a theorem of Kobayashi and Ochiai [26], $r_X \geq \dim X + 1$ if and only if $(X, L) \simeq (\mathbb{P}^{\dim X}, \mathcal{O}_{\mathbb{P}}(1))$, and $r_X = \dim X$ if and only if $(X, L) \simeq (\mathbb{Q}^{\dim X}, \mathcal{O}_{\mathbb{Q}}(1))$.

Fano manifolds of index $\dim X - 1$ and $\dim X - 2$, which are called del Pezzo and Mukai manifolds, respectively, have been classified ([23], [32], [30]).

The method used for those cases (i.e. proving that the linear sistem $|L|$ contains a smooth divisor and constructing a ladder down to the known cases of lower dimensional varieties) does not work for Fano manifolds of index $\dim X - 3$, since there are no results on the existence of a (smooth) divisor in the linear system $|L|$ and, most of all, the classification of Fano fourfolds is very far from being known.

Nevertheless some classification results for Fano manifold of index $\dim X - 3$ and Picard number greater than one are known: by the classification of Fano manifolds of middle index and Picard number greater than one obtained by Wiśniewski and other authors (see [41] for a survey on these results) we have the complete classification of Fano manifolds of index $\dim X - 3$, Picard number greater than one and dimension greater than or equal to six.

Roughly speaking, apart from $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, these varieties have Picard number two, and thus two extremal elementary contractions, and the classification is obtained by a careful study of these contractions and their interplay.

Actually, by a theorem of Wiśniewski [39], there are no Fano manifolds of index $\dim X - 3$ and dimension greater than eight; this theorem is a particular case of a conjecture of Mukai relating the pseudoindex, the dimension and the Picard number of a Fano manifolds:

$$\rho_X(i_X - 1) \leq \dim X.$$

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In [4] it was proved that the conjecture holds for Fano manifolds of dimension five (for lower dimensional cases the result was already known).
However, the information on the Picard number when $\rho_X \geq 3$ is not enough to decide the number and type of the extremal contractions of the variety, i.e. to understand the structure of the cone of curves $\mathrm{NE}(X)$, result that was achieved for Fano fivefolds of pseudoindex greater than one in [18].
The present paper is intended as a first step in going from the table of the cones given in [18] to the actual classification of Fano fivefolds of index two, and it deals with ruled Fano fivefolds, i.e. with triples $(X, Y, \mathcal{E})$ constituted by a Fano fivefold X of index two, a smooth variety Y of dimension four and a rank two vector bundle $\mathcal{E}$ over Y such that $X = \mathbb{P}_Y(\mathcal{E})$.

The paper is organized as follows: in section 2 we collect basic material concerning Fano-Mori contractions, families of rational curves and Fano manifolds; section 3 is dedicated to $\mathbb{P}^{r-1}$-ruled Fano manifolds of index r, i.e. triples as above where $\mathrm{rk}\,\mathcal{E} = r_X = r$, relating the extremal contractions of X and Y.
Section 4 contains some criteria to establish if a $\mathbb{P}^{r-1}$-ruled Fano manifold of index r is a product of another Fano manifold of index r with a projective space $\mathbb{P}^{r-1}$.
In section 5 we begin with the classification problem; as already showed by the table of the cones in [18], the greater is the Picard number, the easier the classification becomes; this allows us to treat the cases $\rho_X \geq 4$ in a broader context, proving two general results on Fano manifolds with large Picard number and only (or almost only) fiber type contractions (propositions 5.1 and 5.2).
The following two sections are dedicated to the case $\rho_X = 3$, and we prove the following

Theorem 1.1. *Let $(X, Y, \mathcal{E})$ be a ruled Fano fivefold of index two with $\rho_X \geq 3$; then either X is a product $\mathbb{P}^1 \times Y$, with Y a Fano fourfold of index two and $\rho_Y = 2$ (for a classification of these manifolds see* [38]*) or X is one of the following:*

(1) $X \simeq Bl_p(\mathbb{P}^4) \times_{\mathbb{P}^3} Bl_p(\mathbb{P}^4)$;
(2) $X \simeq Bl_S(Bl_p(\mathbb{P}^5))$ *with S the strict trasform of a plane $\ni p$;*
(3) *the blow up of $\mathbb{P}^5$ in two non meeting planes;*
(4) *the blow up of a cone in $\mathbb{P}^9$ over the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ along its vertex;*
(5) *the blow up of a general member of $\mathcal{O}(1,1) \subset \mathbb{P}^2 \times \mathbb{P}^4$ along a two dimensional fiber of the second projection.*

In these cases the corresponding pairs $(Y, \mathcal{E})$ are, respectively,

(1) $(Bl_p(\mathbb{P}^4), 2H + E \oplus 3H + E)$, *$E$ exceptional divisor and H pullback on Y of $\mathcal{O}_{\mathbb{P}^3}(1)$;*
(2) $(Bl_l(\mathbb{P}^4), 2H - E \oplus 3H - E)$, *$E$ exceptional divisor and H pullback on Y of $\mathcal{O}_{\mathbb{P}^4}(1)$;*
(3) $(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(1,2) \oplus \mathcal{O}(2,1))$;
(4) $(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(1,1) \oplus \mathcal{O}(2,2))$;
(5) $(\mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2(-1) \oplus \mathcal{O}_{\mathbb{P}^2}) \subset \mathbb{P}^2 \times \mathbb{P}^3, \mathcal{O}(1,1) \oplus \mathcal{O}(1,2))$.

The last section contains the case $\rho_X = 2$, in which we have the following

Theorem 1.2. *Let $(X, Y, \mathcal{E})$ be a ruled Fano fivefold of index two with $\rho_X = 2$; then either X is a product $\mathbb{P}^1 \times \mathbb{Q}^4$, or $\mathbb{P}^1 \times Y$ with Y a Mukai fourfold of Picard number one (see [32]) or X is one of the following:*

(1) $\mathbb{P}_{\mathbb{P}^4}(\mathcal{O}_{\mathbb{P}^4} \oplus \mathcal{O}_{\mathbb{P}^4}(a))$, *with $a = 1$ or $a = 3$;*
(2) $\mathbb{P}_{\mathbb{Q}^4}(\mathcal{O}_{\mathbb{Q}^4} \oplus \mathcal{O}_{\mathbb{Q}^4}(2))$;
(3) $\mathbb{P}_{V_d}(\mathcal{O}_{V_d} \oplus \mathcal{O}_{V_d}(1))$, *with V_d a del Pezzo fourfold of degree $d = 1, \dots, 5$;*
(4) *a general divisor in the linear system $|2\xi|$ in $\mathbb{P}_{\mathbb{P}^3}(\Omega\mathbb{P}^3(3) \oplus \mathcal{O}(1))$;*
(5) *in $G(1,4) \times \mathbb{P}^4$, the intersection of two divisors in the linear system $|\mathcal{O}(1,0)|$ with the flag variety of point and lines in $\mathbb{P}^4$;*
(6) *a $\mathbb{P}^1$-bundle over a Fano fourfold of index one and pseudoindex two or three.*

Our classification is effective, apart from case (6) of theorem 1.2; we point out that it is not known whether a Fano fourfold as in case (6) (i.e. a Fano fourfold of Picard number one without a line) exists or not, and its existence (or non existence) constitutes a very hard problem.

2. Background material

2.1. **Extremal contractions.** Let X be a smooth complex Fano variety of dimension n and let K_X be its canonical divisor. By Mori's Cone Theorem the cone of effective 1-cycles, which is contained in the $\mathbb{R}$-vector space of 1-cyles modulo numerical equivalence, $\mathrm{NE}(X) \subset N_1(X)$, is polyhedral; a face of $\mathrm{NE}(X)$ is called an extremal face and an extremal face of dimension one is called an extremal ray.

To an extremal face $\sigma \subset \mathrm{NE}(X)$ is associated a morphism with connected fibers $\varphi_\sigma : X \to Z$ onto a normal variety, morphism which contracts the curves whose numerical class is in σ; φ_σ is called an extremal contraction or a Fano-Mori contraction, while a Cartier divisor H such that $H = \varphi_\sigma^* A$ for an ample divisor A on Z is called a supporting divisor of the map φ_σ (or of the face σ).

An extremal contraction associated to an extremal ray is called an elementary contraction; an extremal ray R is called numerically effective, and the associated contraction is said to be of fiber type, if $\dim Z < \dim X$; otherwise the ray is called non nef and the contraction is birational; the terminology is due to the fact that, if R is a non nef ray, there exists an irreducible divisor which has negative intersection number with curves in R.

We usually denote with $\mathrm{Exc}(\varphi_\sigma) := \{x \in X \mid \dim \varphi_\sigma^{-1}(\varphi_\sigma(x)) > 0\}$ the exceptional locus of φ_σ; if φ_σ is of fiber type then, of course, $\mathrm{Exc}(\varphi_\sigma) = X$.

If the codimension of the exceptional locus of an elementary birational contraction is equal to one, the ray and the contraction are called divisorial, otherwise they are called small.

Definition 2.1. An elementary fiber type extremal contraction $\varphi : X \to Z$ is called a scroll (respectively a quadric fibration) if there exists a φ-ample line bundle $L \in \mathrm{Pic}(X)$ such that $K_X + (\dim X - \dim Z + 1)L$ (respectively $K_X + (\dim X - \dim Z)L$) is a supporting divisor of φ; we will call conic fibration a quadric fibration such that $\dim X - \dim Z = 1$.

An elementary fiber type extremal contraction $\varphi : X \to Z$ onto a smooth variety Z is called a

$\mathbb{P}$-bundle (respectively quadric bundle) if there exists a vector bundle $\mathcal{E}$ of rank $\dim X - \dim Z + 1$ (respectively of rank $\dim X - \dim Z + 2$) on Z such that $X \simeq \mathbb{P}(\mathcal{E})$ (respectively there exists an embedding of X over Z as a divisor of $\mathbb{P}(\mathcal{E})$ of relative degree 2); we will call conic bundle a quadric bundle such that $\dim X - \dim Z = 1$.
An equidimensional scroll is a projective bundle by [22, Lemma 2.12], while an equidimensional quadric fibration is a quadric bundle by [3, Theorem B].
Some special scroll contractions arise from projectivization of Bănică sheaves (cfr. [13]); in particular, if $\varphi : X \to Z$ is a scroll such that every fiber has dimension $\leq \dim X - \dim Z + 1$, then Z is smooth and X is the projectivization of a Bănică sheaf on Z (cfr. [13, Proposition 2.5]); we will call these contractions special Bănică scrolls.

2.2. **Families of rational curves.** For this subsection our main reference is [27], with which our notation is coherent. Let X be a normal projective variety and let $\mathrm{Hom}(\mathbb{P}^1, X)$ be the scheme parametrizing morphisms $f : \mathbb{P}^1 \to X$; let $\mathrm{Hom}_{bir}(\mathbb{P}^1, X) \subset \mathrm{Hom}(\mathbb{P}^1, X)$ be the open subscheme corresponding to those morphisms which are birational onto their image, and let $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ be its normalization; the group $\mathrm{Aut}(\mathbb{P}^1)$ acts on $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ and the quotient exists.

Definition 2.2. The space $\mathrm{Ratcurves}^n(X)$ is the quotient of $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ by $\mathrm{Aut}(\mathbb{P}^1)$, and the space $\mathrm{Univ}(X)$ is the quotient of the product action of $\mathrm{Aut}(\mathbb{P}^1)$ on $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1$.

Definition 2.3. A family of rational curves is an irreducible component $V \subset \mathrm{Ratcurves}^n(X)$.
Given a rational curve $f : \mathbb{P}^1 \to X$, we will call a family of deformations of f any irreducible component $V \subset \mathrm{Ratcurves}^n(X)$ containing the equivalence class of f.
Given a family V of rational curves, we have the following basic diagram

$$\begin{tikzcd} p^{-1}(V) =: U \arrow[r, "i"] \arrow[d, "p"] & X \\ V & \end{tikzcd}$$

where i is the map induced by the evaluation $ev : \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1 \to X$ and p is the $\mathbb{P}^1$-bundle induced by the projection $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1 \to \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$. We define $\mathrm{Locus}(V)$ to be the image of U in X; we say that V is a covering family if $\overline{\mathrm{Locus}(V)} = X$.
If $L \in \mathrm{Pic}(X)$ is a line bundle, we will denote by $L \cdot V$ the intersection number of L and a general member of the family V. Finally, given a family $V \subseteq \mathrm{Ratcurves}^n(X)$, we denote by V_x the subscheme of V parametrizing rational curves passing through x.

Definition 2.4. Let V be a family of rational curves on X. Then V is unsplit if it is proper.

Example 2.5. *Let R_i be an extremal ray and C_i a curve whose numerical class belongs to R_i and whose anticanonical degree is minimal among curves whose class is in R_i; C_i is often called a* minimal extremal rational curve.
Denote by R^i an irreducible component of $Ratcurves^n(X)$ containing C_i; then the family R^i is unsplit: indeed, if C_i degenerates into a reducible cycle, its components must belong to the ray R_i, since R_i is

extremal; but in R_i the curve C_i has the minimal intersection with the anticanonical bundle, hence this is impossible.

Proposition 2.6. [27, IV.2.6] *Let X be a smooth projective variety and V a family of rational curves. Assume that V is unsplit and x is any point in* $\mathrm{Locus}(V)$. *Then*

(a) $\dim X - K_X \cdot V \leq \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1$;
(b) $-K_X \cdot V \leq \dim \mathrm{Locus}(V_x) + 1$.

This last proposition, in case V is the unsplit family of deformations of a minimal extremal rational curve, gives the fiber locus inequality:

Proposition 2.7. *Let φ be a Fano-Mori contraction of X and let $E = \mathrm{Exc}(\varphi)$ be its exceptional locus; let S be an irreducible component of a (non trivial) fiber of φ. Then*

$$\dim E + \dim S \geq \dim X + l - 1,$$

where

$$l = \min\{-K_X \cdot C \mid C \text{ is a rational curve in } S\}.$$

If φ is the contraction of a ray R, then $l(R) := l$ is called the length of the ray.

Let X be a smooth variety, $V^1, \ldots, V^k$ unsplit families of rational curves on X and $Z \subset X$.

Definition 2.8. We denote by $\mathrm{Locus}(V^1, \ldots, V^k)_Z$ the set of points that can be joined to Z by a connected chain of k cycles belonging respectively to the families $V^1, \ldots, V^k$.
We denote by $\mathrm{ChLocus}_m(V^1, \ldots, V^k)_Z$ the set of points that can be joined to Z by a connected chain of at most m cycles belonging to the families $V^1, \ldots, V^k$.

Definition 2.9. We define a relation of rational connectedness with respect to $V^1, \ldots, V^k$ on X in the following way: x and y are in $\mathrm{rc}(V^1, \ldots, V^k)$-relation if there exists a chain of rational curves in $V^1, \ldots, V^k$ which joins x and y, i.e. if $y \in \mathrm{ChLocus}_m(V^1, \ldots, V^k)_x$ for some m.

To the $\mathrm{rc}(V^1, \ldots, V^k)$-relation we can associate a fibration, at least on an open subset.

Theorem 2.10. [17],[27, IV.4.16] *There exist an open subvariety $X^0 \subset X$ and a proper morphism with connected fibers $\pi : X^0 \to T^0$ such that*

(a) *the $\mathrm{rc}(V^1, \ldots, V^k)$-relation restricts to an equivalence relation on X^0;*
(b) *the fibers of π are equivalence classes for the $\mathrm{rc}(V^1, \ldots, V^k)$-relation;*
(c) *for every $t \in T^0$ any two points in $\pi^{-1}(t)$ can be connected by a chain of at most $2^{\dim X - \dim T^0} - 1$ cycles in $V^1, \ldots, V^k$.*

Definition 2.11. In the above assumptions, if π is the constant map, we will say that X is $\mathrm{rc}(V^1, \ldots, V^k)$-connected.

For other properties of $\mathrm{Locus}(V^1, \ldots, V^k)_Z$ and $\mathrm{ChLocus}_m(V^1, \ldots, V^k)_Z$ we refer to [4] and [18].

2.3. **Fano manifolds and projective bundles.**

Lemma 2.12. *Let X be a Fano manifold and $p : X \to Y$ an elementary contraction onto a smooth variety such that every fiber of p is a projective space of dimension r. Denote by $R_{\mathcal{E}}$ the extremal ray of $NE(X)$ corresponding to p. Then*

(a) *Y is a Fano manifold with pseudoindex $i_Y \geq i_X$;*
(b) *if $i_Y = i_X$ and $f : \mathbb{P}^1 \to Y$ is a rational curve of degree i_Y, then $f^*\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus r+1}$;*
(c) *if $NE(X) = \langle R_{\mathcal{E}}, R_1, \ldots, R_k\rangle$, then $NE(Y) = \langle p(R_1), \ldots, p(R_k)\rangle$.*

Proof. Y is a Fano manifold by [28, Corollary 2.9]; the assertion on the pseudoindex and part (b) are proved in [16, Lemma 2.5], while part (c) is contained in the proof of [40, Lemma 3.1]. □

Lemma 2.13. *Let X be a Fano manifold of pseudoindex $i_X \geq 2$ and let $\varphi : X \to Y$ be an elementary contraction which is equidimensional with one dimensional fibers. Then there exists a rank two vector bundle $\mathcal{E}$ on Y such that $X = \mathbb{P}_Y(\mathcal{E})$.*

Proof. By [2, Theorem 3.1 (ii)] Y is smooth and $\varphi : X \to Y$ is a conic bundle. It follows that $-K_X \cdot f = 2$ for every fiber f of φ, therefore f can not be reducible or nonreduced, being $i_X \geq 2$. By lemma 2.12 (a) Y is a Fano manifold; in particular its Brauer group is trivial, hence there exists a rank two vector bundle $\mathcal{E}$ on Y such that $X = \mathbb{P}_Y(\mathcal{E})$. □

The fact that cone of curves of a Fano manifold is polyhedral and generated by a finite number of extremal rays easily leads to the following

Lemma 2.14. [15, Lemme 2.1] *Let X be a Fano manifold and D an effective divisor on X. Then there exists an extremal ray $R \subset NE(X)$ such that $D \cdot R > 0$.*

which, combined with lemma 2.13, gives

Corollary 2.15. *Let X be a Fano manifold of pseudoindex $i_X \geq 2$, $R \subset NE(X)$ an extremal ray and D an effective divisor on X such that no curve in D has numerical class belonging to R. If $D \cdot R > 0$, then the contraction associated to R, $\varphi_R : X \to Y$ is a $\mathbb{P}^1$-bundle.*

Proof. Let F be any fiber of φ_R; the intersection $D \cap F$ has to be zero dimensional, otherwise D would contain a curve whose numerical class is in R. It follows that φ_R is equidimensional with one dimensional fibers and we can apply lemma 2.13. □

The following lemma will be of frequent use in our proofs:

Lemma 2.16. *Let T be a smooth threefold of Picard number one, $\mathcal{F}$ a rank two vector bundle on T and $Y = \mathbb{P}_T(\mathcal{F})$; assume that Y is a Fano manifold of pseudoindex $i_Y \geq 2$. Then, if Y is not a product $Y = \mathbb{P}^1 \times T$, we have either $T \simeq \mathbb{P}^3$ or $T \simeq \mathbb{Q}^3$.*

Proof. By lemma 2.12 (a), T is a Fano threefold of pseudoindex $i_T \geq i_Y \geq 2$; in particular, by the classification of Fano threefolds, T admits an unsplit covering family V_T of rational curves of degree i_T.

If $i_Y = i_T$, then, by lemma 2.12 (b), the restriction of $\mathcal{F}$ to any curve of V_T splits as $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(a)$, so, by [12, Proposition 1.2], $\mathcal{F}$ is decomposable and $Y \simeq \mathbb{P}^1 \times T$.
Otherwise $i_T \geq 3$ and, by the classification of Fano threefolds, either $T \simeq \mathbb{Q}^3$, or $T \simeq \mathbb{P}^3$. □

Finally we prove two lemmata which ensure that, in some cases, a fibration in projective spaces is a projective bundle.

Lemma 2.17. *Let $p : Y \to B$ be a morphism from a smooth variety to a smooth curve, such that $\rho(Y/B) = 1$ and the general fiber of p is a projective space; then there exists a vector bundle $\mathcal{F}$ of rank $= \dim Y$ on B such that $Y = \mathbb{P}_B(\mathcal{F})$ and p is the natural projection.*

Proof. Over an open Zariski subset U of B the morphism p is a projective bundle; indeed over a curve C a fibration in projective spaces is a projective bundle, since the obstruction lies in $H^2(C, \mathcal{O}^*) = 0$ (see [20]). By taking the closure in Y of a hyperplane section of p defined over the open set U we get a global relative hyperplane section divisor (we use $\rho(Y/B) = 1$) hence p is a projective bundle globally by [22, Lemma 2.12]. □

Lemma 2.18. *Let X be a Fano manifold and $p : X \to S$ be an elementary contraction associated to an extremal ray of length $\dim X - 1$ onto a surface S. Then S is smooth and there exists a rank $\dim X - 1$ vector bundle $\mathcal{F}$ over S such that $X = \mathbb{P}_S(\mathcal{F})$.*

Proof. Since p is elementary and $\dim S = 2$ then p is equidimensional; by [11, Corollary 1.4] S is smooth.
By adjunction the general fiber of p is a projective space of dimension $\dim X - 2$; over a general hyperplane section of S, φ is a projective bundle by lemma 2.17, whence the locus over which the fiber is not a projective space is discrete in S. We can apply [5, Lemma 3.3] and [22, Lemma 2.12] to obtain that every fiber of φ is a projective space. The surface S is dominated by a Fano manifold, hence is rationally connected; therefore $H^2(S, \mathcal{O}^*) = 0$ and the Brauer group of S is trivial. This implies the existence of a rank $\dim X - 1$ vector bundle $\mathcal{F}$ over S such that $X = \mathbb{P}_S(\mathcal{F})$. □

3. $\mathbb{P}^{r-1}$-ruled Fano manifolds: general properties

Definition 3.1. Let Y be a smooth variety of dimension n, let $\mathcal{E}$ be a vector bundle of rank r on Y and let $X = \mathbb{P}_Y(\mathcal{E})$ be the projectivization of $\mathcal{E}$; assume moreover that X is a Fano manifold. We will call a triple $(X, Y, \mathcal{E})$ as above a $\mathbb{P}^{r-1}$-ruled Fano manifold; if $r = 2$, we will call for short $(X, Y, \mathcal{E})$ a ruled Fano manifold.

Definition 3.2. Let $(X, Y, \mathcal{E})$ be a $\mathbb{P}^{r-1}$-ruled Fano manifold verifying one of the following

1) X has index r;
2) $K_Y + \det \mathcal{E}' = \mathcal{O}_Y$, with $\mathcal{E}'$ an ample twist of $\mathcal{E}$.

We will call such a triple a $\mathbb{P}^{r-1}$-ruled Fano manifold of index r; if $r = 2$, we will call for short $(X, Y, \mathcal{E})$ a ruled Fano manifold of index two.
From now on, unless otherwise stated, we will assume that $\mathbb{P}^{r-1}$-ruled Fano manifold of index r $(X, Y, \mathcal{E})$ are normalized, i.e. $\mathcal{E}$ is ample and $K_Y + \det \mathcal{E} = \mathcal{O}_Y$.

Remark 3.3. The assumptions 1) and 2) are equivalent.

Proof. Let us show first that 1) $\Rightarrow$ 2); let $H \in \mathrm{Pic}(X)$ be the (unique) line bundle such that $-K_X = rH$; by adjunction, if l is a line in a fiber of the projection $p : X \to Y$, then $r = -K_X \cdot l = rH \cdot l$, so H restricts to $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ on the fibers of p. Therefore p_*H is an ample vector bundle of rank r, $\mathcal{E}'$, which differs from $\mathcal{E}$ by a twist with a line bundle in $\mathrm{Pic}(Y)$ and, by the canonical bundle formula

$$\mathcal{O}_X = K_X + rH = p^*(K_Y + \det \mathcal{E}'),$$

hence $K_Y + \det \mathcal{E}' = \mathcal{O}_Y$.
Assume now that 2) holds; for a suitable ample twist $\mathcal{E}' = \mathcal{E} \otimes L$, we have $K_Y + \det \mathcal{E}' = \mathcal{O}_Y$, therefore, by the canonical bundle formula,

$$K_X + r\xi_{\mathcal{E}'} = p^*(K_Y + \det \mathcal{E}') = \mathcal{O}_X,$$

whence $-K_X = r\xi_{\mathcal{E}'}$ and X is a Fano manifold of index r. □

Proposition 3.4. *Let $(X, Y, \mathcal{E})$ be a $\mathbb{P}^{r-1}$-ruled Fano manifold and denote by $R_{\mathcal{E}}$ the extremal ray in $NE(X)$ associated to the bundle projection $p : X \to Y$. There is a one-to-one correspondence*

$$\left\{ \begin{array}{c} \textit{Extremal rays of } NE(X) \textit{ spanning} \\ \textit{a two dimensional face with } R_{\mathcal{E}} \end{array} \right\} \underset{\alpha_Y}{\overset{\alpha_X}{\rightleftarrows}} \left\{ \textit{Extremal rays of } NE(Y) \right\}.$$

If $\theta \subset NE(Y)$ and $\vartheta \subset NE(X)$ are corresponding rays, then we will call them fellow *rays.*

Proof. Let θ be an extremal ray of $\mathrm{NE}(Y)$ and denote by $\varphi_\theta : Y \to W$ the associated elementary contraction; then $\rho(X/W) = 2$ and $-K_X$ is $(\varphi_\theta \circ p)$-ample, so $\varphi_\theta \circ p : X \to W$ is the contraction of a two dimensional extremal face $\sigma \subset \mathrm{NE}(X)$ containing $R_{\mathcal{E}}$. Let ϑ be the extremal ray in σ different from $R_{\mathcal{E}}$; we set $\alpha_Y(\theta) = \vartheta$.

On the other hand, if ϑ is an extremal ray of $\mathrm{NE}(X)$ such that $\sigma = \langle R_{\mathcal{E}}, \vartheta \rangle$ is an extremal face, then the contraction $\psi_\sigma : X \to W$ factors both through the contraction p of $R_{\mathcal{E}}$ and through the contraction $\psi_\vartheta : X \to Z$ of ϑ, hence we have a commutative diagram

(3.4.1)
$$\begin{tikzcd} X \arrow[r, "\psi_\vartheta"] \arrow[d, "p"'] \arrow[dr, "\psi_\sigma"] & Z \arrow[d, "p'"] \\ Y \arrow[r, "\varphi_\theta"'] & W \end{tikzcd}$$

Since Y is a Fano manifold and φ_θ is a surjective morphism with connected fibers, we have that φ_θ is an extremal contraction; moreover, being $\rho(Y/W) = 1$, the contraction is elementary, thus it corresponds to an extremal ray θ. Setting $\alpha_X(\vartheta) = \theta$ we have the desired bijection. □

Lemma 3.5. *Let $(X, Y, \mathcal{E})$ be a $\mathbb{P}^{r-1}$-ruled Fano manifold and let $\theta \subset NE(Y)$ and $\vartheta \subset NE(X)$ be two fellow rays with associated extremal contractions $\varphi_\theta : Y \to W$ and $\psi_\vartheta : X \to Z$, with exceptional loci $\mathrm{Exc}(\varphi_\theta)$ and $\mathrm{Exc}(\psi_\vartheta)$ respectively. Then*

$$p(\mathrm{Exc}(\psi_\vartheta)) \subset \mathrm{Exc}(\varphi_\theta). \tag{3.5.1}$$

Moreover, if x is a point in $\mathrm{Exc}(\psi_\vartheta)$, $(F_\psi)_x$ is the fiber of ψ_ϑ through x and $(F_\varphi)_{p(x)}$ is the fiber of φ_θ through $p(x)$, we have

$$\dim(F_\psi)_x = \dim p((F_\psi)_x) \leq \dim(F_\varphi)_{p(x)}. \tag{3.5.2}$$

Finally, if x_1 is a point in $p^{-1}(p(x)) \cap \mathrm{Exc}(\psi_\vartheta)$ and $(F_\psi)_{x_1}$ is the fiber of ψ_ϑ through x_1, then

$$p(F_\psi)_{x_1} \subset (F_\varphi)_{p(x)}. \tag{3.5.3}$$

Proof. The statements follows from the commutativity of diagram 3.4.1 and the fact that the projection p, being the contraction of an extremal ray different from ϑ, is finite to one on the fibers of ψ_ϑ. □

Corollary 3.6. *Under the assumptions of lemma 3.5, if ψ_ϑ is of fiber type then also φ_θ is of fiber type, while if φ_θ is birational then also ψ_ϑ is birational.*

Lemma 3.7. *Let $(X, Y, \mathcal{E})$ be a $\mathbb{P}^{r-1}$-ruled Fano manifold of index r and let $\theta \subset NE(Y)$ and $\vartheta \subset NE(X)$ be two fellow rays with associated extremal contractions $\varphi_\theta : Y \to W$ and $\psi_\vartheta : X \to Z$. Then there exist an ample vector bundle $\mathcal{E}_\Theta$ on Y and an ample line bundle $L \in \mathrm{Pic}(X)$ such that φ_θ is supported by $K_Y + \det \mathcal{E}_\Theta$ and ψ_ϑ is supported by $K_X + rL$.*

Proof. Pick two ample line bundles $A \in \mathrm{Pic}(W)$ and $B \in \mathrm{Pic}(Z)$. Set $\mathcal{E}_\Theta = \mathcal{E} \otimes \varphi_\theta^* A$; we have $K_Y + \det \mathcal{E}_\Theta = r\varphi_\theta^* A$, so we have only to prove the ampleness of $\mathcal{E}_\Theta$.
The tautological line bundle associated to $\mathcal{E}_\Theta$ on $\mathbb{P}(\mathcal{E}_\Theta) = \mathbb{P}(\mathcal{E}) = X$ is

$$\xi_\Theta = \xi_\mathcal{E} + p^*(\varphi_\theta^* A),$$

hence it is ample, being the sum of an ample line bundle and a nef one.

To prove the second statement observe that $K_X + r\xi_\Theta = p^*(K_Y + \det \mathcal{E}_\Theta) = r(p^*(\varphi_\theta^* A))$; therefore, if $L := \xi_\Theta + \psi_\vartheta^* B$, we have

$$K_X + rL = r(p^*(\varphi_\theta^* A) + \psi_\vartheta^* B) = r\psi_\vartheta^*(p'^* A + B).$$

Moreover L is ample, being the sum of an ample line bundle and a nef one. □

We now analyze some cases in which φ_θ is a special contraction (projective bundle, smooth blow-up, special Bănică scroll), describing the structure of the corresponding contraction ψ_ϑ.

Proposition 3.8. *Let $(X, Y, \mathcal{E})$ be a $\mathbb{P}^{r-1}$-ruled Fano manifold of index r; let $\theta \subset NE(Y)$ and $\vartheta \subset NE(X)$ be two fellow rays and let $\varphi_\theta : Y \to W$ and $\psi_\vartheta : X \to Z$ be the associated contractions. Then*

(a) *if φ_θ is a $\mathbb{P}^{r-1}$-bundle, then ψ_ϑ is a $\mathbb{P}^{r-1}$-bundle;*

(b) *if φ_θ is the blow up of a smooth subvariety of W of codimension $r+1$, then ψ_ϑ is the blow up of a smooth subvariety of Z of codimension $r+1$.*

In both cases, if $H \in \mathrm{Pic}(Y)$ is a line bundle which restricts to $\mathcal{O}_{\mathbb{P}}(1)$ on the fibers of φ_θ, then $\mathcal{E} \otimes H^{-1} = \varphi_\theta^* \mathcal{E}'$, where $\mathcal{E}'$ is a rank r vector bundle on W, and $Z = \mathbb{P}_W(\mathcal{E}')$.

Proof. Denote by $l(\vartheta)$ the length of the extremal ray ϑ; since X is a Fano manifold of index r we have $l(\vartheta) \geq r$.
In case (a), if $x \in X$ is any point in $\mathrm{Exc}(\vartheta)$, $(F_\psi)_x$ is the fiber of ψ_ϑ through x and $(F_\varphi)_{p(x)}$ is the fiber of φ_θ through $p(x)$, by proposition 2.7 and formula 3.5.2 we have

$$r - 1 \leq l(\vartheta) - 1 \leq \dim(F_\psi)_x \leq \dim(F_\varphi)_{p(x)} = r - 1,$$

so ψ_ϑ is an equidimensional contraction with $(r-1)$-dimensional fibers (and thereby of fiber type, by proposition 2.7). By lemma 3.7, there exists an ample $L \in \mathrm{Pic}(X)$ such that ψ_ϑ is supported by $K_X + rL$, and we conclude by [22, Lemma 2.12].
In case (b), by corollary 3.6, since φ_θ is birational, also ψ_ϑ is birational. Then, if $x \in X$ is any point in $\mathrm{Exc}(\vartheta)$, $(F_\psi)_x$ is the fiber of ψ_ϑ through x and $(F_\varphi)_{p(x)}$ is the fiber of φ_θ through $p(x)$, by proposition 2.7 and formula 3.5.2 we have

$$r \leq l(\vartheta) \leq \dim(F_\psi)_x \leq \dim(F_\varphi)_{p(x)} = r,$$

thus ψ_ϑ is equidimensional with fibers of dimension r and, by lemma 3.7, it is supported by $K_X + rL$, for some ample $L \in \mathrm{Pic}(X)$; therefore we can apply [10, Theorem 4.1] to conclude.
In both cases the extremal ray θ has length r, hence $r \geq i_Y$; by lemma 2.12 (a) we have $i_Y \geq i_X$ and, recalling that the pseudoindex i_X is greater or equal than the index $r_X = r$, we have $i_X \geq r$. We conclude that $i_Y = i_X = r$.
By lemma 2.12 (b), for every line l in every fiber of φ_θ we have $\mathcal{E}_l \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$, hence, if $H \in \mathrm{Pic}(Y)$ is a line bundle which restricts to $\mathcal{O}_{\mathbb{P}}(1)$ on the fibers of φ_θ, the vector bundle $\mathcal{E} \otimes H^{-1}$ is trivial on every fiber, so it is the pullback of a rank r vector bundle $\mathcal{E}'$ on W. It is now easy to prove that the induced map $\mathbb{P}_Y(\varphi_\theta^* \mathcal{E}') = X \to \mathbb{P}_W(\mathcal{E}')$ is just ψ_ϑ, whence $Z = \mathbb{P}_W(\mathcal{E}')$. □

Proposition 3.9. *Let $(X, Y, \mathcal{E})$ be a $\mathbb{P}^{r-1}$-ruled Fano manifold of index r; let $\theta \subset NE(Y)$ and $\vartheta \subset NE(X)$ be two fellow rays and let $\varphi_\theta : Y \to W$, $\psi_\vartheta : X \to Z$ be the associated contractions. Then*

(a) *if φ_θ is a $\mathbb{P}^r$-bundle and ψ_ϑ is of fiber type, then ψ_ϑ is a $\mathbb{P}^{r-1}$-bundle;*
(b) *if φ_θ is a $\mathbb{P}^r$-bundle and ψ_ϑ is birational, then ψ_ϑ is the blow up of a codimension $r+1$ subvariety of Z.*

Moreover, in case (a), if $H \in \mathrm{Pic}(Y)$ is a line bundle which restricts to $\mathcal{O}_{\mathbb{P}^r}(1)$ on the fibers of φ_θ, then p^*H restricts to $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ on the fibers of ψ_ϑ; in case (b), the divisor $\mathrm{Exc}(\psi_\vartheta)$ restricts to $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ on the fibers of p.

Proof. Let $\sigma = \langle R_{\mathcal{E}}, \vartheta \rangle \subset \mathrm{NE}(X)$ and let $\psi_\sigma : X \to W$ be the contraction associated to the face σ, which can be factored both as $\varphi_\theta \circ p$ and as $p' \circ \psi_\vartheta$:

$$\begin{tikzcd} X \arrow[r, "\psi_\vartheta"] \arrow[d, "p"'] \arrow[dr, "\psi_\sigma"] & Z \arrow[d, "p'"] \\ Y \arrow[r, "\varphi_\theta"'] & W \end{tikzcd}$$

A fiber F_σ of ψ_σ can thus be viewed as the inverse image via p of a fiber $F_\theta \simeq \mathbb{P}^r$ of φ_θ, $F_\sigma \simeq \mathbb{P}_{F_\theta}(\mathcal{E}_{|F_\theta})$.
The ampleness of the vector bundle $\mathcal{E}$ together with the fact that

$$\det(\mathcal{E}_{|F_\theta}) = (\det \mathcal{E})_{|F_\theta} = (-K_Y)_{|F_\theta} = \mathcal{O}_{\mathbb{P}^r}(r+1)$$

yields that the splitting type of $\mathcal{E}$ on lines of F_θ is constantly $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$; by [21], either $\mathcal{E}_{|F_\theta} \simeq \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus r-1} \oplus \mathcal{O}_{\mathbb{P}^r}(2)$, or $\mathcal{E}_{|F_\theta} \simeq T\mathbb{P}^r$.

In case (a) ψ_ϑ is of fiber type, so also its restriction to $F_\sigma = \psi_\vartheta^{-1}(\psi_\vartheta(F_\sigma))$ is a fiber type contraction, therefore $\mathcal{E}_{|F_\theta} \simeq T\mathbb{P}^r$; it follows that ψ_ϑ is equidimensional and each of its fibers is $\mathbb{P}^{r-1}$. By lemma 3.7, there exists an ample $L \in \mathrm{Pic}(X)$ such that ψ_ϑ is supported by $K_X + rL$, hence, by [22, Lemma 2.12], ψ_ϑ is a $\mathbb{P}^{r-1}$-bundle over Z.
From this description it is clear that, if $H \in \mathrm{Pic}(Y)$ is a line bundle which restricts to $\mathcal{O}_{\mathbb{P}^r}(1)$ on the fibers of φ_θ, then p^*H restricts to $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ on the fibers of ψ_ϑ.

In case (b), if $x \in X$ is any point in $\mathrm{Exc}(\vartheta)$, $(F_\psi)_x$ is the fiber of ψ_ϑ through x and $(F_\varphi)_{p(x)}$ is the fiber of φ_θ through $p(x)$, by proposition 2.7 and formula 3.5.2 we have

$$r \leq l(\vartheta) \leq \dim(F_\psi)_x \leq \dim(F_\varphi)_{p(x)} = r,$$

thus ψ_ϑ is equidimensional with fibers of dimension r and, by lemma 3.7, it is supported by $K_X + rL$, for some ample $L \in \mathrm{Pic}(X)$; therefore, by [10, Theorem 4.1] ψ_ϑ is the blow up of a codimension $r+1$ subvariety of Z.
Let F_ϑ be a fiber of ψ_ϑ and let F_σ be the fiber of ψ_σ containing F_ϑ; the restriction of ψ_σ to this fiber has a non trivial fiber of dimension r, therefore $\mathcal{E}_{|F_\theta} \simeq \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus r-1} \oplus \mathcal{O}_{\mathbb{P}^r}(2)$.
It follows that F_σ is the blow up of $\mathbb{P}^{2r-1}$ along $\mathbb{P}^{r-1}$ and $\mathrm{Exc}(\psi_\vartheta)_{|F_\sigma}$ is the exceptional divisor of this blow up, hence it restricts to $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ on the fibers of p. □

Proposition 3.10. *Let $(X, Y, \mathcal{E})$ be a $\mathbb{P}^{r-1}$-ruled Fano manifold of index r; let $\theta \subset NE(Y)$ and $\vartheta \subset NE(X)$ be two fellow rays and let $\varphi_\theta : Y \to W$ and $\psi_\vartheta : X \to Z$ be the associated contractions. If φ_θ is a special Bănică scroll with general fiber of dimension $r-1$, then also ψ_ϑ is a special Bănică scroll with general fiber of dimension $r-1$. Moreover, if J is a jumping fiber of φ_θ (i.e. a fiber of dimension r), then there is an isomorphism $f : \mathbb{P}^{r-1} \times J \to p^{-1}(J)$ and, for every $x \in \mathbb{P}^{r-1}$, $f(\{x \times J\})$ is a jumping fiber of ψ_ϑ.*

Proof. The general fiber of φ_θ is $r-1$ dimensional, and every fiber of φ_θ has dimension $\leq r$; using formula 3.5.2 , as in the proof of proposition 3.8 we find that the same is true for ψ_ϑ.

By lemma 3.7, the contraction ψ_ϑ is supported by $K_X + rL$ for some ample $L \in \mathrm{Pic}(X)$; we can thereby apply [13, Proposition 2.5] to conclude that ψ_ϑ is a special Bănică scroll.
Let l be a line in a fiber F_θ of φ_θ; since this contraction has length r we have

$$\det(\mathcal{E}_{|F_\theta}) = (\det \mathcal{E})_{|F_\theta} = (-K_Y)_{|F_\theta} = \mathcal{O}_{F_\theta}(r),$$

so the splitting type of $\mathcal{E}$ on l is constantly $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$; it follows that $\mathcal{E}_{|F_\theta} \simeq \mathcal{O}_{F_\theta}(1)^{\oplus r}$. Therefore $p^{-1}(F_\theta) = \mathbb{P}_{F_\theta}(\mathcal{E}_{|F_\theta}) \simeq \mathbb{P}^{r-1} \times F_\theta$; since $p^{-1}(F_\theta) = \psi_\vartheta^{-1}(\psi_\vartheta(p^{-1}(F_\theta)))$ the subvarieties $\{x\} \times F_\theta$ of $\mathbb{P}^{r-1} \times F_\theta$ correspond to fibers of ψ_ϑ.
In particular, if $J \simeq \mathbb{P}^r$ is a jumping fiber of φ_θ, then $p^{-1}(J) = \mathbb{P}_J(\mathcal{E}_{|J}) \simeq \mathbb{P}^{r-1} \times J \simeq \mathbb{P}^{r-1} \times \mathbb{P}^r$ and the restriction $\psi_\vartheta : p^{-1}(J) \to \psi_\vartheta(p^{-1}(J))$ is a fibration in $\mathbb{P}^r$, hence each fiber is a jumping fiber. □

4. Recognizing products

In this section we collect some technical results that we are going to use in order to establish whether a ruled Fano manifold is a product of another Fano manifold with a suitable projective space.

The idea of the following lemma is taken from [12, Lemma 1.2.2].

Lemma 4.1. *Let $(X, Y, \mathcal{E})$ be a $\mathbb{P}^{r-1}$-ruled Fano manifold, and let $R_\mathcal{E} \subset NE(X)$ be the extremal ray corresponding to the bundle projection. Suppose that there exist an open subset $X^0 \subset X$ and a proper morphism $\psi : X^0 \to Z$ onto a variety Z of dimension $r-1$ which does not contract curves of $R_\mathcal{E}$. Then $X \simeq \mathbb{P}^{r-1} \times Y$.*

Proof. Let F be a general fiber of ψ; the dimension of F is $\dim F = \dim X - \dim Z = \dim Y$, therefore F dominates Y, since ψ does not contract curves in the fibers of p.
Denote by $p_F : F \to Y$ the restriction of p to F and consider the pullback $\mathcal{E}_F = p_F^*\mathcal{E}$; denoted by X_F the projectivization $\mathbb{P}_F(\mathcal{E}_F)$, we have a commutative diagram

$$\begin{tikzcd} X_F \arrow[r, "\widetilde{p}_F"] \arrow[d, "\widetilde{p}"'] & X \arrow[d, "p"] \\ F \arrow[r, "p_F"'] & Y \end{tikzcd}$$

By the universal property of the fiber product, $\widetilde{p}$ has a section $s : F \to X_F$ such that $\widetilde{p}_F \circ s$ is the embedding of F into X. Let $\widetilde{F} = s(F)$ be the image of F in X_F; by the canonical bundle formula for X_F we have

$$r\xi_{\mathcal{E}_F} - \widetilde{p}^* \det \mathcal{E}_F = -K_{X_F} + \widetilde{p}^* K_F.$$

Since $\widetilde{p}^* K_F = K_{\widetilde{F}} = (K_{X_F})_{|\widetilde{F}}$, restricting to $\widetilde{F}$ we have $(r\xi_{\mathcal{E}_F} - \widetilde{p}^* \det \mathcal{E}_F)_{|\widetilde{F}} = \mathcal{O}_{\widetilde{F}}$; therefore, using the canonical bundle formula for X,

$$\mathcal{O}_F = (r\xi_\mathcal{E} - p^* \det \mathcal{E})_{|F} = (-K_X + p^* K_Y)_{|F}.$$

It follows that $\mathcal{O}_F = (K_X)_{|F} = p_F^* K_Y$, so p_F is unramified. As Y, being Fano, is simply connected p_F is an isomorphism, hence F is a section of p. To this section it is associated an exact sequence of bundles over Y

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow H \longrightarrow 0 \tag{4.1.4}$$

such that $F \in H^0(\xi_{\mathcal{E}} \otimes p^*\mathcal{E}'^{\vee})$; in particular the normal bundle of F in X is $(\xi_{\mathcal{E}} \otimes p^*\mathcal{E}'^{\vee})_{|F}$.
Pulling back the sequence 4.1.4 to F we obtain an exact sequence of bundles over F

$$0 \longrightarrow p_F^*\mathcal{E}' \longrightarrow p_F^*\mathcal{E} \longrightarrow p_F^* H \longrightarrow 0 \tag{4.1.5}$$

Since F is a general fiber of ψ, its normal bundle in X is trivial; thus we have

$$\mathcal{O}_F^{\oplus r-1} = N_{F/X} = (\xi_{\mathcal{E}} \otimes p^*\mathcal{E}'^{\vee})_{|F}.$$

It follows that $(p^*\mathcal{E}')_{|F} \simeq (\xi_{\mathcal{E}})_{|F}^{\oplus r-1}$; therefore we can rewrite the sequence 4.1.5 as

$$0 \longrightarrow \xi_{\mathcal{E}_F}^{\oplus r-1} \longrightarrow \mathcal{E}_F \longrightarrow p_F^* H \longrightarrow 0. \tag{4.1.6}$$

Recalling that $(\det \mathcal{E})_{|F} = r\xi_{\mathcal{E}|F} = r\xi_{\mathcal{E}_F}$, we have $p_F^* H = \xi_{\mathcal{E}_F}$ and the sequence 4.1.6 splits, because $h^1(F, \mathcal{O}_F) = 0$. Thus $\mathcal{E}_F$ is decomposable as $\xi_{\mathcal{E}_F}^{\oplus r}$ and, being p_F is an isomorphism, also $\mathcal{E}$ is decomposable, as a sum of r copies of H. □

Remark 4.2. In the proof of the lemma, instead of assuming that Y is a Fano manifold, it is enough to assume that Y is simply connected and that $h^1(Y, \mathcal{O}_Y) = 0$.

Corollary 4.3. *Let $(X, Y, \mathcal{E})$ be a $\mathbb{P}^{r-1}$-ruled Fano manifold of index r; assume that $Y \simeq \mathbb{P}^{r-1} \times W$ and denote by π_1 and π_2 the projections of Y onto the factors. Then there exists a vector bundle $\mathcal{E}'$ over W such that $\pi_2^*\mathcal{E}' = \mathcal{E} \otimes \pi_1^*\mathcal{O}_{\mathbb{P}^{r-1}}(-1)$ and $X = \mathbb{P}^{r-1} \times \mathbb{P}_W(\mathcal{E}')$.*

Proof. The projection π_2 is the contraction associated to an extremal ray $\theta \subset \mathrm{NE}(Y)$; let $\vartheta \subset \mathrm{NE}(X)$ be its fellow ray. By proposition 3.8 the contraction associated to ϑ, $\psi_\vartheta : X \to Z$, is a $\mathbb{P}^{r-1}$-bundle and $Z = \mathbb{P}_W(\mathcal{E}')$, with $\mathcal{E} \otimes \pi_1^*\mathcal{O}_{\mathbb{P}^{r-1}}(-1) = \pi_2^*\mathcal{E}'$.
In particular there exists a vector bundle $\mathcal{F}$ over Z such that $(X, Z, \mathcal{F})$ is a $\mathbb{P}^{r-1}$-ruled Fano manifold; we can apply lemma 4.1 to $(X, Z, \mathcal{F})$, taking as ψ the composition $\pi_1 \circ p : X \to \mathbb{P}^{r-1}$. □

Proposition 4.4. *Let $(X, Y, \mathcal{E})$ be a $\mathbb{P}^{r-1}$-ruled Fano manifold of index r. Suppose that there exist $R_1, \dots R_{\rho_Y}$ extremal rays of length r in NE(Y) such that Y is rationally connected with respect to curves in the corresponding families $R^1, \dots, R^{\rho_Y}$ (see example 2.5). Then $X \simeq \mathbb{P}^{r-1} \times Y$.*

Proof. Let C_i be a curve in the family R^i; since $\mathcal{E}$ is ample and $\det \mathcal{E} \cdot C_i = -K_Y \cdot C_i = l(R_i) = r$, denoting by $f_i : \mathbb{P}^1 \to C_i$ the normalization morphism, we have $f_i^*\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$.
Let $X_i = \mathbb{P}^1 \times_Y X = \mathbb{P}_{\mathbb{P}^1}(f_i^*\mathcal{E}) = \mathbb{P}^1 \times \mathbb{P}^{r-1}$ and let G_i be the image of X_i in X.

We have a commutative diagram

$$\begin{tikzcd} X_i \arrow[r, "\bar{f}_i"] \arrow[d, "\bar{p}"'] & X \arrow[d, "p"] \\ \mathbb{P}^1 \arrow[r, "f_i"'] & Y \end{tikzcd}$$

Let $\widetilde{C}_i$ be a section of $\bar{p} : X_i \to \mathbb{P}^1$, let $\Gamma_i = \bar{f}_i(\widetilde{C}_i)$ be its image in X and let V^i be a family of deformations of Γ_i; by the canonical bundle formula we have $-K_X \cdot \Gamma_i = r\xi_{\mathcal{E}} \cdot \Gamma_i = r$, therefore the family V^i is an unsplit family.
Let x be a point of X and y a point of Y; as Y is rationally connected with respect to curves in $R^1, \dots, R^{\rho_Y}$, there exists a chain of curves $C_{i_1}, \dots, C_{i_m}$ in $R^1, \dots, R^{\rho_Y}$ connecting $p(x)$ and y, with $m \leq 2^{\dim Y} - 1$.
Let y_1 be a point in $C_{i_1} \cap C_{i_2}$ and let Γ_{i_1} be a curve in V^{i_1} which is mapped to C_{i_1} and passes through x. The fiber of p over y_1 is contained in G_{i_2}, so there is a minimal section Γ_{i_2} in G_{i_2} which meets Γ_{i_1}; repeating the argument we construct a chain of curves in $V^1, \dots, V^{\rho_Y}$ which joins x with a point of the fiber over y. We have thereby proved that, for every $x \in X$ and for some m, $\mathrm{ChLocus}_m(V^1, \dots, V^{\rho_Y})_x$ dominates Y.

Let $\psi : X^0 \to Z$ be the $\mathrm{rc}(V^1, \dots, V^{\rho_Y})$-fibration; a general fiber F of ψ is an equivalence class for the $\mathrm{rc}(V^1, \dots, V^{\rho_Y})$-relation, thus it contains $\mathrm{ChLocus}_m(V^1, \dots, V^{\rho_Y})_x$ for every point $x \in F$ and every m; then we have $\dim F \geq \dim Y$ and $\dim Z \leq \dim X - \dim F \leq r - 1$.
On the other hand, F cannot contain a curve in a fiber of p, otherwise $R_{\mathcal{E}}$ would be contained in the subvector space of $N_1(X)$ generated by the classes of $V^1, \dots, V^{\rho_Y}$ by [4, Corollary 4.2]. Being $\mathrm{Locus}(R^{\mathcal{E}})_F = X$, this, again by [4, Corollary 4.2], would imply that the class of every curve in X would be contained in the subvector space of $N_1(X)$ generated by the classes of $V^1, \dots, V^{\rho_Y}$, hence $\rho_X = \rho_Y$, a contradiction.
In particular it follows that $\dim F = \dim Y$; therefore $\dim Z = r - 1$ and we can apply lemma 4.1 to $(X, Y, \mathcal{E})$ and ψ to conclude. □

5. Fano manifolds with many fiber type contractions

In this section we will prove that a ruled Fano fivefold of index two and Picard number greater than three is a product. We will derive this conclusion from two more general results concerning Fano manifolds with many fibrations.

Proposition 5.1. *Let X be a Fano manifold of dimension n and pseudoindex $i_X \geq 2$ which has only contractions of fiber type. Then $\rho_X \leq n$. Moreover,*

(1) *if $\rho_X = n$, then $X = (\mathbb{P}^1)^n$;*
(2) *if $\rho_X = n - 1$, then $X = (\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$ or $X = (\mathbb{P}^1)^{n-3} \times \mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2)$.*

Proof. By [40, Theorem 2.2] we have that a Fano manifold of dimension n admits at most n fiber type elementary contractions, and the bound on the Picard number follows. More precisely we have that the cone of curves of X is generated by at most n extremal rays.

We can assume that $n \geq 4$, since for lower dimensions the claimed result follows from the classification of Fano manifolds.
Suppose that $\rho_X = n$; by the discussion above we have $\mathrm{NE}(X) = \langle R_1, \dots, R_n \rangle$. Let $R^1, \dots, R^n$ be the corresponding families of rational curves, as in example 2.5; by [4, Lemma 5.4 (c)] we have

$$n \geq \dim \mathrm{Locus}(R^1, \dots, R^n)_x \geq \sum_{i=1}^{n} (-K_X \cdot R^i - 1) \geq n,$$

forcing $-K_X \cdot R^i = 2$ for every i (recall that $i_X \geq 2$) and $\sum_{i=1}^{n} (-K_X \cdot R^i - 1) = n$. We can therefore apply [33, Theorem 1] to conclude.

Suppose now that $\rho_X = n-1$; let $R_1, \dots, R_{n-1}$ be extremal rays of X which span $N_1(X)$ and let $R^1, \dots, R^{n-1}$ be the corresponding families of rational curves.
Suppose that, among the chosen rays, there exists a ray $R_{i(1)}$ such that the associated contraction $\varphi_{i(1)}$ has a fiber F of dimension greater than one. We claim that for every ray $R_{i(j)} \in \{R_1, \dots, R_{n-1}\}$ different from $R_{i(1)}$ the contraction associated to $R_{i(j)}$ is equidimensional with one dimensional fibers.
Assume by contradiction that there exists an index $i(2)$ such that the contraction associated to $R_{i(2)}$ has a fiber G of dimension ≥ 2.
Consider an irreducible component D of $\mathrm{Locus}(R^{i(3)}, \dots, R^{i(n-1)})_G$, which, by [4, Lemma 5.4 (c)], has dimension

$$\dim D \geq \sum_{j=3}^{n-1} (-K_X \cdot R^{i(j)} - 1) + \dim G \geq n-1.$$

By [4, Lemma 5.1], $N_1(D) = \langle R_{i(2)}, \dots, R_{i(n-1)} \rangle$, therefore we cannot have $D = X$, thus D is an effective divisor in X. We will now derive a contradiction by considering the intersection number of this divisor with the family $R^{i(1)}$.
Suppose first that $D \cdot R^{i(1)} > 0$; in this case D meets F, which has dimension at least two, whence the intersection $D \cap F$ contains a curve, contradicting the fact that curves in $R^{i(1)}$ are numerically independent from curves in D.
Suppose now that $D \cdot R^{i(1)} = 0$ and let $C_{i(1)}$ be a curve of $R^{i(1)}$ meeting D. Since the intersection number is zero, this curve is contained in D, contradicting again the independence of curves in $R^{i(1)}$ from curves in D.

We have thereby proved that X has at least $n-2$ extremal rays whose associated contractions are equidimensional with one dimensional fibers. Let $\varphi_j : X \to Y_j$ be one of these contractions; by lemma 2.13 there exists a rank two vector bundle $\mathcal{E}_j$ on Y_j such that $X = \mathbb{P}_{Y_j}(\mathcal{E}_j)$.
By lemma 2.12 (a), Y_j is a Fano manifold of pseudoindex $i_{Y_j} \geq i_X \geq 2$ and, by part (c) of the same lemma, has only contractions of fiber type, so, by induction on the dimension, $Y_j \simeq (\mathbb{P}^1)^{n-3} \times \mathbb{P}^2$ or $Y_j \simeq (\mathbb{P}^1)^{n-4} \times \mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2)$.
It follows that $i_{Y_j} = 2 = i_X$, hence, by lemma 2.12 (b), the restriction of $\mathcal{E}_j$ to every fiber of a

$\mathbb{P}^1$-bundle contraction of Y_j splits as a sum of two line bundles of the same degree.
Up to twist $\mathcal{E}_j$ with a suitable line bundle in $\mathrm{Pic}(Y_j)$, we can now assume that the restriction of $\mathcal{E}_j$ to any fiber of a $\mathbb{P}^1$-bundle contraction is $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.
In particular $K_{Y_j} + \det \mathcal{E}_j$ is trivial on all the extremal rays of Y_j, hence $K_{Y_j} + \det \mathcal{E}_j = \mathcal{O}_{Y_j}$; by the canonical bundle formula we have $-K_X = 2\xi_{\mathcal{E}_j}$, consequently $(X, Y_j, \mathcal{E}_j)$ is a ruled Fano manifold of index two.
For both possible basis Y_j the ruled Fano manifold $(X, Y_j, \mathcal{E}_j)$ verifies the assumptions of proposition 4.4, so we have $X = \mathbb{P}^1 \times Y_j$. □

Proposition 5.2. *Let X be a Fano manifold of dimension n and pseudoindex $i_X \geq 2$ such that all its elementary contractions but one are of fiber type. Then $\rho_X \leq n-1$, equality holding if and only if $X = (\mathbb{P}^1)^{n-3} \times Bl_p(\mathbb{P}^3)$.*

Proof. We can assume that $n \geq 4$, since for lower dimensions the claimed result follows from the classification of Fano manifolds.
Let R_1 be the birational ray and let $R_2, \ldots, R_{\rho_X}$ be fiber type rays such that $R_1, R_2, \ldots, R_{\rho_X}$ span $N_1(X)$. Let $\varphi_1 : X \to X'$ be the contraction of R_1 and let F be a nontrivial fiber of φ_1; since φ_1 is birational, by proposition 2.7 we have $\dim F \geq 2$.
For every permutation $i(2), \ldots, i(\rho_X)$ of the integers $2, \ldots, \rho_X$, by [4, Lemma 5.4 (c)] we have

$$\dim \mathrm{Locus}(R^{i(2)}, \ldots, R^{i(\rho_X)})_F \geq \dim F + \rho_X - 1,$$

forcing $\rho_X \leq n-1$; moreover, if equality holds, we have $\dim F = 2$ and $X = \mathrm{Locus}(R^{i(2)}, \ldots, R^{i(\rho_X)})_F$.
In particular we note for later use that, since φ_1 is birational and all its nontrivial fibers have dimension $= 2$, $\mathrm{Exc}(\varphi_1)$ is a divisor by proposition 2.7.
Set $T_{i(2)} = \mathrm{Locus}(R^{i(2)})_F$; being $X = \mathrm{Locus}(R^{i(3)}, \ldots, R^{i(\rho_X)})_{T_{i(2)}}$, by [33, Lemma 1] every curve $C \subset X$ is equivalent to a linear combination

$$\alpha \Gamma_{i(2)} + \sum_{k=3}^{\rho_X} \alpha_k R^{i(k)}$$

of a curve $\Gamma_{i(2)}$ in $T_{i(2)}$ and curves in $R^{i(3)}, \ldots, R^{i(\rho_X)}$ with $\alpha \geq 0$. By [18, Corollary 2.23] every curve in $T_{i(2)}$ is numerically equivalent (in X) to a linear combination with positive coefficients of a curve in F (and so whose numerical class is in R_1) and a curve in $R^{i(2)}$; hence we can write C as a combination

$$\alpha_1 R_1 + \alpha_2 R^{i(2)} + \sum_{k=3}^{\rho_X} \alpha_k R^{i(k)},$$

with $\alpha_1, \alpha_2 \geq 0$.
Since this is true for every permutation $i(2), \ldots, i(\rho_X)$, and the decomposition of $[C]$ is unique, we get that $\alpha_k \geq 0$ for all k and $\mathrm{NE}(X) = \langle R_1, R_2, \ldots, R_{\rho_X} \rangle$.

Denote again by F a nontrivial fiber of φ_1 and, for every $i = 2, \dots, \rho_X$ consider an irreducible component D_i of $\mathrm{Locus}(R^2, \dots, \hat{R^i}, \dots, R^{\rho_X})_F$ which, by [4, Lemma 5.4 (c)], has dimension

$$\dim D_i \geq \sum_{j=2}^{\rho_X} (-K_X \cdot R^j - 1) + (K_X \cdot R_i + 1) + \dim F \geq n - 1.$$

By [4, Lemma 5.1] $N_1(D_i) = \langle R_1, \dots \hat{R}_i, \dots, R_{\rho_X} \rangle$, therefore we cannot have $D_i = X$, whence D_i is an effective divisor in X.

As in proposition 5.1 we can now prove that the contraction $\varphi_i : X \to Y_i$, associated to the ray R_i, has one dimensional fibers, since the intersection of this fibers with D_i must be 0-dimensional, hence, by lemma 2.13 there exists a rank two vector bundle $\mathcal{E}_i$ on Y_i such that $X = \mathbb{P}_{Y_i}(\mathcal{E}_i)$.

By lemma 2.14, for at least one index $j \in \{2, \dots, \rho_X\}$ we have $\mathrm{Exc}(\varphi_1) \cdot R_j > 0$; let $\varphi_j : X \to Y_j$ be the contraction associated to the ray R_j.

By lemma 2.12, Y_j is a Fano manifold of pseudoindex $i_{Y_j} \geq 2$; by lemma 3.5 all the extremal contractions of Y_j are of fiber type and, by the same lemma, one of these contractions has two dimensional fibers. We can apply proposition 5.1 to Y_j to get $Y_j \simeq (\mathbb{P}^1)^{n-3} \times \mathbb{P}^2$.

Let $p_1 : Y_j \to \mathbb{P}^1$ be the projection onto the first factor; the projection to the other factors is an extremal elementary contraction $\varphi_\theta : Y_j \to (\mathbb{P}^1)^{n-4} \times \mathbb{P}^2$, associated to a ray $\theta \subset \mathrm{NE}(Y_j)$.

Let $\vartheta \subset \mathrm{NE}(X)$ be the fellow ray of θ; since φ_θ has one dimensional fibers, the same is true for the contraction associated to ϑ, $\psi_\vartheta : X \to Z$. Therefore $\vartheta \neq R_1$, and the associated contraction ψ_ϑ is a $\mathbb{P}^1$-bundle over a smooth Fano variety Z, which has pseudoindex $i_Z \geq i_X \geq 2$.

Consider the following diagram

$$\begin{tikzcd}
X \arrow[r, "\varphi_j"] \arrow[d, "\psi_\vartheta"'] & Y_j \arrow[r, "p_1"] \arrow[d, "\varphi_\theta"] & \mathbb{P}^1 \\
Z \arrow[r] & (\mathbb{P}^1)^{n-4} \times \mathbb{P}^2 &
\end{tikzcd}$$

We can apply lemma 4.1 to X and $\psi = p_1 \circ \varphi_j : X \to \mathbb{P}^1$ and obtain $X \simeq \mathbb{P}^1 \times Z$. It follows that Z has a birational contraction, so, by induction $Z \simeq (\mathbb{P}^1)^{n-4} \times Bl_p(\mathbb{P}^3)$ and $X \simeq (\mathbb{P}^1)^{n-3} \times Bl_p(\mathbb{P}^3)$. □

Corollary 5.3. *Let X be a Fano fivefold of index $r_X \geq 2$ and Picard number $\rho_X \geq 4$. Then*

(1) $X \simeq (\mathbb{P}^1)^5$;
(2) $X \simeq (\mathbb{P}^1)^2 \times \mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2)$;
(3) $X \simeq (\mathbb{P}^1)^2 \times Bl_p(\mathbb{P}^3)$.

Proof. Note that, since $\rho_X \geq 4$, we have $i_X \leq 2$, by [4, Theorem 1.4], hence $r_X = i_X = 2$. By [18, Theorem 1.1], if $\rho_X \geq 4$, then X has at most one birational contraction, and the conclusion follows from propositions 5.1 and 5.2. Note that $(\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$ has been excluded since its index is one. □

6. Proof of theorem 1.1: Classification of the base

In this section we begin the study of ruled Fano fivefolds $(X, Y, \mathcal{E})$ of index two and Picard number three, which is the most complicated case.
We start by considering the possible bases Y such that there exists a ruled Fano fivefold $(X, Y, \mathcal{E})$ as above which is not a product. By lemma 2.12 Y is a Fano fourfold of pseudoindex $i_Y \geq 2$, and $\rho_Y = 2$, since we are assuming $\rho_X = 3$. We will give a complete classification of fourfolds Y as above which have a birational contraction (Proposition 6.1), and a more rough one of the ones with two fiber type contractions (Proposition 6.2). Then, using the criteria for recognizing products previously estabilished, we will show that there are only four possibilities for Y (Proposition 6.3).

Proposition 6.1. *Let Y be a Fano fourfold of pseudoindex $i_Y \geq 2$ and Picard number $\rho_Y = 2$ such that the contraction $\varphi_\theta : Y \to Y'$, associated to one extremal ray $\theta \subset NE(Y)$, is birational. Then Y is one of the following:*

1. *$Bl_p(\mathbb{P}^4)$ with p a point in $\mathbb{P}^4$;*
2. *$Bl_l(\mathbb{P}^4)$ with l a line in $\mathbb{P}^4$;*
3. *$Bl_l(\mathbb{Q}^4)$ with l a line in $\mathbb{Q}^4$;*
4. *$Bl_\Gamma(\mathbb{Q}^4)$ with Γ a conic in $\mathbb{Q}^4$ not contained in a plane $\Pi \subset \mathbb{Q}^4$;*
5. *$\mathbb{P}_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2))$;*
6. *$\mathbb{P}_{\mathbb{Q}^3}(\mathcal{O}_{\mathbb{Q}^3} \oplus \mathcal{O}_{\mathbb{Q}^3}(1))$.*

Proof. The cone of curves of Y is generated by two extremal rays: $\mathrm{NE}(Y) = \langle \theta, \overline{\theta} \rangle$.
The length of every extremal ray on a Fano manifold is clearly greater than or equal to the pseudoindex; moreover, for a birational extremal ray, by proposition 2.7, the length is bounded above by the dimension of the manifold minus one, hence

$$2 \leq l(\theta) \leq 3.$$

If $l(\theta) = 3$, by [8, Theorem 1.1], the associated contraction $\varphi_\theta : Y \to Y'$ is the blow up at a point of a smooth variety Y'; Fano manifolds which are the blow up at a point of a smooth variety are classified in [15, Theorem 1.1], which gives three possible cases. Among these cases only the blow up at a point of the projective space has pseudoindex greater than one, hence we are in case (1).

If $l(\theta) = 2$, by [8, Theorem 5.2], either φ_θ is the blow up of a smooth variety along a smooth curve, or its exceptional locus $\mathrm{Exc}(\varphi_\theta)$ is isomorphic to $\mathbb{P}^3$ or to a (possibly singular) three dimensional quadric and $\varphi_\theta(\mathrm{Exc}(\varphi_\theta))$ is a point.

If $\varphi_\theta : Y \to Y'$ is the blow up of a smooth variety along a smooth curve, we can apply [9, Theorem 1.3] and, recalling that we are assuming $\rho_Y = 2$, we have cases (2), (3) and (4).

If else $\varphi_\theta(\mathrm{Exc}(\varphi_\theta))$ is a point, we consider the contraction $\varphi_{\overline{\theta}} : Y \to T$, associated to the extremal ray $\overline{\theta}$; the effective divisor $\mathrm{Exc}(\varphi_\theta)$ is positive on $\overline{\theta}$ by lemma 2.14, therefore, by corollary 2.15, $\varphi_{\overline{\theta}}$ makes Y a $\mathbb{P}^1$-bundle over T, $Y = \mathbb{P}_T(\mathcal{F})$. We can thus apply lemma 2.16, obtaining that either Y is a product, or T is a projective space or a smooth quadric. The first case has to be excluded since $\mathbb{P}^1 \times T$ does not have a birational contraction; in the second case we note that $\mathcal{F}$ is a Fano bundle on

T, whence we can use the classification in [35], looking for bundles such that their projectivization has pseudoindex ≥ 2 and a birational extremal contraction.
By that classification it turns out that the only possibilities are number (5) and (6) in our list. □

Proposition 6.2. *Let Y be a Fano fourfold of pseudoindex $i_Y \geq 2$ and Picard number $\rho_Y = 2$ with two fiber type extremal contractions. Then Y is one of the following:*

(1) *a product $\mathbb{P}^1 \times W$;*
(2) *a variety whose extremal rays have length 2 and associated contractions with fibers of dimension ≤ 2;*
(3) $\mathbb{P}^2 \times \mathbb{P}^2$;
(4) $\mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2(-1) \oplus \mathcal{O}_{\mathbb{P}^2})$.

Proof. The manifold Y is Fano and has Picard number two, so its cone is spanned by two extremal rays: $\mathrm{NE}(Y) = \langle \theta, \overline{\theta} \rangle$.

Suppose that the contraction associated to one extremal ray, say $\overline{\theta}$, has a three dimensional fiber $F_{\overline{\theta}}$; then, by lemma 2.14, $F_{\overline{\theta}} \cdot \theta > 0$. By corollary 2.15, the contraction of θ, $\varphi_\theta : Y \to W$, makes Y into a $\mathbb{P}^1$-bundle over a smooth threefold W, $Y = \mathbb{P}_W(\mathcal{F})$; by lemma 2.16, either $Y \simeq \mathbb{P}^1 \times W$, or W is $\mathbb{P}^3$ or $\mathbb{Q}^3$.
By the classification given in [35], there are no of Fano bundles over $\mathbb{P}^3$ and $\mathbb{Q}^3$ such that their projectivization is not a product and has two fiber type contractions, one of which has a three dimensional fiber.

Therefore either we are in case (1) or both the contractions of Y have fibers of dimension ≤ 2; this implies that the lengths of the extremal rays are ≤ 3, by proposition 2.7.
Either we are in case (2) or the length of one extremal ray, say θ, is equal to three; again by proposition 2.7 we have that $\varphi_\theta : Y \to W$ is equidimensional with fibers of dimension two.
By lemma 2.18 W is smooth and so, being a smooth surface of Picard number one dominated by a Fano manifold, $W \simeq \mathbb{P}^2$; moreover, by the same lemma $Y = \mathbb{P}_{\mathbb{P}^2}(\mathcal{F})$ for some rank three vector bundle on $\mathbb{P}^2$. In particular $\mathcal{F}$ is a Fano bundle over $\mathbb{P}^2$.
From the classification of such bundles given in [36], recalling that, in our case, the other contraction of Y has length ≥ 2, we are either in case (3) or in case (4). □

Proposition 6.3. *Suppose that there exists a ruled Fano fivefold of index two $(X, Y, \mathcal{E})$ with $\rho_X = 3$ which is not a product with $\mathbb{P}^1$ as a factor. Then Y is one of the following:*

(1) $Bl_p(\mathbb{P}^4)$;
(2) $Bl_l(\mathbb{P}^4)$;
(3) $\mathbb{P}^2 \times \mathbb{P}^2$;
(4) $\mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2(-1) \oplus \mathcal{O}_{\mathbb{P}^2})$.

Proof. Suppose first that Y has a birational contraction; then Y is one of the manifolds listed in proposition 6.1. The varieties (3)-(6) are rationally connected with respect to minimal curves in the extremal rays, which have length two, so, if they are the base of a ruled Fano fivefold $(X, Y, \mathcal{E})$

of index two, then X is a product $\mathbb{P}^1 \times Y$ by proposition 4.4. Therefore, if Y has a birational contraction and X is not a product, Y is either $Bl_p(\mathbb{P}^4)$ or $Bl_l(\mathbb{P}^4)$ (cases (1) and (2) of proposition 6.1).

Suppose now that Y has only fiber type contractions; then, by proposition 6.2, we have four possible cases. To finish the proof we have to rule out cases (1) and (2) of that proposition.

If $Y \simeq \mathbb{P}^1 \times W$, we can apply corollary 4.3 to get that X is a product $\mathbb{P}^1 \times \mathbb{P}_W(\mathcal{E}')$.

We are left with the case of a manifold Y whose extremal rays have length 2 and associated contractions with fibers of dimension ≤ 2. Let θ be one of the rays in $\mathrm{NE}(Y)$, let $\varphi_\theta : Y \to W$ be the associated contraction and let R^θ be the associated family of rational curves; we claim that R^θ is a covering family.
If the general fiber F_θ of φ_θ has dimension one, this follows from proposition 2.6, since $\mathrm{Locus}(R^\theta)_x$ is contained in the fiber of φ_θ through x:

$$\dim \mathrm{Locus}(R^\theta) \geq \dim Y + l(\theta) - 1 - \dim \mathrm{Locus}(R^\theta)_x \geq 4.$$

If else F_θ has dimension two, then, by adjunction, it is a smooth quadric and therefore it is covered by curves in R^θ, which are lines in the quadric.
We can thus consider the $\mathrm{rc}(R^\theta, R^{\overline{\theta}})$-fibration, whose image has to be a point, being $\rho_Y = 2$. It follows that Y is rationally connected with respect to curves in R^θ and $R^{\overline{\theta}}$ and X is a product $\mathbb{P}^1 \times Y$ by proposition 4.4. □

7. Proof of theorem 1.1

In this section we achieve the classification of ruled fivefolds $(X, Y, \mathcal{E})$ of index two and Picard number three, proving theorem 1.1.

First we prove that, if X is not a product, one of the contractions of X is birational (proposition 7.1). We then consider separately the case in which also Y has a birational contraction (proposition 7.2) and the case in which both the contractions of Y are of fiber type (proposition 7.3).

Proposition 7.1. *Let $(X, Y, \mathcal{E})$ be a ruled Fano fivefold of index two with $\rho_X = 3$ such that X has only fiber type contractions. Then X is a product with $\mathbb{P}^1$ as a factor.*

Proof. Since X has only fiber type contractions, the same is true also for Y by corollary 3.6, so, by proposition 6.3, if X is not a product with $\mathbb{P}^1$ as a factor, then Y is either $\mathbb{P}^2 \times \mathbb{P}^2$ or $\mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2(-1) \oplus \mathcal{O}_{\mathbb{P}^2})$.

Case a) $Y \simeq \mathbb{P}^2 \times \mathbb{P}^2$.

The cone of curves of Y is generated by two extremal rays, θ and $\bar{\theta}$, corresponding to the projections $\varphi_\theta, \varphi_{\overline{\theta}} : Y \to \mathbb{P}^2$ Let ϑ and $\overline{\vartheta}$ be the fellow rays of θ and $\overline{\theta}$, respectively, and denote by $\psi_\vartheta : X \to Z$ and $\psi_{\overline{\vartheta}} : X \to \overline{Z}$ the associated contractions. By proposition 3.9 the contractions ψ_ϑ and $\psi_{\overline{\vartheta}}$ are $\mathbb{P}^1$-bundles and $p^*\mathcal{O}_Y(1,1)$ restricts to $\mathcal{O}_{\mathbb{P}^1}(1)$ on the fibers of ψ_ϑ and $\psi_{\overline{\vartheta}}$. Hence there exist two vector bundles $\mathcal{F}$ on Z and $\overline{\mathcal{F}}$ on $\overline{Z}$ such that $(X, Z, \mathcal{F})$ and $(X, \overline{Z}, \overline{\mathcal{F}})$ are ruled Fano

fivefolds of index two.
Since all the contractions of X are of fiber type, the same is true also for Z and $\overline{Z}$, by corollary 3.6. We can apply proposition 6.3 to $(X, Z, \mathcal{F})$ and to $(X, \overline{Z}, \overline{\mathcal{F}})$ and we have for Z and $\overline{Z}$ two possibilities: $\mathbb{P}^2 \times \mathbb{P}^2$ or $\mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2(-1) \oplus \mathcal{O}_{\mathbb{P}^2})$.
In the last case one extremal contraction of Z ($\overline{Z}$) is a special Bănică scroll onto $\mathbb{P}^3$ so, by proposition 3.10, also one contraction of X has to be a special Bănică scroll with jumping fibers, but we have already proved that all the contractions of X are $\mathbb{P}^1$-bundles.
It follows that both Z and $\overline{Z}$ are $\mathbb{P}^2 \times \mathbb{P}^2$. All the extremal rays of X have length two, hence $\xi_{\mathcal{E}}$ restricts to $\mathcal{O}_{\mathbb{P}^1}(1)$ on the fibers of any contraction of X.
Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{P}^2 & & & Z \arrow[dr] & \\
& Y \arrow[ul, "\varphi_\theta"'] \arrow[dl, "\varphi_{\overline{\theta}}"] & X \arrow[l, "p"] \arrow[ur, "\psi_\vartheta"] \arrow[dr, "\psi_{\overline{\vartheta}}"'] \arrow[rr, "\psi_\sigma"] & & \mathbb{P}^2 \\
\mathbb{P}^2 & & & \overline{Z} \arrow[ur] &
\end{tikzcd}$$

The line bundle $\xi_{\mathcal{E}} \otimes p^*\mathcal{O}_Y(-1,-1)$ is trivial on the face σ spanned by ϑ and $\overline{\vartheta}$, and restricts to $\mathcal{O}_{\mathbb{P}^1}(1)$ on the fibers of p, hence $\xi_{\mathcal{E}(-1,-1)} = \xi_{\mathcal{E}} \otimes p^*\mathcal{O}_Y(-1,-1) = \psi_\sigma^*\mathcal{O}_{\mathbb{P}^2}(1)$ is spanned. Equivalently $\mathcal{E}(-1,-1)$ is spanned and $h^0(\mathcal{E}(-1,-1)) = 3$. We thus have a surjective map $\mathcal{O}_Y^{\oplus 3} \to \mathcal{E}(-1,-1) \to 0$, which gives rise to an exact sequence

$$0 \longrightarrow L \longrightarrow \mathcal{O}_Y^{\oplus 3} \longrightarrow \mathcal{E}(-1,-1) \longrightarrow 0;$$

computing the splitting type we find $L \simeq \mathcal{O}_Y(-1,-1)$. The dual bundle $L^\vee$ is thereby ample, therefore, by [24, 12.1.6], the map $L \to \mathcal{O}_Y^{\oplus 3}$ must have a non empty degeneracy locus, whence $X = \mathbb{P}_Y(\mathcal{E}(-1,-1)) \hookrightarrow \mathbb{P}_Y(\mathcal{O}_Y^{\oplus 3})$ is not a $\mathbb{P}^1$-bundle over Y, a contradiction.

Case b) $Y \simeq \mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2(-1) \oplus \mathcal{O}_{\mathbb{P}^2})$.

The cone of curves of Y is generated by two extremal rays: θ, corresponding to the projection $\varphi_\theta : Y \to \mathbb{P}^2$, and $\overline{\theta}$, corresponding to the contraction $\varphi_{\overline{\theta}} : Y \to \mathbb{P}^3$, which is a special Bănică scroll with exactly one jumping fiber $J \simeq \mathbb{P}^2$, which is the section corresponding to the trivial summand of the bundle $T\mathbb{P}^2(-1) \oplus \mathcal{O}_{\mathbb{P}^2}$.
Let ϑ and $\overline{\vartheta}$ be the fellow rays of θ and $\overline{\theta}$, respectively, and denote by $\psi_\vartheta : X \to Z$ and $\psi_{\overline{\vartheta}} : X \to \overline{Z}$ the associated contractions. By proposition 3.9 the contraction $\psi_\vartheta : X \to Z$ is a $\mathbb{P}^1$-bundle, while, by proposition 3.10, the contraction $\psi_{\overline{\vartheta}} : X \to \overline{Z}$ is a special Bănică scroll with a one parameter family of jumping fibers which are sections of p over over J.
Since $\psi_\vartheta : X \to Z$ is a $\mathbb{P}^1$-bundle, there exists a vector bundle $\mathcal{F}$ on Z such that $(X, Z, \mathcal{F})$ is a ruled Fano fivefold of index two. All the contractions of Z are of fiber type by corollary 3.6, so proposition 6.3 applied to $(X, Z, \mathcal{F})$ gives us two possibilities: either $Z \simeq \mathbb{P}^2 \times \mathbb{P}^2$ or $Z \simeq \mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2(-1) \oplus \mathcal{O}_{\mathbb{P}^2})$.
In the first case we conclude as in case a), replacing $(X, Y, \mathcal{E})$ with $(X, Z, \mathcal{F})$, otherwise we consider

the following commutative diagram

$$\begin{tikzcd}
 & J \arrow[d, hook] & p^{-1}(J) \arrow[d, hook] & \\
\mathbb{P}^2 & Y \arrow[l, "\varphi_\theta"'] \arrow[d, "\varphi_{\overline{\vartheta}}"'] & X \arrow[l, "p"'] \arrow[r, "\psi_\vartheta"] \arrow[d, "\psi_{\overline{\vartheta}}"'] & Z \arrow[d] \\
 & \mathbb{P}^3 & \overline{Z} \arrow[l, "p'"] \arrow[r] & \mathbb{P}^3
\end{tikzcd}$$

By proposition 3.10 there is an isomorphism $f : \mathbb{P}^1 \times \mathbb{P}^2 \to p^{-1}(J)$, and the subsets $f(\{x\} \times \mathbb{P}^2)$ are jumping fibers of $\psi_{\overline{\vartheta}}$. In particular the numerical class of every curve in $p^{-1}(J)$ belongs to the face $\langle R_{\mathcal{E}}, \overline{\vartheta}\rangle$. It follows that ψ_ϑ is finite to one on $p^{-1}(J)$, but this is a contradiction since, by lemma 3.5 every jumping fiber of $\psi_{\overline{\vartheta}}$ has to be mapped by ψ_ϑ to a jumping fiber of the contraction $Z \to \mathbb{P}^3$, but this map has only one jumping fiber. □

Proposition 7.2. *Let $(X, Y, \mathcal{E})$ be a ruled Fano fivefold of index two with $\rho_X = 3$ such that both X and Y have a birational contraction. Then, if X is not a product with $\mathbb{P}^1$ as a factor, one of the following happens:*

(1) $X \simeq Bl_p(\mathbb{P}^4) \times_{\mathbb{P}^3} Bl_p(\mathbb{P}^4)$;
(2) $X \simeq Bl_S(Bl_p(\mathbb{P}^5))$ *with S the strict trasform of a plane $\ni p$.*

In these cases the corresponding pairs $(Y, \mathcal{E})$ are, respectively,

(1) $(Bl_p(\mathbb{P}^4), 2H + E \oplus 3H + E)$, *$E$ exceptional divisor and H pullback on Y of $\mathcal{O}_{\mathbb{P}^3}(1)$;*
(2) $(Bl_l(\mathbb{P}^4), 2H - E \oplus 3H - E)$, *$E$ exceptional divisor and H pullback on Y of $\mathcal{O}_{\mathbb{P}^4}(1)$.*

Proof. We assume that X is not a product and that Y has a birational contraction, so, by corollary 6.3, Y is the blow up of $\mathbb{P}^4$ either along a point or along a line.

Case a) $Y = Bl_p(\mathbb{P}^4)$.

Another possible description of Y is $\mathbb{P}_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1))$; let $\theta \subset \mathrm{NE}(Y)$ be the extremal ray corresponding to the $\mathbb{P}^1$-bundle contraction $\varphi_\theta : Y \to \mathbb{P}^3$, let E be the exceptional $\mathbb{P}^3$ and let H be the pullback of $\mathcal{O}_{\mathbb{P}^3}(1)$. Let $\vartheta \subset \mathrm{NE}(X)$ be the fellow ray of θ; by proposition 3.8, the contraction associated to ϑ, $\psi_\vartheta : X \to Z$, is a $\mathbb{P}^1$-bundle, too. Moreover, by the same proposition, since E restricts to $\mathcal{O}_{\mathbb{P}^1}(1)$ on the fibers of φ_θ, we have $\mathcal{E} \otimes (-E) = \varphi_\theta^* \mathcal{E}'$ and $Z = \mathbb{P}_{\mathbb{P}^3}(\mathcal{E}')$.

Since $E_{|E} \simeq \mathcal{O}_{\mathbb{P}^3}(-1)$ and E is a section of φ_θ, we have

$$\mathcal{E}_{|E} = (\varphi_\theta^* \mathcal{E}' \otimes E)_{|E} \simeq \mathcal{E}'(-1).$$

Recalling that $(\det \mathcal{E})_{|E} = (-K_Y)_{|E} = \mathcal{O}_{\mathbb{P}^3}(3)$ and that $\mathcal{E}$ is ample, we see that the splitting type of $\mathcal{E}$ on lines of E is constantly $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$, hence, by [34, Theorem 3.2.3], $\mathcal{E}_{|E}$ is decomposable as $\mathcal{E}_{|E} \simeq \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2)$. It follows that $\mathcal{E}' \simeq \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(3)$, thus $\mathcal{E} \simeq (2H \oplus 3H) \otimes E$.

Case b) $Y = Bl_l(\mathbb{P}^4)$.

Let $\theta \subset \mathrm{NE}(Y)$ be the extremal ray whose associated contraction, $\varphi_\theta : Y \to \mathbb{P}^4$, is the blow up of $\mathbb{P}^4$ along a line. Denote by E the exceptional locus of φ_θ and by H the pullback of the ample generator of $\mathrm{Pic}(\mathbb{P}^4)$.
Let $\vartheta \subset \mathrm{NE}(X)$ be the fellow ray of θ; by proposition 3.8, the associated contraction, $\psi_\vartheta : X \to X'$, is the blow up of a smooth fivefold along a smooth surface.
By the same proposition, since $-E$ restricts to $\mathcal{O}_{\mathbb{P}^2}(1)$ on the fibers of φ_θ, there exists a rank two vector bundle on X' such that $\mathcal{E} \otimes E = \varphi_\theta^* \mathcal{E}'$ and $X' = \mathbb{P}_W(\mathcal{E}')$; by [7, Lemma 2.10] $\mathcal{E}'$ is ample.
The canonical bundle formula for blow ups, $K_Y = \varphi_\theta^* K_{\mathbb{P}^4} + 2E$, combined with the determinant formula, $\det \varphi_\theta^* \mathcal{E}' = \det \mathcal{E} + 2E$, gives

$$\varphi_\theta^*(K_{\mathbb{P}^4} + \det \mathcal{E}') = K_Y + \det \mathcal{E} = \mathcal{O}_Y,$$

whence $K_{\mathbb{P}^4} + \det \mathcal{E}' = \mathcal{O}_{\mathbb{P}^4}$. It follows that $-K_{X'} = 2\xi_{\mathcal{E}'}$ is ample, therefore X' is a Fano manifold and $\mathcal{E}'$ is a rank two Fano bundle on $\mathbb{P}^4$, which, by [1, Main Theorem], is decomposable as $\mathcal{E}' \simeq \mathcal{O}_{\mathbb{P}^4}(a) \oplus \mathcal{O}_{\mathbb{P}^4}(b)$. We can thereby write $\mathcal{E} \simeq (aH - E) \oplus (bH - E)$. Now, recalling that $\mathcal{E}$ is ample and that $K_Y + \det \mathcal{E} = \mathcal{O}_Y$, it is easy to prove that $(a, b) = (2, 3)$. □

Proposition 7.3. *Let $(X, Y, \mathcal{E})$ be a ruled Fano fivefold of index two with $\rho_X = 3$ such that X has a birational contraction but Y has not. Then one of the following happens:*

1. *X is the blow up of a cone in $\mathbb{P}^9$ over the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ along its vertex;*
2. *X is the blow up of $\mathbb{P}^5$ in two non meeting planes;*
3. *X is the blow up of a general member of $\mathcal{O}(1,1) \subset \mathbb{P}^2 \times \mathbb{P}^4$ along a two dimensional fiber of the second projection.*

In these cases the corresponding pairs $(Y, \mathcal{E})$ are, respectively,

1. $(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(1,1) \oplus \mathcal{O}(2,2))$;
2. $(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(1,2) \oplus \mathcal{O}(2,1))$;
3. $(\mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2(-1) \oplus \mathcal{O}_{\mathbb{P}^2}) \subset \mathbb{P}^2 \times \mathbb{P}^3, \mathcal{O}(1,1) \oplus \mathcal{O}(1,2))$.

Proof. First of all it is clear that X cannot be a product $\mathbb{P}^1 \times Y$; by proposition 6.3, recalling that Y has not birational contractions, the only possible cases are $Y \simeq \mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2(-1) \oplus \mathcal{O}_{\mathbb{P}^2})$ or $Y \simeq \mathbb{P}^2 \times \mathbb{P}^2$.

Let $\vartheta \subset \mathrm{NE}(X)$ be an extremal ray associated to a birational contraction $\psi_\vartheta : X \to X'$ and let $\theta \subset \mathrm{NE}(Y)$ be its fellow ray, with associated contraction $\varphi_\theta : Y \to W$.
Denote by E the exceptional locus of $\psi_\vartheta : X \to X'$; if $E \cdot R_{\mathcal{E}} = 0$, then $E = p^* E_Y$ with E_Y an effective divisor on Y. Being E not nef, also E_Y is not nef, and Y has a birational contraction, against the assumptions. Therefore $E \cdot R_{\mathcal{E}} > 0$ and E dominates Y.
The fibers of ψ_ϑ have dimension ≥ 2 by proposition 2.7; then, by lemma 3.5, also the fibers of φ_θ have dimension ≥ 2, hence φ_θ is a $\mathbb{P}^2$-bundle contraction onto $W \simeq \mathbb{P}^2$. By proposition 3.9, ψ_ϑ is the blow up of a smooth surface $S \subset X'$ and, denoted by f a fiber of p, we have $E \cdot f = 1$.
Let y be a point in Y and let $F_y \simeq \mathbb{P}^2$ be the fiber of φ_θ through y; by the proof of proposition 3.9,

$\mathcal{E}_{|F_y} \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$ and $E \cap p^{-1}(F_y)$ is the section corresponding to the $\mathcal{O}_{\mathbb{P}^2}(1)$ summand. In particular the divisor E cannot contain $f = p^{-1}(y)$. It follows that E is a section of p, thus $E \simeq Y$.

Suppose that X' is not a Fano manifold; by [40, Proposition 3.4], E is negative on another extremal ray $\overline{\vartheta} \subset \mathrm{NE}(X)$, hence the exceptional locus of the associated contraction $\psi_{\overline{\vartheta}} : X \to X''$ is contained in E, whence $\psi_{\overline{\vartheta}}$ is birational.
Arguing as above, $\psi_{\overline{\vartheta}} : X \to X''$ is the blow up of a smooth fivefold along a smooth surface, thus its exceptional locus is the divisor E; consequently E has two $\mathbb{P}^2$-bundle structures over smooth surfaces and we have $E \simeq Y \simeq \mathbb{P}^2 \times \mathbb{P}^2$.

Since E is a section of p, there exists an exact sequence

$$0 \longrightarrow \mathcal{O}(a_1, a_2) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(b_1, b_2) \longrightarrow 0$$

such that $E \simeq \xi_{\mathcal{E}} \otimes p^*\mathcal{O}(-a_1, -a_2)$; being $E \cdot \vartheta = E \cdot \overline{\vartheta} = -1$, we have $a_1 = a_2 = 2$; then

$$-1 = E \cdot \vartheta = (1 - a_1) = E \cdot \overline{\vartheta} = (1 - a_2).$$

Recalling that $\det \mathcal{E} = -K_{\mathbb{P}^2 \times \mathbb{P}^2} = \mathcal{O}(3, 3)$, we obtain $b_1 = b_2 = 1$; since $h^1(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(a_1 - b_1, a_2 - b_2)) = h^1(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(1, 1)) = 0$, the above sequences splits, the vector bundle $\mathcal{E}$ is decomposable: $\mathcal{E} \simeq \mathcal{O}(1, 1) \oplus \mathcal{O}(2, 2)$, and we are in case (1).

We can now assume that X' is a Fano manifold; consider the commutative diagram as in 3.4.1

$$\begin{tikzcd} X \arrow[r, "\psi_\vartheta"] \arrow[d, "p"'] & X' \arrow[d, "p'"] \\ Y \arrow[r, "\varphi_\theta"'] & \mathbb{P}^2 \end{tikzcd}$$

Let $x \in \mathbb{P}^2$ be a general point; the fibers $G = p'^{-1}(x)$ and $F = \psi_\vartheta^{-1}(p'^{-1}(x))$ are smooth and, by the commutativity of the diagram, $F = p^{-1}(\varphi_\theta^{-1}(x)) = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$; therefore $G \simeq \mathbb{P}^3$.
By lemma 2.18 there exists a rank four vector bundle $\mathcal{F}$ over $\mathbb{P}^2$ such that $X' = \mathbb{P}_{\mathbb{P}^2}(\mathcal{F})$; in particular $\mathcal{F}$ is a Fano bundle over $\mathbb{P}^2$.
By the canonical bundle formula for blow ups we have

$$-\psi_\vartheta^* K_{X'} = -K_X + 2E = 2(\xi_{\mathcal{E}} + E),$$

whence the index of X' is two. Writing $K_{X'}$ with the canonical bundle formula for projectivizations

$$K_{X'} = -4\xi_{\mathcal{F}} + p'^*(\mathcal{O}_{\mathbb{P}^2}(-3) + c_1(\mathcal{F})),$$

this implies that the first Chern class of $\mathcal{F}$ is odd. By the classification in [36] either $\mathcal{F} \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ or $\mathcal{F} \simeq T\mathbb{P}^2(-1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$.

As for every $x \in \mathbb{P}^2$ the fiber $F_x = \psi_\vartheta^{-1}(p'^{-1}(x))$ is the blow up of $\mathbb{P}^3$ at a point and the fiber $G_x = p'^{-1}(x)$ is a projective space of dimension three, we have that S, the center of the blow-up

ψ_ϑ, is a section of p'; therefore we have an exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}(a) \longrightarrow 0 \tag{7.3.1}$$

such that S is the zero locus of a section of the vector bundle $\xi_{\mathcal{F}} \otimes p'^*\mathcal{G}^\vee$; in particular the conormal bundle $N^*_{S/X'}$ of S is $(p'^*\mathcal{G} \otimes \xi_{\mathcal{F}}^{-1})_{|S}$. Recall that the exceptional divisor E is the projectivization of the conormal bundle of S, i.e. $E \simeq \mathbb{P}_S(N^*_{S/X})$.

If $E \simeq Y \simeq \mathbb{P}^2 \times \mathbb{P}^2$, then $N^*_{S/X}$, hence $\mathcal{G}$ is decomposable. It follows that $h^1(\mathcal{G}(-a)) = 0$, thus the sequence splits and we have $\mathcal{G} \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}$, $\mathcal{F} \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, i.e. S is the section corresponding to the surjection $\mathcal{F} \to \mathcal{O}_{\mathbb{P}^2}(1)$ and it is disjoint from the exceptional divisor of the blow down $X' \to \mathbb{P}^5$. We thereby conclude that X is the blow up of $\mathbb{P}^5$ in two non meeting planes.

Suppose now that $E \simeq Y \simeq \mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2(-1) \oplus \mathcal{O}_{\mathbb{P}^2})$.
Let $\overline{\theta}$ be the extremal ray corresponding to the contraction $\varphi_{\overline{\theta}} : Y \to \mathbb{P}^3$, which is a special Bănică scroll, and let $\psi_{\overline{\vartheta}} : X \to Z$ be the contraction associated to $\overline{\vartheta}$, the fellow ray of $\overline{\theta}$; by proposition 3.10 $\psi_{\overline{\vartheta}}$ is a special Bănică scroll.
Let $\sigma \subset \mathrm{NE}(X)$ be the face spanned by ϑ and $\overline{\vartheta}$; the contraction of this face, call it ψ_σ, factors through the contraction $\psi_\vartheta : X \to X'$ and we have a commutative diagram

$$\begin{tikzcd}
\mathbb{P}^3 & & Z \arrow[dr] & \\
Y \arrow[u, "\varphi_{\overline{\theta}}"] \arrow[d, "\varphi_\theta"'] & X \arrow[l, "p"'] \arrow[ur, "\psi_{\overline{\vartheta}}"] \arrow[rr, "\psi_\sigma"] \arrow[dr, "\psi_\vartheta"'] & & W' \\
\mathbb{P}^2 & & X' \arrow[ll, "p'"] \arrow[ur, "\pi"'] &
\end{tikzcd}$$

The morphism $\pi : X' \to W'$ is the contraction of X' different from the projection onto $\mathbb{P}^2$; since $\dim W' \leq \dim Z < \dim X$, π is a fiber type contraction, so $\mathcal{F} \simeq T\mathbb{P}^2(-1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$ and $W' \simeq \mathbb{P}^3$.
We claim that $E \cdot \overline{\vartheta} = 0$; indeed, if this is not true, then, for every $x \in X$, denoting by $(F_{\overline{\vartheta}})_x$ the fiber of $\psi_{\overline{\vartheta}}$ containing x, we will have

$$\dim \psi_{\overline{\vartheta}}^{-1}(\psi_{\overline{\vartheta}}(\psi_\vartheta^{-1}(\psi_\vartheta((F_{\overline{\vartheta}})_x)))) \geq 3.$$

Denoting by V^ϑ and $V^{\overline{\vartheta}}$ the families of minimal degree rational curves whose numerical class is in ϑ and $\overline{\vartheta}$, respectively, and by $(F_\sigma)_x$ the fiber of ψ_σ containing x we will have

$$(F_\sigma)_x \supset \mathrm{ChLocus}(V^\vartheta, V^{\overline{\vartheta}})_x \supset \psi_{\overline{\vartheta}}^{-1}(\psi_{\overline{\vartheta}}(\psi_\vartheta^{-1}(\psi_\vartheta((F_{\overline{\vartheta}})_x)))),$$

a contradiction, since the general fiber of $\psi_\sigma : X \to \mathbb{P}^3$ is two dimensional.

As we have already noticed, $E = \mathbb{P}(N^*_{S/X'})$ and, since $E \simeq Y$, $N^*_{S/X'} \simeq T\mathbb{P}^2(b-1) \oplus \mathcal{O}_{\mathbb{P}^2}(b)$ for some b. The fact that $E \cdot \overline{\vartheta} = 0$ implies that $b = 0$, so

$$\mathcal{G} \simeq (p'^*\mathcal{G})_{|S} \simeq (\xi_{\mathcal{F}})_{|S} \otimes N^*_{S/X'} \simeq (\xi_{\mathcal{F}})_{|S} \otimes (T\mathbb{P}^2(-1) \oplus \mathcal{O}_{\mathbb{P}^2}) \simeq T\mathbb{P}^2(x-1) \oplus \mathcal{O}_{\mathbb{P}^2}(x)$$

with $x \geq 0$ since $\mathcal{F}$ is nef; by the sequence (7.3.1), we have an injection

$$0 \to T\mathbb{P}^2(x-1) \oplus \mathcal{O}_{\mathbb{P}^2}(x) \to \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2},$$

which forces $x = 0$. It follows that S corresponds to a surjection $\mathcal{F} \to \mathcal{O}_{\mathbb{P}^2} \to 0$, so it is a two dimensional fiber of the special Bănică scroll contraction of X'. □

8. Proof of theorem 1.2

The main idea of the proof of theorem 1.2 is to consider, when possible, a smooth divisor Y' in the linear system of the ample generator of Y, and to study the manifold $X' = \mathbb{P}_{Y'}(\mathcal{E}_{|Y'})$; in order to do that we first establish some relations between the geometry of X and the geometry of X'.

Lemma 8.1. *Let Y be a smooth variety, $L \in \mathrm{Pic}(Y)$ an ample line bundle and $Y' \in |L|$ an effective divisor. Let $\mathcal{E}$ be a rank two vector bundle on Y and denote by $\mathcal{E}_{Y'}$ its restriction to Y'. Then*

a) *if $\mathcal{E}_{Y'}$ is spanned, then $\mathcal{E}$ is nef;*
b) *if $h^i(\mathcal{E}_{Y'}(-jL)) = 0$ for $i = 0, 1$ and every $j \geq 1$, then $H^0(Y, \mathcal{E}) \simeq H^0(Y', \mathcal{E}_{Y'})$.*

Proof. By definition, the nefness of $\mathcal{E}$ is the nefness of its tautological bundle; let $X = \mathbb{P}_Y(\mathcal{E})$ and let $X' = \mathbb{P}_{Y'}(\mathcal{E}_{Y'})$. Since the restriction of $\xi_{\mathcal{E}}$ to X' is spanned, if $\xi_{\mathcal{E}} \cdot C < 0$ for some effective curve C, then $C \cap X' = \emptyset$. By the ampleness of Y' in Y this implies that C is a fiber of the natural projection $p : X \to Y$, but this is impossible since such curves cover X.
To prove b), by the exact sequence

$$0 \longrightarrow \mathcal{E}(-L) \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_{Y'} \longrightarrow 0,$$

we have to show that $h^0(\mathcal{E}(-L)) = h^1(\mathcal{E}(-L)) = 0$, and this follows from [14, Corollary 4.1.6]. □

Proposition 8.2. *Let Y be a smooth variety of Picard number one and dimension ≥ 4, $\mathcal{E}$ a rank two vector bundle on Y, $L \in \mathrm{Pic}(Y)$ an ample line bundle and $Y' \in |L|$ an effective divisor.*
Assume that $\mathcal{E}_{Y'} = \mathcal{E}_{|Y'}$ is spanned and that $|\xi_{\mathcal{E}_{Y'}}|$ defines an extremal contraction $\varphi_{\vartheta'} : X' = \mathbb{P}_{Y'}(\mathcal{E}_{Y'}) \to Z$ associated to an extremal ray $\vartheta' \subset NE(X')$. Then, under the identification $N_1(X') \simeq N_1(X)$, given by the inclusion $i : X' \to X$, we have $NE(X') = NE(X)$.

Proof. Since $\dim Y \geq 4$, by Weak Lefschetz theorem we have $\rho_{Y'} = 1$, hence the cones of curves $\mathrm{NE}(X)$ and $\mathrm{NE}(X')$ have dimension two and, under the identification $N_1(X') \simeq N_1(X)$, they have in common the extremal ray $R_{\mathcal{E}}$ corresponding to the bundle projection. We have therefore to prove ϑ' is extremal in $\mathrm{NE}(X)$, too.
Since $(\xi_{\mathcal{E}})_{|X'} = \xi_{\mathcal{E}_{Y'}}$ is zero on ϑ', if ϑ' is not extremal in $\mathrm{NE}(X)$ we have $\xi_{\mathcal{E}} \cdot C < 0$ for some curve whose class is in $\mathrm{NE}(X) \setminus \mathrm{NE}(X')$. This contradicts the fact that, by lemma 8.1 a), $\mathcal{E}$ has to be nef. □

Corollary 8.3. *Let $(X, Y, \mathcal{E})$ be a ruled Fano fivefold of index two and Picard number $\rho_X = 2$, let L be the ample generator of $\mathrm{Pic}(Y)$, and assume that there exists an effective divisor $Y' \in |L|$ such that $\mathcal{E}_{Y'} = \mathcal{E}_{|Y'}$ is spanned and that $|\xi_{\mathcal{E}_{Y'}}|$ defines an extremal contraction $\varphi_{\vartheta'} : X \to Z$ of fiber type. Then there exists an extremal contraction $\psi_{\vartheta} : X \to Z$ such that $(\psi_{\vartheta})_{|X'} = \varphi_{\vartheta'}$.*

Proof. This assertion follows from [6, Proposition 3.13]. □

Proof of theorem 1.2. By lemma 2.12, Y is a Fano variety of pseudoindex $i_Y \geq i_X = 2$; moreover, since $\rho_X = 2$, we have $\rho_Y = 1$.

If $r_Y = i_Y = 2$, i.e. Y is a Mukai manifold, then, denoted by $\mathcal{O}_Y(1)$ the ample generator of $\mathrm{Pic}(Y)$, by [30, Theorem 1] a general section Y' in $|\mathcal{O}_Y(1)|$ is smooth, and so it is a Fano threefold of index one. By adjunction $X' = \mathbb{P}_{Y'}(\mathcal{E}_{Y'})$ is a Fano manifold, hence we can apply [29, Theorem 8.4] to get $X' = \mathbb{P}^1 \times Y'$.
Up to a twist, we can assume that $\mathcal{E}_{Y'} \simeq \mathcal{O}_{Y'} \oplus \mathcal{O}_{Y'}$; this bundle verifies the assumptions of proposition 8.2, so, by corollary 8.3, there exists an extremal contraction $\psi_\vartheta : X \to \mathbb{P}^1$; by lemma 4.1 we have $X \simeq \mathbb{P}^1 \times Y$.

If $r_Y = i_Y = 3$, i.e. Y is a del Pezzo manifold, we again denote by $\mathcal{O}_Y(1)$ the ample generator of $\mathrm{Pic}(Y)$ and we take a general divisor $Y' \in |\mathcal{O}_Y(1)|$. By adjunction $X' = \mathbb{P}_{Y'}(\mathcal{E}_{Y'})$ is a Fano manifold; by [29, Theorem 8.2] and [37, Proposition 4.2] we have the following possibilities for $(Y', \mathcal{E}_{Y'})$ (here the vector bundles are not normalized as in definition 3.2):

(1) $(V_d, \mathcal{O}_{V_d} \oplus \mathcal{O}_{V_d}(-1))$;
(2) (V_4, restriction of a spinor bundle on $\mathbb{Q}^4$);
(3) (V_5, restriction of the universal bundle on $G(1,4)$).

Case 1 $(Y', \mathcal{E}_{Y'}) \simeq (V_d, \mathcal{O}_{V_d} \oplus \mathcal{O}_{V_d}(-1))$.

By lemma 8.1 b) $H^0(Y, \mathcal{E}) \simeq H^0(Y', \mathcal{E}) \simeq \mathbb{C}$. It follows that $\mathcal{E}$ has a section, s; this section does not vanish on Y', which is ample, whence s can vanish only at points outside Y'. Let x be one of these points and let l be a line through x; $\mathcal{E}(1)$ is ample and $\det \mathcal{E}(1) \simeq \mathcal{O}_Y(3)$, so $\mathcal{E}$ restricts to l as $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, and s cannot vanish on l.
We thereby have a short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow L \longrightarrow 0$$

where, computing the splitting type, we have $L = \mathcal{O}_Y(-1)$; consequently the sequence splits and $\mathcal{E} \simeq \mathcal{O}_Y \oplus \mathcal{O}_Y(1)$.

Case 2 $(Y', \mathcal{E}_{Y'}) \simeq (V_4$, restriction of a spinor bundle on $\mathbb{Q}^4)$.

In case (2), as proved in [37, 4.4], X' has a conic bundle structure $\varphi : X' \to \mathbb{P}^3$, and can be described as a divisor in the flag manifold of lines and points in $G(1,3) \times \mathbb{P}^3$. Indeed, $\varphi_* \xi_{\mathcal{E}_{Y'}(1)} \simeq \Omega\mathbb{P}^3(3)$ and the flag manifold can be identified with the projectivization $\mathbb{P}_{\mathbb{P}^3}(\Omega\mathbb{P}^3(3))$; with this description X' is a divisor in $|2\xi_{\Omega\mathbb{P}^3(3)} - 2\varphi^*\mathcal{O}_{\mathbb{P}^3}(1)|$.

Since $\mathcal{E}$ is spanned on Y' and $|\xi_{\mathcal{E}_{Y'}}|$ defines a fiber type contraction, by corollary 8.3, there exists a contraction $\psi_\vartheta : X \to \mathbb{P}^3$ such that its restriction to X' is the conic bundle contraction $\varphi : X' \to \mathbb{P}^3$. In particular, since the restriction of ψ_ϑ to X' is equidimensional and X' is ψ_ϑ-ample, also ψ_ϑ is equidimensional and, by adjunction, is a quadric bundle contraction.

Let $\mathcal{F} = \psi_{\vartheta *}\xi_{\mathcal{E}(1)}$; $\mathcal{F}$ is a vector bundle of rank four and X embeds in $\mathbb{P}_{\mathbb{P}^3}(\mathcal{F})$ as a divisor of relative degree 2, i.e. $X \in |2\xi_{\mathcal{E}(1)} + \psi_\vartheta^*\mathcal{O}_{\mathbb{P}^3}(x)|$.
The vector bundle $\mathcal{F}$ has $\mathcal{G} = \varphi_*\xi_{\mathcal{E}_{Y'}(1)} \simeq \Omega\mathbb{P}^3(3)$ as a quotient. Indeed, if $x \in \mathbb{P}^3$ is a point and we denote by F and f the fibers of ψ_ϑ and $\psi_{\vartheta|X'} = \varphi$ over x, we have that $\mathcal{G}_x = H^0((\xi_\mathcal{E})_{|f})$ is a quotient of $\mathcal{F}_x = H^0((\xi_\mathcal{E})_{|F})$.
It follows that there exists an exact sequence on $\mathbb{P}^3$:

$$0 \longrightarrow \mathcal{O}(a) \longrightarrow \mathcal{F} \longrightarrow \Omega\mathbb{P}^3(3) \longrightarrow 0.$$

Since $(\xi_{\mathcal{E}(1)})_{|X'} = \xi_{\Omega\mathbb{P}^3(3)}$, $(\psi_\vartheta^*\mathcal{O}(1))_{|X'} = \varphi_{\mathbb{P}^3}^*\mathcal{O}_{\mathbb{P}^3}(1)$ and $X_{|\mathbb{P}(\mathcal{G})} = X'$, we have $x = -2$ and $X \in |2\xi_{\mathcal{E}(1)} - 2\psi^*\mathcal{O}_{\mathbb{P}^3}(1)| = |2\xi_\mathcal{E}|$. By adjunction

$$-2\xi_{\mathcal{E}(1)} = K_X = (K_{\mathbb{P}_{\mathbb{P}^3}(\mathcal{F})} + X)_X = -2\xi_{\mathcal{E}(1)} + \psi^*\mathcal{O}_{\mathbb{P}^3}(c_1(\mathcal{F}) - 6),$$

hence $c_1(\mathcal{F}) = 6$. Computing the degree in the above sequence, we have $a = 1$. Therefore the sequence splits and we have $\mathcal{F} \simeq \Omega\mathbb{P}^3(3) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$.

Case 3 $(Y', \mathcal{E}_{Y'}) \simeq (V_5$, restriction of the universal bundle on $G(1,4))$.

We claim that $\mathcal{E}$ is spanned on Y; to prove the claim we show that $\xi_\mathcal{E}$ is spanned on $X = \mathbb{P}(\mathcal{E})$. Assume that $\bar{x} \in X$ is a base point of $|\xi_\mathcal{E}|$; since $\mathcal{O}_Y(1)$ is very ample, we can find a smooth section $Y'' \in |\mathcal{O}_Y(1)|$ containing $p(x)$. The restriction $(\xi_\mathcal{E})_{|Y''} = \xi_{\mathcal{E}_{Y''}}$ is spanned, so there exists a section of $(\xi_\mathcal{E})_{Y''}$ which does not vanish at x and this section, by lemma 8.1 b), extends to X.
We have thus proved that $\mathcal{E}$ is spanned; again by lemma 8.1 b), $h^0(Y, \mathcal{E}) = h^0(Y'', \mathcal{E}_{Y''}) = 5$ so we have an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_Y^{\oplus 5} \longrightarrow \mathcal{E} \longrightarrow 0$$

which gives an injection $X \to \mathbb{P}^4 \times Y$ and then an injection $X \to \mathbb{P}^4 \times G(1,4)$. We claim that X is the intersection of $p^{-1}(Y)$ with the flag manifold of lines and points in $G(1,4) \times \mathbb{P}^4$. Indeed, given a point $y \in Y$, denoting by Y' a smooth member of $\mathcal{O}_Y(1)$ passing through y, $\mathcal{E}_{Y'}$ is the restriction of the universal bundle of $G(1,4)$, thus the fiber of $\mathcal{E}$ over y is the line parametrized by $y \in G(1,4)$.

If $i_Y = 4$ then, by [31, Theorem 0.1], $Y \simeq \mathbb{Q}^4$. We can apply [1, Theorem 2.4] to get that $\mathcal{E}$ is decomposable (the other bundles have odd c_1, while, in our case, since $\det \mathcal{E} = -K_Y = \mathcal{O}_{\mathbb{Q}^4}(4)$, $c_1(\mathcal{E})$ is even) and we are in case (2) of theorem 1.2.

If $i_Y = 5$ then, by [19, Corollary 0.4] or [25, Theorem 1.1], $Y \simeq \mathbb{P}^4$. We can apply [1, Theorem 2.4] to get that $\mathcal{E}$ is decomposable, hence we are in case (1) of theorem 1.2. Note that only the bundles whose projectivization gives a Fano manifold of index two are considered. □

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