**Blow-ups and Fano manifolds of large pseudoindex**

# BLOW-UPS AND FANO MANIFOLDS OF LARGE PSEUDOINDEX

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**Abstract**

We describe the Kleiman–Mori cones of Fano manifolds of large pseudoindex that admit a structure of smooth blow-up.

## 1. Introduction

Let $X$ be a Fano manifold, *i.e.* smooth complex projective variety whose anticanonical bundle $-K_X$ is ample. A Fano manifold is naturally associated with two invariants: the *index*, $r_X$, defined as the largest integer dividing $-K_X$ in the Picard group of $X$, and the *pseudoindex*, $i_X$, defined as the minimum anticanonical degree of rational curves on $X$. It is known that these invariants satisfy the relations $1 \leq r_X \leq i_X \leq \dim X + 1$ ([10] and [9]). Moreover, the index is related with both the dimension and the Picard number, $\rho_X$, of $X$ by a conjecture of Mukai ([14]) that states: $\rho_X(r_X - 1) \leq \dim X,$ with equality if and only if $X = (\mathbb{P}^{r_X-1})^{\rho_X}$. However, when dealing with Fano manifolds of Picard number grater than one it can happen that the index is equal to one even for simple varieties such as $\mathbb{P}^s \times \mathbb{P}^{s+1}$, so it seems that in studying these varieties the pseudoindex could be a more useful invariant than the index. In particular, the above conjecture has been restated ([5]) by replacing the index with the pseudoindex, and this generalization, under the assumption $i_X > \frac{\dim X}{3}$, has been proved ([18, Theorem 3], [15, Theorem 5.1]; see also [21, Theorem A] and [20, Corollary 4.3] for $i_X \geq \frac{\dim X+2}{2}$). Building on this, a first step to the actual classification of Fano manifolds with $i_X \geq \frac{\dim X+1}{3}$ and $\rho_X \geq 3$ has been treated in [16], where the complete classification of Fano manifolds of pseudoindex $i_X \geq \frac{\dim X+2}{3}$ and Picard number $\rho_X \geq 3$ is given.

In general when the Picard number of the variety is large, namely $\rho_X \geq 4$, the setting is quite easy to be understood; as to next case, namely $\rho_X = 3$, these varieties are more difficult to classify. However, by looking at the proofs in [18]

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and [15], one can see that $X$ is rationally connected with respect to some families of rational curves and that these families have "good" properties. So we can make use of such families of rational curves to study the manifolds we are interested in. Though for $i_X \geq \frac{\dim X+2}{3}$ the classification has been achieved, when $i_X = \frac{\dim X+1}{3}$ things are quite complicated. However, the complete classification both in case $\rho_X \geq 4$, and in case $\rho_X = 3$ if $X$ is rationally connected with respect to three unsplit families of rational curves, one of them having anticanonical degree greater than $i_X$, is settled in [16].

In this paper we reconsider Fano manifods with $i_X \geq \frac{\dim X+1}{3}$ and $\rho_X \geq 3$. Since a natural question coming from the study of Fano manifolds is to investigate Fano manifolds with a "special" extremal contraction, we assume that $X$ admits a structure of smooth blow-up. We prove the following

THEOREM 1.1. *Let $X$ be a Fano manifold of pseudoindex $i_X \geq \frac{\dim X+1}{3} \geq 2$ and Picard number $\rho_X \geq 3$. Assume that $X$ has an extremal ray $R_\sigma$ associated with a smooth blow-up. Then one of the following holds:*

(1) $\rho_X = 4$, $\dim X = 5$, $i_X = \frac{\dim X+1}{3}$ *and* $NE(X) = \langle R_\sigma, R_1, R_2, R_3\rangle$, *where $R_1, R_2$ and $R_3$ are associated with contractions of fiber type.*
(2) $\rho_X = 3$ *and* $NE(X) = \langle R_\sigma, R_1, R_2\rangle$, *where $R_1$ and $R_2$ are associated with contractions of fiber type.*
(3) $\rho_X = 3$, $i_X = \frac{\dim X+1}{3}$ *and* $NE(X) = \langle R_\sigma, R_1, R_2\rangle$, *where $R_1$ is associated with a contraction of fiber type and $R_2$ is associated with a smooth blow-up.*

The paper is organized as follows.
In Section 2 we collect basic material concerning definitions and results on extremal contractions, on families of rational curves and on chains of rational curves on projective manifolds.
In Section 3 we describe the Kleiman–Mori cones of Fano manifolds with $i_X \geq \frac{\dim X+1}{3} \geq 2$ and $\rho_X \geq 3$ that admit a structure of smooth blow-up. We start by recalling in Remark 3.1 that if $X$ is a Fano manifold of pseudoindex $i_X \geq \frac{\dim X+1}{3} \geq 2$ and Picard number $\rho_X \geq 4$, then case (1) in the statement of Theorem 1.1 is achieved by combining [16, Proposition 4.1 and Proposition 5.1]. Therefore in the rest of the section we deal with $\rho_X = 3$. We split the proof in two main cases, according to the existence on $X$ of an unsplit dominating family of rational curves which is positive with respect to the exceptional locus of the given blow-up. If $X$ admits such a family of rational curves, say $V$, we first prove in Lemma 3.2 that X cannot admit an extremal ray associated with a small contraction and that $[V]$ belongs to an extremal ray associated with a contraction of fiber type; then we use these facts to describe the possible Kleiman–Mori cones of $X$ in Theorem 3.3. If otherwise $X$ does not admit such a family of rational

curve, we prove that $X$ admits an extremal ray associated with another blow-up, and we get the description of the Kleiman–Mori of $X$ in Theorem 3.7
Finally, in Section 4 we give examples, from which we see that the result is effective.

## 2. Background material

Let $X$ be a smooth complex projective variety.

DEFINITION 2.1. A *contraction* $\varphi\colon X \to Y$ is a proper surjective map with connected fibers onto a normal variety $Y$. If the canonical bundle $K_X$ is not nef, then the negative part of the closure $\overline{\mathrm{NE}}(X)$ of the cone of effective 1-cycles into the $\mathbb{R}$-vector space of 1-cycles modulo numerical equivalence is polyhedral, by the Cone Theorem. By the Contraction Theorem, every face in this part of the cone, called *extremal face*, is associated with a contraction, called *extremal contraction* or *Fano–Mori contraction*.
An extremal contraction associated with an extremal face of dimension one, *i.e.* with an *extremal ray*, is called an *elementary contraction*; if $\dim Y < \dim X$ then it is called *of fiber type*, otherwise it is called *birational*. If the codimension of the exceptional locus of an elementary birational contraction is equal to one, the contraction is called *divisorial*, otherwise it is called *small*. The *length* of an extremal ray is defined as the minimum anticanonical degree of rational curves whose numerical equivalence class belongs to the ray; a rational curve attaining the length of the ray is called *minimal* curve of the ray.

REMARK 2.2. Fibers of contractions associated with different extremal rays can meet at most at points.

**Notation**. The exceptional locus of a contraction associated with an extremal ray $R$ will be denoted by $\mathrm{Exc}(R)$.

DEFINITION 2.3. A *family of rational curves* $V$ on $X$ is an irreducible component of the scheme $\mathrm{Ratcurves}^n(X)$ (see [11, Definition II.2.11]).
Given a rational curve we will call a *family of deformations* of that curve any irreducible component of $\mathrm{Ratcurves}^n(X)$ containing the point parameterizing that curve.
We define $\mathrm{Locus}(V)$ to be the set of points of $X$ through which there is a curve among those parameterized by $V$; we say that $V$ is a *covering family* if $\mathrm{Locus}(V) = X$ and that $V$ is a *dominating family* if $\overline{\mathrm{Locus}(V)} = X$.
By abuse of notation, given a line bundle $L \in \mathrm{Pic}(X)$, we will denote by $L \cdot V$ the intersection number $L \cdot C$, with $C$ any curve among those parameterized by $V$.
We will say that $V$ is *unsplit* if it is proper; clearly, an unsplit dominating family is covering.

We denote by $V_x$ the subscheme of $V$ parameterizing rational curves passing through a point $x$ and by $\mathrm{Locus}(V_x)$ the set of points of $X$ through which there is a curve among those parameterized by $V_x$. If, for a general point $x \in \mathrm{Locus}(V)$, $V_x$ is proper, then we will say that the family is *locally unsplit*; by Mori's Bend and Break arguments, if $V$ is a locally unsplit family, then $-K_X \cdot V \leq \dim X + 1$. If $X$ admits dominating families, we can choose among them one with minimal degree with respect to a fixed ample line bundle $A$, and we call it a *minimal dominating family.* Such a family is locally unsplit.

Definition 2.4. Let $U$ be an open dense subset of $X$ and $\pi\colon U \to Z$ a proper surjective morphism to a quasi-projective variety; we say that a family of rational curves $V$ is a *horizontal dominating family with respect to* $\pi$ if $\mathrm{Locus}(V)$ dominates $Z$ and curves parameterized by $V$ are not contracted by $\pi$. If such families exist, we can choose among them one with minimal degree with respect to a fixed ample line bundle and we call it a *minimal horizontal dominating family* with respect to $\pi$; such a family is locally unsplit.

Remark 2.5. By fundamental results in [13], a Fano manifold admits dominating families of rational curves; also horizontal dominating families with respect to proper morphisms defined on an open set exist, as proved in [12]. In the case of Fano manifolds with "minimal" we will mean minimal with respect to $-K_X$, unless otherwise stated.

Definition 2.6. We define a *Chow family of rational 1-cycles* $\mathcal{W}$ to be an irreducible component of $\mathrm{Chow}(X)$ parameterizing rational and connected 1-cycles.
We define $\mathrm{Locus}(\mathcal{W})$ to be the set of points of $X$ through which there is a cycle among those parameterized by $\mathcal{W}$; notice that $\mathrm{Locus}(\mathcal{W})$ is a closed subset of $X$ ([11, II.2.3]). We say that $\mathcal{W}$ is a *covering family* if $\mathrm{Locus}(\mathcal{W}) = X$.
If $V$ is a family of rational curves, the closure of the image of $V$ in $\mathrm{Chow}(X)$, denoted by $\mathcal{V}$, is called the *Chow family associated with* $V$.

Remark 2.7. If $V$ is proper, *i.e.* if the family is unsplit, then $V$ corresponds to the normalization of the associated Chow family $\mathcal{V}$.

Definition 2.8. Let $V$ be a family of rational curves and let $\mathcal{V}$ be the associated Chow family. We say that $V$ (and also $\mathcal{V}$) is *quasi-unsplit* if every component of any reducible cycle parameterized by $\mathcal{V}$ has numerical class proportional to the numerical class of a curve parameterized by $V$.

Definition 2.9. Let $V^1, \dots, V^k$ be families of rational curves on $X$ and $Y \subset X$.
We define $\mathrm{Locus}(V^1)_Y$ to be the set of points $x \in X$ such that there exists a

curve $C$ among those parameterized by $V^1$ with $C \cap Y \neq \emptyset$ and $x \in C$. We inductively define $\mathrm{Locus}(V^1, \dots, V^k)_Y := \mathrm{Locus}(V^2, \dots, V^k)_{\mathrm{Locus}(V^1)_Y}$. Notice that, by this definition, we have $\mathrm{Locus}(V)_x = \mathrm{Locus}(V_x)$. Analogously we define $\mathrm{Locus}(\mathcal{W}^1, \dots, \mathcal{W}^k)_Y$ for Chow families $\mathcal{W}^1, \dots, \mathcal{W}^k$ of rational 1-cycles.

**Notation**. We denote by $\rho_X$ the Picard number of $X$, *i.e.* the dimension of the $\mathbb{R}$-vector space $\mathrm{N}_1(X)$ of 1-cycles modulo numerical equivalence. If $\Gamma$ is a 1-cycle, then we will denote by $[\Gamma]$ its numerical equivalence class in $\mathrm{N}_1(X)$; if $V$ is a family of rational curves, we will denote by $[V]$ the numerical equivalence class of any curve among those parameterized by $V$.
If $Y \subset X$, we will denote by $\mathrm{N}_1(Y, X) \subseteq \mathrm{N}_1(X)$ the vector subspace generated by numerical classes of curves of $X$ contained in $Y$; moreover, we will denote by $\mathrm{NE}\,(Y, X) \subseteq \mathrm{NE}(X)$ the subcone generated by numerical classes of curves of $X$ contained in $Y$.

We will make frequent use of the following dimensional estimates:

Proposition 2.10. ([11, IV.2.6]) *Let $V$ be a family of rational curves on $X$ and $x \in \mathrm{Locus}(V)$ a point such that every component of $V_x$ is proper. Then*

(a) $\dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) \geq \dim X - K_X \cdot V - 1$;
(b) $\dim \mathrm{Locus}(V_x) \geq -K_X \cdot V - 1$.

Definition 2.11. We say that $k$ quasi-unsplit families $V^1, \dots, V^k$ of rational curves are numerically independent if, in $\mathrm{N}_1(X)$, we have $\dim\langle [V^1], \dots, [V^k]\rangle = k$.

Lemma 2.12. (*Cf.* [1, Lemma 5.4]) *Let $Y \subset X$ be a closed subset and $V^1, \dots, V^k$ numerically independent unsplit families of rational curves such that $\langle [V^1], \dots, [V^k]\rangle \cap NE(Y, X) = \underline{0}$. Then either $\mathrm{Locus}(V^1, \dots, V^k)_Y = \emptyset$ or*

$$\dim \mathrm{Locus}(V^1, \dots, V^k)_Y \geq \dim Y + \sum -K_X \cdot V^i - k.$$

Definition 2.13. Let $Y \subset X$ be a closed subset, let $V$ be a dominating family of rational curves on $X$ and denote by $\mathcal{V}$ be the associated Chow family; define $\mathrm{ChLocus}(\mathcal{V})_Y$ to be the set of points $x \in X$ such that there exist cycles $\Gamma_1, \dots, \Gamma_m$ with the following properties:

- $\Gamma_i$ belongs to the family $\mathcal{V}$;
- $\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$;
- $\Gamma_1 \cap Y \neq \emptyset$ and $x \in \Gamma_m$,

*i.e.* $\mathrm{ChLocus}(\mathcal{V})_Y$ is the set of points that can be joined to $Y$ by a connected chain of at most $m$ cycles belonging to the family $\mathcal{V}$.

We will use the description of the numerical expression of curves in $\mathrm{ChLocus}(\mathcal{V})_Z$, with $Z \subset X$ a closed subset and $V$ a quasi-unsplit family of rational curves, as stated in [19, Lemma 1.10].

LEMMA 2.14. (Cf. [4, Proof of Lemma 1.4.5] and [20, Lemma 3.2 and Remark 3.3]) *Let $Z \subset X$ be a closed subset and let $V$ be a quasi-unsplit family of rational curves. Then every curve contained in* $\mathrm{ChLocus}(\mathcal{V})_Z$ *is numerically equivalent to a linear combination with rational coefficients*

$$\lambda_V C_V + \lambda_Z C_Z,$$

*with $C_V$ a curve among those parameterized by $V$, $C_Z$ a curve in $Z$ and $\lambda_Z \geq 0$.*

Define a relation of *rational connectedness with respect to* $\mathcal{V}$ on $X$ in the following way: two points $x$ and $y$ of $X$ are in $\mathrm{rc}(\mathcal{V})$-relation if there exists a chain of cycles in $\mathcal{V}$ which joins $x$ and $y$, *i.e.* if $y \in \mathrm{ChLocus}(\mathcal{V})_x$. In particular, $X$ is *rc($\mathcal{V}$)-connected* if we have $X = \mathrm{ChLocus}(\mathcal{V})_x$.

The family $\mathcal{V}$ defines a proper prerelation in the sense of [11, Definition IV.4.6]. This prerelation is associated with a fibration, which we will call the *rc($\mathcal{V}$)-fibration*:

THEOREM 2.15. ([11, IV.4.16], *Cf.* [6]) *Let $X$ be a normal and proper variety and $\mathcal{V}$ a proper prerelation; then there exists an open subvariety $X^0 \subset X$ and a proper morphism with connected fibers $\pi\colon X^0 \to Z^0$ such that*

- *$\langle \mathcal{U} \rangle$ restricts to an equivalence relation on $X^0$;*
- *$\pi^{-1}(z)$ is a $\langle \mathcal{U} \rangle$-equivalence class for every $z \in Z^0$;*
- *$\forall\, z \in Z^0$ and $\forall\, x, y \in \pi^{-1}(z)$, $x \in \mathrm{ChLocus}(\mathcal{V})_y$ with $m \leq 2^{\dim X - \dim Z^0} - 1$.*

Clearly $X$ is $\mathrm{rc}(\mathcal{V})$-connected if and only if $\dim Z^0 = 0$.

Given $\mathcal{V}^1, \ldots, \mathcal{V}^k$ Chow families of rational 1-cycles, it is possible to define a relation of $\mathrm{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-connectedness, which is associated with a fibration, that we will call $\mathrm{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-fibration. The variety $X$ will be called *rc($\mathcal{V}^1, \ldots, \mathcal{V}^k$)-connected* if the target of the fibration is a point.

**Notation**. In the next sections for simplicity we will write $\mathrm{Locus}(V)_x$ to mean $\mathrm{Locus}(V)_x$ for a general point $x \in \mathrm{Locus}(V)$, and $\mathrm{Locus}(V^\alpha, \ldots, V^\beta)_{x_\alpha}$ to mean $\mathrm{Locus}(V^\alpha, \ldots, V^\beta)_{x_\alpha}$ for a general point $x_\alpha \in \mathrm{Locus}(V^\alpha)$, unless otherwise stated.

We end this section by recalling the following general construction.

Construction 2.16. ([18, Construction 1]) Let $X$ be a Fano manifold; let $V^1$ be a minimal dominating family of rational curves on $X$ and consider the associated Chow family $\mathcal{V}^1$. If $X$ is not $\mathrm{rc}(\mathcal{V}^1)$-connected, let $V^2$ be a minimal horizontal dominating family with respect to the $\mathrm{rc}(\mathcal{V}^1)$-fibration, $\pi_1 \colon X \dashrightarrow Z^1$. If $X$ is not $\mathrm{rc}(\mathcal{V}^1, \mathcal{V}^2)$-connected, we denote by $V^3$ a minimal horizontal dominating family with respect to the $\mathrm{rc}(\mathcal{V}^1, \mathcal{V}^2)$-fibration, $\pi_2 \colon X \dashrightarrow Z^2$, and so on. Since $\dim Z^{i+1} < \dim Z^i$, for some integer $k$ we have that $X$ is $\mathrm{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$-connected.

By abuse of notation, we will write $V^i$ instead of $\mathcal{V}^i$ if the family is unsplit.

Remark 2.17. Examples of the above construction are given in [15, Examples 4.2]. Note that at each step the dimension drops at least by $\dim \mathrm{Locus}(V^i)_{x_i}$.

Remark 2.18. Let $X$ be a Fano manifold of dimension $\dim X \geq 3$, pseudoindex $i_X \geq \frac{\dim X+1}{3}$ and Picard number $\rho_X \geq 3$. By looking at the proofs of [15, Theorem 5.1] and [18, Theorem 5], we see that, if one of the families $V^j$ as in Construction 2.16 is not unsplit, then $\dim X = 5$, $i_X = 2$ and $X$ is $\mathrm{rc}(V^1, V^2, \mathcal{V}^3)$-connected.

## 3. Description of the Kleiman–Mori cone of $X$

In this section we describe the Kleiman–Mori cone of a Fano manifold $X$ of pseudoindex $i_X \geq \frac{\dim X+1}{3} \geq 2$ and Picard number $\rho_X \geq 3$ admitting an extremal ray associated with a smooth blow-up and we show Theorem 1.1.

Remark 3.1. Let $X$ be a Fano manifold of pseudoindex $i_X \geq \frac{\dim X+1}{3} \geq 2$ and Picard number $\rho_X \geq 4$. Then we have case (1) in the statement of Theorem 1.1 by combining [16, Proposition 4.1 and Proposition 5.1].

Therefore we are left to deal with $\rho_X = 3$.

*We start by assuming that $X$ admits an unsplit dominating family of rational curves which is positive with respect to the exceptional locus of the blow-up.*

Lemma 3.2. *Let $X$ be a Fano manifold of pseudoindex $i_X \geq \frac{\dim X+1}{3} \geq 2$ and Picard number $\rho_X = 3$. Assume that $X$ has an extremal ray $R_\sigma$ associated with a smooth blow-up. If $X$ admits an unsplit dominating family of rational curves $V$ such that $\mathrm{Exc}(R_\sigma) \cdot V > 0$, then $[V]$ is contained in an extremal ray of $X$.*

*Moreover, any other extremal ray of $X$ is associated either with a contraction of fiber type, or with a smooth blow-up. In the last case, $i_X = \frac{\dim X+1}{3}$ and the non-trivial fibers of each blow-up are $i_X$-dimensional.*

Proof. Let $R$ be an extremal ray of $X$ different from $R_\sigma$ and such that $[V] \notin R$. Denote by $F_R$ any non-trivial fiber of the contraction associated with $R$. Since $V$ is a dominating family and $\mathrm{Exc}(R_\sigma) \cdot V > 0$, we have

$$\mathrm{Exc}(R_\sigma) \cap \mathrm{Locus}(V)_{F_R} \neq \emptyset,$$

hence there exists a fiber $F_\sigma$ of the blow-up associated with $R_\sigma$ such that

$$F_\sigma \cap \mathrm{Locus}(V)_{F_R} \neq \emptyset. \tag{3.2.1}$$

Now, the numerical equivalence class of any curve in $F_\sigma$ belongs to $R_\sigma$, while every curve contained in $\mathrm{Locus}(V)_{F_R}$ is numerically equivalent to a linear combination of a curve among those parameterized by $V$ and a curve in $F_R$; it thus follows that the intersection (3.2.1) is 0-dimensional. So we get

$$\dim X \geq \dim F_\sigma + \dim \mathrm{Locus}(V)_{F_R} \geq \dim F_\sigma + \dim F_R - K_X \cdot V - 1, \tag{3.2.2}$$

where the last inequality follows by Lemma 2.12. By taking into account [22, Theorem 1.1] applied to $R_\sigma$, from (3.2.2) we derive $\dim F_R \leq i_X$. So, by [22, Theorem 1.1] applied to $R$, we get that $R$ is associated either with a contraction of fiber type, or with a divisorial contraction.

In the last case by (3.2.2) we have $\dim X = 3i_X - 1$ and $\dim F_R = i_X = \ell(R)$, the length of $R$, so the contraction associated with $R$ is a smooth blow-up by [2, Theorem 5.1]; moreover, both $R_\sigma$ and $R$ are associted with blow-ups with $i_X$-dimensional non-trivial fibers.

To prove the existence of an extremal ray containing $[V]$, we consider the $\mathrm{rc}(V)$-fibration $\pi\colon X \dashrightarrow Z$. Let $V^\sigma$ a family of deformations of a minimal curve in $R_\sigma$. As $\mathrm{Exc}(R_\sigma) \cdot V > 0$, the family $V^\sigma$ is horizontal and dominating with respect to $\pi$. Moreover, $X$ is not $\mathrm{rc}(V, V^\sigma)$-connected, being $\rho_X = 3$. So $[V]$ and $[V^\sigma]$ lie in an extremal face $\langle R_\sigma, R_1\rangle$ by [7, Lemma 2.4], as we have proved above that $X$ has no small contractions.

Now, let $H$ be the pullback of a very ample line bundle on $Z$. The curves parameterized by $V$ are contracted by $\pi$, so $H \cdot V = 0$; moreover, $H$ is positive outside the indeterminacy locus of $\pi$, so $H \cdot R_\sigma > 0$, since $V^\sigma$ is horizontal and dominating with respect to $\pi$; finally, either $[V] \in R_1$, or the exceptional locus of $R_1$ is contained in the indeterminacy locus of $\pi$. However, the last case cannot occur since $R_1$ is not associated with a small contraction and the indeterminacy locus of $\pi$ has codimension at least 2 in $X$.

Theorem 3.3. *Let $X$ be a Fano manifold of pseudoindex $i_X \geq \frac{\dim X+1}{3} \geq 2$ and Picard number $\rho_X = 3$. Assume that $X$ has an extremal ray $R_\sigma$ associated with a smooth blow-up. If $X$ admits an unsplit dominating family of rational curves $V$ such that $\mathrm{Exc}(R_\sigma) \cdot V > 0$, then one of the following holds:*

(1) *$NE(X) = \langle R_\sigma, R_1, R_2\rangle$, where $R_1$ and $R_2$ are associated with contractions of fiber type;*
(2) *$NE(X) = \langle R_\sigma, R_1, R_2\rangle$, where $R_1$ is associated with a contraction of fiber type and $R_2$ is associated with a smooth blow-up.*

*Moreover, $[V] \in R_1$.*

Proof. In view of Lemma 3.2 we know that $X$ has no small contractions and that it admits an extremal ray $R_1$ such that $[V] \in R_1$, so that the contraction associated with $R_1$ is of fiber type. Moreover, since $\mathrm{Exc}(R_\sigma) \cdot V > 0$, the rays $R_\sigma$ and $R_1$ span an extremal face in $\mathrm{NE}(X)$ by [7, Lemma 2.4]. Now, let $R_2$ be an extremal ray of $X$ which does not belong to $\langle R_\sigma, R_1\rangle$; by Lemma 3.2 the contraction associated with $R_2$ is either of fiber type, or a smooth blow-up.

*We first assume that the contraction associated with $R_2$ is of fiber type and we prove that we are in case* (1) *of the statement.*

Since $X$ has no small contractions, by [7, Lemma 2.4] we have that $R_1$ and $R_2$ are contained in an extremal face of $\mathrm{NE}(X)$. Now, it is enough to show that $R_\sigma$ and $R_2$ lie in an extremal face of $\mathrm{NE}(X)$.

If $\mathrm{Exc}(R_\sigma) \cdot R_2 > 0$, a family of deformations $V^\sigma$ of a minimal curve in $R_\sigma$ is horizontal and dominating with respect to the fibration associated with a family of deformations $V^{R_2}$ of a minimal curve whose numerical equivalence class belongs to $R_2$. Moreover, $X$ is not $\mathrm{rc}(V^{R_2}, V^\sigma)$-connected since $\rho_X = 3$. So $R_\sigma$ and $R_2$ lie in an extremal face of $\mathrm{NE}(X)$, again by [7, Lemma 2.4]. Therefore $\mathrm{NE}(X) = \langle R_\sigma, R_1, R_2\rangle$.

If otherwise $\mathrm{Exc}(R_\sigma) \cdot R_2 = 0$, assume to get a contradiction that there exists an extremal ray, say $R_3$, in the halfspace of $\mathrm{NE}(X)$ which is bounded by $\langle R_\sigma, R_2\rangle$ and does not contain $R_1$. Then $\mathrm{Exc}(R_\sigma) \cdot R_3 < 0$, so the exceptional locus of $R_3$ is contained in $\mathrm{Exc}(R_\sigma)$, hence $R_3$ is associated with a blow-up by Lemma 3.2. Now, let $F_\sigma$, $F_1$, $F_2$ and $F_3$ be the fibers of the contractions associated with $R_\sigma$, $R_1$, $R_2$ and $R_3$, respectively, which contain a point $x \in \mathrm{Exc}(R_3)$. Since fibers of different extremal rays can meet at most at points, we get

$$3i_X - 1 \geq \dim X \geq \dim F_\sigma + \dim F_1 + \dim F_2 + \dim F_3 \geq 4i_X - 2,$$

where the last inequality is due to [22, Theorem 1.1] applied to each extremal ray; it follows that $i_X = 1$, a contradiction. Therefore $\mathrm{NE}(X) = \langle R_\sigma, R_1, R_2\rangle$.

*We can now assume that the contraction associated with $R_2$ is birational and that $R_1$ is the only extremal ray of $X$ whose associated contraction is of fiber type; we prove that we are in case* (2) *of the statement.*

Notice that by Lemma 3.2 we know that the contractions associated with $R_2$ and with any other extremal ray different from $R_1$ are smooth blow-ups.

We show that $R_1$ and $R_2$ lie on an extremal face.

If $\mathrm{Exc}(R_2)\cdot R_1 > 0$, recalling that $X$ has no small contraction by Lemma 3.2, by [7, Lemma 2.4] we get that $R_1$ and $R_2$ are contained in an extremal face of $\mathrm{NE}(X)$.

So we are left to assume that $\mathrm{Exc}(R_2)\cdot R_1 = 0$. Clearly we have $\mathrm{Exc}(R_2)\cdot R_2 < 0$. We can argue as in the proof of [8, Theorem 5.7], so we claim that $\mathrm{Exc}(R_2)\cdot R_\sigma > 0$ and that $\mathrm{NE}(\mathrm{Exc}(R_2)) = \langle R_\sigma, R_1, R_2\rangle$.
Let $V^\sigma$ a family of deformations of a minimal curve in $R_\sigma$. Since $\mathrm{Exc}(R_\sigma)\cdot V > 0$, the family $V^\sigma$ is horizontal and dominating with respect to the $\mathrm{rc}(V)$-fibration, so we can consider the $\mathrm{rc}(V, V^\sigma)$-fibration, whose general fiber $F$ has dimension $\geq 2i_X - 1$ by Lemma 2.12. Let $V^{R_2}$ a family of deformations of a minimal curve in $R_2$. Now, by computing the dimension of $\mathrm{Locus}(V^{R_2})_F$ with Lemma 2.12, we derive $\mathrm{Exc}(R_2) = \mathrm{Locus}(V^{R_2})_F$, so $\mathrm{NE}(\mathrm{Exc}(R_2)) = \langle R_\sigma, R_1, R_2\rangle$. Since an effective divisor cannot be non-positive on the whole $\mathrm{NE}(X)$, the claim follows.

Now, assume by contradiction that $R_1$ and $R_2$ are not contained in an extremal face of $\mathrm{NE}(X)$. Then there exists an extremal ray, say $R_3$, in the halfspace of $\mathrm{NE}(X)$ which is bounded by $\langle R_1, R_2\rangle$ and does not contain $R_\sigma$. It follows that $\mathrm{Exc}(R_2)\cdot R_3 < 0$, so the exceptional locus of $R_3$ is contained in $\mathrm{Exc}(R_2)$, contradicting $\mathrm{NE}(\mathrm{Exc}(R_2)) = \langle R_\sigma, R_1, R_2\rangle$.

Notice that the same argument as for $R_2$ applies to every extremal ray different form $R_1$ and $R_\sigma$; it follows that the only possibility for the Kleiman–Mori cone of $X$ is $\mathrm{NE}(X) = \langle R_\sigma, R_1, R_2\rangle$, so we are in case (2) of the statement.

Remark 3.4. Notice that, in view of Lemma 3.2, the second case in Theorem 3.3 can happen only for Fano manifolds $X$ of Picard number $\rho_X = 3$ and pseudoindex $i_X = \frac{n+1}{3}$.

*Next we assume that $X$ does not admit any unsplit dominating family of rational curves which is positive with respect to the exceptional locus of the blow-up.*

We will make use of the following remark.

Remark 3.5. Let $X$ be a Fano manifold of dimension $\dim X \geq 3$, pseudoindex $i_X \geq \frac{\dim X+1}{3}$ and Picard number $\rho_X = 3$. By looking at the proofs of [15, Theorem 5.1] and [18, Theorem 5], we see that, $X$ is $\mathrm{rc}(\mathcal{V}^1, \mathcal{V}^2, \mathcal{V}^3)$-connected with respect to three families as in Construction 2.16 which turn out to be unsplit, unless $\dim X = 5$, $i_X = 2$ and only the first two families are unsplit, or $X$ is $\mathrm{rc}(\mathcal{V}^1, \mathcal{V}^2)$-connected with respect to two families and only the first family is unsplit.

However, the description of the Kleiman–Mori cone of Fano fivefolds with pseudoindex 2 is given in [7, Theorem 1.1], so we could confine to manifolds of dimension at least six that we know to be $\mathrm{rc}(V^1, V^2, V^3)$-connected with respect to three unsplit families of rational curves. This will be done in Theorem 3.7.

LEMMA 3.6. *Let $X$ be a Fano manifold of pseudoindex $i_X \geq \frac{\dim X+1}{3} \geq 2$. Assume that $X$ is $rc(V^1, V^2, V^3)$-connected with respect to three unsplit families as in Construction 2.16 and that $X$ has an extremal ray $R_\sigma$ associated with a smooth blow-up.*
*If $X$ does not admit any unsplit dominating family of rational curves $V$ such that $\mathrm{Exc}(R_\sigma) \cdot V > 0$, then $V^2$ is not dominating, $X$ is $rc(V^1, V^2, V^\sigma)$-connected, with $V^\sigma$ a family of deformations of a minimal curve in $R_\sigma$, $\langle [V^1], [V^2] \rangle$ is extremal and $-K_X \cdot V^1 = -K_X \cdot V^2 = -K_X \cdot V^\sigma = i_X = \frac{\dim X+1}{3}$.*

PROOF. By construction $V^1$ is dominating, so $\mathrm{Exc}(R_\sigma) \cdot V^1 = 0$. Therefore $\mathrm{Exc}(R_\sigma)$ does not dominate the target of the $\mathrm{rc}(V^1)$-fibration $\pi_1 \colon X \dashrightarrow Z_1$. It follows that $\mathrm{Exc}(R_\sigma)$ does not contain $\mathrm{Locus}(V^2)$, hence $\mathrm{Exc}(R_\sigma) \cdot V^2 \geq 0$.

Let $H_1$ be the pullback of a very ample line bundle on $Z_1$. The curves parameterized by $V^1$ are contracted by $\pi_1$, so $H_1 \cdot V^1 = 0$; moreover, $H_1$ is positive outside the indeterminacy locus of $\pi_1$, so $H_1 \cdot V^2 > 0$, since $V^2$ is horizontal with respect to $\pi_1$, and $H_1 \cdot R_\sigma > 0$, since the indeterminacy locus of $\pi_1$ has codimension at least 2 in $X$ while $\mathrm{Exc}(R_\sigma)$ is a divisor.

Now denote by $V^\sigma$ a family of deformations of a minimal rational curve in $R_\sigma$. We claim that $[V^1]$, $[V^2]$ and $[V^\sigma]$ are numerically independent. Assume to get a contradiction that $[V^\sigma] \in \langle [V^1], [V^2] \rangle$, so that there exist $a, b \in \mathbb{R}$ such that $[V^\sigma] = a[V^1] + b[V^2]$. Now, by intersecting with $\mathrm{Exc}(R_\sigma)$ we obtain $b < 0$, while we have $b > 0$ by intersecting with $H_1$, so we reach a contradiction.

In particular, it follows that the curves of $R_\sigma$ are not contracted by the $\mathrm{rc}(V^1, V^2)$-fibration $\pi_2 \colon X \dashrightarrow Z_2$.

Now we show that $V^2$ is not a dominating family. Assume to get a contradiction that $V^2$ is a dominating family. We can consider $\mathrm{Locus}(V^a, V^b)_{F_\sigma}$, $\{a, b\} = \{1, 2\}$ for a general non-trivial fiber $F_\sigma$ of the contraction associated with $R_\sigma$. Notice that, since $\mathrm{Exc}(R_\sigma) \cdot V^1 = \mathrm{Exc}(R_\sigma) \cdot V^2 = 0$, any curve in $\mathrm{Locus}(V^a, V^b)_{F_\sigma}$ has negative intersection with respect to $\mathrm{Exc}(R_\sigma)$, so $\mathrm{Locus}(V^a, V^b)_{F_\sigma} \subseteq \mathrm{Exc}(R_\sigma)$. So by computing its dimension with Lemma 2.12, we get $\mathrm{Locus}(V^a, V^b)_{F_\sigma} = \mathrm{Exc}(R_\sigma)$.
By repeated applications of Lemma 2.14 the numerical equivalence class of any curve in $\mathrm{Exc}(R_\sigma)$ can be written as a linear combination with nonnegative coefficients of $[V^1]$, $[V^2]$ and $[V^\sigma]$, hence $\mathrm{NE}(\mathrm{Exc}(R_\sigma)) = \langle [V^1], [V^2], [V^\sigma] \rangle$.
Therefore the contraction associated with an extremal ray which is positive with respect to $\mathrm{Exc}(R_\sigma)$ is a $\mathbb{P}^1$-bundle by [17, Corollary 2.15], hence a family of deformations of a minimal curve in this ray is dominating, unsplit and has positive intersection with respect to $\mathrm{Exc}(R_\sigma)$, a contradiction.

Therefore $V^2$ is not a dominating family. So, for a general point $x_2 \in \mathrm{Locus}(V^2)$, we have $\dim \mathrm{Locus}(V^2, V^1)_{x_2} \geq 2i_X - 1$, hence $\dim Z_2 \leq i_X$. Therefore a general non-trivial fiber $F_\sigma$ of the blow-up associated with $R_\sigma$ dominates $Z_2$ and $i_X = \frac{\dim X+1}{3}$. It follows that $X$ is $\mathrm{rc}(V^1, V^2, V^\sigma)$-connected and $X = \mathrm{Locus}(V^2, V^1)_{F_\sigma}$, so $\langle [V^1], [V^2] \rangle$ is extremal by [16, Lemma 3.5]. Moreover, $-K_X \cdot V^1 = -K_X \cdot V^2 = -K_X \cdot V^\sigma = i_X$.

THEOREM 3.7. *Let $X$ be a Fano manifold of pseudoindex $i_X \geq \frac{\dim X+1}{3} > 2$ and Picard number $\rho_X = 3$. Assume that $X$ has an extremal ray $R_\sigma$ associated with a smooth blow-up. If $X$ does not admit any unsplit dominating family of rational curves $V$ such that* $\mathrm{Exc}(R_\sigma) \cdot V > 0$, *then $i_X = \frac{\dim X+1}{3}$ and NE$(X) = \langle R_\sigma, R_1, R_2 \rangle$, where $R_\sigma$ and $R_2$ are associated with smooth blow-ups with nontrivial $i_X$-dimensional fibers and $R_1$ is associated with a contraction of fiber type. Moreover,* $\mathrm{Exc}(R_2) \cdot R_1 > 0$.

PROOF. In view of Remark 3.5 we know that $X$ is $\mathrm{rc}(V^1, V^2, V^3)$-connected with respect to three unsplit families as in Construction 2.16, so by Lemma 3.6 $X$ is $\mathrm{rc}(V^1, V^2, V^\sigma)$-connected, where $V^\sigma$ is a family of deformations of a minimal curve in $R_\sigma$.

Let $R$ be an extremal ray which is positive on $\mathrm{Exc}(R_\sigma)$ and let $F_R$ be any nontrivial fiber of the contraction associated with $R$. Then $R \notin \langle [V^1], R_\sigma \rangle$ and there exists a nontrivial fiber $F_{R_\sigma}$ of the contraction associated with $R_\sigma$ which intersects $F_R$. It follows that $F_R \cap \mathrm{Locus}(V^1)_{F_{R_\sigma}} \neq \emptyset$. On the other hand this intersection cannot have positive dimension since the numerical equivalence class of any curve in $F_R$ belongs to $R$, while every curve contained in $\mathrm{Locus}(V^1)_{F_{R_\sigma}}$ is numerically equivalent to a linear combination of a curve among those parameterized by $V^1$ and a curve in $F_{R_\sigma}$. It follows that

$$\dim F_R \leq \dim X - \dim \mathrm{Locus}(V^1)_{F_{R_\sigma}} \leq i_X \tag{3.7.1}$$

where the last inequality follows by Lemma 2.12. Notice that $R$ cannot be associated with a contraction of fiber type, otherwise $\mathrm{Exc}(R_\sigma) \cdot R$ would be zero. By taking into account [22, Theorem 1.1] applied to $R_\sigma$, we derive $\dim F_R = i_X$ and $i_X = \frac{\dim X+1}{3}$. So, by [22, Theorem 1.1] applied to $R$, we get that $R$ is associated with a divisorial contraction. In particular, $\dim F_R = i_X = \ell(R)$, so the contraction associated with $R$ is a smooth blow-up by [2, Theorem 5.1].

Note that, in view of (3.7.1), this implies $\dim F_{R_\sigma} = i_X$.

Now, if $\mathrm{Exc}(R) \cdot V^1 = 0$, then $\mathrm{Exc}(R) \cdot V^2 < 0$, since $X$ is $\mathrm{rc}(V^1, V^2, V^\sigma)$-connected and $\mathrm{Exc}(R) \cdot V^\sigma > 0$. On the other hand, we can argue as in the first lines of the proof of Lemma 3.6 by replacing $R_\sigma$ with $R$; then we get $\mathrm{Exc}(R) \cdot V^2 \geq 0$. So we have a contradiction.

Therefore $\mathrm{Exc}(R)\cdot V^1 > 0$, so we can conclude by Lemma 3.2 and Theorem 3.3.

## 4. Examples

Let $X$ be a Fano manifold of pseudoindex $i_X \geq \frac{\dim X+1}{3} \geq 2$ and Picard number $\rho_X = 3$. We know from [22, Theorem 1.1] that, if $X$ has an extremal ray associated with a smooth blow-up, the dimension of the non-trivial fibers is greater than or equal to the pseudoindex of $X$.

We start this section with an example of $X$ of pseudoindex $i_X > \frac{\dim X+1}{3}$ admitting an extremal ray associated with a smooth blow-up.

EXAMPLE 4.1. *Consider $X = \mathbb{P}^{i_X-1} \times \mathrm{Bl}_{\mathbb{P}^{i_X-2}}\mathbb{P}^{2i_X-1}$. This is a Fano manifold of pseudoindex $i_X = \frac{\dim X+2}{3} \geq 2$, Picard number $\rho_X = 3$ that admits a blow-up structure.*

REMARK 4.2. Notice that, in view of [16, Theorem], the variety in Example 4.1 is the only Fano manifold of pseudoindex greater than $\frac{\dim X+1}{3}$, Picard number $\rho_X = 3$ that admits a blow-up structure.

Now we give an example of $X$ admitting an extremal ray associated with a smooth blow-up whose non-trivial fibers have dimension greater than the pseudoindex of $X$.

EXAMPLE 4.3. *Consider $X = \mathbb{P}^{i_X-1} \times \mathrm{Bl}_{\mathbb{P}^{i_X-2}}\mathbb{P}^{2i_X}$. This is a Fano manifold of pseudoindex $i_X = \frac{\dim X+1}{3} > 2$, Picard number $\rho_X = 3$ that admits a blow-up structure with fibers of dimension equal to $i_X + 1$.*

We remark that the variety in Example 4.3 is the only Fano manifold of dimension $3i_X - 1$, Picard number $\rho_X = 3$ admitting an extremal ray associated with a smooth blow-up whose fibers have dimension greater than the pseudoindex of $X$. This is proved in the following

PROPOSITION 4.4. *Let $X$ be a Fano manifold of pseudoindex $i_X \geq \frac{\dim X+1}{3} > 2$ and Picard number $\rho_X = 3$. Assume that $X$ has an extremal ray $R_\sigma$ associated with a smooth blow-up. Then the non-trivial fibers of the contraction associated with $R_\sigma$ are $i_X$-dimensional unless $X = \mathbb{P}^{i_X-1} \times \mathrm{Bl}_{\mathbb{P}^{i_X-2}}\mathbb{P}^{2i_X}$.*

PROOF. By [22, Theorem 1.1] we know that the dimension of each non-trivial fiber of the contraction associated with $R_\sigma$ has dimension greater than or equal to $i_X$. Assume that these fibers have dimension greater than $i_X$. By Theorem 1.1 we know that $\mathrm{NE}(X) = \langle R_\sigma, R_1, R_2\rangle$, where $R_1$ is associated with a contraction of fiber type and $R_2$ is associated either with a contraction of fiber type or with a smooth blow-up. However, the last case is ruled out by taking into account Lemma 3.2 and Theorem 3.7.

With no loss of generality we can now assume that $\mathrm{Exc}(R_\sigma) \cdot R_1 > 0$ and we can argue as in the proof of Lemma 3.2. Denote by $F_2$ any fiber of the contraction associated with $R_2$ and denote by $V^1$ a family of deformations of a minimal curve in $R_1$. Since $V^1$ is a dominating family and $\mathrm{Exc}(R_\sigma) \cdot V^1 > 0$, we have

$$\mathrm{Exc}(R_\sigma) \cap \mathrm{Locus}(V^1)_{F_2} \neq \emptyset,$$

hence there exists a fiber $F_\sigma$ of the blow-up associated with $R_\sigma$ such that

$$F_\sigma \cap \mathrm{Locus}(V^1)_{F_2} \neq \emptyset. \tag{4.4.2}$$

Now, the numerical equivalence class of any curve in $F_\sigma$ belongs to $R_\sigma$, while every curve contained in $\mathrm{Locus}(V^1)_{F_2}$ is numerically equivalent to a linear combination of a curve among those parameterized by $V^1$ and a curve in $F_2$; it thus follows that the intersection (4.4.2) is 0-dimensional. So we get

$$\dim X \geq \dim F_\sigma + \dim \mathrm{Locus}(V^1)_{F_2} \geq \dim F_\sigma + \dim F_2 - K_X \cdot V^1 - 1, \tag{4.4.3}$$

where the last inequality follows by Lemma 2.12. By taking into account [22, Theorem 1.1] applied to $R_\sigma$, from (4.4.3) we derive $\dim F_2 \leq i_X - 1$. So, by [22, Theorem 1.1] applied to $R_2$, we get that $\dim F_2 = i_X - 1$. Therefore a general fiber of the contraction associated with $R_2$ dominates the target of the $\mathrm{rc}(V^1, V^\sigma)$-fibration, hence $X$ is $\mathrm{rc}(V^1, V^\sigma, V^2)$-connected, where $V^2$ is a family of deformations of a minimal curve of the contraction associated with $R_2$. Notice that $-K_X \cdot V^1 = -K_X \cdot V^2 = i_X$ and $-K_X \cdot V^\sigma = i_X + 1$, so we get $X = \mathbb{P}^{i_X - 1} \times \mathrm{Bl}_{\mathbb{P}^{i_X-2}} \mathbb{P}^{2i_X}$ by [16, Proposition 5.2].

Next we consider Fano manifolds with two blow-ups structures.

Example 4.5. *Consider* $X = \mathrm{Bl}_{\mathbb{P}^{2i_X-2}}(\mathrm{Bl}_{\mathbb{P}^{i_X-2}}\mathbb{P}^{3i_X-1})$. *This is a Fano manifold of pseudoindex* $i_X = \frac{\dim X + 1}{3} > 2$, *Picard number* $\rho_X = 3$ *that admits two blow-up structures.*

Moreover, we show that

Proposition 4.6. *Let* $X$ *be a Fano manifold of pseudoindex* $i_X = \frac{\dim X+1}{3} > 2$ *and Picard number* $\rho_X = 3$. *Assume that* $X$ *admits two extremal rays associated with smooth blow-ups onto Fano manifolds. Then* $X = \mathrm{Bl}_{\mathbb{P}^{2i_X-2}}(\mathrm{Bl}_{\mathbb{P}^{i_X-2}}\mathbb{P}^{3i_X-1})$.

Proof. In view of Theorem 1.1, we know that $\mathrm{NE}(X) = \langle R, R_\sigma, R_{\sigma'} \rangle$, where $R_\sigma$ and $R_{\sigma'}$ are associated with smooth blow-ups and $R$ is associated with a contraction of fiber type. Moreover, the intersection of the exceptional loci of the two blow-ups is not empty. In fact, if $\mathrm{Exc}(R_\sigma) \cdot R = 0$ (resp., $\mathrm{Exc}(R_{\sigma'}) \cdot R = 0$) then $\mathrm{Exc}(R_\sigma) \cdot R_{\sigma'} > 0$ (resp., $\mathrm{Exc}(R_{\sigma'}) \cdot R_\sigma > 0$) since an effective divisor is positive on an extremal ray, while if $\mathrm{Exc}(R_\sigma) \cdot R > 0$ and $\mathrm{Exc}(R_{\sigma'}) \cdot R > 0$ then $\mathrm{Exc}(R_\sigma) \cdot R_{\sigma'} > 0$ in view of the proof of Theorem 3.3.

Moreover, in view of Theorem 3.7, with no loss of generality we can assume that $\mathrm{Exc}(R_\sigma) \cdot R > 0$.

Denote by $V^R$, resp. $V^\sigma$, resp. $V^{\sigma'}$, a family of deformations of a minimal curve in $R$, resp. $R_\sigma$, resp. $R_{\sigma'}$. Then $V^\sigma$ is horizontal and dominating with respect to the $\mathrm{rc}(V^R)$-fibration and $V^{\sigma'}$ is horizontal and dominating with respect to the $\mathrm{rc}(V^R, V^\sigma)$-fibration. By computing the dimensions with Lemma 2.12, we derive $\dim \mathrm{Locus}(V^R)_{F_\sigma} = 2i_X - 1$, where $F_\sigma$ is any non-trivial fiber of the contraction associated with $R_\sigma$.

Let $\Phi$ be the contraction associated with the extremal face $\langle R, R_\sigma \rangle$ and denote by $F$ a general fiber of $\Phi$. We have the following diagram:

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[dr, "\Phi"'] & X' \arrow[d, "\psi"] \\ & Y, \end{tikzcd}$$

where $\sigma$ is the blow-up associated with $R_\sigma$. Then $F$ contains $\mathrm{Locus}(V^R)_{F_\sigma}$, so it has positive intersection with a nontrivial fiber of $R_{\sigma'}$. It follows that $\dim F = 2i_X - 1$, so $F$ is a Fano manifold of pseudoindex equal to $i_X$ with two extremal rays, one of which is associated with a blow-up with $i_X$-dimensional nontrivial fibers. Therefore $F = \mathrm{Bl}_{\mathbb{P}^{i_X-2}}\mathbb{P}^{2i_X-1}$ by [3, Theorem 1.1]. Then $\psi$ is a contraction of fiber type with $\mathbb{P}^{2i_X-1}$ as general fiber. Therefore $X' = \mathrm{Bl}_{\mathbb{P}^{i_X-2}}\mathbb{P}^{3i_X-1}$ by [3, Theorem 5.1], so $X = \mathrm{Bl}_{\mathbb{P}^{2i_X-2}}(\mathrm{Bl}_{\mathbb{P}^{i_X-2}}\mathbb{P}^{3i_X-1})$.

Remark 4.7. For examples of Fano manifolds of pseudoindex $i_X = \frac{\dim X+1}{3} = 2$ (hence Fano manifolds of dimension 5) we refer to the classification table in [17, Appendix] and to the examples in [7, Section 3].

Next we consider Fano manifolds with two elemetary contractions of fiber type, one of them being associated with an extremal ray of length greater than $i_X$.

Example 4.8. *Consider* $X_1 = \mathbb{P}^{i_X-1} \times \mathrm{Bl}_{\mathbb{P}^{i_X-1}} \mathbb{P}^{2i_X}$ *and* $X_2 = \mathbb{P}^{i_X} \times \mathrm{Bl}_{\mathbb{P}^{i_X-2}} \mathbb{P}^{2i_X-1}$. *These are Fano manifolds of pseudoindex* $i_X = \frac{\dim X+1}{3} \geq 2$*, Picard number* $\rho_X = 3$ *that admits a blow-up structure with fibers of dimension equal to* $i_X$ *and two elemetary contractions of fiber type, one of which being associated with an extremal ray of length greater than* $i_X$*.*

Remark 4.9. Notice that, in view of [16, Proposition 5.2 and Proposition 5.4], the varieties in Example 4.8 are the only Fano manifolds of pseudoindex equal to $\frac{\dim X+1}{3} \geq 2$, Picard number $\rho_X = 3$ that admit a blow-up structure with fibers of dimension equal to $i_X$ and two elemetary contractions of fiber type, one of them being associated with an extremal ray of length greater than $i_X$. Notice also that, if the extremal ray of length greater than $i_X$ is $R_\sigma$, we have $X = \mathbb{P}^{i_X-1} \times \mathrm{Bl}_{\mathbb{P}^{i_X-2}} \mathbb{P}^{2i_X}$ by Proposition 4.4.

An example in which all the extremal rays of $X$ have length equal to $i_X$ is the following:

EXAMPLE 4.10. *Consider* $X = \mathbb{Q}^{i_X} \times \mathrm{Bl}_{\mathbb{P}^{i_X-2}} \mathbb{P}^{2i_X-1}$. *These is a Fano manifolds of pseudoindex* $i_X = \frac{\dim X+1}{3} \geq 2$, *Picard number* $\rho_X = 3$ *that admits a blow-up structure with fibers of dimension equal to* $i_X$ *and two elemetary contractions of fiber type, all of them being associated with extremal rays of length equal to* $i_X$.

Finally, we consider a Fano manifold $X$ admitting an extremal ray associated with a smooth blow-up and Picard number $\rho_X \geq 4$.

EXAMPLE 4.11. *Consider* $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{Bl}_p(\mathbb{P}^3)$. *This is a Fano manifold of pseudoindex* $i_X = \frac{\dim X+1}{3} = 2$, *Picard number* $\rho_X = 4$ *that admits an extremal ray associated wit a blow-up and* 3 *extremal rays associated with contractions of fiber type.*

REMARK 4.12. Notice that, in view of [16, Proposition 4.1 and Proposition 5.1], the variety in Example 4.11 is the only Fano manifold of pseudoindex greater than or equal to $\frac{\dim X+1}{3} \geq 2$, Picard number $\rho_X \geq 4$ that admits a blow-up structure.

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