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Exploring non-prototypical configurations of equivalent areas through inquiring-game activities within DGE

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Inquiring-game activities are inspired to Hintikka’s semantical games and are designed within a Dynamic Geometric Environment (DGE). In this report, we focus on inquiring-game activities on the equivalence of quadrilaterals areas, which were proposed to primary school students in Italy. The goal of these activities is to exploit the dynamicity of semantical games and of DGEs to foster students’ geometrical thinking. Through a design-based approach and a case-study, we highlight how the double dynamicity of the didactical design allows the students to i) explore non-prototypical configurations and ii) reinvest their geometrical knowledge to make sense of the observed relationships between the figural and the numerical register.

Keywords: Non-prototypical figures, inquiring-games, area equivalence, primary school, DGEs.

Introduction and theoretical framework

In primary school the concept of area and its measurement are among the main goals of learning in all countries. Area measurement is usually the first case in which students encounter numerical computation in geometry (Smith & Barrett, 2017): first by counting units, and then by multiplicative computations. Comparing areas of different regions can be perceptively difficult when the shapes are different or they are not prototypical. Usually, students learn to establish the equivalence of figures both by using measurements and by decomposing them into congruent polygons, by cutting and pasting figures, folding papers, and drawing in paper and pencil environment. In this context, algebraic formulas are also introduced. However, research and international tests show that formulas often remain obscure or even ‘black boxes’ to many students (ibid.). Furthermore, students show difficulties in distinguishing between area and perimeter (D’Amore & Fandiño Pinilla, 2006).

In our study, we explore how inquiring-game activities within Dynamic Geometric Environments (DGEs) may foster students’ geometrical reasoning, in particular on area and area equivalence. The study is part of a larger research project in which theorems of elementary geometry are transposed into inquiring-game activities inspired to semantical games (Soldano & Arzarello, 2018). These games are connected with a new form of logic, called ‘logic of inquiry’, developed by the Finnish logician J. Hintikka (Hintikka, 1999). In this logic, Hintikka made use of games of verification/falsification, known as semantical games (Hintikka, 1998), through which he reversed the recursive definition of truth given by Tarski.

For reason of space, we will refer only to the semantical games associated to statements expressed in the form $\exists x \ E y \ S(x, y)$, since we took inspiration from them to design our games. To this end, we can imagine a situation in which a verifier (V) has the goal of showing the truth of the statement while a falsifier (F) has the opposite goal of showing that it is false. F controls variable $x$ and V controls variable $y$. F starts the game by choosing a value $x_0$ for the variable $x$ and then the turn moves to V,
who should find a value \( y_0 \) for \( y \) such that \( S(x_0, y_0) \) is true. In this game, F tries to create difficulties to V by choosing non-typical cases (such as extreme cases). According to Hintikka, the choice of the \( y_0 \) by V is a reliable test of truth if the \( x_0 \) chosen by F forces V to play in the worst-case scenarios. This game between V and F hence creates a dynamic interplay between them, paralleled by a dynamism at the cognitive level. Such dynamism is assumed to trigger students’ inquiring process and foster their understanding of the mathematical situation.

Within DGEs, variables \( x \) and \( y \) are thought as base-points of geometric constructions. Since dragging preserves the critical attributes of robust constructions (Healy, 2000), by moving the base-points new configurations of the same robust shape can be explored. As it is well known from the literature, students tend to identify shapes with the prototypical examples representing the category (Rosch, 1973), e.g., they easily recognize figures when one of their sides is horizontal. Goldenberg and Mason (2008) underline the importance for students of dealing with many and different examples, since they are “a major means for ‘making contact’ with abstract ideas and a major means of mathematical communication” (p. 184) and “the variation in examples can help learners distinguish essential from incidental features” (ibid.). Variation can also produce the enlargement of students’ personal example space (Watson & Mason, 2005), namely the “set of mathematical objects and construction techniques that a learner has access to as examples of a concept while working on a given task” (Sinclair, Watson, Zazkis & Mason, 2011, p.1). DGEs may facilitate students’ accessibility to non-prototypical examples, thanks to the dynamic exploration of the figures. Non-examples serve to clarify “the boundaries or necessary condition of a concept” and can play the role of counterexample if they are used to “show that a conjecture is false” (Watson & Mason, 2005, p. 65). If a geometric property is not robustly constructed, namely the construction does not retain the critical attributes of the properties, DGE offers the possibility to produce both examples and non-examples of the considered property.

Generally, DGEs provide tools for computing an approximate value of the area of the constructed figures. If this value is made visible in the environment, dragging the base-points of a figure produces a double change: one within the figural register of representation and the other within the numerical one. Duval (2006) highlights two different operations with registers of representation, which he calls treatment and conversion. Both are fundamental for the understanding of mathematical concepts: treatments deal with transformations within the same register of representation, while conversions consider more than one register and the relationships between them. Revisiting Duval theory within DGEs, dragging of a robust figure can be interpreted as a treatment made within the figural register, while the coexistence of figural and numerical registers of representation may engage students to inquire their links by making conversions.

Methodology

Following a design-based perspective (DBRC, 2003), we carried out a teaching-experiment in two Italian 5th grade classrooms with the collaboration of their school teachers and of a master’s degree student. We designed a sequence of five DGE inquiring-game activities centered on area equivalence and on the area/perimeter relationship of quadrilaterals. In each activity, pairs of students are invited to play a game in DGE, and to answer to specific questions that guide the inquiry of the geometric
concepts. After all students write their answers to the questions, a classroom discussion is developed under the teacher’s guidance. Usually, two hours are dedicated for each inquiring-game activity (including discussion), for an amount of 10 hours.

In this contribution we present a case study from the first activity, focusing on the area equivalence between rectangles and parallelograms. The inquiring game and the worksheets for students are presented below. In previous lessons, students have explored the area and perimeter of quadrilaterals using paper models (cutting and pasting) and in paper&pencil environment. The collected data consist of the audio and screen capture of one pair of students (that we will call Rose and Lily) during the DGE inquiring-game activity, the completed worksheets from all the pairs of students and the video-recording of the classroom discussion. The collected data are analysed according to a semiotic multimodal approach to mathematics (Arzarello, Paola, Robutti & Sabena, 2009), which focuses on students’ speech, gestures, drawings and actions within the DGE. Through the produced signs it is possible to grasp how students discover and make sense of the geometric property on which the game is based. More precisely our interest is to explore the potentials and limits of the didactic design based on the double dynamicity provided by the game and by the DGE affordances, with respect to students’ geometrical reasoning in non-prototypical configurations.

The inquiring-game activity on parallelogram-rectangle area equivalence

The activity focuses on the relationship between the area of a rectangle and a parallelogram that share one side (named AB): the corresponding game starts from the configuration shown in Figure 1.

![Figure 1: The initial configuration of the parallelogram-rectangle game](image)

Both the parallelogram and the rectangle figures are robust constructions; by moving the free point A, it is possible to change the length of AB and rotate it around the fixed-point B (see Fig. 2a). This move changes also the values of the area of the figures, which are displayed on the screen. By moving point C, it is possible to drag the parallelogram and vary its area while the side AB remains fixed (see Fig. 2b). By dragging point E it is possible to vary the length of EB and the area of the rectangle ABEF. Since the figures share the same side AB, if the heights relative to AB are of the same measure the two figures are equivalent. A key-feature of the design is that this property is not constructed in a robust way, so to allow students to produce examples and non-examples of the property, through the verifier and falsifier’s moves. The dotted straight line (the extension of the side AB) is inserted to
help students to visualize the correct position of the height relative to AB of the parallelogram. All draggable points can be moved in either half-plane identified by the line.

![Figure 2: The effects of dragging point A, C and E](image)

The following table contains the English translation of the rules of the game:

<table>
<thead>
<tr>
<th>Within your pair, choose a verifier and a falsifier.</th>
</tr>
</thead>
<tbody>
<tr>
<td>- The falsifier can move point A or C,</td>
</tr>
<tr>
<td>- The verifier moves point E.</td>
</tr>
<tr>
<td>Each match is made of two moves and the first one is always made by the falsifier.</td>
</tr>
</tbody>
</table>

**GOALS:**

The goal of the verifier is to make Area EBEF equal to Area ABCD, while the goal of the falsifier is to prevent the verifier from reaching the goal.

The player who reaches her/his goal at the end of the verifier’s move wins the match.

**Table 1: Rules of the parallelogram-rectangle game**

According to the rules of the game, the verifier and the falsifier play a semantical game on the following statement: ‘For all positions of point A and C there exists a position of point E such that Area ABEF is equal to Area ABCD’. The verifier can always win, by transforming any non-example of equivalent figure (i.e. rectangle non-equivalent to parallelogram, as in Figure 1) into an example of equivalent one (as shown in Figure 3). In agreement with the teacher we decided to round the area measure to the first decimal place.

![Figure 3: Examples of game configurations showing equivalent areas](image)
The players’ moves produce a double simultaneous treatment: one in the figural register, since each figure is transformed into another one of the same category, and one in the numerical register since also area values change. The following inquiring questions are proposed to students:

<table>
<thead>
<tr>
<th>Answer these questions trying to use geometrical language:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Consider the case in which the verifier is winning. Write here your observations.</td>
</tr>
<tr>
<td>2. Explain why each time the verifier reaches the goal area ABEF is equal to area ABCD.</td>
</tr>
<tr>
<td>3. While playing, have you ever found a situation in which the two areas are zero? If not, play a new match to get it and describe what happened to the figures. If you like, make a drawing.</td>
</tr>
</tbody>
</table>

**Table 2: Questions proposed to students**

The guiding questions are meant to shift students’ attention from the game to the geometric properties on which the game is based. Through the first one we collect students’ observations, without giving them any hint. The second question focuses on area equivalence, that is the geometric property on which the game is based. It is meant to know whether students are able to convert the results observed in the figural and in the numerical registers using the known geometric properties. More precisely, students can provide two mathematically correct explanations: the first relies on the geometric interpretation of the area formula, the second one is made by showing that the two figures are made by the same congruent polygons (decomposition). Primary students are not expected to completely justify the congruence of the figures in a rigorous way, but chunks of arguments are expected to emerge. Finally, the last question requires students to produce a degenerate configuration and is meant both to investigate their conceptions of these cases, and to foster geometrical reasoning in non-prototypical configurations.

**Data analysis**

We report about Rose and Lily’s inquiry in the game and in answering the worksheet questions. According to the teacher, they are medium-level students; Rose is reflective and her mathematical knowledge is a little bit higher then Lily, who is less reflective and is smarter in more practical activities. They play the game for about 40 minutes and then they turn to answer the worksheet questions. During the game it happens that Lily wins all the matches both when she plays the verifier’s role and when she plays the falsifier’s. While playing as falsifier, Lily produces non-prototypical configurations (very big and stretched or very small, see as examples matches 3 and 4 reported in Fig. 4) in which it is very difficult for the verifier reaching the goal (sometimes it is impossible for approximation reasons).

<table>
<thead>
<tr>
<th>3rd match</th>
<th>4th match</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="3rd_match.png" alt="Image" /></td>
<td><img src="4th_match.png" alt="Image" /></td>
</tr>
</tbody>
</table>

**Figure 4: Falsifier ‘winning’ configuration (3rd and 4th matches in Rose and Lily’s game)**

\[
\begin{align*}
\text{Area } ABCD &= 0.5, & \text{Area } ABEF &= 0.1, \\
\text{Area } ABCD &= 62.4, & \text{Area } ABEF &= 54.4
\end{align*}
\]
While playing the verifier’s role, Rose notices that her defeats depend on technical aspects of the DGE, namely on the fact that after zooming out many times area measures are expressed by quite big numbers, even if the figures look as big as usual (see Fig. 4). In this condition, a little dragging produces a huge variation in the numerical values and it is very difficult for the verifier to win. Students’ first observations relate to tool affordances rather than geometric properties. They do not even call the figures with their names until the researcher (who is videotaping them, first author) asks them to do it. In doing so, Rose immediately answers correctly, but Lily needs to drag the configuration and to produce prototypical examples of rectangle and parallelogram. From this moment on, Lily and Rose focus on geometrical features. We report an excerpt in which students are discussing the configuration contained in Fig. 5a:

1. Lily: This (pointing to the parallelogram in Fig. 5a) is bigger than this (pointing to the rectangle in Fig. 5a) but they have the same area
2. Rose: You mean that it has a bigger shape?
3. Lily: Yes, but they have the same area, they are equivalent with different measures.
4. Rose: With different sizes. And why do you believe it happens?
5. Lily: I have no idea… Or maybe here it’s called area, but it means the perimeter… no it’s impossible…
6. Rose: But Lily, if you look at this side here, the one overlapped (pointing to AB), it is equal to this side here (pointing to CD), but this is slopped. This one (pointing to EF) is equal to this one (AB) and this one (CD). Do you think that this height (with her finger on the screen, she is tracing the height relative to AB of the parallelogram, from C to the dotted line) is equal to this one (pointing to AF)?

![Figure 5: Configurations discussed by Lily and Rose](image)

Lily and Rose highlight a perceptual problem that creates a contradiction: the parallelogram looks bigger than the rectangle and this perceptual observation collides with the fact that they have equivalent areas (lines 1-3). Namely, Lily and Rose observe a contradiction between the figural and the numerical registers (Duval, 2006) due to a visualization problem. In order to solve the contradiction, Rose relies on the algebraic formulas of the areas, which she interprets geometrically on the figures (see her words and gestures in line 6, Fig. 5).

Being the figures in opposite half-planes, it is difficult to visually compare their heights relative to AB (see Rose’s last question, line 6). Hence, the researcher suggests considering configurations in
which both figures are in the same half-plane. Following the researcher’s suggestion, the students make a treatment in the figural register and discuss the configuration shown in Fig. 5b:

7. Rose: What does it happen? This one (pointing to the rectangle), the rectangle…

8. Lily: Is the half of the other [the parallelogram]

9. Rose: How is it possible? If we put this triangle (pointing to BCE) here (pointing to ADF), we make the whole rectangle. Instead here we have only a piece of parallelogram (moving her finger on BE). This triangle which lies outside (pointing to ADF) is this triangle (pointing to BCE)! Do you understand, Lily?

When looking at Fig. 5b, Rose does not continue in discussing the heights relative to AB, rather she observes that areas are equivalent since they may be obtained by the same congruent polygons (line 9). The imagined treatment in the figural register, based on decomposing and rearranging the figures, allows Rose solving the previous contradiction between the numerical and the figural register. In this way, she avoids referring to the parallelogram height relative to AB, which is not drawn and is not perceptually easy to seize (because the figure is slanted) nor to compare with the rectangle height. Having shown the equivalence of the figures, Rose moves back to the previous height observation and concludes that “if the area is the same then heights should be equal”.

**Conclusion**

As highlighted in the analysis, it took time for the videotaped students to start a geometric discussion of the game and the researcher had to prompt it through some questions. Anyway, the provided chunk of argumentations revealed that the design worked well: the students made sense of the equivalence between parallelogram and rectangle with arguments that exploited both the areas’ formulas and the decomposition of the figures into congruent polygons.

While playing, students produced non-prototypical examples of parallelograms and rectangles. The non-prototypicalities can be related to different aspects: the figures orientation (the produced configurations do not have the base in a horizontal position); the figures size (the explored area values are very big or very small non-integers); the figures proportions (the produced configurations are stretched and the ratio between their sides is unusual). Moreover, a second order of non-prototypicalities should be considered. It refers to configurations of parallelogram-rectangle equivalence in which the figures belong to opposite half-plane (with respect to line AB) or in which the height of the parallelogram referred to side AB lies on the extension of AB. In these cases, it is more difficult or even impossible for students to imagine the decomposition and rearrangement of the showed figures into each other.

Some non-prototypicalities were not considered in the design and they actually did not help students in linking the game situation and the mathematical property at stake. Rose and Lily do not perceive that the parallelogram is equivalent to the rectangle in a case of second order non-prototypicality: the parallelogram, which was not overlapped by the rectangle, looks bigger than the rectangle (lines 1-3). On a design-based perspective, this result will be considered in the second cycle of experimentation.
The exploration of non-prototypical configurations is the effect of the double dynamicity characterizing the inquiring-game activity, namely the one provided by the verifier/falsifier game and the one given by the DGE affordances. Implications of this double dynamicity seem promising with respect of developing students’ critical thinking and argumentation competences.

**Acknowledgment**

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**References**


