**Parts of Structures**

# Parts of Structures

**Abstract:** We contribute to the ongoing discussion on mathematical structuralism by focusing on a question that has so far been neglected: when is a structure part of another structure? This paper is a first step towards answering the question. We will show that a certain conception of structures, abstractionism about structures, yields a natural definition of the parthood relation between structures. This answer has many interesting consequences; however, it conflicts with some standard mereological principles. We argue that the tension between abstractionism about structure and classical mereology is an interesting result and conclude that the mereology of abstract structures is a subject that deserves further exploration. We also point out some connections between our discussion of the mereology of structures and recent work on non-well-funded mereologies.

**Keywords**: Structuralism, Mereology, Abstraction

## 1. Structures as abstractions

Mathematical structuralism, the view often summarized by the slogan that "mathematics is the science of structures", has received considerable attention in contemporary philosophy of mathematics (see e.g. Benacerraf 1965, Awodey 2014, Shapiro 1997, Resnik 1981, Hellman 1989, Leitgeb 2020a,b). One issue that divides advocates of different brands of structuralism is how to understand talk of structures: are structures special kinds of objects or is talk of structures just a convenient way to talk about systems of objects? *Eliminativist* structuralists like Hellman favour the second option, *non-eliminative* structuralists like Resnik, Shapiro, and Leitgeb the first. In this paper, we offer a new perspective on this debate, focusing on a question that is rarely addressed explicitly: assuming that structures are objects, how should we define the part-whole relation among structures? Our discussion will reveal some fruitful interactions between the debate on mathematical structuralism and recent work on non-standard mereologies (Cotnoir 2010, Cotnoir and Bacon 2012, Kearns 2011, Effingham 2010).

We will see that a certain way to conceive of structures suggests an apparently plausible and natural answer to the question of when a structure is part of another structure. This answer, however, has surprising consequences.

There are many reasons why the question of when a structure is part of another structure is an important one. One is that in the case of a prototypical example of abstract mathematical objects, i.e. classes, there is a standard answer to the question of what are their parts: the parts of a class are its subclasses (Lewis 1991). It is natural to wonder whether something similar holds for structures too: are the parts of a structure its substructures? When is a structure a substructure of a larger structure? Different conceptions of structures might yield different answers to those questions. It might be useful to compare these answers and test their plausibility.

More generally, just as it is natural to claim "[T]here is no room for a view that takes structures seriously as objects and leaves the identity relation between structures indeterminate." (Shapiro 2006, p. 140), one might claim that "there is no room for a view that takes structures seriously as objects and leaves the *parthood* relation between structures indeterminate".

A further reason to find the question of what makes a structure a part of a larger structure interesting is that we already know that one conception of structures has problems explaining their mereological composition. David Lewis' attacked structured universals (Lewis 1986) because they violate standard mereological principles. Hence, the conception of abstract mathematical structures as structured universals conflicts with classical mereology. In this paper, we will not assume that abstract mathematical structures are structured universals but will rather adopt a different conception of abstract mathematical structures. We will show that also this conception is in conflict with classical mereology.[1] This suggests that the problem of accounting for the mereology of structures cannot simply be solved by denying that they are structured universals.

Finally, sometimes mathematicians make some surprising claims about structures. Kurt Gödel, in a passage that we will comment on section 3, claims that some structures are proper parts of themselves. Gödel's claim seems to contradict the standard mereological principle that proper parthood is irreflexive (a proper part is a part that is *distinct* from the whole). We will show that within a certain conception of structures (what we call "abstractionism about structures") there is a natural way to define the parthood relation among structures that vindicates Gödel's claim. However, we also point out that the natural definition of parthood between structures that vindicates Gödel's claim violates standard mereological principles. We submit that this is interesting and good to know.

Let us start by exposing the conception of structures that we provisionally assume as a guide in our search for a definition of parthood among structures. We will label this conception "abstractionism about structures". Shapiro defines a structure as "the abstract form of a system [of objects]" (Shapiro 1997: 74): for him, a system stands to its structure as a letter token stands to its type (Shapiro 1997: 84, see also Linnebo and Pettigrew 2014). The conception of structures as the abstract form of systems gives us a clue about how to understand the relation of parthood between structures. If structures are introduced via a certain abstraction principle, this helps with the problem of defining parthood for structures: every time a domain of abstract objects is introduced by abstraction over a domain of objects $D$, there is a natural way to use a relation on $D$ to define a relation between the newly introduced abstract objects.

[1] See Linnebo 2017, 166-7, for a comparision between the conception of structures that we analyse in this paper (abstractionism about structures) and the view that structures are structured universals.

Let $Abs(x)$, $Abs(y)$ be abstract objects introduced via an abstraction principle of the following form, where $\sim$ is an equivalence relation on a domain of more "concrete" objects, the range of the variables $x$ and $y$:

$$Abs(x) = Abs(y) \leftrightarrow x \sim y$$

A natural way to define a relation on the domain of $Abs(x)$ and $Abs(y)$ is to start with a relation $R$ on the domain of $x$ and $y$ such that $\sim$ is a congruence over $R$ (for all $x, x'$ and $y, y'$: $x \sim x' \wedge y \sim y' \rightarrow (Rxy \leftrightarrow Rx'y')$) and define the relation $R^*$ on the domain of $Abs(x)$ and $Abs(y)$ by setting: $R^*\big(Abs(x), Abs(y)\big) \leftrightarrow R(x, y)$. A standard illustration of this procedure is the case where $D$ is the set of lines, $\sim$ is parallelism (∥), $Abs(a)$ is the direction of line $a$, and the relation of orthogonality among directions ($\perp^*$) is defined from the orthogonality relation among lines ($\perp$):

$$\mathrm{d}(a) \perp^* d(b) \leftrightarrow a \perp b$$

This way of defining the orthogonality relation among directions from the orthogonality relation among lines is admissible precisely because parallelism, the equivalence relation figuring in the abstraction principle for directions ($d(x) = d(y) \leftrightarrow x \parallel y$), is a congruence with respect to orthogonality: the relation does not distinguish between parallel lines, so if $x \parallel x'$ and $y \parallel y'$, then $x \perp y \leftrightarrow x' \perp y'$.

Here is the abstraction principle encapsulating the conception of structures as abstract forms of systems (Linnebo and Pettigrew 2014 and Awodey 2014):

*Frege's Abstraction*: $[S] = [S']$ if and only if $S$ is isomorphic to $S'$[2]

$S$ and $S'$ are systems of objects, i.e. sequences $\langle D, \mathrm{R}_1, \dots, \mathrm{R}_\mathrm{n} \rangle$ where $D$ is a set and each $R_i$ is a relation on $D$ (Linnebo and Pettigrew 2014); $[S]$ denotes the structure of $S$; $S$ is isomorphic to $S'$ if there is a bijective function $f: D \rightarrow D'$such that for all $x \dots, x_n \in D$ and all $R_i$: $R_i(x_1, \dots, x_n) \leftrightarrow R_i'(fx_1, \dots, fx_n)$.

## 2. Parts of structures as structures of parts

To define parthood among structures using the method exposed above, we need to find a relation among systems such that isomorphism is a congruence with respect to that relation.

To simplify our discussion, from now on we will deal with just one kind of systems of objects: *graphs*, i.e. ordered pairs $\langle D, R \rangle$, where $D$ is a set and $R$ is a binary relation on $D$.[3] A natural way to define

[2] See Linnebo and Pettigrew (2014: 274) on how to formulate *Frege's abstraction* consistently, avoiding the Burali-Forti paradox.
[3] Graphs are important systems of objects. Other systems of objects, like lattices, can be represented by a graph, their Hasse diagram. See also Leitgeb (2020a, 2020b).

parthood among graphs is to let the parts of a graph $G$ be the subgraphs of $G$ ($G' = \langle D', R' \rangle$ is a subgraph of $G = \langle D, R \rangle$ just in case $D' \subseteq \mathrm{D}$ and $R' \subseteq R$). An alternative choice is to identify the parts of $G$ with the induced subgraphs of $G$ ($G' = \langle D', R' \rangle$ is an induced subgraph of $G = \langle D, R \rangle$ just in case $D' \subseteq \mathrm{D}$ and $R'$ is the restriction of $R$ on $D'$). The problem is that neither the relation of being a subgraph nor the relation of being an induced subgraph are such that isomorphism is a congruence with respect to them.[4]

Luckily, a close cousin of the relation of (induced) subgraph is such that isomorphism is a congruence with respect to that relation: the relation of *embedding* ($\hookrightarrow$). To see how embedding relates to the notion of being an induced subgraph, define embedding in this way: G is embeddable in G' iff G is isomorphic to an induced subgraph of G'.[5] If take embedding as the relation for systems of objects in terms of which we define parthood among structures following the procedure described above, we obtain:

$$(PS)\ [S] \sqsubseteq [S'] \leftrightarrow \mathrm{S} \hookrightarrow S'$$

Here is an interesting perspective on $PS$. Go back to the idea that the parts of a graph are its induced subgraphs. S is embeddable in S' exactly when S is isomorphic to (i.e., has the same structure as) an induced subgraph of S'. We can then paraphrase $(PS)$ in this way: *the parts of the structure of a system are the structures of the parts of that system*.

## 3. Conflict with orthodox mereology

$(PS)$ is based on a natural idea: structures are abstractions from isomorphic systems, so relations among structures should be defined in terms of relations among systems with respect to which isomorphism is a congruence. The definition at least respects the standard assumption that parthood is transitive, given that embedding is transitive. Yet, it is easy to see that the definition of parthood among structures based on $(PS)$ violates many of the standard mereological axioms.

Parthood ($\sqsubseteq$) is traditionally assumed to be anti-symmetric: ($x \sqsubseteq y \wedge y \sqsubseteq x \rightarrow x = y$). Embedding, on the other hand, is not antisymmetic: an infinite binary tree $T$ is embeddable in $T + T$, the graph that contains two copies of $T$, and $T + T$ is embeddable in $T$, but $T$ is not isomorphic to $T + T$. This means that $[T + T] \sqsubseteq [T]$ and $[T] \sqsubseteq [T + T]$, but $[T + T] \neq [T]$.

---

[4] Suppose we take two structures S and T, such that S is a (induced) subgraph of T. Moreover, suppose that S is isomorphic to S' and T is isomorphic to T'. It's easy to see that S' is not necessarily a (induced) subgraph of T'. Therefore, the relation of being an induced subgraph is not a congruence with respect to isomorphism.

[5] This definition of embedding is equivalent to the standard one ($S$ is embeddable in $S'$ if there is an injective function $f: D \rightarrow D'$such that for all $x \ldots, x_n \in D$ and all $R_i$: $R_i(x_1, \ldots, x_n) \leftrightarrow R_i'(fx_1, \ldots, fx_n)$). To see that embedding is a congruence with respect to isomorphism (i.e. that $S' \simeq S \hookrightarrow \mathrm{T} \simeq \mathrm{T}'$ entails $S' \hookrightarrow T'$) simply note that if $f$ is an isomorphism from $S'$ to $S$, $f'$ is an embedding from $S$ to $T$ and $f''$is an isomorphism from $T$ to $T'$, then $f'' \circ f' \circ f$ is an embedding from $S'$ to $T'$.

Moreover, weak supplementation ($x \sqsubset y \rightarrow \exists z(z \sqsubseteq y \wedge \neg z \circ x)$ fails if we adopt $PS$.[6] Let $S$ be an omega-sequence (a graph isomorphic to the natural numbers in their natural order): assuming $PS$, the structure of any initial finite segment of S is a proper part of the structure of the S; and yet any part of the structure of S overlaps the structure of a finite segment of S. For similar reasons, strong supplementation $\left(x \not\sqsubseteq y \rightarrow \exists z(z \sqsubseteq x \wedge \neg z \circ y)\right)$ fails too: an omega-sequence is not embeddable in one of its finite initial segments, so the structure of the omega sequence is not part of the structure of any finite initial segment; however, any part of the structure of the omega-sequence has at least one part in common with the structure of any initial segment, because given any two initial segments of an omega-structure, one is embeddable in the other.[7]

That $(PS)$ introduces a highly non-standard definition of parthood might count as a reason against adopting it. However, it is also possible to use $(PS)$ to account for some puzzling claims that are sometimes made concerning a structure and its parts. Here is a quote from Gödel:

> "Nor is it self-contradictory that a proper part should be identical (not merely equal) to the whole, as is seen in the case of structures in the abstract sense. The structure of the series of integers, e.g., contains itself as a proper part" (Gödel 1990: 130)

Note that Gödel is explicitly talking about a structure that is "not merely equal" to one of its proper parts, but identical to it. How is this possible, if a proper part is by definition a part that is not identical to the whole? A small change in $(PS)$ provides a way to make sense of Gödel's claim.

$$(PS^{*})\ [S] \sqsubset [S'] \leftrightarrow S \hookrightarrow^{*} S'$$

The notation $S \hookrightarrow^{*} S'$ is used to indicate that the embedding from $S$ to $S'$ is not an isomorphism. The idea behind $(PS^{*})$ is to take proper parthood as a primitive relation and define the proper parts of $[S]$ as the structures of systems of objects that are isomorphic to an induced proper subgraph of that system. Identity among structures is defined by *Frege's abstraction*: the structures of systems $S$ and $S'$ are identical just in case $S$ and $S'$ are isomorphic. If we adopt $(PS^{*})$ and *Frege's abstraction* (see above), we can vindicate Gödel's claim: the natural numbers in their natural order, $\mathbb{N}$, are isomorphic to $\mathbb{N} - \{0\}$, hence $[\mathbb{N} - \{0\}] = [\mathbb{N}]$; on the other hand, $[\mathbb{N} - \{0\}] \sqsubset [\mathbb{N}]$, because $\mathbb{N} - \{0\}$ is (isomorphic to) a proper subgraph of $\mathbb{N}$. A proper part is identical to the whole, as Gödel said.

[6] $x \circ y =_{df} \exists z(z \sqsubseteq x \wedge z \sqsubseteq y)$

[7] In presence of antysimmetry, strong supplementation entails weak supplementation, so the failure of weak supplementation entails the failure of strong supplementation. However, given that in the present context antysimmetry fails, we are not entitled to infer the failure of strong supplementation from the failure of weak supplementation.

## 4. A trilemma

A natural definition of parthood among structures $(PS)$ is incompatible with classical mereology. There are two ways to react to this result:

a. Reject $(PS)$
b. Accept that the mereology of "structures in the abstract sense" is highly unorthodox

A rejection of $(PS)$ can be motivated in two different ways: (i) one might argue with the problem with $(PS)$ is abstractionism about structures, the view discussed in section 1 according to which structures are abstractions from systems of objects, introduced via an abstraction principle, and relations among structures should be defined in terms of relations among systems of objects with respect to which isomorphism is a congruence; (ii) one might accept abstractionism about structures and reject the choice of embedding as the base relation, i.e. the relation $R$ among systems of objects such that parthood among structures should be defined as $R^*$ following the procedure described in section 1. So, we end up with a trilemma:

i. Reject abstractionism about structures
ii. Reject embedding as the base relation
iii. Accept an unorthodox mereology for structures

We will briefly consider these options and point out that each has a cost.

Some philosophers will be inclined to choose option (i). John Burgess, for instance, has claimed that:

> "Sometimes [pure structures] are confused with isomorphism types, but this is a mistake: An isomorphism type is no more a special kind of system than a direction is a special kind of line" (Burgess 1999: 286-7)

However, as Linnebo and Pettigrew (2014: 274): "this is only partially right: in many cases an 'isomorphism type' can, in an entirely natural way, be regarded as 'a special kind of system'." Moreover, it should be kept in mind that *Frege's abstraction* has a lot be said in its favor, as Linnebo and Pettigrew also stress (see also Wigglesworth 2021 for a defense of *Frege's abstraction*). In addition, it should be noted that other accounts of 'structures in the abstract sense' (to borrow Gödel's expression) face problems too (see for instance Linnebo 2017: 166-7). In light of this, the strengths and weaknesses *Frege's abstraction* should be carefully compared. The purpose of the present discussion is to highlight that *Frege's abstraction* is in tension with classical mereology. This might be useful to know, when comparing abstractionism about structures with other conceptions of structures, like Shapiro's *ante-rem* structuralism, according to which structures do not depend for their existence on the existence of systems of objects exemplifying them, but rather on the presence of coherent sets of axioms.

Abstractionism about structures and the Hilbert-inspired view that consistency of an axiomatized theory is the only requirement for the existence of a structure described by such a theory are two forms of *ontological minimalism* (see Linnebo 2018, section 1.2, for a definition of ontological minimalism and a comparison between the two views). Both are attractive options. Our only claim is that knowing that abstractionism is in tension with classical mereology might help to compare it with rival views and suggest a further line of inquiry: seeing how different conceptions of structures fare in defining the parthood relation between structures. In sum: if the tension with classical mereology that we highlighted is a reason to abandon abstractionism about structures, then the present paper has indeed contributed to the current debate on mathematical structuralism.

Option (ii) is an attempt to retain abstractionism about structures and define parthood among structures from a base relation among systems different from embedding. The challenge for anyone choosing this option is to find a plausible candidate for replacing embedding as base relation.

Option (iii) is to accept that the mereology of structures is highly unorthodox. We have already noticed that a consequence of $(PS)$ is that parthood is not antisymmetric. This might not be a fatal problem: "the anti-symmetry postulate can hardly be regarded as constitutive of the basic meaning of 'part'" (Varzi 2019, see also Kearns 2011). Some authors have recently worked on "non-well-founded mereology" (see Cotnoir 2010, Cotnoir & Bacon 2012, Obojska 2013). The fact that a natural notion of part for mathematical structures is not well-founded could make the case for non-well-founded mereology much more compelling than it currently is. Indeed, the examples provided in the literature of a non-antisymmetric parthood relation might sound exotic and *ad hoc*. See this example:

> Imagine a cube, with each side measuring 10m, made of a homogeneous substance. Not only do we take it back to a time that it previously existed at, but we use a shrinking machine and miniaturize by a factor of 100.We then remove a cube-shaped portion, with edges measuring 10cm, from the earlier, larger version of the cube and replace that portion with the miniaturized future version (which now fits perfectly). The cube is now a proper part of itself at that time. (Effingham 2010: 335)

It is useful to know that, once we accept $PS$ and *Frege's abstraction*, the case for non-well-founded mereology can be supported by examples of a different kind: the cases, discussed above, of pairs of mathematical structures that can each be embedded in the other without being isomorphic. It should still be noted that the axioms for non-well-founded mereology discussed by Cotnoir & Bacon 2012 include strong supplementation, which, as discussed, fails in the mereology for structures based on $PS$. This shows that the kind of mereology that $PS$ produces is *highly* unorthodox.[8] One might wonder

[8] It might be worth noticing that mereology for structures based on $PS$ validates some composition principles. The disjoint union of two graphs $G_1, G_2$ is the graph $G_1 \cup^* G_2$ obtained by taking two isomorphic copies of the two graphs, $G_1^*, G_2^*$, whose domains are disjoint, letting the domain of $G_1 \cup^* G_2$ be the union of the domains of $G_1^*$ and $G_2^*$, and the binary relation of $G_1 \cup^* G_2$ be the union of the binary relations of $G_1^*$ and $G_2^*$. The structure of $G_1 \cup^* G_2$ contains as parts the parts of the structure

whether a relation that invalidates so many of the standard mereological axioms can still be regarded as a relation of parthood.

## 5. Conclusions

In this paper, we made a first step in addressing the issue of the mereology of abstract mathematical structures. Instead of addressing in full generality the question of what are the parts of a mathematical structure, we focused on a sub-question: when is a structure part of another structure? We contributed to addressing this question with a negative result. We have presented an account of the parthood relation among mathematical structures based on an apparently natural idea: structures are the common form of isomorphic systems and relations among structures should be obtained from relations among systems over which isomorphism is a congruence. Our contribution has been to highlight that taking this idea seriously and using embedding as the relation between systems of objects from which the parthood relation between structures is abstracted yields a theory of parthood for structures that violates many of the standard mereological axioms. Either we reject abstractionism about structures, or we replace embedding with another relation, or we must accept a deviant mereology. This result poses an important constraint on attempts to define the parthood relation among structures and suggests a line of inquiry: test whether the tension with classical mereology that besets abstractionism about structures can be solved by alternative conceptions of structures or whether we should really take seriously the idea of developing a highly non-standard mereology for mathematical structures.

## References


Awodey, Steve, 2014. 'Structuralism, Invariance, Univalence'. *Philosophia Mathematica*, 22(1): 1-11.

Benacerraf, Paul (1965). What numbers could not be. *Philosophical Review* 74 (1):47-73.

Burgess, John, 1999. 'Book Review. Stewart Shapiro. Philosophy of Mathematics: Structure and Ontology. OUP 1997'. *Notre Dame Journal of Formal Logic*, 40(2): 283-291.

Cotnoir, Aaron, 2010. 'Anti-Symmetry and Non-Extensional Mereology'. *Philosophical Quarterly*, 60 (239): 396-405.

Cotnoir, Aaron & Bacon, Andrew, 2012. 'Non-Wellfounded Mereology'. *The Review of Symbolic Logic*, 5 (2): 187-204.

Effingham, Nikk, 2010. 'Mereological Explanation and Time Travel'. *Australasian Journal of Philosophy*, 88(2): 333-345.

Gödel, Kurt, 1990. *Collected Works. Volume 2* (Eds. Salomon Feferman et al.). Oxford University Press.


of $G_1$ and the parts of the structure of $G_2$ , because any graph embeddable in either $G_1$ or $G_2$ is embeddable in $G_1 \cup^* G_2$. In this sense, $[G_1 \cup^* G_2]$ can be considered an upper bound of the pair $\{[G_1],[G_2]\}$, even though this terminology is strictly speaking improper since the notion of upper bound is standardly defined with relation to a partial order and the relation of parthood based on $PS$ is not a partial order. Note that every induced subgraph of $G_1 \cup^* G_2$ is the (disjoint) union of a subgraph of $G_1$ and a subgraph of $G_2$ , hence for any graph $G$, $[G]$ overlaps $[G_1 \cup^* G_2]$ if and only if $[G]$ overlaps either $[G_1]$ or $[G_2]$.

Kearns, Stephen, 2011. 'Can a thing be part of itself?'. American Philosophical Quarterly, 87(1).
Ketland, Jeffrey, 2006. 'Structuralism and the Identity of Indiscernibles'. *Analysis*, 66(4): 303-315.
Hellman, Geoffrey (1989). *Mathematics Without Numbers: Towards a Modal-Structural Interpretation*. Oxford University Press.
Leitgeb, Hannes & Ladyman, James, 2008. 'Criteria of Identity and Structuralist Ontology'. *Philosophia Mathematica*, 16(3): 388-396.
Hannes Leitgeb (2020a), On Non-Eliminative Structuralism. Unlabeled Graphs as a Case Study, Part A, *Philosophia Mathematica*, 28 (3): 317–346, https://doi.org/10.1093/philmat/nkaa001
Hannes Leitgeb (2020b), On Non-Eliminative Structuralism. Unlabeled Graphs as a Case Study, Part B, *Philosophia Mathematica*; 29 (1): 64-87, https://doi.org/10.1093/philmat/nkaa009
Lewis, David, 1991. *Parts of Classes*. Oxford: Basil Blackwell.
Linnebo, Øystein & Richard, Pettigrew, 2014. 'Two Types of Abstraction for Structuralism'. *The Philosophical Quarterly*, 64(255): 267-283.
Linnebo, Øystein (2017). *Philosophy of Mathematics*. Princeton, NJ: Princeton University Press.
Obojska, Lidia, 2013. 'Some Remarks on Supplementation Principles in Absence of Antisymmetry'. *Review of Symbolic Logic*, 6(2): 343-347.
Macbride, Frazer (ed.) (2006). *Identity and Modality*. Oxford, Oxford University Press.
Resnik, Michael D. (1981). Mathematics as a science of patterns: Ontology and reference. *Noûs* 15 (4):529-550.
Shapiro, Stewart, 1997. *Philosophy of Mathematics: Structure and Ontology*. Oxford University Press.
Shapiro, Stewart 2006. 'Structure and identity', in MacBride, 2006, pp. 109–145.
Varzi, Achille, 2019. 'Mereology', *The Stanford Encyclopedia of Philosophy* (Spring 2019 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/spr2019/entries/mereology/>.
Wigglesworth, J. (2021) Non-eliminative Structuralism, Fregean Abstraction, and Non-rigid Structures. *Erkenntnis* 86, 113–127 https://doi.org/10.1007/s10670-018-0096-3