COX RINGS OF ALGEBRAIC STACKS

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20 April 2024

COX RING OF AN ALGEBRAIC STACK

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ABSTRACT. We give a proper definition of the multiplicative structure of the following rings: Cox ring of invertible sheaves on a general algebraic stack; Cox ring of rank one reflexive sheaves on a normal and excellent algebraic stack.

We show that such Cox rings always exist and establish its (non-)uniqueness in terms of an Ext-group. Moreover, we compare this definition with the classical construction of a Cox ring on a variety. Finally, we give an application to the theory of Mori dream stacks.

INTRODUCTION

Let $\mathcal{X}$ be an algebraic stack, e.g. a variety over some field. The *Cox ring of line bundles* on $\mathcal{X}$ as an $H^0(\mathcal{X}, \mathcal{O})$-module is

$$\mathsf{R}_{\mathrm{Pic}}(\mathcal{X}) = \bigoplus_{\mathcal{L} \in \mathrm{Pic}(\mathcal{X})} H^0(\mathcal{X}, \mathcal{L}).$$

As for varieties, the question about the multiplicative structure is delicate, and in the articles [7, 8] the first and second named authors glossed over this technicality when speaking about the Cox ring of algebraic stacks.

The purpose of the paper is to give a very general definition of a Cox ring, that is a multiplicative structure on $\mathsf{R}_{\mathrm{Pic}}(\mathcal{X})$. The following main theorem summarises our results; see Theorem 2.7.

Theorem A. *Let $\mathcal{X}$ be an algebraic stack. Then a Cox ring of line bundles on $\mathcal{X}$, also called a* Pic-*Cox ring* $\mathsf{R}_{\mathrm{Pic}}(\mathcal{X})$, *exists and the set of isomorphism classes of* Pic-*Cox rings is in bijection to*

$$\mathrm{Ext}^1(\mathrm{Pic}(\mathcal{X}), H^0(\mathcal{O}_{\mathcal{X}}^*)).$$

In particular a Pic-*Cox ring is unique up to isomorphism if*

- $\mathrm{Pic}(\mathcal{X})$ *is free; or*
- $H^0(\mathcal{O}_{\mathcal{X}}^*) = \Bbbk^*$ *for an algebraically closed field* $\Bbbk$.

In the case that $\mathcal{X}$ is a noetherian, normal and excellent algebraic stack, it makes sense to speak about the Cox ring of reflexive sheaves of rank one. We denote the group of isomorphism classes of such sheaves by $\mathrm{Ref}_1(\mathcal{X})$. We obtain an analogous statement for the Cox ring of such sheaves; see Theorem 2.14.

Theorem B. *If $\mathcal{X}$ is a noetherian, normal and excellent algebraic stack then a Cox ring of reflexive sheaves of rank one*

$$\mathsf{R}_{\mathrm{Ref}}(X) = \bigoplus_{\mathcal{L} \in \mathrm{Ref}_1(\mathcal{X})} H^0(\mathcal{X}, \mathcal{L}),$$

also called a Ref_1*-Cox ring, exists and the set of isomorphism classes of* Ref_1*-Cox rings is in bijection to*

$$\mathrm{Ext}^1(\mathrm{Ref}_1(\mathcal{X}), H^0(\mathcal{O}_{\mathcal{X}}^*)).$$

In particular a Ref_1*-Cox ring is unique up to isomorphism if*

- $\mathrm{Ref}_1(\mathcal{X})$ *is free; or*
- $H^0(\mathcal{O}_{\mathcal{X}}^*) = \Bbbk^*$ *for an algebraically closed field* $\Bbbk$.

Under the conditions on $\mathcal{X}$, we show this theorem by restricting to the regular locus $\mathcal{U} \subseteq \mathcal{X}$, which induces an isomorphism of Ref_1-Cox rings of $\mathcal{X}$ and Pic-Cox rings of $\mathcal{U}$.

We note that our construction does not need that the Picard group (or the group of reflexive sheaves of rank one) is finitely generated, although also the classical construction in [2] does not really rely on it. Additionally the classical construction assumes that $H^0(\mathcal{O}_{\mathcal{X}}^*) = \Bbbk^*$ for an algebraically closed field $\Bbbk$. We see here clearly that this becomes important for the uniqueness of the Cox ring, and not for its existence.

Another point to remark is that no kind of regularity is needed on $\mathcal{X}$ to state and prove the results about the Pic-Cox ring. Actually in this paper we will not even assume that $\mathcal{X}$ is an algebraic stack, just a category fibered in groupoids admitting an fpqc atlas, called a *pseudo-algebraic fibered category* (see Definition 1.1).

The main ingredient in the proof of the above results and in the study of multiplicative structures on $\mathsf{R}(X)$ is the notion of a *family of sheaves*: if G is an abelian group then a G-family of sheaves is a collection of quasi-coherent sheaves $(\mathcal{F}_g)_{g \in G}$ together with morphisms

$$\mathcal{O}_{\mathcal{X}} \to \mathcal{F}_0 \text{ and } \mathcal{F}_g \otimes \mathcal{F}_{g'} \to \mathcal{F}_{g+g'}$$

satisfying certain compatibility conditions (see Definition 1.2). With a G-family $\mathcal{F}$, one can associate

$$\mathcal{R}_{\mathcal{F}} = \bigoplus_g \mathcal{F}_g \text{ and } \mathsf{R}_{\mathcal{F}} = \bigoplus_g H^0(\mathcal{X}, \mathcal{F}_g)$$

which we call the *Cox sheaf* and *Cox ring* of $\mathcal{F}$ and which are a sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras and a $H^0(\mathcal{O}_{\mathcal{X}})$-algebra, respectively.

The Ext-groups in the two theorems above parametrize the possible ways to put a structure of a family of sheaves on the collections $(\mathcal{L})_{\mathcal{L} \in \mathrm{Pic}(\mathcal{X})}$ and $(\mathcal{F})_{\mathcal{F} \in \mathrm{Ref}_1(\mathcal{X})}$, respectively.

In particular, if $\mathcal{X}$ is a quasi-compact algebraic stack, there exists a Pic-Cox sheaf of algebras

$$\mathcal{R}_{\mathrm{Pic}}(\mathcal{X}) = \bigoplus_{\mathcal{L} \in \mathrm{Pic}(\mathcal{X})} \mathcal{L}$$

such that $H^0(\mathcal{R}_{\mathrm{Pic}}(\mathcal{X})) = \mathsf{R}_{\mathrm{Pic}}(\mathcal{X})$. Moreover, if $\mathcal{X}$ is a Noetherian, normal and excellent algebraic stack, there exists a Ref_1-Cox sheaf of algebras

$$\mathcal{R}_{\mathrm{Ref}}(\mathcal{X}) = \bigoplus_{\mathcal{F} \in \mathrm{Ref}_1(\mathcal{X})} \mathcal{F}$$

such that $H^0(\mathcal{R}_{\mathrm{Ref}}(\mathcal{X})) = \mathsf{R}_{\mathrm{Ref}}(\mathcal{X})$.

The crucial point is that a G-family of line bundles can be thought of as a torsor under the dual diagonalisable group scheme $\mathsf{D}(G)$. All the results are then consequence of constructions and known results about torsors.

Finally, as an application, we obtain the following result for a Mori dream space (see Theorem 3.8 and Remark 3.9):

Theorem C. *Let X be a Mori dream space over an algebraically closed field $\Bbbk$ of characteristic zero. Then*

$$\mathscr{X}^{can} = \left[\mathrm{Spec}_X \mathscr{R}_{\mathrm{Ref}}(X) \Big/ \mathsf{D}(\mathrm{Cl}(X)) \right].$$

is a Mori dream quotient stack in the sense of [8], that is,

$$\mathscr{X}^{can} = \left[\mathrm{Spec}\, \mathsf{R}_{\mathrm{Pic}}(\mathscr{X}^{can}) \setminus V \Big/ \mathsf{D}(\mathrm{Pic}(\mathscr{X}^{can})) \right],$$

where V is locus defined by the irrelevant ideal.

Moreover, let $\pi \colon \mathscr{X}^{can} \to X$ the structure map. Then

$$\mathrm{Ref}_1(\mathscr{X}^{can}) = \mathrm{Pic}(\mathscr{X}^{can}) = \mathrm{Cl}(X) \ \text{and} \ \pi_* \mathscr{R}_{\mathrm{Pic}}(\mathscr{X}^{can}) \simeq \mathscr{R}_{\mathrm{Ref}}(X)$$

so that $\mathsf{R}_{\mathrm{Pic}}(\mathscr{X}^{can}) = \mathsf{R}_{\mathrm{Ref}}(X)$.

This generalises [8, Prop. 2.9] where we used the additional assumption that $H^0(\mathrm{Spec}\, \mathsf{R}_{\mathrm{Ref}}(X), \mathscr{O}^*) = \Bbbk^*$ and $\mathrm{Pic}(\mathrm{Spec}\, \mathsf{R}_{\mathrm{Ref}}(X)) = 0$. We point out that, in [7] and [8], $\mathscr{X}^{can}$ is the starting point to build Mori dream stacks by so-called root constructions. Consequently, this assumption can be dropped in further results there, see Remark 3.9.

The paper is divided as follows. In Section 1, we give a very general definition of a Cox ring, introducing the language of G-families of sheaves, where G is an abelian group. In the case of families of line bundles, this notion is equivalent to torsors, see Subsection 1.2. Proposition 1.34 is the key lemma of our construction: it gives a criterion for a G-family of line bundles to induce a family of a quotient G/H. We apply this proposition in Section 2, where we first show the existence of a Cox ring of line bundles using a free resolution of the Picard group. Moreover, we measure the (non-)uniqueness of Cox rings with an Ext-group. Both are content of Theorem 2.3. In the case that it makes sense to speak about reflexive sheaves of rank one, we obtain the same result, see Subsection 2.2. In Subsection 2.3, we compare our construction with the classical one given in [2]. The upshot is that our key lemma is already present there. Finally, in Section 3 we give the application to Mori dream spaces.

1. Families of sheaves, torsors and their Cox rings

Definition 1.1. [13, Definition 1.2] A category fibered in groupoids $\mathscr{X}$ is called *pseudo-algebraic* if it admits an fpqc atlas $U \to \mathscr{X}$ from a scheme U, that is a map such that, for all maps $V \to \mathscr{X}$ from a scheme, the base change $V \times_{\mathscr{X}} U \to V$ is an fpqc covering of algebraic spaces.

Moreover $\mathscr{X}$ is called *quasi-compact* if it admits an fpqc atlas from a quasi-compact scheme.

For pseudo-algebraic fibered categories, the notions of quasi-coherent sheaves, vector bundles, line bundles, their pullbacks and pushforwards are well defined and well behaved. For an introduction to the subject, we refer to [12, §4.1] or [13, §1].

The idea is that a quasi-coherent sheaf on a pseudo-algebraic fibered category $\mathcal{X}$ assigns in a compatible way to any object of $\mathcal{X}$ over a scheme S a quasi-coherent sheaf on S. Therefore $\mathcal{X}$ can be thought of as a sort of parameter space for quasi-coherent sheaves.

Note that this notion of sheaves extends the classical one for schemes and algebraic stacks. The pseudo-algebraic condition guarantees that the category $\mathrm{QCoh}(\mathcal{X})$ of quasi-coherent sheaves is an abelian category (see [13, Proposition 1.4]) and other nice properties.

In this section we consider a pseudo-algebraic fibered category $\mathcal{X}$, for example a variety or an algebraic stack. We describe a general procedure that associates with a family of sheaves on $\mathcal{X}$ with certain properties an $H^0(\mathcal{O}_\mathcal{X})$-algebra of sections of them, that we call the *Cox ring* associated with the family. Moreover, we show existence of Cox rings of line bundles using the language of torsors and we discuss their uniqueness. Finally, we extend these results to Cox rings of reflexive sheaves of rank 1.

1.1. **Families of sheaves.**

Definition 1.2. Let G be an abelian group. A *G-family of sheaves* $\mathcal{F}$ on $\mathcal{X}$ is a collection of quasi-coherent and finitely presented sheaves $\mathcal{F}_g$ on $\mathcal{X}$ for $g \in G$ together with

- an isomorphism $\xi_0 \colon \mathcal{O}_\mathcal{X} \to \mathcal{F}_0$ and
- bilinear morphisms $\xi_{g,g'} \colon \mathcal{F}_g \times \mathcal{F}_{g'} \to \mathcal{F}_{g+g'}$ for all $g, g' \in G$, such that the morphisms

$$\omega_{g,g'} \colon \mathcal{F}_g \to \mathcal{H}om_\mathcal{X}(\mathcal{F}_{g'}, \mathcal{F}_{g+g'}), \quad x \mapsto \xi_{g,g'}(x, -)$$

are isomorphisms,

We require that the following diagrams are commutative for all $g, g', g'' \in G$:

- "associativity":

$$
\begin{array}{ccc}
\mathcal{F}_g \times \mathcal{F}_{g'} \times \mathcal{F}_{g''} & \xrightarrow{\xi_{g,g'} \times \mathrm{id}} & \mathcal{F}_{g+g'} \times \mathcal{F}_{g''} \\
{\scriptstyle \mathrm{id} \times \xi_{g',g''}} \downarrow & & \downarrow {\scriptstyle \xi_{g+g',g''}} \\
\mathcal{F}_g \times \mathcal{F}_{g'+g''} & \xrightarrow{\xi_{g,g'+g''}} & \mathcal{F}_{g+g'+g''}
\end{array}
$$

- "commutativity":

$$
\begin{array}{ccc}
\mathcal{F}_g \times \mathcal{F}_{g'} & \xrightarrow{\mathrm{swap}} & \mathcal{F}_{g'} \times \mathcal{F}_g \\
{\scriptstyle \xi_{g,g'}} \downarrow & & \downarrow {\scriptstyle \xi_{g',g}} \\
\mathcal{F}_{g+g'} & \xrightarrow{\mathrm{id}} & \mathcal{F}_{g'+g}
\end{array}
$$

- "unit":

$$
\begin{array}{ccc}
\mathcal{F}_g & \xrightarrow{\ \mathrm{id}\ } & \mathcal{F}_g \\
{\scriptstyle \mathrm{id}\,\times\xi_0}\searrow & & \nearrow{\scriptstyle \xi_{g,0}} \\
& \mathcal{F}_g \times \mathcal{F}_0 &
\end{array}
$$

We denote a G-family by $\mathcal{F} = (\mathcal{F}, \xi)$ or by $\mathcal{F}\colon G \to \mathrm{QCoh}(\mathcal{X})$.

Remark 1.3. In Definition 1.2 one can also directly use morphisms $\xi_{g,g'}\colon \mathcal{F}_g \otimes \mathcal{F}'_g \to \mathcal{F}_{g+g'}$ and change the commutative diagrams accordingly.

Remark 1.4. Let $\mathcal{F}$ be a G-family of sheaves on $\mathcal{X}$. Note that

$$
\mathcal{F}_g \xrightarrow{\ \omega_{g,-g}\ } \mathcal{H}om(\mathcal{F}_{-g}, \mathcal{F}_0) \xrightarrow{\ \mathcal{H}om(-,\xi_0^{-1})\ } \mathcal{H}om(\mathcal{F}_{-g}, \mathcal{O}_{\mathcal{X}}) = \mathcal{F}_{-g}^{\vee}
$$

is an isomorphism and the composition $\mathcal{F}_g \to \mathcal{F}_{-g}^{\vee} \to \mathcal{F}_g^{\vee\vee}$ is the bidual map. Thus, all the sheaves $\mathcal{F}_g$ are reflexive.

If $\mathcal{F}_h$ is invertible for some $h \in G$ then the maps $\xi_{g,h}$ induce isomorphisms $\mathcal{F}_g \otimes \mathcal{F}_h \to \mathcal{F}_{g+h}$ for all $g \in G$.

Definition 1.5. Let $\mathcal{F}$ be a G-family and $\mathcal{G}$ an H-family together with a group homomorphism $\alpha\colon G \to H$. A *morphism of families* $\phi\colon \mathcal{F} \to \mathcal{G}$ *with respect to* α consists of $\{\phi_g\colon \mathcal{F}_g \to \mathcal{G}_{\alpha(g)}\}_{g \in G}$ such that the following diagrams commute for all $g, g' \in G$:

$$
\begin{array}{ccc}
& \mathcal{O}_{\mathcal{X}} & \\
{\scriptstyle \xi_0^{\mathcal{F}}}\swarrow & & \searrow{\scriptstyle \xi_0^{\mathcal{G}}} \\
\mathcal{F}_0 & \xrightarrow{\ \phi_0\ } & \mathcal{G}_0
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{F}_g \times \mathcal{F}_{g'} & \xrightarrow{\ \phi_g \times \phi_{g'}\ } & \mathcal{G}_{\alpha(g)} \times \mathcal{G}_{\alpha(g')} \\
{\scriptstyle \xi_{g,g'}^{\mathcal{F}}}\downarrow & & \downarrow{\scriptstyle \xi_{\alpha(g),\alpha(g')}^{\mathcal{G}}} \\
\mathcal{F}_{g+g'} & \xrightarrow{\ \phi_{g+g'}\ } & \mathcal{G}_{\alpha(g+g')}
\end{array}
$$

One can check that such a $\phi\colon \mathcal{F} \to \mathcal{G}$ is also compatible with the remaining three diagrams in the definition of a G-family.

We call a morphism of families $\phi\colon \mathcal{F} \to \mathcal{G}$ with respect to $\alpha = \mathrm{id}_G$ just a *morphism of G-families*.

The definition above allows us to speak of the category of families on a given space $\mathcal{X}$.

Definition 1.6. Let G be an abelian group and $\mathcal{F}$ be a G-family of sheaves. The $H^0(\mathcal{O}_{\mathcal{X}})$-module

$$
\mathsf{R}_{\mathcal{F}} := \bigoplus_{g \in G} H^0(\mathcal{F}_g)
$$

is called the *Cox ring* associated with $\mathcal{F}$. The ring structure is given by

$$
H^0(\mathcal{F}_g) \times H^0(\mathcal{F}_{g'}) \xrightarrow{\ H^0(\xi_{g,g'})\ } H^0(\mathcal{F}_{g+g'})
$$

which turns $\mathsf{R}_{\mathcal{F}}$ into a commutative G-graded $H^0(\mathcal{O}_{\mathcal{X}})$-algebra with structure morphism $H^0(\mathcal{O}_{\mathcal{X}}) \xrightarrow{\ \xi_0\ } H^0(\mathcal{F}_0)$.

Definition 1.7. We define the *Cox sheaf of algebras* as
$$\mathscr{R}_{\mathscr{F}} = \bigoplus_{g \in G} \mathscr{F}_g$$
using the G-family structure.

Remark 1.8. There is always a morphism of $H^0(\mathscr{O}_{\mathscr{X}})$-algebras
$$\mathsf{R}_{\mathscr{F}} \to H^0(\mathscr{R}_{\mathscr{F}})$$
and it is an isomorphism if G is finite or $\mathscr{X}$ is quasi-compact, that is it has an fpqc atlas $U \to \mathscr{X}$ from a quasi-compact scheme U.

Remark 1.9. If $f \colon \mathscr{Y} \to \mathscr{X}$ is a morphism of fibered categories then $f^*\mathscr{F} = (f^*\mathscr{F}_g, f^*\xi_0, f^*\xi_{g,g'})_{g,g' \in G}$ is in general not a G-family: the problem is that $f^*\mathscr{F}_u \to \mathscr{H}om_{\mathscr{Y}}(f^*\mathscr{F}_v, f^*\mathscr{F}_{u+v})$ may fail to be an isomorphism.

If all the sheaves involved are invertible or the map $f \colon \mathscr{Y} \to \mathscr{X}$ is flat, then $f^*\mathscr{F}$ is a G-family of sheaves on $\mathscr{Y}$ and we have a G-graded $H^0(\mathscr{O}_{\mathscr{X}})$-linear map $\mathsf{R}_{\mathscr{F}} \to \mathsf{R}_{f^*\mathscr{F}}$.

Remark 1.10. The assignment $\mathscr{F} \mapsto \mathsf{R}_{\mathscr{F}}$ mapping a family $\mathscr{F}$ to its Cox ring is functorial, i.e. given a morphism of families $\phi \colon \mathscr{F} \to \mathscr{G}$ with respect to a group homomorphism α, there is an induced ring homomorphism $\mathsf{R}_\phi \colon \mathsf{R}_{\mathscr{F}} \to \mathsf{R}_{\mathscr{G}}$.

1.2. **Generalities on torsors.** We recall here the classical definition of torsor over a scheme and the (less classical) definition of torsors over a fibered category.

Definition 1.11. Let R be a ring and G be an affine group scheme over R such that $\mathsf{G} \to \operatorname{Spec} R$ is flat (hence faithfully flat). Let X be a scheme over R. A G-*torsor* over X is a faithfully flat morphism $P \to X$ with an action of G on P as a scheme over X such that it is locally trivial in the fpqc-topology.

We denote by with $\mathscr{B}_R\mathsf{G}$, or just $\mathscr{B}\mathsf{G}$ for short, the stack of G-torsors over R.

Remark 1.12. The local triviality means that there exists a fpqc-covering $\{U \to X\}_{U \in \mathfrak{U}}$ of X and a G-equivariant isomorphism
$$P_{|U} \simeq \mathsf{G} \times U, \quad \forall U \in \mathfrak{U}.$$
It is equivalent to $\mathsf{G} \times P \to P \times_X P, (h, p) \mapsto (hp, p)$ being an isomorphism.

Definition 1.13. Let G be a flat and affine group scheme over R. Let $\mathscr{X}$ be a category fibered in groupoids over R. A G-*torsor* over $\mathscr{X}$ is a morphism $\mathscr{P} \to \mathscr{X}$ of categories fibered in groupoids over R such that, for all morphisms $X \to \mathscr{X}$ from a scheme, the base change $\mathscr{P} \times_{\mathscr{X}} X \to X$ is a G-torsor over X and all those torsors are compatible in the obvious way.

Notice that, since we are restricting ourselves to affine group schemes, torsors are affine maps.

Remark 1.14. A G-torsor over $\mathscr{X}$ is the same as a morphism of fibered categories $\mathscr{X} \to \mathscr{B}_R\mathsf{G}$. In this case $\mathscr{P} \to \mathscr{X}$ is the base change along the trivial torsor $\operatorname{Spec} R \to \mathscr{B}_R\mathsf{G}$.

Remark 1.15. If $\mathcal{X}$ is an algebraic stack over R (resp. a scheme, a pseudo-algebraic fibered category) then so is the total space of a G-torsor.

Remark 1.16. Let G be a flat an affine group scheme over R. If $\mathcal{Y} \to \mathcal{X}$ is a representable map of fibered categories over R, e.g. an affine map, an action of G on $\mathcal{Y}$ over $\mathcal{X}$ consists in the following data: for all objects $T \to \mathcal{X}$ an action of G (over T) on the algebraic space $\mathcal{Y} \times_{\mathcal{X}} T$ which are compatible in an obvious way. Here, considering only actions relative to $\mathcal{X}$, we want to avoid to talk about action of groups on fibered categories in general.

If $\mathcal{Y} \to \mathcal{X}$ is a representable map with an action of G, then we can form the quotient "stack" (the stackiness is only relative to the base) $[\mathcal{Y}/\mathsf{G}] \to \mathcal{X}$. Its central property is that for an object $T \to \mathcal{X}$ and $\mathcal{Y}_T = \mathcal{Y} \times_{\mathcal{X}} T$ we have

$$[\mathcal{Y}/\mathsf{G}] \times_{\mathcal{X}} T \simeq [\mathcal{Y}_T/\mathsf{G}]$$

and that $\mathcal{Y} \to [\mathcal{Y}/\mathsf{G}]$ is a G-torsor.

A proper definition of the category fibered in groupoids $[\mathcal{Y}/\mathsf{G}]$ is the following. An object of $[\mathcal{Y}/\mathsf{G}]$ over an R-scheme T is a pair (u, v) where $u \in \mathcal{X}(T)$ and v is a section of $[\mathcal{Y}_u/\mathsf{G}] \to T$, where $\mathcal{Y}_u \to T$ is the base change of $\mathcal{Y} \to \mathcal{X}$ along $u \colon T \to \mathcal{X}$.

If the action of G on $\mathcal{Y}$ is free, that is, it is free on all the $\mathcal{Y}_T$, then $[\mathcal{Y}_T/\mathsf{G}]$ equals the quotient sheaf $\mathcal{Y}_T/\mathsf{G}$. In this situation we will simply write $\mathcal{Y}/\mathsf{G}$ in place of $[\mathcal{Y}/\mathsf{G}]$. This is coherent with the idea that if $\mathcal{Y}$ is a collection of sheaves indexed by objects of $\mathcal{X}$ (more precisely a pseudo-functor) then $\mathcal{Y}/\mathsf{G}$ is the collection of quotient sheaves.

Remark 1.17. A G-torsor is a faithfully flat and affine map $\mathcal{P} \to \mathcal{X}$ together with an action of G over $\mathcal{X}$. Moreover the action of G is free. If H is a flat and affine subgroup of G in particular we can consider the quotient $\mathcal{P}/\mathsf{H} \to \mathcal{X}$, which, according to Remark 1.16, coincides with $[\mathcal{P}/\mathsf{H}] \to \mathcal{X}$.

We warn the reader that such notation, in some concrete situation, may lead to confusing notation, which we will try to avoid. For example $\operatorname{Spec} R \to \mathcal{B}_R \mathsf{G}$ is a G-torsor, but we would never write $\operatorname{Spec} R/\mathsf{G}$ for $[\operatorname{Spec} R/\mathsf{G}] = \mathcal{B}_R \mathsf{G}$.

Remark 1.18. Let $\alpha \colon \mathsf{H} \to \mathsf{G}$ be a map of flat and affine group schemes over R. Then there is an induced functor $\mathcal{B}\alpha \colon \mathcal{B}\mathsf{H} \to \mathcal{B}\mathsf{G}$ such that, for any H-torsor $\mathcal{Q}$, there is a H-equivariant map $\mathcal{Q} \to \mathcal{B}\alpha(\mathcal{Q})$ universal among maps from $\mathcal{Q}$ to a G-torsor.

Moreover, recall that any (equivariant) homomorphism between torsors is automatically an isomorphism. In particular, if $\mathcal{Q}$ is an H-torsor and $\mathcal{P}$ a G-torsor over a common base $\mathcal{X}$, and $\mathcal{Q} \to \mathcal{P}$ an H-equivariant map over $\mathcal{X}$, then the induced map $\mathcal{B}\alpha(\mathcal{Q}) \to \mathcal{P}$ is an isomorphism.

Remark 1.19. Let $\alpha \colon \mathsf{G} \to \mathsf{K}$ be a morphism of affine group schemes over R and assume that G is flat over R. Set also $\mathsf{H} = \ker(\alpha)$, which is a closed subscheme of G, hence affine.

Then α is faithfully flat if and only if H is flat over R and α is an fpqc epimorphism. Indeed if α is an fpqc epimorphism then the monomorphism $\mathsf{G}/\mathsf{H} \to \mathsf{K}$ is an epimorphism and hence an isomorphism. Moreover $\mathsf{G} \to \mathsf{G}/\mathsf{H}$ is an H-torsor, hence affine and faithfully flat if H is flat.

In this situation $\mathsf{G}/\mathsf{H} \simeq \mathsf{K}$ we have

$$\mathcal{B}\alpha(\mathcal{Q}) = \mathcal{Q}/\mathsf{H}$$

for a G-torsor $\mathcal{Q}$ (over some base). Indeed G/H acts on $\mathcal{Q}/\mathsf{H}$ and this space is locally trivial, turning it into a torsor. As there is a G-equivariant map $\mathcal{Q} \to \mathcal{Q}/\mathsf{H}$ we can conclude that $\mathcal{B}\alpha(\mathcal{Q}) = \mathcal{Q}/\mathsf{H}$ by Remark 1.18.

The content of the following lemma and corollary is already known, see for example [1, Lem. 1.17].

Lemma 1.20. *Let G be a flat affine group scheme over a ring R, $\mathcal{X}$ a pseudo-algebraic fibered category over R, $\mathcal{P}$ a G-torsor over $\mathcal{X}$ and $\alpha\colon \mathsf{H} \to \mathsf{G}$ a subgroup. Then H-torsors $\mathcal{Q}$ together with an H-equivariant map $\mathcal{Q} \to \mathcal{P}$ over $\mathcal{X}$ (equivalently an isomorphism $\mathcal{B}\alpha(\mathcal{Q}) \simeq \mathcal{P}$) correspond to sections of $\mathcal{P}/\mathsf{H} \to \mathcal{X}$. Given such a section the corresponding map $\mathcal{Q} \to \mathcal{P}$ fits into the following fibre square*

(1)
$$\begin{array}{ccc} \mathcal{Q} & \longrightarrow & \mathcal{P} \\ {\scriptstyle /\mathsf{H}}\downarrow & & \downarrow{\scriptstyle /H} \\ \mathcal{X} & \longrightarrow & \mathcal{P}/\mathsf{H} \end{array}$$

In particular $\mathcal{P}$ is induced by an H-torsor if and only if $\mathcal{P}/\mathsf{H} \to \mathcal{X}$ has a section.

Proof. If $\mathcal{Q}$ is an H-torsor and $\mathcal{Q} \to \mathcal{P}$ is H-equivariant, then $\mathcal{X} = \mathcal{Q}/\mathsf{H} \to \mathcal{P}/\mathsf{H}$ is a section and

$$\mathcal{Q} \to \mathcal{X} \times_{\mathcal{P}/\mathsf{H}} \mathcal{P}$$

is a morphism of H-torsors and therefore an isomorphism. Conversely, the $\mathcal{Q}$ defined in the statement is an H-torsor over $\mathcal{X}$ and the map $\mathcal{Q} \to \mathcal{P}$ is H-equivariant. $\square$

Corollary 1.21. *Let $1 \to \mathsf{H} \xrightarrow{\alpha} \mathsf{G} \xrightarrow{\beta} \mathsf{G}/\mathsf{H} \to 1$ be an exact sequence of flat affine group schemes over R (in particular H is a normal subgroup of G). Then there is a Cartesian diagram*

$$\begin{array}{ccc} \mathcal{B}\mathsf{H} & \longrightarrow & \operatorname{Spec} R \\ {\scriptstyle \mathcal{B}\alpha}\downarrow & & \downarrow \\ \mathcal{B}\mathsf{G} & \xrightarrow{\mathcal{B}\beta} & \mathcal{B}(\mathsf{G}/\mathsf{H}) \end{array}$$

In other words, an H-torsor corresponds to a G-torsor together with a trivialization of the induced G/H-torsor.

Proof. This is a direct consequence of Lemma 1.20 because $\mathcal{B}\beta(\mathcal{P}) \simeq [\mathcal{P}/\mathsf{H}]$ (see Remark 1.19). $\square$

Definition 1.22. A G-*equivariant quasi-coherent sheaf* on $\mathcal{X}$ is a quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ together with an action of G. By this we mean that for all objects $x\colon T \to \mathcal{X}$ the group G acts on $x^*\mathcal{F}$ and all those actions are compatible. We denote by $\operatorname{QCoh}^G(\mathcal{X})$ the category of G-equivariant quasi-coherent sheaves.

Remark 1.23. Let G be a flat affine group scheme and $\mathcal{X}$ be a pseudo-algebraic fibered category over R. Then

$$\mathrm{QCoh}(\mathcal{B}_{\mathcal{X}}\mathsf{G}) \simeq \mathrm{QCoh}^{\mathsf{G}}(\mathcal{X}).$$

This follows from the same result for schemes. Taking ring objects, this equivalence extends also to an equivalence

$$\mathrm{AlgQCoh}(\mathcal{B}_{\mathcal{X}}\mathsf{G}) \simeq \mathrm{AlgQCoh}^{\mathsf{G}}(\mathcal{X})$$

from the category of quasi-coherent sheaves of algebras on $\mathcal{B}_{\mathcal{X}}\mathsf{G}$ to the category of G-equivariant sheaves of algebras on $\mathcal{X}$.

By applying the relative spectrum (which is a contravariant equivalence) we also obtain an equivalence

$$\mathrm{Aff}(\mathcal{B}_{\mathcal{X}}\mathsf{G}) \to \mathrm{Aff}^{\mathsf{G}}(\mathcal{X})$$

from the category of affine maps over $\mathcal{B}_{\mathcal{X}}\mathsf{G}$ to the category of affine maps over $\mathcal{X}$ together with an action of G over $\mathcal{X}$. This equivalence maps an affine map $\mathcal{Z} \to \mathcal{B}_{\mathcal{X}}\mathsf{G}$ to the base change $\mathcal{Z} \times_{\mathcal{B}_{\mathcal{X}}\mathsf{G}} \mathcal{X} \to \mathcal{X}$. A quasi-inverse is obtained by mapping an affine map $\mathcal{Y} \to \mathcal{X}$ with an action of G to $[\mathcal{Y}/\mathsf{G}] \to \mathcal{B}_{\mathcal{X}}\mathsf{G}$ (see Remark 1.16).

For $\alpha\colon \mathsf{G} \to \mathsf{H}$ of flat affine group schemes over R, denote by $\mathcal{B}\alpha$ also the map

$$(\mathcal{B}\alpha)_{\mathcal{X}}\colon \mathcal{B}_{\mathcal{X}}\mathsf{G} = \mathcal{B}\mathsf{G} \times \mathcal{X} \xrightarrow{\mathcal{B}\alpha \times \mathrm{id}_{\mathcal{X}}} \mathcal{B}\mathsf{H} \times \mathcal{X} = \mathcal{B}_{\mathcal{X}}\mathsf{H}.$$

Moreover, we will use from Remark 1.23 the composition

$$\hat{\alpha}_*\colon \mathrm{AlgQCoh}^{\mathsf{H}}(\mathcal{X}) \xrightarrow{\sim} \mathrm{AlgQCoh}(\mathcal{B}_{\mathcal{X}}\mathsf{H}) \xrightarrow{\mathcal{B}\alpha_*} \mathrm{AlgQCoh}(\mathcal{B}_{\mathcal{X}}\mathsf{G}) \xrightarrow{\sim} \mathrm{AlgQCoh}^{\mathsf{G}}(\mathcal{X}).$$

Proposition 1.24. *Let $1 \to \mathsf{H} \xrightarrow{\alpha} \mathsf{G} \xrightarrow{\beta} \mathsf{G}/\mathsf{H} \to 1$ be a short exact sequence of flat affine group schemes over R. Let $\mathcal{X}$ be a pseudo-algebraic fibered category over R. Then for $\mathcal{R} \in \mathrm{AlgQCoh}^{\mathsf{H}}(\mathcal{X})$ and $\mathcal{S} := \hat{\alpha}_*\mathcal{R}$, there is an isomorphism*

$$[\mathrm{Spec}_{\mathcal{X}} \mathcal{R}/\mathsf{H}] \simeq [\mathrm{Spec}_{\mathcal{X}} \mathcal{S}/\mathsf{G}]$$

over $\mathcal{B}_{\mathcal{X}}\mathsf{G}$.

Proof. The map $\mathcal{B}\alpha\colon \mathcal{B}_{\mathcal{X}}\mathsf{H} \to \mathcal{B}_{\mathcal{X}}\mathsf{G}$ is affine, because its base change along (the relative) fpqc covering $\mathcal{X} \to \mathcal{B}_{\mathcal{X}}\mathsf{G}$ is $(\mathsf{G}/\mathsf{H}) \times \mathcal{X} \to \mathcal{X}$ (the map $\mathsf{G}/\mathsf{H} \to \mathcal{B}\mathsf{H}$ is given by the H-torsor $\mathsf{G} \to \mathsf{G}/\mathsf{H}$).

Let $\overline{\mathcal{R}}$ and $\overline{\mathcal{S}}$ be the sheaves of algebras over $\mathcal{B}_{\mathcal{X}}\mathsf{H}$ and $\mathcal{B}_{\mathcal{X}}\mathsf{G}$, respectively, corresponding to $\mathcal{R}$ and $\mathcal{S}$. By construction, the maps

$$p\colon [\mathrm{Spec}_{\mathcal{X}} \mathcal{R}/\mathsf{H}] \to \mathcal{B}_{\mathcal{X}}\mathsf{H} \text{ and } q\colon [\mathrm{Spec}_{\mathcal{X}} \mathcal{S}/\mathsf{G}] \to \mathcal{B}_{\mathcal{X}}\mathsf{G}$$

are affine and

$$\overline{\mathcal{R}} = p_*\mathcal{O}_{[\mathrm{Spec}_{\mathcal{X}} \mathcal{R}/\mathsf{H}]} \text{ and } \overline{\mathcal{S}} = q_*\mathcal{O}_{[\mathrm{Spec}_{\mathcal{X}} \mathcal{S}/\mathsf{G}]}.$$

Moreover the affine map

$$[\mathrm{Spec}_{\mathcal{X}} \mathcal{R}/\mathsf{H}] \xrightarrow{p} \mathcal{B}_{\mathcal{X}}\mathsf{H} \xrightarrow{\mathcal{B}\alpha} \mathcal{B}_{\mathcal{X}}\mathsf{G}$$

corresponds to

$$(\mathcal{B}\alpha \circ p)_*\mathcal{O}_{[\mathrm{Spec}_{\mathcal{X}} \mathcal{R}/\mathsf{H}]} = q_*\mathcal{O}_{[\mathrm{Spec}_{\mathcal{X}} \mathcal{S}/\mathsf{G}]}.$$

Thus $\mathcal{B}\alpha \circ p$ and q are isomorphic as affine maps over $\mathcal{B}_{\mathcal{X}}\mathsf{G}$ as required. $\square$

1.3. **Families of line bundles as torsors by diagonalisable group schemes.** In this section, we translate between families of line bundles and torsors. This is a very general result which apply to any pseudo-algebraic fibered category $\mathcal{X}$. For this reason we are going to consider diagonalizable groups as group scheme over $\operatorname{Spec}\mathbb{Z}$ and apply the results of Subsection 1.2 with $R = \mathbb{Z}$: the category $\mathcal{X}$ is defined over $\mathbb{Z}$ in the sense that $\mathcal{X} \to \operatorname{Sch} = \operatorname{Sch}/\mathbb{Z}$ is the structure morphism.

Definition 1.25. Let G be an abelian group. We denote by

$$\mathsf{D}(G) := \operatorname{Spec}\mathbb{Z}[G]$$

the *Cartier dual group scheme* of G over $\mathbb{Z}$.

Remark 1.26. Observe that $\mathsf{D}(G)$ co-represents the functor

$$\operatorname{Sch} \to \operatorname{AbGrps}, \ \ X \mapsto \operatorname{Hom}_{\operatorname{Grps}}(G, H^0(\mathcal{O}_X^*)).$$

Indeed for any ring A one has

$$\operatorname{Hom}_{\operatorname{Grps}}(G, A^*) \simeq \operatorname{Hom}_{\mathbb{Z}\text{-Algs}}(\mathbb{Z}[G], A).$$

Remark 1.27. The association $G \longmapsto \mathsf{D}(G)$ defines a contravariant, fully faithful and exact functor from the category of abelian groups to the category of affine group schemes over $\mathbb{Z}$.

Example 1.28. The multiplicative group scheme is $\mathbb{G}_m := \mathsf{D}(\mathbb{Z}) = \operatorname{Spec}\mathbb{Z}[t, t^{-1}]$.

Remark 1.29. Let G be an abelian group and $\mathcal{X}$ be a pseudo-algebraic fibered category. Then $\operatorname{QCoh}(\mathcal{B}_{\mathcal{X}}\mathsf{D}(G))$ is equivalent to the category of G-graded quasi-coherent sheaves: if $\mathsf{D}(G)(R) \ni \chi\colon G \to R^*$ and $t\colon \operatorname{Spec}R \to \mathcal{X}$ then

$$\mathcal{R} = \bigoplus_g \mathcal{R}_g \text{ and } \chi \cdot x = \chi(g)x \text{ for } x \in t^*\mathcal{R}_g.$$

Hence we can extend the equivalences of Remark 1.23 to include also the category of G-graded quasi-coherent sheaves of algebras on $\mathcal{X}$, which we denote by $G\text{-}\operatorname{Gr-AlgQCoh}(\mathcal{X})$:

$$
\begin{array}{ccc}
G\text{-}\operatorname{Gr-AlgQCoh}(\mathcal{X}) & \stackrel{\sim}{=\!=\!=} & \operatorname{AlgQCoh}(\mathcal{B}_{\mathcal{X}}\mathsf{D}(G)) \\
\wr \| & & \| \wr \\
\operatorname{Aff}^{\mathsf{D}(G)}(\mathcal{X}) & \stackrel{}{=\!=\!=\!=\!=}_{\sim} & \operatorname{Aff}(\mathcal{B}_{\mathcal{X}}\mathsf{D}(G))
\end{array}
$$

where the vertical equivalences are contravariant.

Proposition 1.30. *Let G be an abelian group, $\mathcal{X}$ be a pseudo-algebraic fibered category and $\pi\colon \mathscr{P} \to \mathcal{X}$ be an affine map with an action of $\mathsf{D}(G)$ corresponding to the G-graded sheaf of algebras*

$$\pi_*\mathcal{O}_{\mathscr{P}} = \mathcal{R} = \bigoplus_g \mathcal{R}_g.$$

Then $\mathscr{P} \to \mathcal{X}$ is a $\mathsf{D}(G)$-torsor if and only if all maps

$$\mathcal{O}_{\mathcal{X}} \to \mathcal{R}_0 \text{ and } \mathcal{R}_g \otimes \mathcal{R}_h \to \mathcal{R}_{g+h}$$

are isomorphisms. In this case all sheaves $\mathcal{R}_g$ are invertible on $\mathcal{X}$.

Proof. This is essentially [3, Exposé I, §4.7.3 & Exposé VIII, §4.1]. We give the argument for the convenience of the reader.

Recall that an $\mathsf{D}(G)$-action on $\mathscr{P}$ corresponds to a map $\mathscr{P} \times \mathsf{D}(G) \to \mathscr{P} \times_{\mathcal{X}} \mathscr{P}$ or equivalently to a map $\eta\colon \mathscr{R} \otimes \mathscr{R} \to \mathscr{R} \otimes \mathbb{Z}[G]$. Note that η splits into the graded pieces

$$\mathscr{R}_g \otimes \mathscr{R}_{g'} \to \mathscr{R}_{g+g'} \otimes \mathbb{Z}e_{g'}.$$

In particular $\mathscr{P} \times \mathsf{D}(G) \to \mathscr{P} \times_{\mathcal{X}} \mathscr{P}$ is an isomorphism if and only if all maps $\eta_{g,h}\colon \mathscr{R}_g \otimes \mathscr{R}_h \to \mathscr{R}_{g+h}$ are isomorphisms.

We can therefore assume that the above two equivalent conditions hold for our algebra $\mathscr{R}$.

"$\Longleftarrow$" In order to show that $\mathscr{P} \to \mathcal{X}$ is a $\mathsf{D}(G)$-torsor it is enough to show that $\pi_* \mathscr{O}_{\mathscr{P}} = \mathscr{R}$ is faithfully flat.

As $\mathscr{R}_g \otimes \mathscr{R}_{-g} \simeq \mathscr{R}_0 \simeq \mathscr{O}_{\mathcal{X}}$, one can show by an elementary argument that all sheaves $\mathscr{R}_g$ and therefore $\mathscr{R}$ are flat $\mathscr{O}_{\mathcal{X}}$-modules. As $\mathscr{R}_0 \simeq \mathscr{O}_{\mathcal{X}}$, the ring $\mathscr{R}$ is faithfully flat.

"$\Longrightarrow$" Conversely, if $\mathscr{P} \to \mathcal{X}$ is a $\mathsf{D}(G)$-torsor, the sheaf $\mathscr{R}$ is fpqc locally on $\mathcal{X}$ isomorphic to $\mathbb{Z}[G]$ as G-graded algebra. In particular all graded pieces $\mathscr{R}_g$ are fpqc locally invertible. By descent all sheaves $\mathscr{R}_g$ are therefore invertible.

In particular $\operatorname{Spec}\mathscr{R}_0 \to \mathcal{X}$ is a cover of degree 1, which is known to be an isomorphism, that is, $\mathscr{O}_{\mathcal{X}} \to \mathscr{R}_0$ is an isomorphism. $\qquad\square$

The next step in this section is a translation of Corollary 1.21 into the language of families of line bundles.

Proposition 1.31. *Let G be an abelian group. The assignment $\mathscr{F} \mapsto \mathscr{R}_{\mathscr{F}}$ mapping a G-family to its Cox ring defines an equivalence between the category of G-families of line bundles on $\mathcal{X}$ and the category $\mathscr{B}\mathsf{D}(G)(\mathcal{X})$ of $\mathsf{D}(G)$-torsors over $\mathcal{X}$.*

If $\alpha\colon H \to G$ is a map of abelian groups then the functor

$$\mathscr{B}\mathsf{D}(\alpha)\colon \mathscr{B}\mathsf{D}(G)(\mathcal{X}) \to \mathscr{B}\mathsf{D}(H)(\mathcal{X})$$

maps a G-family $\mathscr{F} = (\mathscr{F}_g)_{g \in G}$ to the H-family $(\mathscr{F}_{\alpha(h)})_{h \in H}$.

Proof. The first part is just Proposition 1.30. For the second, given a G-family $\mathscr{F}$ there is an H-graded map

$$\bigoplus_{h \in H} \mathscr{F}_{\alpha(h)} \to \bigoplus_{g \in G} \mathscr{F}_g$$

By Remark 1.18 it follows that $\operatorname{Spec}(\bigoplus_{h \in H} \mathscr{F}_{\alpha(h)})$ is the induced $\mathsf{D}(H)$-torsor, as required. $\qquad\square$

Example 1.32. For $G = \mathbb{Z}$, we have $\mathscr{B}\mathsf{D}(G) = \mathscr{B}\mathbb{G}_m$ and elements of $\mathscr{B}\mathbb{G}_m(\mathcal{X})$ correspond to $\mathbb{Z}$-families of line bundles, which are completely determined by a line bundle on $\mathcal{X}$. So $\mathscr{B}\mathbb{G}_m(\mathcal{X})$ is equivalent to the category of line bundles on $\mathcal{X}$ with isomorphisms as arrows.

More generally, if $G = \mathbb{Z}^{(I)}$ for a set I it follows that $\mathsf{D}(G) \simeq \mathbb{G}_m^I$ and $\mathscr{B}\mathsf{D}(G) \simeq (\mathscr{B}\mathbb{G}_m)^I$. So, by Proposition 1.30, the category of G-families of line bundles on $\mathcal{X}$ is equivalent to the category of collections of line bundles indexed by I, where arrows between them are collections of isomorphisms. The correspondence takes a G-family to its evaluation on a basis of G.

Remark 1.33. If $\alpha\colon H \to G$ is a map of abelian groups we get maps $\mathsf{D}(\alpha)\colon \mathsf{D}(G) \to \mathsf{D}(H)$ and $\mathscr{B}\mathsf{D}(\alpha)\colon \mathscr{B}_\mathcal{X}\mathsf{D}(G) \to \mathscr{B}_\mathcal{X}\mathsf{D}(H)$.

One can check directly that $(\mathscr{B}\mathsf{D}(\alpha))^*$ and $(\mathscr{B}\mathsf{D}(\alpha))_*$ acts on $\mathsf{D}(G)$-equivariant quasi-coherent sheaves interpreted as G-graded sheaves as

$$\mathscr{F} = \bigoplus_{h \in H} \mathscr{F}_h \longmapsto \mathscr{B}\mathsf{D}(\alpha)^*(\mathscr{F}) = \bigoplus_{g \in G}\left(\bigoplus_{h \in \alpha^{-1}(g)} \mathscr{F}_h\right)$$

and

$$\mathcal{G} = \bigoplus_{g \in G} \mathcal{G}_g \longmapsto \mathscr{B}\mathsf{D}(\alpha)_*(\mathcal{G}) = \bigoplus_{h \in H} \mathcal{G}_{\alpha(h)},$$

respectively.

If $\mathscr{F}$ or $\mathcal{G}$ are sheaves of algebras then $\mathscr{B}\mathsf{D}(\alpha)^*(\mathscr{F})$ or $\mathscr{B}\mathsf{D}(\alpha)_*(\mathcal{G})$ are sheaves of algebras in the obvious way, respectively.

If $\alpha\colon H \hookrightarrow G$ is injective with cokernel G/H, so that $\mathsf{D}(G/H)$ is the kernel of the quotient map $\mathsf{D}(\alpha)\colon \mathsf{D}(G) \to \mathsf{D}(H)$, then

$$\mathscr{B}\mathsf{D}(\alpha)_*(\mathcal{G}) = \mathcal{G}_{|H} := \bigoplus_{h \in H} \mathcal{G}_{\alpha(h)} = \mathcal{G}^{\mathsf{D}(G/H)}.$$

Proposition 1.34. *Let $0 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} G/H \to 0$ be an exact sequence of abelian groups and $\mathcal{X}$ a pseudo-algebraic fiber category. A G/H-family of line bundles on $\mathcal{X}$ corresponds naturally to a G-family $\mathscr{F}$ of line bundles on $\mathcal{X}$ together with a trivialization of $\mathscr{F}_{|H}$.*

Proof. This is just a translation of Corollary 1.21 using Proposition 1.31. □

Remark 1.35. In the situation of Proposition 1.34, we make the induced G/H-family explicit, using Remark 1.33. A trivialisation of the H-family $\mathscr{F}_{|H}$ is a collection of isomorphisms $\xi_h\colon \mathscr{F}_h \to \mathcal{O}_\mathcal{X}$ for $h \in H$ such that the morphism

$$\xi\colon (\mathcal{R}_\mathscr{F})^{\mathsf{D}(G/H)} = \mathcal{R}_{\mathscr{F}_{|H}} = \bigoplus_{h \in H} \mathscr{F}_h \to \mathcal{O}_\mathcal{X}$$

is an $\mathcal{O}_\mathcal{X}$-algebra morphism. The $\mathsf{D}(G/H)$-torsor induced by $\mathscr{F}$ and the trivialization ξ is given by the fibre product in (1) of Lemma 1.20. The corresponding G/H-graded sheaf of algebras $\mathcal{R}$ is therefore given by

$$\mathcal{R} = \mathcal{R}_\mathscr{F} \otimes_{\mathcal{R}_{\mathscr{F}_{|H}}} \mathcal{O}_\mathcal{X}.$$

More explicitly, by defining the ideal

$$\mathcal{I} := \ker(\xi) = \langle \xi_h^{-1}(1) - 1 \mid h \in H \rangle < \mathcal{R}_{\mathscr{F}_{|H}},$$

we obtain that

$$\mathcal{R} = \mathcal{R}_\mathscr{F} \otimes_{\mathcal{R}_{\mathscr{F}_{|H}}} \mathcal{O}_\mathcal{X} = \mathcal{R}_\mathscr{F}/\mathcal{I}\mathcal{R}_\mathscr{F}.$$

Finally, for all $u \in G/H$,

$$(\mathcal{R}_\mathscr{F}/\mathcal{I}_\xi \mathcal{R}_F)_u = \frac{\bigoplus_{g \in \beta^{-1}(u)} \mathscr{F}_g}{\mathcal{I}_\xi \bigoplus_{g \in \beta^{-1}(u)} \mathscr{F}_g},$$

where $\beta\colon G \to G/H$ is the projection.

2. Existence and uniqueness of Cox rings

In this section, we first show existence and uniqueness of Cox rings of invertible sheaves. From this we obtain analogous results for the Cox rings of reflexive sheaves. Finally, we compare these results to the construction in [2].

2.1. **Cox ring of invertible sheaves.**

Proposition 2.1. *Let G be an abelian group. Then there is a short exact sequence $0 \to K_1 \to K_0 \to G \to 0$ with K_1 and K_0 free abelian groups.*

Proof. Consider $K_0 = \mathbb{Z}^{(G)}$ and the natural surjection $\mathbb{Z}^{(G)} \to G$. Its kernel K_1 is free, since any subgroup of a free abelian group is free, see, for example, [9, Thm. I.7.3]. $\square$

If $\mathcal{X}$ is a pseudo-algebraic fibred category, then we denote by $\overline{\mathrm{Pic}(\mathcal{X})}$ the full subcategory of $\mathrm{QCoh}(\mathcal{X})$ of invertible sheaves and with $\mathrm{Pic}(\mathcal{X})$ the set of their isomorphism classes.

For $\mathcal{F}$ a G-family of line bundles on $\mathcal{X}$, we denote by

$$\alpha_{\mathcal{F}} \colon G \to \mathrm{Pic}(\mathcal{X}), \quad g \mapsto [\mathcal{F}_g]$$

the morphism induced by $\mathcal{F}$, which is a group homomorphism. We say that a G-family $\mathcal{F}$ *lifts* a group homomorphism $\mu \colon G \to \mathrm{Pic}(\mathcal{X})$ if $\mathcal{F}_g \simeq \mu(g)$ for all $g \in G$.

Lemma 2.2. *Let K be a free abelian group and $\alpha \colon K \to \mathrm{Pic}(\mathcal{X})$ be a group homomorphism. Then the morphism α lifts to a unique K-family up to isomorphism.*

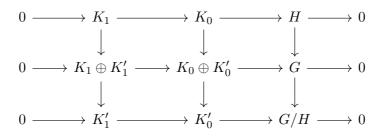*Proof.* This follows immediately from Example 1.32. $\square$

Theorem 2.3. *Let $H < G$ be abelian groups and $\mathcal{F}$ be an H-family of line bundles on $\mathcal{X}$. For a group homomorphism $\mu \colon G \to \mathrm{Pic}(\mathcal{X})$ which extends $\alpha_{\mathcal{F}} \colon H \to \mathrm{Pic}(\mathcal{X})$ holds:*

- *there is a G-family of line bundles $\mathcal{G}$ lifting μ and extending $\mathcal{F}$, that is with $\mathcal{G}_{|H} \simeq \mathcal{F}$;*
- *the set of isomorphism classes of G-families extending $\mathcal{F}$ and lifting μ is isomorphic to*

$$\mathrm{Ext}^1(G/H, H^0(\mathcal{O}_{\mathcal{X}}^*)).$$

Proof. We first show existence. Let $0 \to K_1 \to K_0 \to H \to 0$ and $0 \to K_1' \to K_0' \to G/H \to 0$ be free resolutions of H and G/H, respectively (see Proposition 2.1). By the horseshoe lemma, these two resolutions fit together to form a commutative diagram, where also the columns are short

exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_1 & \longrightarrow & K_0 & \longrightarrow & H & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1 \oplus K_1' & \longrightarrow & K_0 \oplus K_0' & \longrightarrow & G & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1' & \longrightarrow & K_0' & \longrightarrow & G/H & \longrightarrow & 0
\end{array}
$$

Using the surjection $K_0 \to H$, we can turn the H-family $\mathcal{F}$ into a K_0-family $\tilde{\mathcal{F}}$. Now extend $\tilde{\mathcal{F}}$ to a $K_0 \oplus K_0'$-family $\tilde{\mathcal{G}}$, lifting the homomorphism $K_0 \oplus K_0' \to G \to \mathrm{Pic}(\mathcal{X})$ as in Lemma 2.2. Both K_0-families $\tilde{\mathcal{G}}_{|K_0}$ and $\tilde{\mathcal{F}}$ lift the same homomorphism $K_0 \to \mathrm{Pic}(\mathcal{X})$. Thus we can choose an isomorphism $\tilde{\mathcal{G}}_{|K_0} \simeq \tilde{\mathcal{F}}$, again by Lemma 2.2. Using this isomorphism we also get that $\tilde{\mathcal{G}}_{|K_1} \simeq \tilde{\mathcal{F}}_{|K_1}$ are isomorphic to the trivial K_1-family. Since $\tilde{\mathcal{G}}_{|K_1'}$ lifts the trivial morphism $K_1' \to \mathrm{Pic}(\mathcal{X})$, it is isomorphic to the trivial K_1'-family as well. The two trivializations just defined give an isomorphism of $\tilde{\mathcal{G}}_{|(K_1 \oplus K_1')}$ and the trivial $(K_1 \oplus K_1')$-family. Applying Proposition 1.34 to $\tilde{\mathcal{G}}$, there is a G-family $\mathcal{G}$. By construction, $\mathcal{G}_{|H} \simeq \mathcal{F}$.

We now prove the second part of the statement. Fix one G-family $\mathcal{G}$ extending $\mathcal{F}$ and lifting μ. Given another lifting $\mathcal{G}'$, the G-family $\mathcal{N} := \mathcal{G}' \otimes \mathcal{G}^{-1}$ extends the trivial H-family and lifts the trivial morphism $\mathcal{N} \colon G \to \mathrm{Pic}(\mathcal{X})$. Thus we may assume that $\mu \colon G \to \mathrm{Pic}(\mathcal{X})$ is trivial, so $\mathcal{G}_g \simeq \mathcal{O}_{\mathcal{X}}$ for all $g \in G$. Moreover by Proposition 1.34, we may even assume that $H = 0$.

Consider now the exact sequences of groups

$$(2) \qquad\qquad 0 \to K \to \mathbb{Z}^{(G-0)} \to G \to 0.$$

It is easy to show that K is generated by the elements $e_{g,h} = e_g + e_h - e_{g+h}$, where we set $e_0 := 0$. Note that by our assumption on the G-family $\mathcal{G}$, the structure morphisms $\xi_{g,h}$ are elements of $H^0(\mathcal{O}_{\mathcal{X}})^*$ and satisfy the following relations: $\xi_{g,0} = 1$, $\xi_{g,h} = \xi_{h,g}$ and $\xi_{g,h}\xi_{g+h,t} = \xi_{h,t}\xi_{g,h+t}$.

A direct computation on the relations between the $e_{g,h}$ shows that $\xi \colon K \to H^0(\mathcal{O}_{\mathcal{X}})^*$, $e_{g,h} \mapsto \xi_{g,h}$ is a group homomorphism. Let $\mathcal{G}'$ be another G-family such that $\mathcal{G}'_g = \mathcal{O}_{\mathcal{X}}$ and $\eta \in \mathrm{Hom}(K, H^0(\mathcal{O}_{\mathcal{X}})^*)$ its structure morphisms. An isomorphism between $\mathcal{G}$ and $\mathcal{G}'$ is a collection of $\mu_g \in H^0(\mathcal{O}_{\mathcal{X}})^*$ for all $0 \neq g \in G$ such that

$$\xi(e_{g,h})\mu_g\mu_h\mu_{g+h}^{-1} = \eta(e_{g,h}),$$

which follows from the definition of a morphism and the identifications we made. In other words, an isomorphism is given by $\mu \in \mathrm{Hom}(\mathbb{Z}^{(G-0)}, H^0(\mathcal{O}_{\mathcal{X}})^*)$ such that

$$\xi(e_{g,h})\mu(e_{g,h}) = \eta(e_{g,h}).$$

Since the $e_{g,h}$ generates K, the above relation just means $\xi\mu_{|K} = \eta$ in $\mathrm{Hom}(K, H^0(\mathcal{O}_{\mathcal{X}})^*)$.

In conclusion the set of isomorphism classes of G-families lifting the trivial morphism $G \to \mathrm{Pic}(\mathcal{X})$ is in one-to-one correspondence with the quotient of $\mathrm{Hom}(K, H^0(\mathcal{O}_\mathcal{X})^*)$ by the subgroup of elements which extends to $\mathbb{Z}^{(G-0)}$. But applying $\mathrm{Hom}(-, H^0(\mathcal{O}_\mathcal{X})^*)$ to the sequence (2) we get an exact sequence

$$\mathrm{Hom}(\mathbb{Z}^{(G-0)}, H^0(\mathcal{O}_\mathcal{X})^*) \to \mathrm{Hom}(K, H^0(\mathcal{O}_\mathcal{X})^*) \to \mathrm{Ext}^1(G, H^0(\mathcal{O}_\mathcal{X})^*) \to 0$$

which ends the proof. $\qquad\square$

Remark 2.4. By the above proposition all G-families extending a given H-family $\mathcal{F}$ and lifting a given group homomorphism μ are isomorphic if

- G/H is free; or
- the abelian group $H^0(\mathcal{O}_\mathcal{X}^*)$ is divisible (which is equivalent to injective), for instance the group of units of an algebraically closed field,

as in both cases the Ext^1-group in question is zero.

Corollary 2.5. *Let* $\mu\colon G \to \mathrm{Pic}(\mathcal{X})$ *be a group homomorphism. Then the set of isomorphism classes of G-families lifting μ is in bijection with* $\mathrm{Ext}^1(G, H^0(\mathcal{O}_\mathcal{X}^*))$. *In particular there is a G-family of line bundles lifting* μ.

Definition 2.6. *Let $\mathcal{X}$ be a pseudo-algebraic fibered category. A Pic-*Cox ring* $\mathsf{R}_{\mathrm{Pic}}(\mathcal{X})$ of $\mathcal{X}$ is the Cox ring associated with a $\mathrm{Pic}(\mathcal{X})$-family lifting the identity $\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X})$. Analogously, we define a Pic-*Cox sheaf* $\mathcal{R}_{\mathrm{Pic}}(\mathcal{X})$ of $\mathcal{X}$.*

Putting together Theorem 2.3 and Remark 2.4, we obtain the following theorem.

Theorem 2.7. *Let $\mathcal{X}$ be a pseudo-algebraic fibered category. Then $\mathcal{X}$ admits a Pic-Cox ring $\mathsf{R}_{\mathrm{Pic}}(\mathcal{X})$ and the set of isomorphism classes of such Cox rings is in bijection to*

$$\mathrm{Ext}^1(\mathrm{Pic}(\mathcal{X}), H^0(\mathcal{O}_\mathcal{X}^*)).$$

In particular a Pic-Cox ring is unique up to isomorphism if

- $\mathrm{Pic}(\mathcal{X})$ *is free; or*
- $H^0(\mathcal{O}_\mathcal{X}^*)$ *is divisible, e.g. equal to $\Bbbk^*$ for an algebraically closed field $\Bbbk$.*

Corollary 2.8. *Let $\mathcal{X}$ be a pseudo-algebraic fibered category over an algebraically closed field $\Bbbk$. If $H^0(\mathcal{O}_\mathcal{X})^* = \Bbbk^*$ (e.g. if $\mathcal{X}$ is proper, reduced and connected), then the Pic-Cox ring $\mathsf{R}_{\mathrm{Pic}}(\mathcal{X})$ is well-defined up to graded isomorphisms, that is there exists a $\mathrm{Pic}(\mathcal{X})$-family lifing $\mathrm{Pic}(\mathcal{X}) \xrightarrow{\mathrm{id}} \mathrm{Pic}(\mathcal{X})$ and it is unique up to isomorphism.*

Example 2.9. Let us show an example of a variety X over a field $\Bbbk$ with non isomorphic Pic-Cox rings. We will pick an affine variety, in which case non ($\mathrm{Pic}(X)$-graded) isomorphic Pic-Cox rings just means non isomorphic $\mathrm{Pic}(X)$-family lifting $\mathrm{Pic}(X) \xrightarrow{\mathrm{id}} \mathrm{Pic}(X)$. By Theorem 2.3, their isomorphism classes are parametrized by

$$\mathrm{Ext}^1(\mathrm{Pic}(X), H^0(\mathcal{O}_X)^*).$$

Moreover we will pick an affine variety with $\mathrm{Pic}(X) \simeq \mathbb{Z}/d\mathbb{Z}$ with $d \geq 2$. In this case we must show that

$$\mathrm{Ext}^1(\mathrm{Pic}(X), H^0(\mathcal{O}_X)^*) \simeq H^0(\mathcal{O}_X)^*/(H^0(\mathcal{O}_X)^*)^d \neq 0$$

Consider an irreducible homogeneous polynomial $F \in \Bbbk[x_0, \ldots, x_n]$ of degree $d \geq 2$ and consider the complement Y of the hypersurface defining by F inside $\mathbb{P}^n_\Bbbk$. One has $\mathrm{Pic}(Y) = \mathbb{Z}/d\mathbb{Z}$. If $(\Bbbk^*)^d \neq \Bbbk^*$, then $X = Y$ has the desired property.

In the case that $\Bbbk$ is algebraically closed (and therefore $(\Bbbk^*)^d = \Bbbk^*$), we can choose $X = Y \times \mathbb{G}_m$, where $\mathbb{G}_m = \mathrm{Spec}\,\mathbb{Z}[t, t^{-1}]$. By [5, Ex. III.12.6], we get that $\mathrm{Pic}(X) \simeq \mathrm{Pic}(Y) \times \mathrm{Pic}(\mathbb{G}_m) \simeq \mathbb{Z}/d\mathbb{Z}$ is still cyclic of order d. But $t \in H^0(\mathcal{O}_X)^*$ is not a d-th power.

2.2. Cox ring of reflexive sheaves.

In this section, we extend Theorem 2.3 for families of reflexive sheaves of rank one. This will require additional conditions on $\mathcal{X}$.

If $\mathcal{X}$ is an algebraic stack, we denote with $\overline{\mathrm{Ref}_1(\mathcal{X})}$ the subcategory of $\mathrm{QCoh}(\mathcal{X})$ of reflexive sheaves of rank 1 (on the generic points of an atlas) and with $\mathrm{Ref}_1(\mathcal{X})$ the set of their isomorphism classes.

From now on let $\mathcal{X}$ be a Noetherian, normal and excellent algebraic stack and $\mathcal{U}$ be its regular locus which is open and dense. Notice that the complement of $\mathcal{U}$ in $\mathcal{X}$ has codimension at least 2. We refer to [11, §6] for the notion of codimension for algebraic stacks.

Lemma 2.10. *Let $j \colon \mathcal{V} \hookrightarrow \mathcal{X}$ be an open immersion whose complement has codimension at least 2. Then the restriction of sheaves*

$$j^* \colon \overline{\mathrm{Ref}_1(\mathcal{X})} \to \overline{\mathrm{Ref}_1(\mathcal{V})}, \quad \mathcal{L} \mapsto \mathcal{L}_{|\mathcal{V}}$$

is an equivalence of categories with j_* its inverse. In particular,

$$\overline{\mathrm{Ref}_1(\mathcal{X})} \to \overline{\mathrm{Ref}_1(\mathcal{U})} = \overline{\mathrm{Pic}(\mathcal{U})}, \quad \mathcal{L} \mapsto \mathcal{L}_{|\mathcal{U}}$$

is well defined and an equivalence. Moreover, $\mathrm{Ref}_1(\mathcal{X})$ is an abelian group isomorphic to $\mathrm{Pic}(\mathcal{U})$: given $\mathcal{L}, \mathcal{L}' \in \mathrm{Ref}_1(\mathcal{X})$ their product is $(\mathcal{L} \otimes \mathcal{L}')^{\vee\vee}$ and the inverse of $\mathcal{L}$ is $\mathcal{L}^\vee$.

Proof. Let us show that j^* is essentially surjective. Let $\mathcal{L} \in \overline{\mathrm{Pic}(\mathcal{V})}$. Since $j_*\mathcal{L}$ is a quasi-coherent sheaf with $(j_*\mathcal{L})_{|\mathcal{V}} \simeq \mathcal{L}$, from [10, Cor. 15.5] there exists a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ such that $\mathcal{F}_{|\mathcal{V}} \simeq \mathcal{L}$. Then the sheaf $\mathcal{F}^{\vee\vee}$ is reflexive by [6, Cor 1.2] and restricts to $\mathcal{L}$.

By [6, Prop 1.6] and the hypothesis on the codimension it follows that $\mathcal{L} \to j_*j^*\mathcal{L}$ is an isomorphism for reflexive sheaves on $\mathcal{X}$. In particular $j_* \colon \overline{\mathrm{Ref}_1(\mathcal{V})} \to \overline{\mathrm{Ref}_1(\mathcal{X})}$ is well defined and a quasi-inverse of $j_* \colon \overline{\mathrm{Ref}_1(\mathcal{X})} \to \overline{\mathrm{Ref}_1(\mathcal{V})}$.

For the second part one has $\overline{\mathrm{Ref}_1(\mathcal{U})} = \overline{\mathrm{Pic}(\mathcal{U})}$ because $\mathcal{U}$ is regular. The last statement is an easy consequence of [6, Cor 1.2]. $\square$

Corollary 2.11. *Let $j \colon \mathcal{U} \hookrightarrow \mathcal{X}$ be the inclusion of the regular locus. The restriction induces an equivalence between the category of G-families of sheaves on $\mathcal{X}$ and the category of G-families of invertible sheaves on $\mathcal{U}$.*

Proof. By Remark 1.9, the restriction preserves families as $j \colon \mathcal{U} \hookrightarrow \mathcal{X}$ is flat. Let $\mathcal{F}$ be a G-family of invertible sheaves on $\mathcal{U}$. Pushing forward $\mathcal{G} = j_*\mathcal{F}$

we get all the data of a family of sheaves on $\mathcal{X}$. The only condition missing is the fact that $\mathcal{G}_g \to \mathcal{H}om_{\mathcal{X}}(\mathcal{G}_{g'}, \mathcal{G}_{g+g'})$ is an isomorphism. Since this map restrict to the corresponding map for the family $\mathcal{F}$, it is enough to observe that

$$\mathcal{H}om_{\mathcal{X}}(\mathcal{G}_{g'}, \mathcal{G}_{g+g'}) \simeq j_* \, \mathcal{H}om_{\mathcal{U}}(\mathcal{F}_{g'}, \mathcal{F}_{g+g'})$$

is a reflexive sheaf of rank 1. $\qquad\square$

Corollary 2.12. *Let X be a Noetherian, normal and excellent scheme. Then*

$$\mathrm{Ref}_1(X) \simeq \mathrm{Cl}(X) \simeq \mathrm{Pic}(U)$$

where $\mathrm{Cl}(X)$ is the group of Weil divisors modulo equivalence and U is the regular locus of X.

Proof. Indeed one has $\mathrm{Cl}(X) = \mathrm{Cl}(U) = \mathrm{Pic}(U)$ and $\mathrm{Ref}_1(X) = \mathrm{Pic}(U)$ by Corollary 2.11. $\qquad\square$

Definition 2.13. Let $\mathcal{X}$ be a noetherian, normal and excellent algebraic stack. A Ref_1-*Cox ring* $\mathsf{R}_{\mathrm{Ref}}(\mathcal{X})$ of $\mathcal{X}$ is a Cox ring associated with a $\mathrm{Ref}_1(\mathcal{X})$-family lifting the identity of $\mathrm{Ref}_1(\mathcal{X})$. Analogously, we define a Ref_1-*Cox sheaf* $\mathscr{R}_{\mathrm{Ref}}(\mathcal{X})$ of $\mathcal{X}$.

As an application we get the following theorem.

Theorem 2.14. *If $\mathcal{X}$ is a noetherian, normal and excellent algebraic stack then Theorem 2.7 and Corollary 2.8 hold if we replace Pic by Ref_1.*

Proof. One has $\mathrm{Pic}(\mathcal{X}) = \mathrm{Pic}(\mathcal{U})$ and $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})^* = H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}})^*$. So everything follows from Corollary 2.11. $\qquad\square$

We end this subsection by a comparison between Ref_1-Cox sheaf and Pic-Cox sheaf. Recall that for an H-torsor $\mathscr{P}$ over a pseudo-algebraic fibered category $\mathcal{X}$, we have $\mathcal{X} = [\mathscr{P}/\mathsf{H}]$. In particular for the Pic-Cox ring we get that

$$\mathcal{X} = \left[\mathrm{Spec}_{\mathcal{X}} \, \mathscr{R}_{\mathrm{Pic}}(\mathcal{X}) \Big/ \mathsf{D}(\mathrm{Pic}(\mathcal{X})) \right].$$

Proposition 2.15. *Let $\mathcal{X}$ be a Noetherian, normal and excellent algebraic stack and assume that $T = \mathrm{Ref}_1(\mathcal{X})/\mathrm{Pic}(\mathcal{X})$ is finite. Then there is a factorization*

$$[\mathrm{Spec}_{\mathcal{X}} \, \mathscr{R}_{\mathrm{Ref}}(\mathcal{X})/\mathsf{D}(\mathrm{Ref}_1(\mathcal{X}))] \to \mathcal{Y} \to \mathcal{X},$$

where the first map is finite and $\mathcal{Y} \to \mathcal{X}$ is a (relative) $\mathsf{D}(T)$-gerbe. In particular, the above map is proper.

Proof. Note that we can realize $\mathscr{R}' := \mathscr{R}_{\mathrm{Pic}}(\mathcal{X})$ as a graded subsheaf of algebras of $\mathscr{R} := \mathscr{R}_{\mathrm{Ref}}(\mathcal{X})$, since $\mathrm{Pic}(\mathcal{X}) \subseteq \mathrm{Ref}_1(\mathcal{X})$. There is an exact sequence

$$1 \to \mathsf{D}(T) \to \mathsf{D}(\mathrm{Ref}_1(\mathcal{X})) \to \mathsf{D}(\mathrm{Pic}(\mathcal{X})) \to 1$$

and, moreover, we have a Cartesian diagram

$$
\begin{array}{ccc}
\mathrm{Spec}_{\mathcal{X}} \, \mathscr{R} & \xrightarrow{\quad\alpha\quad} & \mathrm{Spec}_{\mathcal{X}} \, \mathscr{R}' \\
\downarrow & & \downarrow \\
\left[\mathrm{Spec}_{\mathcal{X}} \, \mathscr{R} \Big/ \mathsf{D}(\mathrm{Ref}_1(\mathcal{X})) \right] & \xrightarrow{\;\overline{\alpha}\;} & \left[\mathrm{Spec}_{\mathcal{X}} \, \mathscr{R}' \Big/ \mathsf{D}(\mathrm{Ref}_1(\mathcal{X})) \right] =: \mathcal{Y}
\end{array}
$$

because α is $\mathsf{D}(\mathrm{Ref}_1(\mathcal{X}))$-equivariant. In particular, the induced map $\overline{\alpha}$ is finite, if α is finite.

To see that α is finite, we can actually assume that $\mathcal{X}$ is affine (but keeping the original groups $\mathrm{Pic}(\mathcal{X})$ and $\mathrm{Ref}_1(\mathcal{X})$). If we consider $\mathcal{R}$ as an $\mathcal{R}'$-module, then it is generated by $\mathcal{O}_{\mathcal{X}}$-generators of the $\mathcal{R}_q$, where q runs through a system of generators of T in $\mathrm{Ref}_1(\mathcal{X})$. For this note that if $q \in \mathrm{Ref}_1(\mathcal{X})$ and $t \in \mathrm{Pic}(\mathcal{X})$ then $\mathcal{R}_{t+q} \simeq \mathcal{R}'_t \otimes \mathcal{R}_q$ by Remark 1.4. Now consider the Cartesian diagram

$$
\begin{array}{ccc}
\mathcal{Y} = [\mathrm{Spec}_{\mathcal{X}}\, \mathcal{R}'/\mathsf{D}(\mathrm{Ref}_1(\mathcal{X}))] & \longrightarrow & \mathcal{B}\mathsf{D}(\mathrm{Ref}_1(\mathcal{X})) \\
\downarrow & & \downarrow \\
\mathcal{X} = [\mathrm{Spec}_{\mathcal{X}}\, \mathcal{R}'/\mathsf{D}(\mathrm{Pic}(\mathcal{X}))] & \longrightarrow & \mathcal{B}\mathsf{D}(\mathrm{Pic}(\mathcal{X}))
\end{array}
$$

The equation in the bottom left corner holds since $\mathrm{Spec}_{\mathcal{X}}\, \mathcal{R}'$ is a $\mathsf{D}(\mathrm{Pic}(\mathcal{X}))$-torsor over $\mathcal{X}$. Hence $\mathcal{Y} \to \mathcal{X}$ is a base change of $\mathcal{B}\mathsf{D}(\mathrm{Ref}_1(\mathcal{X})) \to \mathcal{B}\mathsf{D}(\mathrm{Pic}(\mathcal{X}))$, which is a relative $\mathsf{D}(T)$-gerbe.

For the last claim we observe that $\mathsf{D}(T) \to \mathrm{Spec}\,\mathbb{Z}$ is proper. Indeed it is separated because the diagonal of $\mathsf{D}(T)$ is (fppf) locally of the form $\mathsf{D}(T) \to \mathrm{Spec}\,\mathbb{Z}$, which is finite. Moreover $\mathcal{B}\mathsf{D}(T) \to \mathrm{Spec}\,\mathbb{Z}$ is a universal homeomorphism, hence universally closed. $\qquad\square$

2.3. Divisorial approach for the Cox ring.
In this section, we show that our construction of a Cox ring is a generalisation of the contruction in [2, §1.4] for varieties.

Let X be an integral, Noetherian, normal scheme and choose a short exact sequence

$$
0 \to K_1 \to K_0 \to \mathrm{Cl}(X) \to 0
$$

of abelian groups where K_0 and, therefore, K_1 are free, where $\mathrm{Cl}(X)$ is the group of Weil divisors modulo equivalence. Recall that $\mathrm{Cl}(X) \simeq \mathrm{Ref}_1(X)$ and they are both isomorphic to $\mathrm{Cl}(U) \simeq \mathrm{Pic}(U)$, where U is the regular locus of X (see Corollary 2.12).

We can lift $K_0 \to \mathrm{Cl}(X)$ to a map $E \colon K_0 \to \mathrm{WDiv}(X), k \mapsto E_k$, where $\mathrm{WDiv}(X)$ is the group of Weil divisors on X. This defines a K_0-family $\mathcal{G}$ of reflexive sheaves of rank one with components

$$
\mathcal{G}_k = \mathcal{O}_X(E_k),
$$

where the multiplication is given by the usual multiplication of rational sections in $\Bbbk(X)$. The Cox sheaf of algebras associated with the K_0-family $\mathcal{G}$ is

$$
\mathcal{S}(X) := \mathcal{R}_{\mathcal{G}} = \bigoplus_{k \in K_0} \mathcal{G}_k
$$

Note that $\mathcal{S}(X)$ is the sheaf of divisorial algebras for the surjection $K_0 \twoheadrightarrow \mathrm{Cl}(X)$, as introduced in [2, §1.3.1].

By construction $\mathcal{G}_{|K_1}$ is trivial. A trivialization $\xi_k \colon \mathcal{G}_k \to \mathcal{O}_X$ is induced by a group homomorphism

$$
\zeta \colon K_1 \to \Bbbk(X)^*
$$

such that $\mathrm{div}(\zeta(k)) = E_k$, so that $\xi_k(f) = f/\zeta(k)$. Consider the ideal

$$\mathcal{I} = \langle \zeta(k) - 1 \mid k \in K_1 \rangle \subseteq \bigoplus_{k \in K_1} \mathcal{O}_X(E_k)$$

where $\zeta(k) \in \mathcal{O}_X(E_k)$, more precisely $\mathcal{O}_X(E_k) = \zeta(k)\mathcal{O}_X$ and

$$\mathcal{R} = \mathcal{R}_{\mathcal{G}}/\mathcal{I}\mathcal{R}_{\mathcal{G}}.$$

This gives exactly the Cox sheaf $\mathcal{R}(X)$ as constructed in [2, §1.4.2]. We claim that $\mathcal{R}$ is the Ref_1-Cox sheaf $\mathcal{R}_{\mathrm{Ref}}(X)$. Note that here $\mathrm{Ref}_1(X) = \mathrm{Cl}(X)$.

We make use of Corollary 2.11. Set $\mathcal{F}$ for the K_0-family of line bundles on the regular locus U restriction of $\mathcal{G}$. In particular $\mathcal{R}_{\mathcal{F}} = (\mathcal{R}_{\mathcal{G}})_{|U}$. The trivialization of $\mathcal{G}_{|K_1}$ defines a trivialization of $\mathcal{F}_{|K_1}$.

By Remark 1.35 we obtain a $\mathrm{Pic}(U)$-family $\overline{\mathcal{F}}$ of line bundles on U and therefore a $\mathrm{Cl}(X)$-family $\overline{\mathcal{G}}$ of reflexive sheaves on X such that $\overline{\mathcal{G}}_{|U} = \mathcal{F}$. Again by Remark 1.35, we have

$$(\mathcal{R}_{\overline{\mathcal{G}}})_{|U} \simeq \mathcal{R}_{\overline{\mathcal{F}}} \simeq \mathcal{R}_{|U}$$

In order to conclude that $\mathcal{R}_{\overline{\mathcal{G}}} \simeq \mathcal{R}$, it is enough to notice that $\mathcal{R}$ is $\mathrm{Cl}(X)$-graded and each graded piece is isomorphic to a graded piece of $\mathcal{R}_{\mathcal{G}}$ and therefore reflexive of rank 1.

More precisely we have

$$(3) \qquad (\mathcal{R}_{\mathcal{G}})_k = \mathcal{G}_k = \mathcal{O}_X(E_k) \simeq \mathcal{R}_{\pi(k)} \text{ for } k \in K_0$$

where $\pi \colon K_0 \to \mathrm{Cl}(X)$.

3. Applications to Mori dream stacks

In this section, let $\Bbbk$ be an algebraically closed field of characteristic zero. Let X be a normal integral scheme of finite type over $\Bbbk$ such that X has only constant invertible functions and a finitely generated divisor class group.

As in Subsection 2.3, let $\mathcal{R}(X) := \mathcal{R}_{\mathrm{Ref}}(X)$ be the Ref_1-Cox sheaf and $\mathcal{S}(X)$ the sheaf of divisorial algebras appearing in the construction of $\mathsf{R}(X) := \mathsf{R}_{\mathrm{Ref}}(X)$, after choosing some surjection $K_0 \twoheadrightarrow \mathrm{Cl}(X)$ and a lift $K_0 \to \mathrm{WDiv}(X)$.

Proposition 3.1. *In the above situation, $\mathcal{S}(X) \twoheadrightarrow \mathcal{R}(X)$ induces an equivalence of quotient stacks*

$$\left[\mathrm{Spec}_X \mathcal{R}(X) \Big/ \mathsf{D}(\mathrm{Cl}(X)) \right] \simeq \left[\mathrm{Spec}_X \mathcal{S}(X) \Big/ \mathsf{D}(K_0) \right].$$

Proof. We apply Proposition 1.24 to the short exact sequence

$$1 \to \mathsf{D}(\mathrm{Cl}(X)) \xrightarrow{\alpha} \mathsf{D}(K_0) \to \mathsf{D}(K_1) \to 1$$

and to $\mathcal{R} = \mathcal{R}(X)$. By Remark 1.33 and (3) of Subsection 2.3 we can conclude that $\hat{\alpha}_* \mathcal{R} = \mathcal{S}(X)$, so that the statement follows. $\qquad\square$

Lemma 3.2. *In the situation above, the ring $\mathsf{S}(X)$ is a unique factorization domain and $\mathrm{Spec}_X \mathcal{S}(X)$ is a locally factorial scheme of finite type over $\Bbbk$.*

If X has affine diagonal (e.g. separated), then $\mathrm{Spec}_X \mathcal{S}(X) \to \mathrm{Spec}\,\mathsf{S}(X)$ is an open immersion whose complement has codimension at least 2.

Proof. As $\mathsf{S}(X)$ is constructed using a surjection $K_0 \twoheadrightarrow \mathrm{Cl}(X)$, the ring $\mathsf{S}(X)$ is a unique factorization domain by [2, Thm. 1.3.3.3]. Moreover, since X is $\mathbb{Q}$-factorial, $\mathrm{Spec}_X \mathscr{S}(X) \to X$ is of finite type, see [2, Proposition 1.3.2.3].

Now assume that X has affine diagonal. By construction $V = \mathrm{Spec}_X \mathscr{S}(X)$ and $Y = \mathrm{Spec}\,\mathsf{S}(X)$ have the same global sections, because

$$H^0(V, \mathcal{O}_V) = H^0(X, \mathscr{S}(X)) = \mathsf{S}(X).$$

By [2, Corollary 1.3.4.6] the scheme V is quasi-affine, hence $V \hookrightarrow Y$ is an open immersion.

Assume by contradiction that the complement of V in Y has codimension lower than 2. This means that there exists a prime ideal P of $\mathsf{S}(X)$ of height 1 such that $P \notin V$. Since $\mathsf{S}(X)$ is UFD, the prime P is principal, say $P = (g)$, which implies that $V \subseteq \mathrm{Spec}\,\mathsf{S}(X)_g$. Since $H^0(\mathcal{O}_V) = H^0(\mathcal{O}_Y) = \mathsf{S}(X)$, it would follow that g is invertible, which is not the case.

Now we prove in general that $V = \mathrm{Spec}_X \mathscr{S}(X)$ is a locally factorial scheme. When X has affine diagonal, this follows from the fact that V is an open subset of $\mathrm{Spec}\,\mathsf{S}(X)$. To see this, denote by $\pi\colon \mathrm{Spec}_X \mathsf{S}(X) \to X$ the structure morphism. If U is an open affine subset of X one has that $\mathscr{S}(X)_{|U}$ is again a sheaf of divisorial algebras for U, because $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is surjective. In particular

$$\mathrm{Spec}_U(\mathscr{S}(X)_{|U}) = \pi^{-1}(U)$$

is locally factorial as required.   □

With Proposition 3.1 we can compare Mori dream spaces and stacks in the spirit of [8, Prop. 2.9].

Definition 3.3. A *Mori dream space* over $\Bbbk$ is an integral and normal scheme X of finite type over $\Bbbk$ such that:

- X is $\mathbb{Q}$-factorial with affine diagonal (e.g. X separated),
- $H^0(X, \mathcal{O}_X^*) = \Bbbk^*$,
- $\mathrm{Cl}(X) = \mathrm{Ref}_1(X)$ is finitely generated as a $\mathbb{Z}$-module,
- its Ref_1-Cox ring $\mathsf{R}_{\mathrm{Ref}}(X)$ is finitely generated as a $\Bbbk$-algebra.

Remark 3.4. This definition is slightly more general than in [7, 8], where we assumed a Mori dream space to be separated.

Note that for a Mori dream space its Cox sheaf $\mathscr{R}_{\mathrm{Ref}}(X)$ is well defined up to (graded) isomorphism. In particular it does not depend on the choices made in Subsection 2.3.

Definition 3.5. Let X be a Mori dream space. We call

$$\mathcal{X}^{can} := \left[\mathrm{Spec}_X \mathscr{R}_{\mathrm{Ref}}(X) \Big/ \mathsf{D}(\mathrm{Cl}(X)) \right]$$

the *canonical MD-stack* of X.

Lemma 3.6. *Let X be a Mori dream space, U its regular locus and $\pi\colon \mathcal{X}^{can} \to X$ be the structure morphisms. Then $\mathcal{X}^{can}$ is a normal stack of finite type over $\Bbbk$, $\mathrm{Ref}_1(\mathcal{X}^{can}) = \mathrm{Pic}(\mathcal{X}^{can})$ and the complement of $U = \pi^{-1}(U)$ in $\mathcal{X}^{can}$ has codimension at least 2.*

Proof. From Lemma 3.2 we see that $\mathcal{X}^{can}$ is normal and of finite type.

Recall that $\mathrm{Cl}(X) = \mathrm{Ref}_1(X)$ by Corollary 2.12. Let $0 \to K_1 \to K_0 \xrightarrow{\alpha} \mathrm{Cl}(X) \to 0$ be a short exact sequence with K_0, K_1 free abelian groups. By Proposition 3.1, $\mathcal{X}^{can}$ is isomorphic to the quotient stack

$$\left[\mathrm{Spec}_X \, \mathcal{S}(X) \Big/ \mathsf{D}(K_0) \right],$$

where as before $\mathcal{S}(X)$ is the sheaf of divisorial algebras appering in the construction of $\mathsf{R}_{\mathrm{Ref}}(X)$.

Since $\mathrm{Spec}_X \, \mathcal{S}(X)$ is a locally factorial scheme by Lemma 3.2, by Corollary 2.12 all reflexive sheaves of rank 1 on $\mathrm{Spec}_X \, \mathcal{S}(X)$ are invertible. In particular if we denote by $f \colon \mathrm{Spec}_X \, \mathcal{S}(X) \to \mathcal{X}^{can}$ the structure map, which is a smooth atlas, for all $\mathcal{L} \in \mathrm{Ref}_1(\mathcal{X}^{can})$ the sheaves $f^*\mathcal{L}$ and, by descent, $\mathcal{L}$ itself are invertible. Thus $\mathrm{Ref}_1(\mathcal{X}^{can}) = \mathrm{Pic}(\mathcal{X}^{can})$.

Set $p \colon \mathrm{Spec}_X \, \mathcal{S}(X) \to X$. As $\mathrm{Ref}_1(X) = \mathrm{Pic}(U)$, the restriction of p to U is a $\mathsf{D}(K_0)$-torsor and therefore $\pi^{-1}(U) \to U$ is an isomorphism. By [2, Proposition 1.3.2.8] the inverse image of U on $\mathrm{Spec}_X \, \mathcal{S}(X)$ has complement of codimension at least 2. As $\mathrm{Spec}_X \, \mathcal{S}(X) \to \mathcal{X}^{can}$ is a smooth atlas, by definition we have that $U = \pi^{-1}(U)$ has complement in $\mathcal{X}^{can}$ of codimension at least 2. $\square$

Remark 3.7. From Lemma 3.2 we see that $\mathcal{X}^{can}$ is not just normal, but it has a smooth atlas $\mathrm{Spec}_X \, \mathcal{S}(X) \to \mathcal{X}^{can}$ with $\mathrm{Spec}_X \, \mathcal{S}(X)$ locally factorial.

In general, this does not imply that $\mathrm{Spec}_X \, \mathcal{S}(X)$ is smooth, in which case $\mathcal{X}^{can}$ would be smooth and therefore the canonical stack of X in the sense of [4, §4.1].

For instance consider $X = \mathrm{Spec}\, A$ with

$$A = \Bbbk[x_1, \ldots, x_n]/(x_1^2 + \cdots + x_n^2) \text{ for } n \geq 5.$$

The algebra A is UFD by the Klein-Nagata theorem, but not regular because the Jacobian criterion fails. In particular $\mathrm{Cl}(X) = \mathrm{Pic}(X) = 0$ and $H^0(X, \mathcal{O}_X)^* = A^* = \Bbbk^*$, since A is a $\mathbb{N}$-graded domain and therefore an invertible element of A has to be homogeneous of degree 0.

In conclusion X is a Mori dream space and $\mathcal{X}^{can} = X$ is not regular.

Theorem 3.8. *Let X be a Mori dream space and $\mathcal{X}^{can}$ its canonical MD-stack. Then the structure morphism $\pi \colon \mathcal{X}^{can} \to X$ induces maps*

$$\overline{\mathrm{Pic}}(\mathcal{X}^{can}) = \overline{\mathrm{Ref}_1}(\mathcal{X}^{can}) \xleftarrow{\quad\longrightarrow\quad} \overline{\mathrm{Ref}_1}(X)$$

$$\mathcal{F} \longmapsto \pi_*\mathcal{F}$$

$$(\pi^*\mathcal{G})^{\vee\vee} \longleftarrow\!\shortmid \; \mathcal{G}$$

which are well defined and quasi-inverses of each other.

In particular, we get that $H^0(\mathcal{X}^{can}, \mathcal{O}^*_{\mathcal{X}^{can}}) = H^0(X, \mathcal{O}_X) = \Bbbk^*$. Moreover, we have

$$\pi_* \mathscr{R}_{\mathrm{Pic}}(\mathcal{X}^{can}) \simeq \mathscr{R}_{\mathrm{Ref}}(X) \text{ and } \mathsf{R}_{\mathrm{Pic}}(\mathcal{X}^{can}) \simeq \mathsf{R}_{\mathrm{Ref}}(X).$$

Proof. Let $i \colon U = \pi^{-1}(U) \to \mathcal{X}^{can}$ be the open immersion and $j = \pi i \colon U \to X$ the inclusion.

The equality $\mathrm{Ref}_1(\mathcal{X}^{can}) = \mathrm{Pic}(\mathcal{X}^{can})$ and the fact that U has complement of codimension at least 2 in $\mathcal{X}^{can}$ follow from Lemma 3.6.

From Lemma 2.10 we have that

$$i_*\colon \overline{\mathrm{Ref}}_1(U) \to \overline{\mathrm{Ref}}_1(\mathcal{X}^{can}) \text{ and } j_*\colon \overline{\mathrm{Ref}}_1(U) \to \overline{\mathrm{Ref}}_1(X)$$

are equivalences, quasi-inverses of the corresponding restrictions. It follows that

$$\pi_*\colon \overline{\mathrm{Ref}}_1(\mathcal{X}^{can}) \to \overline{\mathrm{Ref}}_1(X)$$

is an equivalence. Moreover if $\mathcal{G} \in \overline{\mathrm{Ref}}_1(X)$ then $(\pi^*\mathcal{G})^{\vee\vee}$ is a reflexive sheaf whose restriction to U is $\mathcal{G}_{|U}$.

The last part of the statement follows from Corollary 2.11. $\square$

Remark 3.9. As for any pseudo-algebraic fibered category, we find that

$$\mathcal{X}^{can} = \left[\mathrm{Spec}_{\mathcal{X}^{can}} \, \mathcal{R}_{\mathrm{Pic}}(\mathcal{X}^{can}) \Big/ \mathsf{D}(\mathrm{Pic}(\mathcal{X}^{can})) \right].$$

So by its very definition, we get that $\mathrm{Spec}_{\mathcal{X}^{can}} \, \mathcal{R}_{\mathrm{Pic}}(\mathcal{X}^{can}) = \mathrm{Spec}_X \, \mathcal{R}_{\mathrm{Ref}}(X)$. Moreover, by [2, Prop. 1.6.3.3], $\mathrm{Spec}_X \, \mathcal{R}_{\mathrm{Ref}}(X) \hookrightarrow \mathrm{Spec} \, \mathsf{R}_{\mathrm{Ref}}(X)$ is an open immersion, whose complement is given by the *irrelevant ideal* which is of codimension at least 2.

Additionally by Proposition 2.15, the map $\mathcal{X}^{can} \to X$ is proper. Hence if X is separated, so is $\mathcal{X}^{can}$.

Therefore, by the last part of Theorem 3.8, we get for separated X that $\mathcal{X}^{can}$ is a MD-quotient stack in the sense of [8], under the hypothesis that $H^0(\mathrm{Spec} \, \mathsf{R}_{\mathrm{Pic}}(\mathcal{X}^{can}), \mathcal{O}^*) = H^0(\mathrm{Spec} \, \mathsf{R}_{\mathrm{Ref}}(X), \mathcal{O}^*) = \Bbbk^*$.

This was shown already in [8, Prop. 2.9] with two additional assumptions: namely that $H^0(\mathrm{Spec} \, \mathsf{R}_{\mathrm{Ref}}(X), \mathcal{O}^*) = \Bbbk^*$ (again) and $\mathrm{Pic}(\mathrm{Spec} \, \mathsf{R}_{\mathrm{Ref}}(X)) = 0$.

So Theorem 3.8 allows to weaken the definition of MD-quotient stack $\mathcal{X}$ in [8]: instead of asking for $H^0(\mathrm{Spec} \, \mathsf{R}_{\mathrm{Pic}}(\mathcal{X}), \mathcal{O}^*) = \Bbbk^*$, the weaker condition $H^0(\mathcal{X}, \mathcal{O}^*) = \Bbbk^*$ is sufficient.

As another consequence, the assumptions on $\mathrm{Spec} \, \mathsf{R}_{\mathrm{Ref}}(X)$ can be dropped in all statements in [8], which involve [8, Prop. 2.9]. Most notable in the main theorem [8, Thm. 3.2] about lifting maps between Mori dream spaces, and [8, Thm. 4.3] where a classification of Mori dream quotient stacks is given in terms of root constructions (which is already a generalisation of the corresponding result in [7]).

References

[1] Marco Antei, Indranil Biswas, Michel Emsalem, Fabio Tonini and Lei Zhang, *Nori fundamental gerbe of essentially finite covers and Galois closure of towers of torsors*, Selecta Mathematica 25 (2019), 18, also `arXiv:1706.00739`.

[2] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen and Antonio Laface, *Cox Rings*, Cambridge University Press (2014), first chapter also `arXiv:1003.4229`.

[3] Michel Demazure and Alexandre Grothendieck (eds.), *Séminaire de Géométrie Algébrique du Bois Marie - Schémas en groupes - (SGA 3) - vol. 1*, Springer (1970), also `https://webusers.imj-prg.fr/~patrick.polo/SGA3`.

[4] Barbara Fantechi, Étienne Mann and Fabio Nironi, *Smooth toric Deligne-Mumford stacks*, J. Reine Angew. Math. 648, 201–244 (2010), also `arXiv:0708.1254`.

[5] Robin Hartshorne, *Algebraic Geometry*, Springer (1977).

[6] Robin Hartshorne, *Stable Reflexive Sheaves*, Math. Ann. 254 (1980), 121–176.

[7] Andreas Hochenegger and Elena Martinengo, *Mori dream stacks*, Math. Z. 280 (2015), 1185–1202, also `arXiv:1403.7984`.

[8] Andreas Hochenegger and Elena Martinengo, *Maps of Mori dream spaces*, J. Pure Appl. Algebr. 222 (2018), 1287 – 1305, also arXiv:1605.06789.
[9] Serge Lang, *Algebra*, Springer (2011).
[10] Laurent Moret-Bailly and Gerard Laumon, *Champs algébriques*, Springer (1999).
[11] Brian Osserman, *Relative dimension of morphisms and dimension for algebraic stacks*, J. of Algebra 437 (2015), 52–78, also arXiv:1305.6328.
[12] Mattia Talpo and Angelo Vistoli, *Infinite root stacks and quasi-coherent sheaves on logarithmic schemes*, Proc. Lond. Math. Soc. 117 (2018), 1187–1243, also arXiv:1410.1164.
[13] Fabio Tonini, *Sheafification functors and Tannaka's reconstruction*, arXiv:1409.4073.

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