S -matrix algebra of the AdS2×S2 superstring

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The S-matrix algebra of the $\mathbf{AdS_2 \times S^2}$ superstring

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Abstract

In this paper we find the Yangian algebra responsible for the integrability of the $AdS_2 \times S^2 \times T^6$ superstring in the planar limit. We demonstrate the symmetry of the corresponding exact S-matrix in the massive sector, including the presence of the *secret* symmetry. We give two alternative presentations of the Hopf algebra. The first takes the usual canonical form, which, as the relevant representations are long, leads to a Yangian representation that is not of evaluation type. After investigating the relationship between co-commutativity, evaluation representations and the shortening condition, we find an alternative realisation of the Yangian whose representation is of evaluation type. Finally we explore two limits of the S-matrix. The first is the classical r-matrix, where we re-discover the need for a *secret* symmetry also in this context. The second is the simplifying zero-coupling limit. In this limit, taking the S-matrix as a generating R-matrix for the Algebraic Bethe Ansatz, we obtain an effective model of free fermions on a periodic spin-chain. This limit should provide hints to the *one-loop* anomalous dimension of the mysterious superconformal quantum mechanics dual to the superstring theory in this geometry.

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1 Introduction

The remarkable impact integrability has had on the solution of string theory in the $AdS_5 \times S^5$ background [1] motivates trying to apply the same strategy to the other less supersymmetric string backgrounds which are still integrable [2]. One of these is indeed the $AdS_2 \times S^2 \times T^6$ background with Ramond-Ramond fluxes in Type II superstring theory, which preserves 8 supersymmetries. One way to generate this background is by taking the near-horizon limit of various intersecting brane configurations in Type IIA/B supergravity, related by T-dualities in the T^6 directions [3]. The dual "field" theory [4] is thought to be either a superconformal quantum mechanics, or a chiral two-dimensional CFT [5]. AdS_2 holography, and $AdS_2 \times S^2$ in particular, is an interesting open problem that can be approached from several distinct directions (see for instance [6] for some recent accounts). Given the reduced dimensionality, it would be tempting to regard it as the simplest example where one can test the AdS/CFT duality, but instead it turns out to be one of the most mysterious.

The $AdS_2 \times S^2$ (coset) part of the background is conveniently encoded into a Metsaev-Tseytlin [7] type action [8], based on the quotient

$$\frac{PSU(1,1|2)}{SO(1,1) \times SO(2)} .$$

The algebra $\mathfrak{psu}(1,1|2)$ admits a $\mathbb{Z}_4$ automorphism, which is traditionally the key to the super-coset model being classically integrable. Indeed, this is what happens in the $AdS_5 \times S^5$ [9] and $AdS_3 \times S^3 \times M^4$ [10] cases. In the AdS_2 case one can truncate the Green-Schwarz action [11] to the coset degrees of freedom, however there is no choice of κ-symmetry gauge which decouples the coset from the remaining fermions [12]. The integrability of the Green-Schwarz action for the full ten-dimensional background has been shown up to quadratic order in fermions [12–14].

In our previous paper [15], we have utilised the symmetries of the system and its conjectured quantum integrability to determine the exact S-matrix for the worldsheet scattering of magnon excitations, taking the light-cone gauge-fixed [16] $AdS_2 \times S^2 \times T^6$ action to infinite length. This S-matrix describes the scattering above the BMN vacuum [17], which is a point-like string travelling at the speed of light on a great circle of S^2. The light-cone gauge-fixed Lagrangian [18] is highly non-trivial and breaks two-dimensional Lorentz symmetry. Only the quadratic action preserves the Lorentz group, and describes $2+2$ (bosons+fermions) massive plus $6+6$ massless modes. The massive bosonic excitations are associated to the transverse directions in $AdS_2 \times S^2$, while the massless ones are associated to the T^6 directions.

By following a procedure which has been successful in AdS_5 [19, 20] and AdS_3 [21–23], in [15] we fixed (up to an overall factor) the S-matrix for the excitations transforming under the $\mathfrak{psu}(1|1)^2 \ltimes \mathbb{R}$ symmetry of the BMN vacuum. In order to do that, we relaxed the level-matching condition and postulated the presence of two central extensions, while simultaneously deforming the coproduct in the standard fashion [24]. The resulting massive S-matrix satisfies the Hopf-algebra crossing relation [25], and is unitarity so long as the overall factor (dressing phase) satisfies a certain constraint. We also studied the near-BMN expansion under certain assumptions for the dressing phase, finding consistency with the perturbative computations [18, 26].

The main difference with the AdS_5 and AdS_3 cases is that the representations which scatter are *long*, and there is no shortening condition to be interpreted as the magnon dispersion relation. Furthermore, because of reducibility of the tensor product representation, the S-matrix depends on an undetermined function, which we fixed by imposing the Yang-Baxter equation. Similar features were observed in [27] for long representations in AdS_5, and in the Pohlmeyer reduction of $AdS_2 \times S^2$ superstrings [28]. Finally, the S-matrix enjoys an accidental $U(1)$ symmetry under which only the fermionic excitations are charged, and which is connected to the presence of T^6 [18]. This $U(1)$ allows for the existence of a pseudo-vacuum state, and could be instrumental to derive the Bethe equations conjectured in [12] from our S-matrix.

Although it is not completely clear what representation one should adopt for the massless modes [3, 18], in [15] we took the approach advocated in [29] for the AdS_3 case, and assumed that the massless representations and the corresponding S-matrix are the *zero-mass / finite h* limit of the corresponding massive ones (cf. [30]), at least as far as the $\mathfrak{psu}(1|1)^2 \ltimes \mathbb{R}$ building block is concerned. We obtained in this way the limiting S-matrices for all the choices of left and right chiralities, and discovered that there exists a canonical Yangian for the massless sector.

In this paper, we obtain several results on the algebraic structure of the exact S-matrix of the system, therefore deepening our understanding of the associated spectral problem. Our aim is to explore the Yangian symmetry for the massive sector. The Yangian relevant to the AdS_5 S-matrix was found in [31], while for AdS_3 it was found in [22] in separate sectors, and a larger version encompassing both left and right algebras was discovered in [32]. The approach we will follow is based on the RTT formulation [33], which was first applied to the AdS_5 S-matrix in [34], and later to the AdS_3 S-matrix in [35].

As discussed above, the crucial new feature in the case of interest is that the representations are long. As a result the canonical realisation of the Yangian, similar in spirit to those in [31, 22], results in a representation that is not of evaluation type. This is also a feature for long representations in the AdS_5 case [27]. After investigating this realisation, we find a new alternative realisation of the Yangian that does lead to a representation of evaluation type. This gives more control over the symmetry and its action on one-particle states. Indeed the evaluation representation is the most natural physical manifestation of Yangians in integrable scattering problems. Furthermore, this new realisation and the original one are contained with in a larger family of realisations originating from a symmetry of the restricted Yangian algebra.

Besides the $\mathcal{Y}(\mathfrak{su}_c(2|2))$ Yangian, the S-matrix of the $AdS_5 \times S^5$ superstring admits an additional infinite tower of conserved charges, which constitute the so called *secret symmetry* of the model [36, 37] (see [38] for a review). Such symmetries are present in several other parts of the correspondence, for example in the pure spinor sigma model [39], scattering amplitudes [40] and Wilson loops [41]. Recently, secret symmetries were also found for the AdS_3 superstring [35], providing further evidence of their universal nature in the AdS/CFT framework.

We will begin in section 2 with a brief summary of the key properties of Yangians that will be necessary for the following exposition. In section 3 we use the RTT formulation to construct the Yangian algebra underlying the integrability of the massive sector, including the *secret* symmetry. Two distinct realisations of this symmetry are given, the first of which is close in spirit to that used in the AdS_5 and AdS_3 cases [34, 35] and in general leads to a representation not of evaluation type, while the second is a new realisation leading to a representation that is of evaluation type. We then discuss the issue of evaluation representations in detail, demonstrating a relation to shortening (massless) condition.

In section 4 we explore the strong- and weak-coupling limits. First, we perform a study of the classical r-matrix, and discover that the need for the *secret* symmetry, based on the residue-analysis at the simple pole in the spectral-parameter plane, is present also in this context. Second, we study an effective Bethe ansatz in the simplifying limit of zero coupling, in which the problem reduces to a standard rational spin-chain in each copy of the symmetry algebra. In this way we obtain the spectrum of free fermions on a periodic chain. This should represent an entry-point to the leading-order (traditionally dubbed *one-loop*) anomalous dimension for the composite operators of the mysterious superconformal quantum mechanics, supposed to be dual to the superstring theory.

2 Yangians and integrability

We start by reviewing some of the underlying formalism of Yangians and their various realisations. Here we focus on the details that will be relevant for us when investigating the Yangian of the massive sector of the $AdS_2 \times S^2 \times T^6$ superstring.

Yangians: Drinfeld's First Realisation. If $\mathfrak{g}$ is a Lie superalgebra, its *Yangian* $\mathcal{Y}(\mathfrak{g})$ can be obtained via a quantum deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_u)$, where $\mathfrak{g}_u$ is the polynomial loop algebra of $\mathfrak{g}$ in the variable u. More precisely,

$$\mathcal{Y}(\mathfrak{g}) = \bigcup_{m\in\mathbb{N}} \mathcal{Y}_m(\mathfrak{g}) \ , \qquad \mathcal{Y}_m(\mathfrak{g}) = \text{span}\, \mathfrak{J}^A_{(m)} \ , \tag{2.1}$$

where the vector space $\mathcal{Y}_m(\mathfrak{g})$ corresponds to the m-th *level* of the Yangian $\mathcal{Y}(\mathfrak{g})$. In the so-called *Drinfeld's first realisation*, the generators $\mathfrak{J}^I_{(m)}$ fulfil the graded commutation relations

$$\big[\mathfrak{J}^A_{(m)}, \mathfrak{J}^B_{(n)}\big\} = f^{AB}{}_C\, \mathfrak{J}^C_{(m+n)} \;, \qquad m+n = 0,1 \;, \tag{2.2}$$

with $f^{AB}{}_C$ being the structure constants of $\mathfrak{g}$. Equations (2.2), together with an appropriate set of *Serre relations*, determine $\mathcal{Y}(\mathfrak{g})$ uniquely. Notice that $\mathcal{Y}_0(\mathfrak{g}) \equiv \mathfrak{g}$.

Yangians enjoy the translational symmetry

$$\mathfrak{J}^A_{(0)} \to \mathfrak{J}^A_{(0)} \;, \qquad \mathfrak{J}^A_{(1)} \to \mathfrak{J}^A_{(1)} + \lambda\, \mathfrak{J}^A_{(0)} \;, \qquad \lambda \in \mathbb{C} \;, \tag{2.3}$$

which can be clearly seen from (2.1).

A special type of representation is the *evaluation representation* π_E:

$$\pi_E\big(\mathfrak{J}^A_{(m)}\big) = u^m\, \pi\big(\mathfrak{J}^A_{(0)}\big) \;, \tag{2.4}$$

where u is called the *spectral parameter*, and π is a chosen representation of $\mathfrak{g}$. However, π_E might not exist for all representations π. One reason possible reason for this is that inserting π_E into the Serre relations actually imposes strong constraints back on π itself, which may or may not be satisfied.

Yangians and integrable systems. Quantum systems exhibiting Yangian symmetry are integrable, as a consequence of the fact that the Yangian allows to construct a solution to the Yang-Baxter equation which controls the inverse scattering problem (and often the S-matrix of the excitations) [42]. Indeed, let $R \in A \otimes A$ be a scattering matrix, with $A = \mathcal{Y}(\mathfrak{g})$ being the Hopf superalgebra whose coproduct $\Delta : A \to A \otimes A$ defines the action of the conserved charges on two-particles states. If A and R are such that

$$\begin{aligned} &\Delta^{\rm op}(a)\, R = R\, \Delta(a) && \forall a \in A \;, && \text{(quasi co-commutativity)},\\ &(\Delta \otimes \mathbb{1})(R) = R_{13} R_{23}, && (\mathbb{1} \otimes \Delta)(R) = R_{13} R_{12} \;, && \text{(quasi-triangularity)}, \end{aligned} \tag{2.5}$$

where $\Delta^{\rm op} = (\sigma \circ \Delta)$ is the opposite coproduct, σ the graded permutation operator and the subscripts 1,2,3 indicate the copy of A in the triple tensor product, then R obeys the *quantum Yang-Baxter equation* (QYBE)

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \tag{2.6}$$

(the hallmark of integrability) and the *crossing symmetry* equations. This is the content of a famous theorem of Drinfeld, proving the crucial role played by quantum groups in producing solutions to the QYBE with the desired symmetry properties [43].

RTT Realisation of the Yangian. Given a scattering matrix $R_{12}(u,v)$ that is a function of two spectral parameters and satisfies the QYBE, one can extract the symmetries of the system using the RTT realisation of the corresponding Yangian [33, 34]:

$$R_{12}(u,v)\, \mathcal{T}_{13}(u) \mathcal{T}_{23}(v) = \mathcal{T}_{23}(v) \mathcal{T}_{13}(u)\, R_{12}(u,v) \;. \tag{2.7}$$

The monodromy matrix $\mathcal{T}(u)$ plays the role of a generating function for the Yangian charges, which are in turn symmetries of R. We shall now summarise the main features of this construction: see [34] for further details. We will focus on the $\mathfrak{gl}(n|n)$ case, the one of interest.

Let $\{e^A{}_B\}$ be the standard basis for the $\mathfrak{gl}(n|n)$ Lie superalgebra. That is the matrices $e^A{}_B$ are such that their only non-vanishing entry is $(-)^{[B]}$ in row A, column B. The symbol $[A]$ stands for the Graßmann grading of the index A. $\mathcal{T}(u)$ can then be written as follows:

$$\mathcal{T}(u) = \sum_{A,B} (-)^{[B]} \, e^B{}_A \otimes \mathbb{T}^A{}_B(u) \ , \qquad \mathbb{T}^A{}_B : \mathbb{C} \to \mathcal{Y}(\mathfrak{gl}(n|n)) \ . \tag{2.8}$$

Assuming that $\mathbb{T}^A{}_B(u)$ is holomorphic in a neighbourhood of $u = \infty$, the asymptotic expansion

$$\mathbb{T}^A{}_B(u) = \sum_{l \in \mathbb{N}} u^{-l} \, \mathbb{T}^A_{l-1B} \ , \tag{2.9}$$

is well defined. At this point, one finds that particular combinations of the $\mathbb{T}^A_{l-1B}$ can be engineered to define Drinfeld's first realisation of $\mathcal{Y}(\mathfrak{gl}(n|n))$. In particular, defining

$$\mathbb{U}^{[B]} \delta^A_B = \mathbb{T}^A_{-1B} \ , \qquad \mathbb{J}^A_{0B} = \mathbb{U}^{-[B]} \, \mathbb{T}^A_{0B} \ , \qquad \mathbb{J}^A_{1B} = \mathbb{U}^{-[B]} \, \mathbb{T}^A_{1B} - \frac{1}{2} \mathbb{J}^A_{0C} \, \mathbb{J}^C_{0B} \ , \tag{2.10}$$

$\mathbb{J}_0$ and $\mathbb{J}_1$ span $\mathfrak{gl}(n|n)$ and $\mathcal{Y}_1(\mathfrak{gl}(n|n))$, respectively. The central element $\mathbb{U}$ is the *braiding factor*, and represents a deformation of the co-algebra structure. $\mathbb{U} = \mathbb{1}$ represents the undeformed case.

The key observation is that the R-matrix is a particular representation of $\mathcal{T}$, namely, $R(u, v) = (\mathbb{1} \otimes \pi_v) \, \mathcal{T}(u)$, where π_v indicates a representation depending on the spectral parameter v. Therefore, the coefficients in the Laurent expansion of $R(u, v)$ can be understood as $\mathbb{T}^A_{l-1B}$ for the RTT realisation of the underlying symmetry in the representation of interest. This can then be recast in the form of Drinfeld's first realisation using (2.10). The (representation-independent) graded commutation relations for the $\mathbb{J}^A_{mB}$ are obtained from those for the $\mathbb{T}^A_{l-1B}$, by plugging (2.8) and (2.9) into (2.7) and expanding with respect to both u and v.

Finally, the complete Hopf algebra structure of $\mathcal{Y}(\mathfrak{gl}(n|n))$, in particular, the coproducts and antipodes, can be recovered from the RTT realisation of the Yangian. First, the fusion relation

$$\Delta(\mathbb{T}^A{}_B(u)) = \mathbb{T}^A{}_C(u) \otimes \mathbb{T}^C{}_B(u) \ , \tag{2.11}$$

descending from the R-matrix fusion relations, provides the coproducts for the individual generators via the same expansion: indeed, expanding $\mathbb{T}^A{}_B(u)$ in inverse powers of u gives

$$\begin{aligned} &\Delta(\mathbb{U}) = \mathbb{U} \otimes \mathbb{U} \ , \qquad \Delta(\mathbb{T}^A_{0B}) = \mathbb{T}^A_{0B} \otimes \mathbb{U}^{[B]} + \mathbb{U}^{[A]} \otimes \mathbb{T}^A_{0B} \ , \\ &\Delta(\mathbb{T}^A_{1B}) = \mathbb{T}^A_{1B} \otimes \mathbb{U}^{[B]} + \mathbb{U}^{[A]} \otimes \mathbb{T}^A_{1B} + \mathbb{T}^A_{0C} \otimes \mathbb{T}^C_{0B} \ . \end{aligned} \tag{2.12}$$

This can be used to derive the coproducts for $\mathbb{J}_{0,1}$:

$$\begin{aligned} &\Delta(\mathbb{U}) = \mathbb{U} \otimes \mathbb{U} \ , \qquad \Delta(\mathbb{J}^A_{0B}) = \mathbb{J}^A_{0B} \otimes \mathbb{1} + \mathbb{U}^{[A]-[B]} \otimes \mathbb{J}^A_{0B} \\ &\Delta(\mathbb{J}^A_{1B}) = \mathbb{J}^A_{1B} \otimes \mathbb{1} + \mathbb{U}^{[A]-[B]} \otimes \mathbb{J}^A_{1B} \\ &\qquad + \frac{1}{2} \mathbb{U}^{[C]-[B]} \mathbb{J}^A_{0C} \otimes \mathbb{J}^C_{0B} - \frac{1}{2} (-)^{([A]+[C])([B]+[C])} \mathbb{U}^{[A]-[C]} \mathbb{J}^C_{0B} \otimes \mathbb{J}^A_{0C} \ . \end{aligned} \tag{2.13}$$

Second, the antipode Σ is a graded linear anti-homomorphism satisfying

$$\Sigma \left[\mathbb{T}^A{}_C(u) \right] \mathbb{T}^C{}_B(u) = \mathbb{T}^A{}_C(u) \Sigma \left[\mathbb{T}^C{}_B(u) \right] = \delta^A_B \ , \qquad \Sigma \left[XY \right] = (-)^{[X][Y]} \Sigma \left[Y \right] \Sigma \left[X \right] \ . \tag{2.14}$$

By expanding:

$$\begin{aligned} &\Sigma \left[\mathbb{T}^A_{-1B} \right] = \mathbb{U}^{-[B]} \delta^A{}_B \ , \qquad \Sigma \left[\mathbb{T}^A_{0B} \right] = -\mathbb{U}^{-[A]-[B]} \mathbb{T}^A_{0B} \ , \\ &\Sigma \left[\mathbb{T}^A_{1B} \right] = -\mathbb{U}^{-[A]-[B]} \mathbb{T}^A_{1B} + \mathbb{U}^{-[A]-[B]-[C]} \mathbb{T}^A_{0C} \mathbb{T}^C_{0B} \ , \end{aligned} \tag{2.15}$$

which in turn can be used to determine the antipodes for $\mathbb{J}_{0,1}$

$$
\begin{aligned}
&\Sigma\left[\mathbb{U}\right] = \mathbb{U}^{-1} \;, \qquad \Sigma\left[\mathbb{J}^A_{0B}\right] = -\mathbb{U}^{[B]-[A]}\mathbb{J}^A_{0B} \;, \\
&\Sigma\left[\mathbb{J}^A_{1B}\right] = -\mathbb{U}^{[B]-[A]}\mathbb{J}^A_{1B} + \frac{1}{2}\mathbb{U}^{[B]-[A]}[\mathbb{J}^A_{0C}, \mathbb{J}^C_{0B}\} \;.
\end{aligned} \tag{2.16}
$$

3 RTT realisation for the $\mathcal{Y}(\mathfrak{gl}_c(1|1))$ Yangian

In this section we employ the techniques of [33, 34] to construct the RTT realisation for the $\mathcal{Y}(\mathfrak{gl}_c(1|1))$ Yangian. The starting point is the $\mathfrak{su}_c(1|1)$ R-matrix of [15], which describes the scattering of one bosonic state $|\phi\rangle$ and one fermionic state $|\psi\rangle$ transforming in a long representation of the centrally-extended algebra $\mathfrak{su}_c(1|1)$

$$
\{\mathbb{Q},\mathbb{Q}\} = 2\,\mathbb{P} \;, \qquad \{\mathbb{S},\mathbb{S}\} = 2\,\mathbb{K} \;, \qquad \{\mathbb{Q},\mathbb{S}\} = 2\,\mathbb{H} \;. \tag{3.1}
$$

The explicit form of the representation is given by

$$
\begin{aligned}
&\mathbb{Q}|\phi\rangle = a|\psi\rangle \;, \qquad \mathbb{Q}|\psi\rangle = b|\phi\rangle \;, \qquad \mathbb{S}|\phi\rangle = c|\psi\rangle \;, \qquad \mathbb{S}|\psi\rangle = d|\phi\rangle \;, \\
&\mathbb{P}|\Phi\rangle = P|\Phi\rangle \;, \qquad \mathbb{K}|\Phi\rangle = K|\Phi\rangle \;, \qquad \mathbb{H}|\Phi\rangle = H|\Phi\rangle \;, \qquad |\Phi\rangle \in \{|\phi\rangle, |\psi\rangle\} \;.
\end{aligned} \tag{3.2}
$$

where the eigenvalues of the central elements $\mathbb{P}$, $\mathbb{K}$ and $\mathbb{H}$ are given by

$$
P = ab \;, \qquad K = cd \;, \qquad 2H = ad + bc \;, \tag{3.3}
$$

as a consequence of the algebra relations (3.1). There are no further conditions on the central elements for the algebra to close and hence this 2-dimensional representation is long.

For generic values of P, K and H the tensor product of two of these representations gives a 4-dimensional representation that is fully reducible into two 2-dimensional representations.[1] Therefore, the R-matrix acting on the tensor product is fixed up to two functions by demanding invariance under the symmetry

$$
\Delta^{\mathrm{op}}(\mathbb{J})\,R = R\,\Delta(\mathbb{J}) \;, \qquad \mathbb{J} = \{\mathbb{Q},\mathbb{S},\mathbb{P},\mathbb{K},\mathbb{H}\} \;. \tag{3.4}
$$

Here Δ is the coproduct, while Δ^{op} is the opposite coproduct defined in (2.5). As usual for integrable systems arising in the context of the AdS/CFT correspondence the coproduct is deformed through the introduction of an abelian generator $\mathbb{U}$

$$
\begin{aligned}
&\Delta(\mathbb{Q}) = \mathbb{Q}\otimes\mathbb{1} + \mathbb{U}\otimes\mathbb{Q} \;, \quad \Delta(\mathbb{S}) = \mathbb{S}\otimes\mathbb{1} + \mathbb{U}^{-1}\otimes\mathbb{S} \;, \quad \Delta(\mathbb{U}) = \mathbb{U}\otimes\mathbb{U} \;, \\
&\Delta(\mathbb{P}) = \mathbb{P}\otimes\mathbb{1} + \mathbb{U}^2\otimes\mathbb{P} \;, \quad \Delta(\mathbb{K}) = \mathbb{K}\otimes\mathbb{1} + \mathbb{U}^{-2}\otimes\mathbb{K} \;, \quad \Delta(\mathbb{H}) = \mathbb{H}\otimes\mathbb{1} + \mathbb{1}\otimes\mathbb{H} \;.
\end{aligned} \tag{3.5}
$$

To admit an R-matrix the coproducts of the central elements should be co-commutatitve, i.e. $\Delta^{\mathrm{op}}(\mathbb{C}) = \Delta(\mathbb{C})$. This relates the central charges $\mathbb{P}$ and $\mathbb{K}$ to the braiding factor $\mathbb{U}$:

$$
\mathbb{P} = \frac{h}{2}(1-\mathbb{U}^2) \;, \qquad \mathbb{K} = \frac{h}{2}(1-\mathbb{U}^{-2}) \;, \tag{3.6}
$$

where without loss of generality we have taken the constants of proportionality to be equal. In the following we will refer to the algebra (3.1) with these relations (3.6) imposed as the restricted

[1] At the special (massless) point $H^2_{\mathrm{tot}} - P_{\mathrm{tot}}K_{\mathrm{tot}} = 0$ the tensor product is still reducible but no longer decomposable. Here the subscript "tot" indicates that these are the eigenvalues of the central charges acting on the tensor product state.

algebra. While the symmetry only constrains the R-matrix up to two functions, demanding that the QYBE is solved (along with imposing various physical requirements such as crossing symmetry and a sensible strong coupling limit) fixes the R-matrix up to a single overall factor.

For reference we will quote the necessary details of the $\mathfrak{su}_c(1|1)$ R-matrix from [15]. The R-matrix in terms of the usual Zhukovsky variables [19, 44]

$$\frac{x^+}{x^-} = U^2 \ , \qquad x^+ - \frac{1}{x^+} - x^- + \frac{1}{x^-} = \frac{iH}{h} \ , \qquad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{iM}{h} \ . \tag{3.7}$$

Here

$$M = \frac{ad - bc}{2} = \sqrt{H^2 - PK} \ , \tag{3.8}$$

is unconstrained as the representation (3.2) is long. The representation parameters a, b, c and d are given in are given in terms of the Zhukovsky variables in [15] – we set the parameter α used there, which controls the normalisation of the bosonic state relative to the fermionic state, to one. The R-matrix is then given by

$$\begin{aligned} R\,|\phi_x\phi_y\rangle &= S_1\,|\phi_x\phi_y\rangle + Q_1\,|\psi_x\psi_y\rangle \ , \qquad & R\,|\psi_x\psi_y\rangle &= S_2\,|\psi_x\psi_y\rangle + Q_2\,|\phi_x\phi_y\rangle \ , \\ R\,|\phi_x\psi_y\rangle &= T_1\,|\phi_x\psi_y\rangle + R_1\,|\psi_x\phi_y\rangle \ , \qquad & R\,|\psi_x\phi_y\rangle &= T_2\,|\psi_x\phi_y\rangle + R_2\,|\phi_x\psi_y\rangle \ , \end{aligned} \tag{3.9}$$

with $x^\pm$ the kinematic variables associated to the first representation and $y^\pm$ to the second. The parameterising functions are given by [15]

$$\begin{aligned} S_1 &= \sqrt{\frac{x^+y^-}{x^-y^+}}\frac{x^- - y^+}{x^+ - y^-}\frac{1+s_1}{2}P_0 \ , & S_2 &= \frac{1+s_2}{2}P_0 \ , \\ T_1 &= \sqrt{\frac{y^-}{y^+}}\frac{x^+ - y^+}{x^+ - y^-}\frac{1+t_1}{2}P_0 \ , & T_2 &= \sqrt{\frac{x^+}{x^-}}\frac{x^- - y^-}{x^+ - y^-}\frac{1+t_2}{2}P_0 \ , \\ Q_1 = Q_2 &= -\frac{i}{2}\sqrt[4]{\frac{x^-y^+}{x^+y^-}}\frac{\eta_x\eta_y}{x^+ - y^-}\frac{f}{x^-x'^+}P_0 \ , & R_1 = R_2 &= -\frac{i}{2}\sqrt[4]{\frac{x^+y^-}{x^-y^+}}\frac{\eta_x\eta_y}{x^+ - y^-}P_0 \ , \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} f &= \frac{\sqrt{\frac{x^+}{x^-}}(x^- - \frac{1}{x^+}) - \sqrt{\frac{y^+}{y^-}}(y^- - \frac{1}{y^+})}{1 - \frac{1}{x^+x^-y^+y^-}}P_0 \ , \quad & s_1 &= \frac{1 - \frac{1}{x^+y^-}}{x^- - y^+}f \ , \quad & s_2 &= \frac{1 - \frac{1}{x^-y^+}}{x^+ - y^-}f \ , \\ \eta_x &= \sqrt{i(x^- - x^+)} \ , \quad \eta_y = \sqrt{i(y^- - y^+)} \ , & t_1 &= \frac{1 - \frac{1}{x^-y^-}}{x^+ - y^+}f \ , & t_2 &= \frac{1 - \frac{1}{x^+y^+}}{x^- - y^-}f \ . \end{aligned} \tag{3.11}$$

As discussed above, there is an overall factor P_0 that is not fixed by the considerations of symmetry and as such will not be relevant for the following analysis. For concreteness we will fix P_0 such that $S_1 = 1$ following [34] and so that the expansion of the R-matrix takes the form outlined in section 2.

Finally, let us observe that the $AdS_2 \times S^2$ worldsheet S-matrix underlying the scattering of the massive modes is built from the tensor product of two copies of this centrally-extended $\mathfrak{su}(1|1)$ R-matrix.

3.1 The $\mathcal{Y}(\mathfrak{su}_c(1|1))$ and $\mathcal{Y}(\mathfrak{gl}_c(1|1))$ Yangians and their RTT realisation

The centrally extended $\mathcal{Y}(\mathfrak{su}_c(1|1))$ Yangian is defined by the graded commutation relations

$$\{\mathbb{Q}_m, \mathbb{Q}_n\} = 2\,\mathbb{P}_{m+n} \ , \qquad \{\mathbb{S}_m, \mathbb{S}_n\} = 2\,\mathbb{K}_{m+n} \ , \qquad \{\mathbb{Q}_m, \mathbb{S}_n\} = 2\,\mathbb{H}_{m+n} \ , \tag{3.12}$$

which extend the algebra (3.1). Here, $m, n \geq 0$ indicate the level of the corresponding generator, with $\mathbb{Q}_0$, $\mathbb{S}_0$, $\mathbb{P}_0$, $\mathbb{K}_0$ and $\mathbb{H}_0$ playing the role of the original generators in (3.1).

It is worth noting that, in contrast to ordinary situations, the infinite-dimensional algebra (3.12) contains infinitely many finite-dimensional $\mathfrak{su}_c(1|1)$ subalgebras. Indeed, the set of generators $\mathbb{Q}_{\hat{m}}$, $\mathbb{S}_{\hat{n}}$, $\mathbb{P}_{2\hat{m}}$, $\mathbb{K}_{2\hat{n}}$ and $\mathbb{H}_{\hat{m}+\hat{n}}$ forms a subalgebra for all $\hat{m}, \hat{n} \geq 0$. This is due to the fact that the generators on the right-hand side of the relations (3.12) are central.

Another consequence of this, again in contrast to usual, is that for an arbitrary representation one cannot generate the whole infinite dimensional algebra by (anti) commuting a finite set of generators. However, considering the natural lift of the 2-dimensional representation (3.2)

$$
\begin{aligned}
&\mathbb{Q}_m|\phi\rangle = a_m|\psi\rangle \ , \quad \mathbb{Q}_m|\psi\rangle = b_m|\phi\rangle \ , \quad \mathbb{S}_m|\phi\rangle = c_m|\psi\rangle \ , \quad \mathbb{S}_m|\psi\rangle = d_m|\phi\rangle \ , \\
&\mathbb{P}_m|\Phi\rangle = P_m|\Phi\rangle \ , \quad \mathbb{K}_m|\Phi\rangle = K_m|\Phi\rangle \ , \quad \mathbb{H}_m|\Phi\rangle = H_m|\Phi\rangle \ , \quad |\Phi\rangle \in \{|\phi\rangle, |\psi\rangle\} \ ,
\end{aligned} \tag{3.13}
$$

it is relatively easy to see that the relations

$$
\begin{aligned}
2H_m &= a_0 d_m + b_0 c_m = a_1 d_{m-1} + b_1 c_{m-1} \ , \quad 2P_m = a_0 b_m + b_0 a_m = a_1 b_{m-1} + b_1 a_{m-1} \ , \\
2H_m &= c_0 b_m + d_0 a_m = c_1 b_{m-1} + d_1 a_{m-1} \ , \quad 2K_m = c_0 d_m + d_0 c_m = c_1 d_{m-1} + d_1 c_{m-1} \ ,
\end{aligned} \tag{3.14}
$$

which follow from the graded commutation relations (3.12), can be used to solve recursively for the higher-level representation parameters given their values at level 0 and 1.

As we will see, one property that does carry down from the higher-dimensional cases is that the $\mathfrak{su}_c(1|1)$ R-matrix (3.9), (3.10), (3.11) exhibits an additional family of symmetries, $\mathbb{B}_n$, $n \geq 1$, known as *bonus* or *secret*. These symmetries enhance the $\mathcal{Y}(\mathfrak{su}_c(1|1))$ Yangian to some *indented* Yangian-like quantum group we call $\mathcal{Y}(\mathfrak{gl}_c(1|1))$, which contains all the generators in (3.12) along with $\mathbb{B}_n$ (not including $\mathbb{B}_0$).

To construct the RTT realisation of the Yangian we introduce a spectral parameter u defined in terms of $x^\pm$, and similarly v for $y^\pm$. We shall use two different definitions for the spectral parameter, but in both cases the expansion of $x^\pm$ in powers of u^{-1} will take the form

$$
x^\pm = u + \mathcal{O}(1) \ . \tag{3.15}
$$

Once this expansion has been specified, we follow the construction outlined in section 2. That is we expand the R-matrix $R(x^\pm, y^\pm)$ in inverse powers of one of the two spectral parameters, say u,[2] and from the resulting Laurent coefficients extract a series of generators J_m whose graded commutation relations reproduce those of the underlying infinite-dimensional symmetry algebra. We can then define abstract generators $\mathbb{J}_m$, of which J_m are a representation. At this point we can construct generators satisfying (3.12) along with their coproducts and antipodes. It should be noted that this construction automatically leads to the restricted Yangian, for which, in addition to (3.6), the higher-level central charges $\mathbb{P}_m$ and $\mathbb{K}_m$ are defined in terms of lower-level central elements. We refer the reader to [34, 35] for a more complete discussion.

3.1.1 Canonical representation

Let us start by considering the canonical spectral parameter and a Hopf algebra structure that is close in spirit to that used in the AdS_5 and AdS_3 cases [34, 35]. The spectral parameter is

[2] For definiteness we will assume that $M(x^+, x^-) = 2$. For the second set of kinematical variables $y_\pm$, in terms of which we find the symmetry generators, we will not assume anything and consequently the symmetries we find hold for any M in (3.8). If we leave $M(x^+, x^-)$ unfixed it simply appears as an overall factor in the generators, for example Q_m , $S_m \sim M(x^+, x^-)^{\frac{1}{2}}$, and hence gives no new information.

given by

$$u = \frac{1}{2}\big(x^+ + \frac{1}{x^+} + x^- + \frac{1}{x^-}\big) \ , \tag{3.16}$$

such that (assuming $M(x^+,x^-)=2$) the expansions of $x^\pm$ are

$$x^\pm = u \pm \frac{i}{h} - \frac{1}{u} \pm \frac{i}{hu^2} + \mathcal{O}(u^{-3}) \ . \tag{3.17}$$

Following the procedure outlined above we then identify the following combinations[3]

$$\begin{aligned}
\mathbb{Q}_0 &= \sqrt{i}h\,\mathbb{J}^2_{01} \ , & \mathbb{Q}_1 &= \sqrt{i}h\,\mathbb{J}^2_{11} - \frac{i}{2}\left(\mathbb{1}+\mathbb{U}^2\right)\sqrt{-i}h\,\mathbb{J}^1_{02} \ , \\
\mathbb{S}_0 &= \sqrt{-i}h\,\mathbb{J}^1_{02} \ , & \mathbb{S}_1 &= \sqrt{-i}h\,\mathbb{J}^1_{12} + \frac{i}{2}\left(\mathbb{1}+\mathbb{U}^{-2}\right)\sqrt{i}h\,\mathbb{J}^2_{01} \ , \\
\mathbb{H}_0 &= \frac{ih}{2}\left(\mathbb{J}^1_{01}-\mathbb{J}^2_{02}\right) \ , & \mathbb{H}_1 &= \frac{ih}{2}\left(\mathbb{J}^1_{11}-\mathbb{J}^2_{12}\right) - \frac{ih}{4}\left(\mathbb{U}^2-\mathbb{U}^{-2}\right) \ , \\
\mathbb{P}_0 &= \frac{h}{2}\left(\mathbb{1}-\mathbb{U}^2\right) \ , & \mathbb{P}_1 &= -i\left(\mathbb{1}+\mathbb{U}^2\right)\mathbb{H}_0 \ , \\
\mathbb{K}_0 &= \frac{h}{2}\left(\mathbb{1}-\mathbb{U}^{-2}\right) \ , & \mathbb{K}_1 &= i\left(\mathbb{1}+\mathbb{U}^{-2}\right)\mathbb{H}_0 \ ,
\end{aligned} \tag{3.18}$$

which satisfy the defining commutation relations (3.12). Evaluated in the representation arising from the expansion of R-matrix, the level-0 generators coincide with those used in [15].

Coproducts. The level-0 coproducts are given in (3.5), while the level-1 coproducts can be constructed from (2.13) and read

$$\begin{aligned}
\Delta(\mathbb{Q}_1) &= \mathbb{Q}_1\otimes\mathbb{1} + \mathbb{U}\otimes\mathbb{Q}_1 \\
&\qquad - \frac{i}{h}\mathbb{Q}_0\otimes\mathbb{H}_0 + \frac{i}{h}\mathbb{U}\mathbb{H}_0\otimes\mathbb{Q}_0 + \frac{i}{h}\mathbb{U}^2\mathbb{S}_0\otimes\mathbb{P}_0 - \frac{i}{h}\mathbb{U}^{-1}\mathbb{P}_0\otimes\mathbb{S}_0 \ , \\
\Delta(\mathbb{S}_1) &= \mathbb{S}_1\otimes\mathbb{1} + \mathbb{U}^{-1}\otimes\mathbb{S}_1 \\
&\qquad + \frac{i}{h}\mathbb{S}_0\otimes\mathbb{H}_0 - \frac{i}{h}\mathbb{U}^{-1}\mathbb{H}_0\otimes\mathbb{S}_0 - \frac{i}{h}\mathbb{U}^{-2}\mathbb{Q}_0\otimes\mathbb{K}_0 + \frac{i}{h}\mathbb{U}\mathbb{K}_0\otimes\mathbb{Q}_0 \ , \\
\Delta(\mathbb{H}_1) &= \mathbb{H}_1\otimes\mathbb{1} + \mathbb{1}\otimes\mathbb{H}_1 + \frac{i}{h}\mathbb{U}^{-2}\mathbb{P}_0\otimes\mathbb{K}_0 + \frac{i}{h}\mathbb{U}^2\mathbb{K}_0\otimes\mathbb{P}_0 \ ,
\end{aligned} \tag{3.19}$$

where $\mathbb{P}_0$ and $\mathbb{K}_0$ are defined in terms of lower-level central elements in (3.18). Indeed inserting these definitions, the coproduct for $\mathbb{H}_1$ becomes manifestly co-commutative as expected.

The coproducts for $\mathbb{P}_1$ and $\mathbb{K}_1$ can be obtained from the graded commutation relations

$$\Delta(\mathbb{P}_1) = \frac{1}{2}\{\Delta(\mathbb{Q}_0),\Delta(\mathbb{Q}_1)\} \ , \qquad \Delta(\mathbb{K}_1) = \frac{1}{2}\{\Delta(\mathbb{S}_0),\Delta(\mathbb{S}_1)\} \ . \tag{3.20}$$

If $\mathbb{P}_1$ and $\mathbb{K}_1$ are defined in terms of lower-level central elements as in (3.18) we find that these coproducts are also co-commutative as required. Moreover, we can compute the coproducts for the level-2 central charges

$$\Delta(\mathbb{P}_2) = \frac{1}{2}\{\Delta(\mathbb{Q}_1),\Delta(\mathbb{Q}_1)\} \ , \quad \Delta(\mathbb{K}_2) = \frac{1}{2}\{\Delta(\mathbb{S}_1),\Delta(\mathbb{S}_1)\} \ , \quad \Delta(\mathbb{H}_2) = \frac{1}{2}\{\Delta(\mathbb{Q}_1),\Delta(\mathbb{S}_1)\} \ . \tag{3.21}$$

Doing so, we find that for co-commutativity of $\Delta(\mathbb{P}_2)$ and $\Delta(\mathbb{K}_2)$ we require

$$\begin{aligned}
\mathbb{P}_2 &= -i\left(\mathbb{1}+\mathbb{U}^2\right)\mathbb{H}_1 - \frac{1}{2h}\left(\mathbb{1}-\mathbb{U}^2\right)\left(\mathbb{H}_0^2-\mathbb{P}_0\mathbb{K}_0\right) \ , \\
\mathbb{K}_2 &= i\left(\mathbb{1}+\mathbb{U}^{-2}\right)\mathbb{H}_1 - \frac{1}{2h}\left(\mathbb{1}-\mathbb{U}^{-2}\right)\left(\mathbb{H}_0^2-\mathbb{P}_0\mathbb{K}_0\right) \ ,
\end{aligned} \tag{3.22}$$

[3]Here the branch cut is chosen such that $\sqrt{i} = e^{\frac{i\pi}{4}}$ and $\sqrt{-i} = e^{-\frac{i\pi}{4}}$.

where the normalisation is fixed by matching with the expansion of the R-matrix,[4] while the coproduct $\Delta(\mathbb{H}_2)$ is automatically co-commutative upon using the definitions of $\mathbb{P}_{0,1}$ and $\mathbb{K}_{0,1}$ in (3.18).

With these definitions of the generators, one can show that the representation of the Yangian arising from the expansion of the R-matrix is in general not evaluation. However, if the eigenvalues of the central elements satisfy

$$H_0^2 - P_0 K_0 = 0 \ , \tag{3.23}$$

which has the interpretation as a (massless) shortening condition, we find that the representation does become of evaluation type, in agreement with [15]. We will return to this issue in the following sections.

Crossing. Using the general expression for the antipodes of (2.16) we can derive the antipode for the generators of interest

$$\begin{aligned} &\Sigma\left[\mathbb{Q}_m\right] = -\mathbb{U}^{-1}\mathbb{Q}_m \ , && \Sigma\left[\mathbb{S}_m\right] = -\mathbb{U}\mathbb{S}_m \ , && \\ &\Sigma\left[\mathbb{H}_m\right] = -\mathbb{H}_m \ , && \Sigma\left[\mathbb{P}_m\right] = \mathbb{K}_m \ , && m = 0, 1 \ , \end{aligned} \tag{3.24}$$

i.e. it is involutive for the generators of $\mathcal{Y}_{0,1}(\mathfrak{su}_c(1|1))$.

Secret symmetry. While $\mathbb{J}^1_{01} + \mathbb{J}^2_{02}$ is central, the combination

$$\mathbb{B}_1 = -\frac{ih}{2}\left(\mathbb{J}^1_{11} + \mathbb{J}^2_{12}\right) \ , \tag{3.25}$$

satisfies

$$[\mathbb{B}_1, \mathbb{Q}_0] = \mathbb{Q}_1 + i\left(\mathbb{1} + \mathbb{U}^2\right)\mathbb{S}_0 \ , \qquad [\mathbb{B}_1, \mathbb{S}_0] = -\mathbb{S}_1 + i\left(\mathbb{1} + \mathbb{U}^{-2}\right)\mathbb{Q}_0 \ , \tag{3.26}$$

The coproduct reads

$$\Delta(\mathbb{B}_1) = \mathbb{B}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{B}_1 - \frac{i}{2h}\mathbb{U}^{-1}\mathbb{Q}_0 \otimes \mathbb{S}_0 - \frac{i}{2h}\mathbb{U}\mathbb{S}_0 \otimes \mathbb{Q}_0 \ , \tag{3.27}$$

while the antipode for the secret symmetry is

$$\Sigma\left[\mathbb{B}_1\right] = -\mathbb{B}_1 - \frac{i}{h}\mathbb{H}_0 \ . \tag{3.28}$$

As in the AdS_5 and AdS_3 case, the antipode is not an involution when acting upon the secret symmetry.

3.1.2 Co-commutativity and shortening condition

Let us now investigate what happens if we try to impose evaluation representation onto the Hopf algebra structure described in section (3.1.1). In particular, we will show that this demonstrates the existence of representations for which one of the higher-level central charges is not co-commutative and hence does not admit an R-matrix.

[4] It is worth recalling that the coproducts for the central charges arising from the RTT realisation of the Yangian are co-commutative by construction, and hence, up to a normalisation, $\mathbb{P}_2$ and $\mathbb{K}_2$ have to take this form when evaluated in the representation arising from the expansion of the R-matrix.

In evaluation representation we have

$$J_n = u^n J_0 \qquad \forall\ J \in \{Q, S, H, P, K\} \ , \tag{3.29}$$

which manifestly satisfies the algebra relations (3.12). As discussed above, in order to have a co-commutative coproduct for all the level-0 central charges one can impose

$$P_0 = \frac{h}{2}(1 - U^2) \ , \qquad K_0 = \frac{h}{2}(1 - U^{-2}) \ , \tag{3.30}$$

where we recall that the constant h is independent on the representation space of the coproduct. Similarly, for the level-1 central charges for co-commutativity one can impose

$$P_1 = -i(1 + U^2)H_0 \ , \qquad K_1 = i(1 + U^{-2})H_0 \ , \tag{3.31}$$

It then follows that for a representation of evaluation type the spectral parameter is given by

$$u = -\frac{2i}{h}\frac{1 + U^2}{1 - U^2}H_0 \ , \tag{3.32}$$

where we use H_m, P_m, K_m and U to denote both the generator in evaluation representation and its eigenvalue, as it is always clear from context which is meant.

At this point, we compute the coproduct of the level-2 central charge P_2 using

$$\Delta(P_2) = \frac{1}{2}\{\Delta(Q_1), \Delta(Q_1)\} \ . \tag{3.33}$$

The expression one obtains is rather lengthy, however it simplifies considerably if one takes the antisymmetric combination

$$\delta P_2 \equiv \Big(\Delta(P_2) - \Delta^{\mathrm{op}}(P_2)\Big) \ , \tag{3.34}$$

which is precisely the quantity that determines whether the coproduct is co-commutative or not.

There are two notable contributions to δP_2, coming from two separate pieces of the coproduct of P_2. The first contribution comes from the part of $\Delta(P_2)$ arising when the S_0 generators in $\Delta(Q_1)$ meet among themselves in the anti-commutator (3.33). Upon antisymmetrisation these terms contribute

$$-\frac{1}{2h}\big((1 - U^2) \otimes (1 - U^2)\big)\big(P_0K_0 \otimes 1 - 1 \otimes P_0K_0\big) \ , \tag{3.35}$$

to δP_2. In fact this would be the only surviving term had we set $H_0 = u = 0$ in both representation spaces. In principle this could already be enough to conclude that there exist representations with non co-commutative P_2. Nevertheless, it is instructive to continue.

The remaining terms reduce to

$$\frac{1}{2h}\big((1 - U^2) \otimes (1 - U^2)\big)\big(H_0^2 \otimes 1 - 1 \otimes H_0^2\big) \tag{3.36}$$

and, combining all contributions together, we obtain

$$\delta P_2 = \frac{1}{2h}\big((1 - U^2) \otimes (1 - U^2)\big)\big((H_0^2 - P_0K_0) \otimes 1 - 1 \otimes (H_0^2 - P_0K_0)\big) \ . \tag{3.37}$$

This means that we can achieve co-commutativity if we demand that

$$H_0^2 - P_0K_0 = \text{constant} \ , \tag{3.38}$$

where the constant does not depend on the representation space. The relation (3.38) is nothing else than the known shortening condition, which is in this way reinterpreted as the condition that makes the central charges' coproduct co-commutative at higher levels (similarly to what the Serre relations do for the canonical part of the Yangian).

3.1.3 Evaluation representation

The Hopf algebra structure discussed in section 3.1.1, which was motivated by similar constructions in the AdS_5 and AdS_3 cases, turned out to give a representation of the Yangian that was not of evaluation type. It turns out that there is an alternative Hopf algebra structure we can put on the same infinite-dimensional algebra, such that the representation arising from the expansion of the R-matrix is evaluation.

To do this we introduce a new spectral parameter

$$u = \frac{1}{4}\big(1+\sqrt{\frac{x^+}{x^-}}\big)^2\big(x^- + \frac{1}{x^+}\big) = \frac{1}{4}\big(1+\sqrt{\frac{x^-}{x^+}}\big)^2\big(x^+ + \frac{1}{x^-}\big) , \tag{3.39}$$

such that the expansions of $x^\pm$ (again assuming that $M(x^+,x^-)=2$) are given by

$$x^\pm = u \pm \frac{i}{h} - \big(1+\frac{1}{4h^2}\big)\frac{1}{u} \pm \frac{i}{h\,u^2} + \mathcal{O}(u^{-3}) , \tag{3.40}$$

Let us now define the following (as an alternative to (3.18)) combinations of generators

$$\begin{aligned}
&\mathbb{Q}_0 = \sqrt{ih}\,\mathbb{J}^2_{01} , && \mathbb{Q}_1 = \sqrt{ih}\,\mathbb{J}^2_{11} - i\mathbb{U}\sqrt{-ih}\,\mathbb{J}^1_{02} ,\\
&\mathbb{S}_0 = \sqrt{-ih}\,\mathbb{J}^1_{02} , && \mathbb{S}_1 = \sqrt{-ih}\,\mathbb{J}^1_{12} + i\mathbb{U}^{-1}\sqrt{ih}\,\mathbb{J}^2_{01} ,\\
&\mathbb{H}_0 = \frac{ih}{2}\left(\mathbb{J}^1_{01} - \mathbb{J}^2_{02}\right) , && \mathbb{H}_1 = \frac{ih}{2}\left(\mathbb{J}^1_{11} - \mathbb{J}^2_{12}\right) - \frac{ih}{2}\left(\mathbb{U} - \mathbb{U}^{-1}\right) ,\\
&\mathbb{P}_0 = \frac{h}{2}\left(\mathbb{1} - \mathbb{U}^2\right) , && \mathbb{P}_1 = -\frac{i}{2}\left(\mathbb{1} + \mathbb{U}\right)^2\mathbb{H}_0 ,\\
&\mathbb{K}_0 = \frac{h}{2}\left(\mathbb{1} - \mathbb{U}^{-2}\right) , && \mathbb{K}_1 = \frac{i}{2}\left(\mathbb{1} + \mathbb{U}^{-1}\right)^2\mathbb{H}_0 ,
\end{aligned} \tag{3.41}$$

which also satisfy the defining commutation relations (3.12). Again, evaluated in the representation arising from the expansion of R-matrix, the level-0 generators coincide with those used in [15].

Coproducts. The level-0 coproducts are given in (3.5), while the level-1 coproducts can be constructed from (2.13) and read

$$\begin{aligned}
\Delta(\mathbb{Q}_1) &= \mathbb{Q}_1 \otimes \mathbb{1} + \mathbb{U} \otimes \mathbb{Q}_1\\
&\quad - \frac{i}{h}\mathbb{Q}_0 \otimes \mathbb{H}_0 + \frac{i}{h}\mathbb{U}\mathbb{H}_0 \otimes \mathbb{Q}_0 + i\mathbb{U}\mathbb{S}_0 \otimes (\mathbb{1} - \mathbb{U}) - i(\mathbb{1} - \mathbb{U}) \otimes \mathbb{U}\mathbb{S}_0 ,\\
\Delta(\mathbb{S}_1) &= \mathbb{S}_1 \otimes \mathbb{1} + \mathbb{U}^{-1} \otimes \mathbb{S}_1\\
&\quad + \frac{i}{h}\mathbb{S}_0 \otimes \mathbb{H}_0 - \frac{i}{h}\mathbb{U}^{-1}\mathbb{H}_0 \otimes \mathbb{S}_0 - i\mathbb{U}^{-1}\mathbb{Q}_0 \otimes (\mathbb{1} - \mathbb{U}^{-1}) + i(\mathbb{1} - \mathbb{U}^{-1}) \otimes \mathbb{U}^{-1}\mathbb{Q}_0 ,\\
\Delta(\mathbb{H}_1) &= \mathbb{H}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{H}_1\\
&\quad - \frac{ih}{2}\big(\mathbb{U} \otimes \mathbb{U} - \mathbb{U}^{-1} \otimes \mathbb{U}^{-1} - (\mathbb{U} - \mathbb{U}^{-1}) \otimes \mathbb{1} - \mathbb{1} \otimes (\mathbb{U} - \mathbb{U}^{-1})\big) .
\end{aligned} \tag{3.42}$$

Here $\Delta(\mathbb{H}_1)$ is written in a manifestly co-commutative form. To check that these coproducts, along with those in (3.5), obey the graded commutation relations (3.12) one needs to use the definitions of $\mathbb{P}_0$ and $\mathbb{K}_0$ in (3.41).

The coproducts for $\mathbb{P}_1$ and $\mathbb{K}_1$ can be obtained from the graded commutation relations (3.20). If $\mathbb{P}_1$ and $\mathbb{K}_1$ are defined in terms of lower-level central elements as in (3.41) we find that these coproducts are also co-commutative as required. Furthermore, we can compute the coproducts for the level-2 central charges using (3.21). Co-commutativity of $\Delta(\mathbb{P}_2)$ and $\Delta(\mathbb{K}_2)$ then requires

$$\begin{aligned}\mathbb{P}_2 &= -2i\mathbb{U}\mathbb{H}_1 - \frac{1}{2h}\left(\mathbb{1} - \mathbb{U}^2\right)\mathbb{H}_0^2 \ , \\ \mathbb{K}_2 &= 2i\mathbb{U}^{-1}\mathbb{H}_1 - \frac{1}{2h}\left(\mathbb{1} - \mathbb{U}^{-2}\right)\mathbb{H}_0^2 \ ,\end{aligned} \tag{3.43}$$

where normalisations are fixed by matching with the expansion of the R-matrix, while the coproduct $\Delta(\mathbb{H}_2)$ is automatically co-commutative by the definitions of $\mathbb{P}_{0,1}$ and $\mathbb{K}_{0,1}$ in (3.41).

From (3.41) we find that the representation of the Yangian arising from the R-matrix expansion is indeed of evaluation type with spectral parameter (3.39)

$$u = -\frac{i}{h}\frac{1+U}{1-U}H_0 \ . \tag{3.44}$$

Crossing. Using the general expression for the antipodes of (2.16) we can derive the antipode for the generators of interest. These turn out to be the same as for the canonical case, i.e. (3.24).

Secret symmetry. While $\mathbb{J}^1_{01} + \mathbb{J}^2_{02}$ is central, the combination

$$\mathbb{B}_1 = -\frac{ih}{2}\left(\mathbb{J}^1_{11} + \mathbb{J}^2_{12}\right) \ , \tag{3.45}$$

satisfies

$$[\mathbb{B}_1, \mathbb{Q}_0] = \mathbb{Q}_1 + \frac{i}{2}\left(\mathbb{1} + \mathbb{U}\right)^2 \mathbb{S}_0 \ , \qquad [\mathbb{B}_1, \mathbb{S}_0] = -\mathbb{S}_1 + \frac{i}{2}\left(\mathbb{1} + \mathbb{U}^{-1}\right)^2 \mathbb{Q}_0 \ . \tag{3.46}$$

The coproduct reads

$$\Delta(\mathbb{B}_1) = \mathbb{B}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{B}_1 - \frac{i}{2h}\mathbb{U}^{-1}\mathbb{Q}_0 \otimes \mathbb{S}_0 - \frac{i}{2h}\mathbb{U}\mathbb{S}_0 \otimes \mathbb{Q}_0 \ , \tag{3.47}$$

while the antipode for the secret symmetry is

$$\Sigma\left[\mathbb{B}_1\right] = -\mathbb{B}_1 - \frac{i}{h}\mathbb{H}_0 \ . \tag{3.48}$$

As before, the antipode is not an involution when acting upon the secret symmetry.

It is worth highlighting that the modification of the combinations in (3.41) has altered the commutation relations involving $\mathbb{B}_1$, changing the tail.

3.1.4 Freedom in the realisation of the Yangian

The two different Hopf algebra structures described in sections 3.1.1 and 3.1.3 are indicative of a larger possible freedom, which we will now describe. Let us consider the defining graded commutation relations (3.12), but in particular focus on the restricted form in which $\mathbb{P}_m$ and $\mathbb{K}_m$ are defined in terms of lower-level central charges.

Motivated by the definitions of $\mathbb{P}_{0,1,2}$ and $\mathbb{K}_{0,1,2}$ in (3.18), (3.22), (3.41), (3.43), we postulate that the following relation is true for all levels:

$$\mathbb{U}^{-1}\mathbb{P}_m = -\mathbb{U}\mathbb{K}_m \ . \tag{3.49}$$

If we now consider the following redefinitions:

$$
\begin{aligned}
&\tilde{\mathbb{Q}}_0 = \mathbb{Q}_0 \ , \qquad \tilde{\mathbb{S}}_0 = \mathbb{S}_0 \ , \qquad \tilde{\mathbb{P}}_0 = \mathbb{P}_0 \ , \qquad \tilde{\mathbb{K}}_0 = \mathbb{K}_0 \ , \qquad \tilde{\mathbb{H}}_0 = \mathbb{H}_0 \ , \\
&\tilde{\mathbb{Q}}_m = \mathbb{Q}_m + \sum_{k=0}^{m-1} y_{q_{m,k}} \mathbb{Q}_{m-k} + z_{q_{m,k}} \mathbb{S}_{m-k} \ , \qquad \tilde{\mathbb{S}}_m = \mathbb{S}_m + \sum_{k=0}^{m-1} y_{s_{m,k}} \mathbb{S}_{m-k} + z_{s_{m,k}} \mathbb{Q}_{m-k} \ , \\
&\tilde{\mathbb{P}}_m = \mathbb{P}_m + \delta_m^{\mathbb{P}} \ , \qquad \tilde{\mathbb{K}}_m = \mathbb{K}_m + \delta_m^{\mathbb{K}} \ , \qquad \tilde{\mathbb{H}}_m = \mathbb{H}_m + \delta_m^{\mathbb{H}} \ , \qquad m > 1 \ ,
\end{aligned} \tag{3.50}
$$

where $y_{q,s}$ and $z_{q,s}$ are functions of the braiding factor $\mathbb{U}$. We are interested in finding a set of these functions such that the algebra relations (3.12) and (3.49) are still satisfied by the new generators. Indeed such a solution exists and is given by

$$
\begin{aligned}
y_{m,k} &= y_{q_{m,k}} = y_{s_{m,k}} = \frac{1}{2}\binom{m}{k}\left((y+z)^{m-k} + (y-z)^{m-k}\right) \ , \\
z_{m,k} &= -i\mathbb{U}^{-1} z_{q_{m,k}} = i\mathbb{U} z_{s_{m,k}} = \frac{1}{2}\binom{m}{k}\left((y+z)^{m-k} - (y-z)^{m-k}\right) \ , \\
\delta_m^{\mathbb{P}} &= \sum_{k=0}^{m-1} y_{m,k} \mathbb{P}_k + i\mathbb{U} z_{m,k} \mathbb{H}_k \ , \qquad \delta_m^{\mathbb{K}} = \sum_{k=0}^{m-1} y_{m,k} \mathbb{K}_k - i\mathbb{U}^{-1} z_{m,k} \mathbb{H}_k \ , \\
\delta_m^{\mathbb{H}} &= \sum_{k=0}^{m-1} y_{m,k} \mathbb{H}_k + i\mathbb{U} z_{m,k} \mathbb{K}_k = \sum_{k=0}^{m-1} y_{m,k} \mathbb{H}_k - i\mathbb{U}^{-1} z_{m,k} \mathbb{P}_k \ .
\end{aligned} \tag{3.51}
$$

I.e. the freedom is parameterised by two functions, y and z, of the braiding factor $\mathbb{U}$. The freedom parameterised by y is a generalisation of the symmetry (2.3). For a representation of evaluation type, its effect is to shift the spectral parameter by $y(U)$.

The redefinitions (3.50) will modify many of the relations underlying the Hopf algebra structure, including the relations between $\mathbb{P}_m$ and $\mathbb{K}_m$ and the lower-level central charges, the coproducts of the generators and the commutation relations involving the secret symmetry $\mathbb{B}_1$. If we demand that the antipode structure (3.24) is preserved, we find

$$
\Sigma(y(\mathbb{U})) = y(\mathbb{U}) \ , \qquad \Sigma(z(\mathbb{U})) = z(\mathbb{U}) \ . \tag{3.52}
$$

These relations are solved by functions symmetric in $\mathbb{U}$ and $\mathbb{U}^{-1}$.

Observing that mapping between the Hopf algebra structures in sections 3.1.1 and 3.1.3 precisely takes the form given above in (3.50) and (3.51), we investigate what happens if we take a more general ansatz for the level-1 Yangian supercharges

$$
\begin{aligned}
&\mathbb{Q}_0 = \sqrt{ih}\, \mathbb{J}_{01}^2 \ , &&\mathbb{S}_0 = \sqrt{-ih}\, \mathbb{J}_{02}^1 \ , \\
&\mathbb{Q}_1 = \sqrt{ih}\, \mathbb{J}_{11}^2 + i\mathbb{U} z(\mathbb{U}) \sqrt{-ih} \mathbb{J}_{02}^1 \ , &&\mathbb{S}_1 = \sqrt{-ih}\, \mathbb{J}_{12}^1 - i\mathbb{U}^{-1} z(\mathbb{U}) \sqrt{ih} \mathbb{J}_{01}^2 \ ,
\end{aligned} \tag{3.53}
$$

with

$$
z(\mathbb{U}) = z(\mathbb{U}^{-1}) \ , \tag{3.54}
$$

to preserve the antipode structure. By anti-commuting $\mathbb{Q}_1$ and $\mathbb{S}_1$ we obtain the central charges

$$
\begin{aligned}
&\mathbb{P}_1 = -\frac{i}{2}\left(\mathbb{1} - 2\mathbb{U} z(\mathbb{U}) + \mathbb{U}^2\right) \mathbb{H}_0 \ , &&\mathbb{P}_2 = 2i\mathbb{U} z(\mathbb{U})\, \mathbb{H}_1 - h^{-2}\left(\mathbb{H}_0^2 - h^2(1 - z(\mathbb{U})^2)\right) \mathbb{P}_0 \ , \\
&\mathbb{K}_1 = \frac{i}{2}\left(\mathbb{1} - 2\mathbb{U}^{-1} z(\mathbb{U}) + \mathbb{U}^{-2}\right) \mathbb{H}_0 \ , &&\mathbb{K}_2 = -2i\mathbb{U}^{-1} z(\mathbb{U})\, \mathbb{H}_1 - h^{-2}\left(\mathbb{H}_0^2 - h^2(1 - z(\mathbb{U})^2)\right) \mathbb{K}_0 \ , \\
&\mathbb{H}_1 = \frac{ih}{2}\left(\mathbb{J}_{11}^1 - \mathbb{J}_{12}^2\right) + \frac{ih}{2}\left(\mathbb{U} - \mathbb{U}^{-1}\right) z(\mathbb{U}) \ .
\end{aligned} \tag{3.55}
$$

We can now ask for what choices of the function $z(\mathbb{U})$ we can have an evaluation type representation of the Yangian. In particular, this would imply the following two relations

$$P_0\,P_2 = P_1^2\ , \qquad P_0\,H_1 = P_1\,H_0\ . \tag{3.56}$$

Combining (3.55) with the conditions just above reveals that a necessary requirement for an evaluation type representation is

$$U^2(1-z(U)^2)(H_0^2 - P_0K_0) = 0\ . \tag{3.57}$$

There are two cases solutions of interest to this condition. The first is

$$H_0^2 - P_0K_0 = 0\ , \tag{3.58}$$

which can be interpreted as a (massless) shortening conditon. This is indeed consistent with our findings in section 3.1.1. If we admit long representations, as in the context of the $\mathfrak{su}_c(1|1)$ R-matrix (3.9), (3.10), (3.11), a consistent evaluation representation demands

$$z(\mathbb{U}) = \pm\mathbb{1}\ . \tag{3.59}$$

Indeed, the choice $z(\mathbb{U}) = -\mathbb{1}$ was the representation analysed in section 3.1.3.

Finally, let us observe that the generator

$$\mathbb{B}_1 = -\frac{ih}{2}\left(\mathbb{J}^1_{11} + \mathbb{J}^2_{12}\right)\ , \tag{3.60}$$

now satisfies

$$\begin{aligned}[][\mathbb{B}_1, \mathbb{Q}_0] &= \mathbb{Q}_1 + \frac{i}{2}\left(\mathbb{1} - 2\mathbb{U}z(\mathbb{U}) + \mathbb{U}^2\right)\,\mathbb{S}_0\ ,\\ [\mathbb{B}_1, \mathbb{S}_0] &= -\mathbb{S}_1 + \frac{i}{2}\left(\mathbb{1} - 2\mathbb{U}^{-1}z(\mathbb{U}) + \mathbb{U}^{-2}\right)\,\mathbb{Q}_0\ .\end{aligned} \tag{3.61}$$

Choosing

$$z(\mathbb{U}) = \frac{1}{2}\left(\mathbb{U} + \mathbb{U}^{-1}\right)\ , \tag{3.62}$$

we see that $\mathbb{P}_1$ and $\mathbb{K}_1$ in (3.55), along with the tails in (3.61) vanish. It is therefore natural to ask if the existence of this choice (3.62) is related to the existence of the secret symmetry.

4 Strong and weak coupling expansions

4.1 Strong coupling expansion and the classical r-matrix

As was done in the AdS_5 case [45, 37] it is instructive to study the so-called *classical r-matrix* of the system. This can be obtained by expanding the quantum R-matrix

$$R = \mathbb{1}\otimes\mathbb{1} + h^{-1}r + O\left(h^{-2}\right)\ , \tag{4.1}$$

at strong coupling. In standard quantum group theory, the knowledge of the classical r-matrix and of its Lie bi-algebra structure allows one to reconstruct the quantum group underlying the exact problem. This is still an open problem for AdS superstrings, nevertheless much can be learnt from this exercise.

4.1.1 Parameterisation and loop algebra

Following [46] we introduce $\zeta = h^{-1}$ and the spectral paramter z

$$x^{\pm} = z\left(\sqrt{1-\frac{\zeta^2}{\left(z-\frac{1}{z}\right)^2}} \pm \frac{i\zeta}{z-\frac{1}{z}}\right) , \tag{4.2}$$

where as before we assume $M(x^+, x^-) = 2$. Expanding the representations of the generators we find

$$\begin{aligned}
&U = \exp(i\zeta\mathfrak{D}) = \mathbb{1} + i\zeta\mathfrak{D} + \mathcal{O}(\zeta^2) , \\
&P_0 = -i\mathfrak{D} + \mathcal{O}(\zeta) , \qquad K_0 = i\mathfrak{D} + \mathcal{O}(\zeta), \qquad H_0 = \mathfrak{H}_0 + \mathcal{O}(\zeta) , \\
&Q_0 = \mathfrak{Q}_0 + \mathcal{O}(\zeta) , \qquad S_0 = \mathfrak{S}_0 + \mathcal{O}(\zeta) , \\
&Q_1 = \zeta^{-1}Q_1 + \mathcal{O}(\zeta) , \quad S_1 = \zeta^{-1}\mathfrak{S}_1 + \mathcal{O}(\zeta) , \quad B_1 = \zeta^{-1}\mathfrak{B}_1 + \mathcal{O}(1) .
\end{aligned} \tag{4.3}$$

The $\zeta \to 0$ limit of the spectral parameter (3.39) is

$$u = \frac{1}{4}\big(1+\sqrt{\frac{x^+}{x^-}}\big)^2\big(x^- + \frac{1}{x^+}\big) = \frac{1}{4}\big(1+\sqrt{\frac{x^-}{x^+}}\big)^2\big(x^+ + \frac{1}{x^-}\big) \to z + z^{-1} . \tag{4.4}$$

In what follows, it is convenient to perform the rescaling

$$u \to \frac{u}{2i} , \tag{4.5}$$

such that the limiting generators $\mathfrak{J}$ are in the evaluation representation with

$$\mathfrak{J}_m = u^m \mathfrak{J}_0 , \qquad u = \frac{1}{2i}(z+z^{-1}) \equiv -i\mathfrak{H}_0\mathfrak{D}^{-1} . \tag{4.6}$$

The non-trivial commutation relations for these $\mathfrak{J}_m$ read

$$\begin{aligned}
&\{\mathfrak{Q}_m, \mathfrak{Q}_n\} = -\{\mathfrak{S}_m, \mathfrak{S}_n\} = 2\mathfrak{H}_{m+n-1} , \qquad &&\{\mathfrak{Q}_m, \mathfrak{S}_n\} = 2\mathfrak{H}_{m+n} , \\
&[\mathfrak{B}_m, \mathfrak{Q}_n] = \mathfrak{Q}_{m+n} + \mathfrak{S}_{m+n-1}, &&[\mathfrak{B}_m, \mathfrak{S}_n] = -\mathfrak{S}_{m+n} + \mathfrak{Q}_{m+n-1} .
\end{aligned} \tag{4.7}$$

4.1.2 Classical r-matrix for the deformed $\mathfrak{gl}(1|1)_{u,u^{-1}}$

The classical limit of the R-matrix gives the classical r-matrix, whose non-trivial entries are[5]

$$\begin{aligned}
&r_{14} = r_{41} = -\frac{i\sqrt{\frac{z_1{}^2}{z_1{}^2-1}}\sqrt{\frac{z_2{}^2}{z_2{}^2-1}}}{z_1z_2-1} , \qquad &&r_{23} = r_{32} = \frac{z_1z_2-1}{z_1-z_2}r_{14} , \\
&r_{22} = \frac{iz_1{}^2\left(z_2{}^2-1\right)}{\left(z_1{}^2-1\right)(z_1-z_2)(z_1z_2-1)} , &&r_{33} = \frac{i\left(z_1{}^2-1\right)z_2{}^2}{\left(z_2{}^2-1\right)(z_1-z_2)(z_1z_2-1)} , \\
&r_{44} = \frac{i\left(z_1{}^2\left(\left(z_1{}^2-4\right)z_2{}^2+z_2{}^4+1\right)+z_2{}^2\right)}{\left(z_1{}^2-1\right)\left(z_2{}^2-1\right)(z_1-z_2)(z_1z_2-1)} .
\end{aligned} \tag{4.8}$$

The residue at $z_2 = z_1$ is

$$\mathrm{Re}\, r_{|z_2\to z_1} = f(z_1)\,(\mathbb{1}\otimes\mathbb{1} - \mathfrak{C}) = \frac{iz_1^2}{1-z_1^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} , \tag{4.9}$$

[5] In this section we label the spectral parameters with integers, for example z_1, z_2, z_3 and so on.

where $\mathfrak{C}$ is the Casimir operator for the $\mathfrak{gl}(1|1)$ tensor algebra.

The classical r-matrix admits the following expression in terms of the generators of the $\mathfrak{gl}(1|1)$ algebra

$$r = \frac{1}{u_2 - u_1}\Big(\frac{1}{2}\mathfrak{Q}_0 \otimes \mathfrak{S}_0 - \frac{1}{2}\mathfrak{S}_0 \otimes \mathfrak{Q}_0 - \frac{u_1}{u_2}\mathfrak{B}_0 \otimes \mathfrak{H}_0 - \frac{u_2}{u_1}\mathfrak{H}_0 \otimes \mathfrak{B}_0 \\ + \frac{1 + u_1^2 + u_2^2}{u_1 u_2}\,\mathfrak{H}_0 \otimes \mathfrak{H}_0\Big)\ . \tag{4.10}$$

The same matrix can then be rewritten as an element in the tensor product of two copies of the loop algebra $\mathfrak{gl}(1|1)_{u,u^{-1}}$

$$r = r_{\mathfrak{psu}(1|1)} - \sum_{n=0}^{\infty}\Big(\tilde{\mathfrak{B}}_{n+1} \otimes \mathfrak{H}_{-n-2} + \mathfrak{H}_{n-1} \otimes \tilde{\mathfrak{B}}_{-n} - \mathfrak{H}_{n-1} \otimes \mathfrak{H}_{-n-2}\Big)\ , \tag{4.11}$$

where

$$r_{\mathfrak{psu}(1|1)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\mathfrak{Q}_n \otimes \mathfrak{S}_{-n-1} - \mathfrak{S}_n \otimes \mathfrak{Q}_{-n-1}\right)\ , \qquad \tilde{\mathfrak{B}}_n = \mathfrak{B}_n - \mathfrak{H}_n\ . \tag{4.12}$$

The peculiarity of (4.11) is that it is representation independent, and can therefore be taken as a candidate for the *universal* classical r-matrix in the AdS_2 case.

4.2 Weak coupling limit and Bethe equations

In this section we study the leading-order weak-coupling ($h \to 0$) term in the R-matrix, extracting from it a set of Bethe equations. These equations should relate to the leading-order first-level nested Bethe equations one would in principle obtain from the spin-chain Hamiltonian of the putative dual superconformal quantum mechanics that is meant to live on the boundary of AdS_2. In AdS_5 parlance, this would be called the *one-loop* nearest-neighbour spin-chain Hamiltonian [47].

The advantage of restricting to this limit is that we can avoid one crucial complication present when dealing with the full R-matrix. To admit a pseudo-vacuum we need to take the tensor product of two copies of the centrally-extended $\mathfrak{su}(1|1)$ R-matrix. (This tensor product is the one relevant for building up the $AdS_2 \times S^2$ worldsheet S-matrix [15]). The corresponding pseudo-vacuum is a specific fermionic linear combination of the states in the two copies, with a definite charge under a certain $U(1)$ quantum number. The corresponding $U(1)$ symmetry does not act in a well-defined way on the individual copies. Performing the algebraic Bethe-ansatz procedure starting from the full pseudo-vacuum is at the moment an open issue.

Dealing with the individual copies, which do not admit a pseudo-vacuum in their own right, could in principle be approached by adapting alternative methods (such as, for instance, the one of Baxter operators). However, the limit $h \to 0$ switches off the most unconventional entries of the R-matrix and allows for the existence of a pseudo-vacuum separately in each copy. Moreover, it drastically simplifies all the remaining entries, allowing for an almost straightforward treatment.

4.2.1 Weak coupling R-matrix

The R-matrix up to order h has the form

$$R_{12} = R_{12}^{(0)} + h\, R_{12}^{(1)}\ , \tag{4.13}$$

where

$$R^{(0)}_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & B_{12} & C_{12} & 0 \\ 0 & C_{12} & D_{12} & 0 \\ 0 & 0 & 0 & E_{12} \end{pmatrix} , \qquad R^{(1)}_{12} = \begin{pmatrix} 0 & 0 & 0 & A_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{12} & 0 & 0 & 0 \end{pmatrix} , \tag{4.14}$$

with the parameterising functions given by

$$\begin{aligned} A_{12} &= \frac{4i(u_1-u_2)(u_1u_2-1)}{(1+u_1^2)(1+u_2^2)(u_1-u_2-2i)} e^{-\frac{i}{4}(p_1-p_2)} , \\ B_{12} &= \frac{u_1-u_2}{u_1-u_2-2i} e^{-\frac{i}{2}p_1} , \\ C_{12} &= \frac{2i}{u_1-u_2-2i} e^{-\frac{i}{4}(p_1-p_2)} , \\ D_{12} &= \frac{u_1-u_2}{u_1-u_2-2i} e^{\frac{i}{2}p_2} , \\ E_{12} &= \frac{u_1-u_2+2i}{u_1-u_2-2i} e^{-\frac{i}{2}(p_1-p_2)} . \end{aligned} \tag{4.15}$$

The maps between $x^{\pm}$, u and p are

$$x^{\pm} = \frac{1}{2h}\Big(\frac{\mathrm{cn}(\frac{p}{2},-4h^2)}{\mathrm{sn}(\frac{p}{2},-4h^2)} \pm i\Big)\Big(1+\mathrm{dn}(\frac{p}{2},-4h^2)\Big) , \qquad u = \cot\frac{p}{4} . \tag{4.16}$$

The factors of e^{ip_k} in (4.14) and (4.15) are the result of a Drinfeld twist [48]. The presence of such a twist was already observed in the same scaling limit of the AdS_5 R-matrix [1]. Indeed, R_{12} can be written as

$$R_{12} = T_{21}\tilde{R}_{12}T_{12}^{-1} , \tag{4.17}$$

where

$$T_{12} = \mathrm{diag}\big(e^{\alpha(p_1+p_2)}, e^{-\frac{i}{4}p_2+\beta(p_1+p_2)}, e^{(\beta-\frac{i}{4})p_1+(\beta-\frac{i}{2})p_2}, e^{(\alpha+\frac{i}{4})p_1+(\alpha-\frac{i}{4})p_2}\big) ,$$

with α and β arbitrary coefficients. The entries of $\tilde{R}_{12}$ are given by

$$\begin{aligned} \tilde{A}_{12} &= \frac{4i(u_1-u_2)(u_1u_2-1)}{(1+u_1^2)(1+u_2^2)(u_1-u_2-2i)} , \\ \tilde{B}_{12} &= \frac{u_1-u_2}{u_1-u_2-2i} , \\ \tilde{C}_{12} &= \frac{2i}{u_1-u_2-2i} , \\ \tilde{D}_{12} &= \frac{u_1-u_2}{u_1-u_2-2i} , \\ \tilde{E}_{12} &= \frac{u_1-u_2+2i}{u_1-u_2-2i} , \end{aligned} \tag{4.18}$$

with the same associations of letters to entries as in (4.13) and (4.14).

Taking $h=0$, the entries of $\tilde{R}_{12}$ involving $\tilde{A}_{12}$ drop out, and we recover the $\mathfrak{gl}(1|1)$ R-matrix, written down in its canonical rational form.

4.2.2 Bethe ansatz and twist

As we have seen, at leading order the R-matrix reduces to the canonical rational (Yangian) R-matrix R_{can} of $\mathfrak{gl}(1|1)$, decorated by a Drinfeld twist. Let us rewrite the twist as

$$T_{12} = t^{ij}_{12}\, E_{ii}\otimes E_{jj} \;=\; e^{i[\underline{i}\,p_1+\underline{j}\,p_2]}\, E_{ii}\otimes E_{jj} , \tag{4.19}$$

where E_{ij} are unit matrices, i.e. 1 in row i, column j and zero everywhere else, and we denote by $\underline{i}$ the numerical coefficient multiplying the momentum in the respective spaces of the T_{12} matrix. Similarly, we write the canonical R-matrix as

$$R_{\mathrm{can}\,12} = r_{ij}^{kl}(p_1, p_2)\, E_{ki} \otimes E_{jl} \;\equiv\; r_{12}{}_{ij}^{kl}\, E_{ki} \otimes E_{jl} \;, \tag{4.20}$$

such that at leading order the R-matrix reads

$$R_{12} = T_{21}\, R_{\mathrm{can}\,12}\, T_{12}^{-1} \;. \tag{4.21}$$

We are now ready to write down the monodromy matrix. Denoting the auxiliary space with the label 0, we have

$$\begin{aligned} M &= R_{01}\, R_{02} \cdots R_{0N} \\ &= T_{10}\, R_{\mathrm{can}01}\, T_{01}^{-1}\, T_{20}\, R_{\mathrm{can}02}\, T_{02}^{-1} \cdots T_{N0}\, R_{\mathrm{can}0N}\, T_{0N}^{-1} \;. \end{aligned} \tag{4.22}$$

Plugging the explicit expressions (4.19) and (4.20) into (4.22) we obtain

$$\begin{aligned} M = t_{10}^{i_1 j_1}\, t_{20}^{i_3 k_1}\, t_{30}^{i_5 k_2} \cdots r_{01}{}_{k_1 j_2}^{j_1 i_1}\, r_{02}{}_{k_2 j_4}^{k_1 i_3}\, r_{03}{}_{k_3 j_6}^{k_2 i_5} \cdots (t_{01}^{k_1 j_2})^{-1} (t_{02}^{k_2 j_4})^{-1} (t_{03}^{k_3 j_6})^{-1} \cdots \\ E_{j_1 k_N} \otimes E_{i_1 j_2} \otimes E_{i_3 j_4} \otimes E_{i_5 j_6} \cdots \;. \end{aligned} \tag{4.23}$$

Now we insert the dependence of the twist on the momentum. Most of the factors appearing in the auxiliary space cancel, such that we are left with

$$\begin{aligned} M = e^{i[\underline{j_1} - \underline{k_N}]p_0}\, e^{i[\underline{i_1} - \underline{j_2}]p_1}\, e^{i[\underline{i_3} - \underline{j_4}]p_2}\, e^{i[\underline{i_5} - \underline{j_6}]p_3} \cdots r_{01}{}_{k_1 j_2}^{j_1 i_1}\, r_{02}{}_{k_2 j_4}^{k_1 i_3}\, r_{03}{}_{k_3 j_6}^{k_2 i_5} \cdots \\ E_{j_1 k_N} \otimes E_{i_1 j_2} \otimes E_{i_3 j_4} \otimes E_{i_5 j_6} \cdots \;. \end{aligned} \tag{4.24}$$

We can therefore define a new set of states and matrices

$$\widetilde{E}_{ab} \equiv e^{i[\underline{a} - \underline{b}]p}\, E_{ab} \;, \qquad |\underline{a}\rangle \equiv e^{i\underline{a}p}\, |a\rangle \;, \tag{4.25}$$

such that we still have

$$\widetilde{E}_{ab}\, \widetilde{E}_{cd} \;=\; \delta_{bc}\, \widetilde{E}_{ad} \;, \qquad \widetilde{E}_{ab}\, |\underline{c}\rangle = \delta_{bc}\, |\underline{a}\rangle \;, \qquad \langle \underline{a} | \underline{b} \rangle = \delta_{ab} \;. \tag{4.26}$$

Using these new vectors and unit matrices, the expression for M becomes indistinguishable from the canonical one. Consequently the Bethe ansatz reduces to the standard one, except with the vectors $|\underline{a}\rangle$ now appearing in the wave functions: in particular, the Bethe equations for M magnons will be

$$\Big(\frac{u_k - i}{u_k + i} \Big)^L = (-1)^{M-1} \;. \tag{4.27}$$

In the effective model obtained from this limit, excitations propagate as free fermions on a periodic one-dimensional lattice.

4.2.3 Bethe equations via the algebraic Bethe ansatz

To conclude this discussion of the weak coupling limit let us recall how the algebraic Bethe ansatz procedure works for the standard $\mathfrak{gl}(1|1)$ rational R-matrix, which, as we have just shown, is relevant in the $h \to 0$ limit. As in section 2, we define $\mathcal{T}(u)$ as in [34]

$$\mathcal{T}(u) = \sum_{A,B} (-)^{[B]}\, e^B{}_A \otimes T^A{}_B(u) \;, \tag{4.28}$$

with

$$\begin{aligned} A(u) &= T^1{}_1(u)\ , & B(u) &= -T^2{}_1(u)\ , \\ C(u) &= -T^1{}_2(u)\ , & D(u) &= T^2{}_2(u)\ . \end{aligned} \tag{4.29}$$

The RTT equations determine the commutation relations for A, B, C and D. In particular, we will need

$$\begin{aligned} A(\lambda)B(\mu) &= f(\mu,\lambda)B(\mu)A(\lambda) + g(\mu,\lambda)B(\lambda)A(\mu)\ , \\ D(\lambda)B(\mu) &= h(\lambda,\mu)B(\mu)D(\lambda) + k(\lambda,\mu)B(\lambda)D(\mu)\ , \\ B(\lambda)B(\mu) &= -q(\lambda,\mu)B(\mu)B(\lambda)\ , \end{aligned} \tag{4.30}$$

where

$$\begin{aligned} f(\mu,\lambda) &= R_{11}(\mu,\lambda)/R_{33}(\mu,\lambda)\ , \\ g(\mu,\lambda) &= R_{23}(\mu,\lambda)/R_{33}(\mu,\lambda)\ , \\ h(\lambda,\mu) &= R_{44}(\lambda,\mu)/R_{33}(\lambda,\mu)\ , \\ k(\lambda,\mu) &= -R_{23}(\lambda,\mu)/R_{33}(\lambda,\mu)\ , \\ q(\lambda,\mu) &= R_{44}(\lambda,\mu)/R_{33}(\lambda,\mu)\ . \end{aligned} \tag{4.31}$$

The standard rational monodromy matrix admits a pseudo-vacuum state Ω, such that

$$A(\lambda)\Omega = \alpha(\lambda)\Omega\ , \qquad C(\lambda)\Omega = 0\ , \qquad D(\lambda)\Omega = \delta(\lambda)\Omega\ . \tag{4.32}$$

This implies that Ω is an eigenstate of the transfer matrix

$$t(\lambda) = (-)^{[I]}T^I{}_I(u) = A(\lambda) - D(\lambda)\ . \tag{4.33}$$

The Bethe equations arise from requiring that the M-magnon state

$$\Phi(\mu_1,\ldots,\mu_M) = \prod_{i=1}^{M} B(\mu_i)\,\Omega\ , \tag{4.34}$$

is an eigenstate of $t(\lambda)$. For instance, for $M=2$ one gets

$$\frac{\alpha(\mu_1)}{\delta(\mu_1)} = \frac{h(\mu_1,\mu_2)\,k(\lambda,\mu_1)}{f(\mu_2,\mu_1)\,g(\mu_1,\lambda)}\ . \tag{4.35}$$

Substituting in the entries of the rational (weak coupling) R-matrix, we see that (4.35) is the same as (4.27). This is the expected result for the $\mathfrak{gl}(1|1)$ R-matrix[6] (see for instance [44, 49]).

5 Conclusions

In this paper, we have performed a series of studies on the conjectured exact S-matrix for the massive excitations of the $AdS_2 \times S^2 \times T^6$ superstring. This S-matrix encodes the integrability of the quantum problem, and is supposed to be the first step towards the complete solution of the theory in the "planar" limit (no joining or splitting of strings). This in turn is expected to provide information on the spectrum of the elusive superconformal quantum mechanics, which should be holographically related to the superstring in this background.

Our main results are as follows:

[6] Let us note that this would actually be true for both the twisted and the untwisted R-matrix.

- By employing the technique of the RTT realisation, we have found the presence of Yangian symmetry for the massive sector, and given two alternative presentations – both in the spirit of Drinfeld's second realisation [50] – along with the map relating them. We have studied the Yangian coproduct, and found the conditions under which we can have a consistent evaluation representation. In order to ascertain these requirements, we studied the co-commutativity of the higher central charges, which is a necessary condition for the existence of an R-matrix. We discovered that shortening is one way to have a consistent evaluation representation, exactly as it was noticed in AdS_5 [27]. However, we demonstrated explicitly that there is a second route, which crucially for the $AdS_2 \times S^2$ superstring holds for long representations.

- We also found, as in the higher dimensional cases, a secret symmetry, which is present only at level 1 of the Yangian and higher. This confirms the ubiquitous presence of this symmetry in all the known manifestations of integrability in AdS/CFT.

- We have studied the *classical r-matrix* of the problem, and rediscovered from its analytic structure the need for the extra $\mathfrak{gl}$ type (secret) generator. This is also similar to the situation in the AdS_5 case.

- We have taken the first steps towards a derivation of the Bethe equations starting from our S-matrix (inverse scattering method). At zero coupling, we discovered that our S-matrix becomes (up to a twist which is easily dealt with) two copies of the standard rational $\mathfrak{gl}(1|1)$ R-matrix. This allows one to define a pseudo-vacuum in each copy individually, and enormously simplifies the problem. Taking the zero-coupling limit of the S-matrix as a generating R-matrix for the Algebraic Bethe Ansatz, we obtain an effective model of free fermions on a periodic spin-chain. Let us note that this should relate to the would-be *one-loop* result of the first nested level in the Bethe ansatz of the putative spin-chain, describing the superconformal quantum mechanics dual to the superstring.

There are a number of future directions which we plan to explore:

- It seems that we are re-discovering many of the features of the AdS_5 (and, to a certain extent, the AdS_3) Yangian. In the AdS_2 case, however, the algebra is small enough that we are able to say a more, especially in terms of alternative presentations. This means that we may hope to find the complete Drinfeld second realisation and derive the universal R-matrix through a suitable ansatz. This would also help to understand the higher dimensional cases. In turn, we would then be able to finally construct the much sought after exotic quantum group, which should quantise the classical r-matrix algebra, and would prove the algebraic integrability of the system.

- The most urgent challenge is probably to derive the full set of Bethe (Beisert-Staudacher) equations for the spectral problem. They encode the information of the planar anomalous dimensions in the dual theory, and would hence provide vital information regarding the nature of the holographic dual to the AdS_2 superstring theory. The simplifying assumption of zero coupling of course eliminates those entries which are responsible for the full pseudo-vacuum being a mixed state in the two copies. Therefore, a more sophisticated technique, rather than the simple algebraic Bethe ansatz computation we have performed here, might be required. This should tie in with a thorough off-shell worldsheet analysis in the spirit of [51, 29].

- Further directions include studying D-branes in this background, and performing a boundary integrability analysis as recently done in [52]. Also, it would be illuminating to continue the perturbative and unitarity analyses of [18, 26, 53, 54], obtaining further information on the dressing phase and the dispersion relation.

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No data beyond those presented and cited in this work are needed to validate this study.

References

[1] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, L. Freyhult, N. Gromov and R. A. Janik *et al.*, "Review of AdS/CFT Integrability: An Overview," Lett. Math. Phys. **99** (2012) 3 [arXiv:1012.3982] • G. Arutyunov and S. Frolov, "Foundations of the $AdS_5 \times S^5$ Superstring. Part I," J. Phys. A **42**, 254003 (2009) [arXiv:0901.4937]

[2] K. Zarembo, "Strings on Semisymmetric Superspaces," JHEP **1005** (2010) 002 [arXiv:1003.0465] • L. Wulff, "Superisometries and integrability of superstrings," JHEP **1405** (2014) 115 [arXiv:1402.3122]

[3] I. R. Klebanov and A. A. Tseytlin, "Intersecting M-branes as four-dimensional black holes," Nucl. Phys. B **475** (1996) 179 [arXiv:hep-th/9604166] • A. A. Tseytlin, "Harmonic superpositions of M-branes," Nucl. Phys. B **475** (1996) 149 [arXiv:hep-th/9604035] • M. J. Duff, H. Lu and C. N. Pope, "$AdS_5 \times S^5$ untwisted," Nucl. Phys. B **532** (1998) 181 [arXiv:hep-th/9803061] • H. J. Boonstra, B. Peeters and K. Skenderis, "Brane intersections, anti-de Sitter space-times and dual superconformal theories," Nucl. Phys. B **533** (1998) 127 [arXiv:hep-th/9803231] • J. Lee and S. Lee, "Mass spectrum of D=11 supergravity on $AdS_2 \times S^2 \times T^7$," Nucl. Phys. B **563** (1999) 125 [arXiv:hep-th/9906105]

[4] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2** (1998) 231 [arXiv:hep-th/9711200] • E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2** (1998) 253 [arXiv:hep-th/9802150]

[5] A. Strominger, "AdS_2 quantum gravity and string theory," JHEP **9901** (1999) 007 [arXiv:hep-th/9809027] • G. W. Gibbons and P. K. Townsend, "Black holes and Calogero models," Phys. Lett. B **454** (1999) 187 [arXiv:hep-th/9812034] • J. M. Maldacena, J. Michelson and A. Strominger, "Anti-de Sitter fragmentation," JHEP **9902** (1999) 011 [arXiv:hep-th/9812073] • C. Chamon, R. Jackiw, S.-Y. Pi and L. Santos, "Conformal quantum mechanics as the CFT_1 dual to AdS_2," Phys. Lett. B **701** (2011) 503 [arXiv:1106.0726]

[6] A. Almheiri and J. Polchinski, "Models of AdS_2 Backreaction and Holography," [arXiv:1402.6334] • M. Heinze, B. Hoare, G. Jorjadze and L. Megrelidze, "Orbit method quantization of the AdS_2 superparticle," J. Phys. A **48** (2015) 31, 315403 [arXiv:1504.04175]. • O. Lunin, "Bubbling geometries for $AdS_2 \times S^2$," [arXiv:1507.06670] • O. Lechtenfeld and S. Nampuri, "A Calogero formulation for four-dimensional black-hole micro states," [arXiv:1509.03256]

[7] R. R. Metsaev and A. A. Tseytlin, "Type IIB superstring action in $AdS_5 \times S^5$ background," Nucl. Phys. B **533** (1998) 109 [arXiv:hep-th/9805028]

[8] J.-G. Zhou, "Super 0-brane and GS superstring actions on $AdS_2 \times S^2$," Nucl. Phys. B **559** (1999) 92 [arXiv:hep-th/9906013] • N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov and B. Zwiebach, "Superstring theory on $AdS_2 \times S^2$ as a coset supermanifold," Nucl. Phys. B **567** (2000) 61 [arXiv:hep-th/9907200]

[9] I. Bena, J. Polchinski and R. Roiban, "Hidden symmetries of the $AdS_5 \times S^5$ superstring," Phys. Rev. D **69** (2004) 046002 [arXiv:hep-th/0305115]

[10] A. Babichenko, B. Stefański, Jr. and K. Zarembo, "Integrability and the AdS_3/CFT$_2$ correspondence," JHEP **1003** (2010) 058 [arXiv:0912.1723]

[11] M. B. Green and J. H. Schwarz, "Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory," Phys. Lett. B **149** (1984) 117.

[12] D. Sorokin, A. Tseytlin, L. Wulff and K. Zarembo, "Superstrings in $AdS_2 \times S^2 \times T^6$," J. Phys. A **44** (2011) 275401 [arXiv:1104.1793]

[13] A. Cagnazzo, D. Sorokin and L. Wulff, "More on integrable structures of superstrings in $AdS_4 \times \mathbb{C}\mathrm{P}^3$ and $AdS_2 \times S^2 \times T^6$ superbackgrounds," JHEP **1201** (2012) 004 [arXiv:1111.4197]

[14] L. Wulff, "On integrability of strings on symmetric spaces," [arXiv:1505.03525]

[15] B. Hoare, A. Pittelli and A. Torrielli, "Integrable S-matrices, massive and massless modes and the $AdS_2 \times S^2$ superstring," JHEP **1411** (2014) 051 [arXiv:1407.0303]

[16] C. G. Callan, Jr., H. K. Lee, T. McLoughlin, J. H. Schwarz, I. Swanson and X. Wu, "Quantizing string theory in $AdS_5 \times S^5$: Beyond the pp wave," Nucl. Phys. B **673** (2003) 3 [arXiv:hep-th/0307032] • S. Frolov, J. Plefka and M. Zamaklar, "The $AdS_5 \times S^5$ superstring in light-cone gauge and its Bethe equations," J. Phys. A **39** (2006) 13037 [arXiv:hep-th/0603008] • G. Arutyunov, S. Frolov and M. Zamaklar, "Finite-size Effects from Giant Magnons," Nucl. Phys. B **778** (2007) 1 [arXiv:hep-th/0606126]

[17] D. E. Berenstein, J. M. Maldacena and H. S. Nastase, "Strings in flat space and pp waves from $\mathcal{N} = 4$ Super Yang Mills," JHEP **0204** (2002) 013 [arXiv:hep-th/0202021]

[18] J. Murugan, P. Sundin and L. Wulff, "Classical and quantum integrability in AdS_2/CFT$_1$," JHEP **1301** (2013) 047 [arXiv:1209.6062] • M. C. Abbott, J. Murugan, P. Sundin and L. Wulff, "Scattering in AdS_2/CFT$_1$ and the BES Phase," JHEP **1310** (2013) 066 [arXiv:1308.1370]

[19] N. Beisert, "The $\mathfrak{su}(2|2)$ dynamic S-matrix," Adv. Theor. Math. Phys. **12** (2008) 948 [arXiv:hep-th/0511082]

[20] G. Arutyunov, S. Frolov and M. Zamaklar, "The Zamolodchikov-Faddeev algebra for $AdS_5 \times S^5$ superstring," JHEP **0704** (2007) 002 [arXiv:hep-th/0612229]

[21] R. Borsato, O. Ohlsson Sax and A. Sfondrini, "A dynamic $\mathfrak{su}(1|1)^2$ S-matrix for AdS_3/CFT$_2$," JHEP **1304** (2013) 113 [arXiv:1211.5119]

[22] R. Borsato, O. Ohlsson Sax, A. Sfondrini, B. Stefański, Jr. and A. Torrielli, "The all-loop integrable spin-chain for strings on $AdS_3 \times S^3 \times T^4$: the massive sector," JHEP **1308** (2013) 043 [arXiv:1303.5995]

[23] B. Hoare and A. A. Tseytlin, "Massive S-matrix of $AdS_3 \times S^3 \times T^4$ superstring theory with mixed 3-form flux," Nucl. Phys. B **873** (2013) 395 [arXiv:1304.4099] • B. Hoare, A. Stepanchuk and A. A. Tseytlin, "Giant magnon solution and dispersion relation in string theory in $AdS_3 \times S^3 \times T^4$ with mixed flux," Nucl. Phys. B **879** (2014) 318 [arXiv:1311.1794]

[24] C. Gomez and R. Hernandez, "The magnon kinematics of the AdS/CFT correspondence," JHEP **0611** (2006) 021 [arXiv:hep-th/0608029] • J. Plefka, F. Spill and A. Torrielli, "On the Hopf algebra structure of the AdS/CFT S-matrix," Phys. Rev. D **74** (2006) 066008 [arXiv:hep-th/0608038]

[25] R. A. Janik, "The $AdS_5 \times S^5$ superstring worldsheet S-matrix and crossing symmetry," Phys. Rev. D **73** (2006) 086006 [arXiv:hep-th/0603038]

[26] R. Roiban, P. Sundin, A. Tseytlin and L. Wulff, "The one-loop worldsheet S-matrix for the $AdS_n \times S^n \times T^{10-2n}$ superstring," JHEP **1408** (2014) 160 [arXiv:1407.7883] • P. Sundin and L. Wulff, "The $AdS_n \times S^n \times T^{10-2n}$ BMN string at two loops," [arXiv:1508.04313]

[27] G. Arutyunov, M. de Leeuw and A. Torrielli, "On Yangian and Long Representations of the Centrally Extended $\mathfrak{su}(2|2)$ Superalgebra," JHEP **1006** (2010) 033 [arXiv:0912.0209]

[28] M. Grigoriev and A. A. Tseytlin, "Pohlmeyer reduction of $AdS_5 \times S^5$ superstring sigma model," Nucl. Phys. B **800** (2008) 450 [arXiv:0711.0155] • B. Hoare and A. A. Tseytlin, "Towards the quantum S-matrix of the Pohlmeyer reduced version of $AdS_5 \times S^5$ superstring theory," Nucl. Phys. B **851**, 161 (2011) [arXiv:1104.2423] • R. Shankar and E. Witten, "The S Matrix of the Supersymmetric Nonlinear Sigma Model," Phys. Rev. D **17** (1978) 2134. • K.-i. Kobayashi and T. Uematsu, "S-matrix of $\mathcal{N} = 2$ supersymmetric Sine-Gordon theory," Phys. Lett. B **275** (1992) 361 [arXiv:hep-th/9110040]

[29] R. Borsato, O. Ohlsson Sax, A. Sfondrini and B. Stefański, Jr., "All-loop worldsheet S-matrix for $AdS_3 \times S^3 \times T^4$," [arXiv:1403.4543] • R. Borsato, O. Ohlsson Sax, A. Sfondrini and B. Stefański, Jr., "The complete $AdS_3 \times S^3 \times T^4$ worldsheet S-matrix," [arXiv:1406.0453] • A. Sfondrini, "Towards integrability for AdS_3/CFT$_2$," [arXiv:1406.2971] • T. Lloyd, O. Ohlsson Sax, A. Sfondrini and B. Stefański, Jr., "The complete worldsheet S-matrix of superstrings on $AdS_3 \times S^3 \times T^4$ with mixed three-form flux," Nucl. Phys. B **891** (2015) 570 [arXiv:1410.0866] • R. Borsato, O. O. Sax, A. Sfondrini and B. Stefański, Jr., "The $AdS_3 \times S^3 \times S^1$ worldsheet S-matrix," [arXiv:1506.00218]

[30] A. B. Zamolodchikov and A. B. Zamolodchikov, "Massless factorized scattering and sigma models with topological terms," Nucl. Phys. B **379** (1992) 602. • P. Fendley, H. Saleur and A. B. Zamolodchikov, "Massless flows, 2. The Exact S-matrix approach," Int. J. Mod. Phys. A **8** (1993) 5751 [arXiv:hep-th/9304051]

[31] N. Beisert, "The S-matrix of AdS/CFT and Yangian symmetry," PoS SOLVAY (2006) 002 [arXiv:0704.0400]

[32] V. Regelskis, "Yangian of AdS_3/CFT$_2$ and its deformation," [arXiv:1503.03799]

[33] A. Molev, M. Nazarov and G. Olshansky, "Yangians and classical Lie algebras," Russ. Math. Surveys **51** (1996) 205 [arXiv:hep-th/9409025]

[34] N. Beisert and M. de Leeuw, "The RTT-Realization for the Deformed $\mathfrak{gl}(2|2)$ Yangian," [arXiv:1401.7691]

[35] A. Pittelli, A. Torrielli and M. Wolf, "Secret Symmetries of Type IIB Superstring Theory on $AdS_3 \times S_3 \times M_4$," [arXiv:1406.2840]

[36] T. Matsumoto, S. Moriyama and A. Torrielli, "A Secret Symmetry of the AdS/CFT S-matrix," JHEP **0709** (2007) 099 [arXiv:0708.1285]

[37] N. Beisert and F. Spill, "The Classical r-matrix of AdS/CFT and its Lie Bialgebra Structure," Commun. Math. Phys. **285** (2009) 537 [arXiv:0708.1762]

[38] M. de Leeuw, T. Matsumoto, S. Moriyama, V. Regelskis and A. Torrielli, "Secret Symmetries in AdS/CFT," Phys. Scripta **02** (2012) 028502 [arXiv:1204.2366]

[39] N. Berkovits and A. Mikhailov, "Nonlocal Charges for Bonus Yangian Symmetries of Super-Yang-Mills," JHEP **1107** (2011) 125 [arXiv:1106.2536]

[40] N. Beisert and B. U. W. Schwab, "Bonus Yangian Symmetry for the Planar S-Matrix of $\mathcal{N} = 4$ Super Yang-Mills," Phys. Rev. Lett. **106** (2011) 231602 [arXiv:1103.0646]

[41] H. Munkler, "Bonus Symmetry for Super Wilson Loops," [arXiv:1507.02474].

[42] V. G. Drinfeld, "Hopf algebras and the quantum Yang-Baxter equation," Sov. Math. Dokl. **32** (1985) 254 [Dokl. Akad. Nauk Ser. Fiz. **283** (1985) 1060]. • V. G. Drinfeld, "Quantum groups," J. Sov. Math. **41** (1988) 898 [Zap. Nauchn. Semin. **155** (1986) 18].

[43] A. Torrielli, "Yangians, S-matrices and AdS/CFT," J. Phys. A **44** (2011) 263001 [arXiv:1104.2474]

[44] N. Beisert, "An $SU(1|1)$-invariant S-matrix with dynamic representations," Bulg. J. Phys. **33S1**, 371 (2006) [arXiv:hep-th/0511013] • N. Beisert, "The Analytic Bethe Ansatz for a Chain with Centrally Extended $\mathfrak{su}(2|2)$ Symmetry," J. Stat. Mech. **0701** (2007) P01017 [arXiv:nlin/0610017]

[45] A. Torrielli, "Classical r-matrix of the $\mathfrak{su}(2|2)$ SYM spin-chain," Phys. Rev. D **75** (2007) 105020 [arXiv:hep-th/0701281] • S. Moriyama and A. Torrielli, "A Yangian double for the AdS/CFT classical r-matrix," JHEP **0706** (2007) 083 [arXiv:0706.0884]

[46] G. Arutyunov and S. Frolov, "On $AdS_5 \times S^5$ String S-matrix," Phys. Lett. B **639** (2006) 378 [arXiv:hep-th/0604043]

[47] J. A. Minahan and K. Zarembo, "The Bethe ansatz for $\mathcal{N} = 4$ superYang-Mills," JHEP **0303** (2003) 013 [arXiv:hep-th/0212208] • O. O. Sax, A. Sfondrini and B. Stefański, Jr., "Integrability and the Conformal Field Theory of the Higgs branch," JHEP **1506** (2015) 103 [arXiv:1411.3676]

[48] V. G. Drinfeld, "Quasi Hopf algebras," Alg. Anal. **1N6** (1989) 114. • S. M. Khoroshkin and V. N. Tolstoi, "Twisting of quantum (super)algebras: Connection of Drinfeld's and Cartan-Weyl realizations for quantum affine algebras," [arXiv:hep-th/9404036]

[49] V. Kazakov, A. S. Sorin and A. Zabrodin, "Supersymmetric Bethe ansatz and Baxter equations from discrete Hirota dynamics," Nucl. Phys. B **790** (2008) 345 [arXiv:hep-th/0703147] • S. Belliard and E. Ragoucy, 'Nested Bethe ansatz for "all" closed spin chains," J. Phys. A **41** (2008) 295202 [arXiv:0804.2822]

[50] V. G. Drinfeld, "A new realization of Yangians and quantized affine algebras," Sov. Math. Dokl. **36** (1988) 212. • F. Spill and A. Torrielli, "On Drinfeld's second realization of the AdS/CFT $\mathfrak{su}(2|2)$ Yangian," J. Geom. Phys. **59** (2009) 489 [arXiv:0803.3194]

[51] G. Arutyunov, S. Frolov, J. Plefka and M. Zamaklar, "The Off-shell Symmetry Algebra of the Light-cone $AdS_5 \times S^5$ Superstring," J. Phys. A **40** (2007) 3583 [arXiv:hep-th/0609157]

[52] A. Prinsloo, "D1 and D5-brane giant gravitons on $AdS_3 \times S^3 \times S^3 \times S^1$," JHEP **1412** (2014) 094 [arXiv:1406.6134] • A. Prinsloo, V. Regelskis and A. Torrielli, "Integrable open spin-chains in AdS_3/CFT_2," arXiv:1505.06767]

[53] T. Klose, T. McLoughlin, R. Roiban and K. Zarembo, "Worldsheet scattering in $AdS_5 \times S^5$," JHEP **0703** (2007) 094 [arXiv:hep-th/0611169] • N. Rughoonauth, P. Sundin and L. Wulff, "Near BMN dynamics of the $AdS_3 \times S^3 \times S^3 \times S^1$ superstring," JHEP **1207** (2012) 159 [arXiv:1204.4742]. • P. Sundin and L. Wulff, "Worldsheet scattering in AdS_3/CFT_2," JHEP **1307** (2013) 007 [arXiv:1302.5349] • B. Hoare and A. A. Tseytlin, "On string theory on $AdS_3 \times S^3 \times T^4$ with mixed 3-form flux: tree-level S-matrix," Nucl. Phys. B **873** (2013) 682 [arXiv:1303.1037] • P. Sundin, "Worldsheet two- and four-point functions at one loop in AdS_3/CFT_2," Phys. Lett. B **733** (2014) 134 [arXiv:1403.1449]. • P. Sundin and L. Wulff, "One- and two-loop checks for the $AdS_3 \times S^3 \times T^4$ superstring with mixed flux," J. Phys. A **48** (2015) 10, 105402 [arXiv:1411.4662]. • L. Bianchi and M. S. Bianchi, "Worldsheet scattering for the GKP string," [arXiv:1508.07331]

[54] L. Bianchi, V. Forini and B. Hoare, "Two-dimensional S-matrices from unitarity cuts," JHEP **1307** (2013) 088 [arXiv:1304.1798] • O. T. Engelund, R. W. McKeown and R. Roiban, "Generalized unitarity and the worldsheet S-matrix in $AdS_n \times S^n \times M^{10-2n}$," JHEP **1308** (2013) 023 [arXiv:1304.4281] • L. Bianchi and B. Hoare, "$AdS_3 \times S^3 \times M^4$ string S-matrices from unitarity cuts," JHEP **1408** (2014) 097 [arXiv:1405.7947]