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A REMARK ON THE QUATERNIONIC MONGE-AMPÈRE EQUATION ON FOLIATED MANIFOLDS

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Abstract. Pursuing the approach in [12] we study the quaternionic Monge-Ampère equation on HKT manifolds admitting an HKT foliation having corank 4. We show that in this setting the quaternionic Monge-Ampère equation has always a unique solution for every basic datum. This approach includes the study of the equation on SU(3).

1. Introduction

A hypercomplex manifold is a $4n$-dimensional smooth manifold M equipped with a triple of complex structures (I, J, K) satisfying the quaternionic relation

$$IJ = -JI = K\,.$$

A Riemannian metric g on a hypercomplex manifold is called *hyperhermitian* if it is compatible with each I, J, K. A hyperhermitian metric g induces the *HKT form*

$$\Omega = \omega_J + i\omega_K\,,$$

where ω_J and ω_K are the fundamental forms of (g, J) and (g, K), respectively. The form Ω belongs to $\Lambda^{2,0}_I$, is related to the metric g by the formula

$$\Omega(Z, W) = 2g(JZ, W)\,, \quad \text{for every } Z, W \in \Gamma(T^{1,0}_I M) \tag{1}$$

and satisfies

$$J\Omega = \bar\Omega\,, \tag{2}$$

$$\Omega(Z, J\bar Z) > 0\,, \quad \text{for every } Z \in \Gamma(T^{1,0}_I M)\,. \tag{3}$$

Viceversa every $\Omega \in \Lambda^{2,0}_I$ satisfying (2) and (3) determines a metric g via (1).

A hyperhermitian manifold (M, I, J, K, g) is called HKT if its HKT form Ω satisfies

$$\partial\Omega = 0\,,$$

where ∂ is with respect to I. HKT structures have been studied intensively in the last years for mathematical and physical reasons (see e.g. [3, 5, 11, 14, 15, 16, 18, 21, 23, 24, 25] and the references therein).

This note focuses on the following conjecture introduced by Alesker and Verbisky in [4]:

Conjecture [Alesker, Verbitsky]. *Let (M, I, J, K, g) be a compact* HKT *manifold. For every $F \in C^\infty(M)$ there exists a unique $(\varphi, b) \in C^\infty(M) \times \mathbb{R}_+$ such that*

$$(\Omega + \partial\partial_J\varphi)^n = b\,\mathrm{e}^F\Omega^n\,, \quad \Omega + \partial\partial_J\varphi > 0\,, \quad \int_M \varphi \mathrm{Vol}_g = 0\,. \tag{4}$$

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Equation (4) is called the *quaternionic Monge-Ampère* equation and involves the operator $\partial_J = J^{-1}\bar\partial J$. Note that the form $\Omega_\varphi := \Omega + \partial\partial_J\varphi$ belongs to $\Lambda^{2,0}_I$ for every $\varphi \in C^\infty(M)$ and the condition $\Omega_\varphi > 0$ means that Ω_φ satisfies (3). Since Ω_φ satisfies (2) for every $\varphi \in C^\infty(M)$, the condition $\Omega_\varphi > 0$ implies that Ω_φ determines a new HKT metric g_φ. Hence the conjecture of Alesker and Verbitsky is the natural counterpart of the classical Calabi conjecture in the realm of HKT geometry. Even if the conjecture is still open there are some partial results suggesting that it should be true: solutions to the equation are unique [4]; solutions to the equation satisfy a C^0 a priori estimate [2, 4, 20]; the conjecture is true under some extra assumptions [1, 6, 10, 12].

The research of the present paper moves from our previous paper [12] where we studied the Alesker-Verbitsky conjecture on some principal torus bundles over a torus. When the bundles considered in [12] are regarded as T^4-bundles over a T^4, then the equation reduces to the classical Poisson equation on the base [12, Remark in Section 2]. Here we generalize the construction to foliated HKT manifolds, where the foliation replaces the role of the fiber. More precisely we consider the following setting:

We say that a foliation $\mathcal{F}$ on an HKT manifold (M, I, J, K, g) is an *HKT foliation* if

$$T_x\mathcal{F} \text{ is } (I_x, J_x, K_x)\text{-invariant for every } x \text{ in } M,$$

where $T\mathcal{F}$ denotes the vector bundle induced by $\mathcal{F}$. A function f is called *basic* with respect to a foliation $\mathcal{F}$ if $X(f) = 0$ for every $X \in \Gamma(\mathcal{F})$, where $\Gamma(\mathcal{F})$ is the space of smooth sections of $T\mathcal{F}$. We denote by $C^k_B(M)$ the space of real C^k basic functions on $(M, \mathcal{F})$. Our main result is the following:

Theorem 1. *Let (M, I, J, K, g) be a compact* HKT *manifold and let $\mathcal{F}$ be an HKT foliation of real corank 4 on M. Then the quaternionic Monge-Ampère equation (4) has a unique solution for every basic datum $F \in C^\infty_B(M)$. Moreover the solution is necessarily basic.*

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2. Proof of Theorem 1

It is well-known that solutions to the quaternionic Monge-Ampère equation (4) on a compact HKT manifold are in general unique. This can, for instance, be observed as follows:
let $(\varphi_1, b_1), (\varphi_2, b_2)$ be two solutions to (4) with $b_1 \geq b_2$. Setting $\Omega_i = \Omega + \partial\partial_J\varphi_i$ we have that

$$\partial\partial_J(\varphi_1 - \varphi_2) \wedge \sum_{k=0}^{n-1} \Omega_1^k \wedge \Omega_2^{n-1-k} = \Omega_1^n - \Omega_2^n = (b_1 - b_2)\mathrm{e}^F\Omega^n \geq 0\,.$$

On the left hand-side we have a second order linear elliptic operator without free term applied to $\varphi_1 - \varphi_2$ and from the maximum principle and the fact that φ_1, φ_2 have zero mean it follows $\varphi_1 = \varphi_2$. Hence we have also $b_1 = b_2$ and the uniqueness follows.

Now we consider the framework of Theorem 1: let (M, I, J, K, g) be a compact HKT manifold equipped with an HKT foliation $\mathcal{F}$ of real corank 4 and consider the quaternionic Monge-Ampère equation

$$(\Omega + \partial\partial_J\varphi)^n = b\,\mathrm{e}^F\Omega^n\,, \quad \Omega + \partial\partial_J\varphi > 0\,, \quad \int_M \varphi \mathrm{Vol}_g = 0\,, \quad F \in C^\infty_B(M). \tag{5}$$

We have the following

Lemma 2. *Let $\varphi \in C^2_B(M)$. Then*

$$\frac{(\Omega+\partial\partial_J\varphi)^n}{\Omega^n}=\Delta\varphi+Q(\nabla\varphi,\nabla\varphi)+1\,,$$

where Δ is the Riemannian Laplacian of g and $Q\in\Gamma(T^*M\otimes T^*M)$ is negative semi-definite.

Proof. Since $\mathcal{F}_x$ is I_x-invariant for every $x\in M$, then $T\mathcal{F}\otimes\mathbb{C}$ splits as $T\mathcal{F}\otimes\mathbb{C}=T^{1,0}\mathcal{F}\otimes T^{0,1}\mathcal{F}$. Let $\{Z_1,\dots,Z_{2n}\}$ be a local g-unitary frame with respect to I such that

$$\langle Z_3,\dots,Z_{2n}\rangle=\Gamma(T^{1,0}\mathcal{F})\,.$$

Let us denote the conjugate $\bar Z_r$ by $Z_{\bar r}$ for every $r=1,\dots,2n$ and suppose

$$J(Z_{2k-1})=Z_{\overline{2k}}\,,\quad \text{for every } k=1,\dots,n.$$

These assumptions imply that the HKT form of g takes its standard expression

$$\Omega=Z^{12}+Z^{34}+\cdots+Z^{(2n-1)(2n)}$$

where $\{Z^1,\dots,Z^{2n}\}$ is the dual coframe to $\{Z_1,\dots,Z_{2n}\}$ and by Z^{ij} we mean $Z^i\wedge Z^j$.

We can write

$$[Z_r,Z_s]=\sum_{k=1}^{2n}B^k_{rs}Z_k\,,\qquad [Z_r,\bar Z_s]=\sum_{k=1}^{2n}\left(B^k_{r\bar s}Z_k+B^{\bar k}_{r\bar s}Z_{\bar k}\right)\,,$$

for some functions $\{B^k_{rs},B^k_{r\bar s},B^{\bar k}_{r\bar s}\}$.

For a basic function φ we have

$$\partial_J\varphi=-J\bar\partial\varphi=-J\left(Z_{\bar 1}(\varphi)Z^{\bar 1}+Z_{\bar 2}(\varphi)Z^{\bar 2}\right)=Z_{\bar 1}(\varphi)Z^2-Z_{\bar 2}(\varphi)Z^1;$$

and

$$\begin{aligned}\partial\partial_J\varphi=&\sum_{k=1}^{2n}(Z_kZ_{\bar 1}(\varphi)Z^{k2}-Z_kZ_{\bar 2}(\varphi)Z^{k1})+\sum_{r<s}(-Z_{\bar 1}(\varphi)B^2_{rs}Z^{rs}+Z_{\bar 2}(\varphi)B^1_{rs}Z^{rs})\\=&\sum_{k=1}^{2n}(Z_kZ_{\bar 1}(\varphi)Z^{k2}-Z_kZ_{\bar 2}(\varphi)Z^{k1})+\sum_{r<s}(Z_{\bar 2}(\varphi)B^1_{rs}-Z_{\bar 1}(\varphi)B^2_{rs})Z^{rs}\\=&(Z_1Z_{\bar 1}(\varphi)+Z_2Z_{\bar 2}(\varphi))Z^{12}+\sum_{k=3}^{2n}(Z_kZ_{\bar 1}(\varphi)Z^{k2}-Z_kZ_{\bar 2}(\varphi)Z^{k1})+\sum_{r<s}(Z_{\bar 2}(\varphi)B^1_{rs}-Z_{\bar 1}(\varphi)B^2_{rs})Z^{rs}\,,\end{aligned}$$

Since $\mathcal{F}$ is a foliation, $B^1_{rs}=0=B^2_{rs}$ for $2<r<s$, thus

$$\begin{aligned}\partial\partial_J\varphi=&(Z_1Z_{\bar 1}(\varphi)+Z_2Z_{\bar 2}(\varphi))Z^{12}+\sum_{k=3}^{2n}\sum_{l=1}^{2n}(B^l_{k\bar 1}Z_l(\varphi)Z^{k2}+B^{\bar l}_{k\bar 1}Z_{\bar l}(\varphi)Z^{k2}-B^l_{k\bar 2}Z_l(\varphi)Z^{k1}-B^{\bar l}_{k\bar 2}Z_{\bar l}(\varphi)Z^{k1})\\&+(Z_{\bar 2}(\varphi)B^1_{12}-Z_{\bar 1}(\varphi)B^2_{12})Z^{12}+\sum_{s=3}^{2n}(Z_{\bar 2}(\varphi)B^1_{1s}-Z_{\bar 1}(\varphi)B^2_{1s})Z^{1s}+\sum_{s=3}^{2n}(Z_{\bar 2}(\varphi)B^1_{2s}-Z_{\bar 1}(\varphi)B^2_{2s})Z^{2s}\\=&(Z_1Z_{\bar 1}(\varphi)+Z_2Z_{\bar 2}(\varphi)+B^1_{12}Z_{\bar 2}(\varphi)-B^2_{12}Z_{\bar 1}(\varphi))Z^{12}\\&+\sum_{k=3}^{2n}(B^1_{1k}Z_{\bar 2}(\varphi)-B^2_{1k}Z_{\bar 1}(\varphi)+B^1_{k\bar 2}Z_1(\varphi)+B^{\bar 1}_{k\bar 2}Z_{\bar 1}(\varphi)+B^2_{k\bar 2}Z_2(\varphi)+B^{\bar 2}_{k\bar 2}Z_{\bar 2}(\varphi))Z^{1k}\\&+\sum_{k=3}^{2n}(B^1_{2k}Z_{\bar 2}(\varphi)-B^2_{2k}Z_{\bar 1}(\varphi)-B^1_{k\bar 1}Z_1(\varphi)-B^{\bar 1}_{k\bar 1}Z_{\bar 1}(\varphi)-B^2_{k\bar 1}Z_2(\varphi)-B^{\bar 2}_{k\bar 1}Z_{\bar 2}(\varphi))Z^{2k}\,.\end{aligned}$$

By setting

$$P_k(\nabla\varphi) = B^1_{1k} Z_{\bar 2}(\varphi) - B^2_{1k} Z_{\bar 1}(\varphi) + B^1_{k\bar 2} Z_1(\varphi) + B^{\bar 1}_{k\bar 2} Z_{\bar 1}(\varphi) + B^2_{k\bar 2} Z_2(\varphi) + B^{\bar 2}_{k\bar 2} Z_{\bar 2}(\varphi)\,,$$
$$Q_k(\nabla\varphi) = B^1_{2k} Z_{\bar 2}(\varphi) - B^2_{2k} Z_{\bar 1}(\varphi) - B^1_{k\bar 1} Z_1(\varphi) - B^{\bar 1}_{k\bar 1} Z_{\bar 1}(\varphi) - B^2_{k\bar 1} Z_2(\varphi) - B^{\bar 2}_{k\bar 1} Z_{\bar 2}(\varphi)\,,$$

we obtain

$$\begin{aligned}\frac{(\Omega+\partial\partial_J\varphi)^n}{\Omega^n} =& 1 + Z_1 Z_{\bar 1}(\varphi) + Z_2 Z_{\bar 2}(\varphi) + B^1_{12} Z_{\bar 2}(\varphi) - B^2_{12} Z_{\bar 1}(\varphi)\\ &+ \sum_{j=3}^{n} \left(P_{2j}(\nabla\varphi) Q_{2j-1}(\nabla\varphi) - P_{2j-1}(\nabla\varphi) Q_{2j}(\nabla\varphi)\right)\,.\end{aligned}$$

Since the Nijenhuis tensor of J vanishes we have

$$\begin{aligned}0 &= [Z_1, Z_{\bar 1}] + J[Z_1, JZ_{\bar 1}] + J[JZ_1, Z_{\bar 1}] - [JZ_1, JZ_{\bar 1}]\\ &= \sum_{k=1}^{2n} \left(B^k_{1\bar 1} Z_k + B^{\bar k}_{1\bar 1} Z_{\bar k} + B^k_{12} JZ_k + B^{\bar k}_{\bar 2\bar 1} JZ_{\bar k} - B^k_{\bar 2 2} Z_k - B^{\bar k}_{\bar 2 2} Z_{\bar k}\right)\end{aligned}$$

and thus

$$\begin{cases} B^{2k-1}_{1\bar 1} - \overline{B^{2k}_{\bar 2\bar 1}} - B^{2k-1}_{\bar 2 2} = 0\,,\\ B^{2k}_{1\bar 1} + \overline{B^{2k-1}_{\bar 2\bar 1}} - B^{2k}_{\bar 2 2} = 0\,,\end{cases} \qquad k = 1, \dots, n\,.$$

Moreover, since $B^{2k-1}_{1\bar 1}, B^{2k}_{1\bar 1}, B^{2k-1}_{2\bar 2}, B^{2k}_{2\bar 2}$ are purely imaginary, for $k = 1$ we deduce

$$\begin{cases} B^{\bar 2}_{\bar 2\bar 1} = B^1_{1\bar 1} + B^1_{2\bar 2} = -B^{\bar 1}_{\bar 1 1} - B^{\bar 1}_{\bar 2 2} = -B^2_{21}\,,\\ B^{\bar 1}_{\bar 2\bar 1} = -B^2_{1\bar 1} - B^2_{2\bar 2} = B^{\bar 2}_{\bar 1 1} + B^{\bar 2}_{\bar 2 2} = -B^1_{21}\,,\end{cases}$$

but then B^2_{21} and B^1_{21} are both real and purely imaginary, yielding

$$\begin{cases} B^1_{21} = 0\,,\\ B^2_{21} = 0\,,\\ B^1_{1\bar 1} + B^1_{2\bar 2} = 0\,,\\ B^2_{1\bar 1} + B^2_{2\bar 2} = 0\,.\end{cases}$$

Writing $X_r = \mathrm{Re}(Z_r)$ and $Y_r = \mathrm{Im}(Z_r)$ for $r = 1, 2$, we see that

$$Z_r Z_{\bar r}(\varphi) = (X_r + iY_r)(X_r - iY_r)(\varphi) = X_r X_r(\varphi) + i[Y_r, X_r](\varphi) + Y_r Y_r(\varphi)$$

and also

$$0 = \sum_{k=1}^{2n} (B^k_{1\bar 1} + B^k_{2\bar 2}) Z_k(\varphi) = ([Z_1, Z_{\bar 1}] + [Z_2, Z_{\bar 2}])(\varphi) = 2i([Y_1, X_1] + [Y_2, X_2])(\varphi)$$

so that

$$Z_1 Z_{\bar 1}(\varphi) + Z_2 Z_{\bar 2}(\varphi) = X_1 X_1(\varphi) + Y_1 Y_1(\varphi) + X_2 X_2(\varphi) + Y_2 Y_2(\varphi) = \Delta\varphi\,.$$

Furthermore, from the vanishing of the Nijenhuis tensor it easy to observe that

$$Q_{2j-1}(\nabla\varphi) = -\overline{P_{2j}(\nabla\varphi)}\,, \qquad Q_{2j}(\nabla\varphi) = \overline{P_{2j-1}(\nabla\varphi)}\,.$$

Thus we finally obtain

$$\frac{(\Omega+\partial\partial_J\varphi)^n}{\Omega^n} = 1 + \Delta\varphi - \sum_{k=3}^{2n} |P_k(\nabla\varphi)|^2\,.$$

The claim then follows by setting

$$Q(\nabla\varphi,\nabla\varphi)=-\sum_{k=3}^{n}|P_k(\nabla\varphi)|^2\,. \qquad \square$$

From the previous lemma it follows that under our assumptions for a basic function F equation (4) reduces to

$$\Delta\varphi+Q(\nabla\varphi,\nabla\varphi)+1=b\,\mathrm{e}^F\,,\qquad \int_M\varphi\mathrm{Vol}_g=0\,. \tag{6}$$

We then focus on this last equation in the general setting of a compact Riemannian manifold.

In order to prove existence of solutions to (6) we need to show some a priori estimates. The C^0 bound is obtained by using the Alexandrov-Bakelman-Pucci estimate as done by Błocki in the case of the complex Monge-Ampère equation [8]. The precise result we need is the following theorem due to Székelyhidi

Theorem 3 (Proposition 10 in [22])**.** *Let $B_1(0)\subseteq\mathbb{R}^m$ denote the unit ball centered at the origin. Assume that $u\in C^2(\mathbb{R}^m)$ satisfies $u(0)+\varepsilon\le\min_{\partial B_1(0)}u(x)$. Then there exists a constant C_m depending only on m such that*

$$\varepsilon^m\le C_m\int_{\Gamma_\varepsilon}\det(D^2u)\,,$$

where D^2u is the Hessian of u and

$$\Gamma_\varepsilon=\left\{x\in B_1(0)\mid u(y)\ge u(x)+\nabla u(x)\cdot(y-x),\,\forall y\in B_1(0),\,|\nabla u(x)|<\frac{\varepsilon}{2}\right\}.$$

This theorem allows us to reduce the C^0 estimate to an L^p estimate by using the following

Theorem 4 (Weak Harnack Inequality, Theorem 8.18 in [13])**.** *Let $R>0$ and fix an integer $m>2$. Assume $u\in C^2(\mathbb{R}^m)$ is non-negative on $B_R(0)$ and such that $\Delta u(x)\le f(x)$ for some $f\in C^0(\mathbb{R}^m)$ and all $x\in B_R(0)$. Consider $1\le p<m/(m-2)$, and $q>m$. Then there exists a positive constant $C=C(m,R,p,q)$ such that*

$$r^{-m/p}\|u\|_{L^p(B_{2r}(0))}\le C\left(\inf_{x\in B_r(0)}u(x)+r^{2-2m/q}\|f\|_{L^{q/2}(B_R(0))}\right),$$

for any $0<r<R/4$.

Lemma 5. *Let (M,g) be a compact Riemannian manifold, $Q\in\Gamma(T^*M\otimes T^*M)$ be negative semi-definite and $F\in C^0(M)$. If $(\varphi,b)\in C^2(M)\times\mathbb{R}_+$ satisfies*

$$\Delta\varphi+Q(\nabla\varphi,\nabla\varphi)+1=b\,\mathrm{e}^F\,,\qquad\int_M\varphi\mathrm{Vol}_g=0\,,$$

then there exists a positive constant C depending only on M, g, $\|Q\|_{C^0}$ and F such that

$$\|\varphi\|_{C^0}\le C\,,\qquad b\le C\,.$$

Proof. First of all we bound the constant b. At a maximum point p of φ we have $b\,\mathrm{e}^{F(p)}-1\le 0$ and thus $b\le\mathrm{e}^{-F(p)}\le\|\mathrm{e}^{-F}\|_{C^0}$ so that the constant b is bounded.

Let $x_0\in M$ be a point where φ achieves its minimum and consider a coordinate chart centered at x_0. Without loss of generality we may identify this chart with a ball $B_1(0)\subseteq\mathbb{R}^m$ of unit radius, where $m=\dim(M)$. Fix $\varepsilon>0$ and define

$$u(x)=\varphi(x)-\max_M\varphi+\varepsilon\,|x|^2\,.$$

Applying Theorem 3 to u we have

$$\varepsilon^m \leq C_m \int_{\Gamma_\varepsilon} \det(D^2 u)\,. \tag{7}$$

We aim to prove that D^2u is bounded on Γ_ε. Differentiating u we see that

$$\nabla u = \nabla\varphi + 2\varepsilon x\,, \qquad D^2u = D^2\varphi + 2\varepsilon I_m\,,$$

where I_m is the identity matrix. As a consequence u satisfies the following equation

$$\Delta u - 2m\varepsilon + Q(\nabla u - 2\varepsilon x, \nabla u - 2\varepsilon x) + 1 = b\,\mathrm{e}^F\,.$$

Set, for instance, $\varepsilon = 1$. Now, since on Γ_1 the Hessian D^2u is non-negative, by the arithmetic-geometric mean inequality we deduce

$$\begin{aligned}\det(D^2u(x)) &\leq \left(\frac{\Delta u(x)}{m}\right)^m \leq \left|Q(\nabla u(x) - 2x, \nabla u(x-2x) + 2m - 1 + b\,\mathrm{e}^{F(x)}\right|^m \\ &\leq \left(\|Q\|_{C^0}\,|\nabla u(x) - 2x|^2 + 2m + 1 + b\left|\mathrm{e}^{F(x)}\right|\right)^m \\ &\leq \left(\frac{5}{2}\,\|Q\|_{C^0} + 2m + 1 + b\left|\mathrm{e}^{F(x)}\right|\right)^m \leq C\,,\end{aligned}$$

for any $x \in \Gamma_1$, where $C > 0$ is a uniform constant.

Now we observe that

$$u(x) \leq u(0) - \nabla u(x)\cdot(-x) \leq u(0) + \frac{1}{2}\,, \qquad \text{for every } x \in \Gamma_1\,,$$

which implies

$$\varphi(x) - \max_M \varphi + |x|^2 \leq \varphi(0) - \max_M\varphi + \frac{1}{2} = \min_M \varphi - \max_M\varphi + \frac{1}{2}\,, \qquad \text{for every } x \in \Gamma_1\,,$$

and therefore

$$\max_M\varphi - \min_M\varphi \leq \max_M\varphi - \varphi(x) + \frac{1}{2}\,, \qquad \text{for every } x \in \Gamma_1\,.$$

It follows that for every $p \geq 1$ we have

$$\left(\max_M\varphi - \min_M\varphi\right)|\Gamma_1|^{1/p} \leq \left\|\max_M\varphi - \varphi + \frac{1}{2}\right\|_{L^p(\Gamma_1)} \leq \left\|\max_M\varphi - \varphi + \frac{1}{2}\right\|_{L^p(B_1(0))}\,.$$

Combining this with (7) and the fact that $\int_M \varphi = 0$, we have

$$\|\varphi\|_{C^0} \leq \max_M\varphi - \min_M\varphi \leq |\Gamma_1|^{-1/p}\left\|\max_M\varphi - \varphi + \frac{1}{2}\right\|_{L^p(B_1(0))} \leq C\left\|\max_M\varphi - \varphi\right\|_{L^p(B_1(0))}\,,$$

for every $p \geq 1$. In conclusion we only need to prove an L^p estimate for $\max_M\varphi - \varphi$ to obtain the desired estimate on φ. Since Q is negative semidefinite we see from the equation that

$$\Delta\varphi \geq b\,\mathrm{e}^F - 1 \geq C\,\mathrm{e}^F - 1\,,$$

where we used that b is uniformly bounded. This entails that $\Delta(\max_M\varphi - \varphi) \leq 1 - C\,\mathrm{e}^F$, and applying Theorem 4 to $\max_M\varphi - \varphi$ with, $1 \leq p \leq m/(m-2)$, $q = 2m$, $r = 1/2$ and $R = 2$ we infer

$$\left\|\max_M\varphi - \varphi\right\|_{L^p(B_1(0))} \leq C\left(\inf_{B_{1/2}(0)}\left(\max_M\varphi - \varphi\right) + \frac{1}{2}\left\|1 - C\,\mathrm{e}^F\right\|_{L^m(B_2(0))}\right) \leq C\,,$$

as required. □

For higher order bounds we need to recall the following two results:

Theorem 6 (Theorem 3.1, Chapter 4 [19])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded connected open subset. Consider a semilinear elliptic equation of the following type*

$$\Delta u + a(x, u, \nabla u) = 0\,,$$

where the function $a(x, u, p)$ is measurable for $x \in \bar{\Omega}$ and arbitrary $u \in \mathbb{R}$, $p \in \mathbb{R}^n$ and satisfies

$$(1 + |p|) \sum_{i=1}^{n} |p_i| + |a(x, u, p)| \leq \mu(|u|)(1 + |p|)^m\,,$$

for some $m > 1$ and some non-decreasing continuous function $\mu\colon [0, +\infty) \to \mathbb{R}$. Let $u \in C^2(\Omega)$ be a solution of the given equation, then, for any connected open subset $\Omega' \subset \Omega$ there exists a constant $C > 0$ depending only on $\|u\|_{C^0(\Omega)}$, $\mu(\|u\|_{C^0(\Omega)})$, m and $d(\Omega', \partial\Omega)$ such that

$$\|u\|_{C^1(\Omega')} \leq C\,.$$

Theorem 7 (Theorem 6.1, Chapter 4 [19])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded connected open subset. Consider a semilinear elliptic equation of the following type*

$$\Delta u + a(x, u, \nabla u) = 0\,,$$

where the function $a(x, u, p)$ is measurable for $x \in \bar{\Omega}$ and arbitrary $u \in \mathbb{R}$, $p \in \mathbb{R}^n$ and satisfies

$$\|a\|_{C^0(\Omega)} < \mu_1\,,$$

for some constant $\mu_1 < \infty$. Let $u \in C^2(\Omega)$ be a solution of the given equation such that

$$\|\nabla u\|_{C^0(\Omega)} < C\,,$$

then there exists $\alpha \in (0, 1)$ depending only on $\|\nabla u\|_{C^0(\Omega)}$ and μ_1 such that $\nabla u \in C^{0,\alpha}(\Omega, \mathbb{R}^n)$. Moreover, for any connected open subset $\Omega' \subset \Omega$ there exists a constant $C > 0$ depending only on $\|\nabla u\|_{C^0(\Omega)}$, μ_1 and $d(\Omega', \partial\Omega)$ such that

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C\,.$$

We can then establish the higher order a priori estimates for solutions to (6).

Lemma 8. *Let (M, g) be a compact Riemannian manifold, $Q \in \Gamma(T^*M \otimes T^*M)$ and $F \in C^0(M)$. If $(\varphi, b) \in C^2(M) \times \mathbb{R}_+$ satisfies*

$$\Delta\varphi + Q(\nabla\varphi, \nabla\varphi) + 1 = b\,\mathrm{e}^F\,,$$

then there exists a positive constant C depending only on M, g, $\|\varphi\|_{C^0}$, b, $\|Q\|_{C^0}$ and F such that

$$\|\Delta\varphi\|_{C^0} \leq C\,.$$

Proof. As an application of Theorem 6 with $a = Q + 1 - b\,\mathrm{e}^F$, $m = 2$, and $\mu \equiv \|Q\|_{C^0} + b\|\mathrm{e}^F\|_{C^0} + \sqrt{n} + 1$ we have that there exists a constant $C > 0$ such that

$$\|\nabla\varphi\|_{C^0} \leq C\,.$$

Then from the equation we have

$$\|\Delta\varphi\|_{C^0} \leq b\,\mathrm{e}^F + 1 + \|Q(\nabla\varphi, \nabla\varphi)\|_{C^0} \leq C\,,$$

and the claim follows. □

Lemma 9. *Let (M,g) be a compact Riemannian manifold, $Q \in \Gamma(T^*M \otimes T^*M)$ and $F \in C^{k,\beta}(M)$. If $(\varphi, b) \in C^2(M) \times \mathbb{R}_+$ solves*

$$\Delta\varphi + Q(\nabla\varphi, \nabla\varphi) + 1 = b\,\mathrm{e}^F\,,$$

then there exists $\alpha \in (0,\beta)$ such that φ in $C^{k+2,\alpha}(M)$ and there is a positive constant $C > 0$ depending only on M, g, b, $\|\nabla\varphi\|_{C^0}$, $\|Q\|_{C^0}$ and F such that

$$\|\varphi\|_{C^{k+2,\alpha}} \leq C\,.$$

Proof. Lemma 8 implies $\|\nabla\varphi\|_{C^0} \leq C$ and we can apply Theorem 7 choosing the constant $\mu_1 = \|Q\|_{C^0} + b\|\mathrm{e}^F\|_{C^0} + 1$ and deduce that there exist $\alpha \in (0,1)$ and a constant $C > 0$ such that

$$\|\varphi\|_{C^{1,\alpha}} \leq C\,.$$

Then the equation implies the estimate $\|\Delta\varphi\|_{C^{0,\alpha}} \leq C$, which can be improved to a $C^{2,\alpha}$ estimate for φ using Schauder theory by assuming $\alpha < \beta$. Then $\varphi \in C^{2,\alpha}(M)$ and by a bootstrapping argument the claim follows. □

Now, we prove that equation (6) is always solvable.

Proposition 10. *Let (M,g) be a compact Riemannian manifold, $Q \in \Gamma(T^*M \otimes T^*M)$ be negative semi-definite and $F \in C^{k,\beta}(M)$. Then equation*

$$\Delta\varphi + Q(\nabla\varphi, \nabla\varphi) + 1 = b\,\mathrm{e}^F\,, \qquad \int_M \varphi \mathrm{Vol}_g = 0$$

admits a solution $(\varphi, b) \in C^{k+2,\alpha}(M) \times \mathbb{R}_+$ for $\alpha \in (0,\beta)$.

Proof. Let $F \in C^{k,\beta}(M)$ and consider the set

$$S := \left\{t' \in [0,1] \;:\; (*_t) \text{ has a solution } (\varphi_t, b_t) \in C^{4,\alpha}(M) \times \mathbb{R}_+ \text{ for } t \in [0,t']\right\}\,,$$

where

$$(*_t) \qquad \Delta\varphi_t + Q(\nabla\varphi_t, \nabla\varphi_t) = b_t\,\mathrm{e}^{tF} - 1\,, \qquad \int_M \varphi_t \mathrm{Vol}_g = 0$$

and $\alpha \in (0,\beta)$ is fixed.

S in not empty since the pair $(\varphi_0, b_0) = (0,1)$ solves $(*_t)$ for $t = 0$ and, therefore, $0 \in S$. In order to prove the statement we need to show that S is open and closed in $[0,1]$.

To show that S is open we apply, as usual, the inverse function theorem between Banach spaces. Let $\hat{t} \in S$ and $(\varphi_{\hat{t}}, b_{\hat{t}})$ be solution of $(*_{\hat{t}})$, let B_1 and B_2 be the Banach spaces

$$B_1 := \left\{\psi \in C^{4,\alpha}(M) \;:\; \int \psi \mathrm{Vol}_g = 0\right\}\,, \quad B_2 := C^{2,\alpha}(M)$$

and let $\Psi\colon B_1 \times \mathbb{R}_+ \to B_2$ be the operator

$$\Psi(\psi, a) := \log\left(\frac{\Delta\psi + Q(\nabla\psi, \nabla\psi) + 1}{a}\right)\,.$$

The differential $\Psi_{*|(\varphi_{\hat{t}}, b_{\hat{t}})}\colon B_1 \times \mathbb{R} \to B_2$ is

$$\Psi_{*|(\varphi_{\hat{t}}, b_{\hat{t}})}(\eta, c) = \frac{\Delta\eta + 2Q(\nabla\eta, \nabla\varphi_{\hat{t}})}{b_{\hat{t}}\,\mathrm{e}^{\hat{t}F}} - \frac{c}{b_{\hat{t}}}\,.$$

Since $T\colon \eta \mapsto \Delta\eta + 2Q(\nabla\eta, \nabla\varphi_{\hat{t}})$ is a second order linear elliptic operator without terms of degree zero, by the maximum principle its kernel is the set of constant functions on M. Moreover, since

T has the same principal symbol of the Laplacian operator it has index zero. Denoting with T^* the formal adjoint of T we then have

$$\dim\ker(T^*) = \dim\operatorname{coker}(T) = \dim\ker(T) - \operatorname{ind}(T) = 1\,.$$

Let $\rho \in C^{2,\alpha}(M)$. Equation

$$\Delta\eta + 2Q(\nabla\eta, \nabla\varphi_{\hat t}) = c\,\mathrm{e}^{\hat t F} + \rho\, b_{\hat t}\,\mathrm{e}^{\hat t F} \tag{8}$$

is solvable if and only if its right hand side is orthogonal to $\ker(T^*)$, or equivalently to a generator of $\ker(T^*)$. This can always be accomplished by a suitable choice of the constant c and, therefore, there always exists a solution $\eta \in C^{4,\alpha}(M)$ to (8). Moreover the solution η is unique in B_1 because $\int_M \eta \mathrm{Vol}_g = 0$. The differential $\Psi_{*|(\varphi_{\hat t}, b_{\hat t})}$ is then an isomorphism and it follows by the inverse function theorem that the operator Ψ is locally invertible around $(\varphi_{\hat t}, b_{\hat t})$, implying that there exists $\varepsilon > 0$ such that for $t \in [\hat t, \hat t + \varepsilon)$ equation $(*_t)$ can be solved.

Next we prove that S is closed. Let $\{t_j\}$ be a sequence in S converging to some $t \in [0,1]$ and consider the corresponding solutions $(\varphi_j, b_j) = (\varphi_{t_j}, b_{t_j})$ to $(*_{t_j})$. In view of Lemma 5 the families $\{\|\varphi_j\|_{C^0}\}$, $\{b_j\}$ are uniformly bounded from above. Moreover, Lemmas 8 and 9 imply that the family $\{\varphi_j\}$ is uniformly bounded in $C^{k+2,\alpha}$-norm. Consequently, by Ascoli-Arzelà Theorem, up to a subsequence, φ_j converges to some $\varphi_t \in C^{k+2,\alpha}(M)$ in $C^{k+2,\alpha}$-norm and b_j converges to some $b_t \in \mathbb{R}$. b_t is in fact positive since from the equation we deduce that the sequence b_j is uniformly bounded from below by a positive quantity. The pair (φ_t, b_t) solves $(*_t)$ and the closedness of S follows. □

We are ready to prove Theorem 1.

Proof of Theorem 1. In view of Lemma 2 for every $\varphi \in C^\infty_B(M)$ we have

$$\frac{(\Omega + \partial\partial_J\varphi)^n}{\Omega^n} = \Delta\varphi + Q(\nabla\varphi, \nabla\varphi) + 1\,,$$

where Δ is the Riemannian Laplacian of g and $Q \in \Gamma(T^*M \otimes T^*M)$ is negative semi-definite.

Let $F \in C^\infty_B(M)$. Proposition 10 implies that equation

$$\Delta\varphi + Q(\nabla\varphi, \nabla\varphi) + 1 = b\,\mathrm{e}^F\,, \qquad \int_M \varphi \mathrm{Vol}_g = 0$$

has a solution $(\varphi, b) \in C^\infty(M) \times \mathbb{R}_+$. We observe that since F is basic, then φ is necessarily basic too. Indeed by setting

$$\Psi(\psi) = \Delta\psi + Q(\nabla\psi, \nabla\psi) + 1$$

we have that for every $X \in \Gamma(\mathcal{F})$ condition $X(F) = 0$ implies

$$0 = X(\Psi(\varphi)) = \Psi_{*|\varphi}(X(\varphi))$$

and since $\Psi_{*|\varphi}$ is a linear elliptic operator without free term, by the maximum principle $X(\varphi)$ must be constant and then necessarily zero. Hence (φ, b) solves the quaternionic Monge-Ampère equation (4) and the claim follows. □

3. An explicit example: SU(3)

In this section we observe that Theorem 1 can be applied for instance to study the quaternionic Monge-Ampère equation on SU(3) endowed with Joyce's hypercomplex structure, which we are about to describe.

The Lie algebra of SU(3)

$$\mathfrak{su}(3)=\left\{\begin{pmatrix} D & f\\ -\bar f^t & -\mathrm{tr}(D)\end{pmatrix}\ :\ D\in\mathfrak{u}(2) \text{ and } f\in\mathbb{C}^2\right\}$$

splits in

$$\mathfrak{su}(3)=\mathfrak{b}\oplus\mathfrak{d}\oplus\mathfrak{f}$$

where

- $\mathfrak{d}\cong\mathfrak{sp}(1)$ is the space of matrices with zero f and $\mathrm{tr}(D)$;
- $\mathfrak{f}$ consists of matrices with zero D;
- $\mathfrak{b}\cong\mathbb{R}$ is the set of diagonal matrices commuting with $\mathfrak{d}$.

We have

$$[\mathfrak{b},\mathfrak{d}]=0\,,\quad [\mathfrak{b},\mathfrak{f}]=\mathfrak{f}\,,\quad [\mathfrak{d},\mathfrak{f}]=\mathfrak{f}\,,\quad [\mathfrak{f},\mathfrak{f}]=\mathfrak{b}\oplus\mathfrak{d}\,,\quad [\mathfrak{d},\mathfrak{d}]=\mathfrak{d}\,.$$

In particular $\mathfrak{b}\oplus\mathfrak{d}$ is a subalgebra of $\mathfrak{su}(3)$ and induces a foliation $\mathcal{F}$ on SU(3). The HKT structure of SU(3) can be described in terms of the "standard" basis of $\mathfrak{su}(3)$

$$\{X_1,\dots,X_8\}$$

given by

$$X_1=\begin{pmatrix} i&0&0\\0&i&0\\0&0&-2i\end{pmatrix},\quad X_2=\begin{pmatrix} i&0&0\\0&-i&0\\0&0&0\end{pmatrix},\quad X_3=\begin{pmatrix} 0&1&0\\-1&0&0\\0&0&0\end{pmatrix},\quad X_4=\begin{pmatrix} 0&i&0\\i&0&0\\0&0&0\end{pmatrix},$$
$$X_5=\begin{pmatrix} 0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\quad X_6=\begin{pmatrix} 0&0&i\\0&0&0\\i&0&0\end{pmatrix},\quad X_7=\begin{pmatrix} 0&0&0\\0&0&1\\0&-1&0\end{pmatrix},\quad X_8=\begin{pmatrix} 0&0&0\\0&0&-i\\0&-i&0\end{pmatrix}.$$

Following Joyce's paper [17], the hypercomplex structure on SU(3) is then defined by the following relations:

- on $\mathfrak{b}\oplus\mathfrak{d}=\langle X_1,X_2,X_3,X_4\rangle$ as $IX_1=X_2$, $IX_3=X_4$, $JX_1=X_3$, $JX_2=-X_4$;
- on $\mathfrak{f}=\langle X_5,X_6,X_7,X_8\rangle$ as $Iv=[X_2,v]$, $Jv=[X_3,v]$, $Kv=[X_4,v]$ for every $v\in\mathfrak{f}$.

According to [9, 7] all invariant hypercomplex structures on compact Lie groups are obtained from Joyce's construction. In view of these results, up to equivalence, this is the unique hypercomplex structure on SU(3). Moreover the standard metric g that makes $\{X_1,\dots,X_8\}$ into an orthogonal frame is HKT with respect to (I,J,K). Hence the foliation $\mathcal{F}$ induced by $\mathfrak{b}\oplus\mathfrak{d}$ is (I,J,K)-invariant and all the assumptions of Theorem 1 are satisfied. Therefore the quaternionic Calabi-Yau equation on $(\mathrm{SU}(3),I,J,K,g)$ can be solved for every $\mathcal{F}$-basic datum F.

As an example we show explicitly how the proof of Lemma 2 works in the case of SU(3). Let $\{X^1,\dots,X^8\}$ be the dual basis of $\{X_1,\dots X_8\}$ and let

$$Z^1=-\frac12(X^1+iX^2)\,,\quad Z^2=\frac12(X^3+iX^4)\,,\quad Z^3=\frac12(X^5+iX^6)\,,\quad Z^4=\frac12(X^7+iX^8)$$

be the induce unitary coframe with respect to (g, I). Since the only non-zero brackets of vectors in $\{X_1, \dots, X_8\}$ are

$$[X_5, X_6] = X_1 + X_2\,, \qquad [X_7, X_8] = X_2 - X_1\,, \qquad [X_3, X_4] = 2X_2\,,$$
$$\frac{1}{2}[X_2, X_4] = [X_5, X_7] = -[X_6, X_8] = -X_3\,,$$
$$\frac{1}{2}[X_2, X_3] = -[X_5, X_8] = -[X_6, X_7] = X_4\,,$$
$$\frac{1}{3}[X_1, X_6] = [X_2, X_6] = -[X_3, X_7] = -[X_4, X_8] = -X_5\,,$$
$$\frac{1}{3}[X_1, X_5] = [X_2, X_5] = -[X_3, X_8] = [X_4, X_7] = X_6\,,$$
$$\frac{1}{3}[X_1, X_8] = -[X_2, X_8] = -[X_3, X_5] = -[X_4, X_6] = X_7\,,$$
$$\frac{1}{3}[X_1, X_7] = -[X_2, X_7] = -[X_3, X_6] = [X_4, X_5] = -X_8\,,$$

we have

$$\partial Z^1 = 0\,, \qquad \partial Z^2 = 2Z^{12} + 2Z^{34}\,, \qquad \partial Z^3 = (1+3i)Z^{13}\,, \qquad \partial Z^4 = (1-3i)Z^{14}\,.$$

For a basic function φ we have $Z_1(\varphi) = Z_2(\varphi) = 0$, where $\{Z_1, \dots, Z_4\}$ is the dual of $\{Z^1, \dots, Z^4\}$, and thus we obtain

$$\begin{aligned}\partial\partial_J\varphi &= -\,\partial J\left(Z_{\bar 3}(\varphi)\, Z^{\bar 3} + Z_{\bar 4}(\varphi)\, Z^{\bar 4}\right) = \partial\left(Z_{\bar 3}(\varphi)\, Z^4 - Z_{\bar 4}(\varphi)\, Z^3\right)\\ &= \left(Z_3 Z_{\bar 3}(\varphi) + Z_4 Z_{\bar 4}(\varphi)\right) Z^{34} - \left(Z_1 Z_{\bar 4}(\varphi) + (1+3i) Z_{\bar 4}(\varphi)\right) Z^{13}\\ &\quad + \left(Z_1 Z_{\bar 3}(\varphi) + (1-3i) Z_{\bar 3}(\varphi)\right) Z^{14} + Z_2 Z_{\bar 3}(\varphi)\, Z^{24} - Z_2 Z_{\bar 4}(\varphi) Z^{23}\end{aligned}$$

which, using

$$[Z_1, Z_{\bar 4}] = (1-3i)Z_{\bar 4}\,, \qquad [Z_1, Z_{\bar 3}] = (1+3i)Z_{\bar 3}\,, \qquad [Z_2, Z_{\bar 3}] = -2Z_4\,, \qquad [Z_2, Z_{\bar 4}] = 2Z_3\,,$$

simplifies to

$$\partial\partial_J\varphi = \left(Z_3 Z_{\bar 3}(\varphi) + Z_4 Z_{\bar 4}(\varphi)\right) Z^{34} - 2Z_{\bar 4}(\varphi)\, Z^{13} + 2Z_{\bar 3}(\varphi)\, Z^{14} - 2Z_4(\varphi)\, Z^{24} - 2Z_3(\varphi) Z^{23}\,.$$

Taking into account that the HKT form is

$$\Omega = Z^{12} + Z^{34}$$

we obtain

$$\frac{(\Omega + \partial\partial_J\varphi)^2}{\Omega^2} = 1 + Z_3 Z_{\bar 3}(\varphi) + Z_4 Z_{\bar 4}(\varphi) - 4\,|Z_3(\varphi)|^2 - 4\,|Z_4(\varphi)|^2$$

in accordance with Lemma 2.

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