Perfect fluid geometries in Rastall's cosmology

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The equation of state of an ultrarelativistic perfect fluid is obtained as a necessary condition for a perfect fluid space-time in Rastall's cosmology.

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1. Introduction

It is well known the importance of a perfect fluid description of gravitational sources in General Relativity[3]. It has been also stressed the relevance of compatibility of perfect-fluid solutions with modified or extended theories of gravity[1].

In this note we point out some cosmological features emerging from the request that the Rastall model[11] describe a perfect fluid geometry of space-time.

We shortly recall some features of the model introduced by Rastall in 1972; we refer to this paper for the relevant issues and stress that the chosen signature for the underling Lorentzian manifold will be $(-,+,+,+)$, in accordance with such a source.

Let then $Lor\boldsymbol{X}$ be the open submanifold of the subbundle of symmetric tensors constituted by regular metrics with the above choice of the Lorentzian signature. An induced fibered chart on $J^2 Lor\boldsymbol{X}$ has local coordinates $(x^\alpha, g_{\alpha\beta}, g_{\alpha\beta,\gamma}, g_{\alpha\beta,\gamma\nu})$ and we can define other local coordinates $g^{\mu\nu}$ and so on by the relation $g_{\alpha\mu}g^{\mu\nu} = \delta^\nu_\alpha$.

As well known Einstein equations read

$$G_{\mu\nu} \doteq R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa_{GR} T_{\mu\nu} \,,$$

where $\kappa_{GR} = \frac{8\pi G}{c^4}$; the Ricci tensor is obtained from a metric connection, so that $R_{\mu\nu} = R_{\mu\nu}(j^2 g)$ and the scalar curvature R has to be intended as $R(j^2 g) = g^{\alpha\beta} R_{\alpha\beta}(j^2 g)$. These equations naturally imply the conservation of energy-momentum tensor as a consequence of the Bianchi identities.

The Rastall model for gravity coupled with matter governed by an energy-momentum tensor satisfying (with λ a suitable non-null dimensional constant)

$$\hat{T}^\mu_{\nu;\mu} = \lambda R_{,\nu}$$

where the comma denotes the partial derivative, and by field equations

$$R_{\mu\nu} - \frac{1}{2} \left(1 - 2\kappa_r \lambda\right) R g_{\mu\nu} = \kappa_r \hat{T}_{\mu\nu} \,, \tag{1}$$

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where κ_r is a dimensional constant to be determined in order to give the right Poisson equation in the static weak-field limit. This model introduces an energy-momentum tensor fulfilling the requirement $\nabla_\mu \hat{T}^\mu_\nu \doteq \hat{T}^\mu_{\nu;\mu} \neq 0$ without violating the Bianchi identities; for a 4-vector, say a_ν, $\hat{T}^\mu_{\nu;\mu} = a_\nu$ and $a_\nu \neq 0$ on curved spacetime, but $a_\mu = 0$ on flat spacetime (in agreement with Special Relativity). Of course, in general, $\kappa_r \neq \kappa_{GR}$. Taking the trace of Eqs.(1) gives us the so-called structural or master equation $(4\kappa_r\lambda - 1)\,R = \kappa_r\hat{T}$ $(\kappa_r\lambda \neq \frac{1}{4})$.

By this relation the Rastall equations (1) can be recast in the form of Einstein equations. Indeed, one can immediately write $G_{\mu\nu} = \kappa_r S_{\mu\nu}$, where $S_{\mu\nu} \doteq \hat{T}_{\mu\nu} - \frac{\kappa_r\lambda}{4\kappa_r\lambda-1} g_{\mu\nu}\hat{T}$. By construction, this new energy-momentum tensor is conserved, *i.e.* $S^\mu_{\nu;\mu} = 0$. Furthermore, (if u_μ is a 4-velocity with $g^{\mu\nu}u_\mu u_\nu = -1$) by assuming $\hat{T}_{\mu\nu} = -(\rho+p)u_\mu u_\nu - p g_{\mu\nu}$, *i.e.* the source is a perfect isentropic fluid with energy density ρ and pressure p, we can explicitly work out an expression for $S_{\mu\nu}$. The source turns out to be still a perfect fluid, provided we redefine its energy density and pressure.[11]

2. Perfect fluid space-times in Rastall's cosmology

Rastall's cosmology has been recently object of attention in scientific community; some papers are concerned with the querelle whether Rastall's stress-energy tensor corresponds to an artificially isolated part of the physical conserved stress-energy or not[2,7,10,15], some others are concerned with the derivation of a related variational principle (*i.e.* with the question how to get a - eventually modified - Rastall model which could be derived by a Lagrangian and physically different from Einstein gravity), see *e.g.* Refs. 12, 14, 13.

We study this model from the point of view of the compatibility of its geometry with a perfect fluid description of space-time. Indeed, as a gravity model it provides a non minimal coupling between metric and matter and we try to understand how this is related with this compatibility requirement.

Indeed it is well known that by suitably introducing a matter density μ, the energy density of space-time $\bar{\rho}$ can be defined as a function of μ satisfying $\bar{p} = \mu\bar{\rho}' - \bar{\rho}$ where the prime denotes derivation with respect to μ; (this is equivalent with asking whether we can derive the tensor $R_{\mu\nu} \equiv \tilde{T}_{\mu\nu}$ from a variational principle, *i.e.* with looking for a Lagrangian density depending on $\rho(\mu)$ from which $\tilde{T}_{\mu\nu}$ can be derived variationally; such kind of space-times appear *e.g.* in the dual Lagrangian description of the Ricci tensor for barotropic perfect relativistic fluids[6], see also Refs. 4, 5, 8).

We therefore have

$$\tilde{T}_{\alpha\beta} = \bar{\rho}(\mu) g_{\alpha\beta} - \mu\bar{\rho}'(\mu)[g_{\alpha\beta} + u_\alpha u_\beta]\,,$$

with trace $\tilde{T} = 4\bar{\rho} - 3\mu\bar{\rho}'$; moreover $\bar{p} = \mu^2 \frac{d}{d\mu}(\frac{\bar{\rho}}{\mu}) = \mu\bar{\rho}' - \bar{\rho}$.

2.1. *Ultrarelativistic equation of state from a perfect fluid geometry*

By requiring the perfect fluid Ricci tensor to be *also* a Rastall's geometry the following identity should hold true.

$$\begin{aligned}\tilde{T}_{\alpha\beta} &\equiv \hat{T}_{\alpha\beta} - \frac{1}{2}\frac{2\tilde{\lambda}-1}{4\tilde{\lambda}-1} g_{\alpha\beta}\hat{T} \\ &= [\rho - \mu\rho' - \frac{1}{2}\frac{2\tilde{\lambda}-1}{4\tilde{\lambda}-1}(4\rho - 3\mu\rho')]g_{\alpha\beta} - \mu\rho' u_\alpha u_\beta \\ &\equiv (\bar{\rho} - \mu\bar{\rho}')g_{\alpha\beta} - \mu\bar{\rho}' u_\alpha u_\beta = -\bar{p} g_{\alpha\beta} - (\bar{\rho} + \bar{p})u_\alpha u_\beta \,. \end{aligned} \tag{2}$$

The above identity leads to

$$\rho' = \bar{\rho}' \,.$$

By inserting this and deriving with respect to μ we get

$$\frac{2\tilde{\lambda}-1}{4\tilde{\lambda}-1}(4\rho' - 3(\mu\rho')') = 0 \,.$$

Let us now exclude the case $2\tilde{\lambda} - 1 = 0$. We see that a necessary condition for the Ricci tensor of a nontrivial Rastall's geometry of matter to be a perfect fluid is

$$4\rho' - 3(\mu\rho')' = 0 \,.$$

i.e.

$$\rho' - 3p' = 0 \,.$$

the equation of state is therefore

$$p = \frac{1}{3}\rho + const.(\mu) \,,$$

i.e. if we denote α the integration constant (a constant function of μ) (uniquely determined by initial data)

$$p - \alpha = \frac{1}{3}\rho$$

In particular it is easy to get that $-3\alpha = \rho - 3p = \tilde{T}$, *i.e.* $\alpha = -\frac{1}{3}\tilde{T}$

By redefining the pressure $\pi = p - \alpha = p + \frac{1}{3}\tilde{T}$ the equation of state is the one of an ultrarelativistic perfect fluid with pressure π and energy density ρ.

$$\pi = \frac{1}{3}\rho \iff p = \frac{1}{3}(\rho - \tilde{T})$$

i.e. with a rescaled pressure or equivalently a *energy density rescaled by the trace of the energy-momentum tensor*. This relation gives us a model with constant (with respect to the matter density μ) energy-tensor trace $\tilde{T}$ and consequently constant (with respect to μ) scalar curvature.

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