Fully non-linear elliptic equations on compact manifolds with a flat hyperkähler metric

FULLY NON-LINEAR ELLIPTIC EQUATIONS ON COMPACT MANIFOLDS WITH A FLAT HYPERKÄHLER METRIC

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Abstract. Mainly motivated by a conjecture of Alesker and Verbitsky, we study a class of fully non-linear elliptic equations on certain compact hyperhermitian manifolds. By adapting the approach of Székelyhidi [60] to the hypercomplex setting, we prove some a priori estimates for solutions to such equations under the assumption of existence of $\mathcal{C}$-subsolutions. In the estimate of the quaternionic Laplacian, we need to further assume the existence of a flat hyperkähler metric. As an application of our results we prove that the quaternionic analogue of the Hessian equation and Monge-Ampère equation for $(n-1)$-plurisubharmonic functions can always be solved on compact flat hyperkähler manifolds.

1. Introduction

A *hypercomplex manifold* is a smooth manifold M of real dimension $4n$ equipped with a triple of complex structures (I, J, K) satisfying the quaternionic relations

$$IJ = -JI = K\,.$$

A Riemannian metric g on a hypercomplex manifold (M, I, J, K) is said to be *hyperhermitian* if it is Hermitian with respect to each of I, J, K. Any hyperhermitian metric induces a 2-form

$$\Omega_0 = \omega_J + i\omega_K$$

where ω_J and ω_K are the fundamental forms of (g, J) and (g, K) respectively. The form Ω_0 is of type $(2, 0)$ with respect to I, satisfies the *q-real* condition $J\Omega_0 = \overline{\Omega_0}$ (here J acts on Ω_0 as $J\Omega_0(\cdot,\cdot) = \Omega_0(J\cdot, J\cdot)$) and is *positive* in the sense that $\Omega_0(X, JX) > 0$ for every nowhere vanishing real vector field on M. (M, I, J, K, g) is hyperkähler if and only if Ω_0 is closed, while it is called HKT (*hyperkähler with torsion*) if Ω_0 is ∂-closed, where ∂ is taken with respect to I. The geometry of HKT manifolds is widely studied in the literature (see e.g. [5, 8, 9, 25, 30, 31, 40, 42, 47, 58, 59, 65, 66, 67] and the reference therein).

Fix a q-real $(2, 0)$ form Ω, a smooth map $\varphi\colon M \to \mathbb{R}$ on a hyperhermitian manifold (M, I, J, K, g) is called *quaternionic Ω-plurisubharmonic* if

$$\Omega_\varphi := \Omega + \partial\partial_J\varphi \text{ is positive}\,,$$

where

$$\partial_J := J^{-1}\bar\partial J$$

is the *twisted differential operator* introduced by Verbitsky [65], being $\bar\partial$ the conjugate of ∂. Animated by the study of "canonical" HKT metrics, in analogy to the Calabi conjecture [17]

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proved by Yau in [69], Alesker and Verbitsky proposed in [6] to study the *quaternionic Monge-Ampère equation*:

$$\Omega_\varphi^n = b\,\mathrm{e}^H \Omega_0^n \tag{1}$$

on a compact HKT manifold, where $H \in C^\infty(M,\mathbb{R})$ is given, while $(\varphi, b) \in C^\infty(M,\mathbb{R}) \times \mathbb{R}_+$ is the unknown. Even if the solvability of the quaternionic Monge-Ampère equation is still an open problem in its general form, several partial results are available in the literature [2, 3, 4, 6, 10, 21, 26, 27, 57, 71]. A nice geometric application of the solvability of equation (1) is the existence of a unique balanced metric $\tilde g$ on a compact HKT manifold (M, I, J, K, g) with holomorphically trivial canonical bundle with respect to I such that the form $\tilde\Omega$ induced by $\tilde g$ belongs to the class $\{\Omega + \partial\partial_J\varphi\}$ (see [67]). From this point of view, equation (1) is the "quaternionic counterpart" of the complex Monge-Ampère equation in Kähler geometry and balanced HKT metrics play the role that Calabi-Yau metrics play in Kähler geometry.

Following the parallelism between Hermitian and hyperhermitian geometry it is quite natural to enlarge the study of the quaternionic Monge-Ampère equation to a general set of fully non-linear elliptic equations on hypercomplex manifolds. Here we adapt the description given by Székelyhidi in [60] to the hypercomplex setting.

In the current paper we consider hypercomplex manifolds (M, I, J, K) which are locally isomorphic to $\mathbb{H}^n$. Unlike complex manifolds, where the integrability of the complex structure guarantees that every point has a neighborhood biholomorphic to an open subset of $\mathbb{C}^n$, for hypercomplex manifolds the integrability of the hypercomplex structure is not enough to ensure that (M, I, J, K) is locally isomorphic to the standard flat space. These manifolds were first introduced by Sommese in [54] and are today called *locally flat* since they can be characterized as hypercomplex manifolds having the curvature of the Obata connection [51] identically zero. We recall that the Obata connection ∇ is the unique torsion free connection on a hypercomplex manifold (M, I, J, K) that preserves the hypercomplex structure, i.e.

$$\nabla I = \nabla J = \nabla K = 0\,.$$

On a *locally flat* hypercomplex manifold (M, I, J, K), we can locally find real coordinates $\{x_p^r\}$, $p = 0, 1, 2, 3$, $r = 1, \dots n$, such that (I, J, K) takes the standard form. Setting

$$q^r := \sum_{p=0}^{3} x_p^r\, e_p\,,$$

where, in order to simplify the notation, we denote the unit quaternions $1, i, j, k$ with e_0, e_1, e_2, e_3, we can define the $\mathbb{H}^n$-valued function $(q^1, \dots, q^n)$, which we refer to as *quaternionic coordinates*. We can then introduce the *quaternionic derivatives* ∂_{q^r} and $\partial_{\bar q^r}$ acting on a smooth $\mathbb{H}$-valued function u as

$$\partial_{q^r} u := \partial_{x_0^r} u\, e_0 - \sum_{i=1}^{3} \partial_{x_i^r} u\, e_i\,, \quad \partial_{\bar q^r} u := \sum_{i=0}^{3} e_i\, \partial_{x_i^r} u\,.$$

The operators ∂_{q^r} and $\partial_{\bar q^s}$ commute, but they do not satisfy the Leibniz rule. Using the vector fields ∂_{q^r} and $\partial_{\bar q^r}$ we can locally regard every q-real $(2,0)$-form Ω on M as a hyperhermitian matrix $(\Omega_{\bar r s})$, i.e. as a $n \times n$ quaternionic matrix lying in $\mathrm{Hyp}(n,\mathbb{H}) = \{H \in \mathbb{H}^{n,n} \mid H = H^*\}$, where $H^* = {}^t\bar H$. Moreover, for a smooth real-valued function φ on M, the matrix associated to $\Omega_\varphi = \Omega + \partial\partial_J\varphi$ is $(\Omega^\varphi_{\bar r s}) = (\Omega_{\bar r s} + \frac14 \partial_{\bar q^r}\partial_{q^s}\varphi)$. The matrix $\mathrm{Hess}_{\mathbb{H}}\varphi := (\varphi_{\bar r s}) = (\frac14 \partial_{\bar q^r}\partial_{q^s}\varphi)$ is usually called the *quaternionic Hessian* of φ.

Now we can describe the class of equations we take into account in the present paper. Let (M, I, J, K, g) be a compact locally flat hyperhermitian manifold and let Ω be a fixed q-real $(2,0)$-form on M (Ω is not necessarily the $(2,0)$-form induced by g). For a smooth real function φ on M let $\Omega_\varphi := \Omega + \partial\partial_J\varphi$ and $A^r_s = g^{\bar{j}r}\Omega^\varphi_{\bar{j}s}$. The matrix (A^r_s) defines a hyperhermitian endomorphism of TM with respect to the metric g, i.e. $A = g^{-1}A^*g$. Note that in general, for quaternionic matrices one does not have (right) eigenvalues in the usual sense, rather conjugacy classes of them. However for hyperhermitian matrices there is a single real eigenvalue in each conjugacy class. Therefore, we consider the function $\lambda\colon \mathrm{Hyp}(n,\mathbb{H}) \to \mathbb{R}^n$ which associates to a matrix A the n-tuple of its eigenvalues $\lambda(A)$.

We can then consider an equation of the following type

$$F(A) = h\,, \tag{2}$$

where $h \in C^\infty(M,\mathbb{R})$ is given and $F(A) = f(\lambda(A))$ is a smooth symmetric operator of the eigenvalues of A. Here $f\colon \Gamma \to \mathbb{R}$, where Γ is a proper convex open cone in $\mathbb{R}^n$ with vertex at the origin which is symmetric (i.e. it is invariant under permutations of the λ_i's) and contains the positive orthant

$$\Gamma_n = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \lambda_i > 0,\, i = 1, \dots, n\}\,.$$

We further require that $f\colon \Gamma \to \mathbb{R}$ satisfies the following assumptions:

C1) $f_i := \frac{\partial f}{\partial \lambda_i} > 0$ for all $i = 1, \dots, n$ and f is a concave function.

C2) $\sup_{\partial\Gamma} f < \inf_M h$, where $\sup_{\partial\Gamma} f = \sup_{\lambda_0\in\partial\Gamma} \limsup_{\lambda\to\lambda_0} f(\lambda)$.

C3) For any $\sigma < \sup_\Gamma f$ and $\lambda \in \Gamma$ we have $\lim_{t\to\infty} f(t\lambda) > \sigma$.

Assumption C1 ensures that equation (2) is elliptic when φ is Γ-*admissible*, i.e.

$$\lambda\left(g^{\bar{k}r}(\Omega_{\bar{k}s} + \varphi_{\bar{k}s}\right) \in \Gamma\,.$$

Assumption C2 says that the level sets of f never touch the boundary of Γ, which also ensures that (2) is non-degenerate and then uniformly elliptic once we have established the C^2 estimate.

An analogue framework was firstly considered by Caffarelli, Nirenberg and Spruck [16] in $\mathbb{R}^n$ and later by Li [48], Urbas [64], Guan [32, 33] and Guan and Jiao [34] on Riemannian manifolds. Székelyhidi [60] studied this framework in Hermitian Geometry for elliptic equations and Phong and Tô [52] for parabolic equations. Székelyhidi's work has been recently generalized in [19, 41] to the almost Hermitian setting.

Our main result is the following:

Theorem 1. *Let (M, I, J, K, g) be a compact flat hyperkähler manifold, Ω a q-real $(2,0)$-form, and $\underline{\varphi}$ a $\mathcal{C}$-subsolution of (2). Then there exist $\alpha \in (0,1)$ and a constant $C > 0$, depending only on (M, I, J, K, g), Ω, h and $\underline{\varphi}$, such that any Γ-admissible solution φ to (2) with $\sup_M \varphi = 0$ satisfies the estimate*

$$\|\varphi\|_{C^{2,\alpha}} \leq C\,.$$

In the above statement by $\mathcal{C}$-subsolution of (2) we mean that

$$\text{for every } x \in M \text{ the set } \left(\lambda\left(g^{\bar{j}r}(\Omega_{\bar{j}s} + \underline{\varphi}_{\bar{j}s})\right) + \Gamma_n\right) \cap \partial\Gamma^{h(x)} \text{ is bounded}\,,$$

where for any $\sigma > \sup_{\partial\Gamma} f$, Γ^σ denotes the convex superlevel set $\Gamma^\sigma = \{\lambda \in \Gamma \mid f(\lambda) > \sigma\}$.

We remark that the assumption of admitting a flat hyperkähler metric in particular implies that (M, I, J, K) is locally flat.

As an application of Theorem 1 we first have the solvability of the quaternionic Hessian equation on hyperhermitian manifolds admitting a flat hyperkähler metric.

Let $(M, I, J, K, g, \Omega_0)$ be a compact hyperhermitian manifold where Ω_0 is the $(2,0)$-form induced by g, fix $1 \leq k \leq n$ and let Ω be a q-real $(2,0)$-form which is k-positive in the sense that

$$\frac{\Omega^i \wedge \Omega_0^{n-i}}{\Omega_0^n} > 0 \qquad \text{for every } i = 1, \dots, k\,. \tag{3}$$

Let $\mathrm{QSH}_k(M, \Omega)$ be the set of continuous functions φ such that Ω_φ is a k-positive q-real $(2,0)$-form in the sense of currents. Then the *quaternionic Hessian equation* is defined as

$$\frac{\Omega_\varphi^k \wedge \Omega_0^{n-k}}{\Omega_0^n} = b\,\mathrm{e}^H, \qquad \varphi \in \mathrm{QSH}_k(M, \Omega)\,, \tag{4}$$

where $H \in C^\infty(M, \mathbb{R})$ is the datum and $(\varphi, b) \in C^\infty(M, \mathbb{R}) \times \mathbb{R}_+$ is the unknown. Equation (4) reduces to the quaternionic Monge-Ampère equation for $k = n$ and to the classical Poisson equation for $k = 1$. Moreover equation (4) is the analogue of the real and complex Hessian equations (see, e.g., [18, 19, 20, 38, 39, 43, 44, 48, 53, 64, 68, 70] and the references therein) in the quaternionic setting. The constant b is uniquely determined by

$$b = \frac{\int_M \Omega_\varphi^k \wedge \Omega_0^{n-k} \wedge \bar\Omega_0^n}{\int_M \mathrm{e}^H \Omega_0^n \wedge \bar\Omega_0^n}\,.$$

Applying Theorem 1 we solve equation (4) on compact flat hyperkähler manifolds:

Theorem 2. *Let $(M, I, J, K, g, \Omega_0)$ be a compact flat hyperkähler manifold and Ω a q-real k-positive $(2,0)$-form. Then the quaternionic Hessian equation*

$$\frac{\Omega_\varphi^k \wedge \Omega_0^{n-k}}{\Omega_0^n} = b\,\mathrm{e}^H\,, \qquad \int_M \varphi\, \Omega_0^n \wedge \bar\Omega_0^n = 0\,, \qquad \varphi \in \mathrm{QSH}_k(M, \Omega)\,,$$

has a unique smooth solution $(\varphi, b) \in C^\infty(M, \mathbb{R}) \times \mathbb{R}_+$ for every $H \in C^\infty(M, \mathbb{R})$.

From Theorem 2 we recover as a special case the result of Alesker [2], where the quaternionic Monge-Ampère equation is solved on compact flat hyperkähler manifolds. We note that during the proof of Theorem 1 the a priori estimates, except for the C^2-estimate, are obtained without assuming anything about the closure of Ω_0 and this suggests that it is worth studying the quaternionic Hessian equation on non-HKT hyperhermitian manifolds.

Our second application is the quaternionic Monge-Ampère equation for $(n-1)$-quaternionic plurisubharmonic functions. Let $(M, I, J, K, g, \Omega_0)$ be a compact hyperhermitian manifold and Ω_1 be a positive q-real $(2,0)$-form. We say that a C^2 function φ on M is $(n-1)$-quaternionic plurisubharmonic with respect to Ω_1 and Ω_0 if the $(2,0)$-form $\Omega_1 + \frac{1}{n-1}[(\Delta_g\varphi)\Omega_0 - \partial\partial_J\varphi]$ is pointwise positive, where Δ_g is the quaternionic Laplacian with respect to g (see section 2 for more details). We also refer to Harvey and Lawson [36, 37] for more general notions of plurisubharmonicity. The *quaternionic Monge-Ampère equation for $(n-1)$-quaternionic plurisubharmonic functions* is written as

$$\left(\Omega_1 + \frac{1}{n-1}\big[(\Delta_g\varphi)\Omega_0 - \partial\partial_J\varphi\big]\right)^n = b\,\mathrm{e}^H \Omega_0^n\,, \qquad \Omega_1 + \frac{1}{n-1}\big[(\Delta_g\varphi)\Omega_0 - \partial\partial_J\varphi\big] > 0\,. \tag{5}$$

Here the constant b is uniquely determined by

$$b = \frac{\int_M \left(\Omega_1 + \frac{1}{n-1}\big[(\Delta_g\varphi)\Omega_0 - \partial\partial_J\varphi\big]\right)^n \wedge \bar\Omega_0^n}{\int_M \mathrm{e}^H \Omega_0^n \wedge \bar\Omega_0^n}\,.$$

Equation (5) is the analogue of the complex Monge-Ampère equation for $(n-1)$-plurisubharmonic functions, introduced and studied by Fu-Wang-Wu [23, 24], it is a kind of Monge-Ampère–type equation. More related works can be found in [19, 41, 62, 63] and the references therein.

Theorem 3. *Let* $(M, I, J, K, g, \Omega_0)$ *be a compact flat hyperkähler manifold and* Ω_1 *a q-real positive* $(2,0)$*-form. Then there is a unique solution* $(\varphi, b) \in C^\infty(M,\mathbb{R}) \times \mathbb{R}_+$ *to the equation*

$$\begin{cases} \left(\Omega_1 + \frac{1}{n-1}\left[(\Delta_g\varphi)\Omega_0 - \partial\partial_J\varphi\right]\right)^n = b\, \mathrm{e}^H \Omega_0^n\,, \\ \Omega_1 + \frac{1}{n-1}\left[(\Delta_g\varphi)\Omega_0 - \partial\partial_J\varphi\right] > 0\,, \quad \sup_M \varphi = 0\,, \end{cases} \tag{6}$$

for every given $H \in C^\infty(M,\mathbb{R})$.

From Theorem 3 we can also obtain Calabi-Yau–type Theorems for quaternionic balanced, quaternionic Gauduchon and quaternionic strongly Gauduchon metrics. We refer the reader to [46, Table 2] for the relevant definitions, which are entirely analogous to the complex case.

Corollary 4. *Let* $(M, I, J, K, g, \Omega_0)$ *be a compact flat hyperkähler manifold and take a quaternionic balanced (resp. quaternionic Gauduchon, quaternionic strongly Gauduchon) metric with induced* $(2,0)$*-form* Ω_2*. Then there is a unique positive constant* b' *and a unique quaternionic balanced (resp. quaternionic Gauduchon, quaternionic strongly Gauduchon) metric with induced* $(2,0)$*-form* $\tilde{\Omega}$*, such that*

$$\tilde{\Omega}^{n-1} = \Omega_2^{n-1} + \partial\partial_J\varphi \wedge \Omega_0^{n-2}\,,$$

for some $\varphi \in C^\infty(M,\mathbb{R})$*, and which solves*

$$\tilde{\Omega}^n = b'\, \mathrm{e}^{H'} \Omega_0^n\,,$$

for any given $H' \in C^\infty(M,\mathbb{R})$.

The paper is organized as follows: sections 2–6 contain the proof of Theorem 1, while in the last section we prove Theorems 2 and 3 and Corollary 4.

More precisely, in section 2 we prove the C^0 a priori estimate for solutions to (2) by using the Alexandroff-Bakelman-Pucci (ABP) method as in [60]. Section 3 deals with the C^0-estimate for the quaternionic Laplacian in terms of the gradient. This estimate is obtained by bounding the highest eigenvalue of the matrix A and here is where we use the assumption of having a flat hyperkähler metric. The Laplacian estimate is then used to perform the *blow-up analysis* in section 4 and reduce the gradient bound to the proof of a *Liouville-type theorem*, which is given in section 5. This yields, in particular, a (non-explicit) bound on the quaternionic Laplacian. Finally in section 6 we conclude the proof of Theorem 1 applying an Evans-Krylov type theorem [22, 45] of which we give two proofs, one in the same spirit of [61], the other by following an argument of Błocki [14] as in Alesker [2].

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2. C^0-estimate

The C^0-estimate for solutions to (2) is obtained by adapting [60, Proposition 10] to our setting and by using the ABP method. This idea was inspired by the C^0-bound of Błocki [13] for the complex Monge-Ampère equation.

On a hyperhermitian manifold (M, I, J, K, g), the *quaternionic Laplacian* of a real function φ is defined by

$$\Delta_g \varphi := n \frac{\partial\partial_J \varphi \wedge \Omega_0^{n-1}}{\Omega_0^n} ,$$

where Ω_0 is the $(2,0)$-form induced by g. This is an elliptic second order linear differential operator. Under the assumption of local flatness, by [10, Lemma 3] we have

$$\Delta_g \varphi = \operatorname{Re} \operatorname{tr}_g(\operatorname{Hess}_{\mathbb{H}} \varphi) = \operatorname{Re} \left(g^{\bar{j}r} \varphi_{\bar{j}r} \right) .$$

Consequently, in quaternionic local coordinates, the quaternionic Laplacian is the sum of the eigenvalues of the quaternionic Hessian with respect to g.

Finally, it will be useful to observe that the domain Γ of f satisfies

$$\Gamma \subseteq \left\{ (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \sum_{i=1}^n \lambda_i > 0 \right\} . \tag{7}$$

As a preliminary step, we prove an L^p-estimate. From here on, we will always denote with C a positive constant that only depends on background data and which may change from line to line.

Lemma 5. *Let (M, I, J, K, g) be a compact locally flat hyperhermitian manifold. If φ is a solution to (2) such that $\sup_M \varphi = 0$, then there exist $p, C > 0$, depending only on the background data, such that*

$$\|\varphi\|_{L^p} \leq C .$$

Proof. From (7) we have $\operatorname{Re} \operatorname{tr}_g(\Omega_\varphi) > 0$, where $\Omega_\varphi = \Omega + \partial\partial_J \varphi$, which in turn translates into a lower bound for the quaternionic Laplacian of φ:

$$\Delta_g \varphi = \operatorname{Re} \operatorname{tr}_g(\Omega_\varphi) - \operatorname{Re} \operatorname{tr}_g(\Omega) \geq -C . \tag{8}$$

An L^1-bound for φ can now be obtained by using the Green operator as in [3]. We give here some details for convenience of the reader. By a quaternionic version of Gauduchon theorem [3, Proposition 2.2], there exists a pointwise strictly positive q-real $(2n, 0)$-form Θ (which might not be holomorphic) such that $\partial\partial_J \left(\Omega_0^{n-1} \wedge \bar{\Theta} \right) = 0$. In addition, we may normalize Θ so that $\int_M \Omega_0^n \wedge \bar{\Theta} = 1$. By [3, Lemma 23], the quaternionic Laplacian admits a non-negative Green function $G(p, q) \geq 0$, namely, for each function u of class C^2 and each point $p \in M$,

$$- \int_{q \in M} G(p, q) \Delta_g u(q) \, \Omega_0^n \wedge \bar{\Theta} = u(p) - \int_M u \, \Omega_0^n \wedge \bar{\Theta} .$$

Choose a point $p \in M$ such that φ attains its maximum at p. Since we assumed $\sup_M \varphi = 0$ we have

$$\|\varphi\|_{L^1} = \int_M (-\varphi) \, \Omega_0^n \wedge \bar{\Theta} = - \int_{q \in M} G(p, q) \Delta_g \varphi(q) \, \Omega_0^n \wedge \bar{\Theta} \leq C \int_{q \in M} G(p, q) \, \Omega_0^n \wedge \bar{\Theta} \leq C .$$

Alternatively an L^p-bound can be obtained by using the weak Harnack inequality as follows. Take an open cover of M made of coordinate balls $B_{2r_i}(x_i)$ such that $\{B_i = B_{r_i}(x_i)\}$ still covers

M. Since φ is non-positive and it satisfies the elliptic inequality (8), the weak Harnack inequality [29, Theorem 9.22] implies

$$\|\varphi\|_{L^p(B_i)} = \left(\int_{B_i} (-\varphi)^p\right)^{1/p} \leq C\left(\inf_{B_i}(-\varphi) + 1\right)$$

where $p, C > 0$ depend only on the cover and the background metric. Since $\sup_M \varphi = 0$ there is at least one index j such that $\inf_{B_j}(-\varphi) = -\sup_{B_j}\varphi = 0$, and thus $\|\varphi\|_{L^p(B_j)} \leq C$. This bound can be extended to all balls B_i such that $B_i \cap B_j \neq \emptyset$, indeed the estimate on $\|\varphi\|_{L^p(B_j)}$ yields an upper bound for $\inf_{B_i}(-\varphi)$ as

$$\inf_{B_i}(-\varphi) \leq \inf_{B_i\cap B_j}(-\varphi) \leq \frac{1}{\mathrm{Vol}(B_i\cap B_j)^{1/p}}\|\varphi\|_{L^p(B_i\cap B_j)} \leq \frac{1}{\mathrm{Vol}(B_i\cap B_j)^{1/p}}\|\varphi\|_{L^p(B_j)}\,.$$

We can now reiterate the argument and in a finite number of steps we will have bound $\|\varphi\|_{L^p(B_i)}$ for each i, and thus also $\|\varphi\|_{L^p(M)}$. □

Proposition 6. *Let (M, I, J, K, g) be a compact locally flat hyperhermitian manifold. If $\underline{\varphi}, \varphi$ are a $\mathcal{C}$-subsolution and a solution to (2) respectively, with $\sup_M \varphi = 0$, then there is a constant $C > 0$, depending only on the background data and the subsolution $\underline{\varphi}$, such that*

$$\|\varphi\|_{C^0} \leq C\,.$$

Proof. Without loss of generality we may assume that $\underline{\varphi} \equiv 0$, otherwise we could modify Ω to simplify the equation. Since $\sup_M \varphi = 0$, we only need to bound $S = \inf_M \varphi$ from below. For convenience, we may assume $S \leq -1$, otherwise we are done.

Since $\underline{\varphi}$ is a $\mathcal{C}$-subsolution there exist $\delta, R > 0$ such that

$$\left(\lambda\left(g^{\bar{j}r}\Omega_{\bar{j}s}\right) - \delta\mathbf{1} + \Gamma_n\right) \cap \partial\Gamma^{h(x)} \subseteq B_R(0)\,, \qquad \text{at every } x \in M\,, \tag{9}$$

where $\mathbf{1} = (1, 1, \ldots, 1)$.

Consider quaternionic local coordinates $(q^1, \ldots, q^n)$ centered at the point where φ attains its minimum S. We may identify such coordinate neighborhood with the open ball of unit radius $B_1 = B_1(0) \subseteq \mathbb{H}^n$ centered at the origin. Let $v(x) = \varphi(x) + \varepsilon|x|^2$ be defined on B_1 for some small fixed $\varepsilon > 0$. Observe that $\inf_{B_1} v = v(0) = \varphi(0) = S$ and $\inf_{\partial B_1} v \geq v(0) + \varepsilon$. These conditions allow us to apply the ABP method (see [60, Proposition 10]) to obtain

$$C_0\varepsilon^{4n} \leq \int_P \det(D^2 v)\,, \tag{10}$$

where $C_0 > 0$ is a dimensional constant,

$$P = \left\{x \in B_1 \mid |Dv(x)| < \frac{\varepsilon}{2},\, v(y) \geq v(x) + Dv(x)\cdot(y - x) \text{ for all } y \in B_1\right\}\,,$$

and Dv, D^2v are the gradient and the (real) Hessian of v. Note that $P \subseteq \{x \in B_1 \mid D^2v(x) \geq 0\}$ and since convexity implies quaternionic plurisubharmonicity (see e.g. [1]), at any point $x \in P$ we have $\mathrm{Hess}_{\mathbb{H}} v(x) \geq 0$. Therefore $\mathrm{Hess}_{\mathbb{H}}\varphi(x) \geq -\frac{\varepsilon}{2}\mathbb{1}$, where $\mathbb{1}$ is the $n \times n$ identity matrix. Choosing ε small enough depending on g and δ, we have

$$\lambda\left(g^{\bar{j}r}(\Omega_{\bar{j}s} + \varphi_{\bar{j}s})\right) \in \lambda\left(g^{\bar{j}r}\Omega_{\bar{j}s}\right) - \delta\mathbf{1} + \Gamma_n\,, \qquad \text{at every } x \in P\,.$$

On the other hand, equation (2) also gives

$$\lambda\left(g^{\bar{j}r}(\Omega_{\bar{j}s} + \varphi_{\bar{j}s})\right) \in \partial\Gamma^{h(x)}\,, \qquad \text{at every } x \in P\,.$$

These two facts, together with (9) imply $|\varphi_{\bar{r}s}| \leq C$ on P and thus also $v_{\bar{r}s} \leq C$. Combining a calculation in [13] with [56, Lemma 2], or alternatively using directly a computation in the proof of [3, Proposition 2.1], at any point $x \in P$ we have

$$\det(D^2 v) \leq 2^{4n} \det(\mathrm{Hess}_{\mathbb{H}}(v))^4 ,$$

where, on the right-hand side, "det" denotes the Moore determinant, introduced in [50] (see also e.g. [1, 7, 58]). Therefore, from (10) we see that

$$C_0 \varepsilon^{4n} \leq C \mathrm{Vol}(P) .$$

The definition of P entails that $v(0) \geq v(x) - Dv(x) \cdot x > v(x) - \varepsilon/2$, i.e. $v(x) < S + \varepsilon/2 < 0$ for all $x \in P$. As a consequence for any $p > 0$

$$\|v\|^p_{L^p(M)} \geq \|v\|^p_{L^p(P)} = \int_P (-v)^p \geq \left| S + \frac{\varepsilon}{2} \right|^p \mathrm{Vol}(P) .$$

From the previous lemma we know that there is a $p > 0$ such that $\|v\|_{L^p}$ is bounded, therefore also $S = \inf_M \varphi$ must be bounded. □

3. Laplacian estimate

This section is devoted to derive a C^0-estimate for the quaternionic Laplacian of solutions to (2) in terms of the squared norm of the gradient. This step is the most involved in terms of calculations and it is here that we use our strongest assumptions to have a locally flat hypercomplex structure and a hyperkähler metric compatible with it.

We follow Székelyhidi [60] and Hou-Ma-Wu [39], which in turn is based on an idea of Chou and Wang [18] for the real Hessian equation. Our restrictive assumptions simplify quite a bit the computations.

As declared in the introduction, let $F(A) = f(\lambda(A))$ be a symmetric function of the eigenvalues of $A_{rs} = g^{\bar{j}r} \Omega^{\varphi}_{\bar{j}s} = g^{\bar{j}r}(\Omega_{\bar{j}s} + \varphi_{\bar{j}s})$. We denote the derivatives of F by

$$F^{rs} = \frac{\partial F}{\partial A_{rs}}, \qquad F^{rs,lt} = \frac{\partial^2 F}{\partial A_{rs} \partial A_{lt}}.$$

Let Q_{rs} be the standard quaternionic coordinates on $\mathbb{H}^{n,n}$ and let E^p_{rs} be the real coordinates underlying Q_{rs}, i.e. $Q_{rs} = E^0_{rs} + E^1_{rs} i + E^2_{rs} j + E^3_{rs} k$. We have the following:

Lemma 7. *The linearization of F at φ is the operator*

$$L(\psi) = \mathrm{Re} \sum_{r,s=1}^{n} F^{rs} g^{\bar{j}r} \psi_{\bar{j}s} .$$

Proof. With respect to the real coordinates E^p_{rs} we decompose a matrix $A \in \mathbb{H}^{n,n}$ as $A^{rs}_p E^p_{rs}$. Define the derivatives $F^{rs}_p := \frac{\partial F}{\partial A^{rs}_p}$ and the matrix $H = (F^{rs})$. For a curve of hyperhermitian matrices A_t with respect to g we have

$$\frac{d}{dt} F(A_t) = \sum_{r,s=}^{n} \sum_{p=0}^{3} F^{rs}_p (A_t)(A'_t)^{rs}_p = \mathrm{Re}\, F^{rs}(A_t)(A'_t)_{rs}$$

Now, for each $\psi \in C^2(M, \mathbb{R})$ and $t \in (-\varepsilon, \varepsilon)$, let $\varphi(t)$ be a curve of Γ-admissible functions in $C^2(M, \mathbb{R})$ with $\varphi(0) = \varphi$ and $\varphi'(0) = \psi$ and set $A_t = g^{-1} (\Omega + \mathrm{Hess}_{\mathbb{H}} \varphi(t))$, then

$$L(\psi) = \frac{d}{dt} F(A_t)\big|_{t=0} = \mathrm{Re}\, F^{rs}(A_0)(A'_0)_{rs} = \mathrm{Re} \sum_{r,s=1}^{n} F^{rs}(A_0) g^{\bar{j}r} \psi_{\bar{j}s} .$$ □

In order to prove the desired bound we will need the following preliminary lemma.

Lemma 8. *Let $\sup_{\partial\Gamma} f < a < b < \sup_\Gamma f$ and $\delta, R > 0$. Then there exists a constant $\kappa > 0$ such that for any $\sigma \in [a,b]$, $B \in \mathrm{Hyp}(n,\mathbb{H})$ satisfying*

$$(\lambda(B) - 2\delta\mathbf{1} + \Gamma_n) \cap \partial\Gamma^\sigma \subseteq B_R(0)\,,$$

$A \in \mathrm{Hyp}(n,\mathbb{H})$ satisfying $\lambda(A) \in \partial\Gamma^\sigma$ and $|\lambda(A)| > R$, we have

$$\text{either} \qquad \operatorname{Re} F^{rs}(A)\,(B_{rs} - A_{rs}) > \kappa \sum_{r=1}^n F^{rr}(A)\,,$$

$$\text{or} \qquad F^{ss}(A) > \kappa \sum_{r=1}^n F^{rr}(A)\,, \qquad \text{for all } s = 1, \dots, n\,.$$

Proof. The lemma follows from the very same argument as [60, Proposition 6] once we have proved a quaternionic analogue of the Schur-Horn theorem. □

Lemma 9 (Quaternionic Schur-Horn Theorem)**.** *Let $\mu = (\mu_1, \dots, \mu_n), \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ be such that $\mu_1 \geq \cdots \geq \mu_n$ and $\lambda_1 \geq \cdots \geq \lambda_n$. There exists a hyperhermitian matrix B with diagonal μ and eigenvalues λ if and only if*

$$\sum_{i=1}^j \mu_i \leq \sum_{i=1}^j \lambda_i\,, \qquad \text{for all } j = 1, \dots, n \qquad \text{and} \qquad \sum_{i=1}^n \mu_i = \sum_{i=1}^n \lambda_i\,. \tag{11}$$

Proof. A hyperhermitian matrix B satisfies the assumptions of the lemma if and only if there exists $C \in \mathrm{Sp}(n)$ such that $B = C^*DC$ where D is the diagonal matrix with diagonal λ. In particular μ is the diagonal of B if and only if $\mu = T\lambda$ where $T = (|c_{rs}|^2)$. Since $C \in \mathrm{Sp}(n)$, the matrix T is doubly stochastic. By the Birkhoff theorem [11] $\mu = T\lambda$, where T is doubly stochastic, if and only if T lies in the convex hull of the set of all permutation matrices. In other words B exists if and only if μ lies in the convex hull of the vectors obtained by permuting the entries of λ, which is known to be equivalent to (11) (see e.g. [35, Theorem 46]). □

Proposition 10. *Let (M, I, J, K, g) be a compact flat hyperkähler manifold. If $\underline{\varphi}, \varphi$ are a $\mathcal{C}$-subsolution and a solution to* (2) *respectively, then there is a constant $C > 0$, depending only on (M, I, J, K), $\|g\|_{C^2}$, $\|h\|_{C^2}$, $\|\Omega\|_{C^2}$, $\|\varphi\|_{C^0}$ and $\underline{\varphi}$, such that*

$$\|\Delta_g \varphi\|_{C^0} \leq C\left(\|\nabla\varphi\|^2_{C^0} + 1\right)\,.$$

Here ∇ denotes the Obata connection on M.

Let us remark that as pointed out by Alesker [2, pp. 204], M admitting a flat hyperkähler metric g compatible with the hypercomplex structure implies that g is parallel with respect to the Obata connection, therefore the Obata connection and the Levi-Civita connection coincide.

We observe that at a point where A is diagonal with distinct eigenvalues we have

- $\lambda_i^{rs} := \frac{\partial \lambda_i}{\partial A_{rs}} = \delta_{ir}\delta_{is}$,
- $\lambda_i^{rs,tl} := \frac{\partial^2 \lambda_i}{\partial A_{rs}\partial A_{tl}} = (1 - \delta_{ir})\frac{\delta_{is}\delta_{it}\delta_{rl}}{\lambda_i - \lambda_r} + (1 - \delta_{it})\frac{\delta_{il}\delta_{ir}\delta_{st}}{\lambda_i - \lambda_t}$

(see e.g. [28, 55]). Furthermore, since $F(A) = f(\lambda(A))$ for f symmetric, then $F^{rs} = \delta_{rs}f_r$, and since f is concave and satisfies $f_i > 0$ (assumption C1 in the introduction), then F is concave and $\frac{f_r - f_s}{\lambda_r - \lambda_s} \leq 0$. In particular $f_r \geq f_s$ anytime $\lambda_r \leq \lambda_s$. Finally, we observe that by [60, Lemma 9 (b)] for any fixed $x \in M$ there is a constant $\tau > 0$ depending on $h(x)$ such that

$$\sum_{a=1}^n F^{aa}(x) > \tau > 0\,. \tag{12}$$

We will mainly be interested in the largest eigenvalue λ_1 of the matrix A around some fixed point x_0. As pointed out by Székelyhidi [60] in order for $\lambda_1 \colon M \to \mathbb{R}$ to define a smooth function at x_0 we need the eigenvalues to be distinct; to be sure of that, we perturb the matrix A.

At any fixed point $x_0 \in M$ we can perturb A in order to have a matrix with distinct eigenvalues. Indeed, fix quaternionic local coordinates around the point x_0 such that, at x_0, A is diagonal and its eigenvalues satisfy

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \,; \tag{13}$$

take a constant diagonal matrix D whose entries satisfy

$$0 = D_{11} < D_{22} < \cdots < D_{nn} \,.$$

The matrix $\tilde{A} = A - D$ has, at x_0, the eigenvalues

$$\tilde{\lambda}_1 = \lambda_1 \,, \qquad \tilde{\lambda}_i = \lambda_i - D_{ii} \,, \text{ for } i = 2, \ldots, n \,,$$

which are distinct by construction.

We will make use of the linearized operator L defined by $L(u) = 4\mathrm{Re} \sum_{a,b=1}^n F^{ab} g^{\bar{c}a} u_{\bar{c}b}$, where $u_{\bar{c}b} = \frac{1}{4}\partial_{\bar{q}^c}\partial_{q^b} u$. First of all, we prove the following inequality for $L\left(2\sqrt{\tilde{\lambda}_1}\right)$.

Lemma 11. *With respect to quaternionic local coordinates around x_0 such that $(g_{\bar{r}s})$ is the identity at x_0 and $(\Omega^{\varphi}_{\bar{r}s})$ is diagonal at x_0, we have*

$$L\left(2\sqrt{\tilde{\lambda}_1}\right) \geq -\frac{F^{aa}|\Omega^{\varphi}_{\bar{1}1,a}|^2}{2\lambda_1\sqrt{\lambda_1}} - \frac{C\mathcal{F}}{\sqrt{\lambda_1}} \,,$$

where $\mathcal{F} = \sum_{a=1}^n F^{aa}(x_0)$, $\Omega^{\varphi}_{\bar{1}1,a} = \partial_{q^a}\Omega^{\varphi}_{\bar{1}1}$ and $C > 0$ is a positive constant depending only on (M, I, J, K), $\|\Omega\|_{C^2}$ and $\|h\|_{C^2}$.

Proof. We have for the perturbed matrix $\tilde{A}_{rs} = A_{rs} - D_{rr}\delta_{rs} = g^{\bar{j}r}\Omega^{\varphi}_{\bar{j}s} - D_{rr}\delta_{rs}$ at the point x_0 where $(g_{\bar{r}s})$ is the identity and A (and thus (F^{rs})) is diagonal

$$L\left(2\sqrt{\tilde{\lambda}_1}\right) = 8\mathrm{Re}\, F^{ab}\left(\sqrt{\tilde{\lambda}_1}\right)_{\bar{a}b} = 2F^{aa}\sum_{p=0}^{3}\left(\sqrt{\tilde{\lambda}_1}\right)_{x_p^a x_p^a} = F^{aa}\sum_{p=0}^{3}\left(\frac{\tilde{\lambda}_{1,x_p^a x_p^a}}{\sqrt{\lambda_1}} - \frac{\tilde{\lambda}^2_{1,x_p^a}}{2\lambda_1\sqrt{\lambda_1}}\right) \,, \tag{14}$$

where the subscript x_p^a denotes the real derivative with respect to the corresponding coordinate. Using the formulas for the derivatives of the eigenvalues we obtain at x_0

$$\begin{aligned}
\tilde{\lambda}_{1,x_p^a} &= \tilde{\lambda}_1^{rs}\tilde{A}_{rs,x_p^a} = \Omega^{\varphi}_{\bar{1}1,x_p^a} \\
\tilde{\lambda}_{1,x_p^a x_p^a} &= \tilde{\lambda}_1^{rs,lt}\tilde{A}_{rs,x_p^a}\tilde{A}_{lt,x_p^a} + \tilde{\lambda}_1^{rs}\tilde{A}_{rs,x_p^a x_p^a} = \sum_{r>1}\frac{\tilde{A}_{r1,x_p^a}\tilde{A}_{1r,x_p^a} + \tilde{A}_{1r,x_p^a}\tilde{A}_{r1,x_p^a}}{\lambda_1 - \tilde{\lambda}_r} + \Omega^{\varphi}_{\bar{1}1,x_p^a x_p^a} \\
&= \sum_{r>1}\frac{A_{r1,x_p^a}A_{1r,x_p^a} + A_{1r,x_p^a}A_{r1,x_p^a}}{\lambda_1 - \tilde{\lambda}_r} + g^{\bar{j}1}\Omega^{\varphi}_{\bar{j}1,x_p^a x_p^a} = 2\sum_{r>1}\frac{|\Omega^{\varphi}_{\bar{r}1,x_p^a}|^2}{\lambda_1 - \tilde{\lambda}_r} + \Omega^{\varphi}_{\bar{1}1,x_p^a x_p^a} \,,
\end{aligned}$$

where we used that the derivatives of D vanish because it is a constant matrix.

Differentiating the equation $F(A) = h$ twice with respect to x_p^1 gives, at the point x_0,

$$\mathrm{Re}\, F^{rs,tl}\Omega^{\varphi}_{\bar{s}r,x_p^1}\Omega^{\varphi}_{\bar{l}t,x_p^1} + F^{rr}\Omega^{\varphi}_{\bar{r}r,x_p^1 x_p^1} = h_{x_p^1 x_p^1} \,. \tag{15}$$

We observe that

$$\sum_{p=0}^{3}\Omega^{\varphi}_{\bar 11,x^a_px^a_p}=\sum_{p=0}^{3}\left(\Omega_{\bar 11,x^a_px^a_p}+\varphi_{\bar 11x^a_px^a_p}\right)=4\Omega_{\bar 11,\bar aa}+4\varphi_{\bar aa\bar 11}=4\Omega_{\bar 11,\bar aa}-4\Omega_{\bar aa,\bar 11}+\sum_{p=0}^{3}\Omega^{\varphi}_{\bar aa,x^1_px^1_p}$$

and thus, by (15) and (12)

$$F^{aa}\sum_{p=0}^{3}\tilde\lambda_{1,x^a_px^a_p}\geq F^{aa}\sum_{p=0}^{3}\Omega^{\varphi}_{\bar 11,x^a_px^a_p}\geq -\mathrm{Re}\,F^{rs,tl}\sum_{p=0}^{3}\Omega^{\varphi}_{\bar rs,x^1_p}\Omega^{\varphi}_{\bar tl,x^1_p}-C\mathcal{F}\geq -C\mathcal{F}$$

where we also used the concavity of F. Finally from (14) we have the desired inequality

$$L\left(2\sqrt{\tilde\lambda_1}\right)\geq -\frac{F^{aa}\sum_{p=0}^{3}(\Omega^{\varphi}_{\bar 11,x^a_p})^2}{2\lambda_1\sqrt{\lambda_1}}-\frac{C\mathcal{F}}{\sqrt{\lambda_1}}\,. \qquad \square$$

Proof of Proposition 10. We have already seen that the Laplacian is bounded from below, as a consequence of (7), therefore it is enough to obtain a bound of the form

$$\frac{\lambda_1}{\|\nabla\varphi\|^2_{C^0}+1}\leq C\,.$$

Define the function

$$G=2\sqrt{\tilde\lambda_1}+\alpha(|\nabla\varphi|^2)+\beta(\varphi)\,,$$

where

$$\alpha(t)=-\frac12\log\left(1-\frac{t}{2N}\right)\,,\qquad N=\|\nabla\varphi\|^2_{C^0}+1\,,$$
$$\beta(t)=-2St+\frac12 t^2\,,\qquad S>\|\varphi\|_{C^0}\,,\ \text{large constant to be chosen later}\,,$$

and $\tilde\lambda_1$ is, as before, the highest eigenvalue of the perturbed matrix $\tilde A$ around a point x_0, which we choose to be a maximum point of G. The derivative of the functions α and β satisfy

$$\frac{1}{4N}<\alpha'(|\nabla\varphi|^2)<\frac{1}{2N}\,,\qquad \alpha''=2(\alpha')^2\,, \tag{16}$$
$$S\leq -\beta'(\varphi)\leq 3S\,,\qquad \beta''=1\,. \tag{17}$$

At x_0 we have $L(G)\leq 0$. Choose quaternionic local coordinates such that $(g_{\bar rs})$ is the identity in the whole neighborhood of x_0 and $(\Omega^{\varphi}_{\bar rs})$ is diagonal at x_0. This is possible because we are assuming g hyperkähler and flat. Then

$$0\geq 4\mathrm{Re}\,F^{ab}G_{\bar ab}=4F^{aa}G_{\bar aa}=F^{aa}\sum_{p=0}^{3}G_{x^a_px^a_p}\,, \tag{18}$$

because F^{ab} is diagonal at x_0. We compute the derivatives of G at x_0:

$$0=G_{x^a_p}=\left(2\sqrt{\tilde\lambda_1}\right)_{x^a_p}+\alpha'\sum_{r=1}^{n}(\varphi_{\bar rx^a_p}\varphi_r+\varphi_{\bar r}\varphi_{rx^a_p})+\beta'\varphi_{x^a_p}\,,$$

$$\begin{aligned}G_{x^a_px^a_p}=&\left(2\sqrt{\tilde\lambda_1}\right)_{x^a_px^a_p}+\alpha''\left(\sum_{r=1}^{n}(\varphi_{\bar rx^a_p}\varphi_r+\varphi_{\bar r}\varphi_{rx^a_p})\right)^2\\&+\alpha'\sum_{r=1}^{n}(\varphi_{\bar rx^a_px^a_p}\varphi_r+2|\varphi_{rx^a_p}|^2+\varphi_{\bar r}\varphi_{rx^a_px^a_p})+\beta''\varphi^2_{x^a_p}+\beta'\varphi_{x^a_px^a_p}\,.\end{aligned}$$

Differentiating the equation $F(A)=h$ yields

$$F^{aa}\Omega^{\varphi}_{\bar a a,x_p^r}=h_{x_p^r}\,,\qquad \text{at } x_0.$$

Using this, Cauchy-Schwarz inequality and (16) we have

$$\begin{aligned}\alpha' F^{aa}\sum_{r=1}^{n}(\varphi_{\bar r\bar a a}\varphi_r+\varphi_{\bar r}\varphi_{r\bar a a}) &= \alpha' F^{aa}\sum_{r=1}^{n}(\varphi_{\bar a a\bar r}\varphi_r+\varphi_{\bar r}\varphi_{\bar a a r})\\ &= \alpha'\sum_{r=1}^{n}\left((h_{\bar r}-F^{aa}\Omega_{\bar a a,\bar r})\varphi_r+\varphi_{\bar r}(h_r-F^{aa}\Omega_{\bar a a,r})\right)\\ &\geq -\frac{C}{N}(N^{1/2}+N^{1/2}\mathcal{F})\geq -C\mathcal{F}\,,\end{aligned}$$

where we used (12) to absorb the constants into $C\mathcal{F}$. Again using (16) we also obtain

$$\begin{aligned}2\alpha' F^{aa}\sum_{r=1}^{n}\sum_{p=0}^{3}|\varphi_{r x_p^a}|^2 &\geq \frac{1}{2N}F^{aa}\sum_{r=1}^{n}\sum_{p,q=0}^{3}\varphi^2_{x_q^r x_p^a}\geq \frac{1}{2N}F^{aa}\sum_{p=0}^{3}\varphi^2_{x_p^a x_p^a}=\frac{8}{N}F^{aa}\varphi^2_{\bar a a}\\ &= \frac{8}{N}F^{aa}(\lambda_a-\Omega_{\bar a a})^2\geq \frac{2}{N}F^{aa}\lambda_a^2-C\mathcal{F}\,,\end{aligned}$$

where, for the last inequality we used that $(a+b)^2\geq \frac{1}{2}a^2-b^2$. Thanks to the last two inequalities, from our main inequality (18) we get

(19)

$$0\geq L\left(2\sqrt{\tilde\lambda_1}\right)+\alpha'' F^{aa}\sum_{p=0}^{3}\left(2\sum_{r=1}^{n}\mathrm{Re}(\varphi_{\bar r x_p^a}\varphi_r)\right)^2+\beta'' F^{aa}|\varphi_a|^2+4\beta' F^{aa}\varphi_{\bar a a}+\frac{2F^{aa}\lambda_a^2}{N}-C\mathcal{F}\,.$$

By $G_{x_p^a}(x_0)=0$ we have

$$\begin{aligned}\alpha'' F^{aa}\left(2\sum_{r=1}^{n}\mathrm{Re}(\varphi_{\bar r x_p^a}\varphi_r)\right)^2 &= 2F^{aa}\left(\frac{\Omega^{\varphi}_{\bar 1 1,x_p^a}}{\sqrt{\lambda_1}}+\beta'\varphi_{x_p^a}\right)^2\\ &\geq 2\varepsilon\frac{F^{aa}(\Omega^{\varphi}_{\bar 1 1,x_p^a})^2}{\lambda_1}-\frac{2\varepsilon}{1-\varepsilon}(\beta')^2F^{aa}\varphi^2_{x_p^a}\,,\end{aligned}\tag{20}$$

where we used the inequality $(a+b)^2\geq \varepsilon a^2-\frac{\varepsilon}{1-\varepsilon}b^2$, which holds for $\varepsilon\in(0,1)$. Summing (20) over p and combining it with Lemma 11 we obtain from (19)

$$0\geq\left(4\varepsilon\sqrt{\lambda_1}-1\right)\frac{F^{aa}|\Omega^{\varphi}_{\bar 1 1,a}|^2}{2\lambda_1\sqrt{\lambda_1}}+\left(\beta''-\frac{2\varepsilon(\beta')^2}{1-\varepsilon}\right)F^{aa}|\varphi_a|^2+4\beta' F^{aa}\varphi_{\bar a a}+\frac{2F^{aa}\lambda_a^2}{N}-C\mathcal{F}\,.\tag{21}$$

Choosing $\varepsilon=1/(18S^2+1)<1$, (17) implies

$$\beta''-\frac{2\varepsilon}{1-\varepsilon}(\beta')^2\geq 0\,.$$

Furthermore, we can assume without loss of generality $\sqrt{\lambda_1}>\frac{1}{4\varepsilon}$ and deduce

$$\left(4\varepsilon\sqrt{\lambda_1}-1\right)\frac{F^{aa}|\Omega^{\varphi}_{\bar 1 1,a}|^2}{2\lambda_1\sqrt{\lambda_1}}\geq 0\,.$$

Then we obtain from (21)

$$0\geq 4\beta' F^{aa}\varphi_{\bar a a}+\frac{2F^{aa}\lambda_a^2}{N}-C\mathcal{F}\,.\tag{22}$$

As before, we can assume $\underline{\varphi} \equiv 0$, otherwise we could choose a suitable background form Ω in order to simplify the equation. Set $B_{rs} = g^{\bar{j}r}\Omega_{\bar{j}s}$ and let $\delta, R > 0$ be such that

$$(\lambda(B) - 2\delta\mathbf{1} + \Gamma_n) \cap \partial\Gamma^{h(x)} \subseteq B_R(0)\,, \qquad \text{at every } x \in M\,,$$

which exist because of the definition of $\mathcal{C}$-subsolution. Supposing $\lambda_1 > R$ we have $|\lambda(A)| > R$ and we can then apply Lemma 8 according to which there exists $\kappa > 0$ such that one of the following two cases occur:

- First case:

$$\operatorname{Re} F^{rs}(A)(B_{rs} - A_{rs}) = -\operatorname{Re}\sum_{r,s=1}^{n} F^{rs}(A) g^{\bar{j}r}\varphi_{\bar{j}s} > \kappa \sum_{r=1}^{n} F^{rr}(A)\,,$$

  i.e. $-F^{aa}\varphi_{\bar{a}a} > \kappa\mathcal{F}$ at x_0, which for a choice of S large enough implies $4\beta' F^{aa}\varphi_{\bar{a}a} - C\mathcal{F} \geq 0$ allowing us to deduce from (22) $0 \geq \frac{2}{N}F^{aa}\lambda_a^2$ which is a contradiction.
- Second case:

$$F^{ss}(A) > \kappa \sum_{r=1}^{n} F^{rr}(A)\,, \qquad \text{for all } s = 1, \dots, n\,,$$

  and in particular $F^{11} > \kappa\mathcal{F}$. Therefore $F^{aa}\lambda_a^2 \geq F^{11}\lambda_1^2 \geq \kappa\mathcal{F}\lambda_1^2$. Moreover, we can assume $F^{aa}\lambda_a \leq F^{aa}\lambda_a^2/(12NS)$ for otherwise we would have $\kappa\mathcal{F}\lambda_1^2 < 12NS\mathcal{F}\lambda_1$ and we would conclude. Then we have

$$4\beta' F^{aa}\varphi_{\bar{a}a} \geq -12SF^{aa}\lambda_a - C\mathcal{F} \geq -\frac{F^{aa}\lambda_a^2}{N} - C\mathcal{F}\,.$$

  Substituting this last inequality into (22) we get

$$0 \geq \kappa\frac{\lambda_1^2}{N^2} - C\,.$$

This gives the bound we were searching for at the maximum point x_0 of G, but by monotony of the square root such bound holds globally, depending additionally on a bound for $\|\varphi\|_{C^0}$. □

Remark. Removing the hypothesis that the metric g is hyperkähler one has to deal with its derivatives. Most of the terms are not an issue and can be easily controlled, however those terms that contain the third derivative of φ seem not to be straightforwardly manageable.

Remark. The function G used in the proof of Proposition 10 is basically the same as the one used in [60], however we replaced the logarithm with the square root, a trick which is inspired by the work of Alesker [2]. It seems that using the square root allows to simplify the argument.

Remark. Under an additional assumption the Laplacian can be controlled linearly by the gradient. Indeed, if we further assume

$$F^{aa}\lambda_a \leq c_0\,, \tag{23}$$

which is the case for the quaternionic Monge-Ampère, the quaternionic Hessian, and the quaternionic Monge-Ampère equation for $(n-1)$-quaternionic plurisubharmonic functions, we obtain the following sharper estimate in the second case above, more precisely, from (22), $F^{11} > \kappa\mathcal{F}$ and (23) we get

$$\begin{aligned} 0 &\geq 4\beta' F^{aa}(\lambda_a - 1) + \frac{2F^{11}\lambda_1^2}{N} - C\mathcal{F} \geq 4\beta' F^{aa}\lambda_a + \frac{2\kappa\lambda_1^2}{N}\mathcal{F} + \big(-4\beta' - C\big)\mathcal{F} \\ &\geq 4\beta' c_0 + \frac{2\kappa\lambda_1^2}{N}\mathcal{F} + \big(-4\beta' - C\big)\mathcal{F} \geq \frac{2\kappa\lambda_1^2}{N}\mathcal{F} + \left(-4\beta' - C + \frac{4\beta' c_0}{\tau}\right)\mathcal{F}, \end{aligned}$$

where we have used $\mathcal{F} \geq \tau > 0$ in the last inequality. Then we have

$$0 \geq 2\kappa \frac{\lambda_1^2}{N} - \left(4\beta' + C - \frac{4\beta' c_0}{\tau}\right),$$

which gives a sharper bound

$$\lambda_1 \leq C(1 + \|\nabla\varphi\|_{C^0}).$$

4. Blow-up analysis

In this section we show that a bound for the gradient of solutions to (2) can be obtained by using a Liouville-type theorem. We adapt the approach of Dinew and Kołodziej [20] to our setting.

We introduce the following:

Definition 12. A continuous function $u\colon \mathbb{H}^n \to \mathbb{R}$ is a *(viscosity)* Γ*-subsolution (resp. supersolution)* if for all $\psi\colon \mathbb{H}^n \to \mathbb{R}$ of class C^2 such that $u - \psi$ has a local maximum (resp. minimum) at p, we have $\lambda(\mathrm{Hess}_{\mathbb{H}}\psi) \in \bar{\Gamma}$ (resp. $\lambda(\mathrm{Hess}_{\mathbb{H}}\psi) \in \mathbb{R}^n \setminus \Gamma$) at p. We say that u is a *(viscosity)* Γ*-solution* if it is both a subsolution and a supersolution.

We show that if the gradient bound for solutions to (2) does not hold, we are able to find a bounded $C^{1,\alpha}$ viscosity Γ-solution $u\colon \mathbb{H}^n \to \mathbb{R}$ with bounded gradient and such that $|\nabla u(0)| = 1$. In particular u is non-constant. In the next section we prove a Liouville-type theorem for this kind of functions, thus yielding a contradiction and showing implicitly that the gradient bound holds.

Let (M, I, J, K, g) be a compact hyperhermitian manifold. Consider a sequence $(\underline{\varphi}_j)_j$, $(\varphi_j)_j$, $(h_j)_j$ of real smooth functions on M and a sequence $(\Omega_j)_j$ of q-real $(2,0)$-forms on M such that $\underline{\varphi}_j$ are $\mathcal{C}$-subsolutions and φ_j, h_j, Ω_j satisfy

$$\begin{cases} F\left(g^{\bar{t}r}((\Omega_j)_{\bar{t}s} + (\varphi_j)_{\bar{t}s})\right) = h_j\,, \\ \sup_M \varphi_j = 0\,, \\ \|\nabla\varphi_j\|_{C^0} \geq j\,. \end{cases} \tag{24}$$

Assume further that $(\underline{\varphi}_j)_j$, $(h_j)_j$ and $(\Omega_j)_j$ are uniformly bounded in C^2-norm.

Set $N_j = \|\nabla\varphi_j\|_{C^0}^2$, $g_j = N_j g$ and let $x_j \in M$ be such that $|\nabla\varphi_j(x_j)|^2 = N_j$ for each $j > 0$. Choose quaternionic local coordinates $(q^1, \ldots, q^n)$ around x_j for $|q^i| < N_j^{1/2}$ such that

$$(g_j)_{\bar{r}s} = \delta_{\bar{r}s} + O(N_j^{-1}|x|)\,, \qquad (\Omega_j)_{\bar{r}s} = O(N_j^{-1})\,, \qquad h_j = h_j(x_j) + O(N_j^{-1}|x|)\,.$$

Then $|\nabla\varphi_j(x_j)|_{g_j}^2 = 1$ and by Propositions 6 and 10 we have in this coordinates

$$\|\varphi_j\|_{C^0} \leq C\,, \qquad |\Delta_g \varphi_j|_{g_j} \leq C\,, \qquad \text{on } B_{N_j^{1/2}}(x_j)\,,$$

where $C > 0$ is uniform in j. It follows by [29, Theorem 8.32] that $(\varphi_j)_j$ is uniformly bounded in $C^{1,\alpha}$-norm for any $\alpha \in (0,1)$. Furhermore, letting $j \to \infty$, we see that Ω_j tends to zero, while g_j tends to the standard Euclidean metric and $(\varphi_j)_{\bar{r}s}$ stays bounded. Therefore

$$\lambda(A_j) = \lambda((\varphi_j)_{\bar{r}s}) + O(N_j^{-1}|x|)\,, \tag{25}$$

where $(A_j)^r_s = g_j^{\bar{t}r}((\Omega_j)_{\bar{t}s} + (\varphi_j)_{\bar{t}s})$.

By Ascoli-Arzelà Theorem we can extract from $(\varphi_j)_j$ a subsequence converging uniformly in $C^{1,\alpha}$ to some $u\colon \mathbb{H}^n \to \mathbb{R}$, moreover, such limiting function satisfies $\|u\|_{C^0} \leq C$, $\|\nabla u\|_{C^0} \leq C$ and $|\nabla u(0)| = 1$. We aim to prove that u is a Γ-solution.

Suppose there exists $\psi \in C^2$, such that $u - \psi$ has a local maximum at some point $p_0 \in \mathbb{H}^n$. By construction of u, for any $\varepsilon > 0$ there are a j large enough, $a \in (-\varepsilon, \varepsilon)$ and a point p_1 with $|p_1 - p_0| < \varepsilon$ such that $\varphi_j - \psi - \varepsilon|x - p_0|^2 + a$ has a local maximum at p_1. As a consequence the quaternionic Hessian of ψ satisfies

$$\mathrm{Hess}_{\mathbb{H}}\psi + \frac{\varepsilon}{2}\mathbb{1} \geq \mathrm{Hess}_{\mathbb{H}}\varphi_j\,, \qquad \text{at } p_1\,,$$

where $\mathbb{1}$ is the $n \times n$ identity matrix. By (25), if j is large enough we see that $\lambda(\mathrm{Hess}_{\mathbb{H}}\psi) \in \Gamma - \varepsilon\mathbf{1}$ at p_1 and letting $\varepsilon \to 0$ we deduce $\lambda(\mathrm{Hess}_{\mathbb{H}}\psi) \in \bar{\Gamma}$ at p_0 because $p_1 \to p_0$. This shows that u is a viscosity Γ-subsolution.

To see that u is also a Γ-supersolution we proceed similarly. Suppose that $u - \psi$ has a local minimum at $p_0 \in \mathbb{H}^n$, then for any $\varepsilon > 0$ there are j large enough, $a \in (-\varepsilon, \varepsilon)$ and $p_1 \in \mathbb{H}^n$ such that $\varphi_j - \psi + \varepsilon|x - p_0|^2 + a$ has a local minimum at p_1. Hence

$$\mathrm{Hess}_{\mathbb{H}}\psi - \frac{\varepsilon}{2}\mathbb{1} \leq \mathrm{Hess}_{\mathbb{H}}\varphi_j\,, \qquad \text{at } p_1\,.$$

By contradiction, suppose $\lambda(\mathrm{Hess}_{\mathbb{H}}\psi(p_1)) \in \Gamma + \frac{5}{2}\varepsilon\mathbf{1}$, then $\lambda(\mathrm{Hess}_{\mathbb{H}}\varphi_j(p_1)) \in \Gamma + 2\varepsilon\mathbf{1}$ and for j large enough (25) we have $\lambda(A_j) \in \Gamma + \varepsilon\mathbf{1}$. By [60, Lemma 9 (a)] it follows that for N_j large enough $\Gamma + N_j\varepsilon\mathbf{1} \subseteq \Gamma^{h_j(p_1)}$ and consequently we deduce

$$N_j\lambda(A_j) \in N_j\Gamma + N_j\varepsilon\mathbf{1} = \Gamma + N_j\varepsilon\mathbf{1} \subseteq \Gamma^{h_j(p_1)}$$

for j sufficiently large. On the other hand, φ_j satisfies (24), i.e.

$$N_j\lambda(A_j) = \lambda\left(g^{\bar{t}r}((\Omega_j)_{\bar{t}s} + (\varphi_j)_{\bar{t}s})\right) \in \partial\Gamma^{h_j(p_1)}\,,$$

which gives a contradiction. Therefore $\lambda(\mathrm{Hess}_{\mathbb{H}}\psi(p_1)) \notin \Gamma + \frac{5}{2}\varepsilon\mathbf{1}$ and letting $\varepsilon \to 0$ we finally obtain $\lambda(\mathrm{Hess}_{\mathbb{H}}\psi(p_0)) \notin \Gamma$ and u is a viscosity Γ-solution.

5. Liouville-type theorem

As in Székelyhidi [60] we can interpret the notion of being a Γ-subsolution (resp. solution) as that of being a viscosity subsolution (resp. solution) of a suitable equation. Indeed, define the function G_0 on the space of hyperhermitian matrices as the function such that

$$\lambda(A) - G_0(A)\mathbf{1} \in \bar{\Gamma},$$

consider the projection $\mathrm{p}\colon \mathbb{R}^{4n,4n} \to \{H \in \mathbb{R}^{4n,4n} \mid I_0HI_0 = J_0HJ_0 = K_0HK_0 = -H\}$

$$\mathrm{p}(H) = \frac{1}{4}(H - I_0HI_0 - J_0HJ_0 - K_0HK_0)\,,$$

where (I_0, J_0, K_0) is the standard hyperhermitian structure on $\mathbb{R}^{4n}$ written in block form as

$$(26) \qquad I_0 = \begin{pmatrix} 0 & -\mathbb{1} & 0 & 0 \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbb{1} \\ 0 & 0 & \mathbb{1} & 0 \end{pmatrix}, \quad J_0 = \begin{pmatrix} 0 & 0 & -\mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & -\mathbb{1} & 0 & 0 \end{pmatrix}, \quad K_0 = \begin{pmatrix} 0 & 0 & 0 & -\mathbb{1} \\ 0 & 0 & -\mathbb{1} & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ \mathbb{1} & 0 & 0 & 0 \end{pmatrix},$$

where $\mathbb{1}$ is the $n \times n$ identity matrix. Then, defining the function G on the space of $4n \times 4n$ symmetric matrices $\mathrm{Sym}(4n, \mathbb{R})$ as $G(H) = G_0(\mathrm{p}(H))$, we have that u is a Γ-subsolution (resp. solution) if and only if it is a viscosity subsolution (resp. solution) of the equation $G(D^2u) = 0$.

Therefore we can take advantage from the known results regarding viscosity subsolutions and solutions (see [15]). In particular we will use the following:

- If $(u_j)_j$ is a sequence of Γ-subsolutions (resp. solutions) converging locally uniformly to u, then u is a Γ-subsolution (resp. solution) as well.
- If u, v are Γ-subsolutions, then $u+v$ is a Γ-subsolution as well.
- A mollification of a Γ-subsolution is again a Γ-subsolution.

We will also need the following comparison result

Lemma 13. *If u is a Γ-solution and v a smooth Γ-subsolution on a bounded open set $U \subseteq \mathbb{H}^n$ such that $u = v$ on ∂U, then $u \geq v$ in U.*

Proof. The very same proof of [60, Lemma 17], which is the analogous result in $\mathbb{C}^n$, can be carried out in our hypothesis. □

The next lemma follows from the same argument as [60, Lemmas 18-19]. The additional case when $\Gamma = \Gamma_n$ is quite easy and can be deduced along the same lines.

Lemma 14. *Suppose $v\colon \mathbb{H}^n \to \mathbb{R}$ is a Γ-solution which is independent of the last variable q_n. Define*

$$\Gamma' = \begin{cases} \Gamma_{n-1} & \text{if } \Gamma = \Gamma_n\,, \\ \Gamma \cap \{x_n = 0\} & \text{if } \Gamma \neq \Gamma_n\,, \end{cases} \tag{27}$$

then Γ' is a symmetric proper convex open cone in $\mathbb{R}^{n-1}$ containing Γ_{n-1} and the function $w(q_1, \ldots, q_{n-1}) = v(q_1, \ldots, q_{n-1}, 0)$ is a Γ'-solution on $\mathbb{H}^{n-1}$.

We remark that in view of (7) every Γ-subsolution is subharmonic.

Proposition 15 (Liouville-type Theorem)**.** *A Lipschitz bounded viscosity Γ-solution $u\colon \mathbb{H}^n \to \mathbb{R}$ with $\|\nabla u\|_{C^0} \leq C$ is constant.*

Proof. The result is proved by induction over n. For $n = 1$ the function u is harmonic and the result is well-known.

Assume now that the result holds for $n-1$ and let us prove it for n. By contradiction we suppose that u is not constant and $\inf_M u = 0$, $\sup_M u = 1$. We adopt the notation of [60] and, for any function $v\colon \mathbb{H}^n \to \mathbb{R}$ we write its mollification

$$[v]_r(q) = \int_{q' \in \mathbb{H}^n} v(q + rq')\psi(q')\,\mathrm{dV}\,,$$

where, here and hereafter, dV denotes the standard volume form in $\mathbb{H}^n$ and $\psi\colon \mathbb{H}^n \to \mathbb{R}$ is a smooth mollifier with support in $B_1(0)$ such that $\psi > 0$ in $B_1(0)$ and $\int_{\mathbb{H}^n} \psi\,\mathrm{dV} = 1$. During the proof we will need to regularize u, considering $u^\varepsilon = [u]_\varepsilon$ for a small $\varepsilon > 0$. Following [20] we use Cartan's Lemma to deduce

$$\lim_{r\to\infty} [u^2]_r(q) = \lim_{r\to\infty} [u]_r(q) = 1\,.$$

For $\rho > 0$ and $r > 0$ consider the set

$$U(\rho, r) = \left\{ q \in \mathbb{H}^n \mid 2u(q) \leq [u^2]_r(q) + [u]_\rho(q) - \frac{4}{3} \right\}.$$

Suppose there are $\rho > 0$, $\varepsilon_j \to 0$, $q_j \in \mathbb{H}^n$, $r_j \to \infty$ and a unit vector $\xi_j \in \mathbb{H}^n$ such that $q_j \in U(\rho, r_j)$ and

$$\lim_{j\to\infty} \int_{B_{r_j}(q_j)} |\bar{\partial}_{\xi_j} u^{\varepsilon_j}|^2 \mathrm{dV} = 0\,, \tag{28}$$

where for any vector $\xi = (\xi_0^1 + \xi_1^1 i + \xi_2^1 j + \xi_3^1 k, \dots, \xi_0^n + \xi_1^n i + \xi_2^n j + \xi_3^n k) \in \mathbb{H}^n$ and any function $w\colon \mathbb{H}^n \to \mathbb{R}$ we use the notation

$$\bar\partial_\xi w = \sum_{r=1}^n \left(\xi_0^r w_{x_0^r} + \xi_1^r w_{x_1^r} i + \xi_2^r w_{x_2^r} j + \xi_3^r w_{x_3^r} k\right) .$$

Composing with rotations and translations, for each j we can take q_j to the origin and assume $\xi_j = q^n/2$, obtaining a sequence $(u_j)_j$ of Γ-solutions satisfying

$$[u_j^2]_{r_j}(0) + [u_j]_\rho(0) - 2u_j(0) \geq \frac{4}{3}\,, \qquad \lim_{j\to\infty} \int_{B_{r_j}(0)} \left|\bar\partial_{\frac{q^n}{2}} u_j^{\varepsilon_j}\right|^2 \mathrm{dV} = 0\,. \tag{29}$$

Since u has bounded gradient, by the Ascoli-Arzelà Theorem, up to a subsequence, $(u_j)_j$ converges locally uniformly to some $v\colon \mathbb{H}^n \to \mathbb{R}$ which must be again a Γ-solution with bounded gradient. Also $u_j^{\varepsilon_j}$ converges to v locally uniformly and working as in [20] we infer that v does not depend on the last variable q^n.

Indeed, if v were not constant along lines with fixed $q' = (q^1, \dots, q^{n-1})$, there would be $a, b \in \mathbb{H}$ and a positive $c \in \mathbb{R}$ such that $v(q_0', a) - v(q_0', b) > 2c$. Since the gradient of v is bounded from above, we could choose δ small enough such that

$$\inf\left\{v(q', q^n) \mid |q' - q_0'| < \delta,\, |q^n - a| < \delta\right\} - \sup\left\{v(q', q^n) \mid |q' - q_0'| < \delta,\, |q^n - b| < \delta\right\} > c\,.$$

Let $\xi \in \mathbb{H}^n$ be the unit vector with last entry $(b-a)/|b-a|$ and all others zero. Let γ be the segment joining $(q', a'), (q', b') \in \mathbb{H}^n$, where $b' - a' = b - a$, $|q' - q_0'| < \delta$, $|a' - a| < \delta$, $|b' - b| < \delta$, then we would have

$$\left|\int_\gamma \bar\partial_\xi v \, d\xi\right| = \left|v(q', b') - v(q', a')\right| > c\,.$$

Cauchy-Schwarz inequality would now give

$$c^2 < \left|\int_\gamma \bar\partial_\xi v\, d\xi\right|^2 \leq \left(\int_\gamma |\bar\partial_\xi v|^2 d\xi\right)\left(\int_\gamma d\xi\right) = |b-a| \int_\gamma |\bar\partial_\xi v|^2 d\xi\,.$$

Let I_1, I_2, I_3 be intervals of length δ all perpendicular to each other and to $[a, b]$ in the q^n-space. Using Fubini's theorem over the set $B(q_0', \delta) \times [a, b] \times I_1 \times I_2 \times I_3$ we would find a strictly positive lower bound for the integral of $|\bar\partial_{q^n/2} v|^2 \mathrm{dV}$. But this would contradict the uniform convergence as the u_j's satisfy (29). Therefore v does not depend on the last variable.

The function $w(q^1, \dots, q^{n-1}) = v(q^1, \dots, q^{n-1}, 0)$ is then a Γ'-solution, thanks to Lemma 14, where Γ' is the cone defined in (27). By the induction hypothesis w is constant and then so is v. But by Cartan's Lemma this contradicts the first of (29) because

$$\frac{4}{3} \leq \lim_{j\to\infty} \left([u_j^2]_{r_j}(0) + [u_j]_\rho(0) - 2u_j(0)\right) = 1 + [v]_\rho(0) - 2v(0) = 1 - v(0) \leq 1$$

as v inherits from u the property that $0 \leq v \leq 1$.

This means that (28) cannot hold, in particular for all $\rho > 0$, there exists $c_\rho > 0$ such that if $r > c_\rho$, for each $q \in U(\rho, r)$, $\varepsilon < c_\rho^{-1}$ and unit vector $\xi \in \mathbb{H}^n$ we must have

$$\int_{B_r(q)} |\partial_\xi u^\varepsilon|^2 \mathrm{dV} > c_\rho\,. \tag{30}$$

Define

$$U'(\rho, r) = \left\{q \in \mathbb{H}^n \mid 2u(q) < [u^2]_r(q) + [u]_\rho(q) - \frac{4}{3}\right\} \subseteq U(\rho, r)\,.$$

We may choose the origin so that $u(0) < 1/12$, and $\rho > 0$ and $r > c$ big enough to have $[u]_\rho(0) > 3/4$ and $[u^2]_r(0) > 3/4$ which can be done by Cartan's Lemma. It follows that $0 \in U'(\rho, r)$.

Since $\partial_{\bar{q}^i}\partial_{q^j}(u^\varepsilon)^2 = 2u^\varepsilon u^\varepsilon_{\bar{i}j} + 2u^\varepsilon_{\bar{i}} u^\varepsilon_j$, proceeding similarly as in [60] we can use (30) to prove that there exists a constant $\delta > 0$ small enough to guarantee that $[(u^\varepsilon)^2]_r - \delta|q|^2$ is a Γ-subsolution over $U'(\rho, r)$. By local uniform convergence also $[u^2]_r - \delta|q|^2$ is a Γ-subsolution. Finally consider

$$U''(\rho, r) = \left\{ q \in \mathbb{H}^n \mid 2u(q) < [u^2]_r(q) - \delta|q|^2 + [u]_\rho(q) - \frac{4}{3} \right\} \subseteq U'(\rho, r)$$

and observe that since $0 \le u \le 1$ this set is bounded. The fact that u is a Γ-solution and yet $[u^2]_r(q) - \delta|q|^2 + [u]_\rho(q) - \frac{4}{3}$ is a smooth Γ-subsolution contradicts the comparison principle of Lemma 13. We conclude that u must be constant. $\square$

6. Proof of Theorem 1

The main theorem follows once we obtain the $C^{2,\alpha}$-estimate. We obtain the desired bound in two ways, by using an analogue of Evans-Krylov theory as developed in Tosatti-Wang-Weinkove-Yang [61] and by adapting the argument of Błocki [14] similarly to what was done by Alesker [2] for the treatment of the quaternionic Monge-Ampère equation.

Proposition 16. *Let (M, I, J, K, g) be a compact locally flat hyperhermitian manifold. If φ is a solution to* (2) *such that $\|\varphi\|_{C^0}$ and $\Delta_g\varphi$ are bounded from above, then there is $\alpha \in (0,1)$ and a constant $C > 0$, depending only on the background data such that*

$$\|\varphi\|_{C^{2,\alpha}} \le C\,.$$

Proof. Let $V = \{H \in \mathbb{R}^{4n,4n} \mid I_0HI_0 = J_0HJ_0 = K_0HK_0 = -H\}$, where (I_0, J_0, K_0) is the standard hypercomplex structure on $\mathbb{R}^{4n}$ as in (26). Consider the real representation of quaternionic matrices $\iota\colon \mathbb{H}^{n,n} \to V$, defined as

$$\iota(A + iB + jC + kD) := \begin{pmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{pmatrix}.$$

The map ι is an isomorphism of real algebras and $\iota(\mathrm{Hyp}(n, \mathbb{H})) = V \cap \mathrm{Sym}(4n, \mathbb{R})$. Let $\mathrm{p}\colon \mathbb{R}^{4n,4n} \to V$ be the projection

$$\mathrm{p}(H) := \frac{1}{4}(H - I_0HI_0 - J_0HJ_0 - K_0HK_0)\,.$$

If we take on $\mathbb{H}^n$ the real coordinates $(x_0^1, \dots, x_0^n, x_1^1, \dots, x_1^n, x_2^1, \dots, x_2^n, x_3^1, \dots, x_3^n)$ underlying the quaternionic coordinates $(q^1, \dots, q^n)$, for a C^2 function $u\colon \mathbb{H}^n \to \mathbb{R}$ we have

$$\iota(\mathrm{Hess}_{\mathbb{H}} u) = 16\mathrm{p}(D^2u)\,.$$

For any point $x_0 \in M$, take a quaternionic coordinate chart centered at x_0 and assume that the domain of the chart contains $B_1(0)$. For any $H \in \mathrm{Sym}(4n, \mathbb{R})$ we have $\iota^{-1}(p(H)) \in \mathrm{Hyp}(n, \mathbb{H})$, therefore

$$\tilde{H}_{rs}(x) = g^{\bar{j}r}(x)(\iota^{-1}(p(H)))_{\bar{j}s}\,, \qquad x \in B_1(0)\,,$$

is hyperhermitian with respect to g.

Define the set

$$\mathcal{E} = \left\{ H \in \mathrm{Sym}(4n, \mathbb{R}) \mid \lambda(\tilde{H}(0)) \in \bar{\Gamma}^\sigma \cap \overline{B_{2R}(0)} \right\},$$

where σ and R are chosen below. $\mathcal{E}$ is compact and also convex by convexity of Γ. Possibly shrinking $B_1(0)$ to a smaller radius $r \in (0,1)$ we may assume that if H lies in a sufficiently close neighborhood U of $\mathcal{E}$, then $\lambda(\tilde{H}(x)) \in \bar{\Gamma}^\sigma \cap \overline{B_{4R}(0)}$ for any $x \in B_1(0)$.

The bound $\Delta_g \varphi \leq C$ implies that σ and R can be chosen so that

$$\lambda\left(g^{\bar{j}r}\left(\Omega_{\bar{j}s} + \varphi_{\bar{j}s}\right)\right) \in \bar{\Gamma}^\sigma \cap \overline{B_R(0)}\,, \qquad \text{on } B_1(0)\,.$$

Therefore, by continuity of g, and possibly shrinking $B_1(0)$ again, for each $x \in B_1(0)$ we have

$$\iota(\Omega_{\bar{r}s}(x)) + 16\mathrm{p}(D^2\varphi(x)) = \iota\left(\Omega_{\bar{r}s}(x) + \varphi_{\bar{r}s}(x)\right) \in \mathcal{E}\,.$$

This discussion and our assumptions on f show that we can apply [61, Theorem 1.2] with

- $F\colon \mathrm{Sym}(4n,\mathbb{R}) \times B_1(0) \to \mathbb{R}$ defined as $F(H,x) = f(\lambda(\tilde{H}(x)))$ for $H \in U$, and extended smoothly to all of $\mathrm{Sym}(4n,\mathbb{R}) \times B_1(0)$;
- $S\colon B_1(0) \to \mathrm{Sym}(4n,\mathbb{R})$ defined as $S(x) = \iota(\Omega_{\bar{r}s}(x))$;
- $T\colon \mathrm{Sym}(4n,\mathbb{R}) \times B_1(0) \to \mathrm{Sym}(4n,\mathbb{R})$ defined as $T(H,x) = 16\mathrm{p}(H)$.

And since $\|\varphi\|_{C^0} \leq C$ we obtain the desired bound $\|\varphi\|_{C^{2,\alpha}} \leq C$ for some $\alpha \in (0,1)$. □

Now we present our second proof.

Proof. Since M is locally flat, we only need to prove the following interior $C^{2,\alpha}$ estimate for $w = \varphi + u$, where $u \in C^\infty_{\mathrm{loc}}(M,\mathbb{R})$ is a local potential for Ω.

Now, $w \in C^4(\mathcal{O})$ satisfies

$$F(w_{\bar{r}s}) = h\,,$$

where $\mathcal{O} \subset \mathbb{H}^n$ is an arbitrary open subset and $h \in C^\infty(\mathcal{O})$. Let $\mathcal{O}' \subset \mathcal{O}$ be a relatively compact open subset. We shall prove that there exist a constant $\alpha \in (0,1)$ depending only on n, h, $\|w\|_{C^0(\mathcal{O})}$, $\|\Delta w\|_{C^0(\mathcal{O})}$ and a constant C depending in addition on $\mathrm{dist}(\mathcal{O},\mathcal{O}')$ such that

$$\|w\|_{C^{2,\alpha}(\mathcal{O})} \leq C\,.$$

There is a difference with respect to the argument of Alesker [2]: the quaternionic Monge-Ampère operator can be written in the divergence form, while this might not be true for more general fully non-linear equations. To overcome this issue we will need a more general version of the weak Harnack inequality for second order uniformly elliptic operators.

Let W be the quaternionic Hessian $(w_{\bar{r}s})$ and define a second order linear operator $\mathcal{D}$ by

$$\mathcal{D}v = \mathrm{Re}\, F^{rs}(W) v_{\bar{r}s}\,.$$

Notice that every $n \times n$ hyperhermitian matrix defines a hyperhermitian semilinear form on $\mathbb{H}^n$. Hence it also determines a symmetric bilinear form on $\mathbb{R}^{4n}$. Let $(a_{ij}) \in \mathrm{Sym}(4n,\mathbb{R})$ be the realization of $(F^{rs}(W))$. Then we can rewrite $\mathcal{D}v$ in the following form

$$\mathcal{D}v = \sum_{r,s=1}^{4n} a_{rs} D_r D_s v\,,$$

Since F is uniformly elliptic on Γ, the operator $\mathcal{D}$ is uniformly elliptic as well.

Let $R > 0$ be such that the open ball B_{2R} of radius $2R$ centered at a point $z_0 \in \mathcal{O}'$ is contained in $\mathcal{O}$. For an arbitrary unitary vector $\xi \in \mathbb{H}^n$, we let Δ_ξ denote the Laplacian on any translate of the quaternionic line spanned by ξ. By virtue of concavity of F, for any unitary vector $\xi \in \mathbb{H}^n$, we have

$$\mathrm{Re}\, F^{rs}(W)\Delta_\xi(w_{\bar{r}s}) \geq \Delta_\xi h\,. \tag{31}$$

Consider the function

$$\hat{w} = \sup_{B_{2R}} \Delta_\xi w - \Delta_\xi w \,.$$

it follows from (31) that $\mathcal{D}\hat{w} \leq -\Delta_\xi h$, where we used the fact $\Delta_\xi(w_{\bar{r}s}) = (\Delta_\xi w)_{\bar{r}s}$.

Then, applying the weak Harnack inequality [29, Theorem 9.22], there exists a positive constant C depending on n, $\|h\|_{C^2(\mathcal{O})}$ and $\|\Delta u\|_{C^0(\mathcal{O})}$ such that

$$\frac{1}{\mathrm{Vol}(B_R)} \int_{B_R} \hat{w} \leq C \left(\inf_{B_R} \hat{w} + R \right).$$

Equivalently, we have

$$\frac{1}{\mathrm{Vol}(B_R)} \int_{B_R} \left(\sup_{B_{2R}} \Delta_\xi w - \Delta_\xi w \right) \leq C \left(\sup_{B_{2R}} \Delta_\xi w - \sup_{B_R} \Delta_\xi w + R \right). \tag{32}$$

Since F is concave on Γ for any pair of $A\,,B \in \mathrm{Hyp}(n,\mathbb{H})$, we have

$$F(B) - F(A) \leq \mathrm{Re}\, F^{rs}(A)(B_{rs} - A_{rs})\,.$$

Choosing $A = W(y)$ and $B = W(x)$ for $x, y \in B_{2R}$, it follows that

$$\mathrm{Re}\, F^{rs}(W(y))(w_{\bar{r}s}(y) - w_{\bar{r}s}(x)) \leq F(W(y)) - F(W(x)) = h(y) - h(x) \leq C\|y - x\| \tag{33}$$

for some positive constant C depending on $\|h\|_{C^1(\mathcal{O})}$.

Now we need the following lemma from matrix theory, which is well-known in the settings of $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{H}^n$ (see e.g. [29, 12, 2]).

Lemma 17. [2, Lemma 4.9] *Let $\lambda, \Lambda \in \mathbb{R}$ satisfy $0 < \lambda < \Lambda < +\infty$. There exist a uniform constant N, unit vectors $\xi_1, \cdots, \xi_N \in \mathbb{H}^n$ and positive numbers $\lambda_* < \Lambda_* < +\infty$, depending only on n, λ, Λ such that any $A \in \mathrm{Hyp}(n,\mathbb{H})$ with eigenvalues lying in the interval $[\lambda, \Lambda]$ can be written in the form*

$$A = \sum_{k=1}^{N} \beta_k \xi_k^* \otimes \xi_k \,, \qquad i.e.\ A_{rs} = \sum_{k=1}^{N} \beta_k \bar{\xi}_{kr} \xi_{ks}\,,$$

for some $\beta_k \in [\lambda_*, \Lambda_*]$.

We apply the previous lemma with $A = (F^{rs}(W))$, obtaining immediately

$$\begin{aligned}\mathrm{Re}\, F^{rs}(W(y))(w_{\bar{r}s}(y) - w_{\bar{r}s}(x)) &= \sum_{k=1}^{N} \beta_k(y) \bar{\xi}_{kr} \xi_{ks} (w_{\bar{r}s}(y) - w_{\bar{r}s}(x)) \\ &= \sum_{k=1}^{N} \beta_k(y)(\Delta_{\xi_k} w(y) - \Delta_{\xi_k} w(x))\end{aligned}$$

for some functions $\beta_k(y) \in [\lambda_*, \Lambda_*]$. By (33), we then have

$$\sum_{k=1}^{N} \beta_k(y)(\Delta_{\xi_k} w(y) - \Delta_{\xi_k} w(x)) \leq C\|y - x\| \qquad \text{for } x, y \in B_{2R}\,. \tag{34}$$

Let us denote

$$M_{k,tR} = \sup_{B_{tR}} \Delta_{\xi_k} w\,, \qquad m_{k,tR} = \inf_{B_{tR}} \Delta_{\xi_k} w\,, \qquad \eta(tR) = \sum_{k=1}^{N} (M_{k,tR} - m_{k,tR})\,,$$

for $t = 1, 2$.

Summing up (32) over ξ_k for $k \neq l$ yields

$$\frac{1}{\mathrm{Vol}(B_R)} \int_{B_R} \sum_{k\neq l} \left(M_{k,2R} - \Delta_{\xi_k} w\right) \leq C(\eta(2R) - \eta(R) + R)\,. \tag{35}$$

Choosing a point $x \in B_{2R}$ at which the infimum $m_{l,2R}$ is attained, by (34) we also know that

$$\Delta_{\xi_l} w(y) - m_{l,2R} \leq \frac{1}{\lambda_*}\left(CR + \Lambda_* \sum_{k\neq l}(M_{k,2R} - \Delta_{\xi_k} w)\right) \tag{36}$$

Integrating (36) on B_R and using (35) yields

$$\frac{1}{\mathrm{Vol}(B_R)} \int_{B_R} (\Delta_{\xi_l} w - m_{l,2R}) \leq C(\eta(2R) - \eta(R) + R)\,.$$

Using (32) again, we then obtain

$$\begin{aligned} \frac{1}{\mathrm{Vol}(B_R)} \int_{B_R} (\Delta_{\xi_l} w - m_{l,2R}) &\geq \frac{1}{\mathrm{Vol}(B_R)} \int_{B_R} (\Delta_{\xi_l} w - M_{l,2R}) + M_{l,2R} - m_{l,2R} \\ &\geq M_{l,2R} - m_{l,2R} - C(M_{l,2R} - M_{l,R} + R) \\ &\geq C(M_{l,R} - m_{l,R}) - (C-1)(M_{l,2R} - m_{l,2R}) - CR\,, \end{aligned}$$

since $m_{k,tR}$ is non-increasing with respect to t. Inserting this last inequality into (35) we get

$$\eta(2R) - \eta(R) \geq C(M_{l,R} - m_{l,R}) - (C-1)(M_{l,2R} - m_{l,2R}) - CR\,,$$

and summing up over l,

$$\eta(R) \leq (1 - 1/C)\eta(2R) + CR.$$

Now applying [29, Lemma 8.23] the proof is complete. □

Proof of Theorem 1. Let (M, I, J, K, g) be a compact flat hyperkähler manifold, $\underline{\varphi}, \varphi \colon M \to \mathbb{R}$ be a $\mathcal{C}$-subsolution and a solution to (2) respectively, with $\sup_M \varphi = 0$. By Proposition 6 we deduce $\|\varphi\|_{C^0} \leq C$. Proposition 10 now implies $\|\Delta_g \varphi\|_{C^0} \leq C(\|\nabla\varphi\|^2_{C^0} + 1)$. The blow-up argument together with the Liouville-type Theorem 15 yield a gradient bound for φ. Therefore $\|\Delta_g \varphi\|_{C^0} \leq C$ and we can deduce from Proposition 16 the desired $C^{2,\alpha}$-estimate $\|\varphi\|_{C^{2,\alpha}} \leq C$, where the constant $C > 0$ only depends on the background data, including $\underline{\varphi}$. □

7. Proof of Theorems 2 and 3

In this section we prove Theorem 2 and Theorem 3 as applications of Theorem 1. For the quaternionic Hessian equation as the cone Γ we consider the k-positive cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_1(\lambda), \dots, \sigma_k(\lambda) > 0\}\,,$$

where $1 \leq k \leq n$ and σ_r is the r-th elementary symmetric function

$$\sigma_r(\lambda) = \sum_{1\leq i_1 < \cdots < i_r \leq n} \lambda_{i_1} \cdots \lambda_{i_r}\,, \qquad \text{for all } \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n\,.$$

Observe that on a locally flat hyperhermitian manifold (M, I, J, K, g) a q-real $(2,0)$-form Ω is k-positive in the sense that it satisfies (3) if and only if $\lambda(g^{\bar{j}r}\Omega_{\bar{j}s}) \in \Gamma_k$.

Moreover, for every $(\lambda_1, \dots, \lambda_n) \in \Gamma_k$ we clearly have

$$\lim_{t\to\infty} \sigma_k(\lambda_1, \dots, \lambda_{n-1}, t) = \infty$$

and by [60, Remark 8] any Γ-admissible function is a $\mathcal{C}$-subsolution. Hence for the quaternionic Hessian equation we easily have existence of a $\mathcal{C}$-subsolution.

Proof of Theorem 2. On Γ_k we define $f = \log \sigma_k$, in order to rewrite the quaternionic Hessian equation as

$$f\left(\lambda\left(g^{\bar{j}r}(\Omega_{\bar{j}s} + \varphi_{\bar{j}s})\right)\right) = h\,,$$

for some positive $h \in C^\infty(M, \mathbb{R})$ depending on H. The function f satisfies conditions C1–C3 stated in the introduction (see e.g. [55]).

We apply the method of continuity. Let $H_0 \in C^\infty(M, \mathbb{R})$ be the function such that

$$\frac{\Omega^k \wedge \Omega_0^{n-k}}{\Omega_0^n} = \mathrm{e}^{H_0}$$

and consider the t-dependent family of equations

$$(*_t) \qquad \frac{\Omega_{\varphi_t}^k \wedge \Omega_0^{n-k}}{\Omega_0^n} = b_t\, \mathrm{e}^{tH+(1-t)H_0}, \quad \varphi_t \in \mathrm{QSH}_k(M, \Omega), \quad t \in [0,1]\,.$$

Let

$$S = \left\{t \in [0,1] \mid (*_t) \text{ has a solution } (\varphi_t, b_t) \in C^{2,\beta}(M, \mathbb{R}) \times \mathbb{R}_+\right\}\,.$$

By our choice of H_0, the pair $(\varphi, b) = (0, 1)$ solves $(*_0)$, hence the set S is non-empty.

Since we assumed Ω to be k-positive $\underline{\varphi} \equiv 0$ is Γ_k-admissible and therefore a $\mathcal{C}$-subsolution. Closedness of S now follows from the $C^{2,\alpha}$-estimate of Theorem 1, a standard bootstrapping argument and the Ascoli-Arzelà Theorem.

Finally, in order to show that S is open, take $t' \in S$ and let $(\varphi_{t'}, b_{t'})$ be the corresponding solution to $(*_{t'})$. Consider the Banach spaces

$$B_1 := \left\{\psi \in C^{2,\beta}(M, \mathbb{R}) \mid \psi \in \mathrm{QSH}_k(M, \Omega)\,,\, \int_M \psi\, \Omega_0^n \wedge \bar{\Omega}_0^n = 0\right\}\,, \qquad B_2 := C^{0,\beta}(M, \mathbb{R})\,,$$

and the linearization of the operator

$$B_1 \times \mathbb{R}_+ \to B_2\,, \qquad (\psi, a) \mapsto \log \frac{\Omega_\psi^k \wedge \Omega_0^{n-k}}{\Omega_0^n} - \log(a)$$

at $(\varphi_{t'}, b_{t'})$, which is

$$L\colon T_{\varphi_{t'}} B_1 \times \mathbb{R} \to B_2\,, \qquad L(\rho, c) = k\frac{\partial\partial_J \rho \wedge \Omega_{\varphi_{t'}}^{k-1} \wedge \Omega_0^{n-k}}{b_{t'}\, \mathrm{e}^{t'H+(1-t')H_0} \Omega_0^n} - \frac{c}{b_{t'}} =: L'(\rho) - \frac{c}{b_{t'}}\,,$$

where

$$T_{\varphi_{t'}} B_1 = \left\{\rho \in C^{2,\beta}(M, \mathbb{R}) \mid \int_M \rho\, \Omega_0^n \wedge \bar{\Omega}_0^n = 0\right\}\,.$$

By the maximum principle the kernel of the operator L' over $C^{2,\beta}(M, \mathbb{R})$ is the set of constant functions. Moreover the principal symbol of L' is self-adjoint and therefore L' has index zero, which implies that its formal adjoint $(L')^*$ has one-dimensional kernel as well. In order to show that L is surjective, let $\zeta \in C^{0,\beta}(M, \mathbb{R})$ and choose $c \in \mathbb{R}$ such that $\zeta + c/b_{t'}$ is orthogonal to $\ker((L')^*)$. By the Fredholm alternative there exists $\rho \in B_1$ such that

$$L'(\rho) = \zeta + c/b_{t'}$$

and the surjectivity of L follows.

By the inverse function theorem between Banach spaces S is open. This proves the existence of a solution to the quaternionic Hessian equation.

Finally we show uniqueness. Suppose $(\varphi_1, b_1), (\varphi_2, b_2)$ are both solutions and assume $b_1 \geq b_2$; then

$$\left(\Omega^k_{\varphi_1} - \Omega^k_{\varphi_2}\right) \wedge \Omega_0^{n-k} \geq 0\,,$$

which can be rewritten as

$$\partial\partial_J(\varphi_1 - \varphi_2) \wedge \left(\sum_{i=0}^{k-1} \Omega^{k-i-1}_{\varphi_1} \wedge \Omega^i_{\varphi_2}\right) \wedge \Omega_0^{n-k} \geq 0\,.$$

Since

$$\varphi \mapsto \frac{\partial\partial_J\varphi \wedge \left(\sum_{i=0}^{k-1} \Omega^{k-i-1}_{\varphi_1} \wedge \Omega^i_{\varphi_2}\right) \wedge \Omega_0^{n-k}}{\Omega_0^n}$$

is a second order linear elliptic operator without free term, by the maximum principle we deduce $\varphi_1 = \varphi_2$ and thus also $b_1 = b_2$. □

Proof of Theorem 3. Similarly as discussed in [60], let T be the linear map given by

$$T(\lambda) = \big(T(\lambda)_1, \dots, T(\lambda)_n\big)\,, \qquad T(\lambda)_k = \frac{1}{n-1}\sum_{i\neq k} \lambda_i\,,$$

for every $\lambda \in \mathbb{R}^n$ and define

$$f = \log \sigma_n(T), \qquad \Gamma = T^{-1}(\Gamma_n)\,.$$

It is straightforward to verify that the above setting satisfies the assumptions C1–C3 in the introduction. Let

$$\Omega := \mathrm{Re}\left(g^{\bar{j}s}(\Omega_1)_{\bar{j}s}\right)\Omega_0 - (n-1)\Omega_1\,.$$

Thus, equation (5) can be written as

$$f(\lambda) = H + \log b\,, \qquad \lambda = \lambda\left(g^{\bar{j}r}(\Omega_{\bar{j}s} + \varphi_{\bar{j}s})\right) \in \Gamma\,.$$

Then, Theorem 3 can be proved by a similar argument of Theorem 2, we give some details here.

We consider the following family of equations for $t \in [0,1]$:

$$\begin{cases} \left(\Omega_1 + \frac{1}{n-1}\big[(\Delta_g\varphi_t)\Omega_0 - \partial\partial_J\varphi_t\big]\right)^n = \mathrm{e}^{tH+(1-t)H_0+c_t}\Omega_0^n\,, \\ \Omega_1 + \frac{1}{n-1}\big[(\Delta_g\varphi_t)\Omega_0 - \partial\partial_J\varphi_t\big] > 0\,, \quad \sup_M \varphi_t = 0\,, \end{cases} \tag{$*$}_t$$

where $H_0 = \log\frac{\Omega_1^n}{\Omega_0^n}$ and $c_t : [0,1] \to \mathbb{R}$ is a path from $c_0 = 0$ to $c_1 = \log b$. Let us define

$$S = \{t \in [0,1] \mid \text{ there exists a pair } (\varphi_t, c_t) \in C^\infty(M, \mathbb{R}) \times \mathbb{R} \text{ solving } (*)_t\ \}\,.$$

Note that $(\varphi_0, c_0) = (0,0)$ solves $(*)_0$ and hence $S \neq \emptyset$. To prove the existence of solutions to (6), it suffices to show that S is both closed and open.

Step 1. S is closed. We first show that $\{c_t\}$ is uniformly bounded. Suppose φ_t achieves its maximum at the point $p_t \in M$, then the maximum principle yields that $\partial\partial_J\varphi_t$ is non-positive at p_t. Combining this with $(*)_t$, we obtain the upper bound for c_t:

$$c_t \leq (-tH + H_0)\,(p_t) \leq C\,,$$

for some C depending only on H, Ω_1 and Ω. The lower bound of c_t can be obtained similarly.

Observe that the positivity of Ω_1 implies that $\underline{\varphi} \equiv 0$ is a $\mathcal{C}$-subsolution of $(*)_t$. Then C^∞ a priori estimates of φ_t follow from Theorem 1. Combining this with the Arzelà-Ascoli theorem, we conclude that S is closed.

Step 2. S is open. Suppose there exists a pair $(\varphi_{\hat{t}}, c_{\hat{t}})$ satisfies $(*)_{\hat{t}}$. We shall prove that when t is close to $\hat{t}$, there exists a pair $(\varphi_t, c_t) \in C^\infty(M, \mathbb{R}) \times \mathbb{R}$ solving $(*)_t$.

First of all, let Θ be a pointwise strictly positive $(2n, 0)$-form with respect to I which is I-holomorphic, namely $\bar\partial\Theta = 0$. Equivalently, $\partial\bar\Theta = \partial_J\bar\Theta = 0$.

For every function $\psi : M \to \mathbb{R}$ of class C^2, we define

$$L_{\hat\varphi}(\psi) := \frac{n}{n-1} \frac{\left((\Delta_g\psi)\Omega_0 - \partial\partial_J\psi\right) \wedge \left(\Omega_1 + \frac{1}{n-1}\left[(\Delta_g\hat\varphi)\Omega_0 - \partial\partial_J\hat\varphi\right]\right)^{n-1}}{\left(\Omega_1 + \frac{1}{n-1}\left[(\Delta_g\hat\varphi)\Omega_0 - \partial\partial_J\hat\varphi\right]\right)^{n}} \,.$$

Since the operator $L_{\hat\varphi}$ is second order elliptic its symbol is self-adjoint, and therefore the index is zero. Then the classical maximum principle yields that

$$\ker(L_{\hat\varphi}) = \{\mathrm{const}\} \,. \tag{37}$$

Denote by $L^*_{\hat\varphi}$ the L^2-adjoint operator of $L_{\hat\varphi}$ with respect to the volume form

$$\mathrm{dvol} = \left(\Omega_1 + \frac{1}{n-1}\left[(\Delta_g\hat\varphi)\Omega_0 - \partial\partial_J\hat\varphi\right]\right)^n \wedge \bar\Theta \,.$$

By the index theorem, we know there is a non-negative function ζ such that

$$\ker(L^*_{\hat\varphi}) = \mathrm{Span}\{\zeta\} \,. \tag{38}$$

It follows from the strong maximum principle that $\zeta > 0$. Up to a constant, we may and do assume

$$\int_M \zeta \,\mathrm{dvol} = 1.$$

Define a Banach space

$$B_1 := \left\{\varphi \in C^{2,\alpha} \mid \lambda\left(g^{\bar j r}(\Omega_{\bar j s} + \varphi_{\bar j s})\right) \in \Gamma, \int_M \varphi\zeta \,\mathrm{dvol} = 0\right\}.$$

It is easy to verify that the tangent space of B_1 at $\hat\varphi$ is given by

$$T_{\hat\varphi}B_1 = \left\{\psi \in C^{2,\alpha}(M, \mathbb{R}) \mid \int_M \psi\zeta \,\mathrm{dvol} = 0\right\} \,.$$

Let us consider the map

$$\tilde H(\varphi, c) = \log \frac{\left(\Omega_1 + \frac{1}{n-1}\left[(\Delta_g\varphi)\Omega_0 - \partial\partial_J\varphi\right]\right)^n}{\Omega_0^n} - c \,,$$

which maps $B_1 \times \mathbb{R}$ to $C^{0,\alpha}$. The linearized operator of $\tilde H$ at $(\hat\varphi, \hat t)$ is given by

$$L_{\hat\varphi} - c : T_{\hat\varphi}B_1 \times \mathbb{R} \to C^{0,\alpha}(M, \mathbb{R}) \,. \tag{39}$$

On the one hand, for any real-valued $h \in C^{0,\alpha}(M)$, there exists a unique real constant c such that

$$\int_M (h + c)\zeta \,\mathrm{dvol} = 0.$$

By (38) and Fredholm theorem, there exists a real function ψ on M such that $L_{\hat\varphi}(\psi) - c = h$. Hence, the map $L_{\hat\varphi} - c$ is surjective. On the other hand, let (ψ_1, c_1) be a solution of $L_{\hat\varphi}(\psi) - c = 0$. By (38) and Fredholm theorem again, we get $c_1 = 0$. Using (37) and (39), we also obtain $\psi_1 = 0$. Therefore, $L_{\hat\varphi} - c$ is injective.

As a consequence, we conclude that $L_{\hat\varphi} - c$ is bijective. By the implicit function theorem, we know that when $|t - \hat t|$ is small enough, there exists a pair (φ_t, c_t) satisfying

$$\tilde H(\varphi_t, c_t) = tH + (1 - t)H_0.$$

In the general case, when we assume M is a compact manifold which admits a flat hyperkähler metric g compatible with the underlying hypercomplex structure, we may take $\Theta=\Omega^n$ and apply the previous procedure to show existence of solutions to (6).

Uniqueness can be obtained with a very similar technique as in Theorem 2, therefore we omit the proof here. $\square$

Before we move on to the proof of Corollary 4 we need to lay down some preliminaries in linear algebra in order to mimic the proof of [62, Corollary 1.3]. Let (M,I,J,K,g,Ω_0) be a compact hyperhermitian manifold. Let $(z^1,\dots,z^{2n})$ be holomorphic coordinates with respect to I and denote $\Lambda^{p,0}_I(M)$ the space of $(p,0)$-forms with respect to I. Consider the pointwise inner product $\langle\cdot,\cdot\rangle_g$ defined by

$$\langle\alpha,\beta\rangle_g=\frac{1}{p!}g^{r_1\bar s_1}\cdots g^{r_p\bar s_p}\alpha_{r_1\cdots r_p}\overline{\beta_{s_1\cdots s_p}}\,,\qquad\text{for every }\alpha,\beta\in\Lambda^{p,0}_I(M)\,,$$

where any $(p,0)$-form α is locally written as $\alpha=\frac{1}{p!}\alpha_{r_1\cdots r_p}dz^{r_1}\wedge\cdots\wedge dz^{r_p}$ and $(g^{r\bar s})$ is the inverse of the Hermitian matrix $(g_{r\bar s})$ induced by the I-Hermitian metric g.

We will need the following Hodge star-type operator $*\colon\Lambda^{p,0}_I(M)\to\Lambda^{2n-p,0}_I(M)$, defined by the relation

$$\alpha\wedge*\beta=\frac{1}{n!}\langle\alpha,\beta\rangle_g\Omega_0^n\,,\qquad\text{for }\alpha,\beta\in\Lambda^{p,0}_I(M)\,.$$

We fix a point $x_0\in M$ and take holomorphic coordinates $(z^1,\dots,z^{2n})$ with respect to I such that $(g_{r\bar s})$ is the identity at x_0, then we may compute

$$*(dz^{2i-1}\wedge dz^{2i})=dz^1\wedge\cdots\wedge\widehat{dz^{2i-1}}\wedge\widehat{dz^{2i}}\wedge\cdots\wedge dz^{2n}\,. \tag{40}$$

Observe that the Hodge operator sends q-real $(2,0)$-forms to q-real $(2n-2,0)$-forms and vice versa. Recall that, when the hypercomplex structure is locally flat, to any q-real $(2,0)$-form Ω is associated a hyperhermitian matrix $(\Omega_{\bar rs})$, thus, we may define the determinant of Ω as the Moore determinant of $(\Omega_{\bar rs})$. This definition naturally extends to any q-real $(2n-2,0)$-form Φ by setting $\det(\Phi)=\frac{1}{(n-1)!}\det(*\Phi)$. In particular, for any q-real $\Omega\in\Lambda^{2,0}_I(M)$, we have

$$\det(\Omega^{n-1})=\det(\Omega)^{n-1}\,, \tag{41}$$

which can be checked by taking coordinates in which $(\Omega_{\bar rs})$ is diagonal at a given point and using (40). Indeed, the fact that we can choose coordinates that diagonalize both $(g_{r\bar s})$ and $(\Omega_{\bar rs})$ is ensured by [57, Lemma 3]. For any pair of q-real $\chi,\Omega\in\Lambda^{2,0}_I(M)$, we also have

$$\frac{\chi^n}{\Omega^n}=\frac{\det(\chi)}{\det(\Omega)}=\frac{\det(*\chi)}{\det(*\Omega)}\,. \tag{42}$$

A q-real $(2n-2,0)$-form Φ is said to be positive if $\Phi\wedge\Omega>0$ for all positive $(2,0)$-forms Ω. We observe that the Hodge star maps positive $(2,0)$-forms to positive $(2n-2,0)$-forms and conversely. On a locally flat hyperhermitian manifold the $(n-1)^{\text{th}}$ power $\Omega\mapsto\Omega^{n-1}$ is a bijective correspondence between the cone of positive $(2,0)$-forms and the cone of positive $(2n-2,0)$-forms. The proof of this fact is just a matter of linear algebra and it is entirely analogous to the argument in [49, pp. 279-280], therefore we omit it.

Proof of Corollary 4. For starters, we claim

$$\frac{1}{(n-1)!}*\left(\partial\partial_J\varphi\wedge\Omega_0^{n-2}\right)=\frac{1}{n-1}\left[(\Delta_g\varphi)\Omega_0-\partial\partial_J\varphi\right]\,, \tag{43}$$

for any arbitrary function $\varphi \in C^2(M,\mathbb{R})$. It is enough to prove that for every $W \in \Lambda_I^{2n-2,0}(M)$, we have

$$\partial\partial_J\varphi \wedge \frac{\Omega_0^{n-2}}{(n-2)!} \wedge (*W) = (\Delta_g\varphi)W \wedge \Omega_0 - W \wedge \partial\partial_J\varphi.$$

Let $Z = dz^1 \wedge \cdots \wedge dz^{2n}$ for simplicity and fix a point $x_0 \in M$ where Ω_0 takes the standard form

$$\Omega_0 = \sum_{i=1}^{n} dz^{2i-1} \wedge dz^{2i}\,.$$

Without loss of generality, we may assume $W = \widehat{dz^1} \wedge \widehat{dz^2} \wedge dz^3 \wedge \cdots \wedge dz^{2n}$. It is easy to see that

$$W \wedge \Omega_0 = Z\,, \qquad W \wedge \partial\partial_J\varphi = (\varphi_{1\bar{1}} + \varphi_{2\bar{2}})Z\,.$$

As $*W = dz^1 \wedge dz^2$, we obtain

$$\begin{aligned}
\partial\partial_J\varphi \wedge \frac{\Omega_0^{n-2}}{(n-2)!} \wedge (*W) =& \partial\partial_J\varphi \wedge \frac{\Omega_0^{n-2}}{(n-2)!} dz^1 \wedge dz^2 \\
=& \partial\partial_J\varphi \wedge \sum_{i>1} dz^1 \wedge dz^2 \wedge \cdots \widehat{dz^{2i-1}} \wedge \widehat{dz^{2i}} \wedge \cdots \wedge dz^{2n} \\
=& \sum_{i>1} (\varphi_{2i-1\overline{2i-1}} + \varphi_{2i\overline{2i}})Z = (\Delta_g\varphi)Z - (\varphi_{1\bar{1}} + \varphi_{2\bar{2}})Z \\
=& (\Delta_g\varphi)W \wedge \Omega_0 - W \wedge \partial\partial_J\varphi\,,
\end{aligned}$$

as claimed.

From (42) and (43), it follows that

$$\begin{aligned}
\frac{\left(\Omega_1 + \frac{1}{n-1}\left[(\Delta_g\varphi)\Omega - \partial\partial_J\varphi\right]\right)^n}{\Omega_0^n} &= \frac{\det\left(*\left(\Omega_1 + \frac{1}{n-1}\left[(\Delta_g\varphi)\Omega - \partial\partial_J\varphi\right]\right)\right)}{\det(*\Omega_0)} \\
&= \frac{\det\left(\Omega_2^{n-1} + \partial\partial_J\varphi \wedge \Omega_0^{n-2}\right)}{\det(\Omega_0^{n-1})}\,.
\end{aligned}$$

This implies that given a positive $(2,0)$-form Ω_1 and a smooth function H on M, the pair $(\varphi, b) \in C^\infty(M,\mathbb{R}) \times \mathbb{R}_+$ is a solution to (6) if and only if it solves

$$\begin{cases} \det(\Omega_2^{n-1} + \partial\partial_J\varphi \wedge \Omega_0^{n-2}) = b\,\mathrm{e}^H \det(\Omega_0^{n-1})\,, \\ \Omega_2^{n-1} + \partial\partial_J\varphi \wedge \Omega_0^{n-2} > 0\,, \quad \sup_M \varphi = 0\,, \end{cases} \tag{44}$$

where Ω_2 is uniquely defined by

$$\Omega_1 = \frac{1}{(n-1)!} * \Omega_2^{n-1}\,,$$

because the $(n-1)^{\text{th}}$ power is a bijection between the spaces of positive $(2,0)$-forms and positive $(2n-2,0)$-forms.

Now, let $(\varphi, b) \in C^\infty(M,\mathbb{R}) \times \mathbb{R}_+$ be the solution to (6), or equivalently (44), with datum $H = (n-1)H'$. Define $\tilde{\Omega}$ as the unique $(n-1)^{\text{th}}$ root of $\Omega_2^{n-1} + \partial\partial_J\varphi \wedge \Omega_0^{n-2}$. Then it is clear that if Ω_2 is the $(2,0)$-form induced by a quaternionic balanced (resp. quaternionic Gauduchon, quaternionic strongly Gauduchon) metric, then so is $\tilde{\Omega}$. Finally, set $b' = b^{1/(n-1)}$, then using (41) we conclude

$$\frac{\tilde{\Omega}^n}{\Omega_0^n} = \left(\frac{\det(\tilde{\Omega}^{n-1})}{\det(\Omega_0^{n-1})}\right)^{\frac{1}{n-1}} = \left(\frac{\det\left(\Omega_2^{n-1} + \partial\partial_J\varphi \wedge \Omega_0^{n-2}\right)}{\det(\Omega_0^{n-1})}\right)^{\frac{1}{n-1}} = \left(b\, \mathrm{e}^H\right)^{\frac{1}{n-1}} = b'\, \mathrm{e}^{H'} . \qquad \square$$

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