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THE PLANAR ANISOTROPIC *N*-CENTRE PROBLEM AT NEGATIVE ENERGIES

Abstract. We propose a survey of a forthcoming paper, concerning the study of the N-centre problem on the plane. In particular, we associate an anisotropic potential to every centre and our idea is to provide non-collision periodic trajectories in negative energy shells. This will be made through a double technique, which includes both variational and perturbation methods. The chaotic behaviour of the system is finally suggested by the presence of a symbolic dynamics.

A. Introduction

The classical N-centre problem of Celestial Mechanics consists in the study of the dynamics of a moving particle on the plane or in the space, whose trajectory is determined by the gravitational attraction of N fixed centres of mass. It is usually considered as a simplified version of a $(N+1)$-body problem, in which one of the bodies is moving much faster than the others and in which the Coriolis' and centrifugal forces are neglected (see [15] and references therein). Actually, this is not the only reason for which this problem is so interesting and challenging, but it can also be of great interest in Molecular Physics. Indeed, in this model one can replace the gravitational attraction by a Coulombic interaction, but also consider the centres as massive components in atoms (see [16, 21] and references therein).

In a 2 or 3-dimensional Euclidean space and considering radial Kepler-like potentials with the same homogeneity degree, the problem is usually introduced as follows. Let us indicate by $c_1,\dots,c_N \in \mathbb{R}^n$ $(n=2,3)$ the position of the centres, with $m_1,\dots,m_N>0$ their masses and with $x=x(t)\in\mathbb{R}^n$ the position of the test particle at time $t\in\mathbb{R}$. The motion equation is

$$\ddot{x}(t) = -\sum_{j=1}^{N} \frac{m_j(x(t)-c_j)}{|x(t)-c_j|^{\alpha+2}}, \quad \alpha\in[1,2), \tag{1}$$

which is singular whenever $x=c_j$ for some $j=1,\dots,N$ and whose solution verify the energy equation

$$\frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = h,$$

for $h\in\mathbb{R}$. The whole system can be also rephrased in Hamiltonian formalism, introducing the $-\alpha$-homogeneous potential

$$V(x) = \sum_{j=1}^{N} \frac{m_j}{\alpha|x-c_j|^{\alpha}}, \tag{2}$$

and considering the equivalent form of (1)

$$\ddot{x}(t) = \nabla V(x(t)).$$

In particular, the Hamiltonian function of the system will be the function

$$h(x,\dot{x}) = K(\dot{x}) - V(x) = \frac{1}{2}|\dot{x}|^2 - V(x),$$

which is nothing but the total energy of the system.

When $N = 1$, we end up with the classical Kepler problem, which is known to be super-integrable and whose solutions are conic sections. Moreover, Euler showed that the 2-centres problem is analytic integrable through an ellipsoidal change of coordinates and its explicit solutions have been computed by Jacobi in his celebrated work *Vorlesungen über dynamik*. On the other hand, it has been proved (see [5] for the planar case and [6, 17] for the spatial case) that the N-centre problem is not analytically integrable when $N \geq 3$.

As we have said, the system is conservative, so that one can choose to study either negative, positive or zero energy solutions, confined to the energy shell

$$\mathcal{H} = \left\{ (x,\dot{x}) \in (\mathbb{R}^n \setminus \{c_1,\ldots,c_N\}) \times \mathbb{R}^n : \ \frac{1}{2}|\dot{x}|^2 - V(x) = h \right\} \simeq \mathbb{R}^{n-1}.$$

The existence of a symbolic dynamics for positive energies has been proved in [16], where the authors gave a qualitative description of the planar scattering for some problems deriving from the N-centre model. After that, in [15] the author generalized some parts of the previous paper to the spatial case, providing the existence of unbounded trajectories for high energy levels. Concerning the zero-energy case, we refer to [7], where the authors showed the existence of entire parabolic trajectories for the spatial N-centre problem, with prescribed ingoing and outgoing directions. On the other hand, the existence of a symbolic dynamics when $h < 0$ has been established in [20]. In particular, the authors built collision-less periodic solutions of equation (1) glueing together perturbation arcs and variational paths. Introducing a small parameter, all the centres have been confined in a ball and, far from that, the problem turned out to be a perturbation of a particular Kepler problem. We can consider [20] as our starting point, since our goal is to produce a similar result under the presence of anisotropic interactions between the particle and the centres.

Anisotropic phenomena have been largely studied during the last 50 years, mostly for their remarkable interest in physical applications. Considering a moving particle and a single attraction centre, it is possible to introduce anisotropy in the system, defining an appropriate non-radial force field. In this way, the anisotropic Kepler problem was firstly introduced by Gutzwiller in his two celebrated papers [12, 13], in which the author actually exploited this model to better understand some deep relations between classical and quantum mechanics. In particular, he was able to prove the existence of parabolic orbits for a particle driven by a Coulombic potential, reducing the equations of motions to an autonomous system and providing some numerical results. Through the years, this problem was furthermore investigated by several authors, with Devaney among them. Using a technique introduced by McGehee in the study of the collinear 3-body problem ([18, 19]), Devaney gave an exhaustive picture of the main properties of this model (see [8, 9, 10, 11]). To conclude this introduction, zero-energies trajectories for the anisotropic Kepler problem with prescribed ingoing and outgoing directions have been largely studied both in the plane ([3, 14]), but also in higher dimensions ([4]).

B. The problem

Inspired by the previous motivations, we thought to consider an anisotropic version of the N-centre problem and to investigate the existence of periodic orbits for small negative energies. In this way, potential (2) can be properly modified, in order to introduce anisotropy in the system. In particular, our goal is to produce a symbolic dynamics for the N-centre problem, driven by the following potential

$$V(x) = \sum_{j=1}^{N} |x-c_j|^{-\alpha_j} V_j\left(\frac{x-c_j}{|x-c_j|}\right),$$

where $V_j \in \mathcal{C}^2(\mathbb{R}\setminus\{c_j\})$ is a $-\alpha_j$-homogeneous function, with $\alpha_j \in (1,2)$, for every $j=1,\dots,N$. Note that, introducing polar coordinates $x=(r\cos\vartheta, r\sin\vartheta)$ with $r>0$ and $\vartheta\in[0,2\pi)$, every potential V_j can be rewritten as

$$V_j(x) = V_j(r\cos\vartheta, r\sin\vartheta) = r^{-\alpha_j}U_j(\vartheta),$$

where $U_j(\vartheta) \doteq V_j(\cos\vartheta,\sin\vartheta)$ represents the *angular component* of V_j, for every $j=1,\dots,N$. In this setting, we can present here our complete hypotheses on V, which read:

Given $U_1,\dots,U_N\colon \mathbb{S}^1\to\mathbb{R}$ *of class* $\mathcal{C}^2$ *such that*

$$(U)\qquad \forall j=1,\dots,N\ \exists\vartheta_j\in\mathbb{S}^1 \text{ s.t. } U_j(\vartheta)\geq U_j(\vartheta_j)>0 \text{ and } U''(\vartheta_j)>0,$$

define

$$(V)\qquad V(x) = \sum_{j=1}^{N} |x-c_j|^{-\alpha_j} U_j\left(\frac{x-c_j}{|x-c_j|}\right),$$

where $\alpha_j>\bar{\alpha}_j$ *and* $\bar{\alpha}_j=\bar{\alpha}_j(U_j)$ *for every* $j=1,\dots,N$ *(following the notations of [3, 4]).*

Notice that the requirement (U) on every angular potential U_j states that it has to admit at least a non-degenerate minimizer. Under these assumptions, we will study the planar equation

$$\ddot{x}(t) = \nabla V(x(t)),$$

requiring without loss of generality that

$$\alpha_1\leq\alpha_2\leq\ldots\leq\alpha_N.$$

Of course, an energy conservation law still holds, so that we will actually study the system

$$
\begin{cases} \ddot{x}(t) = \nabla V(x(t)) \\ \frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = -h, \end{cases} \tag{3}
$$

with $h > 0$ small. All the solutions of our problem will be confined to the *Hill's region*

$$
\left\{ x \in \mathbb{R}^2 \setminus \{c_1, \dots, c_N\} : \, V(x) \geq h \right\}.
$$

REMARK 1. In the first part of the proof, we aim to reduce our model to a perturbation of an anisotropic Kepler problem with a certain homogeneity degree. In order to do this, the Hill's region has to be big enough so that the particle can be properly far from the singularity set. This fact basically motivates the choice of studying a problem in small negative energy shells.

Our main result is the following

THEOREM 1. *Given $h > 0$ sufficiently small, consider a potential V satisfying hypotheses (U)-(V). Then, there exist infinitely many periodic orbits for problem (3) which are collision-less and whose associated dynamical system admits a symbolic dynamics.*

The proof of this statement is divided in several steps and the main ideas therein employed are collected below:

- **Outer arcs.** We perform a perturbation argument, including all the centres in a small ball and thus reducing the system to a perturbed anisotropic Kepler problem driven by one of the potentials associated to the centres, up to a small parameter. Indeed, around a non-degenerate minimizer of this potential, it is possible to shadow homothetic solutions in order to prove the existence of an outer arc for the perturbed problem. This will be made using a shooting method, which stays stable when the perturbation parameter goes to zero.
- **Inner arcs.** We investigate the existence of fixed-end trajectories inside a ball of chosen radius, taking care of a possible interaction with the centres. This will be made through a minimization of both the Maupertuis' and Lagrange-action's functionals under a suitable topological constraint. Combining together the variational approaches introduced in [3], [4] and [20], we will be finally able to build collision-less arcs which connect two points chosen on the boundary of the ball, up to a threshold on the degree of homogeneity $-\alpha_j$ associated to every centre c_j (see hypotheses (U)-(V)).
- **Glueing pieces.** The idea is to alternate an outer and an inner arc and then to *glue* them together, in order to obtain a closed periodic trajectory. Even if every single trajectory obtained in the two previous steps is smooth, we need to show that such smoothness is preserved in the contact point. This will be made using a broken-geodesics argument, through a finite dimensional reduction.

- **Symbolic dynamics.** We want to show that our model admits a symbolic dynamics, i.e., that our dynamical system is topologically semi-conjugate with a metric space of bi-infinite sequences of symbols under the action of a Bernoulli shift map. In our setting, every symbol is represented by a partition of the centres in two non-trivial subsets and a non-degenerate minimizer of one of the potentials associated to the centres. In particular, given N centres and m non-degenerate minimizers, the number of admissible symbols is $\mathfrak{n} \doteq m(2^{N-1}-1)$. Hence, in order to have a non-trivial symbolic dynamics, we need to require $N \geq 2$ and $m \geq 1$, with one of the two inequalities holding strictly, so that $\mathfrak{n} > 1$.

To conclude this discussion, we want to inform the reader that this paper is mainly concerned with the idea of construction of inner and outer arcs for the problem, since the other two steps of the proof are still in progress and will not be stressed here.

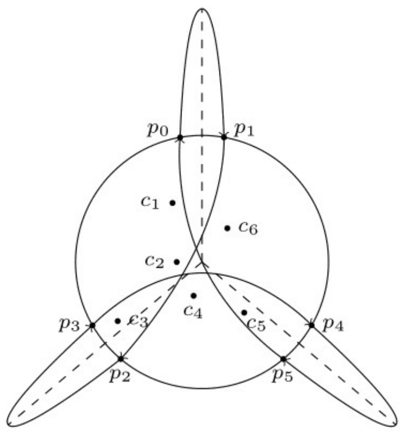

Figure 1: An example of a collision-less periodic trajectory.

C. Outer dynamics

Given $\varepsilon > 0$ and $y \in \mathbb{R}^2 \setminus \{c_1, \ldots, c_N\}$, let us introduce the rescaled potential

$$V_\varepsilon(y) = V_1(y - \varepsilon c_1) + \sum_{j=2}^{N} \varepsilon^{\alpha_j - \alpha_1} V_j(y - \varepsilon c_j).$$

REMARK 2. If we assume without loss of generality that $\max_j |c_j| = 1$, we have that the new centres $c'_j = \varepsilon c_j$ are included in the ball B_ε.

The next proposition shows that we are allowed to study a rescaled version of our initial problem (3), driven by potential V_ε and where the energy $-h$ is normalized to -1.

PROPOSITION 1. *Let $x \in \mathcal{C}^2((a,b);\mathbb{R}^2)$ be a classical solution of*

$$\begin{cases} \ddot{x}(t) = \nabla V(x(t)) \\ \frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = -h, \quad h > 0. \end{cases} \tag{4}$$

Then, in the interval $(h^{\frac{\alpha+2}{2\alpha}}a, h^{\frac{\alpha+2}{2\alpha}}b)$, the function

$$y(t) \doteq h^{1/\alpha} x(h^{-\frac{\alpha+2}{2\alpha}}t)$$

solves the problem

$$\begin{cases} \ddot{y}(t) = \nabla V_\varepsilon(y(t)) \\ \frac{1}{2}|\dot{y}(t)|^2 - V_\varepsilon(y(t)) = -1, \end{cases} \tag{5}$$

where $\varepsilon = h^{1/\alpha}$.

Conversely, if $y \in \mathcal{C}^2((c,d);\mathbb{R}^2)$ is a solution of (5) then, taking $h = \varepsilon^\alpha$, the function

$$x(t) \doteq h^{-1/\alpha} y(h^{\frac{2+\alpha}{2\alpha}}t)$$

is a solution of (4) in the interval $(h^{-\frac{\alpha+2}{2\alpha}}c, h^{-\frac{\alpha+2}{2\alpha}}d)$.

Proof. It is enough to plug $y(t)$ into the equation and to see that even the energy conservation law holds as well. This is a consequence of the fact that if V is $-\alpha$-homogeneous, then the gradient ∇V is $(-\alpha-1)$-homogeneous. $\square$

REMARK 3. We have named this section *Outer dynamics* because our first step is to build a solution arc $y_{ext}(t)$ which solves (5) in a certain interval $[0, T_{ext}]$, satisfying the additional conditions

$$\begin{cases} y(0) = p_0, \ y(T_{ext}) = p_1 \\ |y(t)| > R, \quad \text{for } t \in (0, T_{ext}), \end{cases}$$

where $p_0, p_1 \in \partial B_R(0)$, for some $R > 0$. The choice of R is not arbitrary, since we need at least to make some room between the sphere $\partial B_R(0)$ and the boundary of the Hill's region $\{V_\varepsilon \geq 1\}$ of (5), in which the external solution arc y_{ext} should live. However, if we choose ε small enough, $\{V_\varepsilon \geq 1\}$ contains a ball of radius $R \gg \varepsilon$, so that our perturbation method can finally take place.

The next proposition states that, outside the ball of radius R and centred in the origin, problem (5) is a perturbation of an anisotropic Kepler problem driven by the first potential V_1.

PROPOSITION 2. *Given $\varepsilon > 0$ small enough and $R > \varepsilon$ properly choosen, for every $y \in \mathbb{R}^2$ such that $|y| > R$ we have*

$$V_\varepsilon(y) = |y|^{-\alpha_1} V_1\left(\frac{y}{|y|}\right) + O(\varepsilon^{\alpha_2-\alpha_1}) \quad \textit{as } \varepsilon \to 0^+.$$

Moreover, the potential V_ε is smooth with respect to ε.

Proof. As a sketchy proof, the main idea is to exploit the fact that as $\varepsilon \to 0^+$, if we fix $j \in \{1,\dots,N\}$ and $|y| > R$, for every $\sigma \in \mathbb{R}$ we have

$$|y - \varepsilon c_j|^{-\sigma} = |y|^{-\sigma} + \varepsilon\sigma\frac{\langle y, c_j\rangle}{|y|^{\sigma+2}} + o(\varepsilon) = |y|^{-\sigma} + O(\varepsilon).$$

□

C.1. Shadowing homothetic trajectories

Since the previous perturbation holds, it makes sense to analyse some particular trajectories of the anisotropic Kepler problem, which are called *homothetic trajectories*. In particular, Proposition 2 tells us that when we are far enough from the centres, the leading potential is V_1, i.e., the one with the smallest homogeneity degree α_1. For this reason, we will firstly consider the problem

$$\begin{cases} \ddot{x} = \nabla V_1(x), \quad x \in \mathbb{R}^2 \setminus \{0\} \\ \frac{1}{2}|\dot{x}|^2 - V_1(x) = -1, \end{cases} \tag{6}$$

which, roughly speaking, models the dynamics of the particle in (5) when we are really far from the centres and $\varepsilon \to 0^+$.

Now, given $R > 0$, we look for homothetic solutions of (6), i.e., for

$$x(t) \doteq \lambda(t)\xi, \tag{7}$$

with $\lambda\colon [0,T] \to \mathbb{R}^+$, $T > 0$ and $\xi \in \partial B_R$ such that

$$\lambda(t) > 1 \text{ for every } t \in (0,T), \quad \lambda(0) = 1 = \lambda(T).$$

Plugging (7) into the motion equation $\ddot{x} = \nabla V_1(x)$, and defining the moment of inertia $I(x) = 1/2|x|^2$, we find out that λ solves the equation

$$\ddot{\lambda}(t) = -\frac{\mu}{\lambda^{\alpha_1+1}(t)},$$

while ξ solves

$$\nabla V_1(\xi) + \mu\nabla I(\xi) = 0,$$

for some $\mu > 0$. In other words, we have that the scaling term $\lambda(t)$ of an homothetic orbit for (6) is a solution of a 1-dimensional $-\alpha_1$-Kepler problem, while its motion direction ξ is a *central configuration* for the potential V_1, i.e., a critical point of V_1 constrained to a level surface of I. Note that, comparing this with assumption (U) and using polar coordinates, $\xi = (r\cos\vartheta_\xi, R\sin\vartheta_\xi)$ is a *non-degenerate minimal central configuration* for V_1 if its angular component U_1 satisfies

$$U_1'(\vartheta_\xi) = 0, \quad U_1''(\vartheta_\xi) > 0.$$

Now, a trajectory of equation (7) starts its rectilinear motion in the sphere $\partial B_R(0)$ with a certain velocity v_ξ and, after a time $T > 0$, touches again the sphere with opposite velocity $-v_\xi$. In other words, given $R > 0$ and a central configuration $\xi \in \partial B_R$ for V_1, we can consider the following Cauchy problem

$$\begin{cases} \ddot{x}(t) = \nabla V_1(x(t)) \\ x(0) = \xi, \quad \dot{x}(0) = v_\xi = \frac{1}{R}\sqrt{2(V_1(\xi)-1)}\xi, \end{cases} \tag{8}$$

where the starting velocity is determined by the conservation of energy and that admits as unique solution the homothetic trajectory x described above. Actually, exploiting the transversality of the vector field associated to (8) to the inertial surface $\{I(x) = R^2/2\}$ in the phase space, we can prove the existence of a first-return map, which is defined in a sufficiently small neighbourhood $\mathcal{U} \times \mathcal{V}$ of (ξ, v_ξ). This map gives us information about how a Cauchy problem behaves when we slightly modify the initial conditions in (8). In particular, since we are far from the singularity, the continuous dependence of initial data together with the transversality of the vector field imply that, for every $(x_0, v_0) \in \mathcal{U} \times \mathcal{V}$, there exists a unique solution for

$$\begin{cases} \ddot{x}(t) = \nabla V_1(x(t)) \\ x(0) = x_0, \quad \dot{x}(0) = v_0. \end{cases}$$

Of course, there exists a point $x_1 \in \partial B_R$ depending on the initial velocity v_0 such that, after a time $\bar{T} > 0$, the solution of the previous problem will verify $x(T) = x_1$. We want to point out that, as a preliminary step, our purpose is to prove the existence of an external fixed-end arc, whose starting and arriving points belong to a neighbourhood of ξ on the sphere ∂B_R. For this reason, we need to *switch* from a Cauchy problem to a boundary value problem, so that the piece of external solution will be determined once x_0, x_1 are fixed in the neighbourhood. This is the core of the next result.

THEOREM 2. *Let $\xi \in \partial B_R$ be a minimal non-degenerate central configuration for V_1. Then, there exists a neighbourhood $\mathcal{U}$ of ξ such that, for any $x_0, x_1 \in \mathcal{U}$ with $|x_0| = |x_1| = R$, there exist $\bar{T} > 0$ and a solution $x = x(t)$ of the boundary value problem*

$$\begin{cases} \ddot{x}(t) = \nabla U(x(t)) \\ \frac{1}{2}|\dot{x}(t)|^2 - U(x(t)) = -1 \\ |x(t)| > R \\ x(0) = x_0, \quad x(\bar{T}) = x_1. \end{cases}$$

With some technical modifications and by means of Proposition 2, the previous result can be proved also for potential V_ε, since the shooting technique used in the proof remains stable when $\varepsilon \to 0^+$. In this way, if ε is sufficiently small, we have shown the existence of external solution arcs y_{ext} satisfying the problem

$$\begin{cases} \ddot{y}_{ext}(t) = \nabla V_\varepsilon(y_{ext}(t)) \\ \frac{1}{2}|\dot{y}_{ext}(t)|^2 - V_\varepsilon(y_{ext}(t)) = -1 \\ |y_{ext}(t)| > R \\ y_{ext}(0) = p_0, \quad y_{ext}(T_{ext}) = p_1, \end{cases}$$

for some $T_{ext} > 0$, once p_0, p_1 belong to a neighbourhood on ∂B_R of any minimal non-degenerate central configuration of the leading potential V_1.

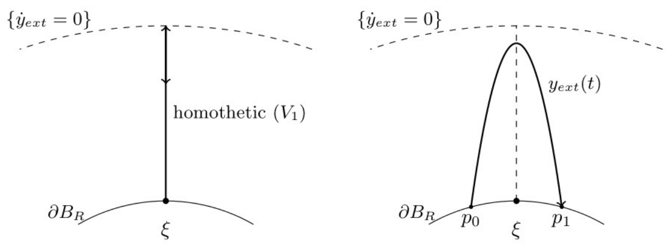

Figure 2: Shadowing Keplerian homothetic trajectories.

D. Inner dynamics

The next step of the proof consists in showing that, for $\varepsilon > 0$ sufficiently small and for any $p_1, p_2 \in \partial B_R$, there exists a solution $y_{int}(t)$ of the following problem

$$\begin{cases} \ddot{y}_{int}(t) = \nabla V_\varepsilon(y_{int}(t)) & t \in [0,T] \\ \dfrac{1}{2}|\dot{y}_{int}(t)|^2 - V_\varepsilon(y_{int}(t)) = -1 & t \in [0,T] \\ |y_{int}(t)| < R & t \in (0,T) \\ y_{int}(0) = p_1, \quad y_{int}(T_{int}) = p_2, \end{cases} \tag{9}$$

for some $T_{int} > 0$.

As previously said, our idea is to adopt a minimization technique in order to infer the existence of a collision-less solution of the fixed-end problem (9). Therefore, let us consider the set of H^1 paths with fixed ends

$$H_{p_1,p_2}([a,b]) \doteq \{u \in H^1([a,b];\mathbb{R}^2) : u(a) = p_1 \text{ and } u(b) = p_2\}$$

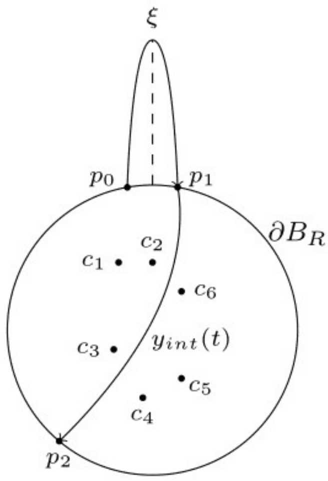

Figure 3: An example of solution of problem (9).

and the Maupertuis' functional $\mathcal{M} : H_{p_1,p_2}([a,b]) \to \mathbb{R}$

$$\mathcal{M}(u) = \frac{1}{2}\int_a^b |\dot{u}|^2\,dt \int_a^b (V_\varepsilon(u)+1)\,dt.$$

The well known *Maupertuis' Principle* assures that a critical point u of $\mathcal{M}$ can be properly reparameterized in order to obtain a solution of (9) (see for instance [2, 1]). Considering a smaller space than $H_{p_1,p_2}([a,b])$, in which we require that every path separates the centres exhibiting a fixed non-trivial partition of them, we can fulfil the hypotheses of the Maupertuis' Principle and thus provide the existence of a solution for (9). Despite that, it could happen that this path u collapses in one of the centres at some instant $\tau \in [0,T]$. In order to avoid that, our idea is to consider the Lagrange-action functional for potential V_ε

$$\mathcal{A}([t_1,t_2];x) = \int_{t_1}^{t_2}\left[\frac{1}{2}|\dot{x}|^2 + V_\varepsilon(x)\right]dt$$

and it is not difficult to prove that an ad-hoc reparameterization of a minimizer u of $\mathcal{M}$ produces a minimizer x of $\mathcal{A}$. At this point, we intend to apply to x a result contained in [3], which, properly rephrased, guarantees that once hypotheses (U)-(V) are satisfied, every fixed-time Bolza minimizer for $\mathcal{A}$ can not collide with the centres.

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