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Exploring an Italian Approach to Lesson Study from a Cultural Perspective for an Improvement in Mathematics Teacher Education

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0 FOREWORD

Lesson Study is a model of professional development that originated in Japan, based on collaboration, action research, and critical reflection on teachers' practices: it is different from the norm of Italian professional development models, but it does not appear too "far removed" from such norms that teachers and institutions could end up "rejecting" it. The Italian school system is characterised by collaboration, and the Ministry of Education has recently called for the creation of collaborative networks of teachers to support their professional development. Lesson Study can be a useful tool for improving teacher professionalism, and consequently the teaching and learning of mathematics in Italian secondary schools, while also answering the institutional request for collaborative networks of teachers.

My doctoral work aims to study the introduction of Lesson Study in the Italian context, identifying and examining the challenges and difficulties that Italian teachers, teacher educators, researchers and institutions may face in adopting and implementing it, in order to identify the cultural and social factors that could hinder its dissemination and propose strategies to overcome at least some of these challenges.

Such challenges may come from Lesson Study being rooted in the Japanese context. Lesson Study originated in Japan around 1880, crossed the sea towards China and Korea in the early 1900 and, although nowadays widely practiced all over the world, it came to the attention of the international research community in mathematics education only around the year 1999. This means that Lesson Study developed for over a century without being scientifically researched. For Japanese teachers Lesson Study is a habit, a working practice shared at an institutional level as well as a cultural and social one, in which teacher education is promoted through sharing and discussion.

Since different social norms might exist in different cultural contexts, understanding Lesson Study, and implementing it in contexts different from Japan, may pose some cultural challenges that need to be addressed. In the case of this dissertation, the Italian secondary-school system, albeit characterised by collaboration, has its own traditions, norms and expectations that may not completely align with the principles and practices of the Japanese school system - and with Lesson Study.

In particular, I focus on the didacticians (researchers engaged in teacher education, as in Jaworski & Potari, 2021) in transposing Lesson Study from Japan to Italy. Didacticians have a primary role in transposing Lesson Study while considering the pre-existent social norms of the target cultural context, yet their role is seldom studied. The transposition of Lesson Study provides a unique opportunity to study the work of didacticians: differently from many professional development models proposed by researchers in mathematics education, Lesson Study is not "invented" by the didacticians. Their role becomes particularly complex as, when researching Lesson Study, didacticians may face multiple tasks: studying Lesson Study as researchers, grasping its essential components that contribute to teacher professional development in Japan; studying the Italian institutional context, identifying the laws, norms and expectations that could foster or hinder the implementation of Lesson Study; adapting Lesson Study to the Italian institutional context, so that it does not contrast with Italian laws, norms and expectations; synthetizing such work of research and adaptation when implementing Lesson Study with Italian teachers. They may adapt Lesson Study to alleviate possible emerging cultural tensions; or may exploit such tensions so that the teachers are invited to reflect on their teaching practices, so becoming more aware of the reasons that supports their teaching action.

My research work is therefore framed within a cultural perspective: Lesson Study is historically situated in the Japanese culture, and its transposition to Italy needs a careful approach. To deal with the cultural aspect of my research, I assume the notion of “practice” as a link between culture and the larger cultural context (Hatano & Inagaki, 1998). To investigate people’s practices, I assume that they are influenced or shaped by diverse institutional conditions and constraints, prevailing in the institution to which people belong (Chevallard & Bosch, 2020).

Successful implementation of Lesson Study in a foreign country ultimately depends on how well it is adapted to the institutional conditions and constraints of a specific cultural context. For that reason I concentrate my study on the institutional conditions and constraints surrounding the implementation of Lesson Study in Italy. To do that, I worked on an institutional *macro*- and *micro*- scope, each with specific research gaps, theoretical frameworks, methodologies, and results.

At the institutional *macro*-scope, I investigated the institutions in which Lesson Study occurs (in Japan) or could develop (in Italy) and compared them to identify conditions and constraints that allow Lesson Study to happen (in Japan) or could either allow or prevent it (in Italy). I noted that while extensive literature was dedicated to comparisons of the Japanese culture and the Western culture at large (with many specific comparisons to the Anglo-American culture), or to the Japanese school system with several Western school systems (again, especially Anglo-American ones), little literature was dedicated to comparisons with the Italian culture and the Italian school system. For the Japanese school system, I relied on recent international literature (e.g., Miyakawa & Winsløw, 2019) and conversations with my co-supervisor Takeshi Miyakawa. For the Italian school system, I relied on institutional documents, national literature (e.g., Capperucci, 2008; Ciarrapico & Berni, 2017), my personal knowledge, conversations with my supervisor Ornella Robutti and with teachers. The so-obtained data were interpreted in the framework of the Anthropological Theory of the Didactic (Chevallard, 1985), specifically the *scale of levels of didactic co-determinacy*[1] (Chevallard, 2002): with this framework, I could picture the complex relations of the factors influencing teachers' practices, which are influenced not only by the teachers' decision, but at a higher level by the society in which the teachers and students are immersed (Fig. 0.1). Similarly, I suggest that professional development practices are influenced by the context in which teachers and their educators are immersed.

Upper levels	*Lower levels*
Humanity	↓↑
↓↑	Discipline
Civilisation	↓↑
↓↑	Domain
Society	↓↑
↓↑	Sector
School	↓↑
↓↑	Theme
Pedagogy	↓↑
↓↑	Question

Figure 0.1 Scale of levels of didactic co-determinacy (Florensa et al., 2018, p. 5)

At the institutional *micro*-scope, I implemented adaptations of Lesson Study with pre-service and in-service secondary school mathematics teachers. The aim of the work at this-scope is twofold: gaining theoretical insights on Lesson Study while providing scientific knowledge on the introduction of Lesson Study in the Italian cultural and institutional context; and fostering collaboration among teachers in the Italian context of teacher professional development. I focused specifically on the role of the didacticians in the transposition of Lesson Study from the Japanese culture to

[1] In the course of the dissertation they are also referred as *levels of-codetermination*.

another. European literature particularly recognises the relevance of this role (e.g., Ponte, 2017), yet little research exists in the specific dynamics that it entails.

The first experiment was carried out with prospective secondary-school teachers. They had no experience in real-classroom teaching, although some had already engaged in mathematics education courses. I chose to work with pre-service teachers to avoid the influence of the "institutional superstructure" of the Italian school system, hoping for less biased data from which I could build design principles for the introduction of Lesson Study to in-service teachers.

A second and third experiments were carried out respectively with lower-secondary school and higher-secondary school mathematics teachers that volunteered to organise Lesson Study in their school.

Since the transposition process of Lesson Study in my research concerns a diversity of practices, the data were analysed using Chevallard's construct of praxeology (Chevallard, 1999). "The anthropological principle states that any human activity can be described in terms of praxeologies" (Bosch et al., 2020, p.xiv): according to the Anthropological Theory of the Didactic, every activity conducted within an institution[2] constitutes of basic "parts" called *tasks*, and when these tasks are repeated regularly, an "way of doing" with regard to the task develops within the institution (Chevallard, 1999). This is particularly true in the school institution: the notion of praxeology was in fact introduced to characterise mathematical practices and knowledge in the classroom or curriculum (*mathematical praxeology*), as well as practices and knowledge of teachers[3] to make the mathematical praxeology emerge in

[2] Or by an individual.

[3] Both in relation to their role in the school, both as individuals.

the classroom (*didactical praxeology*). It is now expanded to include a variety of settings (Chevallard et al., 2022): the teachers' practices outside the classroom such as preparing the lessons, conducting school-based projects, and so forth (*paradidactical praxeologies*); the practices of teacher educators (*teacher education praxeologies*); the practices of researchers (*research praxeologies*); and so on. A *praxeology* is more than a simple practice, or simple theoretical knowledge: it consists of two blocks, the know-how (*praxis*) and the know-why (*logos* - the discourses that justify the know-how). Praxis is made of two elements: a *type of tasks* and a *technique* to solve this type of tasks. Logos is also made by two elements: a *technology* (the discourse that justifies the technique) and a *theory* that supports the technology. Specific to my study is the use of the Meta-Didactical Transposition framework (Arzarello et al., 2014), which stemmed from Chevallard's Anthropological Theory of the Didactic to account for the dynamics that are typical of teacher-education activities which take place with the collaboration of researchers and teachers (Robutti, 2020). Using Meta-Didactical Transposition, the activities and knowledge of teachers and researchers are modelled with the notion of praxeology: they are called *metadidactical praxeologies* because they refer to the knowledge about the didactic system. The metadidactical praxeologies may be referred to in terms of *teachers' praxeologies* and *researchers' praxeologies* (Fig. 0.2). Both praxeologies may evolve overtime, thanks to the interactions between teachers and researchers. Elements of the praxeologies at stake could be transposed from researchers' institution to teachers' institution (and also vice-versa). The result may be a *shared praxeology* (Fig. 0.2) which, in turn, may bring to new praxeologies for each institution. This framework allows investigating the cultural aspect of human practices.

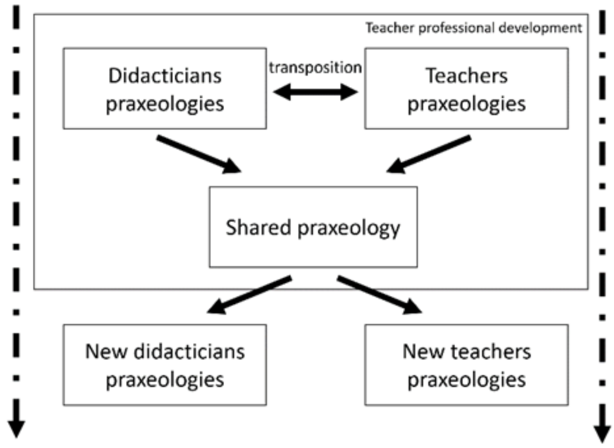

Figure 0.2 The emergence of a shared praxeology in Meta-Didactical Transposition

This doctoral work proposes results on disseminative, educational, methodological and, most important, theoretical levels.

At a disseminative level, the international research community in Mathematics Education has access to the first updated description of the Italian institutional teacher professional development context published in English, after the seminal work of professors Arzarello and Bartolini Bussi in 1998. In a community increasingly aware of the importance of cultural contexts in Mathematics Education research, this will prove crucial in the understanding of Italian Mathematics Education research.

At an educational level, terminological issues in the presentation of Lesson Study to the teachers were highlighted and identified as possible constraints to the implementation of Lesson Study in Italy. This is the first time that this specific issue has been related to Lesson Study, proving a useful source of insight on Lesson Study for the international community. This doctoral work also provides examples of strategies by which such issues can be overcome.

At a methodological level, I propose arguments to the importance of understanding the cultural contexts involved in the research in Mathematics Education, and a method for describing cultural contexts that might be shared by the community of researchers in Mathematics Education.

Most important, at a theoretical level, this doctoral work shows that the synergy of the theoretical frameworks of the Anthropological Theory of the Didactic and of the Meta-Didactical Transposition can guide researchers in the design and analysis of teacher professional development programmes that entails collaboration between researchers and teachers (such as Lesson Study, or others), taking care of interpreting the complexity that this collaboration entails. Specifically, by connecting the analysis of the institutional context with the evolution of the different elements of teachers’ and didacticians’ praxeologies (as suggested by the recent evolution of Meta-Didactical Transposition, MDT.2, Cusi et al., 2023), my doctoral work lays the foundations for a new approach to the study of cultural issues in mathematics education, especially when the problem to be addressed is the transposition - from one cultural context to another - of a culturally rooted practice (be it teaching practice or teacher education practice).

In Chapter 1, I explore the path followed during these years of doctoral research. The principles and characteristics of Lesson Study are examined, highlighting the potential of Lesson Study for improving mathematics teaching and its impact on student learning. The Italian educational context is analysed, with particular attention to the structure of the school system, and policies for education and teacher professional development. This allows us to understand the specificities of the Italian context and to identify any structural and cultural obstacles to the introduction of Lesson Study. I explore different possible approaches to the investigation of Lesson Study, and specify my approach and the research questions. I present the theoretical frameworks that informed the research. Finally, I provide a synthesis of the research work presented in the papers that will follow.

Chapter 2 to 7 are a selection of peer-reviewed articles that constitute the fundamental steps of the doctoral work. Chapters 2 to 4 set the foundation of the research work, while Chapters 5 to 7 develop it in different directions.

Chapter 2 reports the results of the first confrontation of my research with an international context, in which the first steps in adapting the Lesson Study to the Italian context are discussed, and in which a fundamental tool for implementing Lesson Study in Italy, the Lesson Plan, is presented and discussed - from the theoretical perspective of Cultural Transposition.

Chapter 3 describes (through a synthetic and analytical two-pronged approach) the Japanese and Italian institutional system related to the world of school, both from the point of view of school organisation and the teacher education system, with the aim of informing and guiding studies on the introduction of Lesson Study in Italy.

Chapter 4 proposes a theoretical approach informed by the Networking of Theories for the study of Lesson Study as an object of research. The introduction of Lesson Study in the Italian context is then studied through the theoretical frameworks of the Boundary Object, the Semiosphere and Semiotic Mediation. The results, which anticipate a recent research trend, show that the understanding of the Lesson Study cannot prescind from different points of view, both internal and external to the practice, capable of both conducting a meta-analysis of the Lesson Study and investigating its dynamics from within.

Chapter 5 presents the results of the first phase of my research project, my first experiment introducing Lesson Study to prospective teachers. The paper is focused on the dual position of the didacticians (as researchers and as teacher educators) introducing Lesson Study to prospective teachers. We were also able to identify some institutional constraints to the implementation of Lesson Study in the Italian context (mainly on the terminological side), which were the basis of subsequent implementation of Lesson Study with in-service teachers.

Chapter 6 proposes an experiment in which Lesson Study is introduced to in-service higher-secondary school teachers. The paper is focused on the cultural aspects of implementing Lesson Study in the institutional and cultural context of Italian secondary school. The results show that Lesson Study, as a foreign practice, is not so far removed from Italian norms, and teachers can accept it. Moreover, teachers recognize it as an opportunity to confront their teaching practices, to enrich their professional development, resulting in more awareness on their didactical action in and outside the classroom.

Chapter 7 summarizes the results of different Lesson Study experiments in Italy, informed by different theoretical frameworks. In the paper, four examples of “Italian” Lesson Study are presented. The paper showcases the elements that, as didacticians, we identified as necessary to construct an “Italian” Lesson Study coherently with the Italian didactic culture, and the motivations and intentionalities that guided the identification of each element.

1 INTRODUCTION

This introduction is structured as follows:

(1) A brief presentation of the main subject of this dissertation, Lesson Study, with a general frame to introduce the general research problem.

(2) A presentation of the approach to the research problem and of the theoretical frameworks used for the research.

(3) Structure of the dissertation.

1.1 A brief presentation of Lesson Study

Lesson Study is a method of teacher professional development[4] that originated in Japan around 1870. It is based primarily on collaboration among teachers (from the school, from schools in the same district, or on a national basis), and can also involve internal school staff and outside experts (*knowledgeable others*, such as university researchers). It can be described as the process of careful design and discussion of a selected lesson in a real classroom. The goal of Lesson Study is primarily to improve student learning: for this, teachers open themselves to the judgment of peers who, in turn, put their energies into building as constructive a critique as possible. Lesson Study has aroused interest in the international research community of Mathematics Education since 1999, following the TIMSS video study and the book *The Teaching Gap* by Stigler and Hiebert (1999). Since then, it has had a large diffusion in the United States, but there is no shortage of experiments in Canada, the United Kingdom, and other European, East-Asian, and African countries. We will see in the following a historical overview of the context in which Lesson Study emerged, an analysis in detail of Lesson

[4] In the following, we will be using teacher professional development, professional development and teacher education.

Study and its characteristics, a reflection on the reasons for its effectiveness and how Lesson Study has been exported outside Japan. We will then proceed to analyse why we look for introducing Lesson Study in the Italian institutional context.

1.1.1 Historical account of the Japanese educational system

This account takes from Isoda et al. (2007), Halliday (1978), Bouissou (2003) and Caroli & Gatti (2004). Where other references are used, these are indicated in the text.

1.1.1.1 The birth of Japanese public school. The history of the Japanese public educational system began in the fourth year of the *Meiji* period, 1872 of the Gregorian calendar, after the 268 years of the *Edo* period (1600 - 1868), in which education had been an essentially private and individual affair. The Japan of the pre-modern age (called *Kinsei* age, beginning around 1573 with the *Azuchi-Momoyana* period) was a highly class-conscious state, divided into closed social castes, engaged in a policy of total isolation from the outside world[5]. The *Edo* period experienced the very strong development of the mercantile society. This soon strained the economic system and, consequently, the classist social one[6], pushing toward a meritocratic view in which the abilities of the individual counted more than their social class. The education of the nobles was entrusted to educators paid by individual families, while the people had to

[5] An effective description of isolationist Japan can be found in the novel 沈黙 (*Chinmoku*) by Shūsaku Endō, published in Italian by Corbaccio as *Silenzio*.

[6] The merchant class was considered *unproductive*, therefore the last rung of the social ladder: below them, only the *barakumin*, the impure who performed death-related work, the slaves and the *gaijin*, the foreigners. In the Edo era, however, merchants assumed the functions of a central bank, exerting very strong pressure on governments.

make do with *terakoya* (temple schools), the evolution of small educational institutions attached to Buddhist temples. The emergence throughout the nationwide territory of *terakoya* prompted parents of the lower classes to voluntarily invest on their children's education[7].

The Meiji period coincided with another social revolution[8]. Emperor Mutsuhito, the first Japanese emperor with actual political power, sought to break with the past of the shogunate[9] by imposing Shintoism as the state religion and cutting ties with Buddhism and Confucianism. Having failed this attempt, the emperor focused on reforming the educational system starting from 1872. Jon Halliday so illustrates the situation that led to the innovation of the educational system:

> Not only was the traditional Confucian ethical learning despised as narrow and useless, but moral teaching of any kind had been virtually dropped from the school curriculum, and the textbooks prescribed for its teaching were a ludicrous collection of translations of obscure foreign works on ethics and law. (ibid., p. 33)

The reform initiated two major innovations: in practical terms, with the establishment of a normal school in Tokyo (today's Tsukuba University); from the ideological point of view, with newfound openness to Western innovations, with the central government inviting foreign experts so that they could spread Western "modern"

[7] Differently with the Italian situation, always struggling with school dropouts and child labour inside the family.

[8] A shrewd description of Japanese society in the swirling change of the Meiji period can be found in the novel 吾輩は猫である (*Wagahai wa neko de aru*) by Natsume Soseki, published in Italian by Neri Pozza as *Io sono un gatto*.

[9] The shogunate, or *bakufu*, was a form of military government present in Japan since 1185.

knowledge, highly advanced in comparison to that traditionally available in Japan[10]. Japanese mathematics, in particular, was mainly recreational and not institutionalised or regarded as a computational tool (Fukagawa & Rothman, 2008), whereas the textbooks introduced by Western experts were inspired to Klein's approach. Given the immense number of prospective teachers, these experts introduced in Japan the concept of *classroom lecture* - a method then rare even in the West - by conveying to their audience the knowledge of the subject matter while enabling them to learn the new teaching method through observation of the expert. The textbooks brought from the West were also designed in such a way as to enable this dual transmission of knowledge, and the new methodology spread throughout the nation, thanks to the opening of new normal schools throughout Japan. The situation remained stable until 1880, when - due to lack of funds - the normal schools were closed and only the one in Tokyo remained active. The eight years since the start of the reform, however, were enough for the seeds of the new Japanese school to germinate.

1.1.1.2 The first hints of Lesson Study. Between the late 1800s and the first three decades of the 1900s, Japan got acquainted with the educational theories of Johann Heinrich Pestalozzi and John Dewey. Pestalozzi had been known since the beginning of 1880, through a book on his teaching methods, and his theories pushed the Ministry of Education for the establishment of open classes for teachers that encouraged the proposal of new teaching methodologies (in fact, the first examples of *open lessons*,

[10] An interesting account of this contrast is given by the British explorer Isabella Bird in *Unbeaten Tracks in Japan*, which recounts how Bird visited Japan with a Japanese interpreter named Ito, traveling through places that few or no Westerners had seen before, between June and September 1878. ふしぎの国のバード (*Fushigi no Kuni no Bādo*), a manga by Taiga Sassa based on the book, was published starting 2015.

an important element of Lesson Study). The Dewey method is linked to the economic growth of 1920[11], which led to the opening of new private schools, unrelated to the government, and enabled the majority of the population to obtain at least the primary school license. The teachers of such private schools, inspired by Dewey's ideas on *active schooling*, proposed a method based on *peer learning* that put students in charge of their own learning, allowing them to ask their own questions, and to discuss among themselves which ones were worth investigating. For Mathematics in particular, this meant a whole reimagining of the structure of the subject by focusing on the connections between topics. This was the first appearance of the Japanese *problem-solving based teaching*[12], which is today central to the Japanese way of teaching mathematics and which has attracted so much attention from the international scientific community as an illuminating example of the constructivist model of knowledge: a teaching model centred on recognizing the limits of one's own knowledge and on striving to construct new knowledge from one's own strengths.

1.1.1.3 Developments in post-war Japan. World War II marks the end of Modern Japan. We are now in the Showa period (1926-1989), the longest in Japanese history and the most pregnant with radical change. Conservative parties succeeded in rekindling the flame of extremist nationalism, a series of attempted military coups pushed the country into imperialist expansionism and ended up putting in power one of

[11] We are now in the Taisho period (1912 -1926) in which Japan saw the emergence of democratic institutions and the growth of its international influence, factors that together with a series of internal uprisings led to the weakening of the oligarchic structure.

[12] Not to be confused with problem solving as understood in the West (also subject, admittedly, to multiple interpretations: e.g. Santos-Trigo, 2020).

the century's bloodiest war criminals, Hideki Tōjō[13]. While managing not to be influenced by Nazi ideology, Japan was overwhelmed by anti-Semitic and anti-communist currents, ending up with allying with Adolf Hitler's Germany. World War II saw Japan's power expanding in Asia as never before, until the ruinous fall that ended with the tragedy of Hiroshima and Nagasaki. After August 6th, 1945, Japan was a nation in need of ground-up reform, shaken in its certainty of being part of an elite led by a divine emperor, destined to dominate the world[14]: this new awareness pushed toward further modernization of society and toward the end of extremist nationalism[15].

The post-war period saw a radical reform of Japan's school system with the adoption of a fully centralized system and new textbooks for all school levels. The *Japanese National Secondary Mathematics Textbook* written by Shigeru Shimada and published since 1943 was the embodiment of the philosophies of the Japanese way of teaching mathematics, and it was the foundation on which Japan built the system for mathematics education. In the late 1960s, Shimada modernized the model to adapt it to the international trends of mathematics education. Since the 1970s, this evolved into a series of Lesson Studies for the development of new teaching methodologies, a mathematics education system suitable for students: at this time emerged the ideas of *problems with open processes* (that allowed for different methods of solving), *problems*

[13] Estimates of the Nanjing massacre account for 300.000 civilian deaths.

[14] The Emperor of Japan was considered to all intents and purposes a god. This status lapsed overnight with the signing of the unconditional surrender, and Japan went from a great world power ruled by god to a small nation occupied by the United States of America.

[15] Today Japan is part of the G8 and is one of the few countries in the world (if not the only one) without an army - in its place the 自衛隊 (*Jieitai*) or Japan Self-Defense Forces (JSDF). Pacifism is a fundamental pillar of the Japanese Constitution.

with open response (that provided more than one correct solution), *open problems* (from which to start to develop new problems). All of this converged, in the 1980s, in the work of Nobuhiko Nohda who gathered everything under the concept of *open approach*.

1.1.1.4 The role of Lesson Study. In short, from 1860 to the present, mathematics education in Japan has gone through (and is still going through) a constant evolution, in a process of internalisation and reworking of Western content and teaching methods. In this historical and institutional context, Lesson Study has helped to shape the Japanese system of education in at least two ways: firstly, it has created a “diffuse” community of teachers who, while they do not interact directly with each other in most cases, share a common technical language, materials, teaching methods, structures, which are conveyed through both the frequent transfers of staff from location to location and the publication of materials written by teachers in journals dedicated to teaching; secondly, somewhat as an indirect consequence, it has created a kind of balance with the highly centralised institutional system (with the national curriculum published by the MEXT establishing content, times and teaching methods), by which teachers are a much stronger voice in demanding an effective evolution of the education system, creating an ongoing dialogue with both the ministry and the academia.

1.1.2 What is the purpose of Lesson Study?

I have left open the description of what Lesson Study is, how it works, and how it is used. A literal translation of 授業研究 (*jugyō kenkyū*), the term Lesson Study[16]

[16] Where *jugyō* can be translated as instruction, teaching, school, lesson, class, and *kenkyū* as research.

denotes a collaborative process of lesson improvement that is widespread in Japan as a teacher professional development model. It is, along with problem-based teaching, one of the founding pillars of Japanese schooling in general (of mathematics teaching in particular). There are three different types, and for each there are variations depending on the school level and the institutions involved:

- **Scholastic level** organized within the individual school;
- **District level** organized among several schools in the same district with the cooperation of universities and teacher associations;
- **National level** organized by ministries together with universities and professional associations on a national scale.

There are, however, some common features that identify Lesson Study: it is a cycle focused not so much on improving teacher skill as on improving student learning and based on broad goals. We can call the different phases of Lesson Study in this way:

(1) Goals definition
(2) Lesson planning
(3) Research Lesson
(4) Discussion
(5) Reflection

1.1.2.1 Lesson Study in detail. What exactly do the different phases of Lesson Study involve? We have mentioned that there can be differences depending on the school level and the institutions involved, but some elements remain the same. Let’s look at the phases in more detail.

Goals definition: A Lesson Study cycle normally initiates with a meeting in which student learning goals are decided at the school, district or national level. These are not specific learning goals (such as learning to solve algebraic problems, analysing a cycle of epic novels, and so on) but the development of those long-term skills that are considered independent of the individual subject and important within society (such as improving argumentative skills, sharpening critical thinking, and so on). At this stage, teachers must be adept at understanding what the current situation of the students is and how far it is from the goals to be achieved. These elements flow into the choice of the *research topic*, the specific area or question that the Lesson Study team aims to investigate and explore, and which may vary depending on the Lesson Study team and is related to achieving the learning goals.

Lesson planning: Although the essence of Lesson Study is broader in scope, the Lesson Plan is still one of the most important products of the Lesson Study cycle. It is a document on which a Japanese teacher invests much time and energy. There is no official standard, but a typical example of planning includes:

(1) Name of the unit

(2) Objectives

(3) Research topic

(4) Characteristics of the students

(5) Unit learning plan (summarizing, in addition to a description of the lessons that are part of the unit, the connections of the topic with what has been learned previously and what will come in the future, including other courses)

(6) Time planning for the lesson, with a discussion of the reasons that justify the organization of the lesson

(7) Useful materials for observers (e.g., an outline with the classroom arrangement of students)

This collaborative document, which goes by the name *gakushushido-an* (which we can translate as *lesson plan* or *lesson proposal*), is perhaps the most visible product of the work that the teachers put into designing the lesson. Although the third and fourth phases of Lesson Study normally take few hours, the preparation of a Lesson Plan can take months. It is, after all, a document into which teachers pour all their knowledge and with which one tests oneself in front of an audience of peers or experts in the hope of improving collective professionalism.

Research Lesson. Once the Lesson Plan has been determined, it comes time to put it into practice. Where the lesson takes place and who attends varies depending on the school level[17] and the organizational level of Lesson Study.

Figure 1.1 An example of a national Lesson Study (Isoda et al., 2007)

- **School:** in this case the research lesson takes place in a selected class during the last hour of the school day. Pupils who are not involved end class one hour

[17] This also influences the subjects involved. In primary school, each year there is usually one Lesson Study cycle for each subject, while for secondary only for those most problematic for the student (e.g., mathematics).

earlier so that school staff is free to attend the research lesson[18] (the entire teaching staff in the case of elementary school, only teachers of the same subject and those of special subjects for secondary schools)[19]. Many schools require the intervention of an external expert, but their presence is not mandatory.

- **District** and **National:** The research lesson is usually attended by the teachers interested in the subject. The lecturer may be a university researcher, or an expert teacher (even retired ones).

Before the lesson, the lesson plan is distributed, and while the teacher in charge teaches, observers focus their attention on the reactions of the students. It is worth mentioning, here, the structure of a mathematics lesson within a Japanese classroom. The approach we have called *open-ended* is radically different from that traditionally applied in Western classrooms and has aroused as much interest as that aroused by Lesson Study. Whereas in the West the structure of a traditional lesson is based on a purely logical-mathematical approach, namely:

(1) Summary of prior knowledge;
(2) Presentation of new properties;
(3) Demonstration;

[18] Some video examples of lessons conducted in Japanese primary schools (with English subtitles) are available at http://www.criced.tsukuba.ac.jp/math/video/previous/ (last visited on 18/07/2023).

[19] We are not talking about hundreds of people, however, although it is not uncommon to reach fifty attendants: in Japanese schools there is little staff different from teachers, the cleaning being entrusted to students, and much of the auxiliary functions (nursing, for example) are handled by the teachers themselves.

(4) Exercises in applying what has been seen;

and strongly centered on the teacher pouring out their knowledge on the pupils, the Japanese model tests the ability of the students to work independently:

(1) The teacher reviews the previous lesson;
(2) The teacher proposes an open problem[20];
(3) Students work, individually or in groups, while the teacher walks around the desks, observes[21] and sometimes advises single students[22];
(4) The solutions found by the pupils are presented and discussed by the whole class[23];
(5) The teacher summarizes and systematizes what has been learned[24] with a careful planning of the of the blackboard[25];

It is in the success of the second and third phases that we see the teacher’s skill and the goodness of the plan of the lesson: a problem that is too difficult and far from students’ current abilities will lead to have nothing to discuss; failure to anticipate students’ responses means not being able to evaluate and react appropriately to their proposals.

[20] *Hatsumon*, a question posed in such a way as to stimulate students' curiosity.

[21] *Kikan-Junshi*: careful observation of students’ learning processes.

[22] Research shown that Japanese teachers spend little time interacting with the singular students, with stark preference of whole-class instructions (Hatano & Inagaki, 1998).

[23] The Japanese term for this phase is *neriage*.

[24] In what goes by the name of *matome*.

[25] This also has its own name, *bansho*.

Discussion: During the school Lesson Study, after the lesson is over, teachers and observers discuss what happened in the classroom. If an outside expert is present, they are usually the one who leads the discussion, while it is not uncommon for inexperienced teachers to simply observe in order to learn the specific dynamics of Lesson Study. The discussion takes place by analysing the Lesson Plan in relation to the reactions obtained and the student learning observed during the lesson. The goal, in fact, is not to recognize the best or most innovative idea, but the one that achieves an appropriate response with respect to the goals proposed in the lesson plan, and that best promotes learning in the classroom.

As described by Fujii (2016) one of the main misconceptions about Lesson Study is thinking that “it is aimed at the design of the perfect lesson” and therefore includes within it a phase in which the research lesson is presented again in (another) class after making the corrections that came up during the discussion. This is usually not so: Stigler and Hiebert (2016) suggest that this phase is important especially for Western culture, which is less accustomed to long-term planning and therefore needs to see concrete results at the end of a teacher education programme. The presence or absence of a repetition phase heavily influences the unfolding of Lesson Study.

In the case of the district or national Lesson Study, since there is no class to observe, the goal is to create a bridge between the world of research and the world of school practice: researchers propose possible innovations and teachers evaluate whether these are applicable to the reality of the classroom, while at the same time having the opportunity to deepen their own knowledge of the subject. Some characteristics typical of the Japanese culture play a very important role in this phase: the ability to immediately highlighting the weaknesses of a certain behaviour in order to promote long-term improvement; the constructive acceptance of criticism, however humiliating it may be, knowing that others do not speak with malice but that they have the greater good in mind. Teachers and observers, in fact, put aside proverbial Japanese politeness to comment on the lesson in a forthright manner, if necessary, very crudely.

Reflection: The reflection phase, which we might say “closes” a Lesson Study cycle, is one of those hidden parts of the Japanese teacher’s work that have a fundamental importance within this process. It is, in fact, that moment when teachers metabolize what they have learned during the discussion. This reflection on their and others’ work is realized in a written text to be submitted to the principal and which is published at the end of the year, for the benefit of the school and the ministry, in a school magazine. In the early 1990s, for example, the ministry collected more than 4.000 written reports on the Lesson Studies held across the nation (Krainer, 2011).

Among the reasons for the need for written text is the fact that, at the end of the school year, some of the staff may be reassigned to other schools. It is not uncommon for part of the teaching staff or the principal themselves to change school every five years. If Japanese teachers do not have freedom of choice on which school to teach in, it is essential that they pass on their own knowledge to newcomers and have a chance to pick up that of their new destination. The individual Japanese school may not be a stable organism, but it becomes one when embedded in an organized complexity (Ohtani, 2009).

1.1.2.2 What makes it so effective? This description is not enough to understand what makes Lesson Study such an effective teacher education model to the point that it has been one of the fundamental pillars of the Japanese school for nearly a century. We can nonetheless try to highlight some of the fundamental aspects that might be the reasons for the success of this very particular practice. For example, Krainer (2011) defines the *three Co* - content, community, context - as decisive dimensions for the success of teacher education:

- **Content** meaningful mathematical activities and appropriate reflections on them in support of the growth of teachers' perspectives and teaching practice;
- **Community** sharing of experiences, ideas, beliefs, skills and trusts connecting to learn independently and support others;
- **Context** favourable general conditions toward teacher education.

Krainer finds these three characteristics in Lesson Study practice:

- **Content** four key elements of Lesson Study contribute to the content dimension: it is developed around teachers' interests; it is student-focused; it involves a research lesson (observed in real time as a holistic experience), and it is a reflective process. Teachers learn to listen to students and observe their development; teaching is seen as a complex and deep adventure of mutual interaction and exploration of both the mathematical content and the students' ideas.
- **Community** the key feature of Lesson Study is its collaborative nature. New teachers learn by observing and participating in the lessons of experienced colleagues and improve their own professionalism by seriously engaging in activities that encourage personal growth. Lesson Study also creates a common language, oral and written, building a rich body of shared knowledge over time. Moreover, the community is not limited to teachers and experts; students are also part of it.

- **Context** external experts participating in Lesson Study are rewarded. Principals hire substitutes to allow teachers time to devote to professional development. Lesson Study teams are funded by the school board, the board of education, the municipality, the ministry.

These are stable elements in the Japanese context, even from a time perspective, to the point that it makes sense to speak of a Japanese Lesson Study culture. Notwithstanding that Lesson Study is culturally rooted in Japan, it is important to note that Japanese teaching (and Lesson Study itself) is open to improvements with theories and tools coming from Western cultures, according to the motto *you never stop learning* (to teach). Furthermore, one of the most important features of Japanese Lesson Study is that although it is nationally funded, there is no obligation for schools or teachers to organize it, or participate in it (Krainer, 2011).

C. Lewis (2016) compares the organizational levels of Lesson Study highlighting the synergies that create four conditions that are optimal for its success:

- **Condition 1** *opportunities to learn and implement teaching and curriculum innovations*: during district- and national-scale research lessons, teachers have the opportunity to observe innovations born from research and engage with them to comprehend them better and relate them to student learning. In addition, observing passionate students eager to share their ideas and comfortable in discussing them, can ignite in the teachers the desire to see the same happen in their own classrooms;
- **Condition 2** *feedback to researchers*: university researchers are often called upon by schools to participate in in-school Lesson Studies. On these occasions

they have the opportunity to evaluate the success of the proposed innovations - which is impossible to observe outside of the classroom;

- **Condition 3** *need, among teachers, for innovation of teaching methods and curricular innovation*: Lesson Study creates the time and communities suitable for the experimentation of new didactic practices. In addition, teachers pursue quality research lessons to foster the learning goals set by the school;
- **Condition 4** *emergence of curriculum development skills*: curriculum creation requires a specialized knowledge of mathematics, to which not all teachers have access. Professional mathematicians, on the other hand, may be flawed in their knowledge of the dynamics of the classroom. The collaboration of the two communities allows for a synergy of different skills, so much so that national textbooks are often written in collaboration between professional mathematicians and mathematics teachers.

1.1.3 *Lesson Study in the world*

It may be surprising that most authors mentioned up to this point are not Japanese. As noted by Groves and colleagues (2016) in Japan Lesson Study is so deeply rooted that it has not been, for a very long time, a scientific interest. Quite different, however, is the worldwide situation. As reported by several sources (e.g. Wake, Foster and Swan, 2013; Fujii, 2016; Robutti, Cusi and Clark-Wilson, 2016) Lesson Study has been under the lens of the international community since 1999, when James W. Stigler and James Hiebert published *The Teaching Gap*. In the book they describe the results of the TIMSS 1999 study[26], focusing on the lessons of 8th grade[27] mathematics in the United States, Germany, and Japan. In particular the contents of the chapter *Japan's approach to the improvement of classroom teaching* draw attention not only to Lesson Study but also to the characteristic *open approach*, mentioned earlier. To date, Lesson Study is widely spread throughout the world to the point that there are several associations for its promotion at the international level: most prominent is the World Association of Lesson Study (WALS), which organizes annual conferences (the one in 2022 counted almost 800 participants from 31 countries[28]) and publishes *The International Journal for Lesson and Learning Studies* (IJLLS) in collaboration with Emerald Publishing.

1.1.3.1 Lesson Study as a cultural object We mentioned that until, two decades ago, in Japan Lesson Study was so deeply ingrained in the cultural structure that it was not considered an object for scientific research. This has never been the case for the rest of the world, since 1999, when Western researchers took it on themselves to study Lesson Study: implementing a model with such deep historical roots in contexts different from that in which it originated requires, after all, a profound knowledge and theoretical understanding of the model, so that it can be manipulated without distorting it. Over the years, some issues connected to cultural misunderstandings of the practice accumulated, and only recently researchers have been framing Lesson Study as a complex organism in an even more complex context.

[26] http://www.timssvideo.com/the-study (last visited 2023/10/10)

[27] In Italy it corresponds to the last year of lower secondary school.

[28] https://www.walsnet.org/blog/2022/09/28/thank-you-all-for-a-successful-wals-2022/ (last visited 2023/08/04)

1.1.3.2 Japanese professional development system, a complex organism We have already seen how C. Lewis (2016) emphasizes the synergies of the Japanese cultural and institutional context that surrounds Lesson Study. Despite this, in 2019 Miyakawa and Winsløw noted what White and Lim had already stressed in 2008: that the Western narrative on Lesson Study usually fails to recognize that Lesson Study is but a cog in a larger, more complex system of professional development for teachers, built on the collaboration between different institutional agents.

> In widespread accounts of lesson study, it is presented as an isolated and cyclic activity […] in which the "planning" of a lesson is portrayed as a kind of group work, supported by the sheer study of textbooks. Possibly under the influence of such simplified operationalizations of lesson study, this will often be the situation when lesson study is implemented in other countries. […] the design of a creative and successful lesson often involves several elements of JPI [the Japanese context for professional development] and not just lesson study in the limited sense in which it is presented in parts of the literature. In particular, venues for discussing more tentative ideas for mathematical activities, a variety of media beyond textbooks and curriculum guidelines, and systematic experimentation of hypotheses […] (Miyakawa & Winsløw, 2019, pp. 299-300)

If, on the one side, White & Lim (2008) suggest that the fault partially lies on the narrative coming mainly from Anglo-American countries, on the other side it must be considered that the cultural barrier between the West and Japan (or China, or Korea) was - for a long time - almost impassable. Eastern languages and cultures are, in general, very different from Western ones (for a glimpse of Japanese culture, one can refer to: Boyé Lafayette De Mente's books in English, e.g. De Mente 1993, 2003; Laura Imai Messina's or Tiziano Terzani's books in Italian, e.g. Imai Messina, 2018, Terzani, 1998; or Roland Barthes' beautiful semiotic analysis published in 1970 and edited countless times, e.g. Barthes, 2015). In general, it would seem impossible to completely “understand” one’s own culture (Lotman, 1990), let alone that of others. Even so, efforts in this direction have recently been made by some Japanese researchers, also with the support of Western colleagues. Thus, Asami-Johansson (2020) analyses the context of primary school teacher education in Japan, Finland and Sweden; Clivaz and Miyakawa (2020) analyse the effect that culture can have on the structure of a lesson designed by pre-service primary school teachers in Switzerland and Japan; Shinno & Yanagimoto (2023) focus on the use of Lesson Study for pre-service teacher education in Japan. One of the major contributions to understanding the context of teacher education in Japan comes precisely from Miyakawa & Winsløw (2019), who map the different elements of Japanese teacher education: interestingly, they observe (as also pointed out by C. Lewis in 2016) the unprecedented - for the West - role played by the “media” agents, i.e. all those journals specifically dedicated to mathematics teachers that contribute to the active and continuous dissemination of good teaching practices tested and verified in real contexts. In Miyakawa and Winsløw’s map, Lesson Study is but one cog in a complex system made up of career incentives for teachers most prolific in the professional development of colleagues and the dissemination of knowledge, of state funds and service hours explicitly dedicated to quality training, of schools directly linked to university research centres, of teachers’ associations that promote active participation in the educational dialogue, and, above all, of a community of teachers always ready to welcome colleagues (from the same or other schools) into their classrooms, to question their own way of teaching, with the ultimate aim of improving themselves and others:

> [The Japanese context for professional development] offers a tightly connected system for teachers’ development and creation of shared didactic knowledge, and (in relation to this activity) also for maintaining a vital level of curiosity and “research”

> when it comes to the [mathematics] they teach […]. New ideas […] can be circulated and tested more locally before, possibly, being more widely disseminated in practice research reports, teacher journals and books, textbooks, etc.; it also means that official curricula […] are not just implemented by teachers. The published nature of teachers' main achievements, in terms of building didactic [practices and knowledge], means that they can function as an important resource for the periodic reforms of curricula. (Miyakawa & Winsløw, p. 301)

1.1.3.3 Lesson Study, a complex organ of a complex organism Another problem related to research on Lesson Study concerns the difficulty of considering it in its entirety, in at least two different ways. The first way concerns the fact that, to all intents and purposes, there is no agreement on what all the characteristic elements of Lesson Study are (although some researchers have tried to draw up some of them, e.g. Fujii, 2016 or Seleznyov, 2018). The literature almost always refers to the phases seen in 1.1.2.1, but we also have conceptualisations with fewer phases (e.g., the four phases in C. Lewis, 2016, which merges discussion and reflection into one phase) or with more phases (e.g., the six phases in Buchard, 2017, which also include some "preliminary" aspects to Lesson Study). Fujii (2016), in particular, argues that an overview of Lesson Study is almost impossible for those who are not "culturally immersed" in it:

> despite much research into Lesson Study, the process of creating a lesson plan, as a collaborative effort by teachers, is largely invisible to non-Japanese adopters of Lesson Study (p. 422)

Furthermore, whereas Seleznyov (2018), in listing seven essential components of Lesson Study, states that:

> Japan's systemic approach has embedded LS experience and expertise into the education system, meaning a uniform approach to JLS is much more likely (p. 223)

that is actually not the case. The impression is that Lesson Study is a Platonic idea, rather than something well-articulated and defined, so that even in Japan there are a myriad of variations of it (as already pointed out, after all, in Fernandez & Yoshida, 2004), all of which are unquestionably (at least according to Japanese teachers and researchers) *true Lesson Study*. Lacking a precise formalisation of the practice (which is difficult to achieve, due to the way Lesson Study has evolved), Western Lesson Studies can only be reinterpretations:

> Sometimes, features can be omitted if they just won’t work in the new context. Other times, it will be worth figuring out how to make features work in the face of resistance. Still other times, it will be worth testing various adaptations to find those that can be adopted in the new context with similar effects. (Stigler & Hiebert, 2016, p. 586)

The second way is a direct consequence of the first, and concerns the problem of not seeing the different phases of Lesson Study as a whole, and therefore only trying to implement “bits and pieces” of it:

> There have been various adaptations of Japanese LS in different countries. However, some adaptations of LS outside Japan […] include misunderstanding of core components and/or ideas of Japanese LS. (Huang and Shimizu, 2016, p. 395)

For example, the research topic may be missing, or it is considered necessary to repeat the research lesson until supposed perfection is achieved. Again, citing one of the studies analysed by Huang and Shimizu:

> [...] altering the LS model by removing one or two of the essentials of effective LS -observing student responses and collecting student learning data in classrooms during the lesson - renders the activity something much different from LS. (ibid., p. 395)

as clear as it is that it is not easy to transport such a complex practice outside of its own context.

Goldsmith, Doerr and Lewis (2009) describe three teacher education experiences in the United States of America to understand how teachers improve their teaching practice over time. Alongside the Math and Literacy Project (MLP) and the Turning to the Evidence project, Goldsmith and colleagues present a Lesson Study experiment held in an elementary school. As explained in the article itself:

> Two meetings of each group were coded: a planning meeting that took place before the group had finalized the lesson, and the post-lesson discussion. (ibid., p. 102)

the researchers' focus is on only two phases of Lesson Study. However, it is interesting that the project is an initiative of the school's teachers, not external parties.

Pierce and Stacey (2009) consider Lesson Study as a new strategy for research and professional development but continue:

> The focus for this study was the design of lessons that take advantage of technology that provides graphing, symbolic algebra and dynamic capabilities [...] (ibid., p. 369)

stating that they adapted Lesson Study by focusing on the discussion and development of a single lesson. This was constructed by the research team, while the teachers were only concerned with the implementation within their classrooms. The research lesson was also re-presented in another classroom after being corrected by the researchers, following the discussion had with both observers and students.

Robinson and Leikin (2009) present the results of a Lesson Study cycle held within a larger research project, focusing on a team of five teachers from the primary school. From the data collected by the team:

> We use the following data: (1) lesson planning team session; (2) lesson taught by one teacher and observed by the other members of the team; (3) reflective discussion of the lesson aimed at planning an improved version of the lesson; (4) teaching an improved lesson by the same teacher and observation of the lesson by the same team members; and (5) final reflective discussion. (ibid., p. 490)

All phases (plus the re-teaching phase) would seem to be present. A closer reading of the analysis, however, highlights how the planning phase lacks a research topic and a long-term learning goal, so that the ultimate objective of this Lesson Study is actually that of perfecting the individual lesson.

Verhoef and Tall (2011) investigate the effects of Lesson Study on the training of three teachers secondary school teachers in the Netherlands. In this case, unfortunately, it is difficult to understand whether the structure of Lesson Study is adhered to or what elements are used in the research, although the focus seems to be on the research lesson and its discussion.

1.1.3.4 Risks of Lesson Study The interest of the international community in Lesson Study seems well motivated, and the practice is in general praised for its potential as a tool for teacher education, research and improving student learning. We can cite, for example, Pierce and Stacey (2009):

> Lesson study research proved to provide rich data of a range of types, under surprisingly varied conditions [...] and from all perspectives, including students. (ibid., p. 375)

and:

> Effective professional development did occur. In particular teachers learned about teaching with technology by observing each other and sharing ideas. (ibid., p. 373)

or Huang and Shimizu (2016) when they highlight the dual role of Lesson Study:

> Research documents that LS in mathematics can improve teaching and enhance student learning. Meanwhile, teachers can develop their knowledge for teaching and productive disposition through LS. In addition, the important roles of LS in promoting reform -curriculum have been demonstrated [...] Furthermore, modest evidence reveals that instructional products based on LS are sharable and transferable and LS provides a platform for the interaction between theory and practice [...] (ibid., p. 401)

Runesson, Kullberg and Maunula (2011) discuss how teachers improve comprehension of students' learning processes:

> You could say that the teachers learned about the students' learning and this learning made them able to refine and develop the lesson plan in terms of how to handle the content. (ibid., p. 113)

Groves, Doig, Vale and Widjaja (2016) on the teacher community's positive evaluation of the whole Lesson Study process:

> [...] in hindsight participants realized the value of producing a lengthy, detailed lesson plan [...] They also highly valued the post -lesson discussions [...] Teachers unanimously reported major changes in their pedagogy [...] (ibid., p. 511)

It is the latter who warn us against throwing ourselves headlong into Lesson Study, highlighting the risks of imposing from above a method or so radically different from those commonly accepted by the teaching community. In their article, Groves and colleagues emphasize three essential issues:

- **Adaptation** the first step in adapting Lesson Study is to study it, to better understand its essential characteristics. This is an indispensable step for the teachers involved in the training, as they would otherwise run the risk of underestimating the importance of careful planning, or effective *neriage*;

- **Support** the support received from the community and institutions is an indispensable factor in the success of Lesson Study. There should be no shortage of opportunities for discussion among colleagues, and the teachers must be able to find the necessary time to dedicate to the different phases of Lesson Study. In particular, one of the most problematic aspects is the research lesson, which employs a large part of the school staff, putting a strain on the organization of the school;
- **Sustainable culture** teacher education is a cultural activity. Consequently, the application of Lesson Study, which originated in Japan and is a continuous and incremental process, can be difficult in those contexts that undergo continuous change or demand immediate results.

We have seen that Krainer, Huang and Shimizu, and C. Lewis (among others) suggest that care should be taken in adapting Lesson Study, arguing that its effectiveness lies precisely in the synergy of all its components. Huang and Shimizu (2016) also list a long list of possible issues, including the school environment:

> [...] a successful LS group requires the development of a shared professional culture and participants must feel comfortable with one another. Furthermore, a school and classroom culture that emphasizes the standard examination-oriented evaluation may further constrain teachers from pursuing long -term goals of student learning. (ibid., p. 401)

as also noted by Cajkler, Wood, Norton and Pedder (2014):

> We are encouraged in the belief that lesson study can act as an alternative or complementary model of teachers’ learning. However, this can only occur if certain conditions are in place, for example time and management support, as well

> as freedom from quick-fix expectations associated with what Hargreaves and Fulla (2012) describe as business capital thinking. (ibid., p. 527)

Fujii (2016) stresses the importance of an appropriate national curriculum:

> With a clear curriculum sequence, teachers could identify the value of the research lesson and the unit within the curriculum: by identifying closely related content in former and later grades, teachers can understand why the research lesson is important for later learning. And, identifying similar units or content in earlier grades helps teachers infer what students might do to solve the task, based on their previous learning. (ibid., p. 420)

and indeed, as we can observe in Verhoef and Tall (2011):

> This study reveals the significance of the complex reality of school practice in reference with the powerful claim of curriculum guidelines, study guides based on textbooks, and the attaining of high exam results. (ibid., p. 303)

its absence (or in the presence of a curriculum that is not constructed with the right priorities) can be a destabilizing factor.

An often-insurmountable obstacle would seem to be that dictated by the time factor. Thus, the trainee in Elipane (2011) is not completely satisfied with their training:

> Several factors, such as time (length of the practicum) [...] might have exiled the ST from being able to make sense of the reform -oriented stances that could be envisaged [...]. The ST said, “I wanted to deliver the same lesson again. In this way, I would have the opportunity to reconstruct and rectify my lesson based on the discussions during the hanseikai.” (ibid., p. 312)

and the facilitators in J.M. Lewis (2016) point out that the time factor is linked to so many issues:

> Facilitators addressed the number of issues related to time: time management, time shortage, scheduling, and use of time across all the academic year. It is evident that time is a primary issue for teachers and facilitators in many different contexts. (ibid., p. 534)

exacerbated by the U.S. context that demands rapid results:

> Teacher learning in lesson study is incremental, ongoing, and unfolds over the span of one’s career. This stands in stark contrast to the professional development model dominant in the US, where the expectation is that teachers will learn a complex pedagogical skill in a session or two. [...] Lesson study includes the close reading of student activity during classroom lessons, methodical study of multiple curriculum materials, the joint construction of highly detailed lesson plans, and evidence-based reflections on research lessons that are observed in person. (ibid., p. 539)

Hunter and Back (2011) also point out the difficulties of finding the right temporal dimension for Lesson Study:

> Engagement with the principles of lesson study is neither straightforward nor easy, and learning and change through engaging with this process takes prolonged time periods. (ibid., p. 96)

but they are also convinced of the opportunities behind such hard work:

> However, the findings of this study indicate the potential benefits of engaging in lesson study through the facilitation of awareness of effective mathematics pedagogy and the development of professional knowledge and reflection on teaching practice (ibid., p. 113)

Let us leave the *engagement* factor for last, albeit not least. Lesson Study cannot work if the teachers involved do not understand its usefulness, as Huang and Shimizu (2016) highlight:

> [...] in order to make LS successful, teachers should think of LS as a way to improve their own learning as well as student learning. (ibid., p. 402)

and also evident from Olson (2005), in which we have some participants who, considering Lesson Study a waste of time, fail to achieve any improvement in their own practice. Also interesting is the observation of Pierce and Stacey (2009): at first moment teachers may be wary, but effective preparation can make them put their fears aside and appreciate this particular professional development model.

We conclude, then, with remarks in Stigler and Hiebert (2016) on working to bring Lesson Study beyond Japanese borders. Certainly, this is not an easy task. One should not be discouraged, however. Even the work of adaptation itself can teach us so much, both about Lesson Study itself and about our culture:

> As educators make these decisions and try out alternatives, much is learned about lesson study itself, and about our own cultures and the beliefs and assumptions that constrain us. Lesson study provides educators with a rare opportunity to re-think our views of teaching, and what it takes to improve teaching. (ibid., p. 586)

As Fernandez (2005) also seems to believe:

> In the long run, society will also want to understand how these opportunities to learn compare to those offered by other forms of professional development centered on the examination of practice. This will be necessary to move toward a more nuanced understanding of what teachers can gain from the various ways in which they can examine practice. (ibid., p. 45)

we can have confidence that in time interest in this practice will grow to the point that it will change (for the better) the society itself.

1.1.3.5 What about the opinion of teachers? Unfortunately, articles that have teachers “talking” about the Lesson Study experience are scarce. Opinions are in general positive, as in the case of Bruce and colleagues (2016) both on the experience itself:

> And I feel that I’ve grown more this year than I have in 17, of taking risks myself and trying things in new ways. (ibid., p. 197)

and on the confrontation with peers allowed by the context:

> ...how amazing it feels to feel safe taking a risk and trying to become better and not being afraid to make mistakes. And I think the kids are seeing that, and as a result the kids aren’t afraid to make as many mistakes. (ibid., p. 198)

and/or permitting consideration of issues that would otherwise risk remaining hidden. In this regard, one can cite the comment of a teacher in Groves et. al (2016) that sheds light on a difficult issue that is likely to make Lesson Study in the West undoubtedly less effective than the original:

> What I’ve noticed is, sort of the culture that sits behind it. I can imagine the Japanese post lesson discussions being a lot more, as I said honest, really perhaps a little bit more animated, a bit more challenging. I think we’re very polite here. And I know Japanese are polite, but I have a feeling they’re a bit more blunt in this context. So that really stands out to me, as the culture of where that discussion is happening. (ibid., p. 510)

The cultural issue surrounding Lesson Study, after all, has a long reach.

1.1.3.6 Lesson Study adaptations As we said previously, different cultures and different institutional contexts require different adaptations of Lesson Study. The literature review shows many different forms of Lesson Study adaptation around the world, with some of them most prominent:

- **Chinese Lesson Study** The discussion of Chinese Lesson Study requires some caution, due to some historical uncertainties. It is not clear, in fact, whether the Japanese and Chinese Lesson Studies developed in parallel or whether the Chinese one is a reinterpretation of the "original" Japanese Lesson Study. We can certainly state that it is the philosophically and structurally closest variable to the Japanese Lesson Study, for cultural and institutional reasons. According to some researchers, it is characterised by a strong collaboration between academia and school, which is necessary to meet the challenges imposed by a vast territory such as China (Chen, 2017; Huang et al., 2019).
- **Lesson Study in the USA** The USA were probably the first Western country to implement an adaptation of Lesson Study. Particularly interesting is the extensive research work carried out by Catherine Lewis and colleagues, which resulted in an impressive amount of materials for teachers and administrations (C. Lewis, 2016).
- **Lesson Study in the UK** After the USA, the UK is probably the Western country where Lesson Study is most widespread. The most widely used version is the one proposed by Dudley who published a manual to guide schools in implementing Lesson Study. The UK Lesson Study is perhaps what we can most appropriately consider a "cycle", strongly characterised by the repetition and improvement of the research lesson (Dudley, 2014; Seleznyov, 2018).
- **Lesson Study in Portugal** Lesson Study in Portugal is mainly developed by the UIDEF. Lesson Study with in-service teachers is mainly focused on working with exploratory lessons (e.g. Gomes et al., 2022). Efforts are also toward identification of design principles to integrate Lesson Study for pre-service

teacher professional development (e.g. Ponte, 2018; Martins et al., 2023), and toward internationalisation of research on Lesson Study (as seen in Quaresma et al., 2018).

- **Lesson Study in Switzerland** In Switzerland, the 3LS laboratory (Laboratoire Lausannois Lesson Study) of the Haute école pédagogique Vaud forms Lesson Study teams so that there are two facilitators, a specialist in the subject and a specialist in didactics (Clivaz et al., 2023).
- **Lesson Study in South America** In South-America, Lesson Study is carried out especially in Brazil and Chile. While South-American countries can enjoy great support from the Japanese Ministry of Education and the Japanese community of researchers in mathematics education (with collaborative programmes now running for twenty years, also due to the historical presence of many South-American immigrants in Japan), the greatest challenge to be faced is related to the geographical dimension of the continent (Baldin & Malaspina, 2018, Isoda et al., 2021).

This list is not meant to be exhaustive, and perhaps it cannot be: both because there are Asian countries where the roots of Lesson Study are almost as developed as in Japan (e.g. South Korea), and because the Lesson Study research community is constantly growing, particularly in developing countries (e.g. Adler & Alshwaikh, 2019), and beyond (e.g., Calleja & Camilleri, 2021, or Holden, 2023).

1.1.3.7 Lesson Study variations Interesting experiences of adaptations or approaches to Lesson Study from different angles are encountered in Canada, Sweden, Italy and Japan. Ulla Runesson tells of two different Lesson Study experiences, one for primary school and one for secondary school (Runesson, 2009), this one deepened in a later study (Runesson, Kullberg and Maunula, 2011). The *Learning Study,* as they call it, combines some key features of Lesson Study (particularly the research lesson and discussion) with the iterative structure of design research. As also pointed out in Runesson (2009):

> However, the aim of a Learning Study is not to improve the lessons in a general way, but to improve students' learning of a specific object of learning. (ibid., p. 25)

The focus is thus on the critical factors of student learning and the lesson is discussed and revisited from this perspective. Interestingly, the desire to make it easier for student learning had negative repercussions in both studies, showing that reducing the complexity of activities does not always benefit comprehension. Runesson, Kullberg and Maunula (2011) however, doubt that this revelation would have come without collaborative cyclical work. Bruce, Flynn, Ross and Moss (2011) also experiment with the Learning Study (without mentioning it explicitly, but speaking only of the encounter between Lesson Study and design research) in the Canadian upper secondary school, but focusing on its power as a tool for teacher education and improvement of teaching practice.

Elipane (2011) acknowledges:

> The main objective of doing LS is not being able to come up with the best "lesson"; instead, the lesson just serves as a vehicle toward achieving intersecting goals on the improvement of teaching and student learning, and/or evincing contextual mathematical knowledge for teaching. (ibid., p. 307)

Elipane's goal is to understand how to utilize Lesson Study in preservice teacher education by exploring what skills and competencies need to be developed in order to participate successfully in Lesson Study. For this, Elipane observes a Lesson Study cycle in a secondary school Japanese secondary school from the perspective of a teacher-in-training.

Chapman (2010) describes an inquiry-based teaching experience. Although Lesson Study and Learning Study are not mentioned explicitly, it is evident from reading the article that we are dealing with something very similar. The research grew out of the need of a group of primary school teachers to better explore the new teaching methods demanded by a curriculum reform introduced at the district level:

> Inquiry, as a basis of teaching, is being associated with notions of learner-focused, question-driven, investigation/research, communication, reflection and collaboration. (ibid., p. 361-362)

The role of researchers is to support teachers in exploring this new way of working, without direct intervention in organizing the experience. Interestingly, it is the teachers themselves develop a course of work very similar to that envisaged by Lesson Study, from the identification of a research topic to the planning and observation of an experimental lesson experiment, to its analysis with subsequent reflection phase. The major difference is the study of some videos to deepen the theory behind the new approach to teaching.

Fujii (2016) tries to showcase some aspects of Lesson Study often under-explored at the international level, by analysing the participation of three Japanese elementary school in the *International Math-teacher Professionalization Using Lesson Study* (IMPULS) project[29]. Fujii focuses on the lesson planning phase, highlighting the critical factors that determine the success of a Lesson Study cycle.

[29] The project, funded by the Ministries of Education, Culture, Sports, Science and Technology, has two main objectives: train teachers from around the world on Japanese Lesson Study

Groves, Dolg, Vale and Widjaja (2016) report on the experience of the project *Implementing structured problem-solving mathematics lessons through lesson study* that investigates the elements essential for the successful adaptation of open problem solving and Japanese Lesson Study to the Australian school system. As we will saw previously, Groves and colleagues identify some issues that we could safely take out of context to understand what the difficulties are at the international level.

J.M. Lewis (2016) states:

> Little is known about how teacher developers learn to facilitate professional development where teachers drive much of the process. (ibid., p. 527)

Indeed, one of the main problems with carrying Lesson Study as a top-down project is that... teachers do not know how to do it:

> Despite its ubiquity in policy documents, US teachers and teacher educators have likely never taken part in the kind of teacher-driven, content-rich that the PLC literature calls for. (ibid., p. 528)

J.M. Lewis, therefore, investigates the processes of training what we might call facilitators, teachers who can ensure proper implementation of the novelty of Lesson Study in the school context.

In Italy, Bertolini et al. (2023) began by intersecting two important lines of Italian research, namely that on Lesson Study and that on the theoretical construct of mathematical discussion (Bartolini Bussi, 1998) to devise the Discussion Study, now expanding the idea to multidisciplinary realities:

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and then help them implement it in the school systems of their respective home countries; explore the mechanics of Lesson Study to maximize its impact on the school system.

> In this research, we chose to adopt an original way of using LS as a tool for teacher education, in that we did not focus on the development of a whole lesson within a specific discipline, but rather on the design and implementation of a lesson with a specific transversal didactic format, common to and central to several disciplinary didactics: the class discussion. (Bertolini et al., 2023, p. 2, original in Italian)

1.1.4 Lesson Study in Italy

We now come to the general aim of this dissertation, introducing Lesson Study in Italy. From my first encounter with this practice, I thought Lesson Study would be a useful tool to support the current panorama of teacher education in Italy, which is already teeming of interesting and valuable experiences, as well as a chance to reflect on practices related to the world of teaching, rooted in our culture to the point of becoming what Bourdieu (cited in Chevallard, 2002, p. 11) defines *inconscient scolaire* and justified by what Jullien (2006) defines as *unthoughts*: namely, practices and habits so ingrained in a culture that they seem to be the only possible solution to the challenges faced in teaching, and justified so unconsciously that any reflection on them seems impossible, if not unwarranted.

As observed above, however, we need to carry out a preliminary analysis of the Italian context in order to understand whether there are at least the starting conditions for trying to implant the seed of Lesson Study, and what, if any, difficulties we might face. We will begin with a presentation of the context of teacher professional development in Italy, starting with a historical overview of the collaboration between academia and school for the improvement of teacher education. We will then propose a synthetic presentation of the role of collaboration in Italian schools. Finally, we will discuss why we consider Lesson Study a positive addition to the panorama of teacher professional development in Italy: we will do so by listing some salient features of the

Italian school context, of the present context for teacher professional development, and by proposing some examples of good practices in the Italian context of teacher education.

1.1.4.1 The academia-school collaboration for teacher education in Italy The history of teacher education in Italy is long and varied, mainly due to the political fragmentation that plagued Italy until 1861, year of the unification of the country. For almost another century, moreover, the Italian school underwent many revolutions, which are, however, beyond the scope of this dissertation. Suffice it to say that these revolutions also affected the panorama of teacher education, and that this in particular is undergoing turbulent developments even in the present. Here we would like to summarise, referring mainly to the article by Arzarello & Bartolini Bussi (1998), the periods that have characterised the history of teacher education in Italy in the period from the decades after the Second World War (i.e. around 1960) to the present, with a focus on mathematics teachers.

From this point of view, the history of mathematics teacher education in Italy is intertwined with that of Italian research in mathematics education. We can distinguish two periods in particular: the 1960s to the 1980s, and the 1980s to the present.

In the first period, most of the “official” research in mathematics education was conducted internally at universities by professional mathematicians. Research focused on the “best” logical organisation of mathematical concepts, designed for “generic” teaching situations. The content was elaborated with a preference for adherence to the formal language of mathematics. The teaching action was also designed to address only conceptual difficulties, ignoring any psychological and pedagogical issues. In classroom experiments, the focus was on processes rather than products. The research model was explicitly *top-down*. In parallel, an innovation movement developed in schools in which teachers and teachers' associations were heavily involved. The primary aim of this movement was to produce paradigmatic examples of good teaching practices from specific problem situations. Research problems were formulated from practical problems, and the design of experiments guided by pragmatism. When experiments were carried out in the classroom, much of their success relied on the professionalism of the experimenting teacher, so that the analysis of the experiments was limited: the mere fact that the experiment was successful in a specific school situation was taken as proof of validity. At the very least, the teaching-learning process was somehow explored. Moreover, although the root of the innovation movement was in the schools, some professional mathematicians were involved in the process, suggesting a *bottom-up* research model centred around action research. In both cases, the dissemination was based on the goodwill of the teachers: in the first case, teachers - who would have had access to a better pre-service and in-service education system - should have been able to change the school by their own efforts; in the second case, having good opportunities for collaboration, teachers should have passed on their professional expertise to others.

The second period is characterised by collaboration between the university and the school. The involvement of professional mathematicians in the movement for innovation had laid the foundations for the evolution of the mathematics teacher training model in Italy. Starting in the 70s, in fact, the *Nuclei di Ricerca Didattica* ("Educational Research Teams", *ibid*., p. 246) were created, made up of professional mathematicians and teachers of all grades, who were involved in curricular innovation projects funded by the *Consiglio Nazionale delle Ricerche* (National Research Council) and the Ministry of Education. Many Nuclei became operative in the Universities in the 80s and are still alive today, albeit in a different form.

1.1.4.2 The "products" of collaboration in mathematics education The Nuclei have been integrated in the academia and are today research groups in mathematics education with tenured professors, researchers, and PhD students. Their mission is research in mathematics education, teacher professional development, and engagement of schools in new projects. We can quote: professional development programmes and orienteering approaches for students (e.g. m@t.abel project, see Arzarello et al., 2021, and Piano Nazionale Lauree Scientifiche), design of resources for teacher professional development (e.g. La Matematica per il Cittadino project, see Anichini et al., 2003; 2004; 2006), and models to empower teachers in promoting scientific inquiry and argumentation in the classroom (e.g. Licei Matematici, see Branchetti et al., 2019). The collaboration between researchers and teachers is one of the distinctive features of such programmes (e.g. Cusi & Malara, 2015; Robutti et al., 2020, 2021). Such teachers may be considered "researcher-teachers" (Arzarello & Taranto, 2021), as they are actively involved in action research and collaborate with the researchers. According to Hollingsworth (1995):

> [researcher-teachers] are concerned simultaneously with (a) ways to improve their practices, (b) change the situations in which they work, and (c) understand their practices within the larger society. (ibid., p. 16)

In the Italian context, they act as brokers (Wenger, 1996) of knowledge between the academia and the school. Dissemination of knowledge happens on two synergetic levels: teacher-researchers participate in professional development programs, and later share their knowledge with their peers.

In addition to academia, national associations of professional mathematicians (UMI) and researchers in mathematics education (AIRDM) are also involved in teacher education. As, internationally, the IMU has the ICMI, the UMI has a specific commission dedicated to teaching, the Commissione Italiana per l’Insegnamento della Matematica (CIIM, Italian Commission on Mathematical Instruction). The CIIM has the task of examining the problems concerning mathematics teaching in Italy, at all levels, also with regard to studies and experiences in other countries, and to propose possible solutions to the UMI. Since 2014, UMI-CIIM and AIRDM organise a yearly Summer School for Mathematics Teachers of primary and secondary school.

1.1.4.3 Collaboration in the Italian School Since the post-war period, Italian school has developed around a model of collaboration. As Capperucci (2008) points out, as early as 1974, the Consiglio di Classe (*Class Council*, participated by the teachers of a specific class) and the Collegio dei Docenti (*Teachers’ Council*, participated by the principal and all teachers) and were established, in addition to the pre-existent Consiglio di Istituto (*School Council*, composed of the principal and representatives of teachers, ATA staff, parents and students [30]). Article 5 of Legislative Decree 297/1994 states the objectives of the Class Council:

> The Class Council deals with the general progress of the class, formulates proposals to the Principal for the improvement of the teaching activity, presents proposals for an effective school-family relationship, expresses its opinion on any experimental projects.

[30] School Council exist only in secondary school. In primary school there are the Circle Councils which do not include -for obvious reasons- student representatives.

The Teachers' Council is in charge of the didactic and educational organisation of the school. It has the task of planning, organising, monitoring and evaluating the educational life of the school. The School Council is in charge of drafting the Piano dell'Offerta Formativa (POF, Educational Offer Plan), which is an act that presents the pedagogical, organizational and management choices of the schools in a given area, makes explicit the educational goals, the general objectives relating to educational activities and the resources provided to achieve them (Art. 3 of Presidential Decree No. 275/1999). Between the School Council and the Teachers' Council there should be a synergistic relationship to organize and manage educational and didactic action through planning.

With Law 107/2015, the POF has become the Piano Triennale dell'Offerta Formativa, (PTOF, Three-Year Educational Offer Plan). The PTOF is an expression of the cultural and planning identity of the school and requires educational institutions and teaching staff to be open to the territory, keeping in mind the educational needs expressed by a plurality of stakeholders. The entire school group is involved: the school principal is the holder of relations with institutions and coordinates the work carried out by the different professional components, playing a role of great importance in the network of relationships inside and outside the school; the teachers are responsible for some fundamental choices (the educational objectives, the project groups, the presentation of the PTOF to students and families); ATA staff maintain relations with families, they link projects with budget items, collaborate on the logistical aspects of training; and clearly parents and students, a necessary link between the internal reality of the school and the territory. The School Council verifies the effective use of all resources to ensure the quality of the PTOF. Since 2015, the PTOF also includes guidelines for a school-shared teacher professional development, in accordance with the plan for the improvement of the school.

1.1.4.4 A note on some other critical points for an "Italian" Lesson Study

Now, in the course of this introduction we have seen what are considered to be the essential elements to the success of Lesson Study. We have described some possible reasons why Lesson Study could fit the Italian context(e.g., the culture of collaboration in the school, or the collaboration between school and academia in teacher education). There are, however, some other elements that characterise the Italian school system that may be critical for the implementation of Lesson Study. I will make a few points using Krainer's three Co's seen in the above:

Content A major obstacle is the lack of a common national curriculum more in-depth than that provided by the National Guidelines, a structure that is perhaps essential in the Italian situation, but which ensures too much freedom of movement even within the individual school. There is a lack of adequate textbooks that go beyond the rigid logical-conceptual approach typical of academic mathematics.

Community Looking at the spaces devoted to teachers makes it easy to realize the highly individualistic mentality of the teaching profession: the staff room often consists of a single table, nothing more than momentary support, and most of the work must still necessarily be done at home or in precarious and inadequate spaces. Moreover, in recent years it is evident a breakdown in the relationship of trust between the Ministry and teachers, and teachers and families (Blandino, 2008).

Context Every year the school reports a chronic lack of funds, including in relation to teacher training, which moreover is not well regulated at the national level allowing for large differences (including qualitative) from region to region. The system of evaluating of the effectiveness of schools' teacher education efforts pushes for the achievement of short-range goals range immediately verifiable in a short time[31].

Other considerations need to be made about the Italian teacher professional development system, as well.

1.1.4.5 The current (complex) panorama of Italian teacher professional development

The National Plan for the Professional Development of School Personnel states:

> In-service professional development is not a formal or contractual obligation, it is a professional choice that allows broad cultural, planning, teaching, and research autonomy, within the framework of teaching freedom and scientific innovation. (paragraph 124, article 1, law 107/2015)

However, in-service professional development is compulsory (law 107/2015), there is no minimum number of hours per year, and must be carried out outside of teaching hours. There are no compulsory contents or practices for teacher professional development, and courses are chosen by teachers according to their own needs. The current panorama of Italian teacher professional development is diverse, and difficult to navigate. On paper, teachers have numerous occasions for improving their professionalism. The Ministry attests more than 500 agencies offering professional development opportunities. Universities, academic associations, teachers' associations, and educational companies which fulfil quality standards defined by the Ministry, are registered in a national database and can publish their teacher professional development

[31] The situation is in constant evolution, especially in the last year, thanks to post-Covid European funds PNRR, which let be possible some initiatives for the schools, Lesson Study included.

proposals on a digital platform (S.O.F.I.A.). The in-service professional development 'system" is conceived as a "lifelong learning environment" for teachers (law 107/2015). At national level, proposals come from the national education centre, academic associations, teachers' associations, educational companies. At regional level, regional school offices intervene by supporting, managing, and publicising the proposals. At local level, experienced teachers also offer courses in their school, sometime opens to teachers in the surrounding area. No official account is given on how many teachers participate in professional development. Yet, the impression is that this vastity of opportunities does not correspond to a high-quality offer: the Ministry states that the general quality of professional development programmes is compromised by the "low quality of [some] models and methodologies" (law 107/2015), suggesting that teachers might be easily lost and caught in low quality programmes. The Ministry generally does not provide guidance to orientate in this labyrinth, but it is nevertheless possible to identify - in the recesses of institutional documents - some programmes indicated as high-quality. We will indicate some in the next paragraph.

1.1.4.6* Liceo Matematico*, one "Excellent" (according to VQR[32]) example of professional development coming from research The previous considerations may be discouraging, but the situation is constantly evolving: we have seen, for example, that the Italian school is organised under the banner of collaboration, and so is the trend of collaboration within academia and school.

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[32] The assessment of academic project of dissemination of research in the society.

In addition, the *Piano per la formazione docenti* (Plan for teacher education) provides for the allocation of €325million[33] and the present PNRR that followed the pandemic allocates another €800millions for the period 2023-2027.

Although there are no official indicators that attest to the quality of an educational institution, reference can be made to the investment programmes to find niches of excellence to which funds are entrusted to activate courses and experiments. These can be found mainly in the academia, or in specific research centres such as CARME[34]. Particularly relevant in terms of numbers and scope is the *Piano Lauree Scientifiche* (Plan for Scientific Degrees), a collaborative project of Ministry and academia (which involves about 40 Universities), which aims to increase the number of STEAM graduates through both student orientation and teacher training activities (DM 976/2014, DM 1049/2017, and DSG 1295/2023).

Among these, the national *Licei Matematici* project involves 28 universities in Italy (out of 37 offering a bachelor’s degree in mathematics) and about 200 schools. The project provides that, in the face of schools offering additional hours of mathematics beyond those required by the national curriculum, the teachers of the involved schools (and anyone else who wishes to participate) have the support of the university through a teacher education programme and the proposal of teaching activities to be implemented in the classroom.

Within this national project, the University of Turin has been promoting the project Scuole Secondarie Potenziate in Matematica (SSPM, Secondary Schools Enhanced in Mathematics) since 2016. In 2021, the SSPM project was indicated as

[33] https://www.miur.gov.it/web/guest/piano-per-la-formazione-dei-docenti.

[34] https://www.carme.center/

“Excellent” by the national research quality assessment system (VQR) of the Agenzia Nazionale di Valutazione del Sistema Universitario e della Ricerca (ANVUR, National Agency for the Evaluation of the Academy and Research System). The project involves about 30 schools through an official *Licei Matematici* Memorandum[35]. The SSPM project is distinguished by the inclusion in the teacher education programme of activities aimed at disseminating the results of research in mathematics education. With this in mind, in 2019 the SSPM project was the scene of the first Lesson Study experiment for in-service secondary school teachers in Piedmont and Lombardy.

Other experiments started with primary school teachers in Piedmont (Manolino, 2021), which then brought to a Memorandum with the Ufficio Scolastico Regionale del Piemonte (USR Piemonte, Regional School Office of Piedmont) and the Sovrintendenza agli Studi della Valle d’Aosta (Superintendence of Education of Valle d’Aosta), head of the project Carola Manolino. The Memorandum now involves about 30 schools and 150 teachers of all levels.

1.1.4.7 Lesson Study as a support for the current Italian context of teacher education In 2022, the University of Turin hosted the first national conference in Italy dedicated to Lesson Study, after a number of local seminars (e.g., *CORIRE costruire _ osservare _ riprogettare*[36], of the University of Modena and Reggio Emilia). The aim of the conference was to summarise the experience accumulated by researchers at the University of Turin and to strengthen the network of collaboration between the other research centres involved in Lesson Study, both Italian and foreign, and with the many

[35] Teachers are also allowed to participate outside of the Memorandum: overall, it involves over 400 teachers from more than 200 (primary and secondary) schools.

[36] build _ observe _ redesign

schools involved in the project. The conference was attended by 120 participants (before registration had to be closed due to the limitations of the space in which it was hosted) between teachers and researchers.

In the previous paragraphs we have seen that, in the Italian context, valuable teacher education programmes can get lost in a sea of "low quality models and methodologies". We have also mentioned some of the difficulties that we may encounter in implementing Lesson Study in the Italian context:

(1) The lack of time and space for collaboration among teachers
(2) Limited familiarity with action research and critical reflection on one's own teaching practices
(3) Resistance to change and sharing experiences and difficulties with colleagues
(4) The diversity of disciplines and curricula requiring greater flexibility and adaptability of lesson plans
(5) The needs of students with specific learning disorders (SLD) or special educational needs (SEN) requiring inclusive and personalized teaching.

Whether (1) remains a main concern (Aires, 2023), the contributions presented at the conference indicate a positive sign for the development of Lesson Study in the Italian context. For one, the large number of participants indicates that teachers are not hiding from the challenge of learning to take a 'scientific approach" (2) to their work and are eager to collaborate and exchange experiences with colleagues (3). Furthermore, Lesson Study was reported by the participants as an important support for the management of flexibility, innovation and freedom of teaching (4, Peirone & Vilella,

2023) as well as an important tool to support teachers’ work with SLD and SEN students (5, Peirone, 2023).

Finally, Lesson Study could also respond to some institutional needs mentioned in the *Piano per la formazione docenti* by establishing a collaborative “network of opportunities for professional growth and development for teachers”, law 107/2015 (e.g., Marta, 2023), and an opportunity to further strengthen collaboration between academia and school.

1.1.4.8 Role of Lesson Study in this dissertation As observed, Lesson Study could play a fundamental role in relaunching teachers’ professionalism, while also providing a useful environment for collaboration and reconciliation between the academic and school worlds. The Italian context of collaboration between academia and school for research in mathematics education and for the professional development of Italian teachers seem to be a suitable environment for the implementation of Lesson Study. The Italian school system is (at least from an institutional point of view) organised for collaboration between all members of the school staff. Teachers seem to need a model of work that enables them to share experiences and professionalism with their colleagues. Finally, the Ministry pushes for the creation of professional collaboration networks between teachers, especially aimed at effective teacher education.

Nevertheless, the introduction of a teacher education model from abroad should be treated with care: some specificities of the Italian institutional context may make it suitable for Lesson Study, while others may hinder the implementation of this teacher education model.

At this point it should be clear why the introduction of this methodology should be carefully paved, especially with an eye to *what could be the cultural and institutional characteristics of Italian schools that could be decisive towards a successful introduction of Lesson Study - or towards a failure of this attempt*. This is, broadly, the research problem that I will discuss in this dissertation.

In what follows, I will explain how I approached my research work. Namely, I will frame the cultural issue in which Lesson Study is immersed, and how it has been addressed in the Italian and International contexts; I will then proceed to explain how I built the theoretical framework for this research and specify the research questions.

1.2 Approach to the research problem

In the previous section, we observed different ways of approaching Lesson Study. The introduction of Lesson Study in a context other than the Japanese one would entail various degrees of complexity, the whole spectre from the choice of the mathematical content of one lesson to the difference in cultural values between Italy and Japan. In addition, the researcher may decide to adopt an "internal" perspective, in which they are actively involved in working with teachers, or an "external" perspective as they observe others engaging with Lesson Study.

I am interested in introducing Lesson Study in the Italian context of pre-service and in-service teacher education. To do so, I first sought to understand what might be the characteristics of Italian institutions that could help or hinder the implementation of Lesson Study in the Italian context. I then decided to take a cultural perspective for my research.

1.2.1 The cultural issue

> [Lesson Study] cannot exist devoid of context; it is always a part of something bigger, for example, the school system and the culture (LO, 2019, p. 803)

> [Lesson Study]is a cultural practice that has evolved in a context [Japan] that, almost by definition, supports it. […] not only must educators adapt the routines of lesson study to each new context, but they also must redesign the context into which lesson study is being imported” (Stigler & Hiebert, 2016, pp. 584-585).

1.2.1.1 The cultural issue in the Italian research on Lesson Study Attention to the cultural aspects of translating professional development models has always been the foundation of Italian research, in which attempts at the introduction of Lesson Study have been going on for over a decade (Arzarello et al., 2023; Bartolini Bussi & Ramploud, 2018). Different approaches have been proposed: in the research group of the University of Modena and Reggio Emilia, Lesson Study is seen as an opportunity for teachers to confront with the unthought justification of teaching practices (see Bartolini Bussi et al., 2017); a recent PhD dissertation at the University of Turin proposed a semiotic approach to investigate how Lesson Study influences cultural aspects of the teaching and learning processes of mathematics (Manolino, 2021b); and more recent works tried to understand Lesson Study by means of its culturally conflictual aspects that trigger teachers’ questioning about their own teaching practices and educational intentionality (Ramploud et al., 2022).

The aim of my dissertation is to contribute to the knowledge on the implementation of Lesson Study in Italy, made through an institutional approach: I am going to analyse the characteristics of such institutions (schools, universities…) that could support or hinder the implementation of Lesson Study (we will refer to them as conditions and constraints).

1.2.1.2 The Cultural Issue in the International research on Lesson Study In recent years, other countries have finally shown growing interest in studies that would consider the specificities of the cultural context in which Lesson Study is implemented, to understand cultural conditions for (and constraints to) a successful implementation of the model (even in Japan itself, where Lesson Study is implemented since the 1880's: see Shinno & Yanagimoto, 2023). Interestingly, such studies report as constraints in their context (e.g., Skott, 2022, in the Netherlands and Wolthuis and colleagues, 2022, in Denmark) what Krainer (2011), C. Lewis (2016) and, more recently, Miyakawa & Winsløw (2019) report as conditions for Japan: successful implementations of Lesson Study rely, ultimately, on wide-spread support of the practice at an institutional - cultural - level. Such conditions and constraints are, ultimately, culture-specific, as they are the results of centuries of development of the specific cultural context. Yet, their dissemination can provide the international research community with interesting outlooks for reflection. This suggests that making the cultural context in which research is situated accessible to others is of the utmost importance for researchers in Mathematics Education (Bakker et al., 2021), specifically those involved in cultural practices such as Lesson Study.

1.2.2 *The cultural issue in my research on Lesson Study*

When I started my research work, a hard task was to identify a theoretical framework capable of dealing with the complexity of this situation. Stigler & Hiebert, in 2016, introduced the possibility that Lesson Study, in order to work in a specific context, had to undergo a cultural *adaptation*:

> Sometimes, features can be omitted if they just won't work in the new context. Other times, it will be worth figuring out how to make features work in the face of

> resistance. Still other times, it will be worth testing various adaptations to find those that can be adopted in the new context with similar effects. (Stigler & Hiebert, 2016, p. 586)

However, a theoretical perspective was needed to understand what to keep, what to omit and what to adapt. Incidentally, in the very years when I was starting my PhD research work, the Cultural Transposition perspective was proposed by an Italian research group in mathematics education (Mellone et al., 2018). At the time, I was impressed by the idea of “decentralize the educational practice [in our case of teacher education] of one’s own cultural context through the contact with the educational practices of other cultural contexts” (Mellone et al., 2018, p. 201). I thought it could both guide the study of Lesson Study from a research standpoint (observing Lesson Study through Italian practices would have allowed us to identify features of Lesson Study that otherwise remain “hidden” to Japanese eyes, as they are immersed in their culture of reference), and it could have played an important role in teacher education (observing Italian practices through Lesson Study could have been an opportunity to reflect on practices related to the teacher’s work, rethinking some of them and becoming more aware of others). The idea for Chapter 2 arose from the fact that the CIEAEM71 conference was an opportunity to present a workshop focusing on a “genuine” discussion on Lesson Study with people from different cultures.

However, the Cultural Transposition framework presented in the 2018 article might have functioned as a theoretical landscape (in the sense of Vithal, 2003), but it did not propose a strong analysis methodology to address the research problem (this is no longer the case, e.g. Mellone, 2021 or Ramploud et al., 2022).

I considered that the research should be tackled on two scopes: a *macro*-scope, in which I would have investigated and "compared" Japanese culture and Italian culture, in order to understand what the similarities and differences between them were, and to try to hypothesise how the Italian cultural context and the Lesson Study might relate; a *micro*-scope, in which (also on the strength of the knowledge accumulated on the *macro*-scope), I would have tried to transpose the Lesson Study into the Italian context, and I would have studied how the people involved in the Lesson Study (researchers, teacher educators, and teachers) would relate to the Lesson Study.

To sum up, I was trying to answer the research question stated above:

(1) What could be the cultural and institutional characteristics of Italian schools that could be decisive towards a successful introduction of Lesson Study - or towards a failure of this attempt?

by working on two more narrow research questions:

(a) Macro: How are teacher education practices, such as Lesson Study, influenced from the cultural and institutional context in which they are being implemented?

(b) Micro: How do Italian researchers, teacher educators and teachers relate to a teacher education model - such as Lesson Study - that is foreign to their cultural context?

1.2.2.1 The macro-scope I found myself wondering *what* Culture was, as an object of investigation. At this point I had read several books on Japanese culture and history, and I thought that finding a way to describe culture would help me in my research work: I could describe Japanese culture and Italian culture by identifying some salient features and I could compare them. I was faced with different definitions of culture, such as:

> “any aspect of the ideas, communications, or behaviours of a group of people which give them a distinctive identity and which is used to organise their internal sense of cohesion and membership” (Scollon & Scollon, 1995, p. 127)

or:

> “[t]he system of shared beliefs, values, customs, behaviours, and artefacts that the members of society use to cope with their world and with one another, and that are transmitted […] through learning” (Bates & Plog, in Freimuth, 2006, p. 2).

I soon had to surrender to the fact that there is no shared definition of culture (Spencer-Oatey, 2012), and that creating one was definitely outside my field of expertise. I needed, at the very least, to try to set a fixed point and choose where to start.

But what was more important, then? The fact that Japanese culture is made up of *kata*, of forms (De Mente, 2003)? Or that the ideographic writing of kanji makes the Japanese way of thinking profoundly different from that of the West, where signified and signifier are unconnected (Barthes, 2015)? Their inclination to think of the good of the group before personal good (De Mente, 1993)? Should I have described their culture of impeccable customer service (Stigler & Hieber, 1999)?

There was also the problem that I had not read as much about Italian culture, and I was not sure I could make an effective comparison at such a high level (it was, again, outside of my field of expertise). In short, I had to choose a path to follow, and I thought of narrowing it down to two alternatives. In Chapter 3, a paper written for the CERME12 conference, I use two approaches to describe the cultural and institutional contexts of Japan and Italy.

On the one side, the synthetic approach of Hofstede’s *Dimensions of Culture*. Hofstede defines culture as:

> the collective mental programming of the people in an environment. Culture is not a characteristic of individuals; it encompasses a number of people who were conditioned by the same education and life experience (in de Mooij, 2010, p. 48).

Hofstede identifies five basic value orientations of a certain national culture. These values "are broad preferences for a certain state of affairs (e.g., preferring equality over hierarchy) […] transmitted by the environment […] shaped by the time we hit 10-12 years of age" (https://news.hofstede-insights.com/news/what-do-we-mean-by-culture), are scored over 100 points. Most importantly:

> [t]he country scores on the dimensions are relative, in that we are all human and simultaneously we are all unique. In other words, culture can only be used meaningfully by comparison. (https://hi.hofstede-insights.com/national-culture)

perfectly fitting the comparison work I had envisioned. It seemed to me, however, that this was too reductive a description: could five numbers really define something as complex as the culture of a people? Would I have been able to grasp the *quid* of how an education system worked? Moreover, Hofstede's framework had a strong "entrepreneurial" characterisation, and ethical issues arose (partly addressed in, e.g., Ebaeguin & Stephens, 2014).

I felt I had to follow a more descriptive approach as well. During my university studies, after all, I had well understood that teaching practices and professional development practices are influenced and shaped by the context in which teachers and didacticians are immersed, as we have also stated multiple time during this introduction. Hofstede's numbers alone would not have been able to describe the complex relationships between the Italian and Japanese institutional components. I therefore thought to rely on the notion of

> culture […] and the larger cultural contexts as an influence of the pattern of practices (Hatano & Inagaki, 1998, p. 80)

so that I could expect

> culture to contribute to the forms of acceptable […] teacher professional development programs (Ebaeguin & Stephens, 2014, p.199)

The concept of culture is often linked with the concept of society and organisation, so I needed a framework that would allow me to describe Italian or Japanese society and its organisation (obviously only a small part of it, the one related to the world of mathematics teaching). I found a suitable framework in the *scale of levels of didactic co-determinacy* proposed in Chevallard’s Anthropological Theory of the Didactic (e.g., Chevallard 2022b). With the scale of levels of didactic co-determinacy, it is possible to systematically model the complex relations of the factors influencing teachers (and didacticians) practices, which are influenced not only by the teachers’ or didacticians’ decisions, but at a higher level by the society in which the didacticians, teachers and students are immersed, as shown in Fig. 1.2.

Upper levels	*Lower levels*
Humanity	↓↑
↓↑	Discipline
Civilisation	↓↑
↓↑	Domain
Society	↓↑
↓↑	Sector
School	↓↑
↓↑	Theme
Pedagogy	↓↑
↓↑	Question

Figure 1.2 Scale of levels of didactic co-determinacy (Florensa et al., 2018, p.5)

The upper levels, specifically, refer to the way in which Society understand the rationale of the school institution, and in turn organize the School system and related institutions. This supports the idea that the analysis of the system in which didacticians and teachers work can illuminate on the conditions and constraints that shape their practice and provide a structure to conduct such analysis.

The Anthropological Theory of the Didactic also offered a perfect hook to study the *micro*-scope.

1.2.2.2 The* micro-*scope The *micro*-scope of analysis in this dissertation concerns how people, immersed within institutions, and thus influenced by the demands and conventions of these institutions, relate to Lesson Study. A Lesson Study involves the entire school world: certainly teachers and educators, but also students (who are then the core of the Lesson Study's existence), managers, administrative staff… although the focus of Lesson Study is the improvement of student learning, I chose not to concentrate on that aspect of the research. As I am also a teacher, my primary goal was to ensure that my fellow teachers and I would be provided with a working method that would allow us to break out of our professional isolation and create an environment within which we could collaborate in a purely working context. Consequently, I thought I would focus my attention on the process of implementing Lesson Study in Italian schools, and how Lesson Study could be introduced without teachers identifying it as a body "too foreign" to really assimilate it into their own practices. In this sense I considered essential the figure of the didactician. As a researcher and teacher educator, the didactician bears the greatest responsibility for the introduction of Lesson Study: studying Lesson Study as researchers, trying to grasp its essential components that contribute to teacher professional development in Japan; adapting Lesson Study to the

Italian institutional context, so that it does not contrast with Italian formal laws and informal norms and expectations; synthetize such work of research and adaptation when implementing Lesson Study with Italian teachers. This responsibility is recognised internationally (e.g., Ponte, 2018), yet not many studies can be found on the subject, either in the Lesson Study context or elsewhere (e.g., Goos, 2008).

Not finding many studies dedicated to the figure of the didactician, I was not sure which theoretical framework I could refer to. Again, there are many variables that can be observed, but again for my position as a researcher and teacher, I needed something that would highlight how the work of a didactician cannot fail to consider the contribution of the teachers involved in the teacher education. After all, human knowledge evolves mainly through confrontation with others and, in a kind of micro-perspective of Cultural Transposition, I am convinced that teachers are not the only ones who learn during teacher education. If didacticians are open to the possibility of confrontation, in fact, they too can use this opportunity to reflect on their own practices and learn from teachers how the school system works. Therefore, I needed a theoretical framework that would allow me to consider the practices and knowledge that a person (or a group of persons) enacts to solve a problem related to their work, and above all that would allow me to highlight the evolution of these practices and knowledge over time. Moreover, the framework should have explicitly admitted the possibility that this evolution takes place within a specific institutional context, and also through confrontation with other persons. At the University of Turin, we have been using the theoretical framework of Meta-Didactical Transposition for many years (Arzarello et al., 2012; Arzarello et al., 2014), and its effectiveness in analysing teacher education contexts characterised by a collaborative framework between researchers and teachers is attested. The framework is developed from the Anthropological Theory of the Didactic (which we have already mentioned and will discuss in a moment) and is based on the notion of praxeology (Chevallard, 1999). Praxeology is a model of human activity. A praxeology is made of two blocks, representing the know-how (praxis) and know-why (logos). The praxis refers to a task and to the techniques which can be used to ‘solve” the task, while the logos refers to the discourses that justify the techniques, and the theory that support such discourses. Specifically, the analysis of the logos can be used to study the transformation of knowledge through communication.

Meta-Didactical Transposition, in particular, allows us to observe the professional development of Mathematics teachers as a process considering “the relationships and reciprocal influences of two [didacticians’ and teachers’] communities […] with respect to their professional practices” (Robutti, 2020). The notion of metadidactical praxeology can be used to model, specifically, “the theoretical reflections developed in teacher education activities” (Arzarello et al., 2014, p.354). Metadidactical praxeologies do not refer to classroom practices managed by teachers, but to the practices and reflections that teachers and didacticians develop on the didactical or paradidactical praxeologies (that are object of the teacher education activities). Metadidactical praxeologies are characterized by the reflections resulting from the interaction between teachers and didacticians: specifically, the teachers’ own reflection on their didactical or paradidactical praxeologies, and the consequent reflection of the community of didacticians concerned with the effects of the educational processes they develop. The tasks addressed concern the understanding of teacher professional development programmes. The praxis block refers to the practices developed within the community of inquiry for the management of teacher professional development, while the logos block refers to the theoretical elaboration on the practices adopted. As subsets of metadidactical praxeologies, didacticians praxeologies develop in accordance with certain theoretical frameworks, whether teachers praxeologies develop in accordance with classroom practice.

Another fundamental concept of Meta-Didactical Transposition is that the evolution of (teachers’) praxeologies occurs through the *double dichotomy* that is established between two dialectical levels: the didactical level, that is developed within the classroom, between the personal meaning that students attribute to the teaching situation and its shared scientific meaning; the metadidactical level, that lies between the interpretation given to the classroom dialectic by teachers and that given by didacticians, both based on the beliefs and praxeologies of their respective communities. During professional development, didacticians introduce the teachers to new practices and knowledge. These starts off as external components of the teachers praxeologies that are developed during the professional development programme, and are expected to be reflected on, negotiated and - eventually - internalized by the teachers, becoming internal components of the teachers praxeologies.

In Chapter 5 I highlight that a double dichotomy exists also for the researchers, between the metadidactical level developed during teacher professional development and the research level, developed when they are making sense of the data collected during teacher professional development. This result, however, would have been impossible with only the tools of Meta-Didactic Transposition. In this sense, the contribution of the Anthropological Theory of the Didactic was fundamental. Initially (Chevallard, 1985), the theory dealt with the didactic system of the classroom, mainly observing the teacher’s *didactical transposition* of *savoir savant* (i.e., the mathematics of professional mathematicians) into *savoir enseigné* (i.e., knowledge taught to the students), considering the knowledge to be taught (i.e., the demands coming from the educational institution). Object of study were the mathematical practices and knowledge found in the classroom or textbooks and related to a class of tasks (mathematical praxeology), as well as teacher's practices and knowledge to make the mathematical praxeology emerge in the classroom (didactical praxeology).

Today, the Anthropological Theory of the Didactic has evolved (Chevallard, 2022a) to consider more "didactic systems" besides the one that develops in the classroom in the presence of teachers and students. Two concepts have played a fundamental role in this evolution: of *institution*, understood in a broader sense, including not only the institutions like school or university, but also "any created reality of which people can be members (permanent or temporary)" (Chevallard & Bosch, 2020, p. xxxi), such as class, family, researchers' community, Italian society, and so forth; and of *position* occupied by a person in an institution in relation to an object.

These two concepts allow us to state, for example, that the position of a didactician and a teacher with respect to Lesson Study are different during teacher education. As different as the didacticians' *positions* as teacher educator or as researcher with respect to Lesson Study. As we are able to highlight that didacticians are both *the ones who teach the teachers* and *the ones who research (about teacher professional development)*, we can also affirm the existence of the *double dichotomy* for didacticians, adding a little piece to our knowledge about the mechanisms of their role when dealing with teacher education. That this result arrives in relation to Lesson Study is not insignificant: it is precisely the complexity of the cultural adaptation of this practice that places the didactician in a role that they cannot fulfil unless they use the synergy between the two positions.

1.2.2.3 Is that all? The description given in the last two subsections seems very linear, and it would appear that one can represent the research work as a segmented half-line, as in Fig. 1.3:

Figure 1.3 Ideal representation of the research workflow

Of course, the research path has not been so. The two strands of research (macro, more of literature review, and micro, basically of field experiments) were neither parallel nor consequential, but intersected several times, and had an “alternating” relevance in my working days. Sometimes it was the *macro*-scope that informed the *micro*-scope, other times it was the other way around. Thus, I got an initial idea of what the cultural obstacles were by working with pre-service teachers (who, lacking direct experience in the school world took Lesson Study “face-value” and provided a very genuine response to the difficulties they encountered, where experienced teachers would be able to “juggle” the difficulties, hiding them from me), and this knowledge guided the study of the cultural and institutional factors that could justify what I observed. This cultural work would then allow me to modify my strategies for working with in-service teachers, so that some cultural issues could be resolved and others allowed to emerge, and so on. The workflow can be represented by a gyre:

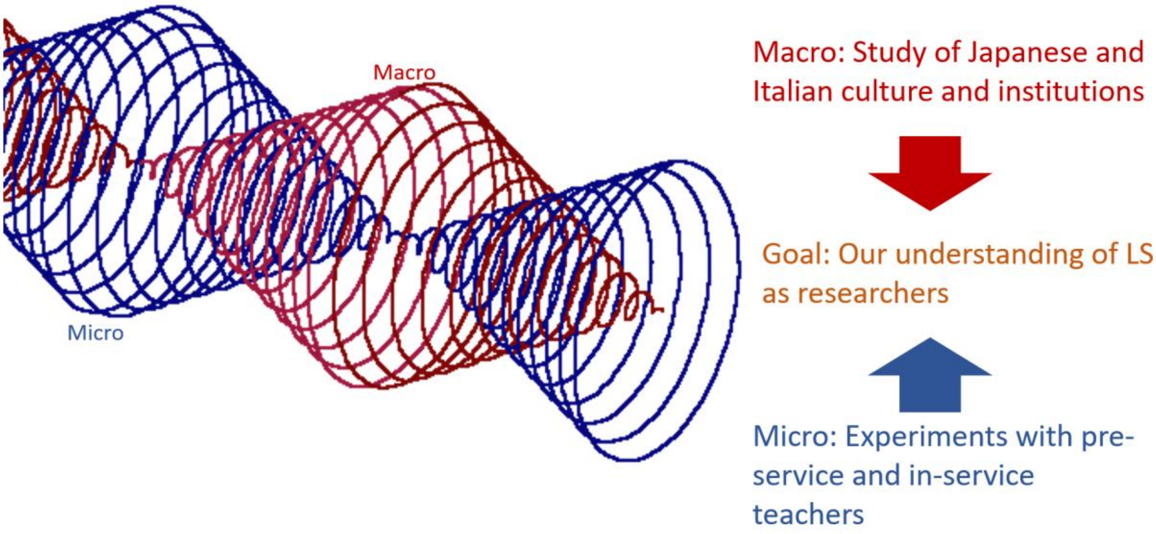

Figure 1.4 Realistic representation of the research workflow

In all this, I was also working with other colleagues on the analysis of data collected in different contexts, reinforcing my idea that the Italian exploration of Lesson Study, although begun in Italy in 2012 (Arzarello et al., 2023), was still in its early stages. Thus, in Chapter 4 dating back to 2020, we hypothesised that understanding Lesson Study could not be achieved by using only one theoretical framework, and that it was necessary to look at Lesson Study as an object of research from multiple standpoints (anticipating Volume 12 Issue 1 of the International Journal for Lesson and Learning Studies entitled precisely Networking theories for understanding and guiding Lesson Study). Chapter 4, in particular, involves the theoretical frameworks of Boundary Crossing, the Semiosphere, and Semiotic Mediation, respectively two "external" and one "internal" framework to Lesson Study. In Chapter 6 we explored the use of the "internal" framework by leveraging teachers' work through it and analysing in-service teacher interactions through the Cultural Transposition framework. Finally, in Chapter 7 we observed through the Cultural Transposition theoretical framework the results obtained from experiments constructed from different theoretical frameworks.

A map that guides through the chapter of the dissertation could therefore be as follows (Fig. 1.5):

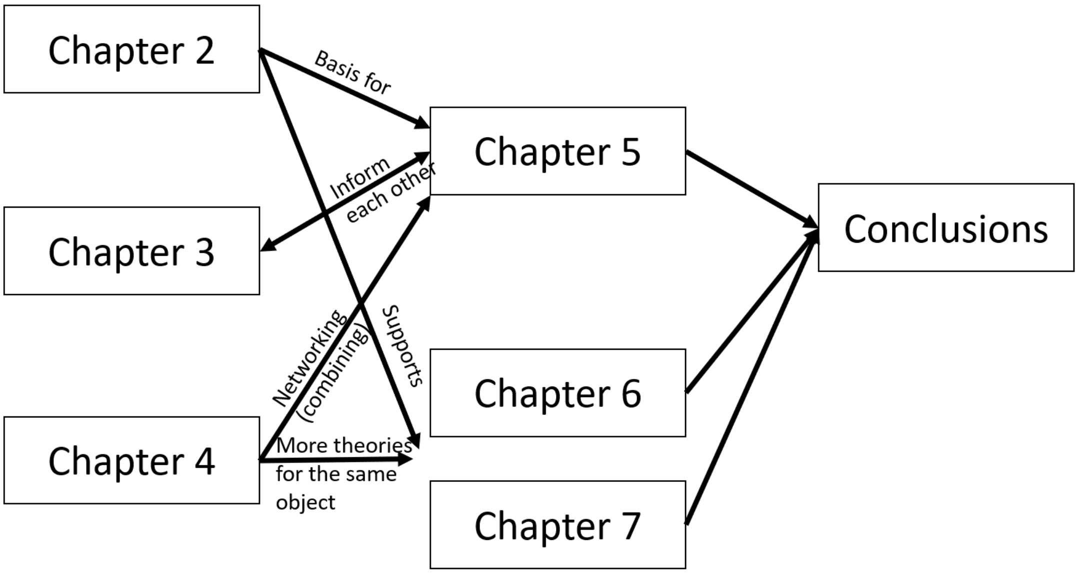

Figure 1.5 Workflow of the dissertation

In what follows, I will therefore proceed to outline the chapters that make up this dissertation step by step. These are papers published after double-blind peer review in international conference proceedings, or in international journals. The only exception is Chapter 7, published in international journal by invitation.

1.3 Structure of the dissertation

In the following, I will present the chapters that compose this dissertation.

Chapter 2, 3 and 4 will set the groundwork for the thesis. Chapter 5, 6 and 7 embody the results of my personal research and of collaboration with my peers on this groundwork.

Chapter 2 is the result of the first international presentation of my work on Lesson Study, a workshop for CIEAEM71 with a dual cultural aim: to discuss, with an international audience, the tentative structure of Lesson Study in the Italian context; to discuss the elements of a Lesson Plan, and collect international experiences of Lesson Planning.

Chapter 3 is an article for CERME12 in which the importance of cultural context in the study of educational systems and research in mathematics education is addressed. In the paper, the Japanese and Italian educational systems are briefly presented and compared to provide an overview of the difficulties that may be encountered in implementing Lesson Study in Italy. An overall view of the similarities between the two contexts is also provided, thus justifying the fact that Lesson Study can still be adapted to the Italian institutional context.

Chapter 4 presents the results of a networked analysis of Lesson Study according to three theoretical frameworks and introduce the idea that Lesson Study cannot possibly be understood if observed by a single standpoint.

Chapter 5 is the paper that presents the results of the first phase of my research project, my first experiment introducing Lesson Study to prospective teachers. The paper is focused on the dual position of didacticians (as researchers and as teacher educators) introducing Lesson Study to prospective teachers. The data are analysed with the theoretical frameworks of the Anthropological Theory of the Didactic and the Meta-Didactical Transposition. We observed the double dichotomy that didacticians experience during teacher professional development, between the metadidactical level and the research level of the elements of their praxeologies. We also observed how the interaction with teachers is an essential component of their work as didacticians. This was the basis for proposing an evolution of the theoretical frameworks involved in the analysis, the Anthropological Theory of the Didactic and the Meta-Didactical Transposition, and to analyse the potential of *combining* the two frameworks (in the sense of the Networking of Theories). We were also able to identify some institutional constraints to the implementation of Lesson Study in the Italian context (mainly on the terminological side), which were the basis of subsequent implementation of Lesson Study with in-service teachers.

Chapter 6 proposes an experiment in which Lesson Study is introduced to in-service higher-secondary school teachers. The paper is focused on the cultural aspects of implementing Lesson Study in the institutional and cultural context of Italian secondary school. The data are analysed with the theoretical framework of Cultural Transposition. We observed how teachers engage with a foreign practice, and how the reflection that arises when dealing with the unknown realises in self-reflection on their own teaching practice. The results show that Lesson Study, as a foreign practice, is not so far removed from Italian norms, and teachers can accept it. Moreover, teachers recognize it as an opportunity to confront their teaching practices, to enrich their professional development, resulting in more awareness on their didactical action in and outside the classroom.

Chapter 7 summarizes the results of different Lesson Study experiments in Italy, informed by different theoretical frameworks. The goal of the paper is to showcase the elements that, as didacticians, we identified as necessary to construct Lesson Study coherently with the Italian didactic culture, and the motivations and intentionalities that guided the identification of each element. The data are analysed with the theoretical framework of Cultural Transposition. In the paper, four examples of "Italian" Lesson Study are presented. The fourth example is a narrative of an experiment that I carried out with in-service higher-secondary school teachers, as part of the second phase of my research project. The goal of the experiment was to investigate how a didactician, aware of the institutional conditions and constraints of implementing Lesson Study in Italy (identified from the experiment described in Chapter 5), acts when working with teachers during teacher professional development and Lesson Study, and the effect that his actions have on the teachers' reactions towards Lesson Study. In particular, the experiment is informed by the *coordinating* (in the sense of the Networking of Theories) of the Meta-Didactical Transposition framework with the scale of levels of didactic co-determinacy proposed by the Anthropological Theory of the Didactic:

- The Meta-Didactical Transposition framework informed on the dynamics by which teachers internalise new practice and knowledge in their own professionalism;
- The scale of levels of didactic co-determinacy informed on the rationale of the institutional conditions and constraints of implementing Lesson Study in Italy.

Tentative results of the experiment show that Lesson Study, as a foreign practice rooted in a different culture, when transposed in another culture needs to be continually discussed and renegotiated by the people who participate in it. This continuous discussion and renegotiation are essential at least until Lesson Study is ‘stabilised”, reasonably in a different form to the one it takes in Japan, lest it be “rejected” when old habits prevail again.

1.4 DECLARATIONS

1.4.1 Declaration of previous publication

This thesis includes the following original papers that have been previously published/accepted for publication/submitted in peer reviewed journals or conference proceedings. I certify that I have obtained a written permission from the copyright owners to include the published materials in my thesis. I certify that the material describes work completed during my registration as a doctoral student at the University of Turin.

Ch.	**Reference**	**Status**
2	Manolino, C., **Minisola, R.**, Robutti, O. & Arzarello, F. (2020). Translating practices for reflecting on ourselves: Lesson Study. In: Di Paola, B. & Palhares, P. (Eds.). *Quaderni di ricerca in didattica: CIEAEM 71*. Palermo: Università di Palermo. pp. 519-525.	Published
3	**Minisola, R.**, & Manolino, C. (2022). Teachers' professional development: A cultural matter. How to describe cultural contexts? In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of European Research in Mathematics Education (CERME12)* (pp. 3650–3657). Free University of Bozen Bolzano, Italy and ERME. https://hal.archives-ouvertes.fr/hal-03748740/	Published
4	Capone, R., Manolino, C. & **Minisola, R.** (2020). Networking of theories for a multifaceted understanding on Lesson Study in the Italian context. In: H. Borko & D. Potari (Eds.). *The Twenty-Fifth ICMI Study: Teachers of Mathematics Working and Learning in Collaborative Groups. Conference Proceedings.* Lisbon: ICMI. pp. 102-109.	Published
5	**Minisola, R.**, Robutti, O., & Miyakawa, T. (submitted) Didacticians introducing Lesson Study for the Professional Development of Prospective Mathematics Teacher	Submitted (as of 2023/09/26)
6	Capone, R., Adesso, M.G., Manolino, C., **Minisola, R.** & Robutti, R. (2023). Culturally crafted Lesson Study to improve teachers' professional development in mathematics: a case study in Italian secondary school.	Published

	Journal of Mathematics Teacher Education. https://doi.org/10.1007/s10857-023-09578-3	
7	Arzarello, F., Bartolini Bussi, M. G., Funghi, S., Manolino, C., **Minisola, R.**, & Ramploud, A. (2023). Del Lesson Studies al Lesson Study italiano: Un Proceso de Transposición Cultural. PARADIGMA, 44(2), 340-375. https://doi.org/10.37618/PARADIGMA.1011-2251.2023.p340-375.id1423	Published

1.4.2 Declaration of co-authorship

I hereby declare that this thesis incorporate material that results from joint research with the following authors:

Author 1 Ornella Robutti

Author 2 Takeshi Miyakawa

Author 3 Ferdinando Arzarello

Author 4 Carola Manolino

Author 5 Maria Giuseppina Bartolini Bussi

Author 6 Roberto Capone

Author 7 Silvia Funghi

Author 8 Alessandro Ramploud

Author 9 Maria Giuseppina Adesso

The extent of my contribution to the manuscript as declared hereafter is based on the following scale:

A. The candidate did the majority of the work independently (67-100%)

B. The candidate has made a substantial contribution (34-66%)

C. The candidate has contributed to the work (0-34%)

Chapter 2: Translating practices for reflecting on ourselves: Lesson Study

Authors: Author 4, Candidate, Author 1, Author 3

Individual elements	**Candidate's contribution**
Formulation/identification of the scientific problem	B
Development of the key methods	B
Planning of the experiments and methodology design	B
Conducting the experimental work	B
Conducting the analysis of data	B
Interpretation of the results	B
Writing of the first draft of the manuscript	B
Finalisation of the manuscript and submission	B

Chapter 3: Teachers' Professional Development: a Cultural Matter. How to Describe Cultural Contexts?

Authors: Candidate, Author 4

Individual elements	**Candidate's contribution**
Formulation/identification of the scientific problem	B
Development of the key methods	A
Planning of the experiments and methodology design	A
Conducting the experimental work	B
Conducting the analysis of data	B
Interpretation of the results	B
Writing of the first draft of the manuscript	A

Finalisation of the manuscript and submission	B

Chapter 4: Networking of Theories for a Multifaceted Understanding on Lesson Study in the Italian Context

Authors: Author 6, Author 4, Candidate

Individual elements	**Candidate's contribution**
Formulation/identification of the scientific problem	B
Development of the key methods	B
Planning of the experiments and methodology design	B
Conducting the experimental work	C
Conducting the analysis of data	B
Interpretation of the results	B
Writing of the first draft of the manuscript	B
Finalisation of the manuscript and submission	B

Chapter 5: Didacticians introducing Lesson Study for the Professional Development of Prospective Mathematics Teacher

Authors: Candidate, Author 1, Author 2

Individual elements	**Candidate's contribution**
Formulation/identification of the scientific problem	A
Development of the key methods	B

Planning of the experiments and methodology design	B
Conducting the experimental work	B
Conducting the analysis of data	B
Interpretation of the results	A
Writing of the first draft of the manuscript	A
Finalisation of the manuscript and submission	B

Chapter 6: Culturally crafted Lesson Study to improve teachers' professional development in mathematics: a case study in Italian secondary school

Authors: Author 6, Author 9, Author 4, Candidate, Author 1

Individual elements	**Candidate's contribution**
Formulation/identification of the scientific problem	C
Development of the key methods	C
Planning of the experiments and methodology design	C
Conducting the experimental work	C
Conducting the analysis of data	B
Interpretation of the results	B
Writing of the first draft of the manuscript	C
Finalisation of the manuscript and submission	B

Chapter 7: From Lesson Studies to an Italian Lesson Study: a cultural transposition process

Authors: Author 3, Author 5, Author 7, Author 4, Candidate, Author 8

Individual elements	**Candidate’s contribution**
Formulation/identification of the scientific problem	C
Development of the key methods	C
Planning of the experiments and methodology design	C
Conducting the experimental work	C
Conducting the analysis of data	C
Interpretation of the results	B
Writing of the first draft of the manuscript	C
Finalisation of the manuscript and submission	C

I certify that I have properly acknowledged the contribution of other researchers to my dissertation, and have obtained written permission from each of the co-author(s) to include the above material in my dissertation.

We, the undersigned, endorse the above stated contribution of work undertaken for each of the co-authored manuscripts contributing to this dissertation:

Co-authors and signatures			
Date	Name	Title	Signature
12/09/2023	Ornella Robutti	Full Professor of Mathematics Education, supervisor	
12/09/2023	Takeshi Miyakawa	Full Professor of Mathematics Education, co-supervisor	
14/09/2023	Ferdinando Arzarello	Professor emeritus of Mathematics Education	
13/09/2023	Carola Manolino	Assistant professor of Mathematics Education	
22/09/2023	Roberto Capone	Assistant professor of Mathematics Education	
20/09/2023	Maria Giuseppina Bartolini Bussi	Professor emeritus of Mathematics Education	
20/09/2023	Silvia Funghi	Assistant professor of Mathematics Education	
22/09/2026	Alessandro Ramploud	Assistant professor of Mathematics Education	
22/09/2023	Maria Giuseppina Adesso	Teacher of Mathematics	

1.4.3 General statement

I certify that, with the above qualification, this dissertation and the research to which it refers are the product of my own work. I declare that, to the best of my knowledge, my dissertation does not infringe upon anyone’s copyright nor violate any proprietary rights and that any ideas, techniques, quotations, or any other material from the work of other people included in my dissertation, published or otherwise, are fully acknowledged in accordance with the standard referencing practices.

Riccardo Minisola

Riccardo Minisola

2 TRANSLATING PRACTICES FOR REFLECTING ON OURSELVES: LESSON STUDY

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Chapter 2 reports the results of the first confrontation of my research with an international context, in which the first steps in adapting the Lesson Study to the Italian context are discussed, and in which a fundamental tool for implementing Lesson Study in Italy, the Lesson Plan, is presented and discussed - from the theoretical perspective of Cultural Transposition. Results show that the task of planning and compiling the Lesson Plan depends on the documentational work that may vary from context to context, for cultural and institutional reasons.

Lesson Study (LS) is a collaborative methodology for teachers' professional development (TPD) rooted in Japan. In LS, a group of teachers and experts collaborates to the detailed planning of a one-hour lesson. The difference between LS and other methodologies is the collaborative foundation of the experience, there is no evaluation on the performance of a single member of the group. We believe that, in the Italian context, LS can be an appropriate tool to answer in an efficient way the Ministry's demands for a "mandatory, permanent and strategic" TPD and for the "establishment of adequate networks for professional collaboration", while maintaining the focus on teachers' needs. Discussing, observing and reflecting on your own and others' practices can help in re-thinking your own professionalism while relating to a community of peers. In the workshop we will work on one of the main tools in LS, the Lesson Plan, and discuss how the encounter of such a foreign tool can lead to self-reflection on one's own practices.

Le Lesson Study (LS) est une méthodologie collaborative pour le développement professionnel des enseignants, enracinée en Japon. Dans la méthodologie LS, un groupe d'enseignants et d'experts collabore à la planification détaillée d'une leçon d'une heure. La différence entre la LS et les autres méthodologies est la base collaborative de l'expérience, il n'y a pas d'évaluation sur la performance d'un seul membre du groupe. Nous croyons que, dans le contexte italien, la LS pourrait être un outil approprié pour répondre de manière efficace aux demandes du Ministère pour une développement professionnel des enseignants "obligatoire, permanente et stratégique" et pour "l'établissement de réseaux adéquats de collaboration professionnelle", tout en maintenant une focalisation sur les besoins des enseignants. Discuter, observer et réfléchir sur ses propres pratiques et sur celles des autres peut aider à repenser sonpropre professionnalisme tout en établissant des relations avec une communauté de pairs. Au cours de cet atelier, nous travaillerons sur l'un des principaux outils du LS, le Lesson Plan, et discuterons de la façon dont la rencontre d'un tel outil étranger peut mener à une réflexion personnelle sur ses propres pratiques.

2.1 Lesson Study

Lesson Study (LS) is a collaborative methodology for teachers' professional development (TPD) rooted in Japan. Since 1999, researchers in TPD and didactics from all over the world have started studying the methodology (Huang & Shimizu, 2016), and since 2003 the Asia-Pacific Economic Cooperation (APEC) has been following its international diffusion. Catherine Lewis, vice-president of WALS, has had an essential role in the world-wide diffusion of LS (Bartolini Bussi & Ramploud, 2018).

In a LS, a group of at least three teachers (which we will call Lesson Study Group; it can include one or more student teachers or university experts) collaborates to the detailed planning of a one-hour lesson, to be taught in one of the teachers' classroom observed by the other teachers, and discussed by the group. The difference between LS and other methodologies is the collaborative foundation of the experience, which leads to the establishment of a sense of diffused responsibility between the members of the group. Moreover, the observation of the lesson indicates LS as a form of action-research. There is no evaluation on the performance of a single member of the group: the focus is on the lesson and the students, not on teachers' individual ability.

The National strategies for Teachers' Professional Development document of the Italian Ministry of Public Education, covering the three-year period 2016 - 2019, stresses the importance of addressing issues such as: teachers' isolation in managing pupils' learning; connecting work and professional development; difficulties in applying in a real-classroom context the didactic innovations proposed by universities. We believe that LS can support the researchers' and teachers' communities in answering the Ministry's demands for a “mandatory, permanent and strategic” TPD and for the “establishment of adequate networks for professional collaboration”, while maintaining the TPD focus on teachers' needs. Discussing, observing and reflecting on your own and others' practices can help in re-thinking your own professionalism while relating to a community of peers. The encounter with others, from this point of view, is one's self-rediscovery (Mellone et al., 2018).

2.2 Italian Lesson Study: ideas from Turin

As shown from the literature review (Fernandez & Yoshida, 2004; Minisola, 2016; Robutti, et al., 2016), LS is generally a three-step cycle aimed at creating a virtuous process in which teachers can grow continuously (Ramploud & Munarini Frenesi, 2015). In its (cultural) transposition to Western cultures - particularly the Italian one - these steps can be defined the "essentials" of LS: establishment of long-term learning goals and lesson planning, implementation and observation of the research lesson, discussion on the lesson. These steps can be repeated, like a life cycle in which each lesson is the foundation for subsequent growth. In the Italian context, the time expected for each step is: at least 2 hours for goals establishing, 2 hours for lesson planning, 1 hour for the lesson, 2 to 4 hours for the discussion. The overall commitment for teachers is predicted in 7 to 10 hours.

Initial findings mandate to clarify that, in LS, a lesson is a specific moment in the classroom routine (i.e. the mathematics lesson in class 3B held from 9 to 10 a.m. on February the 3rd, 2019). The group of lessons dedicated to a specific topic (i.e. continued fractions) is called teaching unit.

The existence of a LS-Group is tied to that of the observed lesson. Even so, the same group of people can participate in more study cycle and establish a stable-over-time community of practice. The aim is to build and institutionalize a collaborative methodology, which can sustain teachers in both their job and professional development, focusing on the new multicultural context we all are living in.

The tentative structure of Lesson Study in the Italian context is:

Definition of long-term educational objectives: LS is a form of action-research, in which teachers collaborate to improve their professionalism in accordance to the context in which they work. The reasons to engage in LS might stem from different teachers’ needs: i.e., difficulties in confronting with certain mathematical topics, improving strategies to involve students, experimenting new didactical methodologies. A research question is formulated by the group in accordance to these needs, and exploring the possible answers is the objective of the LS. Moreover, Italian secondary school teachers have autonomy on defining the educational plan, referring to Indicazioni Nazionali - the national curriculum by Ministry of Education - containing knowledge and competences related to the specific kind of school. Moreover, each school has its Piano Triennale dell’Offerta Formativa or PTOF (“three-year plan of the educational offer”), in which more specific educational objectives are described. Thus, in the first phase of LS in Italy, the teachers choose a teaching unit and the related long-term learning objectives, in accordance to Indicazioni Nazionali and PTOF: these objectives should be relevant to the whole group (i.e. because they are difficult to attain) to promote engagement, and related to the research question(s). One (or two) demonstrating teachers should be chosen, to develop a lesson aimed at a specific context, and to investigate the answer to specific (and yet shared by the group) needs.

Lesson Planning: The demonstrating teacher(s) prepares - on his/her own or working with colleagues - a draft of the Lesson Plan, describing: the class context (such as the general level of knowledge and competencies, or the presence of students with special educational needs: the Italian school is inclusive, meaning that in Italy there are no special schools for students with learning difficulties, physical disabilities or behavioural problems); the teaching unit in which the lesson is inscribed; a proposal for the 1-hour lesson in accordance - as much as possible - to the class' didactic contract. The tentative Lesson Plan is given to the whole LS-Group before the planning meeting, in which the group discusses the details and decides: the phases of the lesson, the time to allocate for each phase, the teacher's requests to students, how the teacher should react to some students' expected reactions, what are the educational aim of each phase, which classroom grouping strategies to apply. The plan is carefully fitted on both the classroom's pupils and the demonstrating teacher's disposition. The group proposes ideas, techniques, strategies, but ultimately it is the demonstrating teachers' choice what to implement and what is not doable in his/her classroom. Observational focuses are established in accordance to the initial aims and to the group's decisions in planning the lesson: i.e. the group might decide to focus on the efficacy of artefacts proposed by the teacher to the class, of the grouping strategies, of the problem structure, etc. The group may decide to use a table to guide the observation using some learning descriptors, and/or to focus on some students considered representative of the classroom situation. In this phase, appointing a secretary to record the discussion, and a moderator to the discussion, is useful in terms of time management.

Lesson implementation and observation: The teacher and the observers enter into the classroom to teach and observe the prepared lesson. A series of preliminary encounters might be necessary to get students accustomed to the presence of other people. The observers are silent and should not influence the class' practice. The presence of all the members of the LS-Group is not necessary, but a video record of the lesson is recommended.

Discussion: LS methodology is focused on the efficacy of the prepared lesson in accordance to the established objectives, not on the ability of the individual teacher.

Before the discussion meeting, the whole LS-Group (including the teacher) has shared and studied the observers’ reports and possible videos. The discussion is opened by the teacher, who shares his/her impressions and observations on what occurred in the classroom. The whole group discusses how to fix what did not work, improve what did, reflect on how to deal with (and consider the possibility of) the unexpected: it is not possible to plan for every instance that may occur. Missing something is a “mistake” of the whole group, even if the teacher in charge of the lesson was the one who had to respond to the unexpected event(s). As different teachers make different kind of expertise available for the group, this is the opportunity for both the demonstrating teacher and the whole group to learn how to manage unexpected situations or improve non-optimal behaviours, absorbing new ideas from others’ experience. This discussion may or may not result in a new “improved” Lesson Plan, which can be taught by the same or another teacher in a different classroom, bringing about a new study cycle. As in the planning phase, choosing a secretary and a moderator is advised.

2.3 Plan for the workshop

The workshop will be organised in the following steps:

(2) A 10 minutes introduction on Lesson Study and the Lesson Plan

(3) The participants will be divided in small groups of 3-4 persons each. The groups should be organised as much as possible according to nationality and school level. Each group will have about 10 minutes to decide, possibly according to the participants’ training needs, which activity they will work on among the proposed ones (differentiated by school level).

(4) 60 minutes will be dedicated to the lesson planning, using the empty Lesson Plan form provided. Each group is asked to keep track of the encountered difficulties, whether they are educational, planning-related, organizational, and also relationship-related difficulties. Since the proposed Lesson Plan structure is adequate to the Italian context, the focus will be especially on those related to the different cultural context the participants are used to.

(5) A 10 minutes meta-discussion on the activity: what are the most striking differences between this planning methodology and those used in the participants' usual contexts? Are there any analogies? Does the encounter with the Lesson Plan, a tool coming from a different cultural and institutional context, bring a reflection on the participants' own practices?

2.4 The Workshop

The workshop lasted 70' and the tentative times were adjusted accordingly: 7' for the introduction, 8' for deciding the teaching material, 45' for lesson planning, 10' for the discussion. 17 people attended the workshop, from a number of nationalities, ages and professional backgrounds. In particular, 7 of them were university students and declared no teaching experience. Audio recording the session was not possible due to GDPR limitations; the presenters noted arguments and comments on a notebook.

During step 1, material on Lesson Study was presented to the attendants. The presentation focused on Lesson Study in relation with the Japanese context, analysed the similarities with the Italian context, and the peculiarities of the latter (i.e.: both contexts focus on the pupils and design educational plans in terms of long-term goals; Japanese teachers mainly work inside schools, Italian ones mainly work at home; Italian school is inclusive, Japanese school is not). A possible adaptation of LS in the Italian context was proposed, and a copy of an empty Lesson Plan (cfr. Appendix) was distributed to each participant. An overview of the proposed structure for the Lesson Plan was discussed, and the participants were proposed the task of the workshop for step 2. The groups resulting by the participants self-organisations were more numerous than expected and heterogeneous, differently from what asked by the presenters. Rearranging them was deemed not necessary. In the case that some of the participants needed support in planning for a lesson, two teaching activities were proposed on different mathematical topics: Heron’s problem and its generalization for high school, exploration on non-planar surfaces for middle school.

At the beginning of step 3, the participants were asked to work and discuss within their groups, and three questions were presented to guide the work: (1) What are the most striking differences between this planning methodology and those used in the participants’ usual contexts? (2) Are there any analogies? (3) Does the encounter with the Lesson Plan, a tool coming from a different cultural and institutional context, bring a reflection on the participants’ own practices? About one third of the participants continued working on the teaching activities; after other 10’ the presenter decided to ask to concentrate on the provided Lesson Plan, as it was the focus of the workshop.

By the beginning of the whole-group discussion (step 4), no one had compiled a Lesson Plan: when asked for a reason, a participant commented “analysing and discussing the Lesson Plan is more interesting, as we didn’t have enough time to study the material”, at which the others agreed. Three participants expressed curiosity about the structure of the lesson proposed in the Lesson Plan, and the presenters explained that it was inspired to Calvani (2014) and Bartolini Bussi et al. (2017).

Participants from France and Spain noted that the philosophy of the Lesson Plan is not far from the work teachers usually do in their context. In Spain, for example, “our teachers do this kind of work for each lesson, and lesson plans are uploaded online for families and others to be consulted”; others from Holland and Switzerland explained that detailed lesson planning is usually mandatory for pre-service teachers during their training but not for in-service ones, and it is usually kept as internal documentation for the school. Two participants from Poland commented that “[detailed lesson planning] is unusual for both prospective and in-service [teachers], especially with all the details proposed in this Lesson Plan”. Spain followed that “estimating the time for each phase, we don’t do that” and suggested a modification in the table to highlight this characteristic of the proposed Lesson Plan.

All the participants agreed that specifying the “educational intentionality” for each phase was “the real peculiarity of this document”. When asked for clarification, one young participant from Poland explained: “I believe that this is very important when doing pre-service training. It makes you aware that everything you do in classroom is relevant for your students”. A participant from Belgium agreed: “it is something you tend to forget even when you are an experienced teacher, so it would be nice to have a reminder every now and then. It makes you aware”. The participant from France concluded: “detailed lesson planning is very difficult no matter the subject, but I believe that in Mathematics it is especially important: Mathematics gives students critical thinking, develops their cognitive abilities, is the basis for their scientific approach. We all need to be very careful when we teach it, we are shaping the future… and we need all the help we can get”.

2.5 Discussion

Designing a detailed lesson plan is no easy task, as proven by the fact that no participant produced a complete Lesson Plan, not even those accustomed to detailed documentational work. Furthermore, the task of planning and compiling the plan for a lesson depends on the documentational work that may vary from context to context, for cultural and institutional reasons.

The proposed Lesson Plan is the result of the reflection on the Italian culture, institutional context, usual practices. It embeds Italian institutional peculiarities, such as the different stance on educational objectives: namely, long-term objectives are presented in the National Guidelines, whereas lesson objectives are decided by each teacher individually. A Lesson Plan fitted to a certain context is not immediately effective: its conscious use requires familiarity with the mathematical knowledge, curriculum, teaching traditions, institutional context.

In this sense, we might say that the workshop evolved unexpectedly. The focus shifted from the analysis of Lesson Plan’s specific steps to its general issue as a design tool in a school. This new focus nourished the participants’ confrontation. The self-organized heterogeneity enriched the discussion, albeit some realism got lost in the transition (namely, no Lesson Plan was compiled), and answered to the needs of the participants.

In conclusion, preparing and studying a detailed plan for an effective Mathematics lesson is perceived both like a challenge and a necessity. Discussing and sharing educational experiences, provided they happen within customary practices of designing and programming, might sustain to collaboratively overcome the perceived challenges of teachers' professionalism. To achieve this, it is necessary that researchers in mathematics education deepen their studies of interaction with teachers in order to improve the collaboration with them in concrete teaching activities.

2.6 Appendix

In this appendix, the empty Lesson Plan used in the workshop is presented. Please note that this is a condensed version. The printed version provides enough space to write. This version of the Lesson Plan is inspired by Bartolini Bussi et al. (2017) and Fernandez & Yoshida (2004).

Lesson Plan

School:
Class:

Description of the class
(classroom composition;prevalent teaching methodologies)

Context
(learning trajectory in which Lesson Study is held):

Educational goals on competences
(in accordance to the National Programme)

Specific learning goals
(in accordance to the National Programme)

Initial situation of the class
(with respect to mathematical competences: goals already achieved, pre-requisites for the lesson)

Organization of the didactic work
(the total duration of the project; the place where it is carried out)

Organization of the teaching unit:
LESSON 1:
LESSON 2:
LESSON …N:

Methods of evaluation
(how it is intended to evaluate the degree of achievement of the goals, how it is intended to evaluate the functioning of the activity on the class group)

RESEARCH LESSON PLAN

Topic/content of the lesson in question
(title)

Goal(s) of the lesson
(Learning goals based on National Programme; punctual and minimum goal of 1h of lesson)

What is the purpose of the observation?
(observational focus)

Presentation of the lesson (mathematics)

Description of the activity	Task and/or teacher's questions	Student reactions and directions for the teacher	Grouping	Time table	Educational Intentionality (the reasons for the choices)
Introduction to the lesson and presentation of the topic	*(summary to the class, by the teacher, of the activities already carried out and specific of the topic of the day)*		❑ Whole class ❑ Small group ❑ Pairs ❑ Individually *(where appropriate, a list of the groups and their reasons for doing so)*		
Homework check (optional)					
Formulation/presentation of the problem of the day					*(Explanations of the problematics to be highlighted)*
Presentation / clarification of the problem of the day					

Working on the problem

Description of the activity	Task and/or teacher's questions	Student reactions and directions for the teacher	Grouping	Time table	Educational Intentionality (the reasons for the choices)
Working on the sub-problem (optional)	*(if the task is complex, it is possible, intentionally, to break down the problem into simpler units)*		❑ Whole class ❑ Small group ❑ Pairs ❑ Individually *(where appropriate, a list of the groups and their reasons)*		
Working on the problem	*(criteria for group composition; methodologies)*				*(reasons for the choice of activities, materials and methods)*

Discussion

Description of the activity	Task and/or teacher's questions	Student reactions and directions for the teacher	Grouping	Time table	Educational Intentionality (the reasons for the choices)
Presentation of the work done by the students	*(how to structure the student's restitution according to activity)*		❑ Whole class ❑ Small group ❑ Pairs ❑ Individually *(where appropriate, a list of the groups and their reasons)*		
Discussion of the various resolution methods					
Exercise (optional)					

Conclusions

Description of the activity	Task and/or teacher's questions	Student reactions and directions for the teacher	Grouping	Time table	Educational Intentionality (the reasons for the choices)
Summary and underlining by the teacher of the main point of the lesson	*(specification of the content and the way in which it is summarised)*		❑ Whole class ❑ Small group ❑ Pairs ❑ Individually *(where appropriate, a list of the groups and their reasons)*		*(justification of the choices made in relation to its own aims and national indications)*
Homework assignment (optional)					
Anticipation of the next topic (optional)					

Analysis of methodologies and materials used
(which ones; organized in what way; motivation of the choice - what mathematical meaning can be approached with this tool or method; what types of activities are possible with this tool or method; what are the characteristics that make it functional to achieve the aim of the lesson)

3 TEACHERS' PROFESSIONAL DEVELOPMENT: A CULTURAL MATTER. HOW TO DESCRIBE CULTURAL CONTEXTS?

Minisola, R., & Manolino, C. (2022). Teachers' professional development: A cultural matter. How to describe cultural contexts? In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of European Research in Mathematics Education (CERME12)* (pp. 3650–3657). Free University of Bozen-Bolzano, Italy and ERME. https://hal.archives-ouvertes.fr/hal-03748740/

Chapter 3 describes (through a synthetic and analytical two-pronged approach) the Japanese and Italian institutional system related to the world of school, both from the point of view of school organisation and the teacher education system, with the aim of informing and guiding studies on the introduction of Lesson Study in Italy. Results show that the two approaches are complementary, and arguments supporting the importance of describing the cultural context in which research is carried out are provided.

Teachers' professional development in collaborative contexts is a growing trend in Mathematics Education research. Particularly, Japanese Lesson Study has seen a great focus on its dissemination around the world. Research shows that Japanese culture is one of the main reasons that makes Lesson Study effective: understanding Lesson Study means understanding the cultural context in which it originated. We attempt to describe the Japanese and the Italian cultural contexts. Since there exist no consensus on what is essential to analyse in order to understand a cultural context, we present two approaches to this description, and consider some advantages and shortcomings. We hope to sprout discussion on the possibility to create guidelines for describing cultural contexts, shared by the community of researchers in Mathematics Education: awareness of beliefs, identity and practice is a sensitive element for successful mathematics teacher professional development.

Keywords: Cultural context, Culture, Japan, Lesson Study, Italy.

3.1 Introduction

Over the last 20 years, teachers' professional development (TPD) in collaborative contexts has received ever-growing attention from the community of researchers in Mathematics Education (Robutti *et al*., 2016), and a recent survey by Bakker and colleagues (2021) confirmed the trend. Among the many different collaborative methodologies for TPD in Mathematics and Mathematics Education, Lesson Study has seen a great deal of research focused on its dissemination (i.e.: Huang *et al*., 2019; Huang & Shimizu, 2016; Quaresma *et al*., 2018). Lesson Study (LS) is a collaborative TPD methodology, part of the Japanese paradidactic infrastructure (Winsløw, 2011) since the 1880s, focused on the co-responsibility in the lesson-planning process of the involved teachers and knowledgeable others (Huang *et al*., 2019).

LS is also the focus of the authors' doctoral dissertations (in progress), which also aim at introducing LS in the context of Italian TPD. During YESS11, TWG1 dedicated to *teacher education and professional development* saw four out of eleven papers focused on LS in different contexts (Italy, Mozambique, Portugal, and Switzerland). In the discussions around the four papers, one question resulted relevant: what is the *cultural context* in which the research takes place? Indeed, being aware of their cultural context is one of the essential competences of mathematics teachers (and researchers) to gain awareness of their beliefs, their identity and their professional practice and to develop their teaching knowledge: a sensitive element for successful mathematics teacher professional development, and a demand evermore necessary and therefore not negligible (Andrews, 2010). Yet, issues arise when we try to address such demand: in the following, we will attempt to describe the Japanese and Italian cultural and institutional context and discuss such issues.

3.2 Literature review

Unsuccessful attempts at translating LS outside of its cultural context (Demir *et al*., 2012; Fernandez *et al*., 2003) suggest that, if LS is not introduced in a cultural context with proper consideration to the differences with the Japanese cultural context, it might be rejected by the institutions. Ebaeguin & Stephens (2014) suggest to address the cultural compatibility of LS. A number of scholars proposed different theoretical lenses to analyse why LS is so widespread in Japan (i.e.: Krainer, 2011; C. Lewis, 2016). We can suppose that LS exists because of the Japanese culture, and Japanese culture is one of the main reasons why LS is effective: LS is a cultural activity (Stigler & Hiebert, 2016). The question arises if maintaining the efficacy of LS across different cultural contexts is feasible.

Despite the rising awareness on the importance of studying cultural contexts and identities to contextualize global trends in Mathematics Education (Bakker *et al*., 2021), the majority of reports on LS around the world seems to depict LS as an isolated practice in the Japanese panorama of TPD practices (Miyakawa & Winsløw, 2019) and seemingly ignores that the Japanese definition of LS is not as clear cut as the American one (Miyakawa & Winsløw, 2013). This suggests that "to develop a deeper understanding of Lesson Study in a post-modern global world, there is a need to seek views beyond those presented from an American perspective" (White & Lim, 2008, p. 915).

Understanding LS means understanding the context in which it originated. At the same time, to introduce LS in a new context, it is essential to know the TPD practices already in existence (Miyakawa & Winsløw, 2013). Yet, there exist no consensus on what is essential to analyse in order to understand a cultural context. What is *culture*? This paper has two aims: to provide arguments to the importance of understanding the cultural contexts involved in the research, and to provide a currently-missing description of the Italian TPD context in the English language. We provide a tentative analysis of the Japanese and the Italian cultural and institutional contexts to guide future studies on LS in Italy, and we also hope to sprout discussion on the possibility to create guidelines for describing cultural contexts that might be shared by the community of researchers in Mathematics Education.

3.3 Theoretical Framework

Culture may be described as "any aspect of the ideas, communications, or behaviours of a group of people which give them a distinctive identity and which is used to organise their internal sense of cohesion and membership" (Scollon & Scollon, 1995, p. 127) or as "[t]he system of shared beliefs, values, customs, behaviours, and artefacts that the members of society use to cope with their world and with one another, and that are transmitted […] through learning” (Bates & Plog, in Freimuth, 2006, p. 2). Anthropologists have not reached a shared definition (Spencer-Oatey, 2012), and proposing one would be outside of our expertise. In fact, our aim is not to propose our own definition but to observe how existing approaches and definitions may interact with learning and teaching processes in Mathematics, particularly in TPD. It is a facet of our doctoral research, especially within a semiotic context (Manolino, 2021b). Here we rely on a *popular* understanding of what *culture* is, as the definition is not central to this paper. In the following, we propose two different approaches to the definition and description of *culture* and *cultural contexts*: the first one is synthetic, the second one is descriptive. We hope to show some advantages and shortcomings of each of them, which should provide a mean to engage in this discussion.

3.3.1 Hofstede’s Dimensions of Culture

Hofstede defines culture as “the collective mental programming of the people in an environment. Culture is not a characteristic of individuals; it encompasses a number of people who were conditioned by the same education and life experience” (in de Mooij, 2010, p. 48). Hofstede identifies basic value orientations of a certain national culture. These values “are broad preferences for a certain state of affairs (e.g., preferring equality over hierarchy) […] transmitted by the environment […] shaped by the time we hit 10-12 years of age” (https://news.hofstede-insights.com/news/what-do-we-mean-by-culture described in five dimensions, scored over 100 points, and represent:

- *Power Distance* Index “the extent to which less powerful members of a society accept and expect that power is distributed unequally” (de Mooij, 2010, p. 75). The higher the score, the more hierarchical a society is.

- *Individualism* vs Collectivism "[…] people looking after themselves and their immediate family only, versus people belonging to in-groups that look after them in exchange for loyalty" (de Mooij, 2010, p. 77). Higher scores indicate individualistic values.
- *Masculinity* vs Femininity "The dominant values in a masculine society are achievement and success; the dominant values in a feminine society are caring for others and quality of life" (de Mooij, 2010, p. 79). Lower score indicates a feminine society. Please note that this label is problematic as it reinforces harmful gender stereotypes, and in the following we will use the alternative "*Tough* vs Tender".
- *Uncertaintly Avoidance* Index "[…] the extent to which people feel threatened by uncertainty and ambiguity and try to avoid these situations" (de Mooij, 2010, p. 82). The higher the score, the less open to changes is a society.
- *Long*-*Term* vs Short-Term Orientation "[…] the extent to which a society exhibits pragmatic future-oriented perspective rather than a conventional historic or short-term point of view" (de Mooij, 2010, p. 85). Lower scores point to a society that prefers short-term planning.

One peculiarity of Hofstede's dimensions is that "[t]he country scores on the dimensions are relative, in that we are all human and simultaneously we are all unique. In other words, culture can only be used meaningfully by comparison" (https://hi.hofstede-insights.com/national-culture).

3.3.2 Levels of Co-Determination

We start from "the notion of [teaching] "practice" as a link between culture […] and the larger cultural contexts" (Hatano & Inagaki, 1998, p. 80). In the Anthropological Theory of the Didactic, practices are described in terms of praxeologies: the know-how (*praxis*) and the know-why (*logos* - the discourses that justify the know-how) related to a task. Chevallard (1985) suggests that teachers' praxeologies are shaped by a plurality of agents (politicians, scholars…) and historical or institutional conditions that defines the boundaries of what teachers can or cannot do, their *noosphere* (the sphere of those who thinks). Chevallard (2002) pictures the complex relations of the factors influencing teachers' praxeologies, which are influenced not only by the teachers' decision, but at a higher level by the society in which the teachers and students are immersed, as shown in Figure 3.1.

Upper levels
Humanity
↓↑
Civilisation
↓↑
Society
↓↑
School
↓↑
Pedagogy
↓↑

Lower levels
↓↑
Discipline
↓↑
Domain
↓↑
Sector
↓↑
Theme
↓↑
Question

Figure 3.1 Scale of levels of co-determination (Florensa et al., 2018, p. 5)

Similarly, we suggest that professional development practices are influences by the context in which teachers and their educators are immersed. It is important to notice that Chevallard's framework does not use the term *culture*.

3.4 Context Analysis

3.4.1 Hofstede's Cultural Dimensions

Ebaeguin and Stephens (2014) suggest that comparing Hofstede's scores might be a starting point for studying the introduction of LS in Australia, as they connect the efficacy of LS in Japan to the Japanese scores. The scores for Japan and Italy according to Hofstede's cultural dimensions can be freely collected from the website https://www.hofstede-insights.com/product/compare-countries/ and are shown in Table 3.1. The labels are simplified because of space constraints.

Table 3.1 Hofstede's score for cultural dimension for Italy and Japan

	Power Distance	Individualism	Tough	Uncertainty Avoidance	Long-Term
Italy	50	76	70	75	61
Japan	54	46	95	92	88

Within Hofstede's description, the two cultures appear different in almost all categories: according to these descriptors, Japanese culture appear less individualistic, more prone to success (and therefore more competitive), less open to "thinking outside of the box", and keener to long-term planning. There are some similarities, as Japanese and Italian cultures seem to have a shared approach to hierarchy.

3.4.2 Levels of Co-Determination

Using Chevallard's didactic transposition lens and moving within the co-determination levels, we attempt to provide a description of the Japanese and Italian institutional contexts. Since many sources are available on the Japanese context, the description will be briefer. The description of the Italian context, on the contrary, will be as detailed as the format allows, inspired to that proposed for Japan by Miyakawa and Winsløw (2019).

Japan is an East-Asian country, influenced by countries of “Confucian Heritage Culture” (Mason, 2014) such as China and Korea. These countries generally share some cultural values that are reflected in the school system, and can be considered part of the Japanese system of school-related beliefs:

> a high regard for education […]; […] the cultivation of the self; a strong work ethic […]; a belief […] that success depends more on effort than on innate capacity […]; respect for teachers […] (Mason, 2014, p. 2).

In Japan, the national curriculum is detailed and rigid. Textbooks are essential for lesson planning. Long-term planning is centralized at the prefectural or school level, so teachers’ attention is focused on learning units and lessons. Classes are homogeneous by level: strict entrance tests are usually required for accessing high schools, while students with disabilities attend special schools. Japanese teachers spend the working day at school, where they have their personal workspace in a room shared with the whole teaching staff: in this space, they prepare lessons and discuss with their colleagues. In-service professional development is compulsory and takes place during working hours (Bartolini Bussi & Ramploud, 2018). LS is only part of Japanese TPD activities, which have many common features: in particular, the practice of open classes with observers is common (Miyakawa & Winsløw, 2013). Participating in optional TPD activities increases teachers’ chances of career advancement (Miyakawa & Winsløw, 2019).

The Italian school system is centralized. Recent reforms (2010 and 2012 respectively for secondary and primary school) stressed the importance of inclusiveness (law 133 and 169/2008). Italian school is structured around the concept of equity, and special schools do not exist: all students are given the same opportunities to reach the same goal, plus aids if needed. The Ministry of Education provides the *Indicazioni Nazionali* (National guidelines), which contain contents and aims for each subject, and its number of hours in a year. These contents are not prescriptive, but at the end of the 8th and 13th grades there are two national exams. Each teacher has the responsibility of the didactical plan for their classes, also according to the *Piano Triennale dell'Offerta Formativa* (Three-year Educational Plan - describing the cultural-pedagogical inspiration and the curricular, extracurricular, didactic and organisational design of the proposed activities). The contents of this document are specific of each school and decided by the collegiality of teachers and school staff.

Freedom of teaching, understood as professional autonomy in carrying out teaching activities and free cultural expression of the teacher, is guaranteed as a constitutional right: Article 33 of the Constitution states "Art and science are free and free is their teaching". Institutionally, the duration of the lesson is 60 minutes. The teacher can have up to three consecutive lessons in the same class, without interruptions. During the lesson, the teacher is usually the only adult figure in the class. The Italian teacher works at school from one to six hours a day, dedicated to classroom lessons. The planning of individual lessons is not part of the working hours, nor there are places in the school dedicated to this activity: the teacher's paradidactic activity takes place in personal and private time and space. There are no compulsory contents or practices for TPD, they are chosen by teachers according to their own needs. In-service TPD is compulsory (law 107/2015), there is no minimum number of hours per year, and must be carried out outside working hours. Teachers' career advancement is based exclusively on seniority, although some economic incentives are given to those that take relevant roles in the school organization (Blandino, 2008; Capperucci, 2008).

On paper, teachers have numerous occasions for improving their professionalism. The Ministry[37] attests more than 500 agencies offering TPD opportunities. Universities, academic associations, teachers' associations, and educational companies which fulfil quality standards defined by the Ministry, are registered in a national database and can publish their TPD proposals on a digital platform (S.O.F.I.A.). The in-service professional development "system" is conceived as a "lifelong learning environment" for teachers and is intended as a "network of opportunities for professional growth and development for teachers" (law 107/2015). At national level, proposals come from the national education centre, academic associations, teachers' associations, educational companies. At regional level, regional school offices intervene by supporting, managing, and publicising the proposals. At local level, experienced teachers also offer courses in their school, sometime opens to teachers in the surrounding area. No official account is given on how many teachers participate in TPD. Yet, the impression is that this vastity of opportunities does not correspond to a high-quality offer: the Ministry states that the quality of TPD programmes is compromised by the general "low quality of models and methodologies" (law 107/2015) suggesting that teachers might be easily lost and caught in low quality programmes. The Ministry does not provide guidance to orientate in this labyrinth.

3.5 Some Reflections

At the end of this section, we can ask ourselves: is this description complete? Did we miss any essential point? Did we provide too much information, and made our description useless? Is this description reliable? Hofstede's synthetic data, for example,

[37] https://www.miur.gov.it/accreditamento-enti-e-qualificazione-associazioni

provides a quick overlook on the differences between two cultures and invite to carefully consider LS introduction in Italy. Yet, this description is problematic as it eliminates complexity, and the risk of overgeneralizing is high. Furthermore, these scores are open to interpretation. The similar scores for PDI might suggest that Italy and Japan have a shared view on hierarchy, but we propose a different interpretation: while Japanese invites consciousness of one's hierarchical position in any social setting and act accordingly (Ebaeguin & Stephens, 2014), Italian culture dislike control and formal supervision. An analysis based on Hofstede's cultural dimensions, in support of our assertions, can be found in Giordanengo's Master Degree dissertation (2020)[38].

3.6 Discussion

It emerges that the Japanese and the Italian cultural and institutional contexts share some similarities and come with a number of differences. Hofstede's cultural dimensions show divergent basic values. However, to consider Hofstede's analysis *complete enough* to understand the similarities and differences between Japan and Italy would be preposterous, and similar numbers can still be interpreted with very different founding values. A detailed look suggests similarities between the two educational contexts, yet striking differences in the institutional and paradidactic organization. We believe this is sufficient to justify the importance of a cultural approach when practices from a cultural context are brought in different contexts. A tentative description of the Italian institutional (school) context was provided as a reference for future studies.

[38] https://sia.unito.it/studenti/intesi/Ricerca_tesi_libera/ricerca_tesi_dettaglio.asp?id_upload=192959&cdl_tesi=&cdl=&matricola=781420

Italian researchers are invited to amend this description, which is certainly lacking details.

This paper answers no research questions, but considering our attempt to respond to the need to provide a description of the Italian institutional context in which students, teachers, and researchers work every day, leaves us with some questions. Are we satisfied with the result? No, we are not: the description misses many details, and we are not sure that what we provided is enough to really understand the context. How is Mathematics as a school subject considered at a cultural level? How is the teacher role considered in each society? Many questions are left unanswered, yet we often hear from reviewers that we should focus on describing just some aspects. How detailed can these descriptions be? Too little or too many information will lead to the same result: little understanding of the cultural context. Can we really achieve this *correct* kind of detail? Again, the answer, in our opinion, is no. The gap is embedded in the notion of cultural context and in any possible analysis of it. A number of scholars (e.g., Lotman, 1990) have declared the impossibility of a full knowledge of culture, as we are embodied in it and in what Jullien calls the *unthoughts* (for a broader understanding, see Mellone *et al.*, 2019). To be aware of these unthoughts may not be enough anymore. What is incumbent on us is to frame our research accordingly, as to provide careful attention to their influence on teaching and learning processes in mathematics.

4 NETWORKING OF THEORIES FOR A MULTIFACETED UNDERSTANDING ON LESSON STUDY IN THE ITALIAN CONTEXT

Capone, R., Manolino, C., & Minisola, R. (2020). Networking of theories for a multifaceted understanding on Lesson Study in the Italian context. In H. Borko & D. Potari (Eds.), *The Twenty- Fifth ICMI Study: Teachers of Mathematics Working and Learning in Collaborative Groups. Conference Proceedings* (pp. 102–109). ICMI.

Chapter 4 proposes a theoretical approach informed by the Networking of Theories for the study of Lesson Study as an object of research. The introduction of Lesson Study in the Italian context is then studied through the theoretical frameworks of the Boundary Object, the Semiosphere and Semiotic Mediation. Results show that the networking of theories for the analysis of the three experiments on different school levels provided a novel understanding on Lesson Study.

Exploring an Italian Approach to Lesson Study from a Cultural Perspective for an Improvement in Mathematics Teacher Education

This paper explores the possibilities offered by combining three theoretical frameworks to observe some experiments in different contexts, in the field of mathematics teachers' professional development, based on the Lesson Study methodology and framed into Cultural Transposition, through the Networking of Theories lens. The three researches share a broader goal: studying what might happen when a "foreign object", such as Lesson Study, is introduced into teachers' practices. The specific goal is to explore how dissimilar theoretical frameworks can highlight different aspects related to the cultural transposition of Lesson Study into the Italian context, and how this cultural transposition can improve, modify or strengthen teachers' practices. Does the confrontation with a foreign culture shed light on which teachers' practices and beliefs are more stable, and which are more malleable and subject to change? Findings provide a positive answer to the applicability of LS in the Italian cultural and didactic context.

Lesson Study, in the last years, has been gaining increased attention in the teachers' professional development research field (Bartolini Bussi & Ramploud, 2018), although in Italy it is not very widespread. This research is part of three research projects in Mathematics Education, two at the Department of Mathematics of the University of Turin and one at the University of Salerno. These studies are rooted in a consolidated Italian culture for a meaningful, long-life teachers' professional development, attentive to the cultural and institutional context. The Cultural Transposition framework (Mellone et al., 2019) stresses the need for a careful approach to the confrontation between practices situated in different cultural contexts. This encounter can fuel a reflection on the reasons behind teaching practices, thus fostering the growth of teachers' professionalism. In this paper we take on this challenge, reflecting on three experiments each with its own peculiarities: they have been conducted with prospective and practicing teachers (both in primary and secondary schools) and they assume three theoretical frameworks (Semiosphere and Semiotic of

Cultures for primary school teachers, Semiotic Mediation for high school mathematics teachers, Boundary Objects for prospective mathematics teachers). We will show how the Networking of these theories enrich the discussion on the seemingly common findings.

4.1 What is Lesson Study?

Lesson Study (LS) is a collaborative methodology for teachers’ professional development rooted in the Confucian Heritage Culture. LS is a three-steps cycle: establishment of long-term learning goals and lesson planning, implementation and observation of a research lesson, discussion on the lesson. These steps can be repeated, like a life cycle in which each lesson is the foundation for new growth. In a LS, a group of at least three practicing teachers and in case some university experts and prospective teachers, collaborate to the detailed planning of a one-hour lesson, which will be taught by one of the practicing teachers in his/her classrooms observed by the others, and discussed by the group. On the one side, LS is a culturally situated methodology and it may not be invariant by translation (in the mathematical sense) between cultural contexts. On the other side, Cultural Transposition proposes “the decentralization of the didactic [and, in our case, teachers’ professional development] practice of a specific cultural context through contact with the didactic practices of different cultural contexts” as a way to bring forward in teachers and researchers the implicit assumptions in which practices are rooted, eventually revisiting them through an enriched point of view (Mellone et al., 2019).

4.2 Italian institutional context and Lesson Study

In the Italian context, teachers' professional development is defined as compulsory, permanent and strategic by the Ministry of Education (law 107/2015), and it is recognised as an opportunity for effective professional growth. The widespread feeling of professional isolation on the part of the teaching community, whose work is becoming increasingly complex from the scientific, humanistic and social points of view, is the main cause for the law to highlight the promotion of collaboration between teachers as a key principle, and encourages professional development in collaboration at the level of the individual school, and at territorial, national and international level. The preparation of quality teachers, as a key mediator of student performances, is not exclusive to the Italian context. It has gained increasing attention in recent years at the international level (OECD, 2009, 2012); in Europe, it is the cornerstone of the Europe 2020 development strategy. Italy already has many teams in which the culture for a meaningful, long-life collaboration in teachers' professional development is deeply rooted: for mathematics, a never-ending tradition started in the 60s with the *Nuclei di Ricerca Didattica* (Arzarello & Bartolini Bussi, 1998), grew with the *Matematica per il Cittadino* project (MIUR, 2001; 2003; 2004) and currently goes on with many local and national projects coordinated by the Ministry of Education (e.g. m@t.abel, Piano nazionale Lauree Scientifiche). LS can be a further support in the struggle to respond to the demands of the institutions, and another support to the research community in the ongoing development of a culture for collaboration between mathematics teachers.

4.3 The three experiments

In the following, we will describe the three experiments designed to understand how to promote, design, and assess relevant collaborative professional development practices for mathematics teachers, each with teachers of different school levels and with different theoretical frameworks.

The first LS experiment is set in a primary school in Piossasco, near Turin. The working group consists in a retired teacher-researcher, four teachers of four different classes, and a researcher. Three complete LS cycles are carried out by three teachers in their 1st-grade classes. The theme of the lesson is the introduction of the ‘plus’ sign and its institutionalization. The goal on children is to understand the concept of addition as the sum of two quantities in its epistemological meaning of putting things together, and relate it to the sign of mathematical language. Later on, another cycle is carried out by the fourth teacher in the 3rd-grade class, in which the activity designed for this LS is part of the educational path that includes the knowledge of weight measurements and the study of state transitions, through experiments with water. The aim is to accompany students in reinvesting their mathematical knowledge and argumentation skills to the transversality of the disciplines. Each teacher implemented the lesson in his or her class. The experiment is observed through a semiotic lens, adapted from the Semiosphere (Lotman, 1990), looking at a space in which we can observe the dynamics that develop among teachers in the design, implementation of the lesson and in the *a posteriori* discussion. A semiotic lens is used to look for "a truth already present [in the teachers, and] that [only] waited to be recognized" (Sedda, 2006). According to Greimas (Ricoeur & Greimas, 2000), the semiotic point of view provides the scientific knowledge that enables us to investigate that spectrum of knowledge "always known" (and in contrast to the spectrum of "I never thought about it"), but not explicit. In particular, through the introduction of a "foreign object" (the LS methodology) in the usual practices and beliefs of Italian teachers, a process of *deconstruction* is carried out (Bosch & Gascon, 2006, p. 53; Mellone et al., 2019) which "exudes" from the noosphere (Chevallard, 2002, p. 9), influencing the levels of didactic co-determination (Chevallard, 2002, p. 10) and the reflection by teachers. The Semiosphere is in itself a research lens based on collaboration, interaction, in fact it is alleged that no semiotic system can culturally function in isolation.

The second experiment describes a LS trial conducted in Salerno, which involves four teachers from a scientific-oriented high school in Avellino and three researchers from the Department of Mathematics of the University of Salerno. The trial is connected to the well-established tradition of designing Learning Units and carrying out activities within the project Liceo Matematico (Capone et al., 2017): groups of teachers are systematically in contact through meetings with university researchers, to implement collective planning of *ex ante* educational activities and *ex post* analysis of processes. For this LS experiment, the theme "tessellations" is chosen for the learning unit "the art of geometry", connecting with natural sciences and art. Semiotic Mediation (Bartolini Bussi & Mariotti, 2008) is the framework that characterizes the experiment. Two artefacts are used. The first one, used in the Engage phase in Inquiry mode, uses as traces: a sentence, two images and a technological tool (smartphone). The situated texts produced by the students are transcribed on appropriately-made observation sheets. In the final part of the sheets, students are asked to formalize their observations in a mathematical text: each group will therefore provide its own "definition of tessellation". The second artefact is used to solve the real problem: in this case, the traces are cardboard polygons, while no technological tools are used. Once again, the students will write their texts on an observation sheet. The mathematical knowledge is expressed through oral communication. All the implemented activities are socio-semiotic, both because they arise from the sharing between teachers and because they are designed taking into account the Vygotsky perspective of knowledge as a shared experience.

The third experiment involves 29 prospective teachers at the Department of Mathematics of the University of Turin. The aim is to find out the reproducible components useful to implement LS with practicing teachers in the Italian context. The prospective teachers have no previous teaching experience. They worked in small groups: each group is required to create a different activity on continued fractions, from which to draw up a Lesson Plan for a 20-minutes lesson. The lesson is to be performed in front of the researchers and the other prospective teachers, and subsequently discussed within the group. We can stress some differences with usual LS contexts: first, LS is usually performed inside schools and participants have some teaching experience; second, lessons usually last one curricular hour; last, LS is a non-evaluative methodology. As LS is a new methodology for both prospective teachers and researchers, the Boundary Object and Boundary Crossing framework is used to analyse how the two communities act to cope with the novelty, and how LS (the Boundary Object) evolves as a consequence (Star, 2010; Akkerman & Bakker, 2011).

4.4 Theoretical Framework

We use part of the Networking of Theories framework (Prediger et al., 2008) as an appropriate (meta)language that makes possible the reference to new conceptual entities connecting our frameworks. In particular, we refer to Radford (2008). He describes a theory as a way of producing understanding and ways of acting based on: *a system P* of basic principles (not a set, for which there is a strong relationship between many of its elements), which includes implicit views and explicit statements outlining the frontier of what will be the universe of the discourse and the research perspective adopted; *a methodology M*, which includes techniques for data collection and data interpretation supported by P, that is, both a theoretical characterization and the very functioning of the methodology (Bernstein, 2000) [the minimum requirements for M are operability and consistency with respect to P.]; *a set Q* of paradigmatic research questions. Table 4.1 describes the Principles, Methodologies and Research Questions for our frameworks:

Table 4.1 The three theoretical frameworks

	Semiosphere	**Semiotic Mediation**	**Boundary Objects**
Principles	Semiosphere: a multi-cultural dynamic space, interwoven with flows of text, processes of elaboration and understanding of meanings generated by individuals as they interact and know each other. Outside of it there can be neither communication nor language. It is the result and the condition for the development of culture (Lotman, 1990). Its characteristic elements are: heterogeneity and dynamism (it is linked to natural or human aspects of social relations, homogeneity is an	At the centre of semiotic mediation is an artefact that embeds mathematical meanings, but is not transparent to embedded meanings. Students, interacting with the mediator, will leave "traces" of their activities (through situated signs). These traces constitute the Semiotic Bundle (Arzarello et al., 2009), that is the dynamic system of signs of various nature (e.g. gestures and words) and of their relationships (e.g. the contemporaneity of a gesture and a word) produced by one or more subjects who interact during the execution of a task.	Boundary as a sociocultural difference leading to discontinuity in action or interaction between communities. Continuity and discontinuity, in the sense that within discontinuity two or more sites are relevant to one another in a particular way. When different communities share a goal, they negotiate a platform at the boundary that allows permeation of practices and preservation of the identity of each community. This tension is harbinger of new knowledge (Akkerman & Bakker, 2011).

	anomalous instance); boundary as one of the main mechanisms of semiotic individuation, a porous membrane that marks the passage between "me and the other"; translation as the primary mechanism of dialogue and knowledge due to the generating asymmetry, characteristic of the space of the Semiosphere.	Learning as a teacher mediated social activity. Roots in the vygotskian cultural approach favouring social knowledge.	Boundary Objects are dynamic objects residing at the boundary, ill-structured with the potential of creating a bridge between the different communities, robust enough to maintain their identity when they become tailored to local use (Star, 2010).
Methodology	Comparison and analysis of texts, considering all the productions, both of the teachers and of the students, as texts. Context analysis. A posteriori analysis of texts, both verbal and written, produced collectively or individually (transcriptions of audio and video, protocols).	Context analysis. A priori analysis of the semiotic potential of the artefact. A posteriori analysis of texts, both verbal and written, produced collectively or individually (transcriptions of audio and video, protocols).	Introduction of the Boundary Object in a community. Analysis of group dynamics and documentation to investigate how they evolve in the interaction with the Boundary Object. Analysis of the evolution of the Boundary Object when communities act on it.
Research Questions	1. How does Cultural Transposition interact with teachers' beliefs and educational practices? 2. Which specific methodological elements, encountered in the experiments in the light of Cultural Transposition, are highlighted by the different theories and their Networking? 3. Which methodological components of LS are relevant to question 2, with respect to the evolution analysed in question 1?		

A networking of different theories can be seen as a set of connections involving at least two theories. A connection depends on at least two parameters: the structure of the theories involved in the connection; the purpose of the connection. In the framework of Prediger, Bikner and Arzarello (2008) the panorama of networking, seen as a dialogue between theory and cultures in multi-theoretical research (Bikner-Ahsbahs & Vohns, 2019), is painted by strategies. Since our intention is not to unify theories, but rather to make them communicate with each other, we focus on the goals of the networking strategies: in 'comparing', it is to discover similarities and differences; in 'contrasting' to highlight differences; in 'coordinating', elements from different theories are chosen and put together to investigate a given research problem. In 'combining', the elements chosen do not necessarily show the coherence observed in coordinating theories (it is rather a 'juxtaposition' of theories - Radford, 2008). In our three researches, with different theoretical frameworks and therefore with principles and methodologies that are not entirely congruent, we answer the same research questions. For this reason, to achieve our goal we will use the strategy of combining theories.

4.5 Findings

Because of space constraints, data supporting these findings will be presented in a future paper. Each theory has its own specificity. Combining three different points of view on similar dynamics, therefore, has the same effect that panting a complex object from three different positions of view can have: it highlights aspects and details, relations between components, dynamics that do not belong to the visual cone of a single observer. With our theoretical frameworks, we focus our attention on a common space: the way in which Cultural Transposition allows us to contextualize and deconstruct (Bosh & Gascon, 2006) teachers' beliefs and practices. The three frameworks conceive, in different ways, LS as an element that interacts with the Italian context and its components.

As a *Boundary Object*, LS is a dynamic object, moving on the boundary between communities of practice. The analysis focuses on the meeting at the boundary between the community of prospective teachers and the one of researchers, which also metaphorically represents the practicing teachers. On the one hand, the two communities collaborated on the reflection on LS, which allowed the researchers to highlight some components of the methodological object that had remained implicit in the brokering of LS from the Japanese cultural community to the Italian one. On the other hand, the researchers were able to observe how LS helped the *boundary crossing* of the prospective teachers towards the practicing teacher’s community, making sense of a different perspective. Moreover, it was possible to observe how the encounter with the robust components of LS (Star, 2010) developed a reflection of the prospective teachers on their own meta-didactic praxeologies (Arzarello et al., 2014), possibly transformed into a hybrid between the long-term approach, typical of the Italian context, and the fine analysis used in LS. The analysis of the evolution of both communities of practice and of the Boundary Object itself, allowed the researchers to highlight the potential of LS as a Boundary Object to trigger some dialogic mechanisms for professional growth indicated by Akkerman & Bakker (2011). The community of researchers developed new praxeologies for the introduction of LS in the Italian institutional context.

Immersing ourselves in the visual cone of the *Semiosphere*, we see how the LS methodology, becoming part of the mathematics class’ semiosphere, allows the deconstruction of practices and beliefs, so producing a new awareness. In particular, it allowed to look at the collaboration between teachers, and thus at the elaboration, exchange, and archiving of mathematical knowledge and professional development, as mutually inclusive continuous texts. The texts are written (e.g. Lesson Plan), graphic (e.g. drawings of students or graphic representations of teachers), oral (e.g. dialogues in the various phases of the LS), technologically mediated (e.g. worksheets, machines), embodied (e.g. interaction in the classroom), institutional (e.g. curricula), local (e.g. specific epistemological, didactic and pedagogical needs), and others. Through their heterogeneity in mutual continuity, the semiospherical texts allow to keep connected aspects that would seem distant. Distance (understood here in a cultural and sfemiospheric sense - Lotman, 1990) would generate a loss of meaning.

Semiotic Mediation allows us to reflect on the importance of the teacher's role in the appropriate choice of artefact linked to its semiotic potential, and on the importance of the teacher's role in the management of discussion and sharing of individual signs; the teacher also seems more aware that better time management involves better class management. In the teaching practice, LS, shared with the whole teaching community, seems to contribute not only to the professional growth of the experimenter teacher, but transfers to the whole community the refined skills and the acquired awareness of their role in guiding the discussions in the classroom. The identification shared by the experimenter teachers with the semiotic potential of the proposed artefact was the necessary background to its use in the classroom. The careful planning of the didactic intervention, of the possible tasks and the didactic organization foreseen into LS allowed to assume the right semiotic perspective to focus on the production of signs and on the process of transformation of these signs. The teacher, after just one LS cycle, becomes more aware of the choice and use of the artefact to make it functional to semiotic mediation.

The combination of the three theoretical lenses allows us to go beyond the single point of view. If the frameworks of Semiosphere and Boundary Object carry out a meta-analysis of LS in relation to the context in which it operates, identifying the points and tools for intervention in a complementary way, Semiotic Mediation provides us with the tools to observe LS in its operational practice, reading and interpreting the didactic action of the teacher, and therefore promoting an analysis of the effectiveness of LS itself as a professional development practice. In this sense, in absolute consonance with the Semiotic Mediation, the lens of the Semiosphere allows us to focus on the patterns of reasoning that the students use, and on the essential components of socialization of reasoning in building meaning. Starting from the three experiments and through the juxtaposition of the three lenses, we were able to highlight some of the specific teaching practices within the Italian context. From an institutional point of view, our teachers have expressed great difficulties in organizing the time in which to do their work and, at the same time, a need for flexibility with respect to the management of time in the classroom. Moreover, in particular due to the cultural and institutional context but also to emotional aspects, teachers feel the need to adequately respond individual students' needs, something possible only by giving the right importance to design, planning, and assessment of teachers' actions. Finally, from a content point of view, we found deeply rooted fears about the management of mathematical misconceptions, alongside the underestimation of the possibilities offered, in this sense, by research in Mathematics Education.

The LS methodology has contributed to this study, thanks to the new perspective with respect to teachers' meta-didactical and collaborative praxeologies (Arzarello et al., 2014), providing them a tool for microanalysis of the phases of the lesson in a context accustomed, for historical and institutional reasons, to the design and analysis of long-term development strategies. The encounter with other people's practices within LS is an opportunity to observe and reflect on one's own different praxeologies. The apparent contrast between a fine lesson planning and the attention to the needs of the individual student accentuated the careful design of the didactic intervention. The detailed and collaborative design of possible tasks and the didactic organization provided by LS have allowed to take the right semiotic perspective to focus on the production of signs and on the process of transformation of these signs.

4.6 Conclusion

In this paper, we have used part of the Networking of Theories framework (Prediger et al., 2008) as a (meta-)language that made possible the connection and harmonization between our three theoretical frameworks (Semiosphere, Semiotic Mediation and Boundary Objects). LS methodology in the Italian context allowed us to closely study the practices of prospective and practicing teachers in collaborative contexts. By combining the three theoretical lenses we have tried to extend the experimentation of LS to different contexts (practicing teachers in primary and secondary school and prospective teachers at university level). The results highlight the collaborative dimension in teaching/learning practices as a possible key for a real reform of teaching, seeking and creating connections between teaching practices of different school segments; the collaboration between school and academia can be an added value towards more conscious teaching practices in the light of the research results. This was possible thanks to a careful cultural transposition of LS into our educational context, which provided a solid methodology for teachers’ collaboration within the institutions: the data of the three different experiments on different school levels from different points of view provided a novel understanding on how to promote, design, and assess relevant professional development practices for mathematics teachers. The combined findings seem to have provided further support to the applicability of LS in our didactic praxologies. Overall, LS seems to be replicable as an effective teachers' professional development practice, suggesting the potential of a not only horizontal collaboration. A training that flourishes from below can be of support to the communities of practicing teachers and a stimulus for prospective teachers. Observing one's own work through the practices of others allows a more conscious reflection on one's own practices, laying the foundations of a modern teachers' professional development.

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5 DIDACTICIANS INTRODUCING LESSON STUDY FOR THE PROFESSIONAL DEVELOPMENT OF PROSPECTIVE MATHEMATICS TEACHER

Minisola, R., Robutti, O., & Miyakawa, T. (submitted) Didacticians introducing Lesson Study for the Professional Development of Prospective Mathematics Teacher.

Chapter 5 presents the results of the first phase of my research project, my first experiment introducing Lesson Study to prospective teachers. The paper is focused on the dual position of the didacticians (as researchers and as teacher educators) introducing Lesson Study to prospective teachers. The results make sense of the complexity of didacticians' practices and knowledge at stake during teacher professional development, and highlight how their dual position influences the evolution of such practices and knowledge.

Lesson study is recognised as a cultural activity and its implementation in a context different from Japan is a complex process. Researchers’ role in this process is assumed to be critical, although this has rarely been investigated. In this paper, we analyse a teaching experiment to introduce Lesson Study into a professional development course for prospective teachers at an Italian university, focusing our investigation on a group of researchers acting as teacher educators (didacticians). Using the Anthropological Theory of the Didactic and Meta-Didactical Transposition frameworks, we investigate their dual position as researchers and as teacher educators. We observe the evolution of the didacticians’ teacher-education praxeology (a model of practice and knowledge) during their interactions with prospective teachers. The results indicate that the didacticians’ teacher-education praxeology is distinguished, shaped, and actively influenced by their research praxeology. The results also imply that coordinating the two theoretical frameworks may guide the design and analysis of teachers’ professional development courses.

Keywords: Lesson Study, Didacticians, Prospective Teacher Professional Development, Anthropological Theory of the Didactic, Meta-Didactical Transposition, Praxeology

5.1 Introduction

This paper is part of a larger project aimed at studying the introduction of Japanese Lesson Study into the context of Italian teacher professional development (TPD), which involved many experiences of collaboration between teachers and between teachers and researchers (Arzarello & Bartolini Bussi, 1998). The institutional demand for collaborative TPD has recently grown but there is little information on long-term collaborative experiences between teachers (Minisola & Manolino, 2022).

In the international research community of mathematics education, Lesson Study (LS) has been acknowledged (Isoda, 2007) as a promising TPD model, promoting collaboration between teachers. There are reports of successful attempts at exporting LS (Huang et al., 2019; Huang & Shimizu, 2016; Quaresma et al., 2018), but others highlight the difficulties of the process (Demir et al., 2012; Fernandez et al., 2003). Most reports on LS originate from Western countries, particularly Anglo-American states (White & Lim, 2008). They appear to consider LS as an isolated practice in the Japanese TPD context, but the reality is different (Miyakawa & Winsløw, 2013, 2019). As LS is a cultural practice, in order to introduce it into a different country, there must be careful consideration of its origin and the context in which it is to be introduced (Stigler & Hiebert, 2016): studies clarify that its efficacy in Japan is largely due to the cultural context (Krainer, 2011; C. Lewis, 2016).

The aim of the larger project is twofold: to obtain theoretical insights on LS while providing scientific knowledge on the introduction of LS in the Italian cultural and institutional context; and to foster collaboration between teachers in the Italian TPD context. The first step, which is reported in this paper, is to analyse how Italian communities of researchers and teachers involved in TPD react to a new element.

In this paper, which reports on the initial results of this project, we focus on the people mediating between LS and the teachers. Some past studies have investigated the facilitators or teacher educators, within the LS community (Restani et al., 2019; Seino & Foster, 2021) and in other contexts (Boles et al., 2020; Schwarts et al., 2021). Few studies have focused on the researchers (Goos, 2014) who play complex roles as researchers and as teacher educators. Accordingly, the purpose of this paper is to investigate the researchers acting also as teacher educators (*didacticians*) (Jaworski & Potari, 2021), who introduced LS into the Italian TPD context. We study the evolution of didacticians' practices and their impact on the TPD course in order to gain insights into the implementation of teachers' LS practices in a context different from Japan.

5.2 Lesson Study and Professional Development

5.2.1 Research on Lesson Study

It is no simple task to export LS: Fernandez and colleagues (2003) recruited Japanese teachers as LS coaches for American teachers, finding that cultural traditions cannot be taught; Demir and colleagues (2012) came across resistance from the teachers because they viewed LS as contradicting their beliefs around teaching. LS exists in Japan largely due to Japanese culture and it might be rejected if cultural aspects are not properly considered in other countries (Stigler & Hiebert, 2016).

Some studies investigate why LS is prominent in Japan: Krainer (2011) considers *the three COs* – content, community, and context – that make TPD practices successful, and recognises them in LS. C. Lewis (2016) proposes four conditions that make LS crucial to Japanese TPD: practising curricular innovations, feedback to curriculum developers, creating demand for innovation, and development of expertise in curriculum development. There are positive experiences of LS for in-service (e.g. Clivaz & Ni Shuilleabhain, 2019) and pre-service (e.g. Nakamura, 2019) TPD in several countries, revealing that LS can take different forms depending on the implementing institution, even in Japan (for primary school, see Fernandez & Yoshida, 2004, for secondary school, see Miyakawa & Winsløw, 2013; for pre-service teachers see Elipane, 2012).

The role of external persons involved in the LS process can also be questioned. In Japan, while LS often involves *knowledgeable others* invited to provide comments on the demonstrated lesson (Fujii, 2019; Takahashi, 2014), their role is not necessarily crucial (Seino & Foster, 2021). LS is introduced *by teachers for teachers*. In contrast, this role may be crucial in other contexts. *Facilitators* can act as a link between researchers and teachers in LS (e.g. J. M. Lewis, 2016). Furthermore, the responsibility of implementing LS in contexts outside of Japan mainly lies with the researchers (Ponte et al., 2018), as in the case of our project. Further investigation is required in order to understand how they handle their role in TPD with LS.

5.2.2 Teacher Professional Development in Italy

Italian TPD is traditionally carried out by researchers collaborating with teachers since the 1970s and '80s, organised into local and national *research groups* of teachers and researchers, financed by the National Research Council, and situated in various universities (Arzarello & Bartolini Bussi, 1998). Many remain today, supported by the Ministry of Education. They perform research in mathematics education, TPD, and engage schools in new projects. There are numerous TPD programmes and design of resources for TPD (e.g. *m@t.abel*, and *Piano Nazionale Lauree Scientifiche, Licei Matematici*) (Arzarello et al., 2021; Branchetti et al., 2019) and the collaboration between researchers and teachers is central to these programmes (e.g. Cusi & Malara, 2015; Robutti et al., 2020, 2021b).

Today, Italian teachers seem to need a model of work that enables sharing experiences and professionalism with their colleagues (Blandino, 2008). The Ministry of Education states that the general quality of professional development programmes is compromised by the “low quality of [some] models and methodologies” (law 107/2015). The Ministry also recognises that quality TPD programmes can be found in the academia, albeit difficult to identify in the vast offer of more than 500 attested agencies (Minisola & Manolino, 2022). Finally, there is greater institutional demand for ‘permanent and strategic’ TPD in ‘collaborative networks’ (law 107/2015).

LS could answer such institutional requests, and it could play a fundamental role in relaunching teachers’ professionalism, while also providing a useful environment for collaboration and reconciliation between the academic and school worlds.

The Italian context of collaboration between academia and school for research in mathematics education and for the professional development of Italian teachers seem to be a suitable environment for the implementation of LS. Since previous Italian studies on LS are contextualised in primary schools (Bartolini Bussi et al., 2020; Bartolini Bussi & Ramploud, 2018), we approach LS in the context of pre-service secondary school teachers.

5.3 Theoretical Framework

5.3.1 Institutional perspective and transposition

We assume that TPD practices are influenced and shaped by the context in which teachers and didacticians are immersed. The term *culture* is significant (Hatano & Inagaki, 1998; Presmeg, 2007), as it is often linked with the concept of society and organisation (e.g., Freimuth, 2006).

To address this cultural aspect of TPD, as well as the complexity involved in implementing foreign practices, we adopt the *institutional perspective* proposed within the Anthropological Theory of the Didactic (ATD; Chevallard, 2019). The notion of *institution* is interpreted in a broader sense, including ‘any created reality of which people can be members’ (Chevallard & Bosch, 2020, p. xxxi). In our case, the institutions involved are the classroom of the TPD course at the university, the Italian community of didacticians, the Italian community of researchers, the Italian community of prospective teachers, and the Japanese community of mathematics teachers.

The implementation of LS in Italy is considered a process of *transposition* of teachers' practices from the Japanese institution of mathematics teachers to the classroom of Italian teachers' TPD, or to the Italian community of prospective teachers. The notion of *transposition* was developed to understand the nature of mathematical knowledge taught or supposed to be taught in the classroom (Chevallard, 1985, 2019), i.e. a transposition of mathematical knowledge from a scholarly institution to a school institution.

In an institution, a person occupies a certain *position* in relation to an object (e.g. a didactician and a prospective teacher occupy different positions with respect to LS in TPD). One hypothesis of this perspective is that people's practices and knowledge are influenced or shaped by different elements, known as *institutional conditions* and *constraints*, prevailing in the *institution* to which they belong, and depending on the *position* they occupy. One of the biggest contributions of ATD was to focus, in the case of mathematics teaching, not only on those in the classroom but also those outside of it, such as cultural elements related to school, society, civilisation, etc. (Bosch & Gascón, 2006). This is also the case for teachers' practices and TPD activities. The institutional perspective of ATD allows for the cultural aspect of human practices to be investigated.

These days, the term *transposition* is used in different contexts, such as Meta-Didactical Transposition (Arzarello et al., 2014; Robutti, 2020), which is based on ATD, and Cultural Transposition (Mellone et al., 2019), which is unrelated to ATD. The former models the practices developed in collaborative projects of teachers and researchers, specifically the evolution of teachers' and didacticians' knowledge. The latter investigates the nature of culturally-situated didactic practices when they are decentralised to another context. Similarly, we investigate the transposition process of LS from Japan to Italy, focusing on the position of the didacticians.

5.3.2 *Praxeology: a model of human activity*

The transposition process of LS concerns several different practices: didacticians in the positions of teacher educators and researchers, prospective teachers in the positions of teachers and learners, and so forth. The dual position of the didacticians is due to the complexity of their practices, and the same applies to teachers. To address this complexity, we adopt the notion of *praxeology* (Chevallard, 2019).

“The anthropological principle states that any human activity can be described in terms of praxeologies” (Bosch et al., 2020, p.xiv). A praxeology is more than just a model of practice or knowledge: it consists of two blocks, know-how (*praxis*) and know-why (*logos* - discourses that justify the know-how). Praxis is made up of two elements: a *type of tasks* and a *technique* to solve this type of tasks. Logos also consists of two elements: a *technology* (the discourse that justifies the technique) and a *theory* (which supports the technology).

This notion was introduced to characterise mathematical practices and knowledge in the classroom or curriculum (*mathematical praxeology*), as well as the practices and knowledge used by teachers (*didactical praxeology*) to bring out mathematical praxeology in the classroom.

Various praxeologies can be identified and differentiated with regard to the introduction of LS in a TPD course at the university. LS concerns different (and interdependent) praxeologies, as suggested by Miyakawa & Winsløw (2013, 2019): those referring to mathematical content (mathematical praxeologies), the teaching of mathematical content in the classroom (didactical praxeologies), teachers' practices outside the classroom (*paradidactical praxeologies*).

In the TPD context, a teacher is in the position of the learner who learns didactical praxeology and/or mathematical praxeology. The teacher's activity as a learner could be modelled by other kinds of praxeologies related to the *professional* learning of the teacher. One specific aspect of TPD consists of the fact that there are teacher educators, who are in the position of educators to support teachers' learning (in terms of LS, this is the role played by the knowledgeable others). Their teaching activity could be modelled by *teacher education praxeology* (Asami-Johansson et al., 2020). Teacher educators may also play the role of researchers in relation to teachers' practices, as in our experimentation where didacticians take on this dual position (in this paper, they study LS at meta-level). The researchers deepen the understanding of the *didactic system*. The knowledge and practice related to the research could be modelled by *research praxeology* (Artigue & Bosch, 2014).

The Meta-Didactical Transposition (MDT) framework (Arzarello et al., 2014; Robutti, 2020) was created to manage the complexity of didacticians' positions in the case of collaborative TPD projects with teachers and didacticians. Using MDT, the activities and knowledge of teachers and didacticians are modelled with the notion of praxeology: they are called *metadidactical praxeologies* as they refer to knowledge about the didactic system. Metadidactical praxeologies can be referred to in terms of *teachers praxeologies* and *didacticians praxeologies* (Fig. 5.1): *teachers praxeologies* develop in accordance to classroom practice, whether *didacticians praxeologies* develop in accordance with certain theoretical frameworks. During professional development, didacticians introduce the teachers to new practices and knowledge. A *double* *dichotomy* is established between two dialectical levels: the *didactical level* is developed within the classroom, between the personal meaning that students attribute to the teaching situation and its shared scientific meaning; the *metadidactical level* lies between the interpretation given to the classroom dialectic by teachers and that given by didacticians, both based on the beliefs and praxeologies of their respective institutions. The meta-didactical level arises from the tensions that develop from the encounter between the components of the praxeologies promoted by the didacticians and those of teachers, and vice-versa. Both praxeologies may evolve over time, through interactions between teachers and didacticians, with elements of these praxeologies being transposed from didacticians’ institutions to teachers’ institutions and vice versa. The result may be a *shared praxeology* (Fig. 5.1) in the context of TPD. The hypothesis of MDT is that a *shared praxeology* is achievable over time, as more and more elements of didacticians praxeologies and teachers praxeologies are shared. Because didacticians and teachers belong to different institutions, at the end of a TPD programme these *shared* elements will lead to new praxeologies internal to each institution.

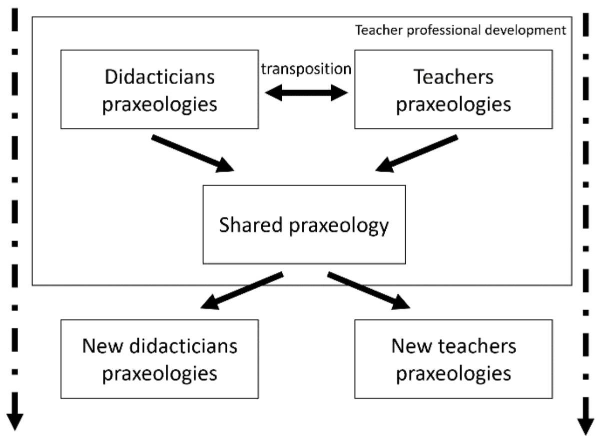


Figure 5.1 The emergence of a shared praxeology in MDT.

These praxeologies (Fig. 5.2) aim to clarify the different practices and knowledge related to the didacticians’ and teachers’ work, and TPD. In this paper, we investigate with a theoretical background of these different kinds of praxeologies a case of TPD organised by the didacticians for prospective teachers, expecting to provide better understanding of the above-mentioned complexity in terms of the praxeologies.

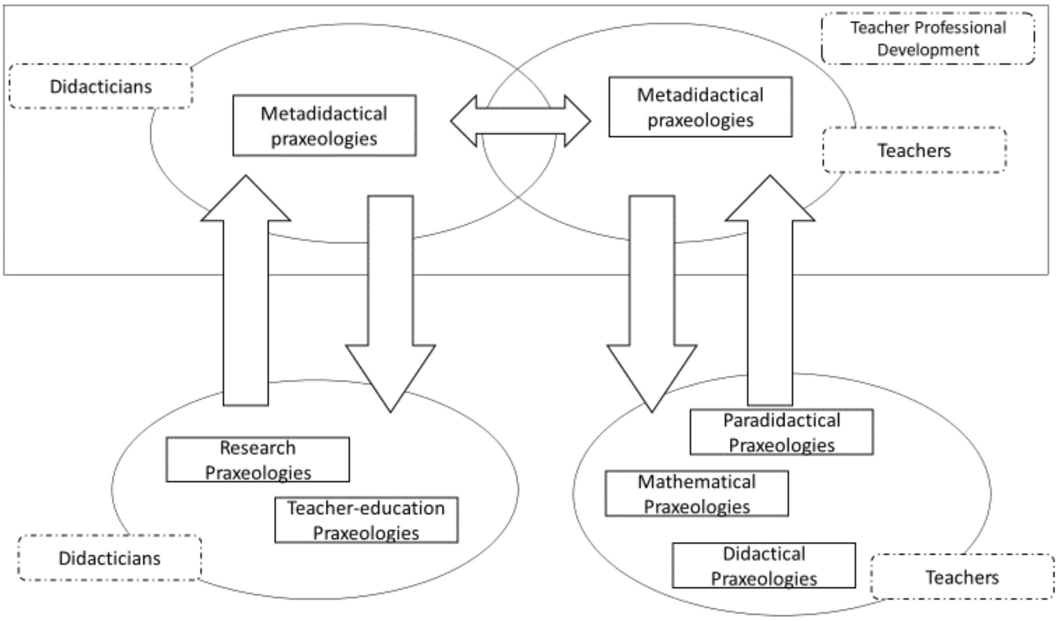


Figure 5.2 A summary of the praxeologies of Didacticians and Teachers.

5.3.3 Research Questions

The purpose of this paper is to investigate the complex positions of the didacticians as they critically examine their practice and generate new knowledge on the introduction of LS in the context of TPD. We analyse the evolution of their praxeologies and the impact of this evolution on the TPD course. Therefore, we will address the following research questions (RQ):

(1) How do the praxeologies of didacticians evolve in the process of implementing LS, through the interaction with prospective teachers?

(2) How do teacher-education praxeologies and research praxeologies of didacticians reciprocally influence and shape their evolution?

A teaching experiment was planned to introduce LS to prospective teachers within a TPD course at an Italian university, as an exploratory study of the introduction of LS into different institutional contexts. The aim was to identify positive and critical aspects of our approach to the implementation of LS in a TPD course. A qualitative analysis was conducted on the data collected during this TPD course.

5.4 Implementation of Lesson Study

5.4.1 Context: the EMAS course

The experiment was carried out within the *Elementary Mathematics from an Advanced Standpoint* (EMAS) course for prospective secondary school teachers at master’s degree level at a university in Italy, in 2018/19. This 48-hour course focuses on *continued fractions*: 30 hours are dedicated to continued fractions from the epistemological and historical standpoints, 16 to didactical approaches, and 2 to another project. In the 16 hours, prospective teachers are encouraged to use their acquired mathematical knowledge to design a teaching activity to introduce continued fractions in one class (grade 6 to 11).

Our experiment was carried out during these 16 hours by two of the paper's authors (the first author is a PhD Student, the second and third authors are professors; the third author substantially contributed to revising the paper and to the data analysis) supported by another PhD student (who is not author of the paper). They all acted as didacticians. 29 prospective secondary school teachers were enrolled on the course and participated in the experiment. They had no experience of real classroom teaching, although some had already engaged in mathematics education courses, and they were aware that their participation in the experiment would be evaluated (as opposed to the non-evaluative context of LS, this is expected to produce some sort of bias in the data).

Up until the 2017/18 academic year, the prospective teachers on the EMAS course had been asked to design *individually* an activity in a written document and submit it for evaluation by the didacticians. In the 2018/19 academic year, they were asked, for the first time, to: design *collaboratively* an activity and a 20-minute lesson based on it, and enact the lesson in front of their peers as a mock lesson.

5.4.2 Details of the experiment

The 16 hours of didactic approach in the TPD course consist of a lecture, teachers' activities (design of teaching activities and mock lessons) and another lecture. The design of the prospective teachers' activities within this course relies on the Japanese LS process. The epistemological and historical introduction to continued fractions formed part of the study of teaching materials (*kyōzaikenkyū*). One implementation of the research lesson was included in the cycle. For the materials to include in the course, examples of LS implementations in Europe (e.g., Dudley, 2014) and in Italy (Bartolini Bussi & Ramploud, 2018) were considered while designing the experiment. The following differences in the respective context were found: Dudley's implementation requires three repetitions of the lesson, incompatible with the time constraints of a University course, and the structure of the lesson in the UK greatly differs from the one of Italian lessons (in general, lesson structure depend on the institutional structure, see Hatano & Inagaki, 1988; for Italian examples, see Calvani, 2014); Bartolini's implementation is inspired by Chinese LS, it heavily relies on facilitators, and has been tested mainly with pre-service and in-service primary school teachers, whose preparation is not content-specific and lesson structure heavily differs from secondary school. Therefore, the didacticians decided against proposing materials adapted from other LS implementations. After further deliberation, the didacticians also decided not to provide examples of Lesson Plans: this was motivated by the desire of creating a structure for Lesson Plans suitable for future experiments, building from the Lesson Plans collected from the prospective teachers.

During the first 1-hour lecture (*Lecture 1*), the didacticians introduced Japanese LS with a set of six slides (*Slides Set 1*). These slides contained information on the Japanese historical and institutional context, and the LS (one slide). They were developed on the basis of the author’s master thesis (2016) which referenced Fujii (2016), Huang and Shimizu (2016), Isoda et al. (2007) and C. Lewis (2016) for LS, and Halliday (1978) for the Japanese historical context. For example, Fig. 5.3 (a) shows the slide on the Japanese historical context, and Fig. 5.3 (b) shows a picture of the staff room of a Japanese high-school used to exemplify the collaborative nature that permeates Japanese teaching culture, Fig. 5.3 (c) shows the slide introducing the process of LS.

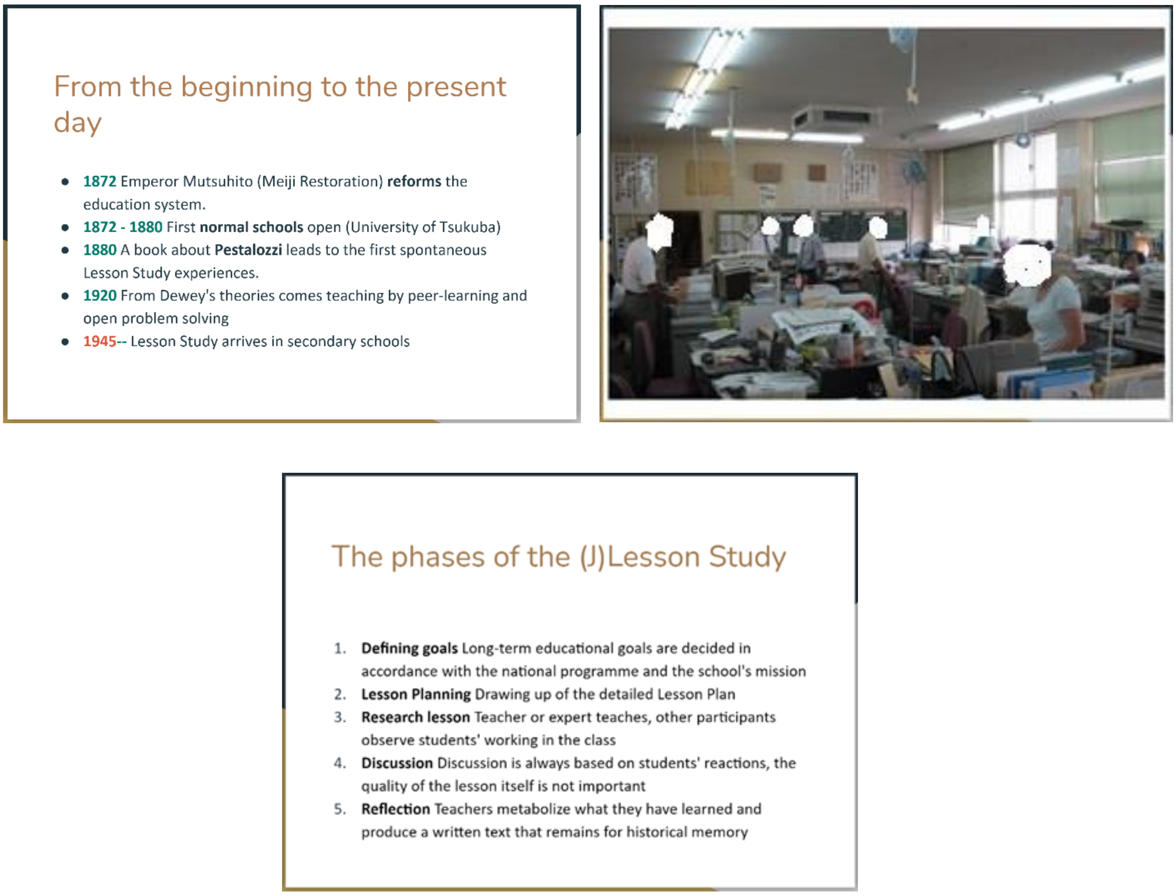


Figure 5.3 Slide 3/6 (a), Slide 5/6 (b) and Slide 6/6 (c) from Slides Set 1 (translated into English. The picture in Slide 5/6 of a staff room at Onizuka Middle School in Karatsu, Saga, Japan, was found on Wikipedia, where it was uploaded by user MC MasterChef with licence CC BY-SA 2.5. The picture was edited for the purpose of this paper, to hide teachers' faces).

During Lecture 1, the prospective teachers were divided into eight self-organised groups to work collaboratively in a LS setting. In this experiment, the five phases in Fig. 5.3 (c) were merged into three, and the prospective teachers were asked to work as follows:

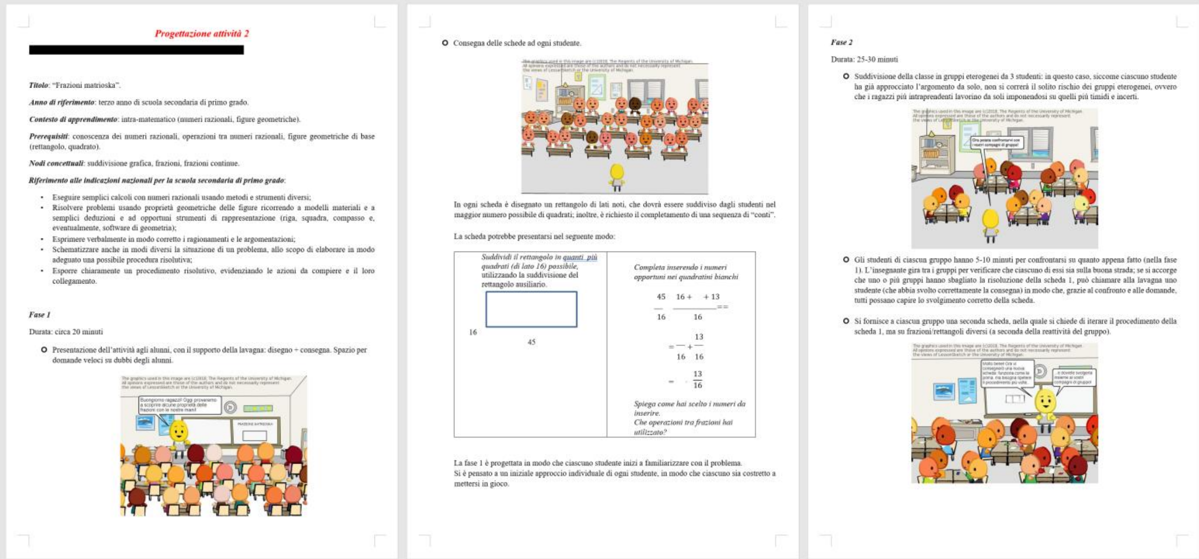
- *Planning phase* (1-2): study and design a teaching activity on continued fractions, and write an *activity report* with a lesson plan for a mock lesson.
- *Implementing phase* (3): teach and observe this mock lesson in front of their peers and didacticians.

- *Reflecting phase* (4-5): collaboratively discuss within the group the efficacy of the lesson and individually reflect on the discussion.

The planning and reflecting phases were organised autonomously by the prospective teachers outside of the course hours, without supervision or intervention by the didacticians. The implementing phase took place at the university, during the course hours, and was supervised by the didacticians.

The *activity report* is a written document containing a description of a teaching activity to be carried out in the classroom (usually over a long period, ranging from 6 to 12 hours) and is commonly used as a resource for national TPD projects in Italy (e.g. *m@t.abel,* in Arzarello et al., 2021). It consists of a short introduction followed by: the task(s) for the students, an extended description of the teaching activity phases, and the corresponding teaching strategies. Examples of activity reports were given to the prospective teachers via the *m@t.abel* repository.

Each group produced an activity report: an example is given in Fig. 5.4. This is the activity report prepared by Group 1, entitled *Matrioska Fractions* (the other reports contain similar types of information, and some of the data presented in this paper originates from Group 1’s work).



Progettazione attività 2

[redacted]

Titolo: "Frazioni matrioska".

Anno di riferimento: terzo anno di scuola secondaria di primo grado.

Contesto di apprendimento: intra-matematico (numeri razionali, figure geometriche).

Prerequisiti: conoscenza dei numeri razionali, operazioni tra numeri razionali, figure geometriche di base (rettangolo, quadrato).

Nodi concettuali: suddivisione grafica, frazioni, frazioni continue.

Riferimento alle indicazioni nazionali per la scuola secondaria di primo grado:

- Eseguire semplici calcoli con numeri razionali usando metodi e strumenti diversi;
- Risolvere problemi usando proprietà geometriche delle figure ricorrendo a modelli materiali e a semplici deduzioni e ad opportuni strumenti di rappresentazione (riga, squadra, compasso e, eventualmente, software di geometria);
- Esprimere verbalmente in modo corretto i ragionamenti e le argomentazioni;
- Schematizzare anche in modi diversi la situazione di un problema, allo scopo di elaborare in modo adeguato una possibile procedura risolutiva;
- Esporre chiaramente un procedimento risolutivo, evidenziando le azioni da compiere e il loro collegamento.

Fase 1

Durata: circa 20 minuti

- Presentazione dell'attività agli alunni, con il supporto della lavagna: disegno + consegna. Spazio per domande veloci su dubbi degli alunni.
- Consegna delle schede ad ogni studente.

In ogni scheda è disegnato un rettangolo di lati noti, che dovrà essere suddiviso dagli studenti nel maggior numero possibile di quadrati; inoltre, è richiesto il completamento di una sequenza di "conti".

La scheda potrebbe presentarsi nel seguente modo:

Suddividi il rettangolo in quanti più quadrati (di lato 16) possibile, utilizzando la suddivisione del rettangolo ausiliario. 16 45	*Completa inserendo i numeri opportuni nei quadratini bianchi* $\frac{45}{16} = \frac{16 + \quad + 13}{16} = = \frac{\quad}{16} + \frac{13}{16} = \quad \frac{13}{16}$ *Spiega come hai scelto i numeri da inserire. Che operazioni tra frazioni hai utilizzato?*

La fase 1 è progettata in modo che ciascuno studente inizi a familiarizzare con il problema.
Si è pensato a un iniziale approccio individuale di ogni studente, in modo che ciascuno sia costretto a mettersi in gioco.

Fase 2

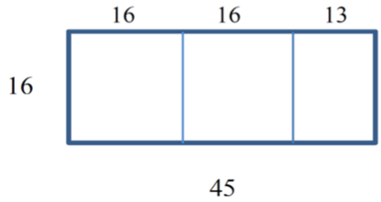
Durata: 25-30 minuti

- Suddivisione della classe in gruppi eterogenei da 3 studenti: in questo caso, siccome ciascuno studente ha già approcciato l'argomento da solo, non si correrà il solito rischio dei gruppi eterogenei, ovvero che i ragazzi più intraprendenti lavorino da soli imponendosi su quelli più timidi e incerti.
- Gli studenti di ciascun gruppo hanno 5-10 minuti per confrontarsi su quanto appena fatto (nella fase 1). L'insegnante gira tra i gruppi per verificare che ciascuno di essi sia sulla buona strada; se si accorge che uno o più gruppi hanno sbagliato la risoluzione della scheda 1, può chiamare alla lavagna uno studente (che abbia svolto correttamente la consegna) in modo che, grazie al confronto e alle domande, tutti possano capire lo svolgimento corretto della scheda.
- Si fornisce a ciascun gruppo una seconda scheda, nella quale si chiede di iterare il procedimento della scheda 1, ma su frazioni/rettangoli diversi (a seconda della reattività del gruppo).

Figure 5.4 The first three of six pages of Group 1's activity report.

The short introduction provides the target school grade (Grade 11), didactical aims according to the national curriculum, resources and artefacts, and prerequisites (rational numbers, and area of rectangles and squares). This activity is intended to connect the process of dividing a rectangle into squares with the algebraic form of a continued fraction: the students divide a given rectangle into squares, using the smaller side of the rectangle as the side for each square, until they are left with a smaller rectangle; they then repeat this process with this smaller rectangle (Fig. 5.5). The sides are chosen so that the generated continued fraction is finite and the process comes to an end.

Divide the new rectangle [the rightmost one] with sides 16 and 13 in as many squares with side 13 as possible; then redraw the starting rectangle and repeat the process for each new rectangle you may find.



Fill in the empty squares with an appropriate number, and complete the process

$$\frac{45}{16}=\frac{32}{16}+\frac{13}{16}=2+\frac{13}{16}=$$

$$=2+\frac{1}{\frac{16}{13}}=2+\frac{1}{\frac{\square+\square}{13}}=2+\frac{1}{\square+\frac{\square}{13}}=\cdots$$

Explain how you chose the numbers.

Which operations between fractions did you use?

Figure 5.5 Transcription of a part of Group 1’s activity report (our English translation). The same task is visible in the centre of Fig. 4

The lesson plan is also a written document, and a specific tool of LS. Its structure may vary, and no concrete examples were proposed to the prospective teachers, for two reasons: a didactical reason, as we expected that prospective teachers would feel overwhelmed by very complicated formats like the one by Bartolini Bussi &Ramploud (2018) (their LS heavily relies on facilitators, therefore it is easier to guide the teachers during TPD); and an experimental reason, as we hoped to draw our own Lesson Plan format by synthetizing the ones autonomously proposed by the prospective teachers. We specified, while assigning the task to the prospective teachers, that the Lesson Plan should at least contain: a detailed description of the task for the students, detailed time planning for each lesson phase, and predictions of possible reactions by students to the teachers’ actions. Our decision revealed problematic, as we will see in the following: no group produced a lesson plan.

The mock lessons were implemented over the course of three meetings: one member of the group acted as implementing teacher, the other members acted as observers, and the teachers from the other groups acted as pupils. At the end of the meeting, the mock lessons were discussed by all the prospective teachers. The didacticians were also present to supervise the process but did not intervene in the discussion of the mock lesson. The meetings were video recorded, but audio from the discussion is inaudible because of technical issues.

After the mock lessons, the didacticians (as researchers and as teacher educators) asked the prospective teachers to fill in an *Anonymous Survey* (AS) to explore the teachers' understanding of LS. The didacticians were interested in investigating why the groups produced no lesson plan, while also studying LS as an object. Two surveys were prepared, one for the implementing teachers and one for the observers (Table 5.1). The survey was anonymous to avoid self-report bias.

Table 5.1 Anonymous Survey

On the Planning phase...		
AS1	How did you feel about the collaboration with your colleagues in the planning phase?	
AS2	*Not relevant to this paper*.	
On the Implementing phase...		
	Observers	Implementing teachers
AS3	Before participating in LS, what did you expect from the implementing phase? Has your opinion changed?	Did you feel that the observers were assessing your performance, or evaluating the activity you developed together?
AS4	If you were to repeat the experience, would you like to be the implementing teacher? Why?	Did you follow the Lesson Plan? If you made changes during the mock lesson, why? Were they discussed with your colleagues?
AS5	What did you focus on during the observation? Did you choose the focus yourself or did you discuss it with the group?	
On the Reflecting phase...		
AS6	During the reflecting phase, were you able to express yourself sincerely? Do you feel that your colleagues did so? Did you feel judged, attacked, or otherwise uncomfortable, when discussing your opinions?	
AS7	*Not relevant to this paper*	

Another 2-hour lecture (*Lecture 2*) was prepared after the above-mentioned process in the course. This lecture was originally aimed at discussing the activity reports, the lesson plans, and the discussions and reflections developed by the different groups after the mock lessons. However, the analysis of the answers to the survey prompted the didacticians to modify its contents: one hour was dedicated to a class discussion on LS, coordinated by the didacticians, and one hour was spent analysing the work carried out up to that point, where they met with each group. The didacticians used a new set of ten slides (*Slides Set 2*), eight of which discussed the phases of LS, stressing the meaning of the term *lesson* in the LS context, the difference with the term *teaching activity*, and detailing what a lesson plan is and how it could be produced (Fig. 5.6); one slide discussed the Japanese institutional context, and one slide proposed some differences between the Japanese and Italian school contexts.

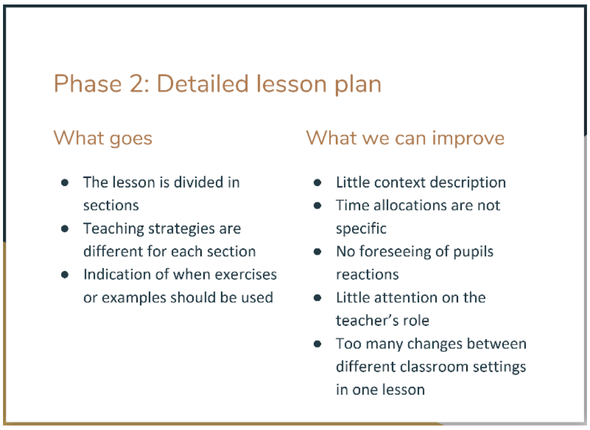


Figure 5.6 Slide 5/10 from Slides Set 2

At the end of the 16 hours, the prospective teachers were asked to fill in the *Final Questionnaire* (FQ, Table 5.2), designed to investigate the prospective teachers’ understanding of LS and their reaction to the discussion on their activity.

Table 5.2 Final Questionnaire

Think about what you have learned about the fundamental aspects of LS and what you have achieved in collaboration with your colleagues:

FQ1	What are your observations regarding the collaborative Planning phase of the teaching activity and mock lesson?
FQ2	What did you learn about the Implementing phase of the lesson?
FQ3	What are the key aspects of the Reflecting phase with colleagues?
The elements of LS that we have experienced are: Activity Design, Lesson Design, Implementation and Observation, Discussion and Reflection.	
FQ4	Which of these elements did you consider most valuable for you, and why?
FQ5	Which are less important and/or less valuable, and why?
FQ6	What would you change about your activity/lesson and its implementation, and why?

5.5 Data Collection and Data Analysis Method

We collected different kinds of data relating to the didacticians and to the prospective teachers (Table 5.3). With regard to the didacticians, we collected the slides and materials (e.g. website, Moodle platform) used in the TPD course and the research report written by them. The research report is a 16-page online document in Italian, written collaboratively during the experiment, containing: research goals, data, and research-related observations. With regard to the prospective teachers, we collected activity reports, answers to the survey and questionnaire, and partially videotaped the mock lessons.

Table 5.3 Data Collection

Dates	**Didacticians**	**Prospective teachers**[a]	**Data**
1-15 Oct	Design of the lecture		*Research report, website, Moodle platform*
Start of the 16 hours of didactical approaches			
16 Oct	Lecture 1		*Slides Set 1*
16-24 Oct	Design of the anonymous survey	Design of activity reports and lesson plans	8 *activity reports*
24 Oct, 14-20 Nov	Mock lessons observation	Mock lessons (teaching and observation)	*Research report*, 2.5 hours of *videos*

9-30 Nov		Fill in the anonymous survey[b]	27 *anonymous surveys*
20 Nov-3 Dec	Analysis of activity reports, observations, survey; design of the final questionnaire		*Research report*
4 Dec	Lecture 2		*Slides Set 2*
18 Dec		Fill in the final questionnaire	26 *final questionnaires*
End of the 16 hours of didactical approaches			
18 Dec-31 Jan	Analysis of questionnaires		*Research report*

[a] The underlined activities took place without supervision by the didacticians, outside the course hours.
[b] Post-lesson discussions took place in the same time period, but we have no related data.

The data analysis focus is twofold: didacticians' praxeologies and prospective teachers' praxeologies. With regard to the didacticians, we carried out two analyses according to the specific moment of the TPD course, prior to and subsequently to the prospective teachers' LS work, to identify the evolution of the didacticians’ praxeologies, which is the focus of RQ1. In the first analysis, we investigated the data related to Lecture 1, and in the second, the data related to Lecture 2. The slides and the materials used in TPD were used to identify mainly elements of the praxis block. The research report was also part of the data for both analyses, mainly to identity elements of the logos block. During this time, we specifically identified the elements of didacticians’ praxeologies related to the TPD course with LS and characterised them in terms of research praxeologies or teacher-education praxeologies. This characterisation is critical to answering RQ2 concerning the relationship between these two kinds of praxeologies.

With regard to the prospective teachers, three analyses were carried out. The first was an analysis of the data prior to their LS activities to identify the prospective teachers' praxeologies expected by the didacticians. This analysis used the same data as those used in the first analysis of the didacticians' praxeologies. The second analysis focused on the data related to their practices during LS to identify their praxeologies in the course: the activity reports and the videos were used to identify elements of the praxis block, whereas the answers to the anonymous survey were used to identify elements of the logos block. The third focused on the answers to the final questionnaires to identify the prospective teachers' level of knowledge after the TPD course, including the LS activities, therefore mainly for elements of the logos block. During this third analysis, elements of the praxis block could only be inferred from the teachers' answers, but it was not possible to observe them in a practical situation. The results of these three analyses allow us to investigate how the prospective teachers' work affects the evolution of didacticians' praxeologies, which is the focus of RQ1.

In these praxeological analyses, we initially explored the elements of praxis block (types of tasks and techniques) in the data and then sought the justifications (technology and theory) for the identified techniques (e.g., for the didacticians we analysed the slides to identify the techniques used for the teacher-education task, and then sought the justifications to such techniques in the research report). The logos block of both didacticians and teachers was used to identify institutional conditions and constraints that justify the praxis block, specifically with respect to LS. The praxeologies of teachers and didacticians are also analysed to identify possible shared elements between the two institutions, and to discuss the possible emergence of a shared praxeology.

The analysis was qualitative. Given the number of participants, proper quantitative analysis was not possible. When useful, quantitative considerations are provided.

5.6 Data analysis results

In this section, quotations from the research report are referred to as RR and sequentially numbered. Responses to the Anonymous Survey and Final Questionnaire are referred to as AS and FQ, the question number (as in Tables 1 and 2) and a sequential number. When describing the praxeologies, we highlight the elements that are more relevant in regards to the research questions, as a complete description would complicate the text more than necessary. The results of the data analysis are summarised in the following diagram (Fig. 5.7).

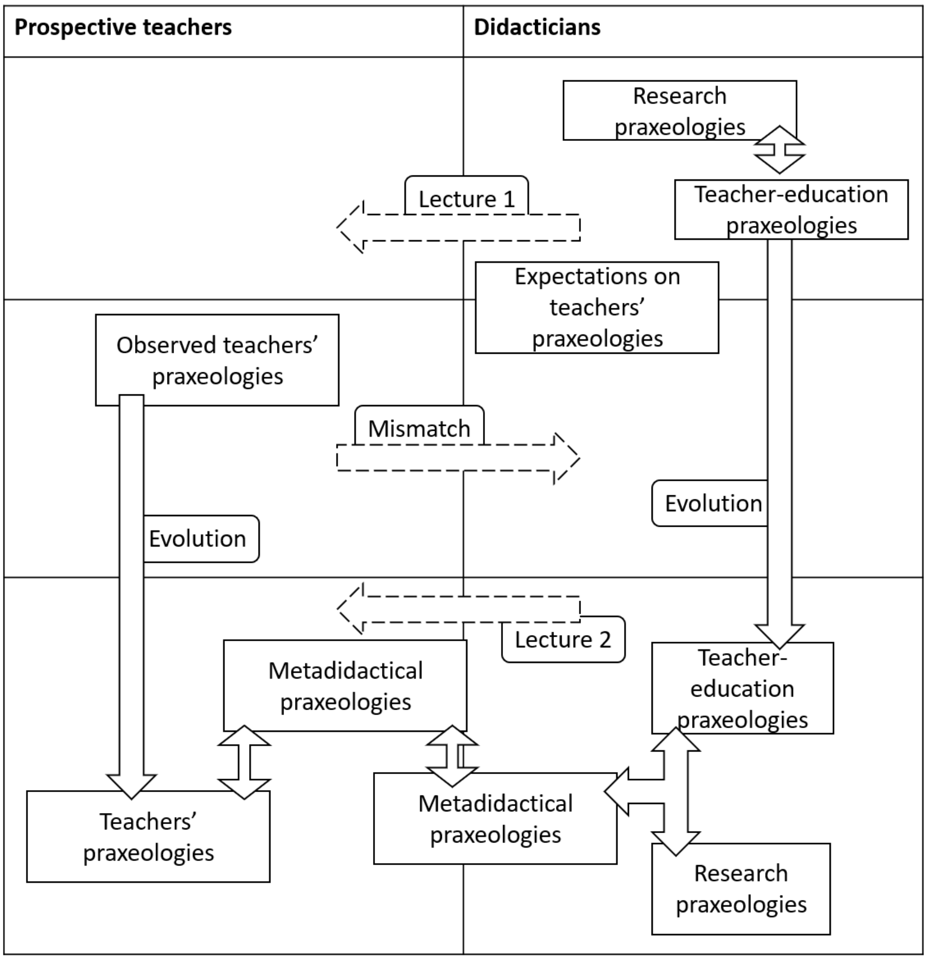


Figure 5.7 A diagram of the data analysis results

5.6.1 *Didacticians' praxeologies: mutual influence of* researcher *and* teacher educator *positions*

The didacticians' types of tasks related to this TPD experiment are: to study LS as an object of research (T1) and to conduct a TPD course (T2). The didacticians are interested in LS itself as an object of study (RR1: 'Both PhD students are interested in studying LS'). Task T1 (for brevity, we refer to *task* rather than *type of tasks* in this paper) is therefore a principal task of a research praxeology, while T2 is a principal task of a teacher education praxeology. We will analyse them below and demonstrate the influence of one on the other.

With regard to research task T1, the main technique is to ‘experiment LS implementation in Italy’, which also includes data collection and data analysis. In fact, the research report refers to the term ‘pre-pilot experiment’. The research praxeology includes other techniques: reviewing the scientific literature related to LS, developing theoretical and methodological frameworks, analysing data…, as described in the research report as well as in this paper. These techniques can be also considered subtasks for accomplishing the main task. In general, to carry out some techniques, several subtasks need to be completed and ‘there exists a dialectical interplay between techniques and types of tasks’ (Chevallard, 2019, p. 85).

The research report also allowed us to identify the elements of the logos block that justify the research technique of LS implementation in Italy for studying LS (T1). The didacticians’ interest was in the cultural aspects of LS. In the research report, we found the following claims:

> RR2 […] an application of this TPD [model] in Italy would be impossible without considering the profound differences between the social, cultural and institutional contexts in the two countries
>
> RR3 To [have] a local group of conscious [didacticians], we tried to observe the critical aspects or potential of this TPD [model] in our context.
>
> RR4 The goal of the experiment was […] to recognise and in future be able to overcome the obstacles that may be encountered when presenting LS to a specific audience.

The term ‘conscious didacticians’ in RR3 means researchers who are aware of the critical aspects of Japanese LS, and teacher educators who are aware that these aspects are critical when implementing LS in the Italian context (RR4). Two elements of the technology that justifies the research technique of LS implementation are: a) an implementation of LS outside of Japan might allow the didacticians to shed light on the cultural specificities or dependencies of LS in its original Japanese context; b) implementing LS in the Italian context enables the identification of aspects of Italian institutions that are critical in its transposition from Japan to Italy. These elements are supported by a theory of the research praxeology: the Cultural Transposition framework (Mellone et al., 2019) and the author's master thesis on LS, which are referred to in the research report.

The task (T2) is carried out by the didacticians – as teacher educators – during a one-hour lecture (Lecture 1) in which they use Slides Set 1 to introduce LS. The analysis of the slides highlights elements of the teacher-education praxeology. Slides Set 1 includes a description of the Japanese historical and institutional context as well as the phases of Japanese LS, as shown in Fig. 5.3 and 5.8. The slide shown in Fig. 5.8 describes the Japanese institutional context, emphasising in bold text some differences between Japan and Italy. Here, we identified one main teacher-education technique, namely teaching LS by way of two kinds of teacher-education techniques: the first directly related to LS, listing and describing the five phases of a LS cycle; the second related to the cultural aspects of LS, i.e. explaining the Japanese context and comparing it with the Italian context.

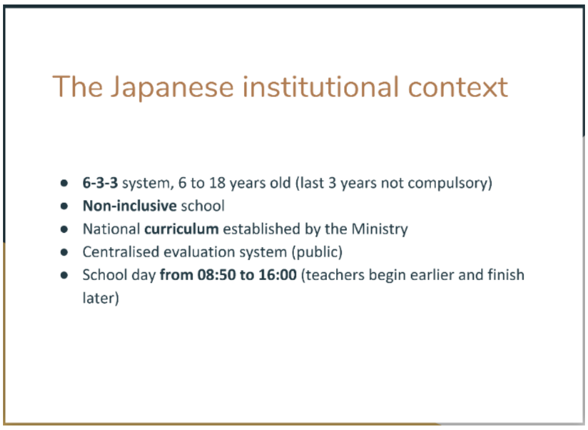


Figure 5.8 Slide 4/6, on the differences between the institutional contexts

The main technique is justified by the institutional demand for collaborative TPD models. With regard to the culture-related technique, the research report identifies elements of the *technology* that justify it, that is, relating the Japanese context with the existing Italian institutional framework is considered crucial for the implementation of teachers’ practices of another cultural context like LS, as shown in RR2.

This demonstrates that the teacher-education praxeology is influenced by the research praxeology. The didacticians consider LS as a suitable response to the institutional demand, and this knowledge originates from the literature review, which can be modelled as part of the logos block of the research praxeology. Due to the knowledge concerning the importance of culture, which is also modelled as part of the logos block of the research praxeology, the didacticians stress the cultural aspects of LS and the conditions and constraints existing in different institutions with regard to carrying out TPD, the teaching practices which are modelled by the teacher-education praxeology. This knowledge – developed from their practices as researchers – guides the experimental design, which involves developing Lecture 1 and the slides. As a result, the slides used by the didacticians in their position as teacher educators contain many references to the Japanese cultural context: the techniques of the teacher-education praxeology are thus justified by the elements of the logos block of the research praxeology.

5.6.2 *Prospective teachers’ praxeology as expected by the didacticians: an analysis from the teaching materials*

The data analysis allows us to identify the didacticians’ expectations on the prospective teachers’ practices during the TPD course. This analysis focused on the research report and the slides, together with resources from the Moodle platform and the course website. The general task for the prospective teachers is ‘to participate in a LS implemented in the Italian context’. In particular, we were able to identify sub-tasks involved in accomplishing this task, and to observe how the didacticians expected the prospective teachers to work. In the Moodle platform of the course (MOD) the prospective teachers were specifically asked to:

> MOD In groups, design an activity: title, grade, class context, prerequisites, key concepts, didactical aims according to the national curriculum, phases, methodologies, and suggestions for assessment.

In the research report, in the section describing the experimental design, we read:

> RR5 [The prospective teachers] were given a task: […] prepare a 20-minute lesson […] working with LS
> RR6 Despite [the differences between the Italian and Japanese context, and the differences between a real school situation and the course context] we have also chosen to stimulate them [the prospective teachers] to keep the following LS elements: attention to time; division into phases; meticulous description of the work in a shared worksheet, lesson plan […]; collegiality, collaboration and sharing.

These items refer to three sub-tasks assigned to the prospective teachers by the didacticians, and to the techniques that the didacticians were expecting from the teachers for each sub-task:

1. The first sub-task is to design a teaching activity and to write an activity report (MOD). The expected technique is to work in groups to plan and choose: school grade, didactical aims according to the national curriculum, resources and artefacts, prerequisites, according to *m@t.abel*

resources (MOD). This sub-task is related to the mathematical topic and the students’ learning.

2. The second sub-task is to plan a mock lesson and to write a lesson plan (RR5). The expected technique is to plan collaboratively and detail every phase of the mock lesson with: educational goals, time needed, role of the teacher, and students’ response (RR6). This sub-task is related to the teaching action and the student’s learning.
3. The third sub-task is for one student per group to teach the mock lesson while the others observe the lesson. The expected technique for one of them is to teach the mock lesson, possibly according to the plan, while the others observe (RR6).

These expectations originate from the didacticians’ position as teacher educators, as teacher education is the main goal of the course. Here, we identified the prospective teachers’ practices and knowledge *on* teachers’ practices inside and outside the classroom. Prospective teachers are learning how to design a teaching activity, how to plan a lesson, and how to teach in the classroom: practices which can be modelled in terms of didactical and paradidactical praxeologies. Furthermore, they are also learning how to carry out LS as a TPD practice. Overall, they can be considered elements of the prospective teachers’ metadidactical praxeologies.

5.6.3 *Prospective teachers’ praxeologies identified in the course: an analysis of the activity reports and mock lessons*

The elements of the praxeology that models the prospective teachers’ practices during the TPD course can be identified by analysing the different kinds of data: activity reports, videos from the mock lessons, and answers to the anonymous survey.

The prospective teachers' praxeology identified in the course corresponds only partially to that expected by the didacticians.

The prospective teachers dealt with the first sub-task as expected by the didacticians. All eight groups produced, over the course of one to two weeks, activity reports that indicate the results of the prospective teachers' work for this sub-task. While the specific technique (process) of producing the report cannot be ascertained due to the data limitation, they were asked to work together in groups, designing tasks for students and identifying target school grade, didactical aims, resources and artefacts, prerequisites, etc., based on the resources provided in the TPD course (e.g. national curriculum, *m@t.abel* resources). An example of an activity report (Group 1) is given in Fig. 5.4 and 5.5, showing that the prospective teachers described the teacher's instruction and students' learning envisioned in the classroom.

The second sub-task was to plan a mock lesson and write a lesson plan. This sub-task overlaps with the first sub-task, where the prospective teachers were asked to design an activity; the second sub-task, however, required a more detailed plan. There was no evidence that the prospective teachers worked on this sub-task, as none of the eight activity reports was accompanied by a lesson plan for the mock lesson. There were traces of planning in some of the activity reports, but these were far from constituting a detailed plan. Only four activity reports contained some time estimations for implementing the activity, and only two of them divided a 1-hour activity (that could fit into one lesson) into 20-minute sections. Therefore, while the prospective teacher who implemented the mock lesson planned the lesson to some extent before teaching, they did not explicitly deal with the second sub-task as expected by the didacticians, and we could not find any details of the plan in the data.

With regard to the third sub-task, one prospective teacher from each group implemented a mock lesson and the others observed it. This sub-task was accomplished but not as expected by the didacticians, as there was no plan to follow. The mock lessons also included different instructions that were not written in the activity report. For example, part of the mock lesson by Group 1 is shown in Fig. 5.9. This teaching activity is entitled *Matryoshka Fractions*, and the matryoshka (Fig. 5.9) was used as a metaphor for continued fractions (a fraction inside another fraction, like a doll inside another doll). The activity report states ‘presentation of the activity to the pupils, with the aid of the blackboard: drawing and task’, without identifying the use of the doll. When asked about the doll, the implementing teacher stated that the doll was her autonomous choice, and that she had not consulted about it with the other members of the group. In six out of the eight groups, there were similar occurrences of resources used in the mock lesson that were not described in the activity reports (or vice versa), with similar justifications.

Figure 5.9 A snapshot from Group 1’s mock lesson

These differences are justified by the prospective teachers. One answer to question AS3 in the anonymous survey reads:

> AS3.1 I like the fact that there is collaboration between teachers to create a common project to be presented to students, but there must be flexibility in adapting the lesson to the class […] according to the characteristics of the teacher (we are people and not machines).

The prospective teachers perceived the lesson plan to be rigid, in contrast with the need to maintain teaching flexibility according to the class context (which can be considered an element of the technology of their didactical praxeology). Elements of the theory that justify the importance of teaching flexibility come from the materials included in the course (i.e. references to *m@t.abel*, material on LS, national curriculum, etc.). In particular, the national curriculum contains non-prescriptive indications about the contents and competencies to be developed in mathematics, and the teachers retain flexibility in choosing the class syllabus (Minisola & Manolino, 2022).

5.6.4 Didacticians' praxeologies after the mock lessons: the influence of research practice on the teacher-education practice

The didacticians' practices and reflections after the mock lessons were analysed using the research report data, the questions designed for the anonymous survey, and Slides Set 2 used in the follow-up lecture. Their first task was to look back on and evaluate the prospective teachers' work. Some comments from the didacticians on the prospective teachers' products can be found in their research report. The didacticians noted a mismatch between the prospective teachers' praxeologies observed in the course and the ones they had expected in response to the task that they had assigned, as highlighted in the following comment from the research report:

RR7 Each group should have produced an activity report and a detailed lesson plan for the lesson to be presented. However, all groups only produced an activity report with a general time indication […].

Based on this evaluation, the didacticians tackled a new task, which was not included in the initial experiment plan, namely to design a survey to gain a better understanding of the prospective teachers' knowledge about LS (RR8).

RR8 […] the research group decided on a survey to investigate how much they [prospective teachers] understood about LS.

This reveals elements of the didacticians' research praxeology: firstly, as researchers, they designed a TPD intervention; they then tested the intervention profiting of their position as teacher educators; and finally, they analysed as researchers what worked or did not work as expected, as highlighted by RR8.

In addition to the tasks of evaluation and of designing a survey, we also identified a task by the didacticians of investigating the mismatch found between the prospective teachers' praxeologies observed in the course and the expected ones. They questioned the extent to which they could communicate LS better:

WN How do we communicate better with teachers?
RR9 Does 'what is meaningful in the world of LS research' have the same meaning in the teachers' community?

The didacticians – as researchers – considered the problem of communicating better with the teachers (WN). *Lesson plan[ning]*, *lesson*, and *[post-lesson] discussion* are technical terms related to LS (Fujii, 2019; Quaresma et al., 2018), which were not defined during Lecture 1, as can be seen in Fig. 5.3 (b). For instance, the meaning of the term 'lesson' as opposed to the word 'activity' and considerations on the lesson plan as a separate entity from the activity report, or in terms of structure or content, were left implicit. Since these terms may have different meanings in different institutions (researchers and teachers, RR9), the researchers conclude that these terms must be defined when working with LS:

> RR10 At macro level, […] we believe there is a need to establish a shared language a priori, particularly concerning the terms ‘lesson’ and ‘activity’.

As a result of these reflections, the didacticians worked on the task of redesigning Lecture 2. The new Slides Set 2 shows how the reflections of the didacticians as researchers influence the following part of the TPD course, i.e. how their position as researchers influences their practices as teacher educators. One hour of Lecture 2 was dedicated to a whole-class new presentation of LS supported by a new set of ten slides (*Slides Set 2*), eight of which, newly designed, presented the phases of LS and detailed what a lesson plan is and how it should be built; two were also contained in Slides Set 1, presenting the Japanese institutional context and proposing some differences between the Japanese and the Italian school context. Fig. 5.6 shows an example from *Slides Set 2*. It describes some of the features of *lesson plans* that were missing from the activity reports. The meaning of the term *lesson* in the LS context was also defined (orally).

From the new resources, we can identify new elements of the didacticians’ teacher-education praxeology. The LS-related technique for task T2 is now: detail each of the five phases of a LS cycle and new resources (such as the lesson plan). The culture-related technique appears to be unchanged.

The research report reveals theoretical elements that support the evolved teacher-education technique. The excerpt from the research report, describing Lecture 2, reads:

> RR11 During this meeting, the phases of LS had to be explained again, focusing in particular on the phases of lesson design and implementation, marking the difference between designing an activity and planning a lesson.

It suggests that the didacticians are more aware of the importance of the terminology, as some terms may have different meanings in different contexts. This can be modelled as an element of the *technology* behind the explanation of the phases of LS. Up to this point, we are identifying teacher-education praxeologies. However, there are some peculiarities due to the influence of the didacticians’ position as researchers, particularly their logos block. As we can see from another excerpt from the same section of the research report:

> RR12 With the support of the data collected up to this point [...]. Based on the analysis of the work […].

This suggest that the findings from experimental data are a new element of the didacticians’ teacher-education *theory*, alongside the ‘Principles of Japanese LS’. This is an aspect specific to the research process, due to the didacticians acting primarily as researchers. It can be observed how the didacticians’ position as researchers influences their practices as teacher educators. The didacticians are aware of a mismatch between the expected outcomes of the TPD course and the observed data. The data are collected and analysed through theoretical lenses that are specific to researchers in mathematics education. Here, the line between the didacticians’ position as teacher educators and the position as researchers is porous: adapting to this mismatch would be a normal teacher-education practice, but how this adaptation is tackled reveals the deeper influence of the research praxeology, as the questions in the anonymous survey are meta-didactic in nature. Therefore, we can observe the synergy between the two positions: as researchers, they investigate the causes of the mismatch in a process of design-based research. They discover that the mismatch may be caused primarily by the different meanings held by some terms in different institutional contexts. The knowledge generated by the research process can be modelled as elements of the logos block for their teacher-education praxeology. This results in new practices as teacher educators. In terms of MDT, this can be modelled as a double dichotomy between the *metadidactic level* of TPD and another level, which can be called the *research level* of the data collected during TPD.

5.6.5 *Prospective teachers' praxeologies after Lecture 2: towards a* shared praxeology

The answers to the final questionnaire can be analysed to identify any evolution, resulting from Lecture 2, of elements of the prospective teachers' praxeology. It can be seen that the prospective teachers' praxeology is now more similar to what was expected by the didacticians: the data infer that the expected *technique* 'to collaboratively plan and detail every phase of the mock lesson with educational goals, time needed, role of the teacher, and students' response' may indeed have emerged. Three answers are presented, to exemplify recurrent reflections by the prospective teachers:

> FQ6.1 If I had to rewrite it [...] I would be much more precise […] full of all those details that we have not reported because for me, an 'implementing teacher', they were already memorised.

> FQ4.2 I found the planning phase very useful because it allowed me to compare myself with the others [and] to understand how to plan a teaching activity when I become a teacher.
> FQ6.3 I would change our lesson plan[…] to make our intentions clear and visible to a reader who […] might otherwise not understand our choices.

In FQ6.1, the prospective teacher describes how a new activity report would be much more detailed than the one initially designed, and then suggests they may be referring to writing a lesson plan, even though the distinction with the activity report is still unclear. FQ4.2, by a different prospective teacher, describes how the planning phase is useful when interacting with other teachers, and how important this phase may be for their future profession. FQ6.3 hints that a detailed lesson plan is useful for sharing information with other teachers. Together, they provide *technologies* associated with different collaborative aspects of the Lesson Plan: the planning itself; and the possibility of sharing it in a collaborative effort of dissemination of good teaching practices. This suggests that teachers and didacticians have a *shared* understanding of LS as an object, albeit with specificities due to their positions.

Other answers to the final questionnaire suggest the impact of the didacticians' praxeologies on the prospective teachers. The following answers are given as examples:

> FQ2.4 I realised that we did not think about some important details, such as the position of the observers beforehand, specific things to be observed, or foreseeing what questions the students could ask during the lesson.
> FQ4.5 I did not fully understand LS until we started commenting [with the didacticians] on all the individual works […]. It was then that I realised the importance of the lesson plan.

FQ2.4 describes how the teachers became aware of the complexity of assuming a research attitude towards their own practice. This is evidence of how the researchers brokered some elements of their metadidactical praxeologies in the teaching community, and these elements are now *shared*. FQ4.5 reveals a newfound understanding of some aspects of LS, and how this could influence the teachers' previous praxeology. FQ2.4 and FQ4.5 also hint at a possible new technology for both the expected techniques related to lesson planning.

5.7 Discussion

5.7.1 Answering the research questions

The purpose of this paper was to investigate the complex twofold position of didacticians in acting as researchers and as teacher educators. The introduction of LS into the Italian TPD context was an opportunity to deepen our knowledge on the didacticians, as we observed that during TPD their positions as researchers and as teacher educators are deeply inter-related and continuously influence each other, and that the interaction with the teachers is an essential component of their work as didacticians.

With regard to RQ1 on the evolution of didacticians' praxeologies, we observed how this evolution could not have happened without the interactions with the prospective teachers, which proved an essential feedback for the didacticians. We note that to accomplish their main task as researchers (to study LS), they designed an experiment to implement LS in a TPD course. Due to their academic role, they also occupied the position of teacher educators, whose main task was to conduct the TPD course. The didacticians' practices and knowledge in terms of teacher-education praxeology evolved, supported by their research praxeology. In the first part of the experiment, they developed a set of teacher-education techniques used to introduce LS, supported by the logos blocks of teacher-education praxeology and of research praxeology. In the second part, these techniques changed, and so did the logos blocks. Between the two parts of the experiment, the didacticians became aware of a mismatch between the prospective teachers’ techniques they had expected and those observed in the course. The identification of this mismatch led the didacticians to investigate the teachers’ knowledge on LS, which is modelled by the logos block of their praxeology, and the impact of their teacher-education techniques on the prospective teachers’ learning of LS. The didacticians investigated their own practices, modelled by teacher-education praxeology: they identified some issues, discussion about how to solve these issues (which causes an evolution of their logos block, specifically the technologies, which are now supported by new theories) and subsequently re-shaped their techniques, along with the TPD course. The method of investigation reveals the influence of research praxeology on teacher-education praxeology, as the questions in the survey are meta-didactic in nature. The dialectic between the two positions of the didacticians (teacher educator and researcher) generates a double dichotomy between the *metadidactic level* and the *research level*, and provides an answer to RQ2 on the reciprocal influence between teacher-education and research praxeologies.

The analysis of the relationship between the didacticians’ dual positions forms the basis of the evolution of their praxeologies, as we find that their teacher-education praxeologies mainly evolve thanks to the knowledge that the didacticians generate in their position as researchers. More specifically, the experiment of implementation of LS described here resembles the process of design-based research (DBR), in which instructional design and educational research are intertwined (Gravemeijer & Prediger, 2019). As we have shown, the didacticians have a teacher-education task (the implementation of LS in the Italian context, instructional design) and a research task (studying LS, educational research). In the two parts of the experiment (corresponding to two cycles of developing, testing and revising in DBR), the didacticians develop teacher-education techniques which are critically examined through the interaction with the prospective teachers. The didacticians test their teacher-education techniques in the first part of the experiment, and revise them (as researchers) by analysing the feedback received from the prospective teachers, which causes an evolution in the logos block of didacticians' praxeology (by way of their didactical practice and in relation to their answers to the anonymous survey). In the second part, they test the revised teacher-education techniques, which are a design result. This will again lead to the revision of the techniques according to the feedback received from the teachers (the answers to the final questionnaire), explored in another paper. Research results were also produced, which will be discussed in the following sub-sections. This was only possible as the didacticians are mainly researchers, and DBR is part of their research practice.

This experiment also confirms the possibility of a convergence process by didacticians and teachers towards a shared terrain. In the second step of the experiment, we observe that the prospective teachers' knowledge of LS is more similar to that of the didacticians than at the beginning of the experiment, so is *shared* by the two institutions albeit with specificities due to their different positions. LS plays a dual role for the prospective teachers and the didacticians. For the Italian prospective teachers, LS is a model of teachers' collaborative practices which are characterised as a paradidactical praxeology, and an object of learning during the TPD. For the didacticians, LS is an object to be taught in teacher education (a paradidactical praxeology to be transposed), and an object of research. LS becomes an example of the *shared praxeology* theorised by MDT, which we can describe with the metaphor of the asymptote: over the course of TPD, the praxeologies of didacticians and teachers can *share* more and more elements, without ever being the same since didacticians and teachers belong to different institutions.

5.7.2 Implementation of LS

Studies on the adaptations of LS necessary to overcome cultural barriers were quite scarce, at the time of this experiment. This is no more the case (e.g., Huang et al., 2019) and it is interesting to notice that many conclusions from this experiment are in line with the new studies at the time (e.g., Ponte et al., 2018 or Peterson et al., 2019), which suggests that international collaboration is important to this field. However, this experiment demonstrated that local research is essential to understand the reasons why some aspects of LS may require more attention in some contexts that in other, in order to introduce LS successfully. Three specific aspects, related to the Italian context, emerged: activity – lesson; lesson plan; time planning.

With regard to the terms *activity* and *lesson*, we identified their ambiguity among Italian teachers due to the terminologies used in their ordinary practices (i.e. they usually design an activity rather than a lesson). This is the first time the issue has been highlighted in the literature on Italian TPD. This aspect may also be of international relevance: terminology is a crucial cultural factor which could prevent the implementation of LS (or other teachers’ practices) in other contexts. The role played by terminologies of mathematics education is now studied in mathematics education research: e.g. the Lexicon Project (Mesiti et al., 2022) investigates the terminologies used by teachers in different cultural contexts when describing mathematics teaching in the classroom.

This issue is connected to the problems identified with the second aspect, the lesson plan, which is a specific tool of LS. Its importance was not clearly understood in our experiment as the activity report plays a similar role for Italian teachers. The lesson plan may play some additional roles. Firstly, it can allow them to be more aware of their educational choices when planning lessons. Secondly, it can be a tool for gaining awareness of the context in which they are working, and to collaborate and discuss the teaching with colleagues. These aspects of the lesson plan must be clarified to teachers.

The lesson plan leads to another problematic aspect, time planning. Time planning of the lesson may be (e.g., Fernandez & Yoshida, 2004, p.73) or may not be part of Japanese Lesson Plans, but for our didactical goal (that is, making the teachers engage in self-reflection on their habitual teaching habits) it is required. Data suggest that it could be very difficult, for Italian teachers, to time plan a single lesson in detail, as they are not used to meticulous time planning of teaching activities within their institutions. Interacting with LS and its tools may compel teachers to focus on every moment of the lesson. While this feature may be due to the cultural differences in teachers' practices, this would be a challenge that is imposed when transposing practices from another cultural context, and it provides new insight for improving teaching practices as a consequence.

This study alerts us to the importance of understanding, and allowing others to understand, the cultural context in which we work. Making the cultural context accessible to others is a current challenge in mathematics education (Bakker et al., 2021), for many years (e.g. Adler et al., 2005). We need 'to frame our research accordingly, so as to provide careful attention to their [the cultural and institutional constraints] influence on teaching and learning processes in mathematics' (Minisola &

Manolino, 2022, p.8), while making these constraints explicit and accessible to the research community.

5.7.3 *Theoretical contribution*

ATD provided a solid framework for formulating the didacticians’ and prospective teachers’ knowledge and practices in terms of praxeology, and for understanding the influence of institutional constraints on the didacticians and on the prospective teachers’ praxeologies, but it did not exhaust the complexity and specificity of the situation.

This paper suggests a further evolution of the ATD framework (in which teachers are *the ones who teach*), as teachers and didacticians assume different positions. Teachers are the *ones who teach the students,* and they are also *learners of how to teach and how to prepare teaching*. Didacticians are both *the ones who teach the teachers* and *the ones who research* (about TPD). It is worth highlighting that the didacticians, here, are *the ones who teach the teachers about LS,* but in other contexts they may also be *the ones who support the teachers during LS as knowledgeable others*. In this paper they do not take on this position, which would add another layer of complexity. This should be considered for future experiments.

Making sense of the complexity of these positions is a current research problem (Robutti et al., 2016). MDT attempts to integrate them into a wider framework (Arzarello et al., 2014), whereby the praxeologies of didacticians’ and teachers’ institutions are analysed in their evolution. The term *metadidactical praxeologies* identifies reflections *on* didactical (and paradidactical) activity, when teachers and didacticians work together during TPD (Arzarello et al., 2014). We were able to analyse the complex interactions between didacticians and prospective teachers, with respect to the dual positions highlighted by the ATD framework. In particular, we observed the results of the collaboration between didacticians and prospective teachers during TPD: didacticians gain a better understanding of LS, while prospective teachers learn new didactical and paradidactical praxeologies and are exposed to research praxeologies that enable *higher* levels of self-reflection. Specifically, the notion of *positions* proposed by ATD brings us to a novel result in MDT. Until now, the double dichotomy that allows the evolution of the praxeologies had been observed only in the case of teachers, between the didactical level developed in the classroom and the metadidactical level developed during TPD (Pocalana, 2023). The results of our analysis show that a double dicothomy exists also for the didacticians, between the metadidactical level developed during TPD and the research level, developed when they are making sense of the data collected during TPD.

This does not exhaust the complexity of the results. This study shows that:

- on the didacticians’ side the double dicothomy, as suggested by MDT, shows that it is difficult to investigate didacticians’ practices and knowledge by clearly separating their research and teacher-education praxeologies. This suggests the need for an evolved model that strongly accounts for the complex position of the didacticians, and the relationships between research and teacher-education praxeologies;
- on the teachers’ side, it is difficult to investigate the evolution of their practices by isolating what is caused by themselves and what is caused by the relationship with the didacticians. This calls for a model which considers the peculiarity of certain TPD contexts.

MDT indicates such models as *metadidactical praxeologies* for both didacticians and teachers, as they are working *on* the didactic and their dynamics are deeply intertwined. However, the *praxis* and *logos* of didacticians and teachers are usually different, and collecting them under the same name may obscure some details. Moreover, ‘metadidactical’ transposition (i.e. transposition of didactical or paradidactical praxeologies) may also occur if teacher educators, with different praxeologies, were involved. This calls for a finer definition of the dynamics and praxeologies involved in *metadidactical transposition*.

Even so, our findings suggest that the *combining* (in the sense of Prediger et al., 2008) of the ATD and MDT theoretical frameworks may guide researchers in designing and analysing TPD programmes ensuring they interpret their complexity. *Coordinating or even integrating* them goes beyond the scope of this paper, and will be explored in future.

5.8 Conclusions and Perspectives

This study gives us an insight into two aspects that guide the complex position of didacticians in TPD: one is the didacticians’ research outlook; another is cultural, linked to the original context of LS. Implications are suggested for the study and implementation of LS itself through a cultural outlook. Possible synergies of the theoretical frameworks of ATD and MDT were shown, and thanks to these synergies both frameworks were expanded. We are also aware of some limitations. Data collection was limited: future studies could take advantage of other documents to analyse the dynamics internal to the communities. Much is left unsaid on the prospective teachers: future studies should consider the extent to which LS may contribute to their professionalism. Finally, we have questions on LS as a research object. The transposition of LS from didacticians to teachers was investigated, but the process of transposition from Japanese teachers to Italian didacticians is still obscure. Moreover, the role of LS in shaping the relationship between the communities involved should be considered.

Acknowledgements The authors are grateful to Carola Manolino for her contribution in conducting the experiment.

6 CULTURALLY CRAFTED LESSON STUDY TO IMPROVE TEACHERS' PROFESSIONAL DEVELOPMENT IN MATHEMATICS: A CASE STUDY IN ITALIAN SECONDARY SCHOOL

Capone, R., Adesso, M. G., Manolino, C., Minisola, R., & Robutti, O. (2023). Culturally crafted Lesson Study to improve teachers' professional development in mathematics: A case study in Italian secondary school. *Journal of Mathematics Teacher Education*. https://doi.org/10.1007/s10857-023-09578-3

Chapter 6 proposes an experiment in which Lesson Study is introduced to in-service higher-secondary school teachers. The paper is focused on the cultural aspects of implementing Lesson Study in the institutional and cultural context of Italian secondary school. The results provide a new understanding on how to promote and design relevant professional development for mathematics teachers in secondary school, even outside of Italy.

This paper describes a Lesson Study in which in-service mathematics secondary-school teachers, collaborating with researchers, involve grade 10 students in tessellation problems. The data are collected by an experiment carried out in the context of the "Liceo Matematico" project, with three volunteer teachers. The experiment goal was to craft a collaborative design of the research lesson between teachers and researchers. The research aim of the paper is to examine the use of Lesson Study in the institutional and cultural context of Italian secondary school with the use of Cultural Transposition as a theoretical framework. The research is qualitative with idiographic aims, based on video research. The educational aim of the research is to provide a solid basis for a revamped in-service teacher education first in the context of the project, then in curricular context. Semiotic mediation is used to provide, within Lesson Study, the conceptual framework for teachers and researchers collaborative design of the research lesson. The results show that Lesson Study, as a foreign practice, is an opportunity for teachers to confront their teaching practices, to enrich their professional development, resulting in more awareness on their didactical action in and outside the classroom.

Keywords: Lesson Study; Teachers' Professional Development; Semiotics; Semiotic Mediation; Cultural Transposition

6.1 Introduction

In recent years, practicing teachers' professional development has acquired a central position in the Italian and international debate on educational policies (Italian law 107/2015). In-service mathematics teacher education offers tools and opportunities to research, for investigating situations that encourage teachers to break out of routinised practices and move towards re-elaborating and re-planning their teaching (Brophy, 2006). Teachers agree that in-service professional development is the driving force of innovation and that the teaching-learning processes cannot remain static (Vermunt et al., 2019; Weber et al., 2018).

Among the various models of collaboration-based teachers’ professional development, that combine effectiveness with sustainability, the Japanese 授業研究 (Jugyokenkyu), known in the Western world as *Lesson Study* (LS) (Yoshida, 1999), stands out.

Through and thanks to LS, several researchers around the world have identified active changes, consciously implemented by the teachers themselves, in teachers’ beliefs or disposition towards work and learning, in their mathematics knowledge for teaching, and in their teaching practices (i.e., Huang & Shimizu, 2016; Xu & Pedder, 2015). A significant feature is that LS occurs within the school environment, where teachers play a central role, and is based on teachers and researchers collaboration. LS creates a context for teachers’ collective reflection and develops the sense of confidence that is central to teachers’ professional development (Nguyen & Tran, 2022; Pang, 2016). In February 2020, the ICMI Study 25 Conference focused on *Teachers of Mathematics working and learning in collaborative groups*, and LS was very present in the presented research works (both in one plenary lecture and in 14 out of 80 accepted papers cited “Lesson Study” in the title, e.g. Capone et al., 2020; Otaki et al., 2020; Shinno & Yanagimoto, 2020; Skott, 2020). The assumption underlying this ICMI Study is on teachers learning and working through collaboration; however, it can be challenging to investigate and explain how this learning occurs and gather evidence of what and how teachers learn.

In this paper, we present the (to our knowledge) first experiment of LS in Italian secondary school. The main aim of the paper is examining the use of LS in the institutional and cultural context of Italian secondary school to provide a solid basis for a revamped in-service teacher education. The LS experiment has been carried out in the context of the Liceo Matematico project (Capone et al., 2017; Capone, 2022) - a project of teacher professional development in which collaboration is a cornerstone, with 3 volunteer mathematics teachers.

The research is qualitative with idiographic aims, based on video research. Cultural Transposition (Mellone et al., 2019; Ramploud et al., 2022) is the main theoretical framework of the research and gives a tool to interpret data. Semiotic mediation is used to provide, within LS, the conceptual framework for teachers and researchers collaborative design of the research lesson.

6.2 Literature review

LS originated in Japan around 1870, to answer professional development needs for qualified teachers (Isoda et al., 2007), and gained attention within the years thanks to the TIMSS Video Study (Stigler et al., 1999) and the book The Teaching Gap (Stigler & Hiebert, 1999). In the last 30 years, it has been increasingly used around the world with teachers of different grades, including higher education: for example, LS was introduced in the United States thanks to Catherine Lewis' research (2000) and Makoto Yoshida's doctoral thesis (1999), creating an Anglo-American tradition; and in Europe it is present in a number of countries (e.g., Bartolini Bussi et al., 2017; Ní Shúilleabháin, 2018; Ponte et al., 2018; Winsløw et al., 2018). LS has been effectively studied in teachers' professional development almost worldwide (e.g., Manolino, 2021a; Okubo, 2007; Ono & Ferreira, 2010; Presutti, 2022).

6.2.1 Lesson Study's main features

LS gained ground to gradually improve standard classroom practices (Stigler & Hiebert, 2016), and different specific features and conceptualizations of LS or associated forms of activity have developed in many countries, pursuing aims and assuming formats that appear very different from each other (Huang et al., 2019; Quaresma et al., 2018). Even within Japan itself LS can take multiple forms (Miyakawa & Winsløw, 2019). Identifying its essential generic characteristics can give insights when studying this model of professional development. A great deal of work in this sense has been coming from Western-cultured researchers, and the diffusion of LS in European countries, which mainly occurred in the last decade (Quaresma et al., 2018), contributed to its development. Mathematics Education teams all over the world are coming to an identification of the “essential generic” characteristics of LS (Buchard & Martin, 2017) through the detailed review of the existing literature and discussions in the WALS group, albeit there is still no complete academic agreement. Each country, each reality, has local cultural features.

One possible diagram is presented in Fig. 6.1 by Buchard and Martin (2017) from Switzerland: albeit slightly different from most used representation of LS (Lewis, 2000, in the USA), we believe it includes some elements of the process (such as the recruitment of participants or the sharing as an element of teacher education beyond the individual cycle) that are not trivial for the success of LS, especially when it is transposed in a context different from the Eastern one.

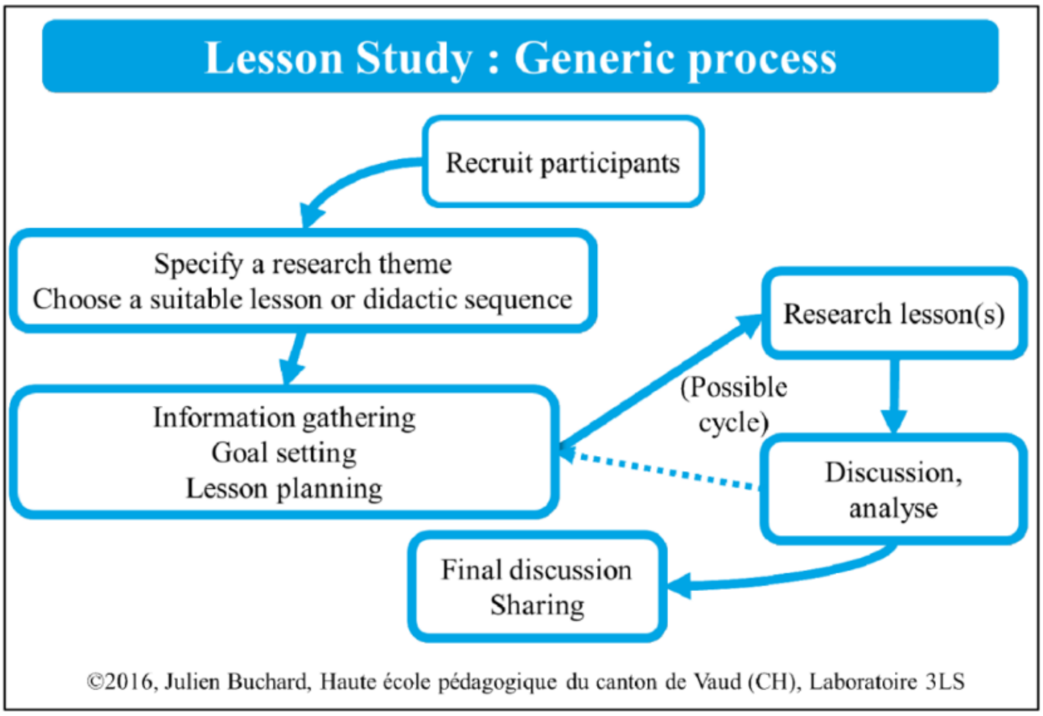


Figure 6.1 The essential Lesson Study features (Buchard & Martin, 2017, p. 13)

A further conceptualisation comes from Seleznyov (2018), who studies LS in the United Kingdom. She points out what she defines "the seven critical components" (ibid., p. 221) of LS:

(1) Identify focus: the LS group compare long-term goals for learning;

(2) Planning: collaborative work leading to the production of a collaboratively written plan (Lesson Plan) for research lesson;

(3) Research lesson: taught by a nominated teacher—the implementing teacher, while the other group members act as observers;

(4) Post-lesson discussion: the LS group meet to formally discuss the evidence gathered;

(5) Repeated cycles of research: new lessons and not revisions nor re-teachings of previous research lessons are planned and taught;

(6) Outside expertise: there is input from knowledgeable other;

(7) Mobilising knowledge: opportunities of teacher learning are created, through observing and networking.

LS is usually modelled as a cycle. Figure 6.2 illustrates the diagram used to present LS to the teachers involved in our experiment, proposed by Joubert et al. (2020, p. 910), from South Africa.

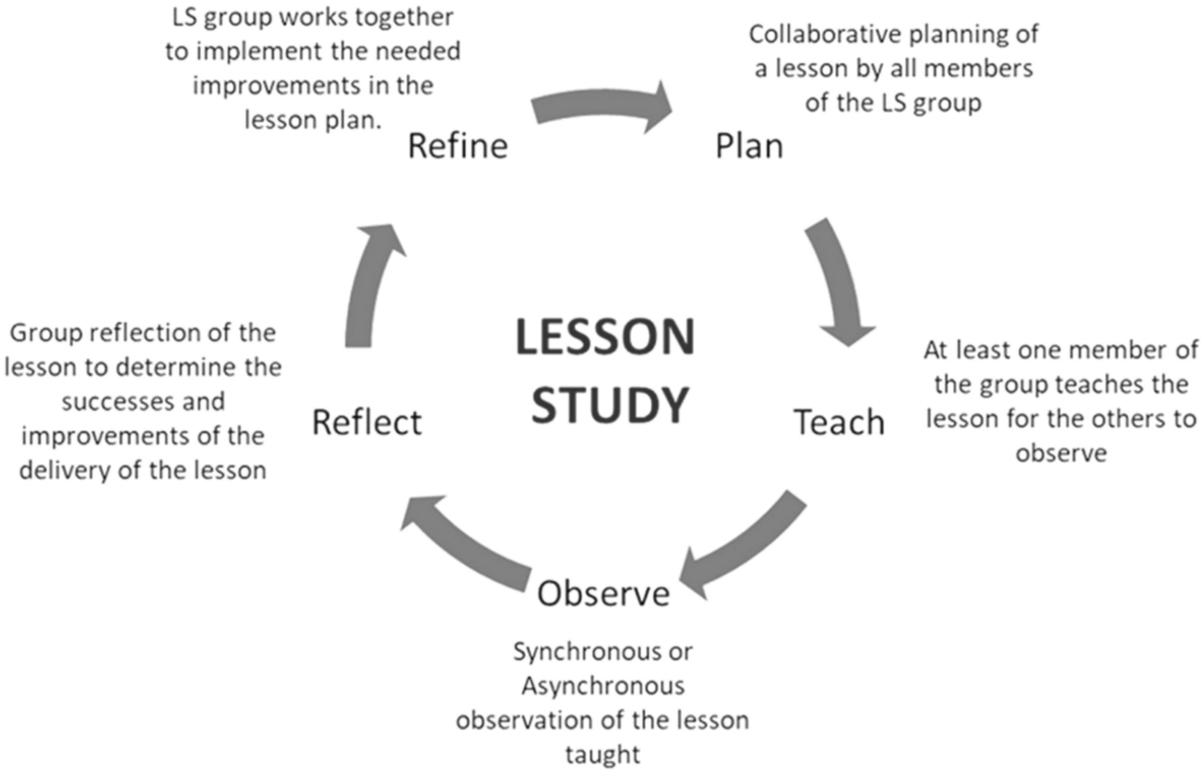


Figure 6.2 The LS cyclical model (Joubert et al., 2020, p. 910)

A number of researchers define LS as a collaborative teachers’ professional development model, focused on the participants’ co-responsibility in the process of planning, implementation, and revision of lessons (in Mathematics, in our case), with exceptional attention to details such as time scheduling, planning of materials and instruments to be used (e.g., Bartolini Bussi et al., 2017; Fernandez & Yoshida, 2004; Huang et al., 2017), characteristics that are rooted in its Confucian origin. European researchers report that, as such, LS has offered to European teachers both a new approach to working on lesson planning, and an opportunity to confront with a different culture, sprouting reflections on the implicit assumptions that surround teaching practices (Clivaz & Miyakawa, 2020; Skott & Møller, 2020). In a review of Western studies on LS, Ponte (2017) from Portugal highlighted these potentialities of LS, while also stressing some difficulties related to the adaptation of LS in a context different from the original one. This study looks to contribute to the knowledge on LS adaptations in Western countries, hoping that the results obtained in the Italian context may prove useful for other contexts as well.

6.2.2 Lesson Study in Italy

As outlined by Ponte and colleagues (2018), LS often starts from a detailed analysis of the teaching difficulties, and teachers research the tools to tackle such difficulties. Because of this, teachers participating in LS may be considered involved in research, without being "qualified" researchers. Hollingsworth (1995) calls them "teacher-researchers": "[they] are concerned simultaneously with (a) ways to improve their practices, (b) change the situations in which they work, and (c) understand their practices within the larger society" (ibid., p. 16). The idea of teachers as researchers (teacher-researchers) has been present in the Italian context since the 1960s (Arzarello & Bartolini Bussi, 1998), and is one of the many reasons that justifies the Italian interest about Lesson Study. In the following, we describe the two main paths through which the Italian research group in Mathematics Education became aware of LS:

(1) The Department of Education and Human Sciences research group at the University of Modena and Reggio Emilia has approached Chinese teaching research experiences in international meetings. Lesson Studies have been

observed in China (Bartolini Bussi et al., 2017). The researchers carried out several LS at the primary school level in Mathematics and Language. From teaching research experiences observed in China, they “identified some conflicts emerging between the cultures of teaching in China and Italy and explored the way to overcome the conflicts”. For example, they “identified some additions to the Lesson Plan, suitable to make LS more compatible with Italian didactical context” (Ramploud et al., 2022, p. 154): one of them is a detailed analysis of classroom context, another - in line with the semiotic mediation framework—is the analysis of materials (Ramploud et al., 2022).

(2) The Department of Mathematics research group at the University of Turin became interested in LS while working on the survey for ICME13 (Robutti et al., 2016). The researchers implemented Japanese and Chinese LS with in-service and prospective primary and secondary school mathematics teachers (Manolino, 2021b).

At the present time, the introduction of LS in the Italian context is in its prime: i.e., we are currently dealing with issues around the management of time (as described in), establishing a shared non-confusing terminology to introduce LS to the teachers (Manolino, 2021b), and so on. Therefore, the characteristics that differentiate Japanese and Chinese LSs are not being considered to their full extent. We could say that Italian researchers are, at the moment, more interested in the general “philosophy” of LS, with special regards to the collaborative setting for planning a lesson in detail (Capone et al., 2020). From these studies, further experiments are being conducted in Piemonte, Lombardia, Valle d’Aosta, and more recently in southern Italy: Naples (Ribeiro et al., 2019) and Salerno (Capone et al., 2022a; this paper). An essential emphasis is provided to the cultural background in which the teaching processes occur (Bartolini Bussi & Martignone, 2013; Mellone et al., 2019).

The LS described in this paper is in continuity with Italian studies and with the contributions presented by the authors at WALS 2019 (https://www.walsnet.org/2019/; Capone et al., 2019), the Second International ACME Symposium on Mathematics Education (https://math.ecnu.edu.cn/academia/acme2019/en/index.html), and ICMI Study 25 (http://icmistudy25.ie.ulisboa.pt/; Capone et al., 2020).

6.2.3 Teachers' professional development in the Italian context and the Liceo Matematico project

As written before, Italy has a long tradition in collaboration between researchers and teachers, which has become a peculiarity of Mathematics Education Research (Arzarello & Bartolini Bussi, 1998), that has recently introduced LS as a topic of investigation (Minisola & Manolino, 2022).

One of the institutional contexts in which this collaboration is carried out is the Liceo Matematico project (https://www.liceomatematico.it/) (involving 25 Universities), characterised both by local teachers' professional development and implementations in schools, and national meetings in seminars/congresses to share issues and resources. At the local level, secondary school teachers collaborate in various ways with researchers, to design, solve, discuss, and implement (mathematical or interdisciplinary) activities for the students (Capone & Faggiano, 2022). The novelties of the educational approach of Liceo Matematico project are manyfold: institutional, because schools and academies involved are officially related by institutional agreements; professional, because of the double role of teachers: as learners in a community of colleagues, in contact with researchers, and as teachers in their classes;

didactical, because of the *mathematics laboratory* approach (Bartolini Bussi & Martignone, 2013) and other teaching practices coming from research studies (e.g., inquiry-based methods, mathematics argumentation). Researchers provide teachers with constant support, via online and face-to-face interactive meetings of professional development, giving them theories, methods, and activities to discuss and implement.

We decided to use the educational context of the Liceo Matematico project to implement LS at secondary school level, because of the already present collaboration among teachers. LS introduces in the community of teachers of this project something more: a structured sequence of collaborative planning and implementing of activities. The results will be useful in a double way: to understand if and how LS can be fruitful in this project and to study its implementation in a more general school and teacher education contexts (out of the project). And moreover, these results can give a more general approach to the introduction of LS, based on a cultural and institutional implementation.

6.3 Theoretical framework

In this section, we present the Cultural Transposition (Mellone et al., 2019; Ramploud et al., 2022) as a theoretical frame for investigating the use of LS in an institutional and cultural context different from the one in which it was created. The Theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008) is also detailed, as it was an important element of the teacher education process: the researchers introduced it to teachers, as a theoretical tool for the planning and teaching phases and as a lens for reading experimental data from the research lesson with the teachers.

6.3.1 Cultural Transposition

Cultural Transposition is not only a theoretical but also an action perspective for teachers and teacher educators. It aims to frame the use of mathematical teaching practices from foreign cultural contexts as an opportunity to question one's own teaching practices and educational intentionality (Mellone et al., 2021), in an "emancipatory" context for teacher learning (Mellone et al., 2020, p. 382). Within this framework, the study is situated in the context of teacher and teacher-educator learning literature (Jaworski & Huang, 2014), where teacher professional development increases teachers' knowledge and awareness (Andriano & Manolino, 2023) while reflecting in (and not only on) their teaching practices in a joint activity, with the aim of improving opportunities for learners of mathematics in the classroom. Here, teacher professional development framed in Cultural Transposition is innovative: culture becomes the trigger for development. Teachers, "interacting with foreignness" (Welsch, 1999, in Barton, 2009, p. 38; Manolino, 2021b) are led to become aware of their *unthoughts* (Jullien, 2006), namely, what is taken for granted and, in a sense, considered mandatory in their practice.

The idea of Cultural Transposition is not to "perform a comparative study or a slavish import-export of mathematics education methodologies and tools between different countries, but rather to open a dialogue between two different cultures in which every thought, meeting the other culture, questions its own unthought", as an emancipatory tool for teacher learning (Mellone et al., 2020, p. 382). An example is described in Di Paola & Buttitta (2020), where variation problems, as one of the most significant problem solving approaches in Chinese schools, is discussed as a useful methodology to let the students discover the relationship between pyramid and cone areas/volumes even in cultural contexts where relations between solids are not usually the focus of the teaching of this topic. In fact, “the transposed teaching practice should embed some of the cultural aspects of its original context, as this would allow teachers to come into contact with different values and beliefs on mathematics teaching/learning” (Mellone et al., 2021, p. 786).

Cultural Transposition (as conceptualised in Ramploud et al., 2022) attempts to hold together a dual purpose: on the one hand, to produce visible changes in teachers’ teaching practices—the primary aim of teacher professional development; on the other hand, to produce teacher professional development practices that is culturally situated—namely, contextual and tailored to teachers’ actual needs.

LS is just one of the possible teacher education models that can be framed in Cultural Transposition. Probably one of the most emblematic, and others have been shown in previous research (transposition of Chinese problems with variation: Mellone et al., 2019; of the El’konin-Davydov curriculum: Mellone et al, 2020; of the Thinking Classroom: Mellone et al., 2021; of the LS in Distance Education during the COVID pandemic: Ramploud et al., 2021; Capone et al., 2022a, Capone et al., 2022b), or for some other practices the need for Cultural Transposition has been postulated for the exportation of practice (transposition of the Reggio Emilia Approach: Landi & Pintus, 2022).

Embedded in the context of LS as an illustrative example, to attempt their dual purpose, Ramploud et al. (2022, p. 150-151) show how the three Key Phases of Cultural Transposition ((KP1): Contact with teaching practices of other cultural contexts; (KP2): deconstruction of teaching practices; and (KP3): Teacher education and development practices) lead to the development of different varieties of LS, always contextualised, as the LS group works. They are called transpositions, i.e. hybrid varieties (in the sense of Ribeiro et al., 2019) of the initial object.

Transposition significantly differs from "cultural adaptation", as meant by authors such as Stigler and Hiebert (2016): the purpose of Cultural Transposition is *not* to achieve the adaptation of LS suitable for the target cultural context. The word "adaptation" conveys the idea of something general, "universal". Rather, what the cultural transposition does is to stress, time after time, one or more hierarchical oppositions of antinomic concepts on which it is chosen to be overthrown, and depending on those decisions, different varieties of LS could be developed suited to the concepts being worked on. Therefore, what Cultural Transposition describes are processes, not adaptations from one static model. Very local ones, that heavily depend on the choices made both, at the input, by the researcher and by the teachers implementing the LS, by the goals of the teacher professional development.

Those processes are carried out in the three Key Phases. KP1 is the encounter with the other/foreignness: teaching practice from another cultural context (here the LS). In KP2, teachers and knowledgeable others identify the deconstructive potential of the foreign practice, i.e. identify some hierarchical oppositions. KP3 is the experimentation of the foreign teaching practice. The three phases constitute an ongoing cyclic process. Throughout these phases it is carried out what Derrida (2002) calls *deconstruction*. The moves of deconstruction are two (Ramploud et al., 2022, p. 149):

(1) "Overthrowing of hierarchical oppositions": addressing the conflict, considering its structure. It is not just a matter of "switching" the hierarchy between two opposite values.

(2) “Hyperanalysis which, in the sign, points to the impossibility/unthinkability of the other”: a continuous movement from one value to its opposite, which is not aimed at reaching a dialectical synthesis between them (the “third term”), but at continuously pointing the other.

The whole process elicits reflection on one’s own practices, prompting a choice of which hierarchies to overthrow and which to maintain, i.e. to become aware of and act on one’s own unthoughts.

The context of the experimentation described in this paper is part of the ongoing cyclic process initiated by the two mentioned research groups (par. 2.2), which are working in the spirit of Cultural Transposition. Researchers activated several processes of deconstruction. We, as well as them, identified in this paper several hierarchical oppositions that can be overthrown keeping the internal articulation of the LS “critical components” almost unchanged. And through hyperanalysis process, we and them also identified some additions, suitable to make LS more compatible with Italian didactical context. An outstanding example is the adoption of a theory, namely the Theory of semiotic mediation, to provide the conceptual framework for teachers and researchers collaborative design of the research lesson. “Despite these changes, LS maintained its deconstructive potential with respect to several hierarchical oppositions” (Ramploud et al., 2022, p. 155), which allowed the reflection of teachers, highlighting new hierarchical opposition to be overthrown, some that have been maintained and others that LS was successful in overthrowing.

6.3.2 Theory of semiotic mediation

The expression “semiotic mediation” refers to the Vygotskian theoretical framework (Vygotsky, 1978). The process of “internalization” is described as a process of building knowledge and modes of thought derived from lived experiences. Two main aspects characterize this process of internalization: an external action, which is essentially social, and an internal action regulated by semiotic processes. These two aspects are interrelated because the systems of signs and the semiotic processes connected to their use are intrinsic to a social or individual activity. In particular, the Vygotskian hypothesis foresees a very close link between technical and psychological tools, thus creating a didactical path that, starting from the use of tools, aims to construct mathematically significant meanings and concepts. In this sense, learning can be considered as a social activity mediated by the teacher (Vygotsky, 1978).

In Fig. 6.3, a scheme of the semiotic mediation, as shown in Bartolini Bussi and Mariotti (2008) is reported.

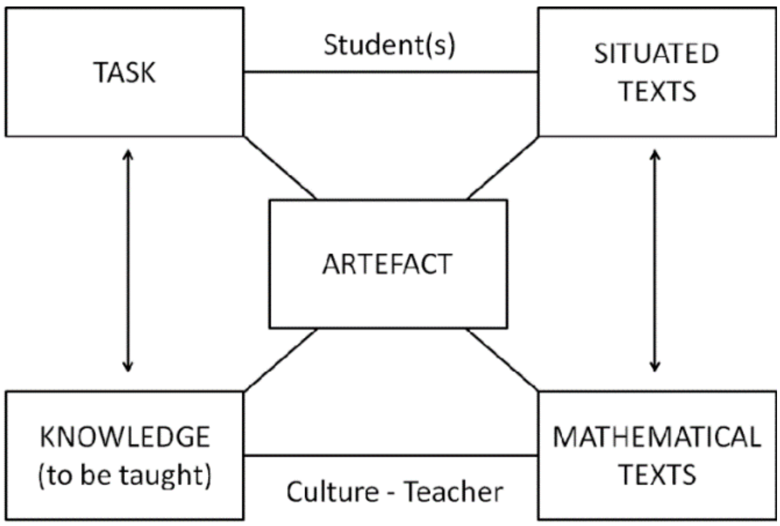


Figure 6.3 The semiotic mediation scheme (Bartolini Bussi & Mariotti, 2008, p. 246)

As Fig. 6.3 highlights, at the heart of semiotic mediation, an artifact embeds mathematical meanings but is not transparent to embedded meanings (Bartolini Bussi &

Mariotti, 2008, p. 246). The teacher gives some tasks to the students. Students, interacting with the teacher-mediator, leave “traces” of their activities (through situated texts). These traces constitute a dynamic system of signs (gestures, words, drawings, sketches,…) and their relationships (e.g., the simultaneity of a gesture with an utterance) produced by one or more subjects who interact during the implementation of a task. Through a social interaction process, the situated texts become mathematical texts, and this cycle leads to the student’s appropriation of mathematics knowledge.

Mathematics knowledge, tasks, and artifacts are the triangle of the semiotic potential of the artifact; it can favour, through concrete and significant actions, formal knowledge acquisition.

Therefore, the semiotic potential of an artifact is given by the set of signs that can arise when used as a tool when the student solves a task or a problem (Bartolini Bussi & Mariotti, 2008). The activities for the students and their organization are decided according to a preliminary analysis of the semiotic potential of artifacts so that the students’ personal meanings can develop as they emerge in-class activities, reaching the status of mathematical meanings, thanks to the teacher’s intervention.

The choice of artifacts is fundamental to the lesson’s success, and the teacher’s role is decisive in this choice (Bartolini Bussi & Mariotti, 2008).

In this LS, the Theory of semiotic mediation is an element of teacher education in the co-planning and teaching phases and a lens for reading experimental data with teachers within the reflecting phase. It is used to frame and then foster the deconstructive process to achieve greater awareness of teachers’ *unthoughts*.

Given this theoretical framework, we can now re-formulate our naïve research questions.

We investigate how, according to the teachers, LS can contribute to the professional development of Italian in-service secondary-school teachers. To do so, we investigate teachers' deconstruction of teaching practices and the *unthoughts* emerging from their interaction with foreignness in professional development. Our re-formulated research questions are:

(1) Given the teachers' interaction with LS, what are the hierarchical oppositions that emerge?

(2) What effects do these hierarchical oppositions have on teachers' professional development?

6.4 Methodology

In this section we present the context of the experiment, with the description of the setting and the participants involved, the data collected and the method.

6.4.1 Setting and participants

The LS experiment presented in this paper was carried out in a scientific-oriented secondary school in Avellino, a suburb of Naples, in the South of Italy. The school is involved in Liceo Matematico project with the University of Salerno.

The confidentiality of participants is maintained through comprehensive pseudonymization.

The headmaster invited two researchers (R_1 and R_2, two of the authors) from the Department of Mathematics at the University of Salerno to hold a teacher-oriented seminar on LS.

Three teachers (T_1, T_2 and T_3) volunteered to experiment with LS in their classes.

T1 has a degree in mathematics with about 10 years of teaching experience. She is a highly motivated teacher who has also proposed as the first to experiment with LS in her classroom. T2 has a degree in mathematics and has been teaching for 15 years. She has dealt extensively with children with special educational needs. T3 has a degree in physics and has a great deal of experience teaching both mathematics and physics. She has been teaching for 38 years and is close to retirement. The two researchers and the three teachers constituted the *project group* and R_1 acted as facilitator. R1 is currently a researcher in mathematics education but has a lot of teaching experience in high school because he taught in high school for 12 years. R2 is also currently a researcher in mathematics education but has about 20 years of teaching experience in high school.

Three different LS cycles were implemented in three grade-10 classes, one for each teacher. T_1 was the first to implement the research lesson (from now on, we will refer to her as the *implementing teacher*).

Grade-10 classes were chosen because in Italy these students are assessed through national standardised tests (INVALSI, 2017) and the project team wanted to work on the issues revealed by the tests (see par. 5.1). Out of all the data collected in the experiment, in this paper we refer as a convenient sample to what was gathered from the implementation of the only first cycle.

6.4.2 Data

Our data, concerning the first LS cycle only, are:

- video recording of the co-planning meetings (lasted a total of 4h), the teaching phase (1h30'), and the reflecting phase (1h50');

- Lesson Plan: a 13-pages document;
- 8 observation grids: each observer (i.e., all members of the project group except T_1) filled out an observation grid twice, the first time in person during the research lesson, the second time by observing the video recording;
- a collection of photos in all the phases;
- 3 questionnaires: the three teachers answered a questionnaire at the end of the reflecting phase.

The tools used for data collection were:

(1) three video cameras, placed initially in 3 different corners of the classroom, one fixed;
(2) the Lesson Plan, designed by the Modena-Reggio Emilia research group (Manolino et al., 2020; Ramploud et al., 2022);
(3) a semi-structured observation grid, jointly designed by the project group;
(4) a final questionnaire, designed by the researchers and filled-in by the teachers.

6.4.3 Method

The research methodology is observational and interpretive (Baker, 2006), with the added value of video research, which allowed researchers to delve into verbal codes (conversational exchanges, oral reflections) and non-verbal, proxemic and interactional codes.

In fact, we approached the “investigator triangulation” (ibid., p. 283). Three authors of this paper independently watched the videos and transcribed the excerpts. All the authors then checked for interpretive validity, i.e. “accuracy in reporting the facts” (ibid., p. 285), by asking for participants’ feedback and checking the use of “lowinference descriptors” (i.e. direct quotes) (ibid., p. 283; see also Adler & Adler, 1994). Finally, all the authors carried out a triangulated discussion on the meaning attributed to what is being studied.

6.5 The case study: an Italian Lesson Study with in-service secondary school teachers

In this section, we present the case study of the first cycle, taught by T_1. After this first cycle, two further LS cycles were planned and implemented with two other classes (not presented here). The first cycle involved twenty-five students.

In the following subsections, all phases of the LS cycle are described in detail.

To introduce LS to teachers, we used the cyclic model proposed by Joubert et al., (2020, Fig. 6.2). One of the reasons for choosing this model is that the observation phase does not coincide with the teaching phase. This served to give value to the two moments of observation: both synchronous, during the teaching phase, and asynchronous, analysing the videos of the research lesson within the reflecting phase.

Compared to the original model, we have made some minor changes (Fig. 6.4). The “plan” phase is called “co-planning” to emphasise the importance of collaborative aspects. Verbs in the gerund are used instead of nouns because they better represent the dynamism of the didactic action.

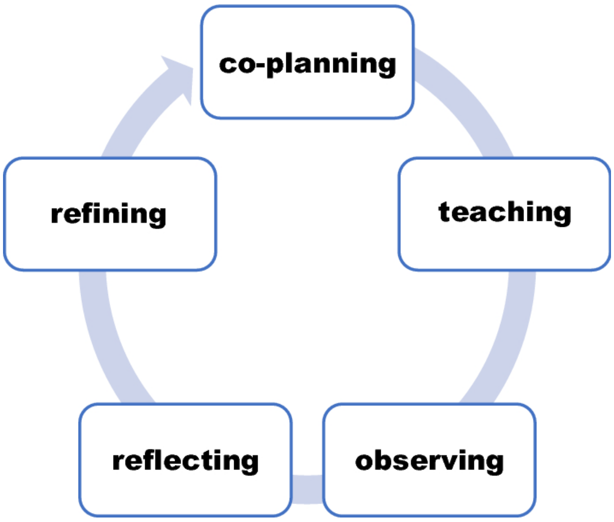


Figure 6.4 The cyclical model used for Lesson Study in the Italian secondary school (adaptation from the Joubert et al. (2020) model of Fig. 6.2)

During the co-planning phase, the project group drafted the Lesson Plan, which includes a detailed description of the class, the activities to be carried out, and the materials to be used, and anticipates student responses, misconceptions, and lesson successes. Research objectives for active professional development are also chosen by the project group. For each activity of the research lesson, the classroom setting, and precise timing are planned and the teachers' educational intentionalities are described.

During the teaching phase, T_1 implemented the lesson. The lesson was video-recorded. By means of observation grids, observations were addressed on the elements chosen by the project group to be discussed during the reflecting phase. In the observing phase, the teacher's actions were observed. In the observing and reflecting phases, each project group component observed the teaching (synchronous observation), then analysed the recordings (asynchronous observation) and filled the grids. In the refining phase, the Lesson Plan was modified: changes were made collectively, based on the criticism that emerged during the reflecting phases to begin a new LS cycle, in which the lesson is taught by another teacher from the same group in a new class.

In the following, all the documents and empirical data are original in Italian language, and they have been translated by the authors.

6.5.1 Co-planning

The co-planning phase took place in three different meetings: the first two in-person at the university (1h30' each) and one via Skype (1h). The first meeting started with the three teachers (T_1, T_2 and T_3) commenting the teacher-oriented seminar on LS conducted by R_1 and R_2 at school upon the invitation of the school headmaster:

> T_1: It seems a very distant teaching system from ours.
> T_2: Such a constructed and planned lesson seems too far removed from our freedom of teaching, guaranteed by law and by the National Curriculum.
> T_3: Thinking of being observed during the lesson… I don't think it is a good thing for the students, but also for us as teachers, who won't feel fully comfortable.
>
> [CO-P1]

During the rest of the meeting, in accordance with the literature on LS (Ponte et al., 2018), the project group focused on identifying students' main difficulties in mathematics. T_1 suggested focusing on Geometry content. The reasons given, with which all other participants agreed, were related to students' difficulties in Geometry at national level (Capone et al., 2022a): it is known from INVALSI (2017), shown in Table 6.1, that the percentage of correct answers in Geometry in national standard tests in grade 10 is the lowest compared to the other content domains.

Table 6.1 INVALSI Report 2017. Percentage of correct answers in different content domains (adapted form INVALSI, 2017, p. 49)

Content domains	**% Correct Answers**
Numbers	49,78
Geometry (Space & Figures)	41,02

Data and Previsions	53,94
Relations and Functions	46,30

Teachers and researchers discussed some possible causes of these failures in Geometry, suggesting different strategies to overcome those difficulties. A discussion arose on the necessity of certain key concepts of the teaching of geometry and their importance for the development of logical thinking (the project group's *goal of the research lesson*). Transcripts of some fragments of the recordings are given:

> T_1: In my opinion, students find more difficulties in Geometry because they have difficulty in reading and interpreting a text and sometimes even in graphically representing what is written.
> R_1: I invite you to read some papers, which describe the difficulty that students encounter within the transition between different registers and representations.
> T_2: I think that we could start from intuitive geometry rather than rational geometry.
> T_1: But proofs are essential, as in the National Curriculum is highlighted!
> R_2: So, the proposal could be to design units that start from intuitive geometry with immediate examples and an effective engagement, and then arrive at a formalization, i.e., to the rational geometry. What do you think?
> T_2: We can try, but it is not effortless.
> T_1: I need to choose a good engagement to motivate my students to study mathematics and to keep their attention during the lesson.

[CO-P2]

The teachers are strongly embedded in their institutional context, indeed they refer to specific topics of the National Curriculum (Indicazioni Nazionali, see Minisola & Manolino, 2022). They are also aware of the students' difficulties in approaching a problem text in general, and specifically in solving geometry problems, also for the difficulty they encounter in working in one representation (what Duval, 2006, calls treatments), or interpreting different representations and their corresponding transformations (what Duval, 2006, calls conversions). The role of the researcher here is to give teachers some hints from the research in Mathematics Education that include Duval and other authors, in order to link their experience coming from their practice to the theoretical studies in the field. Moreover, considering the National Curriculum and the three-year educational plan of the school (Minisola & Manolino, 2022), the teachers decided to plan their LS on Geometry in a long-term perspective, so that LS could be “sustainable”, and not an isolated activity. In fact, according to Italian institutional context, teachers are aware of two equally fundamental elements of the planning: the plan of the research lesson, focused on one/more concepts of Geometry, and the connection of this lesson with a long-term vision, which takes into account the lesson inserted in the year/s (one or more) educational plan. Another important element to consider is the work of teachers, as made not only in collaboration, during the meetings, but also individually in asynchronous mode, to reflect and collect ideas, which later are shared with the colleagues in the project group. Individual reflections are collected in the collaborative phase of planning, as the single teacher’s contributions, which have a role in the construction of the shared plan.

During the second meeting, the teachers cleared the aim to design an activity named *The Art of Geometry*, which collects the short and long-term visions described above. Inside this activity, they selected Tessellation as the topic for the LS research lesson, under the suggestion of T_1. This topic is interdisciplinary, connecting Mathematics, Natural Sciences, and Arts, and for this reason it fits with the Liceo Matematico project’s aims (see par. 2.3).

Coherently with their knowledge on LS, they wrote the Lesson Plan (Authors, 2019) for this topic, considering both the Italian secondary school context and the specific school features. In Table 6.2, the first row of the Lesson Plan is shown.

Table 6.2 A row of the Lesson Plan, as co-planned by teachers and researchers

	Description of the activity	**Grouping and class setting**	**Time**	**Educational intentionality (the reasons for the choices)**
Introduction to the lesson and presentation of the topic	The topic is introduced using an artifact: the students are provided with a card containing a phrase from Hardy, an Escher's artwork, and a hive Students are invited to use their smartphone or tablet (in BYOD mode) to find further examples of tessellations and to	The implementing teacher already arranged the students in small groups of 4/5.Within this setting all the students can look at the Interactive WhiteBoard (IWB) in case of face-to-face explanations	10'	We prefer to immediately use an artifact to "capture" students' attention and then get them into formalization

	grasp the concept of "tessellation"			

While choosing the topic and structure of the Lesson Plan, prepared as in Table 6.2, according to their institutional context, the teachers chose the artifacts to be used by students, and the prerequisites, the general educational goals (long-term vision), the specific learning objectives of the lesson (short-term), and the mathematical competencies associated with them.

The project group's choice for the observing phase (the *purpose of the observation*) was to concentrate on: teacher's communication (posture, tone, gestures, interaction with the class), ability to involve students with the use of scaffolding, and scheduling. The group of teachers and researchers produced a grid of observation (Table 6.3) as a tool to be used in class during the research lesson. During this meeting a criticism also emerged about the presence of the observers during the teaching: the private space of the class becomes a public space.

Table 6.3 Teacher behavioural indicators in the observation grid

Categories	**Behavioral Indicators**
Communication	✔ Does he/she provide students with all the essential elements for the tasks? ✔ Does he/she provide explanations to the students during all the tasks?
Interaction with the class	✔ Does he/she use gestures to support tasks understanding?

	✔ Does he/she use gestures as teaching support?
Class Management	✔ Does he/she participate in the student's activities? ✔ Does he/she intervene in the groups' presentations of their work?
Time management	✔ Does he/she manage the discussion times? ✔ Does he/she act in the scheduled times?

The project group, when writing the Lesson Plan, took into consideration not only the professional need to observe the teacher's teaching behaviour throughout the implementation of the research lesson, but also a number of didactical variables:

- the different students' learning styles;
- the context of the lesson, linked to realistic situations;
- the link between the lesson and the three-year school educational plan;
- the consistency of long-term and short-term learning objectives;
- the artifacts for introducing tessellation.

The artifacts chosen for the research lesson had been examined by the group in relation to their *semiotic potential* for the concepts to be introduced, according to the theory of semiotic mediation. As such, they are described in a specific section of the Lesson Plan.

6.5.2 Teaching

The teaching phase (lasted 1h30', despite a planned duration of 1 h) started with the introduction of the tessellation topic, as required by semiotic mediation (see contextualised scheme, Fig. 6.9). To do that, the implementing teacher, T_1, provided the students with a sheet where three tasks were shown, in reference to different artifacts: a sentence (in Fig. 6.5a); two images (an Escher picture in Fig. 6.5b, and the image of a beehive in Fig. 6.5c); and a technological tool (students’ smartphone in Fig. 6.5d). These artifacts were chosen in the co-planning phase, for their *semiotic potential* and their meanings strictly connected to mathematics meaning, given respectively by the meaning of keywords in the sentence, the repetitive module represented in the two pictures, and the possible images of the same kind searchable on a smartphone. The different registers have been chosen in relation to the stimuli that can convey in the students, to understand the information in the semiotic representation, but also to connect different representations with the same meaning (according to Duval, 2006). For that reason, the students were asked to move from the words to the images.

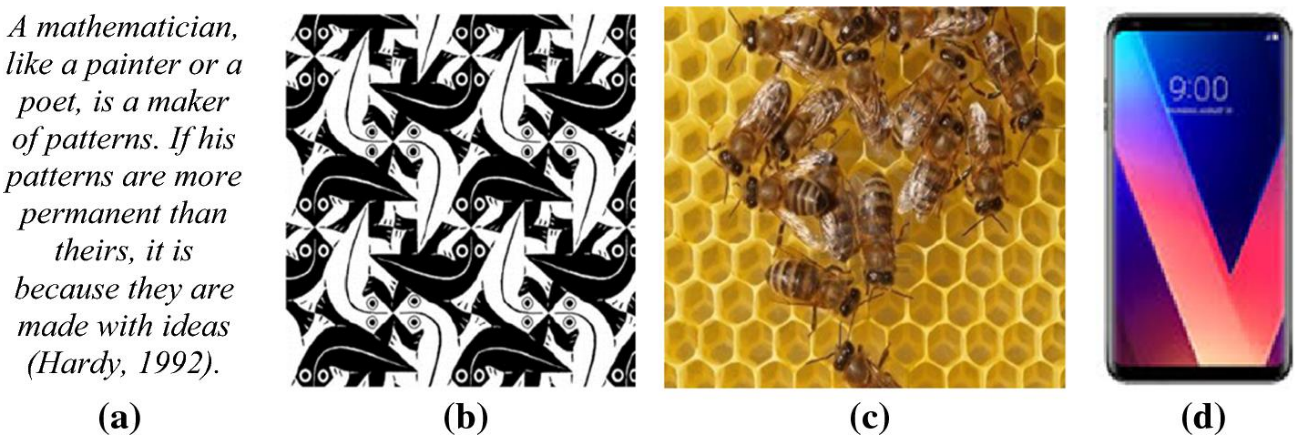


Figure 6.5 The artifacts of the semiotic mediation cycle

The tasks were:

(1) What do you observe concerning the shown images?

(2) *In Hardy's text we read "permanent", what adjective would you use instead of "permanent" to describe the observed images?*

(3) *By using the smartphone, search for some "similar images" in art and in nature and try to represent them.*

About Task 1, in Fig. 6.6, some students answered are shown. These answers are particularly interesting, since the students use the word "repeated", which conveys the idea to reuse the same shape to cover the space.

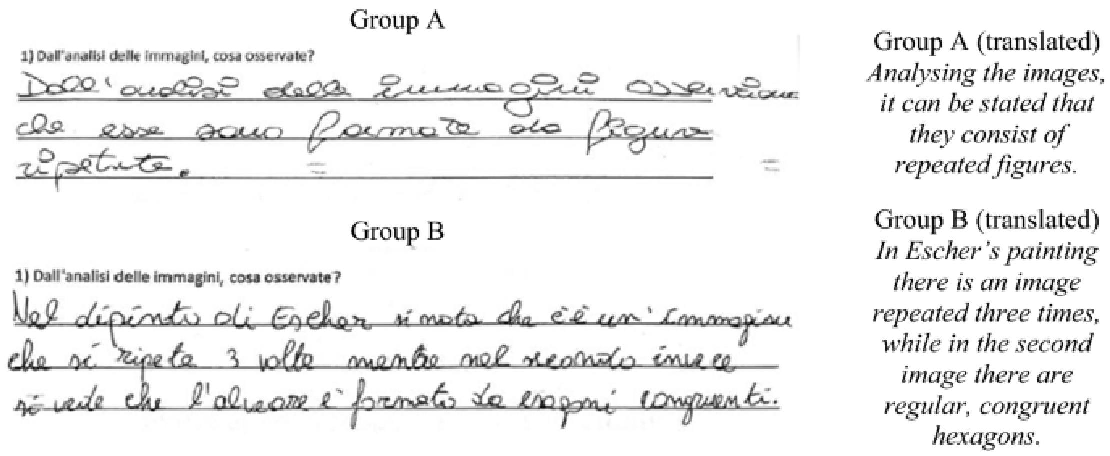


Figure 6.6 Two students' answers to the first task

Again, in the second task, the students answered with similar adjectives: "repeated, regular, perpetual". And in the third task, the students referenced to some websites (eg. https://maretta89.wordpress.com/2010/11/24/le-tassellazioni/) and drew some "repeated and regular images", as in Fig. 6.7.

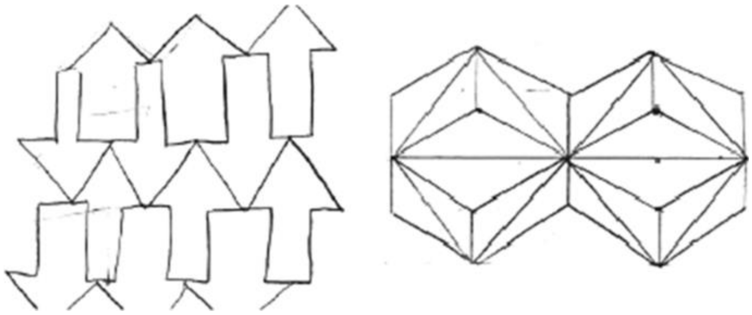


Figure 6.7 Images drawn by students to answer the third task

In coherence with the co-planning and with the semiotic mediation framework, the texts and images produced by the students are *situated texts.* The word "tessellation" was not used during the tasks, and it was introduced by the implementing teacher to move from situated texts to mathematical texts: the students were asked to formalise their observations in a *mathematical text*. Each group provided their own definition of "tessellation". In Fig. 6.8, two examples of students' *mathematical text* (group's definition of tessellation) are shown.

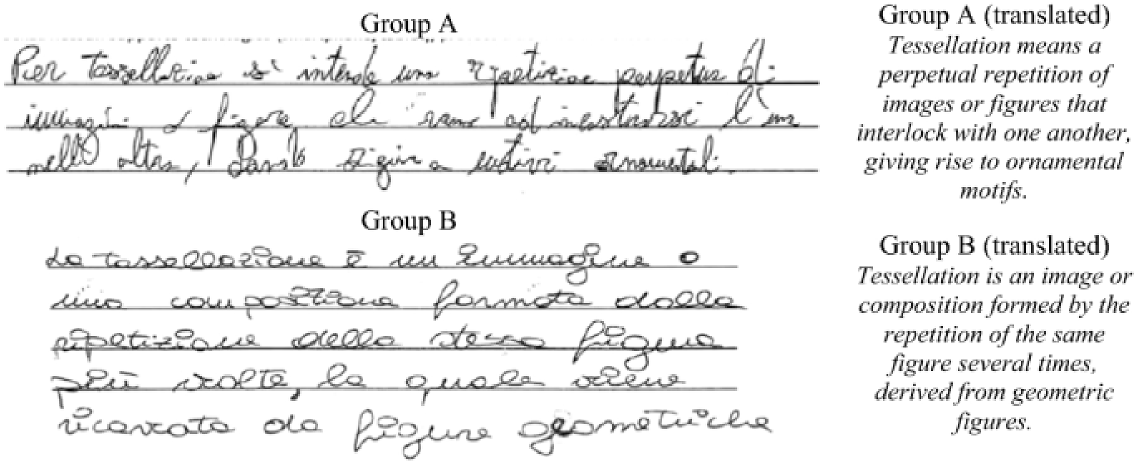


Figure 6.8 Two groups' definitions of tessellation

At the end of the activity, in line with the semiotic mediation scheme (in Fig. 6.9), the students identified the main features of tessellations: repeated, regular figures covering a plane. Indeed, the activity was aimed at the mathematical knowledge of tessellation in agreement with the Italian National Curriculum.

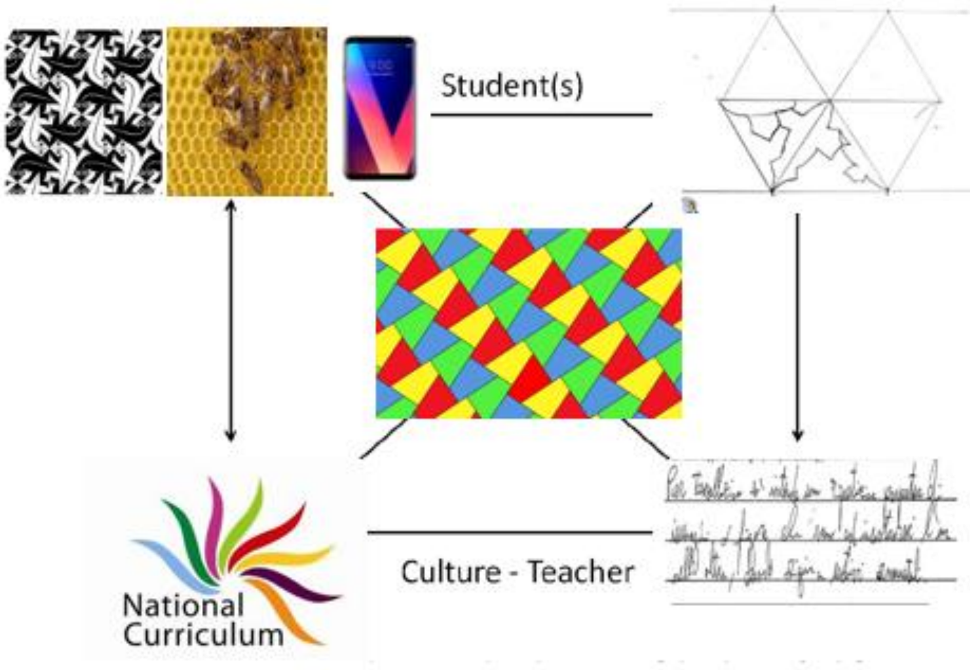


Figure 6.9 The “tessellation” semiotic mediation scheme

The subsequent task was chosen as a crafting reality problem:

> Tessellate your school desk with regular polygons of different types (equilateral triangles, pentagons, hexagons) that you have to build using coloured cards, scissors, pencils, ruler, and compass.

T_1 also provided written instructions on how to use straightedge and compass to realize regular polygons, specifying that smartphones could be used, if necessary, to search for additional information about instructions. An example of tessellation with regular triangles and hexagons is shown in Fig. 6.10, while pentagons tessellation is not possible with desk tessellation, as also shown in the same figure.

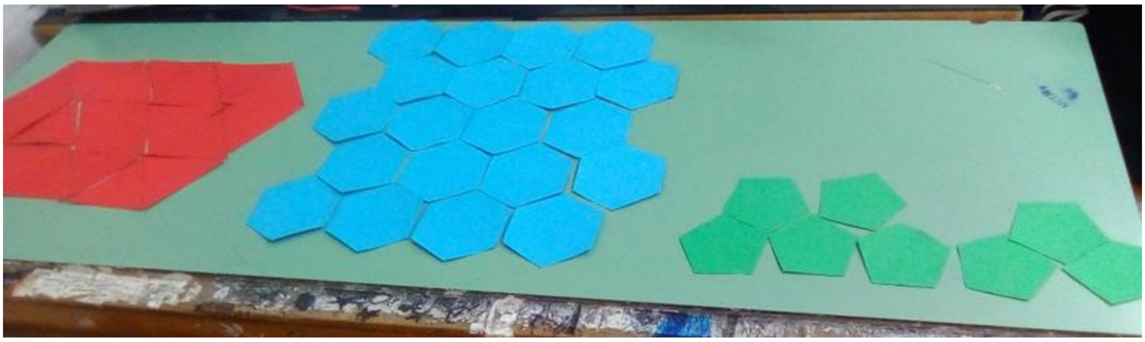
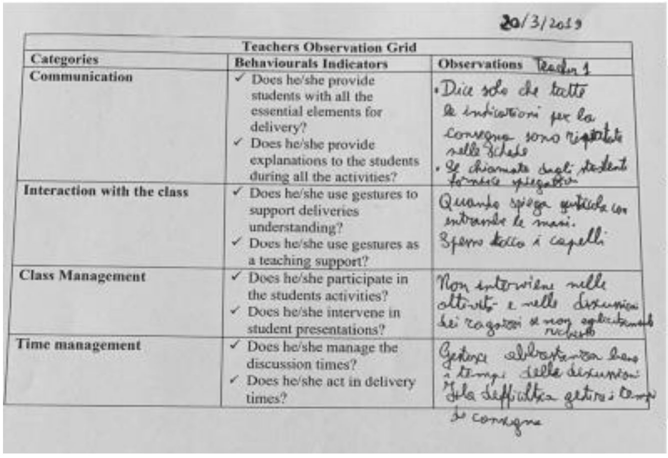


Figure 6.10 A group’s desk tessellation activity with different polygons

The students had to show and discuss the results of the problem with the class. T_1 guided the students to discover that not all tessellations with repeated and regular figures are possible. At the end of the class discussion, she formalised the concept of tessellation using some slides designed in the co-planning phase.

6.5.3 Observing

The observing phase took place at two different times. There was a synchronous observation during the implementation of the research lesson, and an asynchronous observation in which all the members of the project group analyzed the recordings of the student groups. Observers filled out the observation grid both times. As an example, the grid as fulfilled in-presence observation by T_3 is shown in Fig. 6.11.



20/3/2019

Teachers Observation Grid		
Categories	Behaviourals Indicators	Observations Teacher 1
Communication	✓ Does he/she provide students with all the essential elements for delivery? ✓ Does he/she provide explanations to the students during all the activities?	• Dice solo che tutte le indicazioni per la consegna sono riportate nelle schede • Se chiamata dagli studenti fornisce spiegazioni
Interaction with the class	✓ Does he/she use gestures to support deliveries understanding? ✓ Does he/she use gestures as a teaching support?	Quando spiega gesticola con entrambe le mani. Spesso tocca i capelli
Class Management	✓ Does he/she participate in the students activities? ✓ Does he/she intervene in student presentations?	Non interviene nelle attività e nelle discussioni dei ragazzi se non esplicitamente richiesto
Time management	✓ Does he/she manage the discussion times? ✓ Does he/she act in delivery times?	Gestisce abbastanza bene i tempi delle discussioni Ha la difficoltà gestire i tempi di consegna

Translation
T_1 merely says that all instructions for the task are given on the sheets. She only provides some explanations if requested.
She gestures with her hands when she explains. She often touches her hair.
She does not intervene in student activities and discussions unless expressly requested to do so.
She manages the timing of discussions quite well. Has difficulty managing delivery times.

Figure 6.11 Observation grid, as fulfilled by T_3

6.5.4 Reflecting

The reflecting phase took place after two weeks, due to researchers attending a conference and teachers' school commitments. The observing grids, as fulfilled both in a synchronous (in-presence observing phase) and asynchronous (video analysis observing phase) way, have been discussed.

It was noticed that the research lesson did not respect the planned time allocation and some possible causes were analysed. In T_2's observation grid there was written:

> Students waste too much time on the daily problem: each group had to build too many polygons.

[RE1]

From the observational grids and the discussion, the project group deduces that mainly motivations for the time lacking were:

(1) a bad planning:

(2) some teacher's difficulties in managing delivery times;

(3) lacking in organising the materials for the artefacts.

After the video reviewing, the project group observed:

T_1: I recognise that although I realised that the students were exceeding the time limit, I did not interrupt the discussion because I found it interesting how they were discussing with each other. I do not fully agree with these time constraints; I don’t understand how Japanese teachers could interrupt an interesting activity.
T_2: I still think that we designed an activity that was too long. I suggest these changes: arrange the desks before the lesson; establish the groups before the research lesson; avoid students having to cut out the figures during the activity but prepare the materials beforehand.
T_3: We could modify the lesson planning: for example, each group could build just one type of polygon.
T_1: Perhaps we are not used to planning a single hour lesson with so many activities.
R_2: Let’s try the changes suggested by T_2 and T_3, hoping to have a lesson on time.

[RE2]

During the reflecting meeting, concerning the value of jointly designing the activities to be provided to a specific group of students, T_1 stated:

T_1: The discussion between teachers in the choice of the activity was very important; in our teaching practices it is essential to choose a good engage to motivate our students to study mathematics and to keep their attention during the lesson; the laboratorial use of tessellations to introduce concepts of Geometry such as transformations, similitudes and comparison of areas was a nice idea that came out from the sharing of our experiences during many teaching years.

[RE3]

Finally, all the teachers answered a questionnaire, which included the following questions:

(1) *Do you think that the Lesson Study experience has been helpful for your professional development? Write your motivations.*

(2) *What do you think are the strengths and weaknesses of Lesson Study?*

(3) *Do you think that Lesson Study is consistent with the requirements of the National Indications?*

(4) *Do you think the confrontation with researchers in Mathematics Education about the planning of didactic activities was helpful?*

(5) *Do you think your previous experience in collaborative design was significant in experimenting with Lesson Study?*

(6) *Do you think that in Italian teaching, it is important to be able to involve (engage) more and more students in activities from which they generally flee, especially in challenging topics such as Geometry?*

(7) *How important do you think the collaboration between teachers and researchers was choosing an appropriate artifact for effective experimentation of Lesson Study?*

As an empirical data, answering to the question 5, T_2 wrote:

> T_2: During the co-planning phases, we realised that collaborative work is productive. It has been very important that mathematics teachers could discuss the planning of activities, but above all, it was an added value to be able to observe each other; our teaching practices will certainly be enriched and each one of us will bring a piece of the other colleagues in our classroom practices.

[RE4]

About how useful it was to be able to discuss with researchers both in co-planning and teaching phases, T_3 answered to the question 4:

> T_3: The researchers helped us to think about our teaching choices/intentionalities; often these choices are dictated by experience but the understanding that there are

theories of Mathematics Education research that support teaching practices has been an added value of Lesson Study.

[RE5]

6.5.5 *Refining*

Considering the critical issues that emerged during the reflection phase, a revision of the Lesson Plan was carried out. Changes were made to both the tasks and the class discussion, e.g. in the subsequent classes, each group of students had to tessellate the desk with only one specific polygon instead of all four chosen polygons.

The LS cycle started again.

6.6 Theoretical discussion on data

We investigate how, according to the teachers, LS can contribute to the professional development of Italian in-service secondary-school teachers. To do so, we investigated teachers’ deconstruction of teaching practices and the *unthoughts* emerging from their interaction with foreignness in professional development. Our re-formulated research questions were:

(1) Given the teachers’ interaction with LS, what are the hierarchical oppositions that emerge?

(2) What effects do these hierarchical oppositions have on teachers’ professional development?

In order to answer the first research question, we consider the three Key Phases of the Cultural Transposition, referred to as KP1 (Contact with teaching practices of other cultural contexts), KP2 (Deconstruction of teaching practices) and KP3 (Teacher education and development practices).

In this case study, the starting point of phase KP1 is the teacher-oriented seminar on LS conducted by R_1 and R_2 at school upon the invitation of the school headmaster (see par. 4.1). When the researchers emphasised the cyclical model and the LS seven critical components, by showing them videos of Japanese LS examples, many teachers at the school were sceptical. We have seen how even the three teachers (T_1, T_2 and T_3) who voluntarily chose to be involved in LS immediately expressed their doubts (from the transcripts of the first co-planning meeting): "*It seems a very distant teaching system from ours*"; "*seems too far removed from our freedom of teaching*".

The acceptance of the challenge and putting themselves at stake in LS, by the three teachers, triggers the KP2 phase. Some hierarchical oppositions have been overthrown keeping the internal articulation of the LS "critical components" almost unchanged. And through hyperanalysis process, some additions have been identified, to make LS more compatible with Italian didactical context.

The empirical data, collected at the different LS phases and presented in Sect. 5, were analysed and 5 hierarchical oppositions are here highlighted and listed. For each hierarchical opposition, reference is made to at least one empirical representative datum.

6.6.1 HO1. Strict lesson planning/flexible lesson planning; or rather actual planning/customary "a-priori/theoretical" planning.

> T_2: Such a constructed and planned lesson seems **too far removed from our freedom** of teaching, guaranteed by law and by the National Curriculum.

[in CO-P1].

To be caught up in a “*constructed and planned*” planning, creates the perception among teachers that they are losing the freedom that is guaranteed by law in Italy. In Italy, in fact, the curriculum “contain[s] contents and aims for each subject, and its number of hours in a year. These contents are not prescriptive, but at the end of the 8^{th} and 13^{th} grades there are two national exams. Each teacher has the responsibility of the didactical plan for their classes, also according to the *Piano Triennale dell’Offerta Formativa* (Three-year Educational Plan - describing the cultural-pedagogical inspiration and the curricular, extracurricular, didactic and organisational design of the proposed activities). The contents of this document are specific to each school and decided by the collegiality of teachers and school staff (Minisola & Manolino, 2022, p. 5)”.

Also for this reason, teachers are free to choose the purpose of the observation (see the layout of the observation grid, with details of what is the focus of the group’s attention on, in Table 6.3 and Fig. 6.11) and the topic of the research lesson:

> T_2: I think that we could start from intuitive geometry rather than rational geometry.
> T_1: But proofs are essential, as in the National Curriculum is highlighted!
> R_2: So, the proposal could be to design units that start from intuitive geometry with immediate examples and an effective engagement, and then arrive at a formalization, i.e., to the rational geometry. What do you think?

[in CO-P2].

Planning persists as a problem for some teachers, who are not used to detailed planning in a collective manner.

> T_1: I recognise that although I realised that the students were exceeding the time limit, **I did not interrupt the discussion** because I found it interesting how they

> were discussing with each other. **I do not fully agree with these time constraints**; I don't understand how Japanese teachers could interrupt an interesting activity.
> […]
> T_1: Perhaps **we are not used to planning a single hour lesson** with so many activities.

[in RE2].

Given that Japanese teachers would probably never interrupt an interesting discussion or activity when planned, this can be seen as a typical misunderstanding of what the Lesson Plan means. T1's statement points to the "disconnection" between the actual planning situation of working with a specific class and specific students in mind, anticipating student responses, misconceptions, lesson successes, and the customary "a-priori/theoretical" planning situation in teachers' practices, in which teachers broadly plan what they want to do, leaving the details of reaction to the students' answers completely to the "art of the teacher" (Manolino, 2021b, p. 99) and to the day-ahead planning of the teaching materials of the single teacher who enters the classroom (alone).

Teachers, "forced" to plan in detail, are prompted to consciously prioritise (allocate time) certain aspects of the lesson. Indeed, teachers opt to make specific changes in the Lesson Plan:

> T_2: I still think that we designed an activity that was too long. I suggest these changes: arrange the desks before the lesson; establish the groups before the research lesson; avoid students having to cut out the figures during the activity but prepare the materials beforehand.
> T_3: We could modify the lesson planning: for example, each group could build **just one type of polygon**.

[in RE2].

6.6.2 HO2. Teaching as individual work/teaching as collective work

> T_2: […] we realised that collaborative work is productive. It has been very important that mathematics teachers could discuss the planning of activities, […] our teaching practices will certainly be enriched and each one of us will bring a piece of the other colleagues in our classroom practices.

[in RE4]

The T2 sentence “*It has been very important that mathematics teachers could discuss the planning of activities*” emphasises how it is a need for teachers to have the opportunity to discuss content/didactic issues. Teachers are often overwhelmed by logistical, institutional, or bureaucratic demands that detach them from their teaching role. Co-operation in schools often exists linked to educational, logistical, institutional, bureaucratic decisions. A professional development practice that allows teachers to sit at the same table and discuss mathematics contents is a valuable resource. Collaboration engenders this. It spares teachers the familiar feeling of loneliness when faced with teaching/mathematical content issues.

In the co-planning phase, T_1 stated:

> T_1: **I need** to choose a good engage to motivate **my students** to study mathematics and to keep their attention during the lesson.

[in CO-P2].

In the reflecting phase, T_1 stated:

> T_1: The discussion between teachers in the choice of the activity was very important; in **our** teaching practices it is essential to choose a good engage to motivate **our students** to study mathematics and to keep their attention during the lesson; the laboratorial use of tessellations to introduce concepts of Geometry such

as transformations, similitudes and comparison of areas was a nice idea that came out from the sharing of **our** experiences during many teaching years.

[in RE3].

Moving from "*my students*" to "*our students*" was an impressive achievement, as teachers presented themselves as a community of teaching/learning professionals. Teaching as collective work allows for shared responsibility: all the teachers speak in the plural; it is never the fault of a single member of the group if things do not run as planned.

T_2: I still think that **we designed** an activity that was too long. […]
[…]
T_1: Perhaps **we are not used** to planning a single hour lesson with so many activities.

[in RE2].

T_3: The researchers helped **us** to think about **our** teaching choices/intentionalities […].

[in RE5].

6.6.3 HO3. Class as a private space/class as a public space

In contrast to what is customary in China and Japan (Chen, 2017), in Italy it is not usual for more people than just students and the teacher to be present in the classroom, particularly at secondary school. The only exception may be in the presence of the support teacher of students with special needs. This conveys a dual feeling in the teachers: on the one hand of loneliness, and on the other hand of personal freedom and management of the classroom as a personal, private space. Space in which practices are habitual, contextual. Therefore, welcoming colleagues into one's space is seen as a violation. Teachers feel judged, assessed.

Given the common feeling of loneliness, most (at least Italian) teachers would not refuse to partially work collaboratively. Teachers might think that everything is fine in "a-priori" planning together, but then the actual practice will be in their classroom, they will have to make their own choices, as they know their students and what makes themselves feel competent.

> T_3: Thinking of **being observed** during the lesson… I don't think it is a good thing **for the students**, but also **for us as teachers**, who **won't feel fully comfortable**.

[in CO-P1].

LS helps in "overthrows" such a fear. Due to the purpose of the observation, established by the teachers themselves—thus empowered—the preliminary scepticism turns into appreciation. Observation becomes the trigger element for deconstruction. Planning together is no longer the sole moment for everyone. Observation unlocks the space, from private to of the whole group:

> T_2: […] above all, it was **an added value to be able to observe each other**; our teaching practices will certainly be enriched and each one of us will bring a piece of the other colleagues in our classroom practices.

[in RE4].

The added value of a class as a public space is to be enabled to enter the classroom as a peer. It is an uncommon practice, in teachers' context. Above all, it is uncommon with regard to a lesson "of everyone", which was designed by all the teachers, and everybody is responsible. Teachers, before LS, observed each other to learn “from” others, not “with” others. The value of collaboration emerges again; the classroom is a public space for collaboration. Not only collaboration among teachers, but also between the two communities of teachers and researchers, is recognised as an important pillar of teachers’ professional development, especially when supported by the institutional synergy between university and school, as guaranteed in the Liceo Matematico project.

> T_3: The researchers helped us to think about
> our teaching choices/intentionalities […].

[in RE5].

6.6.4 HO4. Teaching as guided by theoretical frameworks/teaching as a-theoretical

According to Chen (2017), a number of cultural beliefs guide the practice of Chinese Lesson Study, two of which are relevant to us at this point: the unity of knowing and doing (知行合一) and the practical reasoning (实践推理) based on specific contexts. “Unity of knowing and doing as reflected in Chinese teachers’ enactment of their understanding about teaching is mediated by their practical reasoning in adopting the most appropriate action in a specific context. […] in terms of their actions in LS, the Chinese teachers enact their understanding of teaching in public lessons through unity of knowing and doing more than conceptual explication. Second, with regard to their thinking about LS, the Chinese teachers use practical reasoning in deliberate practice of repeated teaching through group inquiry and reflection” (Chen, 2017, p. 285). In another way, “Westerners’ way of thinking is inextricable from theoretical frameworks, so for us neither teaching can be conceived without them. […]

What emerged was precisely this impossibility to implement LS without a theoretical framework. For the teaching research group, the designed lesson had no meaning without a semiotic mediation framework, […] [Researchers/teacher educators] maintained this possibility, and the semiotic mediation framework to frame activities with artifacts, for all LS experiments in the following years" (Ramploud et al., 2022, p. 155).

LS emerges as an instance where these two ways meet, i.e. this hierarchical opposition is maintained, but this does not prevent it from bringing out the unthoughts of the teachers.

> T_3: The researchers helped us to think about our teaching choices/intentionalities; often these choices are **dictated by experience** but the understanding that there are **theories of Mathematics Education research that support teaching practices has been an added value of Lesson Study**.

[in RE5].

T_3 voices the added value of LS as a professional development process capable of bringing theory into practice. Western teachers do not know how to think without theory (Winsløw et al., 2018), but often rely on unaware "*experiences*". In LS, research in Mathematics Education, with its explicit and implemented theoretical frameworks, supports teachers' practice. It offers them awareness of their own unthoughts.

6.6.5 HO5. Teaching/Research

The project group's choice for the observing phase (the *purpose of the observation*) was to concentrate on: teacher's communication (posture, tone, gestures, interaction with the class), ability to involve students with the use of scaffolding, and scheduling (see Table 6.3 and Fig. 6.11). The teachers' focus for this LS is therefore all about their own professional development, and in particular the study of their own behaviour in interaction with students. The teaching phase, in the LS, becomes a locus of research. And the Lesson Plan (see the last column of Table 6.2) a locus of disclosure and reflection on one's customary teaching choices/intentionalities [in RE5]. The teaching phase and the Lesson Plan are the trigger elements for deconstruction. This prompts teachers to become more aware of teaching as research. Indeed, how to facilitate the development of teacher competences and which models of teacher education can best meet the needs of today's schools, is a recurring question nowadays and encouraged by the results of the 2018 OECD Teaching and Learning International Survey. If the answer, which can be ascribed to the strand of *participative* research (Nigris et al., 2020), can be found in the many positions that define teachers' professionalism in an attitude of self-reflection and in a "research mentality", this answer can be operationally declined in being protagonists of research actions and activities. Quoting again Chen (2017): "with regard to their thinking about LS, the Chinese teachers use practical reasoning in deliberate practice of repeated teaching through group inquiry and reflection" (p. 285). This is why LS is proposed, but is not yet, in western contexts (and here it triggers deconstruction), a model to foster this protagonism.

6.7 *Final discussion and future perspectives*

Given the teachers' interaction with LS, these are the five hierarchical oppositions that we observed. In fact, we observed that the teachers were able to reflect on the unthought related to the usual practices that are connected with their professionalism. Of the five hierarchical oppositions that emerged, four were overthrown (thus enriching their teaching practices) and one was maintained (resulting in more awareness of their didactic action in and outside the classroom).

The teachers, by taking charge of their own professional development in collaboration with the researchers, have brought out what the LS has achieved on teacher education. Indeed, we see a critical reflection on the functioning of LS processes in the discussion raised by the final questionnaire. This enabled the phase KP3, and answers the second research question.

Going back to our starting goal, we can discuss how this study contribute to the general knowledge on LS. The data collected and studied in these experiments provide a new understanding on how to promote and design relevant professional development for mathematics teachers in secondary school, even outside of Italy. In fact, one may wonder if these results were only possible because of the peculiar cultural, scientific and professional background of the teachers involved in the experiment, but other research in Italy show that this may not be the case. Ramploud et al. (2022) identified mostly the same hierarchical opposition in a parallel analysis of a group of teachers with completely different backgrounds (primary school teachers). Moreover, some features of LS that appeared problematic for the Italian teachers were also identified as problematic in other experiments of LS in Europe (e.g., Ponte, 2018). This lets us wonder if similar results could be found in other cultural context, and what that would entail for the implementation of LS in another country. LS is an opportunity—in this case for Italian teachers—to come together with the unthoughts of their teaching practices, which is not an easy feat. This paper shows that it can be a valuable chance for the teachers to become more aware of the reasons that guide their professionalism, that can evolve over time thanks to LS, and that can become a guide for their practices.

Teachers need professional development models that are immersed in the reality of the classroom and foster collaboration, and LS could be part of the answer to their request. In fact, the data show that LS is not “so far removed” from our teaching practices, which could bring teachers to "reject" it. Moreover, it can be effectively supported by the Italian studies in Mathematics Education research. One of the many aspects that we could not investigate in this paper is the effect that such professional development practices can have on the students. The teachers observed encouraging results, but more data is needed on this aspect of the research. We will tackle the issue in a future study.

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7 FROM LESSON STUDIES TO AN ITALIAN LESSON STUDY: A CULTURAL TRANSPOSITION PROCESS

Arzarello, F., Bartolini Bussi, M. G., Funghi, S., Manolino, C., Minisola, R., & Ramploud, A. (2023). Del Lesson Studies al Lesson Study italiano: Un Proceso de Transposición Cultural. *PARADIGMA*, *44*(2), 340–375. https://doi.org/10.37618/PARADIGMA.1011-2251.2023.p340-375.id1423

Chapter 7 summarizes the results of different Lesson Study experiments in Italy, informed by different theoretical frameworks. In the paper, four examples of “Italian” Lesson Study are presented. The paper showcases the elements that, as didacticians, we identified as necessary to construct an “Italian” Lesson Study coherently with the Italian didactic culture, and the motivations and intentionalities that guided the identification of each element. Results show that such requests are correlated to the Italian institutional context, and a constant through the different experiments.

In this article we trace a path that reports the work of Transposition of the Lesson Study (LS) from Eastern to Italian culture through the perspective given by the approach of Cultural Transposition. In it, the encounter with a culture different from one's own is seen as a means capable of raising new and deeper reflections by teachers on their own educational habits and practices. We call 'Italian Lesson Study' (ILS) the product of this process: it is not established a priori, but depends on the requests to participants that the researchers, as teacher educators, have chosen to graft into it. In particular, we present 4 examples of ILS realized in different school grades (1,5,8,9), to highlight how each LS team responded to these requests through their design choices concerning the research Lesson and the critical reflections of the different participants. Starting from this description, in the conclusions we outline some common features and differences among the examples in order to highlight some more general reflections about the possible ways of implementing LS in cultures different from the Eastern ones.

Keywords: Lesson Study. Cultural Transposition. Italy. China and Japon. Mathematics Education.

7.1 The problem of 'cultural implicit factors' in the transposition of the Lesson Study

Japanese and Chinese Lesson Study are fundamentally rooted in their culture of origin. In these contexts, the school has very specific structure and habits, often quite different from those of Western school systems. Specifically, as far as Italy is concerned, the main group of its specificities, compared with other countries' ones, particularly with those in Eastern countries, concerns the school organisation, as listed below:

(1) The Italian *Indicazioni Nazionali* (National Guidelines - MIUR, 2012) indicate only long-term objectives (Minisola & Manolino, 2022), hence Italian teachers are not used to plan a lesson of limited time in detail.

(2) The structure of the school day varies with the order of the school: a structure segmented into ‘lessons’ of 45-60 minutes dedicated to a single subject, different from the previous and the following lesson, is used in secondary schools, where each discipline or group of similar disciplines (e.g., Art; Mathematics and Physics; Music; History and Philosophy, etc.) is taught by a different teacher , but does not exist in primary school, where generally there is only one generalist teacher or at most two teachers per class. From an institutional point of view, the term ‘lesson hour’ is often accompanied by the locution ‘hourly unit’, ‘hour’, or ‘minimum hourly units’. On the other hand, in the daily life of teachers, this term can have a vaguer meaning, some of which overlaps with that of ‘activity’ (Fig. 7.1).

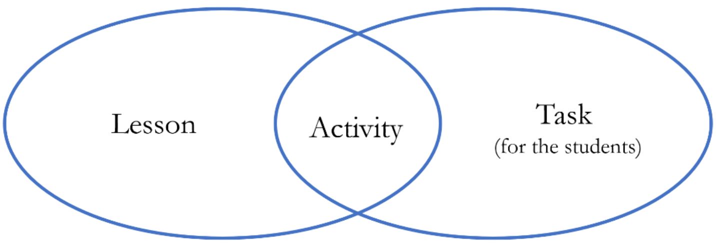

Figure 7.1 Partially overlapping meanings in Italian school terminology

Source: Elaboration by the authors

(3) Freedom of teaching is enshrined in the Italian Constitution: each teacher can choose teaching methods and the order of contents to be presented; as well textbooks are chosen by the teacher according to those that (s)he thinks the best for her/his methods.

(4) Inclusiveness in school is guaranteed by law: students with difficulties attend only regular schools and lessons, possibly with the help of a dedicated teacher (special schools have been abolished since many decades).

(5) Teachers' background varies: at primary school teachers have a pedagogical background; at secondary school mathematics is taught by teachers who have a master in a STEM discipline.

(6) Teachers work mainly alone.

(7) Teaching design is lax when not absent.

(8) Teachers' professional development courses are often theoretical and not centred on everyday teaching practices. Although it is sometimes delivered in the form of workshops, teacher education ultimately leaves it up to the teacher to work out how and when to implement teaching proposals in the classroom. Detailed and context-specific planning is always the responsibility of the individual teacher alone.

These features have deeply puzzled the Italian researchers[39] in mathematics education (Mellone et al., 2021) who in their scientific activities have become in contact

39 In this paper we distinguish between researchers in mathematics education, teacher educators and facilitators: the latter are those responsible for coordinating the work of the LS team; of course, a facilitator can be a researcher. We use also the term *didactician* to indicate the researcher as engaged in a professional development course for teachers, (following Jaworski & Potari, 2021). We prefer this term since in Italy, educator [*educatore*] is a professional figure that is different from the teacher, because his/her role is aimed to reduce the gap between students with special needs and the others; an educator [*educatore*] is not a discipline-expert but an expert in taking care of the human relationship with students (see Ramploud et al., 2021). In the following we will use the Italian word

with the Lesson Study (herein after referred as LS). They seemed to be possible stumbling blocks for a suitable transposition of LS into the Italian schools. In particular, a reflection about this transposition was initiated by the first group of researchers who introduced LS in Italy since 2012, namely the team of M.G. Bartolini Bussi from Reggio Emilia (Bartolini Bussi et al., 2017; Ramploud et al., 2022), to which two other authors of this article belong from the beginning of their Italian LS project. Later, their experience inspired another team of researchers in Turin, three of whom are the other authors of this article (Minisola, 2016; Manolino, 2021). A joint experience so started (Arzarello et al., 2022), which is still continuing and essentially represents the core part of the LS activities in Italy, to which other small groups joined. This paper sketches the main ideas that motivated these experiences featuring, so to say, the Italian LS (ILS henceforth), and illustrates them through some concrete examples of LS experiences in the different school grades, coached by the authors in these last years.

Our approach to LS began participating as observers to public LS realised in Far East countries; starting from our observations, we successively tried to imagine how such a practice could be recontextualized within the Italian school context (see Bartolini Bussi, Sun & Ramploud, 2014; Mellone & Ramploud, 2015; Mellone, Ramploud & Carotenuto, 2021). Our encounter with the "foreign" teaching practice triggered a twofold process.

From the one side, we "recognised" in some components of Eastern LS aspects that were coherent with our experience and to our values as teacher educators: e.g., the need to elicit teachers' reflective approach to teaching practice; as well what we call

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when we refer to this specific professional figure, in order to distinguish it from researchers in mathematics education, teacher educators and facilitators.

‘didattica laboratoriale’ (laboratory teaching) practice. This practice consists of an elaboration of the notion of laboratory, which goes back to Emma Castelnuovo (2006) and was further developed in a professional development program for teachers, developed from 2001 to 2012 with the support of the Italian Ministry of Education and of the professional association of Italian Mathematicians (UMI). According to it (Arzarello et al., 2012), the mathematics laboratory is not necessarily a physical place different from the classroom; instead, it is seen as a structured set of activities for the construction of meanings of mathematical objects. According to this view, the mathematics laboratory environment is in some way comparable to that of a Renaissance workshop, where apprentices learned by doing and seeing, communicating with each other and with teachers. With an evocative metaphor we can say that the classroom becomes like a polyphony performance in a concert hall (Bartolini Bussi, 1996), where the students are the musicians, the teacher is the conductor and the materials that are used are the musical instruments. The construction of meanings cannot reside solely in the instrument, nor can it emerge solely from the interaction between student and instrument. Meaning lies in the purposes for which the tool is used, in the plans that are made for using the tool; moreover, the appropriation of meaning also requires individual reflection on the objects of study and the proposed activities.

Another feature of the ‘didattica laboratoriale’, which we found coherent with the Eastern LS, was the fact that in its development teachers had to keep a logbook, where they reported with some detail the design of their activities in the classroom, what had happened during their concrete development with students, and their comments about the problems they had encountered in it. Of course, the logbook is not the same as the documentation in the LS but surely has some affinity with it, so that -

we as researchers thought - it could be an aspect of the ‘didattica laboratoriale’ which could improve an easier transposition of the LS into the Italian schools.

From the other side, reasoning by contrast about Eastern LS, we also recognized conflicting aspects with respect to our school culture: e.g., focus on short-term goals, detailed lesson planning, teamwork, teacher education through participation in real lessons. A small survey we developed in 2019 with some teachers confirmed that these can be real discordant features in Italian teachers. The data from this experiment highlighted the aforementioned issue on the meanings of the terms “lesson” and “activity”, suggesting an overlap in the meaning of the two terms (see Fig. 7.1). After this, we organized two surveys with different teachers: one with teachers participating in a professional development project of the Department of Mathematics at the University of Turin (44 responses), and another with mathematics teachers at a technical institute and scientific high school in Turin (27 responses). The answers confirmed that, even for in-service secondary school teachers, the meaning of the term “lesson” is not shared with researchers: the majority of the teachers in fact “quite agree” or “very much agree” with the statement “a lesson can occupy several hours spread over several days” (64%) and “completely disagree” or “quite disagree” with the statement “a lesson is something that is articulated in the hours I have in a specific class on a specific day” (70%). This cultural fact is something to be considered when transposing Lesson Study in the Italian context (and which may prove problematic in other contexts as well). Especially as, in the Japanese and Chinese Lesson Study, the word “lesson” assumes a very specific connotation, being a well-defined didactic research object, and which is therefore well "circumscribed" in space and time (the 'research lesson' [40] takes place in a very specific class and time). In the Italian context the "lesson" has no well-defined time boundaries and may last much more than one hour. These conflicting aspects of Eastern LS with respect to our teachers' habits, from our viewpoint as teacher educators constituted excellent "triggers" for teacher education from a perspective of "empowerment" of teachers' critical thinking. Indeed, they allowed us to discuss with teachers' issues concerning the usefulness of a detailed lesson planning, the possible advantages in scaffolding long-term goals into short-term ones, or in increasing the occasions for a collaborative lesson design in their activities. This "empowerment" was important to us as teacher educators in our context, because of teachers' freedom of teaching established by the Italian Constitution. Some of these aspects had also emerged as needs in previous teachers' professional learning courses: the teachers themselves, emphasizing the purely theoretical nature of these courses, complained about a lack of reflection on daily teaching practices (as also outlined in the 2018 OECD Teaching and Learning International Survey - OECD, 2020). LS somehow seemed to provide an overall context for working on all these aspects simultaneously. This made LS an extremely interesting teacher education practice for us. However, already from the first experiments, it became evident for us the need to consider LS's cultural situatedness to redefine its purposes and to discuss a suitable "operating space" to act within the Italian cultural context.

Indeed, reflecting on the potential of LS observed in Far East countries, our idea was not that of comparing different cultures "translating" teaching practices, textbooks,

[40] From here on, we refer to the lesson co-planned, implemented and discussed by the LS teams within the Lesson Study context as the 'research lesson'.

etc. from one culture to another, but that of attempting to highlight differences. In fact, these differences should be those bringing out the possible *unthought* of our own culture - we have to be aware that this process concerns *our* culture, our way of living, our perspectives on the world - prompting us to look at ourselves in a new way. In fact, our work is inspired by the reflection of F. Jullien (2006) who writes that “it is not a matter of comparative philosophy, of putting different conceptions in parallel, but of a philosophical dialogue where each thought, in meeting the other, questions its own unthought” (p. 8). Therefore, we gave a particular attention to the peculiar features of Italian school listed above (Bartolini Bussi & Martignone, 2013) to find a suitable way of implementing the LS. This process is called *Cultural Transposition* (CT), which provides a perspective whereby it is possible to “import” foreign teaching practices, using them consistently with the new context, but retaining some of their conflictual aspects with the new cultural context (Ramploud et al., 2022). CT means not to “perform a comparative study or a slavish import-export of mathematics education methodologies and tools between different countries, but rather to open a dialogue between two different cultures in which every thought, meeting the other culture, questions its own unthought” (Mellone et al., 2021, p. 382). Its aim is to exploit the cultural conflictual aspects to trigger teachers’ questioning about their own teaching practices and educational intentionality (for more details see Mellone et al., 2019). We emphasize that in a CT process the choice of conflictual aspects to be retained is not given *a-priori*. Those described in this paper are our choices, based on our goals as teacher educators. Others may make different choices and obtain different “products”. In our case, CT process was related to specific choices regarding the structure of the Lesson Plan to be used, the length of the lesson to be planned, and specific requests to be fulfilled by participating teachers during the various phases of LS: the "product" of this CT process at the Italian level was the ILS. For the aim of this paper, and in order to provide the reader with the interpretative keys to better grasp the necessary cultural transposition from LS to ILS (for more on LS Cultural Transposition, see Ramploud et al., 2022), in Table 7.1 we present two columns: "*Requests*" (left column) and "*Rationale*" (right column). With the expression "*Requests*" we mean those elements that, as didacticians, we identified as necessary to construct LS coherently with the Italian didactic culture; therefore, we requested the teachers participating to ILS experiments to pay specific attention to these elements (this is why we chose to call them "requests"). With the expression "*Rationale*" we indicate the motivations and intentionalities that guided the identification of each "Request".

Table 7.1 Requests characterising ILS and related rationale

Requests	***Rationale***
[1] Request for lesson structured with a part dedicated to student's hands-on activities to solve a given task, followed by class discussion [2] Request for particular attention to the choice of tasks and materials provided to students.	Italian tradition of research on the 'didattica laboratoriale', where there is a specific attention to the use of instruments and to the mathematical discussion (problem solving is coupled with a collective discussion on the mathematical content).
[3] Request for particular attention to the choice of lesson short-term objectives to be achieved within a single lesson.	The Italian *Indicazioni Nazionali* indicate only long-term objectives; Italian teachers are not used to plan a limited time lesson in detail.
[4] Request for teachers' explicit reflection on educational intentionality underpinning their choices for lesson design.	Freedom of teaching enshrined in Italian constitution: each teacher can choose teaching methods and the order of contents to be presented, so that it was useful for us as teacher educators to make those choices explicit to teachers.
[5] Request for the analysis of the classroom context.	Italian school inclusiveness[41] and Italian teachers' freedom of teaching.
[6] Request for structuring the observation process.	Research in the field of pedagogy in Italian context highlighting the need for teachers to learn how to conduct a systematic observation (Braga & Tosi, 1998; Cardarello, 2016).

Source: Elaboration by the authors

[41] Italian school is "inclusive", in the perspective that there are not special schools for students with disabilities or special needs. Therefore, at every school level, the same class can include children with learning difficulties, with physical and mental disabilities, children that are not Italian speakers, and so on.

Starting from these requests we describe four paradigmatic ILS examples: in order to illustrate how each LS team can choose to answer to these requests, in the description of each example we indicate, for design choices and for critical reflections raised by the participants, the request they refer to. These four examples are singular experiences of LS, without referring to cycles of LS’s, except the small reference in the fourth example. Cycles are present in our experiences, but to analyse them we have used a wider frame, whose illustration is beyond the aim of this paper. The interested reader can see Arzarello et al. (2022) for this analysis. We hope that these examples can inspire the Brazilian researchers and teachers to elaborate their own experiences with LS, not translating them passively into their schools but using them to ‘bring out the unthought of their own cultures’, as it has been the case for us.

7.2 First example: LS on difference (1st grade)

Context: This LS was developed in a 1st grade class (second half of the school year). It was composed of 18 students, one of whom with visual impairment.

LS team: The team was composed by 6 primary school teachers, 2 educatori, and 2 didacticians. Among them, 5 teachers were observers, 1 educatore filmed, the other educatore co-conducted the research lesson with the pilot teacher (by ‘pilot teacher’ we mean the teacher who conducts the research lesson); the educatore had already known the class in previous lessons.

7.2.1 A priori analysis

Aim: The aim of the research lesson was introducing students to difference. The pilot teacher was interested in providing students the opportunity to face and solve a problematic situation not only by subtraction but also by strategies exploiting addition (e.g., counting on strategies) or other possibilities (request [4] in Table 7.1). Teacher, in particular, knew ongoing studies on Chinese didactic practices, as one problem multiple solution tasks (Bartolini Bussi et al., 2014; Venkat et al., 2018).

Main idea: The research lesson design was developed around the core idea of providing to each child a set of KAPLA®s greater than 10 and asking them how they could "make equal" two rows of KAPLA®s[42] like in Fig. 7.2 (request [2]).

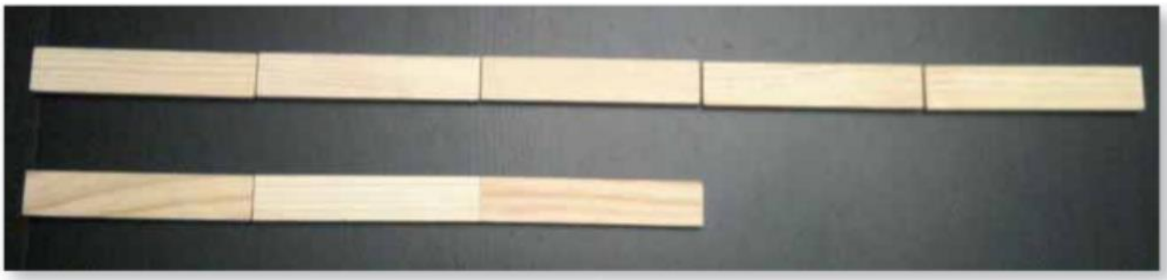

Figure 7.2 Initial configuration of KAPLA® rows shown to the class

Source: Picture taken by the authors

The LS team came to choose this particular task through a discussion highlighting the complexity of a-priori task analysis for teachers (requests [2] and [3]). The participants discussed a lot before finding a "good" question , since in the beginning all the proposed task formulations seemed not really opening to multiple solution, but indicating instead a sort of "preferred" strategy (for instance, teachers argued that the question "How many KAPLA®s of difference there are between the two rows?" could suggest mainly subtractive strategies rather than additive strategies), creating a sort of "hierarchy" between "right" strategies and "wrong" ones. Eventually,

[42] It is a set of wood rectangles having all the same dimensions (see: https://www.kaplaplanks.co.uk/the-kapla-game_i13.html). These wood rectangles can be coloured or not. In our case, see Figure 2, we used the non-coloured set.

a participant formulated the task in this way: «*In my opinion, the way to make them give more solutions is asking “how would you make these two rows equal?”. In this way we could get as answers either taking away 3 from the upper row, or adding 3 to the lower one...* ». This formulation was recognized by all the team as a real multiple solution problem, so it was chosen as the definite task formulation. It is noteworthy that this discussion allowed this teacher to better analyse the possible task formulations in the light of the intentionality at stake, emphasizing the importance of the *a-priori* analysis to elicit teachers’ reflection on their own intentionality (request [4]). Moreover, this choice was aligned with the pilot teacher’s aim to propose a task involving both additive and subtractive solution strategies (requests [2] and [4]). The team expected for the “difference” between the two rows of KAPLA®s to emerge through (at least) three possible solutions:

(1) Removing a certain amount of KAPLA®s (at least, 2 from the longer row);
(2) Adding a certain amount of KAPLA®s (at least, 2 to the shorter row);
(3) Distributing a certain amount KAPLA®s between the two rows (at a minimum, move 1 KAPLA® from the longer row and place it in the shorter row).

After this manipulation with KAPLA®s, the team chose to ask the same question but using a representation of rows of rectangles drawn on paper (see Fig. 7.3). The team argued that the introduction of this different material (paper) could allow students to use different solving strategies with respect to the previous manipulation, i.e. cutting the paper or drawing on it. This change of materials was aimed at highlighting the possibility of multiple solutions among different students, and the possibility for each student to make different choices in relation to different materials (request [2]).

Figure 7.3 Representation on paper given to students in the second part of the activity

Source: Elaboration by the LS team

This manipulation with materials was planned to be organized as a small-group work, even if each child was given his own material. At the end of this part, the team planned to implement a whole-class discussion about different groups' strategies (request [1]), focused especially on students' argumentation in the two cases. The conclusion of the research lesson was planned to be dedicated to a summary by the pilot teacher of the different solving strategies, aimed at pointing out the possibility of having different alternatives for solving a single problem (requests [4] and [3]), but above all at introducing a first possible formalization with symbolic language, such as: 5-2=3 or 5-4=3-2 for subtractive strategies; 3+2=5 or 5+1=3+3 for additive ones; 5-1=3+1 for distributive ones.

7.2.2 Description of the ILS

To collect data about the research lesson development, the team decided to film the lesson placing a camera in a classroom corner, and collecting the 5 observer teachers' field notes (request [6]). The team chose to begin the lesson projecting the photo reported in Fig. 7.2 on the multimedia whiteboard of the classroom, dividing students into 2 groups of 5 and 2 groups of 4, and giving to each child 12 KAPLA®s. To facilitate the work for the visually impaired student, the team chose to cover his desk with a black sheet of paper, to make it easier for him to see the light-coloured KAPLA®s (request [5]). First, the students were asked to reproduce the projected configuration with KAPLA®s; then, the pilot teacher posed the problem of how to make the two rows of KAPLA®s equal (request [2], task). The *educatore,* for this part of the lesson, was going around the class helping students and making questions aimed at supporting students’ reasoning on the task. As anticipated by the team in *a priori* lesson analysis, the task allowed for multiple solutions to emerge within the same group. For instance, of two girls that worked on the same desk, one added 2 KAPLA®s to the shorter row, obtaining two rows of 5 KAPLA®s each, while the other one removed a KAPLA® from the longer row and moved it to the shorter one, obtaining two rows of 4 KAPLA®s each. Then, the pilot teacher proposed to change the material, namely, to work on the representation reported in Fig. 7.3. Each student was given a sheet with this representation. Introducing a different material, students came often up with different solution strategies with respect to what they did with KAPLA®s, even though the two tasks were very similar. Data collected by observers on the work on paper of the same girls mentioned previously revealed that neither of them repeated the strategy used with KAPLA®s: both, in this case, chose to cut the 2 rectangles in the longer row in order to make two rows of 3 rectangles each.

At the end of this small group activity, the teacher dedicated some time to the discussion of solving strategies (request [1]), paying particular attention to students’ argumentations. Then the teacher summarised students’ different solving strategies, pointing out the possibility of having different alternatives for a single problem (request [4]). This recapitulation of different solutions showed the possibility of exposing students to a situation where the individuation of the “difference” could be related not only to a subtraction process, but also to strategies such as counting on or moving an element from the longer row to the shorter one. Moreover, this part of the lesson was dedicated to introducing an initial formalisation of subtractive, additive or "distributive" solving strategies, through the operations of addition and subtraction: e.g., 5-2=3 (subtractive strategy); 3+5=2 (additive strategy); 5-1=3+1 (distributive strategy).

Once the video recordings were shared and watched by participants, they met to discuss and analyse the research lesson. The aim was individuating the strengths and the critical points of the design and identifying crucial points for the future development of class activities. In the discussion, the following crucial reflections emerged:

(1) *Issues of time and of rigid lesson schedule*. Some participants pointed out that in Chinese LS (for example the one described in ICMI study 23), students' work on the proposed tasks is really fast with respect to Italian students' times. The observers suggested to consider reducing the number of proposed tasks in one lesson, to obtain a more profitable time organization (request [3]). These reflections led the team to emphasise the strategic importance of *a priori* analysis, retained very important to adequately implement the lesson as planned (request [3]).

(2) *Importance of* a-priori *lesson analysis to choose materials* (request [2]). The participants realised to have paid so much attention to the analysis of KAPLA®, but to have underestimated the importance of other tools - for instance, the role of the multimedia whiteboard, which was fundamental for tracking recapitulations and argumentations, was considered at first only as part of the classroom setting (request [2]). However, the shift from using KAPLA® to using rectangles representation on paper was retained effective in making each student experience different solution strategies to similar tasks (request [4]).

(3) *Students' group work*. This aspect proved to be directly related to the analysis of materials. In fact, giving to each student their own individual set of KAPLA® and their own rectangle representation did not support group work: most students solved the problem individually, and students' interaction was mainly aimed at borrowing ideas, not at a real collaboration in the solving process (request [1]).

Concerning the importance of time, the team highlighted that also the interaction between the adults involved in the experimentation was conditioned by strict timings and by the (perceived) need to accomplish the initial plan. In particular, the pilot teacher emphasised that, in her opinion, the crucial part of the research lesson was reached in the end, when children's attention was already waning (request [1], overall design of the research lesson structure). This ILS, nevertheless, seemed to the participants a satisfying first attempt, because there were not many downtimes, and the proposed tasks attracted the attention and interest of the entire class. From this perspective, the participation in the activity of all children, even those with particular difficulties, showed that ILS is compatible with an inclusive school (request [5]), as long as the designed task sequence can be adjusted according to students' different times. The most difficult point seemed to be the discussion part, where the teacher can have difficulties in respecting planned timings, because it was difficult for the team to anticipate students' answers and the length or the richness of their argumentations. Therefore, at the end of the discussion meeting, the team identified two possible CT of LS to be chosen in a *priori* analysis: 1) one leaving more space to the discussion with students, aimed at raising important reflections and observations to be further developed in the following activities, less focused on formalization; 2) one focusing more on the core mathematical knowledge formalization. In this second case, the task should be chosen carefully to allow a less extended class discussion and more time for formalization (request [3]).

7.3 *Second example: LS on calculators (5th grade)*

Context: This LS was developed in a 5th grade class of 16 students, some with disabilities or learning disorders, of a school following the “No Schoolbag” model (see Schiedi, 2021). Students were used to work in small groups and in pairs (it is a typical feature of the “No Schoolbag” model: the class setting is with tables for 4/5 people, not individual desks). The LS was carried out in November 2021.

LS team: The team consisted of 5 primary school teachers, 1 didactician and 1 master’s student in mathematics in the role of facilitator. Three out of the five teachers were working in grade-5 classes that year. A support teacher of one of the children with mathematical learning difficulties was also present in the classroom during the research lesson (request [5]), albeit not being part of the team.

7.3.1 A priori analysis

Aim: The research lesson stemmed from the teachers’ reflection about some educational goals to be reached at the end of 5th grade, as indicated into the Italian Indicazioni Nazionali (MIUR, 2012), emphasizing that “The conscious and motivated use of calculators and computers should be encouraged appropriately from the earliest years of primary school, for instance to check the correctness of mental and written calculations and to explore the world of numbers” (p. 60, our translation). The aim of the research lesson was to overcome the prejudice, widely spread among both students and parents, that the calculator should not be used in mathematics education because it facilitates calculations too much. The research lesson was intended to make students realise that the use of calculator not only allows to find the results of operations, as they commonly believe, but that it can enable them to reflect on their way of calculating, facilitating metacognitive reflection.

Main idea: The belief about the detrimental influence of the use of calculators on students’ mathematics learning is not completely without foundation, but the team wanted to provide an example of lesson where the use of calculators cannot exempt students from activating their reasoning to solve the task at stake. In particular, the team thought that asking students to perform, for instance, a division without the use of the division button (i.e., “:”) could completely change students’ beliefs about calculation. Indeed, in this case, in order to find the result of the division, students needed to activate competences related to the different meanings of division, knowledge of the properties of operations and knowledge of our positional and decimal number system.

7.3.2 Description of the ILS

To collect useful data to analyse and discuss the development of the research lesson, the team decided that the researcher would have filmed the lesson with the tablet camera while observing one of the groups. The other groups of students would have been observed by the other team members, taking notes on the spot (request [6]).

In the teachers’ view, a work on the use of calculator could not be reduced to just a one-hour lesson, hence the Lesson Study was part of a broader teaching programme (request [3]). The research lesson was preceded by some activities aimed at making students familiarise with the use of the calculator and the main functions of the buttons. In the research lesson the focus was on two activities inspired by “Matematica

2001" (a compendium of innovative teaching activities and assessment proposals edited by the UMI - Arzarello et al., 2012). The following are the proposed activities with related tasks and educational intentions (request [4]).

Type	See
"On" button	
1	
2	
5	
X	
8	
3	
=	

Operation: ....

Type	See
	0
	8
	86
	86
	2
	24
	62

Operation: ....

Figure 7.4 The two tasks of the "Type and see" activity

Source: Elaboration by the LS team

The first activity was called "Type and see". The 'type/see' pattern was proposed as a tool to understand the difference between what the person does and what the calculator does. This activity was aimed at making students aware of the difference between what the calculator display shows and the way they would hand-write their calculations on paper (request [2]).

On the basis of an initial illustrative example performed by the teacher, showcasing the use of the 'type/see' pattern for computing 17+23, the team proposed the first activity by providing each child with a worksheet with the two tables in Fig. 7.4 and the request to fill in the table step by step and to write down each performed operation. The planned time was 10 minutes. The team chose to make all the students perform the same example at first in order to make every student understand the meaning of the task of "filling in the table" (request [5]). The purpose of the task was in fact to make students reflect on the calculator's actions. The choice to have all the students writing for each step of the calculation was made to avoid dips in attention in the transition from the example to the subsequent problem solving. The LS team chose 17+23 because they considered it simple and understandable as an example: they considered the sum to be the "easiest" operation, and the numbers chosen were considered easy to be summed up without performing written algorithms (request [2], task). In this way students could check whether what they expect by mental calculation corresponded to what they got on the calculator, and thus they could pay their attention to the operation of the calculator. In addition, the numbers were chosen to involve 4 different digits in order to avoid confusion between them (request [2], task). Concerning the first table to be autonomously filled in by students (on the left in Fig. 7.4), it was intended to propose a multiplication not easy to perform with mental calculations (request [2], task). In the second table (on the right in Fig. 7.4), the "see" column was proposed already filled in and the students had to fill in the "type" one. In this way, the team aimed to test whether the students were able to go backwards, following the correlation between sequence of typed buttons and performed operation (request [3]).

The second activity was called "The mischievous calculator". It required performing operations, but for each task the use of the button for the operation at stake was hindered: in this way students must find alternative strategies to compute the considered operation (request [2], task). The 3 proposed tasks were:

(1) 47*3 (use of "*" button hindered). With this task, the team expected students to implement different strategies: using repeated additions, eventually with the commutative property of the addition and some factor decompositions.

(2) 47+7 (use of "7" button hindered). With this task, the teachers meant to explore the students' competence in applying the properties of addition. They purposely

chose to include the digit 7 in both addends to see whether the children used the same strategy for both numbers or they reasoned differently about the two numbers (request [3]).

(3) 250:5 (use of “:” button hindered). The teachers were aware that this prompt was considerably more difficult than the previous two. The team chose not to use division giving non-zero remainder, because they had agreed that this prompt would be an opportunity for further exploration with the calculator (request [3]). The choice of numbers was designed to ensure an easy mental computation.

Students then were asked to write down their reasoning, to choose one spokesperson per group to explain it to the rest of the class, and to justify the solution strategy (request [1]). Figures 5, 6 and 7 show different solutions provided by students' groups to solve the second activity.

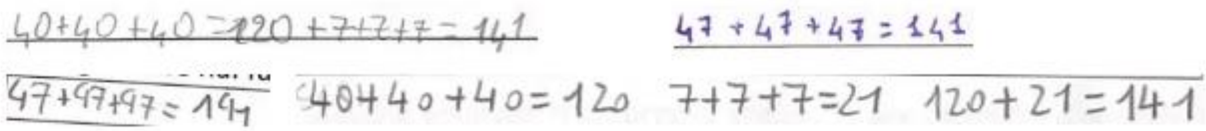

Figure 7.5 First task solutions

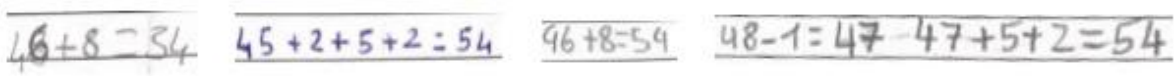

Figure 7.6 Second task solution

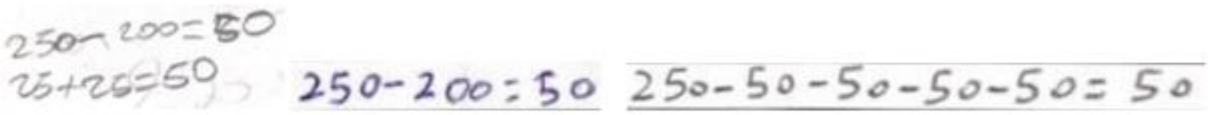

Figure 7.7 Third task solutions (one group did not respond to this task)

Source: Pictures taken by the LS team

To conclude the research lesson, students were asked to answer the following question: “What did we learn today?” (request [3], as the team used this expedient to verify the goals of the lesson). The students' answers can be summarized as follows:

- “There are different ways to get a result; there is not just one solution”;
- “I learned how to find different calculation strategies”;
- “I learned to use the calculator better and I realized that it can be useful to me only if I know what to do”;
- “I discovered that there are multiple methods for each calculation”;
- “I learned how to use the calculator to do calculations without using certain buttons.”

As soon as the lesson ended, the team met to discuss and analyse the research lesson. Several teaching ideas emerged from the discussion. Teachers said they would like to work on them in the future.

Firstly, the teachers reported that a group of students had difficulties with the first activity: they could not complete the second table (on the right in Fig. 7.4). The researcher thus drew attention to the teachers’ habit of proposing to the students’ activities that do not involve backward processes. The researcher encouraged teachers to propose also tasks on backward reasoning. Examples are the “fill in the blanks” such as: ‘...+45= 78’, where students cannot simply apply a known procedure to solve the task but have to reason about the missing quantity to get to 78 starting from 45.

Later, the teachers analysed the second activity and realised that in the planning stage they had not accurately defined where and how students should report the work done in groups, hindering the efficacy of class discussion (request [1]). The spokesperson of the first group wrote the group’s reasoning in the centre of the whiteboard, forcing subsequent speakers to erase it in order to have space for writing their solutions. The teachers argued that, during class discussion, it would be better to divide the board *a priori* into four parts to allow each spokesperson to have their own space to write. Also, having all four solutions written on the board at the same time would have allowed the class to compare them to find similarities and differences (request [2], materials). A second issue emerged regarding the way solution strategies were shared in the class. Each group sequentially presented the reasoning carried out for the three tasks in the second activity, and then passed the turn to the next group. According to the teachers, it was complicated for the pupils to simultaneously follow the strategies applied on the three different problems. The teachers said that it would have been more productive to first reason about the strategies implemented by all four groups on the first task (47*3, use of “*” button hindered), then on the second (47+7, use of “7” button hindered) and finally on the third (250:5, use of “:” button hindered). The teachers then realized the excessive difficulty of this third task. During the research lesson, cognitive conflict emerged due to the presence of too many “5s” within this calculation: the 5 in the dividend (250), the 5 as divisor, and the 5 in the result (50). The students were confused, no longer understanding whether they were talking about the dividend, the divisor, or the result. During the discussion meeting, the teachers realised that in the planning phase they had not paid enough attention to the numbers to be included in the division; if they had chosen a division with all the digits different from each other, the children probably would not have encountered this difficulty (request [2], task). The students probably still would not have been successful in solving the task, given its high difficulty, but they would not have encountered the cognitive obstacle due to the presence of too many equal digits (request [2], task). The teachers also admitted that they had underestimated, or rather disregarded, the fact that this last task would not have allowed the children to visualise the result of the computation, albeit performed in a different manner. The meaning of the division conveyed by this task was in fact that of “by emptying” and not that of “inverse operation of multiplication”, since the last interpretation would have been possible only by knowing in advance the result of the calculation.

The discussion among teachers about students’ difficulties, moreover, highlighted the need to work differently on the properties of operations, which are often presented by teachers in a procedural way without a genuine reflection on their meaning. Internalising the properties of operations is also useful for speeding up mental calculation (request [4]).

Finally, among the team’s reflections that emerged from observers’ note and students’ protocols (see figs. 5, 6 and 7, where students probably used the sign “=” with a procedural meaning) there was the need to work on the meaning of the “=” sign, even before introducing operations in class. Teachers realised that they had not invested enough in getting students to internalise its relational meaning. The discussion highlighted that in Italy, in the team’s view, mathematics teaching in the early years makes students accustomed to writings like a+b=c, where the “=” sign is preceded by whatever operation and followed only by its result, and an easily engendered misconception is that it has a left-to-right reading order. The goal that has emerged from the team is to accustom students to other writings, such as c=a+b or c+d=a+b, from the early years of primary school (request [4]).

Concerning team’s professional development, teachers acknowledged that LS allowed them first and foremost to develop strong reflection on their mathematical content knowledge. Teachers attested that they were not used to discuss about mathematical meanings and contents actually brought (contextualized) into the classroom (requests [1] and [4]). Secondly, in their words, LS allowed teachers to experience “a joint and co-responsible planning”, where they could first-hand experience what it means to plan in detail. Indeed, the pilot teacher said: “Slowly I “got into the mood” of LS when I realized that I could not make some choices on my own. Many things I mistakenly thought I could decide on my own (student group settings, whether observers can intervene, the role of support teachers - request [5], related to teaching freedom), or improvise them (in what words to present the activities to the students), and instead we have to agree and especially justify them - i.e., declare our educational intentions - together” (request [4]). Finally, from further reflection on their experience, the participants retained extremely useful the role of observers (multiple observation - request [6]) during the research lesson. They decided to revise the way of administering the school-assessment tests by hypothesizing the possibility of the presence of observers, in order to detect not only the correctness of the results, but also the processes enacted by the students.

7.4 Third example: LS on problem decoding and solving (8^{th} grade)

Context: This LS was developed in a grade-8 class. It was composed of 17 students, 10 boys and 7 girls, five of whom certified with Mathematics Learning Disabilities (MLD). The class in general is described by the two class teachers (two out of four in the team) as low-medium level. The teachers indicate only three students as high level in mathematics. The LS was carried out in December 2021, which, in Italy, corresponds to the end of the first half of the school year.

LS team: The LS team was composed by 4 teachers and 1 didactician and facilitator. All four teachers taught science and mathematics. Two of them worked in the class where the research lesson was implemented, and one of them was the pilot teacher.

7.4.1 A priori analysis

Aim: The aim of the research lesson - as quoted literally from Lesson Plan - was “to make you [the students] translate the text of a problem into a universal language, the language of Mathematics, that can be understood by anyone (even a foreigner), without having to go back to the original text. Try to avoid using vocabulary, prepositions, verbs, but only a mathematical language that can be understood even by those who do not speak your language” (request [2]). Therefore, the final purpose was to help students reflect on their own communication and interpretative skills, among peers, in decoding and solving mathematical problems (requests [3] and [4]).

Main idea: The research lesson was part of an introduction to literal calculations contextualised in consolidation of problem solving, text comprehension and problem decoding in Euclidean geometry and algebra (request [3]) in a mathematics laboratory setting (request [1]). The team proposed this activity to observe the students dealing with their competence in understanding, decoding and coding a text of a problem, recognising these competences as critical. Students’ manipulation of (pre-)algebraic symbolism often remains anchored to algorithmic ‘as-told-by-the-teacher’ and meaningless resolution procedures (request [4]).

7.4.2 Description of the ILS

The prerequisites for the research lesson are students’ knowledge of the concepts of (1) perimeter and surface area of plane figures, (2) the Pythagorean theorem and (3) basic hints of literal calculations with integer coefficients. The teachers are also aware that, when solving problems (request [1]), Italian students are accustomed to extrapolating the data and explaining the process in the “classical” schematic formulation (with the following key words, identifying the steps of the process): (identify the) *data* - (produce a) *drawing* - (identify the) *question* / *find* (the equation) *and solve* (it)/ (identify the) *unknown* followed by (report the) *solution and* (write the) *answer*.

Since the students in this class were not accustomed to working with peers, efforts were made to encourage group work and cooperative learning (requests [1] and [5]). In particular, some didactic activities were carried out prior to the research lesson: group activities on geometric problems and lessons on introducing literal calculations with integer numbers. Following the research lesson, throughout the course of the year, activities similar to the one carried out in the research lesson, with problems on solid geometry, on the circle and algebra with the literal calculations and equations, are planned in the class teaching plan (request [3]).

For the research lesson, the class was divided into groups of 3 students each. Each group is assigned one of the 7 problems prepared in advance by the team (see Table 7.2 - request [2], task). The problems were invented from scratch by the teachers and designed as “basic”, “intermediate” and “advanced” for the different levels of the students (requests [2] and [5]). One group of students will have to work to pass on to another group all the information needed to solve the assigned problem, using only mathematical language (request [2]).

Table 7.2 The 7 problems for the research lesson

Geometric content	
BASIC	Roberto and Elena compete in a race around a rectangular field measuring 40m * 30m. Starting from one corner, they have to reach the opposite corner. Roberto runs along the edge of the field, while Elena runs along the diagonal of the field. How many metres does Elena run?
INTERMEDIATE	Roberto and Elena compete in a running race around a rectangular field measuring 40m X 30m. Starting from one corner, they have to reach the opposite corner. Roberto runs along the edge of the field, while Elena runs along the diagonal of the field. How many extra metres does Roberto have to run?
ADVANCED	At the entrance to a playground, a rectangular flowerbed of dimensions 8 m and 6 m has been created. Inside the flowerbed is placed a clock inscribed in a rhombus, the latter constructed by

	joining the midpoints of the sides of the rectangle. How much could the diameter of the clock measure?
Algebraic content	
BASIC	In the 3B classroom there are 16 desks and 20 chairs. For an activity, 5 desks and 7 chairs are taken to the basement (where unused furnitures are). At the end of the activity, 7 desks and 4 chairs are returned to 3B. How many desks and chairs are there now in 3B?
INTERMEDIATE	In the 3B classroom there are 16 desks and 20 chairs. For an activity, 5 desks and 7 chairs are taken to the basement. At the end of the activity, 7 desks and 4 chairs are returned to 3B. How many desks and chairs are there now in 3B? How many tennis balls would be needed to silence the chairs in the classroom?
ADVANCED	In the 3B classroom there are 16 desks and 20 chairs. For an activity, 5 desks and 7 chairs are taken to the basement. At the end of the activity, 7 desks and 4 chairs are returned to 3B. How many desks and chairs are there now in 3B? How many tennis balls would be needed to silence the chairs and desks in the classroom? What needs to be done to return to the initial situation?
Extra problem for groups finishing before the set time	
INTERMEDIATE	Marco wants to prepare a chocolate cake for his birthday. The recipe says that 600 g of chocolate is needed. In the supermarket they sell chocolate bars of 250 g each. If each bar consists of 10 squares, how many squares of chocolate does Marco need to make the cake?

Source: Elaboration by the LS team; Translation by the authors

In a second phase of the research lesson, the receiving group will have to solve the problem using only the information provided (i.e., they will not be able to read the original text of the problem - request [2]). After solving the problem, each group returns the solved problem to the sender group.

In the third phase, the two groups come together to discuss the strategies used to translate the problem and understand the data (request [2]). Between phases 1 and 2, the decoded problem can be returned to the sender if it is not understandable and solvable; the first group is not necessarily required to solve the problem, but only to encode the text (request [2]).

For the concluding phase, a large-group discussion on the activity was planned (request [1]) in order to make explicit the teachers’ aim for the activity: i.e., to focus on the decoding and formalisation part, rather than on the problem-solving part (request [3]). However, this phase has never been carried out, due to lack of time.

In order to collect useful data to analyse and discuss the development of the research lesson, the team decided to film the lesson through a tablet camera hold by the didactician, and to collect the notes taken on the spot by the 3 observer teachers present in the classroom (request [6]).

The discussion meeting was held the week following the research lesson. The team met again after watching the video recording. After the pilot teacher’s account of first impressions, it is from the videos that the discussion arose.

For reasons of space, we report here only one example of a reflection made from a video excerpt. It was chosen because it is particularly significant to the aim of the research lesson: it concerns students’ pre-algebraic competence and their spontaneous meta-reflective ability.

M.: So we know that the triangle “times” the circle...
B.: equals...
M.: ...it would have been 4 times 17, right?
M: Ok.
S: So let’s try to do that.
M.: Ah, I used your [calculator], sorry. 4 times 17.
S.: and M.: 68.
S.: But what is 68? [pause] Squares... well, triangle plus circle.
B.: Triangle and circle.
M.: Eh, we have to check...
M.: It will be... it’s possible that it’s like a, I don’t know, 2*ab* squared...
S.: Ah.
M.: either a calculation or a literal thing.
S.: In my opinion, yes.
M.: 68.... So let’s put...
S.: ...triangle and circle.
M.: triangle-circle.
M.: Then we have... circle plus square. Well, we haven’t made the squares yet.
M.: A square must be equal to... 16
S.: ok, um... minus 5... [M repeats and types]
S.: plus 7. [M repeats and types]
M.: And comes 18.
S.: 18 plus 17.
M.: In my opinion I wrote it wrong... Or maybe I made a mistake: you told me to do “less” and I put “plus”.
B.: No, 18 is right. 18 is right!
M.: Is 18 right? Okay! So I wrote right.
S.: So the square...
M.: 18... therefore we have the circle... So, it will then be... circle plus square...
M.: which is equal to... The square did we say 18... and 17?
S.: yyyesss...
M.: So it’s 35 circle-squares...

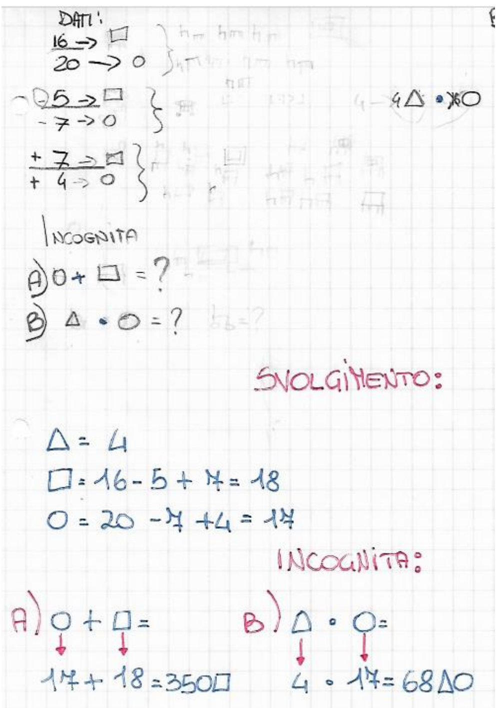

Figure 7.8 The protocol by S, M and B

Source: Picture taken by the authors

The previous is an excerpt from one of the student groups' discussions, while reflecting on the coded text of the intermediate level algebraic content problem. The group consists of two students considered low level (S and M; M also has MLD-certification) and one middle level (B) (request [5]). Fig. 7.8 shows the protocol of this group: the first part (in black) shows the coded text sent by the counterpart sending-group, and the second part (in blue and red) shows the solution proposed by the observed group (receiving-group), according to their interpretation of the problem.

This excerpt was particularly interesting because we can see the students group wondering how to determine the result of their "triangles times circles" calculation. They knew that they had to determine 68 with a label, and that this had to have something to do with circles and triangles. It occurred to M that it could be like 2*ab* squared, as in the computation of the double product of the "binomial square". She meant that, as in the literal calculation in the case of a product (2 "times" *a* "times" *b* = 2*ab*), here too the literal part of 68 must be added and, in this case, it is "4 triangles times 17 circles = 68 triangles-circles". However, the same process was repeated by the students in the case of the sum of circles and squares, heedless of the difference in literal calculation between sum and multiplication. Moreover, this determination would be nonsense if contextualised in the problem-situation of chairs to be soundproofed with tennis balls. For the LS team, having been unable to realise the mathematical discussion, the challenge of using this students' 'misconception' to address the meaning of the literal calculus persisted (request [4]). What do you obtain by multiplying 4 tennis balls (triangles), i.e. one for each chair leg, by the number of chairs (circles); and then what are those 68 objects you get and how are they determined. Or what it means to add desks and chairs together, as well as what it means to add 17*a* and 18*b*.

The teachers realised that, without LS, i.e. without this detailed observation of the videos (request [6]) in which the students discuss, this opportunity would have been lost. Literal calculation would have been taught as a set of procedures, unrelated to the students' experience (requests [1] and [2]). After this analysis, the pilot teacher instead said that she would start from this group misconception to address the issue within the class (request [4]).

7.5 Fourth example: LS on origami geometry (9th grade)

The results of the surveys mentioned in the Introduction were particularly taken into consideration when preparing a professional development meeting with the teachers participating in the professional development course there referred to. As didacticians, we explicated the meaning of the term "lesson" within LS and ILS (in the researchers' notes to the presentation we read "LESSON as something that takes place in a specific place and time frame"), we recalled the ministerial meaning given to the term "lesson" (as a 1-hour segment), specified at several points that the research lesson would take one hour, and that teachers would have to design a one-hour lesson.

Context: This LS was developed in a grade-9 class, composed of 28 students, two of them with MLD. Only few pupils are considered "high achievers". The class is used to work in a mathematics laboratory setting.

LS team: The team consisted of 4 teachers of a scientific-oriented high school with 18 to 35 years of teaching experience. None had experience as researcher or teacher educator. The other team member was a PhD Student, the didactician. The teachers declared that this was their first time collaborating with others to plan something other than the curriculum for the year, which is decided during a meeting of all mathematics teachers of the school at the beginning of the school year. One of them said, "I feel positive about knowing others' experiences, but I also feel some apprehension when they share methodologies without a tradition".

7.5.1 A priori analysis

Aim: The teachers planned an activity for 9th grade on synthetic Geometry, with triangles as main topic. Their main aim was to develop students' abilities around discussion, conjecture, argumentation and proof (request [4]).

Main idea: The main point of the activity is to have the students fold a squared piece of paper following a set of instructions to obtain a triangle (Figure 7.9), and use geometry rules to classify it according to the angles or sides (request [2], task). Citing the Lesson Plan: "folding paper to create figures is a laboratory of intuitive geometry; the students are active on different levels, in a stimulating context to experience and discover mathematical objects. Moreover, paper folding has high learning potential, as it involves visual, manual and thought capabilities; finally, it is accessible and allows one to focus on concepts" (request [2], materials).

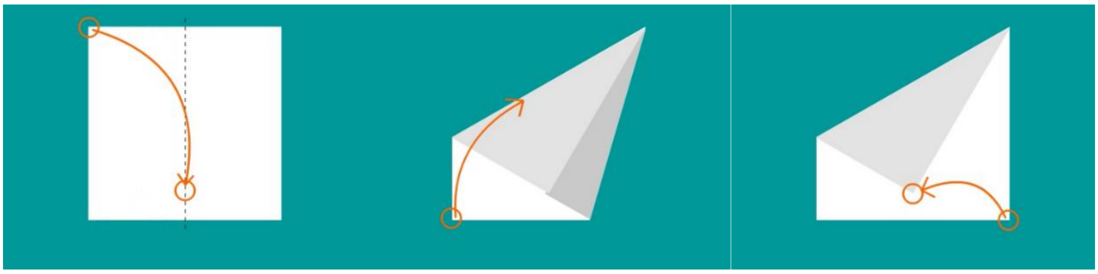

Figure 7.9 Instructions from the video

Source: Snapshots from the video

The activity is divided into three 1-hour lessons (request [3]): in the first one, the "axioms of origami geometry" are introduced, simple loci are constructed (e.g., axis, bisector); in the second one, the students work in small groups to fold the aforementioned squared piece of paper, classify the obtained triangle and discuss their results with the whole class (request [2], task); in the third lesson, students use GeoGebra to reproduce the folding. The second lesson (i.e., the research lesson) was to "have the students come up with conjectures (whether correct or incorrect) and propose arguments to validate them" (request [3]); the teachers decided to observe "the dynamics of homogenous groups, which for us is an unusual setting" and "the ability of selected students to expose and discuss their hypotheses" (request [6]). In the first 10 minutes, the teacher provided the students with the piece of paper, and instructed the students on the folding. To facilitate the students, a video - showing the folding process (Figure 7.9) - was shown on the multimedia whiteboard of the classroom for the first 30 minutes of the lesson (request [2], materials). The students had 20 minutes to work in groups, solve the task and justify their solution. Then, each group was supposed to briefly share and justify their results (10 minutes overall), followed by 10 minutes of whole-classroom discussion of the results (request [1]). As the sharing took more than 10 minutes, the whole-classroom discussion did not take place.

7.5.2 Description of the ILS

The LS cycle took place over four meetings (at least 2 hours each) of the whole team (audio-recorded), plus the research lesson. The first meeting was to share more insights on LS; the second to decide the long-term goals of the LS and define the teaching activity; the third to plan the research lesson; the fourth to discuss the research lesson. Between the second and third meeting, and the third meeting and the research lesson, two undocumented meetings happened between teachers only, in which they defined details of the teaching activity.

Before the first meeting, the teachers emailed to the didactician a document with questions about LS, discussed during the first meeting. The main theme was "time", in particular the time dedicated to the research lesson: "[we] are used to planning lessons of four or five hours. In one hour, it is not possible to get interesting results with the students (request [3]). What if this hour does not go as planned?". During the meeting, a teacher commented: "I know that my lesson will take six hours, and I know the goal of those six hours, but I do not know - explicitly - what is the aim of each of those six hours" (requests [3] and [4]). The didactician noticed that the term "lesson" still had different meaning for him and the teachers. So he provided two reasons to justify the choice of the "1-hour research lesson": by stressing the distinction between "*activity planning* […] something teachers are used to" and "*lesson planning* […] something new which could shed light on some aspect of their work that they did not reflect about"; and by practical reasons (e.g.,: "a 1-hour lesson is easier to prepare […] it is easier to keep the focus on our goals"; or "we want to share our work with other teachers, it is good to also have a standard format").

We can notice that the teachers adopted the shared language. In the second meeting, they noticed that "achieving our goal in one lesson, even in one activity, is impossible. For now, we should assess their argumentation abilities" (request [3]). To this goal, the didactician suggested three different tasks: folding a triangle and categorising it; understanding if cutting a straw in three pieces always gives a triangle; folding a parabola to understand its properties. The teachers mainly discussed the first and third tasks. Finally, they chose the first one to hold the research lesson with grade-9 pupils. The pilot teacher was also chosen. With no further input from the didactician, they structured the activity over three lessons, and the pilot teacher had a focal role in choosing this structure. The goal of the first lesson was “to prepare the setting in which they will work and get them used to manipulating the paper”, while the goal of the third lesson was “to institutionalize the discussions and knowledge from the second lesson” (request [3]). Doing so, they showed that the difference between “activity” and “lesson” was clearer than before.

During the third meeting, the teachers’ worries about the time available for the research lesson emerged again. The pilot teacher seemed the most worried: “one hour is not enough […] we are not ready […] we should move the date further”. This brought some tension in the team, as the date for the research lesson had been decided in the first meeting, and the other members had accommodated their schedule accordingly. Even so, the planning went smoothly: the teachers focused on planning the lesson and the task according to the *Indicazioni Nazionali* (only slightly referenced in the previous meetings), reading them several times during the planning. In particular, the *Indicazioni Nazionali* recommends that the students: “know how to support their argument and can listen to and critically evaluate the others’ arguments; acquire logical rigour when reasoning, identifying problems and finding possible solutions”. Therefore, the goal of the lesson was established as “to have the students come up with conjectures (whether correct or incorrect) and propose arguments to validate them” (request [3]). The teachers noticed that the activity fit well with another recommendation in the *Indicazioni Nazionali*: “the Euclidean approach will not be treated as merely axiomatic”. It was also decided, following a suggestion from the didactician, to organize the pupils in homogenous-by-level groups. The teachers were not used to this way of grouping, as they thought that the low-level ones would have felt uncomfortable due to their inability to contribute to the knowledge. After some discussion, they decided to test the setting (request [4]). The pilot teacher was especially worried about the two students with MLD, and proposed to focus on their performance during the lesson (request [5]). The goal of the observation was twofold: “the dynamics of homogenous groups” and “the ability of selected students to expose and discuss their hypotheses” (request [5] and [6]).

The fourth and final meeting was the same day of the research lesson. It lasted two hours and thirty minutes: two hours to discuss the lesson and thirty minutes to share impressions on LS. The meeting began with an exposition from the pilot teacher, then the team discussed in no particular order. Concerning the goals of the lesson, all teachers considered them only partially achieved: all the students were, in fact, able to produce some level of argumentation and expose to the classroom, but they were not able to listen to others’ arguments. The pilot teacher was surprised by the attitude of the students when presenting their argumentation: they were speaking to the teacher, not to their peers. They concluded that these students probably believed that every exposition to the classroom was, in fact, an oral test from the teacher (request [4] and [5]). The pilot teacher noted that he usually collects ideas from the pupils without focusing on “who contributes”, so the pupils are also not used to being all in charge of contributing (request [4]). Concerning the goals of the observation, the teachers were surprised by the performance of the homogenous groups: the pilot teacher noted that “the low-level students could not contribute much when presenting to the whole class, but they were all very focused inside the group while, usually, they lose focus after few minutes”; he also noted that “she [a girl who had problems integrating into the class] was able to take the lead of her [high-level] group, I think that she felt proud of being part of that group”; finally, he was surprised that the two middle-level groups were able to provide the more complete answers to the task, whether the high-level group could not. He said that this kind of grouping allowed “some usually-hidden dynamics to emerge” and that “I found out some possibilities for working in the classroom that I had forgotten about, or maybe failed me in the past and I decided that they would never work” (request [4] and [5]). The team agreed with him, albeit one teacher said: “Homogenous groups probably would not work with my pupils” (request [5]).

Finally, when discussing LS, all the teachers were impressed by the results of the collaboration: “I loved these moments where we meet and share different experiences”, “it is more productive”, “we do not usually think about such short-term goals, but they are also very important”. On this point, the teachers discussed that “we unconsciously know which are the goals of the lesson, but if we make them explicit they also feel more real, so we think about them more carefully” (request [3]). They also discussed the difficulties that they had in predicting students’ reactions: “I think every teacher believes that they will be able to answer whatever question comes from the students, while they are not”. Another teacher commented, “I particularly fail to predict the ‘silly’ questions, so I do not know how to react”. The main issue, however, was time management, on different levels: the teachers noted that LS cycles should be planned at the beginning of the school year so that it can be better implemented (e.g., more time to articulate the topic in more detail, over the course of more lessons); they also thought that one hour for the research lesson is not enough, using the word “cage” to refer to how they felt in this regard.

7.6 Conclusions

In this paper we have given four examples from all the different school levels (grades 1 and 5 for primary school, grade 8 for lower secondary school and grade 9 for upper secondary school) that we think emblematic of what happens when the Italian teachers are introduced to LS as a tool for teacher education. They concretely illustrate some typical issues that teachers could face in such a teacher education process and that are at the basis of our (Italian) team theoretical elaboration of LS[43]. Our activities with teachers brought us, as didacticians, to work with the notion of Cultural Transposition (CT) in order to elaborate what we called ILS (Italian Lesson Study). As pointed out in the Introduction, CT allows to use foreign teaching practices into a different cultural school context, using them consistently but also dialectically retaining some of the related conflictual aspects. In this process, the didacticians are responsible for facilitating the teachers’ work on the specific identified conflictual aspects, but the validity of such a teacher education process consists in the general chosen design of our experiments: it is a feasible scientific and ethical approach that allows to develop an Italian effective way to LS. In fact, it avoids improvised or not shared changes and it can represent a real improvement of teachers’ professional development, with stable effects on their didactical practices and beliefs. In this final section we sketch a compact picture of some of the results got through our 4 examples using the analysis tools we introduced above, namely the list of the 8 *cultural implicit factors* in the transposition of the Lesson Study, and the *Requests* used to analyse the design choices and the related activities.

First, we reconsider the conflictual elements pointed out in the *cultural implicit factors*’ list. We have verified that these elements are active in the CT process, and we

[43] As highlighted above, apart some rare singular activities, e.g., that of R. Capone (Capone et al., 2023), who however work with us on LS, our research team is the only one working systematically in Italy on LS.

now distinguish two levels[44] among them: (i) a *general background* level, that is a context that influence general didactical attitudes and beliefs of teachers with respect to their mathematical teaching activities when working on LS (factors 1, 3, 4 and 5); (ii) a *specific foreground* level, consequence of the previous one, which determines the effective teachers' practices in their activities during the ILS cycles (factors 2, 6, 7 and 8). It is within this second level that we have observed the main changes in teachers' practical and theoretical approach to their mathematics teaching and professionalism, when in interaction with the LS. On the other hand, with regard to the background level, it is where teachers gain awareness with respect to their practices. For example, the *Indicazioni Nazionali* (factor 1) do not prescribe that teachers develop detailed didactical plans, and the constitutionalized freedom of teaching (factor 3) make it so that the Ministry does not provides a pre-organised succession of contents to be taught nor indications of educational choices; adding to this, the habit in the school is that mathematics teachers of a school meet a couple of times during the year to elaborate the general content of their courses, listing only the main mathematical topics to teach without entering into many didactical details (factors 2, 6 and 7). This causes teachers to experience loneliness in planning their teaching and to perceive lax design as an expression of their professionalism, a proficiency in accommodating to situations. As said above, the combination of these factors resulted in the *requests* we identified as necessary to construct LS coherently with the Italian didactic culture: indeed, in this

[44] This framework could be properly described within Chevallard's Anthropological Theory of Didactics (Bosch & Gascón, 2006; Arzarello et al., 2014), distinguishing between the logos and the praxis in teachers' meta-didactical praxeologies, articulated at different levels of didactical co-determination (Chevallard, 2002). It is beyond the descriptive aim of this paper to elaborate this theoretical analysis.

context, requests 2, 3 and 4 become matter of elaboration of new meanings (which we have tried to describe in detail in each example, both in terms of mathematical and pedagogical knowledge). Despite the tensions that they bring in the teachers' minds, they positively induce teachers to discover 'unthought' parts of their work. What at the beginning appears as an initial 'misunderstanding', with time is elaborated in a positive mood, possibly because of the discussion in the group, or with a proper approach from the didactician and/or the facilitator.

Before sketching some more specific comment on the four groups' examples, it is now important to add some observations on the 6 *requests* (Table 7.1). Some very simple statistical elaborations allow a finer within- and between-comparative analysis of the *requests* (r[n]) with respect to the school grade (labelled here as GR, and numbered according to school grade).

Table 7.3 Percentages of requests (r[n]), normalised to the grade (a) or to the request (b)

	r[1]	r[2]	r[3]	r[4]	r[5]	r[6]	
GR1	16	28	20	24	8	4	100
GR5	12	31	19	19	12	8	100
GR8	20	28	16	16	12	8	100
GR9	4	15	31	23	19	8	100
	13	25	22	21	13	7	

	r[1]	r[2]	r[3]	r[4]	r[5]	r[6]	
GR1	36	32	26	32	17	17	25
GR5	27	36	26	26	25	33	25
GR8	27	14	5	11	17	17	25
GR9	9	18	42	32	42	33	25
	100	100	100	100	100	100	

Table 7.4 Pearson's correlation coefficient between requests

	r[1]	r[2]	r[3]	r[4]	r[5]	r[6]
r[1]	1,00	0,73	-0,96	-0,66	-0,81	-0,29
r[2]		1,00	-0,89	-0,50	-0,84	-0,19
r[3]			1,00	0,70	0,84	0,19
r[4]				1,00	0,21	-0,52
r[5]					1,00	0,66
r[6]						1,00

Source: Elaboration by the authors

Table 7.3a indicates the percentages of Requests used in each analysis, (horizontally) normalised with respect to each grade, with the percentage of incidence of each request in all grades out of the total number of requests (last row in blue). Table 7.3b, instead, contains the same data, but (vertically) normalised with respect to each Request: they show (last column in blue - percentage of incidence) that the four descriptions share a common analysis modality with respect to the different Requests used in it. Table 7.4 contains the correlations between the Requests presence with respect to the different groups (i.e., derived from the columns of Table 7.3a). A positive correlation (r1#r2,r3#r4,r3#r5) indicates some probable complementary affinity between them: for example, the attention to hands-on activities (request [1]) can imply a corresponding attention to the use of materials (request [2]); as well, it is easy that requests concerning a lesson short term objectives (request [3]) are related with teacher's reflection on educational intentionality (request [4]) and the analysis of the classroom context (request [5]). On the contrary, negative values (r1#r3, r1#r4, r1#r5, r2#r3, r1#r4, r2#r5, r4#r6) show that these requests are relatively independent from each other: e.g., the consideration of hands-on activities (request [1]) may pose different problems, i.e. concerning content features, from those put forward by requests concerning the classroom context (request [5]), for example from the point of view of inclusiveness.

After this cross-comparative analysis between the factors and the requests used in our analysis, which sketches the validity and limits of our lens, let us now come to some of the main specific ILS features, which this lens has allowed us to focus in the four examples of the study:

- In GR1, perceived differences with respect to the Eastern LS experiences produce a constructive CT. Teachers' reflection on Chinese children behaviour generates a specific attention on the sequence of tasks in an inclusive perspective: the difficulties of managing the discussion bring to double possibilities of designing a research lesson;

- In GR5, the stimulus to a bigger co-responsibility in planning and an intense attention to cooperative interactions produces a new sensitivity to the importance of small details in the design and explication of educational intentionalities, which was missing before, thus generating also a change in teachers’ mathematical content knowledge.
- In GR8, a fresh attention to the usefulness of observation modalities in the classroom, in particular of students’ group work, can be observed: the focus on mathematical content explodes in the fine observation of a few seconds of dialogue between students and their textual signs, this suggests the need for a new observational posture to teachers.
- In GR9 the decision to work with ‘groups of levels’ generated a higher attention to an unusual activation of observation practices in the class and to a similarly unusual collaborative spirit of co-working among teachers.
- The issue of inclusivity was an unexpected element particularly in GR9, where teaching design is not often linked to the classroom context. The GR9 LS team found ‘unthought’ results from the group-levels: contrary to their expectations, low achievers showed quite good productions, while the best results were got by middle level achievers, since the high-level ones did not perform as teachers expected. All these aspects contributed to the ‘unthought’ phenomenon hinted above: it appeared in different groups and generated processes of change in teachers’ beliefs and practices.

- A main issue of perplexity in many teams concerned the time, which deeply contrasted with the beliefs and practices of most groups. A typical comment, depicting the initial scepticism shared by all groups on the actual feasibility of LS, is the quoted comments of GR9 teachers: “[we] are used to planning lessons of four or five hours. In one hour, it is not possible to get interesting results with the students”. In the final comments they wrote that the issue of time in LS represented a ‘cage’ for them. In this sense, it remains for them one of the most delicate issues not completely solved in the CT. This creates in particular a remarkable difference between secondary school, especially its higher years, and primary school, where this issue triggered productive reflections about the issue of time in teaching (Arzarello et al., 2022). In any case, it remains an open question for researchers. From current investigations it seems that, in higher secondary school, starting the LS design from a main focus on the specific mathematics topics can help in developing an approach to the ‘time issue’ that seems an effective CT of the way ILS can structure the design of tasks according to a shorter time structured model.

Other aspects we observed were induced by a common mood of perceiving the experience with LS as something of unknown and possibly even mysterious. For these reasons all the groups strongly concentrated on the mathematical content to choose for the LS design: all of them considered it something very important, on whose didactical approach they thought to have a good knowledge and strong beliefs and possibly could produce innovative aspects (in particular with respect to requests [1] and [2]) without entering too much into the ‘unknown’ territory of the LS, but elaborating the same a coherent task design. Because of that, the groups generally decided to design tasks, where a possible innovative mathematical content was embedded in new practices. In a sense, the confidence with the mathematical content and the hands-on designed activities allowed the teachers to consider with an open and curious mind the unusual aspects of their didactical activities. These first produced (for them) surprising positive experiences, then a reconsideration of the unusual teaching experiences, and in the end, sometimes, changes in the teaching practices with the consequent elaboration of a fresh terminology ('lesson', 'observation', 'co-design').

Concluding this summary of the ILS with lights and shadows we point out that to deepen the analysis of the CT processes in ILS we think it will be productive to enlarge the analysis developed in Arzarello et al. (2022) through the hybridization construct, also elaborating an investigation focusing on the dynamics of background and foreground cultural implicit factors through the Chevallard's lens of the levels of co-determination (Chevallard, 2002). As pointed out by Winsløw (2011), they can be useful for considering the institutional levels beyond the classroom: from the curricular materials and regulations, to school pedagogy and policies, as well as wider cultural conditions that surround the school and its students. This is the natural landscape, where ILS can be better understood and properly defined.

8 CONCLUSIONS

In this chapter, I summarize the conclusion of my research work. In particular,

(1) I synthetize the results of the previous chapters in relation to the broader research question;

(2) I further discuss the results of this dissertation;

(3) I present future perspectives for the research work.

8.1 The results

8.1.1 Synthesis of the results

At the end of the first part of the introduction, I presented a very broad research question that would direct my research work:

> What could be the cultural and institutional characteristics of Italian schools that could be decisive towards a successful introduction of Lesson Study - or towards a failure of this attempt?

then divided into two sub-questions, one for the work done on the *macro*-scope:

> How are teacher education practices, such as Lesson Study, influenced from the cultural and institutional context in which they are being implemented?

and one for the *micro*-scope:

> How do Italian researchers, teacher educators and teachers relate to a teacher education model - such as Lesson Study - that is foreign to their cultural context?

which represent the complexity of the research work I was engaged in. Each sub-question was decomposed into the research questions, or aims, of the individual papers presented in the previous chapters. The time has now come to summarise the several findings of my research work.

The following table (Table 8.1) summarizes the results of my research work. For each line:

- I recall the proposed research questions for each chapter (where there were no research questions - as in Chapters 3 and 7 - the aim of the paper is recalled);
- I summarise the answers to the research questions of the chapter;
- I present the "key findings" for each paper, namely, the results that are most relevant towards answering the wider research questions of the dissertation.

The key findings are categorised according to the contribution to the answer (it is not a ranking):

(A) disseminative contributions: the expansion of opportunities for the international academic community to access knowledge about the Italian cultural and institutional context, particularly concerning teacher education;

(B) educational contributions: the results regarding the "third mission" of university in Italy[45], which is the impact that research products can have on society at a cultural and social level - in our case, on teacher education;

(C) methodological contributions: about novel research methods, techniques, and approaches developed and employed during the study to deal with cultural issues in research in mathematics education, specifically concerning the transposition of practices from a cultural context to another;

[45] The first mission pertains to the education and training of students, the second mission involves the generation of knowledge through scientific research. The third mission concerns the impact that the university has on society beyond teaching and research.

(D) theoretical contributions: e.g., novel findings within the theoretical framework of Meta-Didactical Transposition, or the in-depth exploration of Networking dynamics among the theoretical frameworks employed in this dissertation.

As these categories are not mutually exclusive, some key findings have multiple labels.

Table 8.1 Synthesis of results and key findings

Ch	**Questions or aim**	**Answers**	**Key Findings**
2	1. What are the most striking differences between this planning methodology and those used in the participants' usual contexts? Are there any analogies? 2. Does the encounter with the Lesson Plan, a tool coming from a different cultural and institutional context, bring a reflection on the participants' own practices?	The task of planning and compiling the Lesson Plan depends on the documentational work that may vary from context to context, for cultural and institutional reasons. The proposed Lesson Plan is the result of the reflection on the Italian culture, institutional context, usual practices. The peculiarity of the Italian Lesson Plan is the request to specify the "educational intentionality" for each phase of the lesson. Preparing and studying a detailed plan for an effective Mathematics lesson is perceived both like a challenge and a necessity.	(B) A Lesson Plan fitted to a certain context is not immediately effective: its conscious use requires familiarity with the mathematical knowledge, curriculum, teaching traditions, institutional context. (B-C) Discussing and sharing educational experiences, provided they happen within customary practices of designing and programming, might sustain to collaboratively overcome the perceived challenges of teachers' professionalism. (C) It is necessary that researchers in mathematics education deepen their studies of interaction with teachers in order to improve the collaboration with them in concrete teaching activities.
3	Aims: - to provide a currently-missing description of the Italian teacher	A tentative description of the Italian institutional context was provided as a reference for future studies.	(C-D) A framework for the description of cultural contexts is constructed. However, it is stressed that having full knowledge of a cultural context

	education context in the English language - to provide arguments to the importance of understanding the cultural contexts involved in the research	The synthetic approach is complemented by the analytic description of the context. Apparent similarities in the Italian and Japanese cultural contexts are complemented by striking differences in the institutional organization. This justifies the importance of a cultural approach when practices from a cultural context are brought in different contexts.	is difficult because we - as people - are embodied in its *unthoughts*. (A-C-D) The influence of *unthoughts* on teaching and learning processes in mathematics is clear. It is necessary to frame our research so that such *unthoughts* can be made accessible to the international research community.
4	1. How does Cultural Transposition interact with teachers' beliefs and educational practices? 2. Which specific methodological elements, encountered in the experiments in the light of Cultural Transposition, are highlighted by the different theories and their Networking? 3. Which methodological components of LS are relevant to question 2, with respect to the evolution analysed in question 1?	Cultural Transposition allows us to contextualize and deconstruct teachers' beliefs and practices, allowing us to closely study the practices of prospective and practicing teachers in collaborative contexts. The frameworks of Boundary Object, Semiosphere and Semiotic Mediation conceive, in different ways, LS as an element that interacts with the Italian context and its components. The results highlight the collaborative dimension in teaching/learning practices as a possible key for a real reform of teaching, seeking and creating connections between teaching practices of different school segments; the collaboration between school and academia can be an added value towards more conscious teaching practices in the light of the research results.	(B-C) The data of the three different experiments on different school levels from different points of view provided a novel understanding on how to promote, design, and assess relevant professional development practices for mathematics teachers. (D) The Networking of Theories provides a (meta-)language that made possible the connection and harmonization between our three theoretical frameworks.

		The LS methodology has contributed to this study, thanks to the new perspective with respect to teachers' praxeologies, providing them a tool for microanalysis of the phases of the lesson in a context accustomed, for historical and institutional reasons, to the design and analysis of long-term development strategies.	
5	1. How do the praxeologies of didacticians evolve in the process of implementing LS, through the interaction with prospective teachers? 2. How do teacher-education praxeologies and research praxeologies of didacticians reciprocally influence and shape their evolution?	We observed that during TPD the didacticians' positions as researchers and as teacher educators are deeply inter-related and continuously influence each other, and that the interaction with the teachers is an essential component of their work as didacticians. The evolution of didacticians' praxeologies could not have happened without the interactions with the prospective teachers, which proved an essential feedback for the didacticians. The teacher-education praxis evolves supported by the logos block of both teacher-education and research praxeologies. The dialectic between the two positions of the didacticians (teacher educator and researcher) generates a *double dichotomy* between the metadidactic level and the research level.	(A-B-C) We highlighted a number of cultural barriers to the implementation of Lesson Study, some in line with other results from the international community, some specific to the Italian context. This further shows the importance of making cultural context (with respective conditions and constraints) accessible to the research community. (C-D) The combining of the Anthropological Theory of the Didactic and Meta-Didactical Transposition theoretical frameworks allows new results within both frameworks. In Anthropological Theory of the Didactic, a number of different positions for teachers and researchers are highlighted. In Meta-Didactical Transposition, the possibility of a convergence process of didacticians and teachers towards a shared praxeology is confirmed. (D) The existence of a double dichotomy also for the researchers, between the metadidactical level and the research level, is also shown.

6	1. Given the teachers' interaction with LS, what are the hierarchical oppositions that emerge? 2. What effects do these hierarchical oppositions have on teachers' professional development?	Five hierarchical oppositions were observed: HO1. Strict lesson planning / flexible lesson planning; HO2. Teaching as individual work / teaching as collective work; HO3. Class as a private space / class as a public space; HO4. Teaching as guided by theoretical frameworks / teaching as a-theoretical; HO5. Teaching / Research. The teachers, by taking charge of their own professional development in collaboration with the researchers, show a critical reflection on the functioning of the Lesson Study processes. This enabled the phase KP3, in which professional development is achieved.	(B-C) The data collected and studied in these experiments provide a new understanding on how to promote and design relevant professional development for mathematics teachers in secondary school, even outside of Italy. (B) Lesson Study is an opportunity for Italian teachers to come together with the unthoughts of their teaching practices. This can be a valuable chance for the teachers to become more aware of the reasons that guide their professionalism, that can evolve over time thanks to LS, and that can become a guide for their practices.
7	Aim: This paper presents the main ideas that motivated the "Italian" Lesson Study, and illustrates them through some concrete examples of Lesson Study experiences in the different school grades.	Specific characteristics of "Italian" Lesson Study are highlighted, through the *requests* of the didacticians to the teachers, and the *rationale* that justifies such requests. The examples presented in the paper show that such requests are correlated to the Italian institutional context, and a constant through the different experiments.	(B) The process of Cultural Transposition allows us to use foreign teaching practices into a different cultural school context, using them consistently but also dialectically retaining some conflictual aspects. (C-D) The analysis points out that, to deepen the analysis of the Cultural Transposition processes in "Italian" Lesson Study, it will be productive to enlarge the analysis through the Networking of Theories, in particular elaborating an investigation focusing on the dynamics of background and foreground cultural implicit factors.

8.1.2 *Answering the research questions*

Let us now relate these results to the research questions. Let us start with the *micro*-scope:

> How do Italian researchers, teacher educators and teachers relate to a teacher education model - such as Lesson Study - that is foreign to their cultural context?

Summarising the previous results, we were able to observe the dynamics by which the didacticians (who are both researchers and teacher educators) and teachers (both prospective and in-service) relate to the Lesson Study model.

The possibility offered by a “foreign” teacher education model to confront the unthoughts of one’s practice is especially highlighted in chapters 2, 5, 6 and 7: the didacticians, for example, have the possibility of discovering unthoughts of their own teacher education practices, such as the explicit definition of certain terms that were taken for granted, when in fact they were defined differently depending on the sub-culture to which they belonged; teachers, as can be seen in chapters 5, 6 and 7, take the opportunity to re-discuss both their own practices, discovering motivations for their actions of which they were unaware, but also to continually re-discuss the “foreign” model, which is not assumed to be the solution to all the problems of teacher professionalism, and is instead an opportunity to learn, to investigate, and to re-discuss, accepting the challenge offered by the “new” to enhance their professionalism and collaboration with colleagues.

These dynamics are particularly emphasised by Lesson Study, a practice that is not so “distant” from what is already experienced by didacticians and teachers as to seem “alien”, but also “different” enough to arouse cultural tensions. Such tensions can be harnessed in the collaboration and dialogue between didacticians and teachers, and the dual dichotomy between the positions at stake (*researchers* and *teacher educators* for the didacticians, *teachers* and *learners* for the teachers) to an evolution of the practices and knowledge of both didacticians and teachers.

In general, what we have seen in the previous chapters can be summarised in Fig. 8.1. The figure is NOT intended to represent a *new* Lesson Study model, but it is meant to illustrate how, in the Italian context (probably in any context, including the Japanese one), the process of introducing and adapting this model can only occur thanks to a constant dialogue between the actors involved, who reflect as much on the model as on themselves.

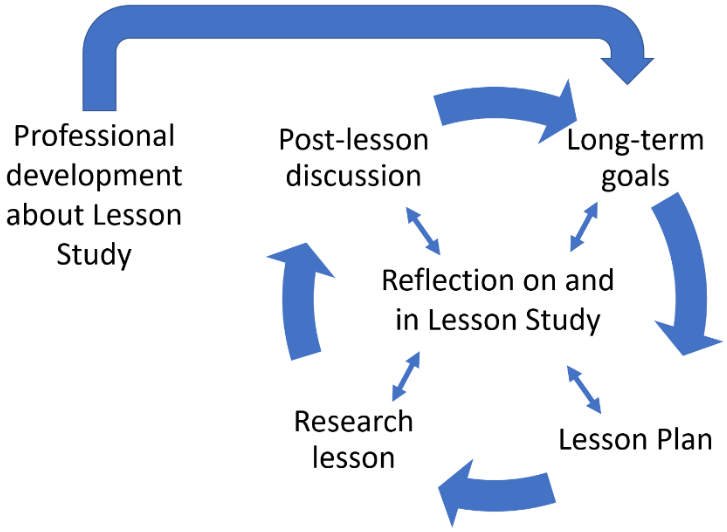

Figure 8.1 Representation of the "Italian" Lesson Study cycle

This brings us to the *macro*-scope:

How are teacher education practices, such as Lesson Study, influenced from the cultural and institutional context in which they are being implemented?

In this dissertation we have clearly observed that teacher education practices are highly dependent on the context in which they are embedded.

The literature review made in the introduction showed how the success of Lesson Study in the Japanese context is due to the fact that the model is in harmony with Japanese cultural values, social norms and institutional structure. This means that Lesson Study cannot be directly transplanted or translated from Japan to Italy, otherwise it could be rejected: not by the institutions, which as we have seen in Chapter 3 can formally decide to adopt practices and models, but especially by the people who occupy *positions* within such institutions.

As Chevallard’s *scale of levels of didactic co-determinacy* suggest, in fact, it is not only belonging to a specific institution (e.g. the school, group of teachers, group of friends) that shapes praxis and logos of a given position, but - at a broader level - also belonging to a given social context (therefore to a number of different institutions) with its needs and norms (which are called conditions and constrains in ATD): these norms are broader than those of the specific institution, and it may happen that, even with the best intentions of the institution at play, these norms may be at odds with each other.

Italian school is certainly not new to such contrasts: for example, the idea of autonomy with the Matura exam established at ministerial level; or the idea of collaboration with the lack of spaces and time suitable for collaborative work (e.g., Blandino, 2008). It is then up to the researchers to act as intermediaries between the needs of the model, those of research, of the teachers, and of the institutions.

This is a massive task that cannot be approached from a single point of view (Chapters 4, 5, 6 and 7), but which, as we have seen in the course of the dissertation, benefits all the people and institutions involved.

At the end of this part of the course we realise that the answer to the initial question:

> What could be the cultural and institutional characteristics of Italian schools that could be decisive towards a successful introduction of Lesson Study - or towards a failure of this attempt?

will necessarily be partial, but that is fair. In this dissertation we have highlighted issues that we could categorise (if it makes sense to do so) as linguistic (Chapters 5, 7), organisational (Chapters 6, 7), and institutional (Chapters 3, 6, 7): namely, cultural issues. Throughout the dissertation, however, we have also highlighted how the Italian context may be favourable for the development and adaptation of an "Italian" Lesson Study that takes these issues into account. Perhaps the "Japanese" essence of Lesson Study will be lost, but… it is no big deal.

What is important, in fact, both as a researcher and as a teacher, is the possibility of implementing a new model, one that brings together what is positive in Italian and foreign experiences, one that brings the world of school closer to the world of research, one that can support teachers who perceive isolation in their professional lives, and one that can ultimately benefit Italian students.

8.2 General discussion on the results

We can now propose a general summary of the results obtained during my research work. This reflection will further extend the synthesis presented in Table 8.1.

At a disseminative level (A), the international research community in Mathematics Education has now access to the first (to our knowledge) updated description of the Italian institutional teacher professional development context published in English, after the glimpse offered by the seminal work of professors Arzarello and Bartolini Bussi in 1998. It is particularly important to note that while Arzarello and Bartolini Bussi's paper focused on the history of Italian research in mathematics education, and its implications on teacher training, the contributions in this dissertation focus on the institutional structure of teacher professional development in Italy, thus constituting a proper map (albeit only sketched out) of the Italian context of teacher professional development. In a community increasingly aware of the importance of cultural contexts in Mathematics Education research, this will prove crucial in the understanding of Italian Mathematics Education research. Smaller descriptions can be found in Chapters 5, 6 and 7, and the finer one in Chapter 2, which mainly contributes to this result.

At an educational level (B), terminological issues in the presentation of Lesson Study to the teachers were highlighted, and identified as possible constraints to the implementation of Lesson Study in Italy. In particular, I identified in the lesson-activity duo a crucial terminological issue rooted in Italian school culture and institutions. The fact that this is the first time (to our knowledge) that this terminological issue is brought to light in the Italian context supports the validity of the Cultural Transposition approach, highlighting the importance of observing one’s own practices through the eyes of other cultures in order to be able to reflect on the motivations justifying such practices (and vice-versa). This result also supports the importance of the introduction of Lesson Study in the Italian context, precisely because it allows teachers to be confronted with such *unthoughts*, stimulating reflection and the adoption of new practices in the design of teaching activities. Looking beyond the Italian context, this is the first time that this specific issue has been related to Lesson Study, even though several research projects have been confronted with the different attitudes to teaching programming in different contexts, compared to the Japanese one. This result might prove a useful source of insight on Lesson Study for the international community.

At a methodological level (C), I propose arguments to the importance of understanding the cultural contexts involved in the research in mathematics education, especially when attempting to "import" from abroad a (research, teaching, or other) practice rooted in a culture other than the culture of the researcher or teacher. Despite the various studies conducted in the field of ethnomathematics, and despite the recent - growing - attention to cultural issues, these are rarely addressed with the appropriate attention (especially, it must be said, in Anglo-American countries, although some recent contributions show that the trend is changing). This lack of attention often results in a lack of methodological tools "internal" to mathematics education that are suitable for dealing with a study of this kind. In this dissertation I propose a method for describing cultural contexts that might be shared by the community of researchers in Mathematics Education. Fundamental, from this point of view, was the synergy between the synthetic point of view - which made it possible to identify the "categories" of the discussion - and the analytical point of view - which made it possible to dissect the aforementioned "categories" of the discussion, as shown in Chapter 3.

Last but not least, at a theoretical level (D), this doctoral work deepens the knowledge on the complex role that the didacticians have in teacher education, and how this role shape and influence didacticians' practices and knowledge, by setting the initial stage for the networking (in the sense of the Networking of Theories framework) of the theoretical frameworks of the Anthropological Theory of the Didactic and of the Meta-Didactical Transposition. Meta-Didactical Transposition branched in 2012 from the Anthropological Theory of the Didactic, to allow for the analysis of those teacher education contexts where the boundaries between didacticians' and teachers' *positions* were not as well defined as they usually are within the Anthropological Theory of the

Didactic. Ten years later, I believe the framework has expanded to the point where it can be considered a stand-alone framework, precisely because of its dynamic approach. Firstly, thanks to the work of Cusi and colleagues (2023) who delve into the dynamics of the *dual dichotomy* by investigating what the “personal” agents may be that assist the assimilation of the external components of teachers’ praxeologies, causing them to become internal components. Secondly, thanks to Pocalana’s inclusion of beliefs in the theoretical framework in a recent (2023) doctoral work, broadening the meaning of *theory* and *logos* usually attributed to the two terms in both the Anthropological Theory of the Didactic and the Meta-Didactical Transposition. In this dissertation I show that didacticians, due to their complex role, live at the boundary between the ‘metadidactic’ and ‘research’ worlds and that it is precisely this being at the boundary that allows for the development of dynamics that support the evolution of their praxeologies. In particular, I showed how *combining* Anthropological Theory of the Didactic and Meta-Didactical Transposition can guide researchers in the design and analysis of teacher education programmes that entails collaboration between researchers and teachers (such as Lesson Study, or others), elaborating the complexity that this collaboration entails.

The sum of the methodological and theoretical results is actually greater than the individual addendums. I believe, in fact, that this dissertation proposes a new approach to the study of cultural issues in mathematics education, especially when the issue to be addressed is the transposition - from one cultural context to another - of a culturally rooted practice (be it classroom practice or teacher education practice). I have shown, in fact, how a single theoretical framework cannot be sufficient to deal with the complexity of the situation. In this dissertation, therefore, I propose that research work should move on two scopes, one macro and one micro, which inform each other on the objects on which to focus.

In particular, in this dissertation we have seen that the analysis of an institutional context may allow us to identify the reasons why a practice such as the Lesson Study may be more or less useful in supporting - for example - teacher education. This analysis justifies and informs the next step, in which we look at how didacticians and teachers relate to Lesson Study. The analysis of these dynamics, which in the case of this dissertation focuses on the didacticians but does not neglect the teachers involved in the experiments, allows us to identify some of the *inconscient scolaire* and the *unthoughts* of a particular institutional context (as indicated by the Cultural Transposition framework). This, in turn, allows us to focus on the analysis of the institutional context, thereby gaining an understanding of the reasons that dictate the practices and unthoughts of both teachers and didacticians, and suggest ways in which these may be addressed.

This last point allows me to reflect on the various issues that remain to be addressed.

8.3 Future perspectives

I now wonder whether I have answered the research questions posed at the beginning of this chapter. The honest answer is: yes, partially. I now try to outline possible future research topics related to the content of this dissertation.

I pointed out in the Introduction that Italian research on Lesson Study is still taking its first steps: although the impression at the end of this phase of my research work is that I have come a long way, I also realise that in the more complex system in which this work fits there is still much to be done. The "formula" for the introduction of Lesson Study in Italy is still a long way from being identified - provided it exists.

While efforts in this direction continue, the time has come for me to wonder what the next contributions I could make towards this goal might be. For each of the levels presented above, therefore, I try here to identify some of the open issues that remain and that need to be addressed by future research work.

At a disseminative level, it must be pointed out that the map drawn in Chapter 3 is still superficial, partly because of the limited space available in the conference proceedings, and partly because of the inherent difficulty in describing a system as complex and constantly evolving as that of teacher education in Italy. In particular, future work should focus on correcting and deepening what has been written so far.

At the educational level, further studies can be promoted in at least two different directions: the direction of teacher education, and the direction of student education. In the direction of teacher education, having identified the terminological issue does not automatically resolve it. We observed to which extent a misunderstanding, by the teachers, of the distinction in meaning of basic terms such as "lesson" and "activity" could trigger a chain reaction, causing a series of bigger misunderstandings of the goals, processes, and tools of Lesson Study. *Misunderstanding* is one of the biggest issues of every kind of communication (Bosco et al., 2006; Verdonik, 2010). Moreover, communication across different languages means navigating the even-bigger issue of *meaning lost in translation* (examples for Mathematics Education in Proulx, 2018 and Ruthven, 2022; for other research fields, see Tiwiyanti & Retnomurti, 2017, or van Nes et al., 2010). In Mathematics Education, the issue has been studied in multilingual contexts such as African countries, where native languages are often overridden by colonialist languages (Kazima, 2008), or New Zealand, where teachers learn in English and are asked to teach in Māori (Trinick et al., 2014). The central role that language plays in Mathematics Educations has long been established (Morgan et al., 2014), as it has been for the issue of intercultural communication (for a general framework, see Ting-Toomey & Dorjee, 2019; for Mathematics Education, see Bartolini Bussi & Martignone, 2013; Källberg & Ryan, 2022; Mesiti et al., 2022; Wang et al., 2023). Even so, the issue of *communication across different languages* of professional development methodologies is seldom explicitly addressed in the literature (Mesiti et al., 2022). Lesson Study, a Japanese model which is mainly disseminated in English, is particularly subject to this issue (White & Lim, 2008). Research efforts can therefore move in diverse directions:

(1) following in the recent trend indicated in Trouche et al. (2023), i.e. exploiting some theoretical frameworks proper to linguistic studies and translation/adaptation from foreign languages (as also seen in Wang et al., 2023), in order to investigate what dynamics could guide the actual “translation” of Lesson Study, including its specific terminology;

(2) investigating, at an institutional level, what the motivations behind this terminological issue are, and using this knowledge to investigate what actions didacticians might take in adapting Lesson Study to “accommodate” this particular terminological issue.

More research on the teachers' specialized knowledge needed to design and carry out Lesson Study activities could also be crucial to the development of teacher education programmes related to Lesson Study. In this sense, the analysis of how mathematical content is transposed during Lesson Study (e.g. how tasks are designed) could prove illuminating.

In the direction of student education, one could argue that this dissertation dedicates little attention to understanding the effect that teacher education through Lesson Study can have on the pupils, which ultimately are the focus of Lesson Study itself. Additional data related to the experiment presented in Chapter 6 suggest that teacher education in mathematics through Lesson Study has a positive impact on the education of students, which is also supported by the Japanese results in OCSE-PISA tests (OECD, 2019a, 2019b, 2019c) and a number of recent research (e.g., Huang et al., 2019; Martins et al., 2023; Warwick et al., 2016). Little research has been carried out on this specific focus in the Italian context (e.g., Capone et al., 2022a, or chapter 7 of this dissertation), possibly because the implementation of Lesson Study is still in the early stages. Future research could focus on the evolution of students’ knowledge in relation to teachers’ knowledge generated in Lesson Study (e.g., Benedict et al., 2023).

At the methodological level, I believe that there is a need for a shared model for describing institutional contexts. For example, we could work towards identifying some “minimum descriptors” that should be included in the presentation of a research paper to enable others to understand the institutional context relevant to the research paper. If not a definite model, at least more effort is needed towards realising that this context must necessarily be described, in some way.

At the theoretical level, after having shown that the evolution of didacticians’ praxeologies can also take place thanks to the double dichotomy that develops between the “research” level and the “metadidactic” level of their work, I would like to investigate the “institutional” agents that influence the dynamics of the double dichotomy, both for didacticians and teachers. This is one of the dimensions of the overall theoretical framework outlined in the previous section that has not been fully explored, as also noted at the end of Chapter 7:

> Future investigations should focus on the dynamics of background and foreground cultural implicit factors.

Building on the theoretical framework presented in this dissertation, Meta-Didactical Transposition by itself does not provide the tools needed to identify such institutional agents. The analysis could be complemented with a specific element of Anthropological Theory of the Didactic, namely the scale of levels of didactic co-determinacy. This would be a step forward of the results presented in the previous chapters: as Meta-Didactical Transposition and Anthropological Theory of the Didactic have *compatible cores*, and the components suggested above are *complementary*, *coordinating* the two theories (in the sense of the Networking of Theories, see Prediger et al., 2008) should be possible.

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