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NEGATIVE DYNAMIC PROGRAMMING WITH NON-ADDITIVELY TIME-SEPARABLE OBJECTIVES

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Negative dynamic programming with non-additively time-separable objectives

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Abstract

A class of intertemporal optimization models characterized by a recursive objective functional obtained as the limit of iterations of the Koopmans aggregator is considered. We focus on negative dynamic programming problems in which aggregators may be unbounded from below and establish existence of an optimal solution under the assumption of strong concavity for the aggregator, both in the deterministic and in the stochastic settings.

Keywords: Negative Dynamic Programming; Non-Additive Recursive Objective; Bellman Equation; Strongly Concave Functionals.
JEL Classification: C61, C65, D81

1 Introduction

Since the axiomatic approach by Koopmans (see [4] and [5]) in the sixties, the advantages of considering recursive utilities that allow a flexible rate of time preference determined endogenously by the underlying consumption stream, which generalize the standard approach based on additively time-separable utilities, have been considered and widely discussed by many authors in the field of dynamic economic models. After an early effort at venturing outside the additively time-separable realm pursued by Mitra [12], who established necessary and sufficient conditions for the existence of optimal solutions when the discount factor varies over time, Lucas and Stokey [9] were the first authors that undertook the road of recovering the whole utility function from an *aggregator* function that represents the fundamental preferences of the agent.

The literature on this subject and on the analysis of the associated models of intertemporal optimization is already large enough. Most contributions assume that aggregators are bounded from below and provide quite satisfactory results (see, for instance, [2], [3], [10] and [1]). Conversely, for aggregators that are unbounded from below, up to our knowledge only few general results are available (see, among others, [8] and [13]).

In the present paper we aim at contributing to fill this gap by studying topics related to dynamic choices when aggregators have negative values and are potentially unbounded from below. In particular, our goal is to extend already established results on negative dynamic

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programming when the return is additively time-separable (see [14]) to problems built through more general aggregators. Our idea is to exploit the strong concavity property of the aggregator to obtain coercivity of the recursive objective functional, which itself triggers the functional analysis procedure to establish existence of optimal solutions in the infinite dimensional setting.

The paper is organized as follows. Section 2 collects some known basics results, together with a few original contributions, on the negative dynamic programming under the hypothesis that the objective function is recursive. Section 3 is dedicated to a detailed analysis for the class of recursive functionals generated by strongly concave aggregators: we show that such a restriction allows to establish our main existence result. In Section 4 we extend the strongly concave recursive problems of Section 3 to a stochastic setting. To help the reading, we gather in the Appendix some mathematical results used in the development of our analysis.

2 Preliminaries

This first part is dedicated to the class of deterministic dynamic programming problems with non-additive objectives. This is realized through the introduction of an aggregator function $W : D \times \mathbb{R} \to \mathbb{R}$, where the one-period constraint D is a subset of $\mathbb{R}^n \times \mathbb{R}^n$. We write $W(x, y, \zeta)$, for $(x, y) \in D$ and $\zeta \in \mathbb{R}$. The special additively time-separable aggregator $W(x, y, \zeta) = u(x, y) + \beta\zeta$ gives rise to standard intertemporal optimization models in reduced form.

Taken W as a primitive, the total return function U, defined over time-unbounded paths, is generated by the aggregator W. Specifically, the functional U must satisfy Koopmans' equation (cf. [4] and [5]) given by

$$U({}_0\mathbf{x}) = W(x_0, x_1, U({}_1\mathbf{x})) . \tag{1}$$

Here ${}_0\mathbf{x} = (x_0, x_1, \ldots)$ denotes a sequence of vectors in $\mathbb{R}^n$ and ${}_1\mathbf{x} = (x_1, x_2, \ldots)$ is its one-time shift.

Below we list the basic hypotheses made about this aggregator.

W. 1 *D is a closed and convex set of $X \times X$, with $\pi_1(D) = X \subseteq \mathbb{R}^n$ where $\pi_1 : X \times X \to X$ is the first projection;*

W. 2 *$W(x, y, \cdot)$ is nondecreasing and continuous over $\mathbb{R}_-$ for each $(x, y) \in D$ and $\lim_{\zeta \to -\infty} W(x, y, \zeta) = -\infty$;*

W. 3 *$W(\cdot, \cdot, \zeta)$ is upper semicontinuous on D for every fixed $\zeta \leq 0$;*

W. 4 $W(\cdot, \cdot, 0) \leq 0$.

Both the set of the states X and the sets $D(x)$ of those states reachable from $x \in X$, i.e., the slices

$$D(x) = \{y : (x, y) \in D\}$$

are not necessarily bounded. This fact, along with the assumption W.4 of non-positivity of the aggregator and the possible lack of monotonicity for $W(\cdot, y, \zeta)$, make this class of recursive problems nonstandard and little covered in literature.[1] One example belonging to this family is Uzawa's [16] aggregator

$$W(x, y, \zeta) = (\zeta - 1)\exp(-u(x, y)) , \tag{2}$$

[1]For additively time-separable aggregators $W(x, y, \zeta) = u(x, y) + \beta\zeta$ the assumption W.4 leads to the so-called negative dynamic programming (see [14] and [13, Sect. 4]). No need to say, by the transformation $C(x, y, \zeta) = -W(x, y, -\zeta)$ we get the dual class of (positive) cost minimization problems.

which satisfies W.1–4 under regularity conditions for the indirect utility u.

Consider the recursive system associated with Koopmans' equation, that is

$$U_{n+1}\left({}_0\mathbf{x}\right) = W\left(x_0, x_1, U_n\left({}_1\mathbf{x}\right)\right) \quad \text{for } x_1 \in D\left(x_0\right) \text{ and } n \geq 0, \tag{3}$$

with $U_0 \equiv 0$, and denote by $\mathbf{D}$ the set of the feasible paths

$$\mathbf{D} = \left\{{}_0\mathbf{x} \in X^\infty : \left(x_t, x_{t+1}\right) \in D \quad \forall t \geq 0\right\}.$$

Analogously, define $\mathbf{D}\left(\overline{x}\right) = \left\{{}_0\mathbf{x} \in \mathbf{D} : x_0 = \overline{x}\right\}$ the set of the feasible paths starting from the initial state $\overline{x} \in X$ at time $t = 0$. We have the following preliminary result on the existence of recursive functionals.

Proposition 1 *Under W.1–4 the sequence of partial returns $U_n\left({}_0\mathbf{x}\right)$ generated by (3) from the initial condition $U_0 \equiv 0$ pointwise converges to an upper semicontinuous function U_∞ that satisfies Koopmans' equation:*

$$U_\infty\left({}_0\mathbf{x}\right) = W\left(x_0, x_1, U_\infty\left({}_1\mathbf{x}\right)\right). \tag{4}$$

In fact, U_∞ is the maximal nonpositive solution to (1).

Proof. The unique delicate point of this proof concerns with the upper semicontinuity of the functions $U_n\left({}_0\mathbf{x}\right)$. As $U_0 \equiv 0$ is trivially upper semicontinuous, let us assume, by induction, that $U_n\left({}_0\mathbf{x}\right)$ is upper semicontinuous. We must thus prove that this implies $U_{n+1}\left({}_0\mathbf{x}\right)$ to be upper semicontinuous as well. Fix the path ${}_0\mathbf{x}$ and take any sequence of paths ${}_0\mathbf{x}^m \to {}_0\mathbf{x}$ pointwise, as $m \to \infty$. Of course, ${}_1\mathbf{x}^m \to {}_1\mathbf{x}$ and $\left(x_0^m, x_1^m\right) \to \left(x_0, x_1\right)$. Since U_n is upper semicontinuous at ${}_1\mathbf{x}$, for every $\lambda > U_n\left({}_1\mathbf{x}\right)$ we have $\lambda > U_n\left({}_1\mathbf{x}^m\right)$ for $m \geq m_0$ and m_0 sufficiently large. By W.2,

$$U_{n+1}\left({}_0\mathbf{x}^m\right) = W\left(x_0^m, x_1^m, U_n\left({}_1\mathbf{x}^m\right)\right) \leq W\left(x_0^m, x_1^m, \lambda\right) \quad \text{if } m \geq m_0.$$

In view of W.3, by taking the limsup we have

$$\limsup_{m\to\infty} U_{n+1}\left({}_0\mathbf{x}^m\right) \leq \limsup_{m\to\infty} W\left(x_0^m, x_1^m, \lambda\right) \leq W\left(x_0, x_1, \lambda\right).$$

This is true for every $\lambda > U_n\left({}_1\mathbf{x}\right)$. Since $W\left(x_0, x_1, \cdot\right)$ is continuous,

$$\limsup_{m\to\infty} U_{n+1}\left({}_0\mathbf{x}^m\right) \leq \lim_{\lambda \downarrow U_n({}_1\mathbf{x})} W\left(x_0, x_1, \lambda\right) = W\left(x_0, x_1, U_n\left({}_1\mathbf{x}\right)\right) = U_{n+1}\left({}_0\mathbf{x}\right),$$

so that $U_{n+1}$is upper semicontinuous. By induction, all the functions U_n are thus upper semicontinuous.

By (3) and W.4,

$$U_1\left({}_0\mathbf{x}\right) = W\left(x_0, x_1, 0\right) \leq 0 = U_0\left({}_0\mathbf{x}\right).$$

From the monotonicity assumption W.2, it follows that $U_n \leq U_{n-1}$ implies $U_{n+1} \leq U_n$. In fact,

$$U_{n+1}\left({}_0\mathbf{x}\right) = W\left(x_0, x_1, U_n\left({}_1\mathbf{x}\right)\right) \leq W\left(x_0, x_1, U_{n-1}\left({}_1\mathbf{x}\right)\right) = U_n\left({}_0\mathbf{x}\right).$$

Therefore, $0 = U_0 \geq U_1 \geq U_2 \geq \cdots$. Hence $\{U_n\}$ is a decreasing sequence of nonpositive functions so that $U_n \downarrow U_\infty$ and the function $U_\infty : \mathbf{D} \to \mathbb{R}_- \cup \{-\infty\}$ turns out to be upper semicontinuous in the product topology. Moreover, taking limits in (3), we get

$$U_\infty\left({}_0\mathbf{x}\right) = \lim_{n\to\infty} U_{n+1}\left({}_0\mathbf{x}\right) = \lim_{n\to\infty} W\left(x_0, x_1, U_n\left({}_1\mathbf{x}\right)\right) = W\left(x_0, x_1, U_\infty\left({}_1\mathbf{x}\right)\right),$$

and thus the return U_∞ satisfies Koopmans' equation.

To end the proof, if V is a nonpositive solution to Koopmans' equation, then $V \leq 0 = U_0$. It follows $V \leq U_n$ for all n, so that $V \leq U_\infty$. ■

A single aggregator may give rise to a multiplicity of solutions to Koopmans' equation (1). Consider, for instance, an additively time-separable aggregator $W(x, y, \zeta) = u(x, y) + \beta\zeta$ with an upper semicontinuous short-run return u and $\beta \in (0, 1)$. The uncountable family of upper semicontinuous recursive functionals V_k defined by

$$V_k({}_0\mathbf{x}) = \sum_{t=0}^{\infty} u(x_t, x_{t+1})\,\beta^t + F_k({}_0\mathbf{x}), \quad \forall k \geq 0$$

are solutions to equation (1), where $F_k({}_0\mathbf{x}) = 0$ if $\limsup_{t\to\infty} |x_t| \leq k$ and $F_k({}_0\mathbf{x}) = -\infty$ elsewhere. Actually, the functionals $V_k({}_0\mathbf{x})$ obey to the equation $V_k({}_0\mathbf{x}) = W(x_0, x_1, V_k({}_1\mathbf{x}))$, as $F_k({}_0\mathbf{x}) = \beta F_k({}_1\mathbf{x})$. Note that the multiplicity cannot be avoided not even by declaring strict concavity for the aggregator.

However, a variant of Blackwell theorem (see Proposition 11 and Corollary 1 in Appendix A.1) tells us that any two different solutions to (1) are far from each other (according to the supnorm distance), provided that

$$|W(x, y, \zeta_1) - W(x, y, \zeta_2)| \leq \beta\,|\zeta_1 - \zeta_2| \tag{5}$$

holds for every $(x, y) \in D$, $\zeta_1, \zeta_2 \in \mathbb{R}_-$ and $0 < \beta < 1$. In other words, if U and V are two different solutions to Koopmans' equation, then

$$\|U - V\|_\infty = \sup_{{}_0x \in \mathbf{D}} |U({}_0\mathbf{x}) - V({}_0\mathbf{x})| = \infty.$$

We refer to Appendix A.1 for more details.

The rather severe restriction W.4 on the sign of the aggregator can be remarkably relaxed by assuming the existence of a positive upper bound $W(x, y, 0) \leq L$. For this purpose, it suffices replacing W.4 with the following assumption.[2]

W. 5 *There is a constant $k > 0$ such that $W(x, y, k) \leq k$ for all $(x, y) \in D$.*

In fact, the new aggregator

$$\widetilde{W}(x, y, \zeta) = W(x, y, \zeta + k) - k$$

satisfies W.4 if and only if W satisfies W.5. It is also easy to see that the relation $U_\infty = \widetilde{U}_\infty + k$ holds between the returns generated by W and $\widetilde{W}$ respectively. Observe that all the other assumptions W.1–2–3 (also W.6–7–8 of the next section) are not affected by the transformation $W \to \widetilde{W}$. Hence, all our results can be extended to such a wider class of aggregators.

A caveat is in order yet: condition W.5 could require a more stringent discounting on the future. To clarify this point, consider the additive aggregator $W(x, y, \zeta) = u(x, y) + \beta\zeta$. If $u(x, y) \leq 0$ then W.4 holds for every $\beta > 0$. While if $u(x, y) \leq L$ with $L > 0$, then W satisfies W.5 if and only if $0 < \beta < 1$ and $k = (1 - \beta)^{-1} L$.

A quite general result along this direction is formulated in the next proposition.

[2] Of course the conditions $\zeta \leq 0$ must be then replaced by $\zeta \in \mathbb{R}$ in this case.

Proposition 2 *Assumption W.5 is fulfilled for any aggregator W such that $W(x,y,0)$ is bounded from above, provided the Lipschitz condition (5) is satisfied. More precisely, if $W(x,y,0) \leq L$ then $W(x,y,k) \leq k$ for any $k \geq (1-\beta)^{-1} L$. The sequence U_n generated by the iterative system (3) from $U_0 = c$, where c is any constant, pointwise converges to the solution U_∞ defined in Proposition 1 and U_∞ is the maximal bounded from above solution to (1).*

Proof. Let $W(x,y,0) \leq L$ with $L > 0$. In view of (5), for $k > 0$, it follows

$$W(x,y,k) - W(x,y,0) \leq \beta k$$

If $k \geq (1-\beta)^{-1} L$ (i.e. $L \leq (1-\beta) k$) then

$$W(x,y,k) \leq W(x,y,0) + \beta k \leq L + \beta k = k$$

and so W.5 is true. Fix now some $k \geq (1-\beta)^{-1} L$, then the sequence U_n generated by (3) from $U_0 = k$ converges monotonically to a solution to (4), say $U_n \downarrow V$. By Proposition 11 the sequences U_n from $U_0 = c$ converge to V as well. If we set $c = 0$, we get $V = U_\infty$.

To complete the proof, let $V_1 \leq M$ where V_1 is a solution to (4) and $M \in \mathbb{R}$. There exists some k such that $V_1 \leq M \leq k$ and $k \geq (1-\beta)^{-1} L$. Applying the monotone operator $\mathbf{T}$, defined in (37), we have $V_1 = \mathbf{T}(V_1) \leq \mathbf{T}(k)$. Iterating, we get $V_1 \leq \mathbf{T}^n(k) \downarrow U_\infty$, as desired. ■

Define now the value functions

$$v(x) = \sup_{{}_0\mathbf{x} \in \mathbf{D}(x)} U_\infty({}_0\mathbf{x}) \tag{6}$$

$$v_n(x) = \sup_{{}_0\mathbf{x} \in \mathbf{D}(x)} U_n({}_0\mathbf{x}), \quad n \geq 1$$

Clearly, $v(x) \in [-\infty, 0]$, while $v_n(x) \in (-\infty, 0]$.

Proposition 3 *Under W.1–4, the value functions v and v_n satisfy the optimality equations*

$$v(x) = \sup_{y \in D(x)} W(x,y,v(y)) \tag{7}$$

$$v_{n+1}(x) = \sup_{y \in D(x)} W(x,y,v_n(y)), \quad n \geq 0.$$

Note that this Proposition still holds if W.4 is replaced by W.5.

Proof. Let us first prove the equality

$$\sup_{{}_1\mathbf{x} \in \mathbf{D}(y)} W(x,y,U_\infty({}_1\mathbf{x})) = W\left(x, y, \sup_{{}_1\mathbf{x} \in \mathbf{D}(y)} U_\infty({}_1\mathbf{x})\right). \tag{8}$$

Under W.2 it holds:

$$\sup_{{}_1\mathbf{x} \in \mathbf{D}(y)} W(x,y,U_\infty({}_1\mathbf{x})) \leq W\left(x, y, \sup_{{}_1\mathbf{x} \in \mathbf{D}(y)} U_\infty({}_1\mathbf{x})\right).$$

If $\sup_{{}_1\mathbf{x} \in \mathbf{D}(y)} U_\infty({}_1\mathbf{x}) = -\infty$, it holds with equality; otherwise, i.e., if $\sup_{{}_1\mathbf{x} \in \mathbf{D}(y)} U_\infty({}_1\mathbf{x}) > -\infty$, for any $\varepsilon > 0$, a path ${}_1\mathbf{x}(\varepsilon) \in \mathbf{D}(y)$ exists such that

$$U_\infty({}_1\mathbf{x}(\varepsilon)) \geq \sup_{{}_1\mathbf{x} \in \mathbf{D}(y)} U_\infty({}_1\mathbf{x}) - \varepsilon.$$

Consequently,

$$\begin{aligned}\sup_{{}_1\mathbf{x}\in\mathbf{D}(y)} W\left(x,y,U_\infty\left({}_1\mathbf{x}\right)\right) &\geq W\left(x,y,U_\infty\left({}_1\mathbf{x}\left(\varepsilon\right)\right)\right) \\ &\geq W\left(x,y,\sup_{{}_1\mathbf{x}\in\mathbf{D}(y)} U_\infty\left({}_1\mathbf{x}\right)-\varepsilon\right)\end{aligned}$$

As $\varepsilon\downarrow 0$, we get the reversed inequality,

$$\sup_{{}_1\mathbf{x}\in\mathbf{D}(y)} W\left(x,y,U_\infty\left({}_1\mathbf{x}\right)\right) \geq W\left(x,y,\sup_{{}_1\mathbf{x}\in\mathbf{D}(y)} U_\infty\left({}_1\mathbf{x}\right)\right);$$

therefore, (8) is true.

To prove the first optimality equation, by (8) we have

$$\begin{aligned}v\left(x\right) &= \sup_{{}_0\mathbf{x}\in\mathbf{D}(x)} W\left(x,x_1,U_\infty\left({}_1\mathbf{x}\right)\right) = \sup_{y\in D(x)}\sup_{{}_1\mathbf{x}\in\mathbf{D}(y)} W\left(x,y,U_\infty\left({}_1\mathbf{x}\right)\right) \\ &= \sup_{y\in D(x)} W\left(x,y,\sup_{{}_1\mathbf{x}\in\mathbf{D}(y)} U_\infty\left({}_1\mathbf{x}\right)\right) = \sup_{y\in D(x)} W\left(x,y,v\left(y\right)\right).\end{aligned}$$

The other optimality equations can be similarly deduced by means of (3). ■

The Bellman equation (7) provides a value function v for each recursive solution U of Koopmans' equation (4). Thus, in general, there may be many solutions to the Bellman equation.[3]

Example 1 *The additively time-separable quadratic aggregator*

$$W\left(x,y,\zeta\right)=-\frac{1}{2}\left(y-mx\right)^2+\beta\zeta$$

is a simple example exhibiting multiple solutions. Clearly $U_\infty\left({}_0\mathbf{x}\right)\leq U_\infty\left({}_0\mathbf{x}^*\right)=0$ *where* ${}_0\mathbf{x}^*$ *is the optimal path generated by the policy* $x_{t+1}=mx_t$*. Hence* $v\left(x\right)\equiv 0$ *is the value function for any* $\beta>0$*.*

If $\beta m^2>1$*, it is easy to check that also the quadratic function*

$$w\left(x\right)=-\frac{\beta m^2-1}{2\beta}x^2$$

solves (7). It is not the unique extra-solution: for instance, the family of linear functions

$$w\left(x\right)=\alpha x+\frac{\alpha^2}{2m\left(m-1\right)}\quad \forall\alpha\in\mathbb{R} \text{ and } m\neq 0,1$$

are solutions to (7) as well, provided that $\beta m=1$*.*

When $W\left(x,y,\cdot\right)$ is strictly increasing, it is easy to check that the optimal solutions to (6), if any, satisfy the Bellman principle: if a path ${}_0\mathbf{x}$ is optimal from x_0, then ${}_1\mathbf{x}$ is optimal from x_1. This in turn implies that optimal paths satisfy the Bellman equation (7).

[3] This kind of phenomena may occur also in bounded problems, as it has been discussed by [1] in the concave and bounded case.

The Bellman operator

$$\mathbf{B} : f \longmapsto \sup_{y \in D(x)} W(x, y, f(y))$$

is clearly monotone and the iterates $v_{n+1} = \mathbf{B} v_n$ decrease,

$$0 = v_0 \geq v_1 \geq \cdots \geq v_n \geq \cdots \geq v, \tag{9}$$

but the convergence $v_n \downarrow v$ may fail. Namely, the sequence v_n could converge to a function $v_\infty \geq v$ which differs from the value function v.

The next proposition, which extends Strauch's result established for additively time-separable aggregators (see [14] and [13, Ch. 4]), claims that if the pointwise limit $v_n \downarrow v_\infty$ of the finite horizon value functions satisfies the Bellman equation, that is $\mathbf{B} v_\infty = v_\infty$, then v_∞ is the value function v of the infinite horizon problem. Unfortunately, without further assumptions, the sufficient condition $\mathbf{B} v_\infty = v_\infty$ is not necessary.

Proposition 4 *Under W.1–4 and the further Lipschitz condition*

$$|W(x, y, \zeta_1) - W(x, y, \zeta_2)| \leq L |\zeta_1 - \zeta_2| \tag{10}$$

for some $L \geq 0$, if the limit function $v_n \downarrow v_\infty$ satisfies the Bellman equation, then $v_\infty = v$.

For example, Uzawa's aggregator (2) satisfies (10) when u is bounded from below.

Proof. Fix $\varepsilon > 0$ and pick a number $\lambda > 0$ so that $L\lambda < 1$ and a number η such that $0 < \eta < \varepsilon (1 - \lambda L)$. As

$$v_\infty(x) = (\mathbf{B} v_\infty)(x) = \sup_{y \in D(x)} W(x, y, v_\infty(y))$$

we can find a feasible sequence ${}_0\mathbf{x}$ such that

$$v_\infty(x_n) \leq W(x_n, x_{n+1}, v_\infty(x_{n+1})) + \lambda^n \eta$$

for every $n \geq 0$.

By substitution and using (10) we get that for all N it holds[4]

$$v_\infty(x_0) \leq U_N({}_0\mathbf{x}) + \eta \sum_{n=0}^{N-1} L^n \lambda^n \leq U_N({}_0\mathbf{x}) + \frac{\eta}{1 - \lambda L} \leq U_N({}_0\mathbf{x}) + \varepsilon.$$

Taking the limit for $N \to \infty$, it follows

$$v_\infty(x_0) \leq U_\infty({}_0\mathbf{x}) + \varepsilon \leq v(x_0) + \varepsilon.$$

As ε is arbitrarily small, we have $v_\infty \leq v$. On the other hand, (9) implies $v_\infty \geq v$. The equality $v_\infty = v$ is thus proved. ■

[4]The proof can be made by induction. We omit lengthy calculations.

Example 2 *The cake eating with logarithmic utility and discounting which depends on the amount of cake, leads to the aggregator*

$$W(x, y, \zeta) = \log(x - y) + \beta(x)\zeta$$

with $X = [0, \overline{x}]$ *and* $D(x) = [0, x]$ *for* $0 \leq x \leq \overline{x}$.

Under $0 \leq \beta(x) \leq \beta < 1$, *conditions W.1-2-3 are true. Also (10) holds in this case. Moreover,* $W(x, y, 0)$ *is bounded from above, as* $W(x, y, 0) \leq \log \overline{x}$. *In view of Proposition 2, thanks to Proposition 1 it follows that the total return function*

$$U_{\infty}({}_0\mathbf{x}) = \sum_{t=0}^{\infty} \left(\prod_{s=0}^{t-1} \beta(x_s) \right) \log(x_t - x_{t+1})$$

is upper semicontinuous.

Whenever the discount factor is constant, i.e., $\beta(x) \equiv \beta$, *calculation in closed form can be easily obtained. Indeed, we have*

$$v_n(x) = A_n \log x + B_n$$

with

$$A_{n+1} = 1 + \beta A_n \quad \text{and} \quad B_{n+1} = \beta B_n + \max_{\xi \in [0,1]} [\log(1 - \xi) + \beta A_n \log \xi],$$

and $A_0 = B_0 = 0$. *This sequence converges to*

$$v(x) = (1 - \beta)^{-1} \log x + (1 - \beta)^{-1} \max_{\xi \in [0,1]} \left[\log(1 - \xi) + \beta (1 - \beta)^{-1} \log \xi \right]$$

which is a fixed point of the Bellman operator $\mathbf{B}$. *Hence* v *is the value function.*

A closer investigation of the problems presented in this section requires some specification for the structure of the dynamic constraint D and/or the growth conditions on the aggregator W. Next example applies Boyd's weighted contraction mapping theorem (see Boyd [2]) for clarifying the domain of the recursive functionals. More precisely, the effective domain of the total return $U_{\infty} : \mathbf{D} \to \overline{\mathbb{R}}$, defined in (4), is generally smaller than $\mathbf{D}$. In the following example, among other things, we provide an insight on the collection of sequences ${}_0\mathbf{x}$ such that $U_{\infty}({}_0\mathbf{x}) > -\infty$.

Example 3 *Suppose that the aggregator obeys to the growth condition*

$$W(x, y, 0) \geq -A(x)(1 + |y|^{\eta}) \tag{11}$$

for all $(x, y) \in D$, *with* $A(x) > 0$ *for all* $x \in X$, *and for some* $\eta > 0$. *Here* $|\cdot|$ *is a norm in* $\mathbb{R}^n$. *Assume further that* W *obeys the Lipschitz condition (5) with* $0 < \beta < 1$. *For a fixed scalar* $\beta < \delta < 1$, *consider the sequence space*

$$\ell_{\eta}(\delta) = \left\{ {}_0\mathbf{x} \in (\mathbb{R}^n)^{\infty} : \Phi_{\delta}({}_0\mathbf{x}) \equiv \sum_{t=0}^{\infty} |x_t|^{\eta} \delta^t < \infty \right\}.$$

The set of feasible paths belonging to the space $\ell_{\eta}(\delta)$ *is*

$$\mathbf{D}(\delta) = \mathbf{D} \cap \ell_{\eta}(\delta),$$

supposed to be nonempty. The total returns $U(_0\mathbf{x})$ *will be understood as functionals* $U:\mathbf{D}(\delta)\to\mathbb{R}_-$*. Endow the space* $\mathbb{R}^{\mathbf{D}(\delta)}$ *of these functionals by the weight*

$$\psi(_0\mathbf{x}) = 1 + \Phi_\delta(_0\mathbf{x})$$

which gives rise to the Banach space $B(\mathbf{D}(\delta);\psi)$ *of the so-called* ψ*-bounded functions (see [2] and [10]), that is, the collection of the functions* $U(_0\mathbf{x})$ *for which*

$$\sup_{_0\mathbf{x}\in\mathbf{D}(\delta)} \frac{|U(_0\mathbf{x})|}{\psi(_0\mathbf{x})} < \infty.$$

We are in a position to apply the weighted contraction theorem (see Proposition 12 in Appendix A.2) to the operator $T : B_-(\mathbf{D}(\delta);\psi)\to\mathbb{R}_-^{\mathbf{D}(\delta)}$ *given by*

$$(TU)(_0\mathbf{x}) = W(x_0, x_1, U(_1\mathbf{x})).$$

Clearly fixed points of T *are solutions to the Koopmans' equation.*

In view of (5), we can write

$$\begin{aligned} T(U-\lambda\psi)(_0\mathbf{x}) &= W(x_0,x_1,U(_1\mathbf{x})-\lambda\psi(_1\mathbf{x})) \geq W(x_0,x_1,U(_1\mathbf{x})) - \lambda\beta\psi(_1\mathbf{x}) \\ &= T(U)(_0\mathbf{x}) - \lambda\beta\psi(_1\mathbf{x}). \end{aligned}$$

Moreover, from the relation $\psi(_1\mathbf{x})\leq\delta^{-1}\psi(_0\mathbf{x})$ *it follows*

$$T(U-\lambda\psi)(_0\mathbf{x}) \geq T(U)(_0\mathbf{x}) - \left(\beta\delta^{-1}\right)\lambda\psi(_0\mathbf{x})$$

and so the (ii) of Proposition 12 is met, as $\beta\delta^{-1}<1$*.*

Now, in view of (11), we have

$$(T\mathbf{0})(_0\mathbf{x}) = W(x_0,x_1,0) \geq -A(x_0)(1+|x_1|^\eta)$$

Hence, the inequality $|(T\mathbf{0})(_0\mathbf{x})| \leq A(x_0)\delta^{-1}\psi(_0\mathbf{x})$ *follows easily. Thus,* $T\mathbf{0}$ *is* ψ*-bounded and so condition (iii) of Proposition 12 is verified.*

Below we list the consequences of Proposition 12. If U_∞ *is the return function (4) then*

$$_0\mathbf{x}\in\bigcup_{\delta>\beta}\mathbf{D}(\delta)\Longrightarrow U_\infty(_0\mathbf{x}) > -\infty.$$

For every fixed $\delta>\beta$*, the restriction of* U_∞ *to the space* $\mathbf{D}(\delta)$ *is the unique solution to Koopmans' equation in the space* $B_-(\mathbf{D}(\delta);\psi)$*. Moreover, the convergence* $U_n\downarrow U_\infty$ *is uniform over every nonempty sublevel* $\Phi_\delta\leq k$*.*

3 Strongly concave aggregators

So far the conditions listed in previous section are in general not enough to guarantee the existence of optimal solutions to such recursive problems, as well as to infer other nice properties. In this section we tackle the particular subclass of concave aggregators having a positive curvature. The following assumptions are added to those of the previous section.

W. 6 $W(\cdot,\cdot,\cdot)$ *is concave on* $D\times\mathbb{R}_-$ *and* $W(\cdot,\cdot,0)$ *is* (α_1,α_2)*-concave*[5] *on* D *with* $\alpha_1,\alpha_2\geq 0$ *and* $\alpha_1+\alpha_2>0$*;*

[5] This is a property of strong concavity illustrated in Definition 1 of Appendix A.3.

W. 7 *there is a scalar* $\underline{\beta} > 0$ *such that*

$$|W(x,y,\zeta_1) - W(x,y,\zeta_2)| \geq \underline{\beta}\,|\zeta_1 - \zeta_2| \tag{12}$$

holds for all $x, y \in D$ *and all* $\zeta_1, \zeta_2 \in \mathbb{R}_-$;

W. 8 *if* $\alpha_1 \cdot \alpha_2 = 0$, *there is a point* $(\bar{x}, \bar{y}) \in D$ *for which* $W(\overline{x}, \overline{y}, 0) \geq W(x,y,0)$ *for every* $(x,y) \in D$.

Remark 1 *When* $\alpha_1 \cdot \alpha_2 > 0$, *the condition W.8 is superfluous since the existence of the point* $(\bar{x}, \bar{y})$ *is assured by W.6 (see Theorem 4 in Appendix A.3).*

Remark 2 *The additional assumption* $0 < \underline{\beta} < 1$ *in W.7 is not restrictive. This qualification is maintained throughout this section.*

Next lemma, related to the strong concavity of the aggregator, plays an important role in the following.

Lemma 1 *Under W.6-7-8, if* $(\bar{x}, \bar{y})$ *is a maximizer of the function* $W(\cdot,\cdot,0)$ *over* D, *then*

$$W(x,y,\zeta) \leq -\frac{1}{2}\alpha_1 \|x - \overline{x}\|^2 - \frac{1}{2}\alpha_2 \|y - \overline{y}\|^2 + \underline{\beta}\zeta$$

for all $(x,y) \in D$ *and* $\zeta \leq 0$.

Proof. As $(\bar{x}, \bar{y})$ is a maximizer of the (α_1, α_2)-concave function $W(\cdot,\cdot,0)$, it follows that (see Proposition 13)

$$\begin{aligned} W(x,y,0) &\leq -\frac{1}{2}\alpha_1 \|x - \overline{x}\|^2 - \frac{1}{2}\alpha_2 \|y - \overline{y}\|^2 + W(\overline{x}, \overline{y}, 0) \\ &\leq -\frac{1}{2}\alpha_1 \|x - \overline{x}\|^2 - \frac{1}{2}\alpha_2 \|y - \overline{y}\|^2 . \end{aligned}$$

According to W.7 we can write

$$\begin{aligned} W(x,y,\zeta) &= W(x,y,\zeta) - W(x,y,0) + W(x,y,0) \\ &\leq -|W(x,y,\zeta) - W(x,y,0)| + W(x,y,0) \\ &\leq \underline{\beta}\zeta - \frac{1}{2}\alpha_1 \|x - \overline{x}\|^2 - \frac{1}{2}\alpha_2 \|y - \overline{y}\|^2 , \end{aligned}$$

which is the desired inequality. ■

We now deduce that the optimization of the recursive functional $U_\infty(x,\cdot) : \mathbf{D}(x) \to \mathbb{R}_- \cup \{-\infty\}$ can be restricted to the Hilbert space $\ell_2(\underline{\beta})$, $\underline{\beta} \in (0,1)$, of the sequences ${}_1\mathbf{x} = (x_1, \ldots, x_t, \ldots)$ for which

$$\|{}_1\mathbf{x}\|^2 = \sum_{t=1}^{\infty} \|x_t\|^2 \underline{\beta}^{t-1} < \infty.$$

W.6 implies that at least one of the scalars α_1 and α_2 of Lemma 1 is strictly positive. Therefore, at least one of the two following inequalities is true:

$$W(x,y,\zeta) \leq -\frac{1}{2}\alpha_1 \|x - \overline{x}\|^2 + \underline{\beta}\zeta, \quad \alpha_1 > 0 \tag{13}$$

$$W(x,y,\zeta) \leq -\frac{1}{2}\alpha_2 \|y - \overline{y}\|^2 + \underline{\beta}\zeta, \quad \alpha_2 > 0. \tag{14}$$

Proposition 5 *For every feasible path* $(x_0, {}_1\mathbf{x})$*, from the initial state* $x_0 \in X$*, we have*

$$U_\infty(x_0, {}_1\mathbf{x}) > -\infty \Longrightarrow {}_1\mathbf{x} \in l_2\left(\underline{\beta}\right).$$

Proof. Suppose that the inequality (13) holds. Then

$$U_\infty({}_0\mathbf{x}) \leq -\frac{1}{2}\alpha_1 \|x_0 - \overline{x}\|^2 + \underline{\beta} U_\infty({}_1\mathbf{x}). \tag{15}$$

By iterating (15) we get easily

$$\begin{aligned} U_\infty({}_0\mathbf{x}) &\leq -\frac{1}{2}\alpha_1 \|x_0 - \overline{x}\|^2 - \frac{\underline{\beta}\alpha_1}{2}\sum_{t=1}^{n} \|x_t - \overline{x}\|^2 \underline{\beta}^{t-1} + \underline{\beta}^n U_\infty({}_n\mathbf{x}) \\ &\leq -\frac{\underline{\beta}\alpha_1}{2}\sum_{t=1}^{n} \|x_t - \overline{x}\|^2 \underline{\beta}^{t-1}, \end{aligned}$$

namely,

$$\frac{\underline{\beta}\alpha_1}{2}\sum_{t=1}^{n} \|x_t - \overline{x}\|^2 \underline{\beta}^{t-1} \leq -U_\infty({}_0\mathbf{x}) < +\infty$$

for every n. Hence,

$$\|{}_1\mathbf{x} - \overline{\mathbf{x}}\|^2 = \sum_{t=1}^{\infty} \|x_t - \overline{x}\|^2 \underline{\beta}^{t-1} < \infty,$$

where $\overline{\mathbf{x}} = (\overline{x}, \overline{x}, \ldots)$. As $\overline{\mathbf{x}} \in \ell_2\left(\underline{\beta}\right)$, it follows that ${}_1\mathbf{x} \in \ell_2\left(\underline{\beta}\right)$.

The case (14) is similar; in fact, in this case we have

$$U_\infty({}_0\mathbf{x}) \leq -\frac{\alpha_2}{2}\sum_{t=1}^{n} \|x_t - \overline{y}\|^2 \underline{\beta}^{t-1}$$

for every n. ■

The strong concavity of assumption W.6 is crucial to get the previous property. Consider, for instance, the aggregator $W(x, y, \zeta) = -(1/2)(y - mx)^2 + \beta\zeta$ of Example 1: W is concave but it fails to be (α_1, α_2)-concave with $\alpha_1 + \alpha_2 > 0$. We have $U({}_0\mathbf{x}) = 0$ for the paths generated by the policy $y = mx$, but the slope m is totally unrelated with the discount factor β.

Next lemma shows that our recursive functionals are *coercive*.[6]

Lemma 2 *Under (13) it holds*

$$U_\infty(x_0, {}_1\mathbf{x}) \leq -\frac{1}{2}\alpha_1 \|x_0 - \overline{x}\|^2 - \frac{1}{2}\underline{\beta}\alpha_1 \|{}_1\mathbf{x} - \overline{\mathbf{x}}\|^2.$$

Likewise, under (14),

$$U_\infty(x_0, {}_1\mathbf{x}) \leq -\frac{1}{2}\alpha_2 \|{}_1\mathbf{x} - \overline{\mathbf{y}}\|^2.$$

[6]Recall that a function $f : V \to \mathbb{R} \cup \{-\infty\}$ defined over a normed space V is called coercive if all its non-empty upper levels $(f \geq \lambda)$ are norm bounded.

Proof. In the first case, consider the new aggregator function

$$\widetilde{W}(x, y, \zeta) = -\frac{1}{2}\alpha_1 \|x - \overline{x}\|^2 + \underline{\beta}\zeta.$$

According to (13) we have $W(x, y, \zeta) \leq \widetilde{W}(x, y, \zeta)$. Denoting by U_n and $\widetilde{U}_n$ the solutions to (3) starting from the initial condition $U_0 = \widetilde{U}_0 = 0$ for the two aggregators W and $\widetilde{W}$, respectively, it is easy to check by induction that $U_n \leq \widetilde{U}_n$ for all n. Actually let $U_n \leq \widetilde{U}_n$. Then

$$\begin{aligned} U_{n+1}({}_0\mathbf{x}) &= W(x_0, x_1, U_n({}_1\mathbf{x})) \leq \widetilde{W}(x_0, x_1, U_n({}_1\mathbf{x})) \\ &\leq \widetilde{W}\left(x_0, x_1, \widetilde{U}_n({}_1\mathbf{x})\right) = \widetilde{U}_{n+1}({}_0\mathbf{x}) \end{aligned}$$

Taking limit we get $U_\infty \leq \widetilde{U}_\infty$. On the other hand,

$$U_\infty(x_0, {}_1\mathbf{x}) \leq \widetilde{U}_\infty(x_0, {}_1\mathbf{x}) = -\frac{1}{2}\alpha_1 \|x_0 - \overline{x}\|^2 - \frac{1}{2}\underline{\beta}\alpha_1 \|{}_1\mathbf{x} - \overline{\mathbf{x}}\|^2$$

as desired. The other case is similar. ■

We can now establish the existence of optimal solutions.

Theorem 1 *Under assumptions W.1–8, the problem*

$$v(x) = \sup_{{}_0\mathbf{x} \in \mathbf{D}(x)} U_\infty({}_0\mathbf{x}) \tag{16}$$

has optimal solutions for every initial state $x \in X$ for which $v(x) > -\infty$.

After the proof below we discuss uniqueness of the optimal solutions as well.

Proof. Proposition 5 implies that we can maximize the upper semicontinuous functional U_∞ on the Hilbert space $l_2(\underline{\beta})$; namely, consider the functional $U_\infty(x, \cdot) : l_2(\underline{\beta}) \to [-\infty, \infty)$.

Let us check, by induction, that the partial functionals $U_n({}_0\mathbf{x})$ are concave. Actually $U_1(x_0, x_1) = W(x_0, x_1, 0)$ is concave by W.6. Let $U_n({}_0\mathbf{x})$ be concave. From the relation $U_{n+1}({}_0\mathbf{x}) = W(x_0, x_1, U_n({}_1\mathbf{x}))$, we have

$$U_{n+1}\left(\vartheta{}_0\mathbf{x}+\overline{\vartheta}{}_0\mathbf{x}'\right) = W\left(\vartheta x_0 + \overline{\vartheta}x_0', \vartheta x_1 + \overline{\vartheta}x_1', U_n\left(\vartheta{}_1\mathbf{x}+\overline{\vartheta}{}_1\mathbf{x}'\right)\right),$$

where $\overline{\vartheta} = 1 - \vartheta$. By W.6,

$$\begin{aligned} U_{n+1}\left(\vartheta{}_0\mathbf{x}+\overline{\vartheta}{}_0\mathbf{x}'\right) &\geq W\left(\vartheta x_0 + \overline{\vartheta}x_0', \vartheta x_1 + \overline{\vartheta}x_1', \vartheta U_n({}_1\mathbf{x}) + \overline{\vartheta}U_n({}_1\mathbf{x}')\right) \\ &\geq \vartheta W(x_0, x_1, U_n({}_1\mathbf{x})) + \overline{\vartheta}W(x_0', x_1', U_n({}_1\mathbf{x}')) \\ &= \vartheta U_{n+1}({}_0\mathbf{x}) + \overline{\vartheta}U_{n+1}({}_0\mathbf{x}'). \end{aligned}$$

Hence, U_{n+1} is concave. Therefore, every U_n is concave and, in turn, the limit U_∞ is concave as well.

Lemma 2 implies that the functional $U_\infty(x, {}_1\mathbf{x})$ is dominated by a quadratic functional $-k\|{}_1\mathbf{x} - \mathbf{u}\|^2$ with $\mathbf{u} \in l_2(\underline{\beta})$. This entails that $U_\infty(x, \cdot)$ is coercive. That is, all the nonempty upper level sets $U_\infty(x, \cdot) \geq \lambda$ are norm bounded (of course also closed, thanks to the upper semicontinuity of the functional). Indeed, the relation

$$-k\|{}_1\mathbf{x} - \mathbf{u}\|^2 \geq U_\infty(x, {}_1\mathbf{x}) \geq \lambda$$

yields $\|{}_1\mathbf{x}-\mathbf{u}\|^2 \leq -\lambda/k$.

Now the proof follows the one of the classical Tonelli Theorem (a cornerstone of infinite dimensional optimization theory). Briefly: endow $l_2\left(\underline{\beta}\right)$ with the weak topology. The upper level sets $U_\infty(x,\cdot)\geq\lambda$ are weakly closed by the Mazur Theorem. Consequently, $U_\infty(x,\cdot)$ turns out to be weakly upper semicontinuous. Since the upper levels are norm bounded, by Alaouglu Theorem the nonempty upper levels are weakly compact. As $v(x)>-\infty$ some upper level is not empty and Weierstrass Theorem concludes the proof. ■

Remark 3 *The same proof leads to the existence of optimal paths for the finite-horizon problems*

$$v_n(x)=\sup_{{}_0\mathbf{x}\in\mathbf{D}(x)} U_n({}_0\mathbf{x}), \quad n\geq 1. \tag{17}$$

In fact, these are a simpler problems because they do not require to employ weak topologies, neither resorting to the property of concavity of the functionals. Observe further that the concavity of U_n and U_∞ imply that the value functions v_n and v are concave.

Uniqueness of optimal solutions is not claimed in Theorem 1. In fact, more regularity conditions on the functionals $U_\infty(x,\cdot):l_2\left(\underline{\beta}\right)\to[-\infty,\infty)$ are required to establish uniqueness; specifically, they must be strongly concave rather than only concave, the latter being the property turning out to characterize them in the proof of the previous theorem under assumption W.6. The following slightly stronger assumption, which replaces W.6, is enough to establish uniqueness of optimal solutions.

W. 9 *$W(\cdot,\cdot,\cdot)$ is $(\alpha_1,\alpha_2,0)$-concave on $D\times\mathbb{R}_-$, with $\alpha_1,\alpha_2\geq 0$ and $\alpha_1+\alpha_2>0$.*

Notice that in the additively time-separable case W.9 coincides with W.6.

Under W.9, through tedious algebra, it can be shown that the functionals $U_\infty(x,\cdot)$: $l_2\left(\underline{\beta}\right)\to[-\infty,\infty)$ are $\left(\alpha_2+\underline{\beta}\alpha_1\right)$-concave over $l_2\left(\underline{\beta}\right)$ for every $x\in X$, so that $U_\infty(x,\cdot)+\frac{1}{2}\left(\alpha_2+\underline{\beta}\alpha_1\right)\|\cdot\|^2$ turns out to be concave, where $\|\cdot\|$ is the norm of $l_2\left(\underline{\beta}\right)$. Hence, a direct application of Theorem 4 in Appendix A.3 provides existence and uniqueness of the optimal solution at once.

Example 4 *The simplest examples of aggregators satisfying W.6 are those related to quadratic functions.*

For example, in the scalar case, the aggregators such that

$$W(x,y,0)=-\frac{1}{2}x^2+\gamma xy-\frac{\delta}{2}y^2+mx+ny$$

turn out to be (α_1,α_2)-concave (with $\alpha_1\cdot\alpha_2>0$), whenever $\gamma^2<\delta$. To see this, it suffices to pick a pair (α_1,α_2) so that $\gamma^2<(\delta-\alpha_2)(1-\alpha_1)$, $0<\alpha_2<\delta$ and $0<\alpha_1<1$.

Likewise, the aggregators for which

$$W(x,y,0)=-\frac{1}{2}x^2+mx+ny$$

are clearly $(\alpha_1,0)$-concave with $\alpha_1>0$. But in this case n must vanish, otherwise $W(x,y,0)$ is unbounded from above and so W.8 is violated.

The multidimensional case is more involved. Let Q be a $2n$-order symmetric matrix, partitioned as

$$Q=\begin{bmatrix} A & B \\ B' & C \end{bmatrix}$$

where A, B, C n-order square matrices. We have

$$W(x, y, 0) = (x, y)' Q(x, y) + m'x + n'y.$$

The aggregator $W(x, y, 0)$ is (α_1, α_2)-concave, with $\alpha_1 \cdot \alpha_2 > 0$ iff C is negative definite and $A - BC^{-1}B'$ is negative definite.

Otherwise, $W(x, y, 0)$ is $(\alpha_1, 0)$-concave, with $\alpha_1 > 0$, iff C is negative semidefinite, $A - BC^{\dagger}B'$ is negative definite and $\mathcal{R}(B') \subseteq \mathcal{R}(C)$. Here $C^{\dagger}$ denotes the Moore-Penrose pseudoinverse of C and $\mathcal{R}(C)$ is the range of the matrix C.

The last example suggests a promising field of investigation in applications related to economic-epidemiological models, as the social planner objective in such models often is to minimize a quadratic social cost associated with the epidemic management (see, for example, [6], and, for a stochastic setting, [7]). Through the transformation $C(x, y, \zeta) = -W(x, y, -\zeta)$ such (positive) cost minimization problems can be handled by means of the results discussed in Sections 3 and 4. We thus believe that testing our methodology in applications of this kind may open a research venue deserving further investigation.

4 Stochastic programming

We now turn our attention to a stochastic version of deterministic problems discussed in the previous sections. It is best to start with the study of the Bellman equation

$$v(x, z) = \sup_{y \in D(x,z)} [W(x, y, z), \mathbf{M}(z, v(y, \cdot))]$$

associated with the stochastic dynamic programming. The first step is that of identifying a suitable space of functions on which let our operator act.

Here $x \in X \subseteq \mathbb{R}^n$ denotes the endogenous variable while $z \in Z$ is the exogenous variable (or shock). The multimapping $D : X \times Z \rightrightarrows X$ is the feasible correspondence. Its graph is defined as

$$\operatorname{Gr} D = \{(x, z, y) \in X \times Z \times X : y \in D(x, z)\}. \tag{18}$$

The function $v : X \times Z \to \mathbb{R}$ is the value function depending on the initial current state (x, z) of the system. The function W is the dynamic aggregator, while $\mathbf{M} : Z \times \mathbb{R}^Z \to \mathbb{R}$ denotes the *stochastic aggregator* (or the *certainty equivalent operator*). Equivalently, one can assign a separate mapping $\mathcal{M} : \mathbb{R}^Z \to \mathbb{R}^Z$ which is related to $\mathbf{M}$ according to $\mathcal{M}(\varphi)(z) = \mathbf{M}(z, \varphi)$ for $\varphi \in \mathbb{R}^Z$.

Alongside the space $X \times Z$, where X is a convex set of $\mathbb{R}^n$ and Z a topological space on which the functions $f(x, z)$ are defined, let us introduce the weighted space $B(X \times Z; w)$ where the weight function is

$$w(x) = 1 + \|x\|^2.$$

A function $f : X \times Z \to \mathbb{R}$ lies in the space $B(X \times Z; w)$ if

$$\|f\|_w = \sup_{(x,z) \in X \times Z} \frac{|f(x, z)|}{1 + \|x\|^2} < \infty.$$

It is well known that $B(X \times Z; w)$ is a Banach space and the convergence in such a space amounts to the uniform convergence on the compact subsets of the space X. Actually, the weight w satisfies $0 < \varepsilon \leq w(x)^{-1} < 1$ over any assigned compact subset of X. Therefore, the norm $\|f\|_w$ is equivalent to the supnorm over each compact subsets of X (see, for instance, [10]).

By $UC_-(X \times Z; w)$ we denote the cone of the negative functions $\varphi \in B_-(X \times Z; w)$ that are upper semicontinuous on $X \times Z$.

As will become clearer later, the relevance of this weighted space is motivated by the following property.

Proposition 6 *All the functions $f : X \times Z \to \mathbb{R}$ satisfying the condition*

$$|f(x,z)| \leq A + B\|x - \bar{x}\|^2 \quad \forall (x,z) \in X \times Z,$$

for some $A, B \geq 0$ and $\bar{x} \in X$, are w-bounded, namely, $f \in B(X \times Z; w)$.

Proof. Let $\|x\| \geq 1$, then

$$\begin{aligned}\frac{|f(x,z)|}{1 + \|x\|^2} &\leq \frac{A + B\|x - \bar{x}\|^2}{1 + \|x\|^2} = \frac{A\|x\|^{-2} + B\left\|\|x\|^{-1}x - \|x\|^{-1}\bar{x}\right\|^2}{1 + \|x\|^{-2}} \\ &\leq A + B(1 + \|\bar{x}\|)^2.\end{aligned}$$

If instead $\|x\| \leq 1$,

$$\frac{|f(x,z)|}{1 + \|x\|^2} \leq \frac{A + B\|x - \bar{x}\|^2}{1 + \|x\|^2} \leq A + B\|x - \bar{x}\|^2 \leq A + B(1 + \|\bar{x}\|)^2.$$

Hence, f is w-bounded. ■

The assumptions we set on the aggregator $W(x, y, z, \zeta)$ are as follows.

B. 1 *Constants $m_1, m_2, \alpha_1, \alpha_2$, with $m_1, m_2, \alpha_1, \alpha_2 > 0$, and $\bar{x}, \overline{y} \in X$ exist such that*

$$-m_2 - n_2\left(\|x\|^2 + \|y\|^2\right) \leq W(x,y,z,0) \leq -m_1 - \alpha_1\|x - \bar{x}\|^2 - \alpha_2\|y - \overline{y}\|^2 \tag{19}$$

for all $(x, y, z) \in \mathrm{Gr}\, D$;

B. 2 *$W(\cdot,\cdot,\cdot,\zeta)$ is upper semicontinuous on $\mathrm{Gr}\, D$ for every fixed $\zeta \leq 0$;*

B. 3 *$W(x, y, z, \cdot)$ is nondecreasing and continuous over $\mathbb{R}_-$ for each $(x, y, z) \in \mathrm{Gr}\, D$ and $\lim_{\zeta \to -\infty} W(x, y, z, \zeta) = -\infty$;*

B. 4 *there is a scalar $0 \leq \overline{\beta} < 1$ such that*

$$|W(x,y,z,\zeta_1) - W(x,y,z,\zeta_2)| \leq \overline{\beta}\,|\zeta_1 - \zeta_2|$$

for all $(x, y, z) \in \mathrm{Gr}\, D$ and $\zeta_1, \zeta_2 \in \mathbb{R}_-$;

B. 5 *$W(x, y, z, \cdot)$ is convex at 0, namely*

$$W(x,y,z,\alpha\zeta) \leq \alpha W(x,y,z,\zeta) + (1 - \alpha) W(x,y,z,0)$$

for all $\alpha \in [0, 1]$ and $\zeta \leq 0$;

B. 6 Gr D *is closed and the correspondence* D *has a deterministic continuous bounded selection, namely, a continuous map* $d : X \to X$ *and a number* N *exist for which* $d(x) \in D(x,z)$ *for all* $(x,z) \in X \times Z$ *and* $\|d(x)\|^2 \leq N$ *for all* $x \in X$.

Regarding to the certain equivalent operator $\mathcal{M} : \mathbb{R}_-^Z \to \mathbb{R}_-^Z$, the assumptions are:[7]

B. 7 $\mathcal{M}(k) = k$, *for* $k \in \mathbb{R}_-$, *and* $\mathcal{M}$ *is monotone and subhomogeneous;*[8]

B. 8 *for every* $f \in UC_-(X \times Z; w)$, *the function* $(z,y) \longmapsto \mathbf{M}(z, f(y,\cdot))$ *is upper semicontinuous over* $Z \times X$.

Next proposition is similar to Lemma 1, though with different assumptions on parameter β and with opposite inequalities.

Proposition 7 *Under B.1 and B.4*

$$-m_2 - n_2\left(\|x\|^2 + \|y\|^2\right) + \overline{\beta}\zeta \leq W(x,y,z,\zeta) \leq -m_1 - \alpha_1\|x-\bar{x}\|^2 - \alpha_2\|y-\overline{y}\|^2 \tag{20}$$

it holds for $(x,y,z) \in$ Gr D *and* $\zeta \leq 0$.

Proof. If $\zeta \leq 0$ it holds

$$|W(x,y,z,0) - W(x,y,z,\zeta)| = W(x,y,z,0) - W(x,y,z,\zeta).$$

In view of B.4 we can write

$$0 \leq W(x,y,z,0) - W(x,y,z,\zeta) \leq -\overline{\beta}\zeta,$$

and using the inequalities (19) we get easily our result. ∎

Proposition 8 *All the pairs of positive functions* $f_1, f_2 \in B(X \times Z; w)$ *of the type*

$$f_1 = A + B\|x-\bar{x}\|^2, \quad f_2 = C + D\|x-\overline{y}\|^2$$

with $A,B,C,D > 0$ *and* $\bar{x}, \overline{y} \in X$, *are linked (see Definition 2 in Appendix A.5).*

Proof. It suffices to show that each function $f_1 = A + B\|x-\bar{x}\|^2$ is linked to every function of the type $C + D\|x\|^2$. That is, it must hold

$$\mu\left(A + B\|x\|^2\right) \leq C + D\|x-\bar{x}\|^2 \leq \lambda\left(A + B\|x\|^2\right)$$

for some $\lambda, \mu > 0$. By studying the signs

$$C + D\|x-\bar{x}\|^2 - \mu\left(A + B\|x\|^2\right) \geq 0$$
$$\lambda\left(A + B\|x\|^2\right) - C - D\|x-\bar{x}\|^2 \geq 0$$

of this two quadratic functions, it is easy to check that they are true for all x, provided $\mu > 0$ is small enough and $\lambda > 0$ is large enough. ∎

The main properties of the Bellman operator

$$(\mathbf{B}f)(x,z) = \sup_{y \in D(x,z)} W(x,y,z,\mathbf{M}(z, f(y,\cdot)))$$

acting on the function $f \in B_-(X \times Z; w)$, is the subject of the next statements.

[7]Constant (deterministic) functions $\varphi(z) = k$ for all $z \in Z$ are usually denoted by k.
[8]Namely, $\mathcal{M}(\alpha f) \leq \alpha\mathcal{M}(f)$ for all $\alpha \in [0,1]$ and $f \in B_-(Z)$.

Lemma 3 *Let* $f \in UC_-(X \times Z; w)$, *then* $\mathbf{B}f \in UC_-(X \times Z; w)$.

Proof. By B.7 the operator $\mathbf{B}$ is monotone and so $\mathbf{B}(f) \leq \mathbf{B}(0)$. But, by (19) in B.1,

$$\mathbf{B}(0)(x,z) = \sup_{y \in D(x,z)} W(x,y,z,0) \leq \sup_{y \in D(x,z)} \left[-m_1 - \alpha_1 \|x - \bar{x}\|^2 - \alpha_2 \|y - \overline{y}\|^2\right] \leq 0 \qquad (21)$$

and so $\mathbf{B}(f) \leq 0$. Given a function $f \in UC_-(X \times Z; w)$, let us show that

$$\rho(x,z,y) = W(x,y,z,\mathbf{M}(z, f(y,\cdot)))$$

is upper semicontinuous over $\operatorname{Gr} D$. Fix a point $(\overline{x}, \overline{y}, \overline{z}) \in \operatorname{Gr} D$ and consider any feasible sequence $(x_n, z_n, y_n) \to (\overline{x}, \overline{y}, \overline{z})$. Let $\lambda > \mathbf{M}(\overline{z}, f(\overline{y}, \cdot))$. The upper semicontinuity assumed in B.8 implies that $\lambda > \mathbf{M}(z_n, f(y_n, \cdot))$ for n large enough. Therefore

$$W(x_n, y_n, z_n, \mathbf{M}(z_n, f(y_n, \cdot))) \leq W(x_n, y_n, z_n, \lambda),$$

so that

$$\begin{aligned}\limsup_{n\to\infty} W(x_n, y_n, z_n, \mathbf{M}(z_n, f(y_n, \cdot))) &\leq \limsup_{n\to\infty} W(x_n, y_n, z_n, \lambda) \\ &\leq W(\overline{x}, \overline{y}, \overline{z}, \lambda).\end{aligned}$$

B.3 implies that

$$\limsup_{n\to\infty} W(x_n, y_n, z_n, \mathbf{M}(z_n, f(y_n, \cdot))) \leq W(\overline{x}, \overline{y}, \overline{z}, \mathbf{M}(\overline{z}, f(\overline{y}, \cdot))),$$

and thus $\rho(x,z,y)$ is upper semicontinuous at $(\overline{x}, \overline{y}, \overline{z})$.

Moreover, by (19), we have

$$\begin{aligned}\rho(x,z,y) &= W(x,y,z,\mathbf{M}(z, f(y,\cdot))) \leq W(x,y,z,0) \\ &\leq -m_1 - \alpha_1 \|x - \bar{x}\|^2 - \alpha_2 \|y - \overline{y}\|^2 \leq -\alpha_2 \|y - \overline{y}\|^2\end{aligned}$$

with $\alpha_2 > 0$. This implies that ρ is coercive with respect y (see Appendix A.4). By invoking Proposition 14, we can infer that $(\mathbf{B}f)(x,z) = \sup_{y \in D(x,z)} \rho(x,z,y)$ is upper semicontinuous. ■

In the following we shall use the notation $[\![f, g]\!]$ to denote the interval of functions between f and g (see Appendix A.5).

Proposition 9

i) *The operator* $\mathbf{B} : B_-(X \times Z; w) \to B_-(X \times Z; w)$ *maps every interval*

$$\left[\!\left[-k - n_2 \|\cdot\|^2, 0\right]\!\right]$$

of $B_-(X \times Z; w)$ *into the interval* $\left[\!\left[-k - n_2 \|\cdot\|^2, -m_1 - \alpha_1 \|\cdot - \bar{x}\|^2\right]\!\right]$, *provided that*

$$k \geq \left(1 - \overline{\beta}\right)^{-1} \left(m_2 + n_2 N \left(1 + \overline{\beta}\right)\right); \qquad (22)$$

ii) *for every* $f \in B_-(X \times Z; w)$, $\mathbf{B}(f) \in \left[\!\left[-k' - n_2 \|\cdot\|^2, 0\right]\!\right]$ *for* $k' > 0$ *large enough;*

iii) $\mathbf{B}$ *is monotone on* $B_-(X \times Z; w)$*;*

iv) $\mathbf{B}$ *is convex at* 0, *i.e.*,

$$\mathbf{B}(\alpha f) \leq \alpha \mathbf{B}(f) + \overline{\alpha} \mathbf{B}(0)$$

for all $f \in B_-(X \times Z; w)$ *and* $\alpha \in [0,1]$.

Proof. (i) From (19) in B.1 we have

$$\mathbf{B}(0)(x,z) = \sup_{y \in D(x,z)} W(x,y,z,0) \leq -m_1 - \alpha_1 \|x - \bar{x}\|^2 . \tag{23}$$

As $\mathbf{B}(f) \leq \mathbf{B}(0)$, it follows $\mathbf{B}(f) \leq -m_1 - \alpha_1 \|\cdot - \bar{x}\|^2$ and this proves a part of (i).

Take the function $\varphi(x,z) = -k - n_2 \|x\|^2$ where k satisfies (22). Clearly

$$\begin{aligned} \mathbf{B}(\varphi)(x,z) &= \sup_{y \in D(x,z)} W(x,y,z,\mathbf{M}(z,\varphi(y,\cdot))) \\ &\geq W(x,d(x),z,\mathbf{M}(z,\varphi(d(x),\cdot))), \end{aligned}$$

where d is the selection of the multimapping D, whose existence is guaranteed by B.6. Moreover, in view of (20) we have

$$\begin{aligned} \mathbf{B}(\varphi)(x,z) &\geq -m_2 - n_2 \left(\|x\|^2 + \|d(x)\|^2\right) + \overline{\beta} \mathbf{M}(z,\varphi(d(x),\cdot)) \\ &\geq -m_2 - n_2 \|x\|^2 - n_2 N + \overline{\beta} \mathbf{M}(z,\varphi(d(x),\cdot)). \end{aligned}$$

On the other hand, $\varphi(d(x),\cdot) = -k - n_2 \|d(x)\|^2 \geq -k - n_2 N$. Therefore,

$$\begin{aligned} \mathbf{B}(\varphi)(x,z) &\geq -m_2 - n_2 \|x\|^2 - n_2 N - \overline{\beta} k - n_2 N \overline{\beta} \\ &\geq -m_2 - n_2 \|x\|^2 - n_2 N - n_2 N \overline{\beta} - k + m_2 + n_2 N \left(1 + \overline{\beta}\right) \\ &= -k - n_2 \|x\|^2 = \varphi(x,z) \end{aligned}$$

where in the second line we are using the inequality $-\overline{\beta} k \geq -k + m_2 + n_2 N \left(1 + \overline{\beta}\right)$, which is equivalent to (22).

Hence, if $f \geq \varphi$, then $\mathbf{B}(f) \geq \mathbf{B}(\varphi) \geq \varphi$ and this concludes the proof of point (i).

(ii) Let $f \in B_-(X \times Z; w)$. Then $f(x,z) \geq -\gamma - \gamma \|x\|^2$ for $\gamma \geq 0$ large enough. Hence,

$$\begin{aligned} \mathbf{B}(f)(x,z) &\geq W(x,d(x),z,\mathbf{M}(z,f(d(x),\cdot))) \\ &\geq -m_2 - n_2 \|x\|^2 - n_2 N + \overline{\beta} \mathbf{M}(z,f(d(x),\cdot)) \\ &\geq -m_2 - n_2 \|x\|^2 - n_2 N - \overline{\beta} \gamma (1 + N), \end{aligned}$$

which proves (ii)

(iii) is easily checked.

(iv) Let $f \in B_-(X \times Z; w)$ and $\alpha \in [0,1]$. Thanks to B.5 and B.7,

$$\begin{aligned} \mathbf{B}(\alpha f) &= \sup_y W(x,y,z,\mathbf{M}(z,\alpha f(y,\cdot))) \leq \sup_y W(x,y,z,\alpha \mathbf{M}(z,f(y,\cdot))) \\ &\leq \sup_y \left[\alpha W(x,y,z,\mathbf{M}(z,f(y,\cdot))) + \overline{\alpha} W(x,y,z,0)\right] \\ &\leq \alpha \sup_y W(x,y,z,\mathbf{M}(z,f(y,\cdot))) + \overline{\alpha} \sup_y W(x,y,z,0) \\ &= \alpha \mathbf{B}(f) + \overline{\alpha} \mathbf{B}(0), \end{aligned}$$

as was to be shown. ■

In order to apply Theorem 6 in Appendix A.5 to establish the existence of a fixed-point, it is convenient to transform the Bellman operator $\mathbf{B}$ into an equivalent operator $\widehat{\mathbf{B}}$ acting on the positive cone $B_+(X \times Z; w)$. To this purpose, it suffices to define its conjugate operator $\widehat{\mathbf{B}}$ given by

$$\widehat{\mathbf{B}}(f) = -\mathbf{B}(-f).$$

Next theorem uses Thompson's metric (see Appendix A.5).

Theorem 2 *Under B.1–8 the Bellman operator* $\mathbf{B}$ *has one and only one fixed point* v^* *in the negative cone* $CU_-(X \times Z; w)$*. The sequence of iterates* $v_{n+1} = \mathbf{B}v_n$ *converges to* v^* *uniformly over the compact sets of* X *for every initial function* $v_0 \in CU_-(X \times Z; w)$*.*

Proof. In view of point (i) of Proposition 9, operator $\widehat{\mathbf{B}}$ sends the interval $[\![0, k + n_2 \|\cdot\|^2]\!]$ into itself for any $k \geq (1 - \overline{\beta})^{-1}(m_2 + n_2 N(1 + \overline{\beta}))$. Moreover, in view of (23), $\widehat{\mathbf{B}}(0) \geq m_1 + \alpha_1 \|\cdot - \bar{x}\|^2$ which is linked to $k + \alpha_1 \|\cdot\|^2$ by Proposition 8. Clearly, the cone $B_+(X \times Z; w)$ is normal. By (ii) and (iii) of Proposition 9 $\widehat{\mathbf{B}}$ is monotone and concave at 0. The existence of a unique attracting fixed point v^* follows from Theorem 6 in Appendix A.5. Note that by (ii) of Proposition 9 this fixed point v^* is unique in the negative cone $B_-(X \times Z; w)$. The last statement follows from Lemma 3. Notice further that

$$-(1 - \overline{\beta})^{-1}(m_2 + n_2 N(1 + \overline{\beta})) - n_2 \|x\|^2 \leq v^*(x, z) \leq -m_1 - \alpha_1 \|x - \bar{x}\|^2$$

holds. ■

Under slight additional assumptions the fixed-point v^* of Theorem 2 is just the value function of the stochastic recursive optimization problem. This requires some more elaboration regarding the sequential description of such a problem.

Let us first consider the finite horizon problems generated recursively by the relation

$$U_{n+1}\left(x, z, \{\pi_t\}_{t\geq 0}\right) = W\left(x, z, \pi_0, \mathbf{M}\left(z, U_n\left(\pi_0, z', \{\pi_t\}_{t\geq 1}\right)\right)\right) \tag{24}$$

with $U_0 \equiv 0$. Here $\{\pi_t\}_{t\geq 0}$ is a stream of feasible contingent plans.[9] More specifically, $\pi_0 \in D(x, z)$ and $x_{t+1} = \pi_t(z^t) = \pi_t(z_1, z_2, \ldots, z_t)$ for every $t \geq 1$, along with the feasibility condition $\pi_t(z^t) \in D(\pi_{t-1}(z^{t-1}), z_t)$.[10] Like in the deterministic case, the associated value functions are defined as

$$v_n(x, z) = \sup\left\{U_n\left(x, z, \{\pi_t\}_{t\geq 0}\right) : \{\pi_t\}_{t\geq 0} \text{ is feasible from } (x, z)\right\}. \tag{25}$$

Recall that a sequence of contingent plans is called *stationary* or *Markov* if it is generated by a *policy* $h : X \times Z \to X$; that is, $\pi_t(z_1, z_2, \ldots, z_t) = h(\pi_{t-1}(z_1, z_2, \ldots, z_{t-1}), z_t)$.

Observe that the plan $\{\overline{\pi}_t\}_{t\geq 0}$ generated by the map $d : X \to X$ of B.6 gives rise to the constant plan $\overline{\pi}_t = d(x)$ for all t. Therefore,

$$v_n(x, z) \geq U_n\left(x, z, \{\overline{\pi}_t\}_{t\geq 0}\right)$$

holds for every n. Then, through the inequality (20) it is not difficult to prove the following lemma, for which we omit a detailed proof.

[9]We must remark that we do not add any assumption of regularity for the mapping $\pi_t : Z^t \to X$.

[10]Notice that in the right-hand side of (24) one should, more correctly, write $\mathbf{M}\left(z, U_n\left(\pi_0, z', \{\sigma\pi_t\}_{t\geq 1}\right)\right)$, where $\sigma\pi_t = \pi_t(z', \cdot)$ is the continuation of π_t after the current shock z'. We maintain the original notation for sake of simplicity.

Lemma 4 *It holds*

$$v_n(x,z) \geq -A_n - n_2 \|x\|^2, \qquad (26)$$

where $A_{n+1} = \overline{\beta} A_n + K$ *with* $K = m_2 + \left(1+\overline{\beta}\right) n_2 N$ *and* $A_1 = m_2 + n_2 N$. *Moreover,* $A_n \uparrow \left(1-\overline{\beta}\right)^{-1} K$.

Clearly (26) implies that the value functions v_n belong to the negative cone $B_-(X \times Z; w)$.
To prove the next result we add a further assumption on the certain equivalent operator $\mathbf{M}$.

B. 9 $\mathbf{M}(z,\cdot)$ *is constant subadditive, i.e.,* $\mathcal{M}(f-k) \geq \mathcal{M}(f) - k$ *for every constant* $k \geq 0$ *and* $f \in \mathbb{R}^Z_-$.

Proposition 10 *Under B.1–9 the value functions of the finite horizon problems (25) satisfy the functional equations:*

$$v_{n+1}(x,z) = \sup_{y \in D(x,z)} W(x,z,y,\mathbf{M}(z, v_n(y,\cdot))).$$

From the last Proposition and Lemma 4 we deduce that the value functions v_n are the functions generated by the Bellman operator $\mathbf{B}$ of Theorem 2. That is, $v_n = \mathbf{B}^n(0)$.

Proof. Fix $(x,z) \in X \times Z$ and the integer n. By definition, for every $\varepsilon > 0$, there exists a sequence of feasible plans $\{\overline{\pi}_t\}_{t\geq 0}$ such that $v_{n+1}(x,z) \leq U_{n+1}\left(x,z,\{\overline{\pi}_t\}_{t\geq 0}\right) + \varepsilon$. We can thus write

$$\begin{aligned} v_{n+1}(x,z) &\leq U_{n+1}\left(x,z,\{\overline{\pi}_t\}_{t\geq 0}\right) + \varepsilon \\ &= W\left(x,z,\overline{\pi}_0,\mathbf{M}\left(z, U_n\left(\overline{\pi}_0, z', \{\overline{\pi}_t\}_{t\geq 1}\right)\right)\right) + \varepsilon \\ &\leq W\left(x,z,\overline{\pi}_0,\mathbf{M}\left(z, v_n\left(\overline{\pi}_0, z'\right)\right)\right) + \varepsilon \\ &\leq \sup_{y \in D(x,z)} W\left(x,z,y,\mathbf{M}\left(z, v_n\left(y, z'\right)\right)\right) + \varepsilon, \end{aligned}$$

and, by letting $\varepsilon \downarrow 0$, we get

$$v_{n+1}(x,z) \leq \sup_{y \in D(x,z)} W\left(x,z,y,\mathbf{M}\left(z, v_n\left(y, z'\right)\right)\right). \qquad (27)$$

Note that $v_{n+1}(x,z) \geq U_{n+1}\left(x,z,\{\pi_t\}_{t\geq 0}\right)$ holds for every feasible sequence of plans $\{\pi_t\}_{t\geq 0}$. Hence,

$$v_{n+1}(x,z) \geq U_{n+1}\left(x,z,\{\pi_t\}_{t\geq 0}\right) = W\left(x,z,\pi_0,\mathbf{M}\left(z, U_n\left(\pi_0, z', \{\pi_t\}_{t\geq 1}\right)\right)\right).$$

Let us now consider plans $\{\pi_t\}_{t\geq 1}$ such that $U_n\left(\pi_0, z', \{\pi_t\}_{t\geq 1}\right) \geq v_n(\pi_0, z') - \varepsilon$. By B.9,

$$\begin{aligned} v_{n+1}(x,z) &\geq W\left(x,z,\pi_0,\mathbf{M}\left(z, v_n\left(\pi_0, z'\right) - \varepsilon\right)\right) \\ &\geq W\left(x,z,\pi_0,\mathbf{M}\left(z, v_n\left(\pi_0, z'\right)\right) - \varepsilon\right). \end{aligned}$$

The continuity hypothesis B.3 implies that

$$v_{n+1}(x,z) \geq W\left(x,z,\pi_0,\mathbf{M}\left(z, v_n\left(\pi_0, z'\right)\right)\right).$$

But this is true for any $\pi_0 \in D(x,z)$, so that

$$v_{n+1}(x,z) \geq \sup_{y \in D(x,z)} W\left(x,z,y,\mathbf{M}\left(z, v_n\left(y, z'\right)\right)\right),$$

which, together with (27), provides the desired result. ■

Like in the deterministic case (see Proposition 1), the total return function U_∞ is defined as the pointwise limit of the partial returns. Also here it holds

$$U_n\left(x, z, \{\pi_t\}_{t\geq 0}\right) \downarrow U_\infty\left(x, z, \{\pi_t\}_{t\geq 0}\right).$$

Unfortunately, it is not straightforward to isolate conditions for the stochastic aggregator $\mathbf{M}$ under which the stochastic functional U_∞ turns out to be recursive, i.e., satisfying

$$U_\infty\left(x, z, \{\pi_t\}_{t\geq 0}\right) = W\left(x, z, \pi_0, \mathbf{M}\left(z, U_\infty\left(\pi_0, z', \{\pi_t\}_{t\geq 1}\right)\right)\right). \tag{28}$$

However, properties weaker than (28) are still true, as established in the following lemma.

Lemma 5 *It holds*

$$U_\infty\left(x, z, \{\pi_t\}_{t\geq 0}\right) \geq W\left(x, z, \pi_0, \mathbf{M}\left(z, U_\infty\left(\pi_0, z', \{\pi_t\}_{t\geq 1}\right)\right)\right) \tag{29}$$

and

$$U_\infty\left(x, z, \{\pi_t\}_{t\geq 0}\right) \leq W\left(x, z, \pi_0, \mathbf{M}_z\left(U_n\left(\pi_0, z', \{\pi_t\}_{t\geq 1}\right)\right)\right) \tag{30}$$

for all n.

Proof. By (24) and B.3,

$$\begin{aligned} U_{n+1}\left(x, z, \{\pi_t\}_{t\geq 0}\right) &= W\left(x, z, \pi_0, \mathbf{M}\left(z, U_n\left(\pi_0, z', \{\pi_t\}_{t\geq 1}\right)\right)\right) \\ &\geq W\left(x, z, \pi_0, \mathbf{M}\left(z, U_\infty\left(\pi_0, z', \{\pi_t\}_{t\geq 1}\right)\right)\right). \end{aligned}$$

As $n \to \infty$, we obtain inequality (29). On the other hand,

$$\begin{aligned} U_\infty\left(x, z, \{\pi_t\}_{t\geq 0}\right) &\leq U_{n+1}\left(x, z, \{\pi_t\}_{t\geq 0}\right) \\ &= W\left(x, z, \pi_0, \mathbf{M}\left(z, U_n\left(\pi_0, z', \{\pi_t\}_{t\geq 1}\right)\right)\right) \end{aligned}$$

for every n, and so (30) is true. ■

Set

$$v_\infty(x, z) = \sup\left\{U_\infty\left(x, z, \{\pi_t\}_{t\geq 0}\right) : \{\pi_t\}_{t\geq 0} \text{ is feasible from } (x, z)\right\}. \tag{31}$$

Theorem 3 *Under B.1–9, the value function v_∞ satisfies the Bellman equation. I.e.,*

$$v_\infty(x, z) = \sup_{y\in D(x,z)} W\left(x, z, y, \mathbf{M}\left(z, v_\infty(y, \cdot)\right)\right)$$

and it coincides with the fixed point $v^* = \mathbf{B}v^*$ of Theorem 2.

Proof. By (29),

$$v_\infty(x, z) \geq U_\infty\left(x, z, \{\pi_t\}_{t\geq 0}\right) \geq W\left(x, z, \pi_0, \mathbf{M}\left(z, U_\infty\left(\pi_0, z', \{\pi_t\}_{t\geq 1}\right)\right)\right) \tag{32}$$

for every feasible plan $\{\pi_t\}_{t\geq 0}$.

Pick a plan $\{\overline{\pi}_t\}_{t\geq 1}$ such that

$$U_\infty\left(\pi_0, z', \{\overline{\pi}_t\}_{t\geq 1}\right) \geq v_\infty(\pi_0, z') - \varepsilon.$$

In view of (32) we can write

$$v_\infty(x,z) \geq W(x,z,\pi_0,\mathbf{M}(z,v_\infty(\pi_0,z')-\varepsilon))$$

By B.9 we have

$$v_\infty(x,z) \geq W(x,z,\pi_0,\mathbf{M}(z,v_\infty(\pi_0,z'))-\varepsilon)$$

Letting $\varepsilon \downarrow 0$ we get

$$v_\infty(x,z) \geq \sup_{y\in D(x,z)} W(x,z,y,\mathbf{M}(z,v_\infty(y,\cdot))). \tag{33}$$

By (30) we have

$$\begin{aligned} U_\infty\left(x,z,\{\pi_t\}_{t\geq 0}\right) &\leq W\left(x,z,\pi_0,\mathbf{M}\left(z,U_n\left(\pi_0,z',\{\pi_t\}_{t\geq 1}\right)\right)\right) \\ &\leq W(x,z,\pi_0,\mathbf{M}(z,v_n(\pi_0,\cdot))) \end{aligned}$$

for every n. Hence,

$$U_\infty\left(x,z,\{\pi_t\}_{t\geq 0}\right) \leq W\left(x,z,\pi_0,\inf_n \mathbf{M}(z,v_n(\pi_0,\cdot))\right)$$

and thus

$$v_\infty(x,z) \leq \sup_{y\in D(x,z)} W\left(x,z,y,\inf_n \mathbf{M}(z,v_n(y,\cdot))\right). \tag{34}$$

Let us show that

$$\inf_n \mathbf{M}(z,v_n(y,\cdot)) \leq \mathbf{M}\left(z,\inf_n v_n(y,\cdot)\right) = \mathbf{M}(z,v^*(y,\cdot)).$$

In fact, from Theorem 2 it follows that $v_n(y,\cdot) \downarrow v^*(y,\cdot)$ uniformly over the space Z. Therefore, given an $\varepsilon > 0$, we can find a v_n such that $v_n(y,\cdot) \leq v^*(y,\cdot)+\varepsilon$. In view of B.9,

$$\mathbf{M}(z,v_n(y,\cdot))-\varepsilon \leq \mathbf{M}(z,v_n(y,\cdot)-\varepsilon) \leq \mathbf{M}(z,v^*(y,\cdot)),$$

that gives

$$\mathbf{M}(z,v_n(y,\cdot)) \leq \mathbf{M}(z,v^*(y,\cdot)),$$

which, in turn, implies $\inf_n \mathbf{M}(z,v_n(y,\cdot)) \leq \mathbf{M}(z,\inf_n v_n(y,\cdot))$; this, by (34), provides the inequality

$$v_\infty(x,z) \leq \sup_{y\in D(x,z)} W(x,z,y,\mathbf{M}(z,v^*(y,\cdot))). \tag{35}$$

Equations (33) and (35) yield

$$\begin{aligned} \sup_{y\in D(x,z)} W(x,z,y,\mathbf{M}(z,v_\infty(y,\cdot))) &\leq v_\infty(x,z) \\ &\leq \sup_{y\in D(x,z)} W(x,z,y,\mathbf{M}(z,v^*(y,\cdot))), \end{aligned}$$

which, by using the operator $\mathbf{B}$, become

$$\mathbf{B}v_\infty \leq v_\infty \leq \mathbf{B}v^* = v^*.$$

From inequality (26) it is straightforward to infer that

$$v_\infty(x,z) \geq U_\infty\left(x,z,\{\pi_t\}_{t\geq 0}\right) \geq -\frac{K}{1-\overline{\beta}} - n_2\|x\|^2,$$

namely, $v_\infty \in B_-(X\times Z;w)$. Therefore, Theorem 2 establishes that $\mathbf{B}^n v_\infty \downarrow v^*$ and thus $v^* \leq \mathbf{B}v_\infty \leq v_\infty \leq v^*$, yielding the desired result. ■

A Appendix

A.1 Blackwell Theorem and unbounded functions

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ and $\overline{\mathbb{R}}^Y$ be the space of all extended valued functions $g : Y \to [-\infty, \infty)$. Here $\operatorname{dom}(g) = \{y \in Y : g(y) > -\infty\}$ denotes the *effective domain* of a function $g \in \overline{\mathbb{R}}^Y$. The collection of the bounded functions on the set Y is denoted by $\mathcal{B}(Y)$.

Proposition 11 *Let $T : \overline{\mathbb{R}}^Y \to \overline{\mathbb{R}}^Y$ be a monotone operator satisfying the "discounting" property:*

$$T(g + c) \leq Tg + \beta c \tag{36}$$

for all $g \in \overline{\mathbb{R}}^Y$ and $c > 0$ and for some $0 \leq \beta < 1$. If $\overline{g}$ is a fixed point of T, then, $\overline{g}$ is the unique fixed point of T in the affine space $\overline{g} + \mathcal{B}(Y)$ and it is globally attracting there.

Proof. Let $T\overline{g} = \overline{g} \in \overline{\mathbb{R}}^Y$. Consider the affine subspace $\overline{g} + \mathcal{B}(Y) \subset \overline{\mathbb{R}}^Y$. Clearly, $\operatorname{dom}(g) = \operatorname{dom}(\overline{g})$ for every $g \in \overline{g} + \mathcal{B}(Y)$. Let us show that $T : \overline{g} + \mathcal{B}(Y) \to \overline{g} + \mathcal{B}(Y)$. Actually, $g \in \overline{g} + \mathcal{B}(Y)$ means $g = \overline{g} + \varphi$ with $\varphi \in \mathcal{B}(Y)$. Hence,

$$\overline{g} - \|\varphi\|_\infty \leq g = \overline{g} + \varphi \leq \overline{g} + \|\varphi\|_\infty .$$

Thanks to the monotonicity and discounting assumptions made on the operator T, it follows that

$$\overline{g} - \beta \|\varphi\|_\infty \leq Tg \leq \overline{g} + \beta \|\varphi\|_\infty .$$

Therefore, $\operatorname{dom}(Tg) = \operatorname{dom}(\overline{g})$ is true for all $g \in \overline{g} + \mathcal{B}(Y)$. Moreover, $Tg - \overline{g}$ is bounded on the domain $\operatorname{dom}(\overline{g})$, so that $Tg \in \overline{g} + \mathcal{B}(Y)$. Consequently, $T : \overline{g} + \mathcal{B}(Y) \to \overline{g} + \mathcal{B}(Y)$. Consider now the conjugate operator $Q : \mathcal{B}(Y) \to \mathcal{B}(Y)$ given by

$$Q(\varphi) = [T(\overline{g} + \varphi) - \overline{g}]|_{\operatorname{dom}(\overline{g})} .$$

Q is clearly monotone, satisfies the discounting property (36) and has the fixed point $\varphi = 0$. According to Blackwell theorem, Q is a contraction and so $\varphi = 0$ is the unique globally attracting point of Q. Passing to the conjugate operator T, we get the desired result. ■

Next corollary is a straightforward consequence.

Corollary 1 *Under the assumptions of Proposition 11, if g_1 and g_2 are two distinct fixed points of T, then either* $\operatorname{dom}(g_1) \neq \operatorname{dom}(g_2)$ *or, if* $\operatorname{dom}(g_1) = \operatorname{dom}(g_2)$, *then*

$$\sup_{y \in \operatorname{dom}(g_1)} |g_1(y) - g_2(y)| = +\infty.$$

Note that, under condition (5), the operator

$$(\mathbf{T}F)({}_0\mathbf{x}) = W(x_0, x_1, F({}_1\mathbf{x})) \tag{37}$$

mapping the space $\mathbb{R}^{\mathbf{D}}$ into itself, satisfies the Lipschitz condition

$$|(\mathbf{T}F_1)({}_0\mathbf{x}) - (\mathbf{T}F_2)({}_0\mathbf{x})| \leq \beta |F_1({}_1\mathbf{x}) - F_2({}_1\mathbf{x})| ,$$

and thus $\mathbf{T} : \mathbb{R}^{\mathbf{D}} \to \mathbb{R}^{\mathbf{D}}$ satisfies the conditions postulated in Proposition 11.

A.2 A negative weighted contraction theorem

We establish here a slight modification of the classical Boyd's result [2] in order to take into account the fact that the functions are constrained to stay into the negative cone. Let $\phi : X \to \mathbb{R}_{++}$ be a strictly positive function on a set X. Let $\mathcal{B}(X;\phi)$ be the Banach space of all the ϕ-bounded functions. Namely,

$$\mathcal{B}(X;\phi) = \left\{ f \in \mathbb{R}^X : \sup_{x \in X} \frac{|f(x)|}{\phi(x)} < \infty \right\}$$

The ϕ-norm is $\|f\|_\phi = \sup_{x\in X} \phi^{-1}(x)\,|f(x)|$. Denote by $\mathcal{B}_-(X;\phi)$ the negative cone of $\mathcal{B}(X;\phi)$.

Proposition 12 *Let $\boldsymbol{T} : \mathcal{B}_-(X;\phi) \to \mathbb{R}^X$ satisfy the conditions:*

i) *$\boldsymbol{T}$ is monotone,*

ii) *there is a scalar $\beta \in [0,1)$ such that*

$$\boldsymbol{T}(f - k\phi) \geq \boldsymbol{T}f - \beta k\phi, \quad \forall f \in \mathcal{B}_-(X;\phi) \text{ and } k \geq 0,$$

iii) *$\boldsymbol{T}0 \in \mathcal{B}_-(X;\phi)$.*

Then

$$\boldsymbol{T} : \mathcal{B}_-(X;\phi) \to \mathcal{B}_-(X;\phi),$$

and

$$\|\boldsymbol{T}f - \boldsymbol{T}g\|_\phi \leq \beta \|f - g\|_\phi, \quad \forall f, g \in \mathcal{B}_-(X;\phi). \tag{38}$$

Proof. By definition,

$$|f(x) - g(x)| \leq \phi(x)\,\|f - g\|_\phi, \quad \forall x \in X.$$

It follows that

$$f(x) - g(x) \geq -|f(x) - g(x)| \geq -\|f - g\|_\phi\,\phi(x).$$

Hence, if $f, g \in \mathcal{B}_-(X;\phi)$,

$$f \geq g - \|f - g\|_\phi\,\phi,$$

so that (i) and (ii) imply

$$\boldsymbol{T}f \geq \boldsymbol{T}g - \beta\|f - g\|_\phi\,\phi,$$

which easily leads to (38).

If $f \in \mathcal{B}_-(X;\phi)$ then $\boldsymbol{T}f \leq \boldsymbol{T}0 \in \mathcal{B}_-(X;\phi)$. Note that $0 \in \mathcal{B}_-(X;\phi)$ and $\boldsymbol{T}0 \in \mathcal{B}_-(X;\phi)$ by assumption (iii). Hence,

$$\|\boldsymbol{T}f\|_\phi \leq \|\boldsymbol{T}f - \boldsymbol{T}0\|_\phi + \|\boldsymbol{T}0\|_\phi \leq \beta\|f - 0\|_\phi + \|\boldsymbol{T}0\|_\phi,$$

and thus $\boldsymbol{T}(\mathcal{B}_-(X;\phi)) \subseteq \mathcal{B}_-(X;\phi)$. ■

A.3 Strong concavity

Hilbert spaces are the natural framework for the concept of strong concavity. Let H be a pre-Hilbert space. A function $f : H \to [-\infty, \infty)$ is called *strongly concave* (or α-concave) if some $\alpha > 0$ exists such that

$$f\left(\lambda x + \bar{\lambda} y\right) \geq \lambda f(x) + \bar{\lambda} f(y) + \frac{1}{2}\alpha\lambda\bar{\lambda}\left\|x - y\right\|^2$$

for all $x, y \in \operatorname{dom} f$ and all $\lambda \in [0, 1]$ (here $\bar{\lambda} = 1 - \lambda$).

Clearly f is α-concave if and only if the function $f + (1/2)\,\alpha\left\|\cdot\right\|^2$ is concave.

When dealing with functions $f(x, y)$ depending on two groups of variables, next definition extends the previous one.

Definition 1 *A function $f : H_1 \times H_2 \to [-\infty, \infty)$, where H_1 and H_2 are two pre-Hilbert spaces, is said to be (α_1, α_2)-concave, with $\alpha_1, \alpha_2 \geq 0$, if*

$$f(x, y) + \frac{1}{2}\alpha_1\left\|x\right\|_1^2 + \frac{1}{2}\alpha_2\left\|y\right\|_2^2$$

is concave over $H_1 \times H_2$.

Of course the condition $\alpha_1 \cdot \alpha_2 > 0$ is equivalent to the property that f is strongly concave on the pre-Hilbert space $H_1 \times H_2$. The weaker assumption $\alpha_1 + \alpha_2 > 0$ is often an acceptable condition for certain purposes.

The following property is also well-known.

Proposition 13 *Let $f : H \to [-\infty, \infty)$ be α-concave. If $x^* \in \arg\max_H f$, then*

$$f(x^*) \geq f(x) + \frac{1}{2}\alpha\left\|x - x^*\right\|^2 \quad \forall x \in H$$

Its extension to (α_1, α_2)-concave functions of the above property is straightforward.

The next remarkable property of existence of optimal solutions for strongly concave functionals requires the completeness of the space. We omit proofs.

Theorem 4 *Let $f : H \to [-\infty, \infty)$ be a function having non-empty effective domain in an Hilbert space H. If f is upper semicontinuous and strongly concave, then:*

i) *there exists a unique point $x^* \in H$ such that $f(x^*) \geq f(x)$ for all $x \in H$;*

ii) *every maximizing sequence[11] $\{x_n\}$ converges to x^*.*

A.4 The max-function

Let $D : X \times Z \rightrightarrows X$ be a correspondence, where $X \subseteq \mathbb{R}^n$ and Z is a topological space. Assume that its graph,

$$\operatorname{Gr} D = \left\{(x, z, y) \in X \times Z \times X : y \in D(x, z)\right\},$$

is closed in $\mathbb{R}^n \times Z \times \mathbb{R}^n$.

[11] A maximizing sequence is any sequence $\{x_n\}$ enjoying the property that $\lim_{n\to\infty} f(x_n) = \sup_{x\in H} f(x)$.

Proposition 14 *Let* $f : \operatorname{Gr} D \to (-\infty, \infty)$ *be upper semicontinuous and coercive with respect to the variable* y.[12] *The max-function*

$$m(x,z) = \sup_{y \in D(x,z)} f(x,z,y)$$

is upper semicontinuous and the sup *is attained for every* (x,z)*. Moreover, the correspondence*

$$H(x,z) = \arg \max_{y \in D(x,z)} f(x,z,y)$$

is compact-valued and closed.

Proof. Fix the point $(\bar{x}, \overline{z}) \in X \times Z$, and consider any sequence $(x_n, z_n) \to (\bar{x}, \overline{z})$ such that $m(x_n, z_n)$ is convergent. Say $m(x_n, z_n) \to \lambda$. As $D(x_n, z_n)$ is closed and $f(x_n, z_n, \cdot)$ is upper semicontinuous and coercive, the sup $m(x_n, z_n)$ is attained. Hence, for every n a point $y_n \in D(x_n, z_n)$ exists so that $f(x_n, z_n, y_n) = m(x_n, z_n)$. Therefore, $\lim_{n\to\infty} f(x_n, z_n, y_n) = \lambda$ and, for every ε, eventually $f(x_n, z_n, y_n) \geq \lambda - \varepsilon$. By assumption, the sequence $\{y_n\}$ is bounded. This implies the existence of a convergent subsequence $(x_{n_k}, z_{n_k}, y_{n_k}) \to (\bar{x}, \overline{z}, \overline{y})$. As $\operatorname{Gr} D$ is closed, $\overline{y} \in D(\bar{x}, \overline{z})$. Hence,

$$m(\bar{x}, \overline{z}) \geq f(\bar{x}, \overline{z}, \overline{y}) \geq \limsup_{k\to\infty} f(x_{n_k}, z_{n_k}, y_{n_k}) = \lim_{k\to\infty} m(x_{n_k}, z_{n_k}) = \lambda.$$

It follows that $\limsup_{n\to\infty} m(x_n, z_n) \leq m(\bar{x}, \overline{z})$ for all the sequences $(x_n, z_n) \to (\bar{x}, \overline{z})$ By definition m is upper semicontinuous at $(\bar{x}, \overline{z})$.

The last claim is a trivial consequence of the latter property. ■

A.5 Thompson metric and contractions

We recall a few results based on the metric introduced by Thompson [15] as a variant of Hilbert's projective metric. More details can be found in [10] and [11].

Consider a normed space V, equipped with a closed pointed[13] and convex cone K. It induces in V the continuous partial order $v \leq w$ if and only if $w - v \in K$.

The cone K is called *normal* if there is some scalar $\gamma > 0$ such that $\|x\| \leq \gamma \|y\|$ for all $0 \leq x \leq y$.

Definition 2 *Two elements* $x, y \in K$ *are* linked *(or comparable) if there are strictly positive scalars* $\alpha, \beta > 0$ *such that* $\alpha x \leq y \leq \beta x$.

Being linked is an equivalence relation that splits K into disjoint components Q.

Definition 3 *The* Thompson metric d_τ *for two linked elements* $0 \neq x, y \in K$ *is*

$$d_\tau(x,y) = \log \max \{M(x \mid y), M(y \mid x)\}$$

where $M(x \mid y) = \inf \{\alpha > 0 : x \leq \alpha y\}$.

In fact, it is not difficult to check that d_τ is a metric on each component Q of the cone K. Most importantly, the following result holds.

[12]This means that, for any c such that the set $f(x,z,y) \geq c$ is nonempty, there is some N such $\|y\| \leq N$ holds for every element of that set.

[13]A pointed cone K is a cone that satisfies the property $x, -x \in K$ implies $x = 0$.

Theorem 5 (Thompson) *If K is a normal and closed cone in a Banach space V, then Thompson's metric d_τ is complete on every component Q of K. Moreover, if in the set Q a sequence d_τ-converges to an element v, then it also norm-converges to v.*

Thanks to this result we can study the existence of fixed point of self-operators $T : V \to V$ by means of such a metric. Next result is a particularly useful isolation of the theorem (see [10] for a proof). Recall that a pointed cone K in a vector space V induces the order $x \leq y$ if $y - x \in K$. Therefore the cone K is the set of positive elements of V and the notation V_+ is also used. The interval $[\![a, b]\!]$ is the collection $a \leq x \leq b$.

Theorem 6 *Let V be a Banach space and $K \equiv V_+$ be closed and normal. Suppose that the operator $T : [\![0, b]\!] \to [\![0, b]\!]$ satisfies the three following conditions:*

i) *T is monotone;*

ii) *T is concave at 0, namely,*

$$T(\alpha x) \geq \alpha T(x) + (1 - \alpha) T(0)$$

for all $x \in [\![0, b]\!]$ and all $\alpha \in [0, 1]$;

iii) *$T(0) = a$ is linked to b.*

Then,

$$d_\tau(T(x), T(y)) \leq \zeta d_\tau(x, y) \quad \forall x, y \in [a, b]$$

where $\zeta = 1 - \mu^{-1} < 1$ and $\mu = M(b \mid a)$. The contraction T has a unique and globally attracting fixed point $\overline{x}$ in the interval $[\![0, b]\!]$, i.e.,

$$\lim_{n\to\infty} \|T^n(x) - \overline{x}\| = 0 \quad \forall x \in [\![0, b]\!].$$

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