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ON THE DISTRIBUTION OF THE ROOTS OF POLYNOMIALS

by F. AMOROSO and M. MIGNOTTE

1. Introduction.

In this paper we are interested in the angular distribution of the roots of univariate polynomials. To explain our results we need to recall some definitions. If $x_1, \ldots, x_N$ is a finite sequence of points in $[0, 2\pi)$, we define the *absolute discrepancy* of this sequence by

$$D(x_1, \ldots, x_N) = \sup_{0 \le \alpha < \beta < 2\pi} \left| \frac{\#\{j; x_j \in [\alpha, \beta)\}}{N} - \frac{\beta - \alpha}{2\pi} \right|,$$

where $\#$ denotes the cardinality of a set. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = a_n \prod_{j=1}^{n} (z - \rho_j e^{i\varphi_j}),$$
$$a_0 a_n \neq 0, \quad \rho_1, \ldots, \rho_n > 0,$$

be a polynomial of degree n with complex coefficients, where $\varphi_j \in [0, 2\pi)$ for $j = 1, \ldots, n$. For $0 \le \alpha < \beta < 2\pi$, put $N(\alpha, \beta) = \#\{j; \varphi_j \in [\alpha, \beta)\}$. We are interested in the distribution of the points $\varphi_1, \ldots, \varphi_n$. With the previous notations,

$$D(\varphi_1, \ldots, \varphi_n) = \sup_{0 \le \alpha < \beta < 2\pi} \left| \frac{N(\alpha, \beta)}{n} - \frac{\beta - \alpha}{2\pi} \right|;$$

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to simplify the notation we put

$$D_P = D(\varphi_1, \ldots, \varphi_n),$$

which we call absolute discrepancy of the roots of P.

The first result on D_P was obtained by Erdös and Turán:

THEOREM A. — *With the above notations, for* $0 \leq \alpha < \beta < 2\pi$, *we have*

$$\left| N(\alpha, \beta) - \frac{\beta - \alpha}{2\pi} n \right| \leq 16 \sqrt{n \log \frac{|P|}{\sqrt{|a_0 a_n|}}},$$

where $|P| = \max_{|z|=1} |P(z)|$.

In other words,

$$D_P \leq 16 \sqrt{\frac{1}{n} \log \frac{|P|}{\sqrt{|a_0 a_n|}}}.$$

The proof of [ET] consists in solving several extremal problems on polynomials, using orthogonal polynomials. A few years later, Ganelius [G] proved a general theorem on conjugate functions and showed that his theorem implies a sharpening of the Erdös-Turán, namely he could replace the constant 16 by $\sqrt{2\pi/k} = 2.5619\ldots$, where $k = \sum_{0}^{\infty} (-1)^{m-1} (2m + 1)^{-2} = 0.915965594\ldots$ is Catalan's constant.

The result of Ganelius is the following:

THEOREM B. — *Let* $F = f + i\tilde{f}$ *be an analytic function on* $\mathbf{D} = \{|z| < 1\}$ *satisfying* $F(0) = 0$. *Suppose that* $f, \tilde{f}$ *are real and* $f < H$, $\partial \tilde{f} / \partial \theta < K$ *on* $\mathbf{D}$ (*). *Then for* $\beta > \alpha$ *and* $\rho < 1$,

$$\left| \tilde{f}(\rho e^{i\beta}) - \tilde{f}(\rho e^{i\alpha}) \right| < 2\pi \sqrt{\frac{\pi}{k}} \cdot \sqrt{HK}.$$

For the convenience of the reader, we briefly explain how Theorem B implies Theorem A. Let us consider the polynomial

$$Q(z) = \prod_{j=1}^{n} (1 - z \cdot e^{-i\varphi_j}).$$

(*) Here and in the sequel we often identify the complex variable z with $\rho e^{i\theta}$.

As remarked by Schur,

$$\rho_j \left|1 - \frac{e^{it}}{\rho_j e^{i\varphi_j}}\right|^2 \geq \left|1 - e^{i(t-\varphi_j)}\right|^2.$$

Hence

$$|Q| \leq \frac{|P|}{\sqrt{|a_0 a_n|}}.$$

Now let

$$f(z) = \frac{1}{\pi}\log|Q(z)|, \qquad \tilde{f}(z) = \frac{1}{\pi}\sum_{j=1}^{n} \operatorname{Arg}\left(1 - ze^{-i\varphi_j}\right)$$

and observe that the function $F(z) = f + i\tilde{f}$ is analytic on $\mathbf{D}$ and satisfies $F(0) = 0$. We have $f \leq \frac{1}{\pi}\log|Q|$ and $\partial\tilde{f}/\partial\theta < n/(2\pi)$. Moreover, it is easily seen that $\tilde{f}$ takes the boundary value

$$\frac{n\theta}{2\pi} - N(0,\theta) + C(Q),$$

where

$$C(Q) = \frac{n}{2} - \sum_{j=1}^{n}\left\{\frac{\varphi_j}{2\pi}\right\}.$$

Henceforth Theorem B gives

$$D_P < \sqrt{\frac{2\pi}{k}} \cdot \sqrt{\frac{1}{n}\log|Q|} \leq \sqrt{\frac{2\pi}{k}} \cdot \sqrt{\frac{1}{n}\log\frac{|P|}{\sqrt{|a_0 a_n|}}}.$$

Theorem B was sharpened much later in [M], where, under the same hypotheses, it is proved that

$$\left|\tilde{f}(\rho e^{i\beta}) - \tilde{f}(\rho e^{i\alpha})\right| < 2\pi\sqrt{\frac{\pi}{k}\tilde{H}K},$$

where

$$\tilde{H} = \frac{1}{2\pi}\int_{-\pi}^{\pi} u^+(e^{i\theta})d\theta \leq \max u^+$$

and $u^+ = \max\{u, 0\}$.

Let us define $\tilde{h}(P) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+|P(e^{i\theta})|d\theta$. Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+|Q(e^{i\theta})|d\theta \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+\frac{|P(e^{i\theta})|}{\sqrt{|a_0 a_n|}}d\theta,$$

Mignotte's result leads to a version of Erdös-Turán's theorem where $\log\frac{|P|}{\sqrt{|a_0 a_n|}}$ is replaced by $\tilde{h}(\frac{P}{\sqrt{|a_0 a_n|}})$. It is worth remarking that $\tilde{h}(P)$

can be much smaller than $\log|P|$: for a discussion on the relations between these two measures, see [A].

In Section 3 we give a new (very short) proof of the result of [M], using a theorem of Kolmogorov on conjugate functions (see Section 2 for definitions and properties of conjugate and harmonic functions).

Recently, Blatt obtained a sharpening of Theorem A for square-free polynomials. He proved the following result:

THEOREM C. — *Let $P(z)$ be a monic polynomial of degree n with all its roots z_j on the unit circle. Assume that*

$$|P| \leq A \quad \text{and} \quad |P'(z_j)| \geq \frac{1}{B}, \quad j = 1, \ldots, n, \tag{1.1}$$

for some constants A, $B > 1$. Then,

$$D_P \leq c\frac{(\log n)\log C_n}{n},$$

where c is some (non computed) absolute constant and $C_n = \max\{A, B, n\}$.

A similar statement holds for polynomials vanishing only on $[-1, 1]$. Totik improved this last result on $[-1, 1]$ by replacing $\log n$ with $\log(n/\log C_n)$, provided that $\log C_n \leq n/2$.

As noticed in Blatt's paper, Theorem C is a direct consequence of the following:

THEOREM D. — *Let $P(z)$ be a monic polynomial of degree n with all its roots on the unit circle. Then*

$$D_P \leq C\frac{\log n}{n} \max_{|z|\geq 1+n^{-8}} \big|\log|P(z)| - n\log|z|\big|. \tag{1.2}$$

Since $|z^{-n}P(z)| = |P(1/\overline{z})|$, inequality (1.2) is equivalent to

$$D_P \leq C\frac{\log n}{n}\big|\log|P(z)|\big|_{1/(1+n^{-8})}.$$

In Section 4 we give a short and simple proof of the following theorem on conjugate functions:

THEOREM E. — *Let f be a real harmonic function on $\mathbf{D} = \{|z| < 1\}$ and let assume that its conjugate function $\tilde{f}$ satisfies $\partial\tilde{f}/\partial\theta \leq K$ on $\mathbf{D}$. Then, for any $r \in [1/2, 1)$*

$$|\tilde{f}| \leq \frac{6}{\pi}\left(\log\frac{2}{1-r}\right)|f|_r + 4\sqrt{3}K\frac{1-r}{r}.$$

If we choose as before $f = \frac{1}{\pi}\log|P|$, we obtain the following improvement of Theorem D:

THEOREM D′. — *Let $P(z)$ be a polynomial of degree n with all its roots on the unit circle. Then*

$$D_P \leq \frac{12}{\pi^2}\left(\log\frac{2}{1-r}\right)\big|\log|P|\big|_r + \frac{8\sqrt{3}}{\pi}\cdot\frac{1-r}{r}, \quad r \in [1/2, 1].$$

This result implies the following improved version of Blatt's theorem:

THEOREM C′. — *Let $P(z)$ be a polynomial satisfying the assumptions of Theorem C. Then*

$$D_P \leq 13 \max\left\{1, \log\frac{2n}{\log C_n}\right\}\frac{\log C_n}{n}.$$

The previous assertion is trivial if $\log C_n > \frac{n}{2}$. Assume that (1.1) holds and suppose $\log C_n \leq \frac{n}{2}$. We apply theorem D′ choosing $r = 1 - \frac{\log C_n}{n}$. By the maximum principle $\log^+|P|_r \leq \log^+ A$, while by the Lagrange interpolation formula

$$1 = \sum_{z_j,\, P(z_j)=0} \frac{P(z)}{P'(z_j)(z-z_j)}$$

we have $\log^-|P|_r \leq \log^+\frac{nB}{1-r}$. Hence

$$\big|\log|P|\big|_r \leq \max\left\{\log^+ A, \log^+\frac{nB}{1-r}\right\} \leq \log C_n + \log\frac{n}{1-r} = \log\frac{n^2 C_n}{\log C_n}.$$

Theorem D′ gives

$$D_P \leq \frac{12}{n\pi^2}\left(\log\frac{2n}{\log C_n}\right)\log\frac{n^2C_n}{\log C_n} + \frac{16\sqrt{3}}{\pi}\frac{\log C_n}{n}$$
$$\leq 13\max\left\{1, \log\frac{2n}{\log C_n}\right\}\frac{\log C_n}{n}.$$

We notice that the conformal mapping $z \mapsto \frac{1}{2}\left(z + \frac{1}{z}\right)$ which sends the unit circle onto $[-1, 1]$ can be used to get similar results on the distribution of the roots of a polynomial vanishing only on $[-1, 1]$.

In Section 5 we consider the problem of finding an upper bound for the maximum modulus of a polynomial depending on its degree and on the discrepancy. We prove the following theorem:

THEOREM F. — *Let f be a real harmonic function on $\mathbf{D} = \{|z| < 1\}$ such that $f(0) = 0$ and $\partial\tilde{f}/\partial\theta \leq K$ on $\mathbf{D}$. Let also*

$$\Delta = \max_{z,w\in\mathbf{D}} |\tilde{f}(z) - \tilde{f}(w)|.$$

Then

$$\sup_{\mathbf{D}} f \leq \frac{\Delta}{\pi}\left(3 + \log\frac{2\pi K}{\Delta}\right).$$

By applying the previous result to the function $f = \dfrac{1}{\pi}\log|P|$ we obtain:

COROLLARY. — *Let $P(z)$ be a polynomial of degree n with all its roots on the unit circle and such that $P(0) = 1$. Then*

$$\log|P| \leq nD_P(3 + \log 1/D_P).$$

Finally, in Section 6 we discuss an extremal example.

2. Some results from harmonic analysis.

In this section we recall some basic facts on harmonic analysis. The standard reference of all definitions and results is the book of P. Koosis ([K]).

Let f be a 2π-periodic real function on in $L_1(-\pi,\pi)$. Then its Hilbert transform

$$\tilde{f}(\theta) = \int_{-\pi}^{\pi} \frac{f(\theta - t)}{2\tan t/2}\,dt$$

exists and is finite almost a.e. (= almost everywhere). We call $\tilde{f}$ the conjugate function of f. Although $\tilde{f}$ does not belongs to $L_1(-\pi,\pi)$ in general, we have the following theorem of Kolmogorov (as improved by Davis) which is very important for our purposes.

THEOREM 2.1. — *Let $f \in L_1(-\pi,\pi)$ be a 2π-periodic real function and let $\tilde{f}$ be its conjugate. Then, for any positive λ,*

$$\mu\{\theta \in [0,2\pi); |\tilde{f}| > \lambda\} < \frac{\pi^2}{8k\lambda}\int_{-\pi}^{\pi} |f(\theta)|\,d\theta$$

where k is Catalan's constant. In this inequality, the constant $\pi^2/8$ is the best possible.

Kolmogorov's proof gave no information about the best constant, which was obtained much later by Davis [D]. Also, see Baernstein [Ba] for another proof.

A real function $f(z)$ on the open disk $\mathbf{D} = \{|z| < 1\}$ is harmonic if it is the real part of a function $F(z)$ analytic on $\mathbf{D}$. We notice that F is unique to within an additive constant. The harmonic conjugate of f is the real function $\tilde{f}$ such that $f + i\tilde{f}$ is analytic and $\tilde{f}(0) = 0$. Given a function on $\mathbf{D}$, we often use the notation $f(r,\theta) = f\left(re^{i\theta}\right)$. Let $f = \Re F$ with F analytic on $\mathbf{D}$. Then the non-tangential limits

$$f(\theta) := \lim_{\substack{\varphi \to \theta \\ r \to 1^-}} f(r,\varphi), \qquad \tilde{f}(\theta) := \lim_{\substack{\varphi \to \theta \\ r \to 1^-}} \tilde{f}(r,\varphi)$$

exist a.e. if F belongs to the Hardy space H_1, i.e. if

$$\sup_{0 \le r < 1} \int_{-\pi}^{\pi} \left|F\left(re^{i\theta}\right)\right| d\theta < +\infty.$$

Let $p \in (1,\infty)$. By a theorem of Riesz,

$$\sup_{0 \le r < 1} \int_{-\pi}^{\pi} |f(r,\theta)|^p \, d\theta < +\infty \tag{2.1}$$

if and only if

$$\sup_{0 \le r < 1} \int_{-\pi}^{\pi} |\tilde{f}(r,\theta)|^p \, d\theta < +\infty. \tag{2.2}$$

Therefore, if (2.1) or (2.2) holds for some $p > 1$, then $f + i\tilde{f} \in H_1$. In particular, if f or $\tilde{f}$ are bounded, then $f + i\tilde{f} \in H_1$. If the non-tangential limit $f(\theta)$ exists, it is called the boundary value of f and similarly for $\tilde{f}$.

Let, for $r \in [0,1)$ and $\theta \in R$,

$$\mathbf{K}(\rho,\theta) = \frac{1-\rho^2}{1 - 2\rho\cos\theta + \rho^2}, \qquad \tilde{\mathbf{K}}(\rho,\theta) = \frac{2\rho\sin\theta}{1 - 2\rho\cos\theta + \rho^2}$$

be the Poisson kernel and the conjugate Poisson kernel. Then for any real harmonic function f we have the Poisson representations

$$f(\rho,\varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{K}(\rho/r,\theta) f(r,\varphi-\theta) \, d\theta \tag{2.3}$$

and

$$\tilde{f}(\rho,\varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\mathbf{K}}(\rho/r,\theta) f(r,\varphi-\theta) \, d\theta \tag{2.4}$$

which hold for $0 \le \rho < r < 1$ and $\varphi \in R$. If $f + i\tilde{f} \in H_1$, then (2.3) and (2.4) still hold for $r = 1$.

Let $g \in L_1(-\pi, \pi)$ be a 2π-periodic real function and let $\tilde{g}$ its conjugate function. Then

$$f(\rho, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{K}(\rho, \theta) g(\varphi - \theta) \, d\theta$$

is harmonic and its harmonic conjugate is

$$\tilde{f}(\rho, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\mathbf{K}}(\rho/r, \theta) g(\varphi - \theta) \, d\theta.$$

Assume further $\tilde{g} \in L_1(-\pi, \pi)$. Then the boundary values $f(\theta)$ and $\tilde{f}(\theta)$ both exist and

$$f(\theta) = g(\theta), \qquad \tilde{f}(\theta) = \tilde{g}(\theta) \quad \text{a.e.}$$

We also recall the following elementary inequalities which hold for all $\rho \in [0, 1)$ and all θ:

$$0 < \frac{1 - \rho}{1 + \rho} \leq \mathbf{K}(\rho, \theta) \leq \frac{1 + \rho}{1 - \rho} \tag{2.6}$$

and

$$|\tilde{\mathbf{K}}(\rho, \theta)| \leq \frac{2\rho}{1 - \rho^2}. \tag{2.7}$$

Moreover, we notice that

$$\int \mathbf{K}(\rho, \theta) \, d\theta = 2 \operatorname{arctg} \left(\frac{1 + \rho}{1 - \rho} \cdot \operatorname{tg} \frac{\theta}{2} \right) + \text{constant}. \tag{2.8}$$

In particular, this implies

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{K}(\rho, \theta) \, dt = 1. \tag{2.9}$$

For the conjugate kernel, we have

$$\int \tilde{\mathbf{K}}(\rho, \theta) \, d\theta = \log(1 - 2\rho \cos \theta + \rho^2) + \text{constant}, \tag{2.10}$$

so that

$$\int_{-\pi}^{\pi} |\tilde{\mathbf{K}}(\rho, \theta)| \, d\theta = 4 \log \frac{1 + \rho}{1 - \rho}. \tag{2.11}$$

3. Ganelius' theorem *via* Kolmogorov.

We begin this section by a very elementary lemma.

LEMMA 3.1. — *Let $g\colon \mathbf{R} \to \mathbf{R}$ be a 2π-periodic function and suppose that there exists a constant K such that*

$$g(\varphi+\varepsilon) \le g(\varphi) + \varepsilon K,$$

for any $\varphi \in R$ and any $\varepsilon > 0$. Assume further that for any positive number λ the set

$$E_\lambda = \{\theta \in [0, 2\pi); |g(\theta)| > \lambda\}$$

satisfies

$$\mu(E_\lambda) < \frac{C}{\lambda},$$

where μ is Lebesgue measure and C is some positive constant. Then

$$\max |g| \le 2\sqrt{CK}.$$

Moreover,

$$|g^+| + |g^-| \le 2\sqrt{2CK}.$$

Proof. — Put $\lambda = \sqrt{CK}$ and $A = 2\sqrt{CK}$. We first want to prove that $|g(\varphi)| \le A$ for any $\varphi \in \mathbf{R}$.

If $\varphi \notin E_\lambda$ then $|g(\varphi)| \le \lambda$ and we have nothing to prove. If $\varphi \in E_\lambda$, since $\mu(E_\lambda) < C\lambda^{-1}$, there exists $\varepsilon_1 > 0$ such that $\varepsilon_1 \le C\lambda^{-1}$ and $\varphi - \varepsilon_1 \notin E_\lambda$, hence

$$g(\varphi) \le g(\varphi - \varepsilon_1) + \varepsilon_1 K \le \lambda + \frac{CK}{\lambda} = A.$$

In the same way, there exists $\varepsilon_2 > 0$ such that $\varepsilon_2 \le C\lambda^{-1}$ and $\varphi + \varepsilon_2 \notin E_\lambda$, which implies

$$g(\varphi) \ge g(\varphi+\varepsilon_2) - \varepsilon_2 K \ge -\lambda - \frac{CK}{\lambda} = -A.$$

This proves the first assertion. To prove the second one consider the sets

$$E_\lambda^+ = \{\theta \in [0, 2\pi); g^+(\theta) > \lambda\} \qquad \text{and} \qquad E_\lambda^- = \{\theta \in [0, 2\pi); g^-(\theta) > \lambda\}.$$

For any $\lambda > 0$ and any φ, $\psi \in \mathbf{R}$, the preceding argument leads to

$$g^+(\varphi) + g^-(\psi) \le \lambda + K\mu(E_\lambda^+) + \lambda + K\mu(E_\lambda^-) \le 2\lambda + K\mu(E_\lambda) \le 2\lambda + \frac{KC}{\lambda},$$

and the choice $\lambda = \sqrt{CK/2}$ gives the second assertion. This concludes the proof. □

We denote by $|f|_r$ the sup of $|f(z)|$ on $|z| = r$ and by $|f|$ the sup of $|f(z)|$ on $|z| = 1$. When f is real-valued we define the *span* of f by the formula

$$\Delta(f) = |f^+| + |f^-|,$$

where $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$. Trivially, $\Delta(f) \leq 2|f|$.

If we apply Kolmogorov's Theorem 2.1 and Lemma 3.1 (with $g = \tilde{f}$), we get:

THEOREM 3.1. — *Let $f \in L_1(-\pi, \pi)$ be a 2π-periodic real function and let $\tilde{f}$ be its conjugate. Suppose that there exists a positive constant K such that*

$$\tilde{f}(\varphi + \varepsilon) \leq \tilde{f}(\varphi) + \varepsilon K,$$

for any $\varphi \in T$ and any $\varepsilon > 0$. Let also

$$\tilde{H} = \frac{1}{4\pi} \int_{-\pi}^{\pi} |f(\theta)| \, d\theta.$$

Then,

$$|\tilde{f}| \leq \pi\sqrt{\frac{2\pi}{k}} \cdot \sqrt{\tilde{H}K} \quad \text{and} \quad \Delta(\tilde{f}) \leq 2\pi\sqrt{\frac{\pi}{k}} \cdot \sqrt{\tilde{H}K},$$

where k is Catalan's constant.

One may notice that this result is essentially the same as the refinement of Ganelius theorem published in [M]. In fact, denote by the same letter f the real harmonic function on D whose boundary value coincide with f almost everywhere. Then, $\int_{-\pi}^{\pi} f(\theta)\, d\theta = f(0)$. Hence, if we further assume $f(0) = 0$, we have

$$\tilde{H} = \frac{1}{4\pi} \int_{-\pi}^{\pi} |f(\theta)| \, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^+(\theta) \, d\theta \leq \max f^+.$$

4. On Blatt's theorem.

Let g be a real harmonic function on $\mathbf{D}$ and assume that there exists a constant K such that $\partial g/\partial\theta < K$ on $\mathbf{D}$. The function $\rho \to |g|_\rho$ in general does not satisfy Lipschitz's condition. As an example, consider $g = \operatorname{Arg}(1 - z)$. However, we have:

LEMMA 4.1 ("Turn-growth lemma"). — *Let g be a real harmonic function on $\mathbf{D}$ and assume that there exists a constant K such that $\partial g/\partial\theta < K$ on $\mathbf{D}$. Then, for any $\rho \in [0,1)$,*

$$|g| \leq 3|g|_\rho + 4\sqrt{3}K\frac{1-\rho}{1+\rho}.$$

Proof. — Let $\varepsilon = 2\,\mathrm{arctg}\left(\sqrt{3}\frac{1-\rho}{1+\rho}\right) \in (0,\pi)$. Then, by (2.8)

$$\frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} \mathbf{K}(\rho,\theta)\,d\theta = \frac{2}{\pi}\,\mathrm{arctg}\left(\frac{1+\rho}{1-\rho}\cdot \mathrm{tg}\,\frac{\varepsilon}{2}\right) = \frac{2}{3} \tag{4.1}$$

and, by (2.9)

$$\frac{1}{2\pi}\int_{\varepsilon<|\theta|\leq\pi} \mathbf{K}(\rho,\theta)\,d\theta = 1-\frac{2}{3} = \frac{1}{3}. \tag{4.2}$$

Now assume $|g| = -g(\varphi)$ for some $\varphi \in \mathbf{R}$ (otherwise $|g| = |g^+|$ and a similar argument applies). Since g is bounded on $|z| \leq 1$, Poisson's Formula (2.3) applies and we have, by (4.2),

$$\begin{aligned} -|g|_\rho \leq g(\rho,\varphi+\varepsilon) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{K}(\rho,\theta)g(\varphi+\varepsilon-\theta)\,d\theta \\ &\leq \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} \mathbf{K}(\rho,\theta)g(\varphi+\varepsilon-\theta)\,d\theta + \frac{1}{3}|g|. \end{aligned} \tag{4.3}$$

By our assumption we have $g(\varphi+\varepsilon-\theta) \leq g(\varphi)+K(\varepsilon-\theta)$ for $\theta \leq \varepsilon$. Moreover $\mathbf{K}(\rho,\theta) > 0$, whence, by (4.1),

$$\frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} \mathbf{K}(\rho,\theta)g(\varphi+\varepsilon-\theta)\,d\theta \leq \frac{2}{3}\left(g(\varphi)+K\varepsilon\right) - \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} \mathbf{K}(\rho,\theta)\theta\,d\theta.$$

Since $\theta \mapsto \theta\mathbf{K}(\rho,\theta)$ is odd we have $\int_{-\varepsilon}^{\varepsilon}\mathbf{K}(\rho,\theta)\theta\,d\theta = 0$ and we obtain

$$\frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} \mathbf{K}(\rho,\theta)g(\varphi+\varepsilon-\theta)\,d\theta \leq \frac{2}{3}(-|g|+K\varepsilon).$$

Now (4.3) gives

$$-|g|_\rho \leq \frac{2}{3}\left(-|g|+K\varepsilon\right)+\frac{1}{3}|g| = \frac{2}{3}K\varepsilon - \frac{1}{3}|g|$$

and, since $\varepsilon \leq 2\sqrt{3}\frac{1-\rho}{1+\rho}$,

$$|g| \leq 3|g|_\rho + 4\sqrt{3}K\frac{1-\rho}{1+\rho}. \qquad \square$$

The next lemma is an easy consequence of Poisson's formula.

LEMMA 4.2. — *Let f be a real harmonic function on $|z| < 1$ and let $0 < \rho < r < 1$. Then,*

$$|\tilde{f}|_\rho \leq \frac{2}{\pi}\left(\log\frac{r+\rho}{r-\rho}\right)|f|_r.$$

Moreover, if $f + i\tilde{f} \in H_1$, we also have

$$|\tilde{f}|_\rho \leq \frac{2\rho}{1-\rho^2}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|\,d\theta.$$

Proof. — By Poisson's formula (2.4)

$$\tilde{f}(\rho,\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{\mathbf{K}}(\rho/r,\theta)f(r,\varphi-\theta)\,d\theta.$$

It follows by (2.11) that

$$|\tilde{f}|_\rho \leq \frac{2}{\pi}\left(\log\frac{r+\rho}{r-\rho}\right)|f|_r.$$

Assume now $f+i\tilde{f} \in H_1$. Then Poisson's formula (2.4) still holds for $r = 1$ and we find

$$\tilde{f}(\rho,\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{\mathbf{K}}(\rho,\theta)f(\varphi-\theta)\,d\theta.$$

Therefore, by (2.7),

$$|\tilde{f}(\rho,\varphi)| \leq \frac{2\rho}{1-\rho^2}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|\,d\theta. \qquad \square$$

Lemma 4.1 and the first part of Lemma 4.2 lead to Theorem E announced in the introduction:

THEOREM 4.1. — *Let f be a real harmonic function on $\mathbf{D}$ and let assume that its conjugate function $\tilde{f}$ satisfies $\partial\tilde{f}/\theta < K$ on $\mathbf{D}$. Then, for any $r \in [1/2, 1)$*

$$|\tilde{f}| \leq \frac{6}{\pi}\left(\log\frac{2}{1-r}\right)|f|_r + 4\sqrt{3}K\frac{1-r}{r}.$$

Proof. — Let ρ, r such that $0 \leq \rho < r < 1$. From Lemmas 2 and 3 we obtain

$$|\tilde{f}| \leq \frac{6}{\pi}\left(\log\frac{r+\rho}{r-\rho}\right)|f|_\rho + 4\sqrt{3}K\frac{1-\rho}{1+\rho}.$$

Now choose $\rho = 2r - 1$. $\square$

The second part of Lemma 4.2 leads to an elementary proof (with a worse constant) of Ganelius-Mignotte's theorem:

THEOREM 4.2. — *Let f be a real harmonic function on* $\mathbf{D}$ *such that* $\partial\tilde{f}/\partial\theta < K$ *on* $\mathbf{D}$. *Then*

$$|\tilde{f}| \leq 4\sqrt{3\sqrt{3}} \cdot \sqrt{\tilde{H}K}$$

where

$$\tilde{H} = \frac{1}{4\pi}\int_{-\pi}^{\pi} |f(\theta)|\, d\theta.$$

Proof. — From Lemma 4.1 (with $g = \tilde{f}$) and Lemma 4.2 (since $\tilde{f}$ is bounded, $f + i\tilde{f} \in H_1$) we obtain

$$|\tilde{f}| \leq \frac{12\rho\tilde{H}}{1-\rho^2} + 4\sqrt{3}K\frac{1-\rho}{1+\rho}.$$

Let α, $\beta > 0$ and $u(\rho) = \dfrac{\alpha\rho}{1-\rho^2} + \dfrac{\beta(1-\rho)}{1+\rho}$. Then $\inf_{0<\rho<1} u(\rho) \leq \sqrt{\alpha\beta}$. In fact, if $\beta \leq \alpha$ we have $u(0) = \beta \leq \sqrt{\alpha\beta}$; otherwise $\rho_0 = 1 - \sqrt{\dfrac{\alpha}{\beta}} \in (0,1)$ and $u(\rho_0) = \sqrt{\alpha\beta}$. Using this remark with $\alpha = 12\tilde{H}$ and $\beta = 4\sqrt{3}K$ we obtain

$$|\tilde{f}| \leq \sqrt{6\tilde{H} \cdot 4\sqrt{3}K} = 4\sqrt{3\sqrt{3}\tilde{H}K} < 9.119 . \sqrt{\tilde{H}K}. \qquad \square$$

5. Upper bounds for $\max f$.

The aim of this section is to give an upper bound for the maximum of an harmonic function f such that $\partial\tilde{f}/\partial\theta$ is bounded on D.

THEOREM 5.1. — *Let f be a real harmonic function on* $\mathbf{D}$ *such that* $f(0) = 0$ *and* $\partial\tilde{f}/\partial\theta < K$ *on* $\mathbf{D}$ *for some* $K > 0$. *Then,*

$$\sup_{\mathbf{D}} f \leq \frac{\Delta(\tilde{f})}{\pi}\left(3 + 2\log\frac{2\sqrt{3}\pi K}{\Delta(\tilde{f})}\right).$$

Proof. — Let $\varphi \in \mathbf{R}$ and let $\rho \in (0,1)$. We apply Poisson's formula (2.4) to the harmonic function $\tilde{f}$. Since $f(0) = 0$ we have $\tilde{\tilde{f}} = -f$.

Moreover, since $\tilde{f}$ is bounded, $f+i\tilde{f} \in H_1$ and (2.4) still holds with $r=1$. By using (2.11) we get

$$(5.1)\ f(\rho,\varphi) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{\mathbf{K}}(\rho,\theta)\tilde{f}(\varphi-\theta)\,d\theta$$
$$= \frac{1}{2\pi}\int_{0}^{\pi} \tilde{\mathbf{K}}(\rho,\theta)\big(\tilde{f}(\varphi+\theta)-\tilde{f}(\varphi-\theta)\big)\,d\theta \le \frac{2\Delta(\tilde{f})}{\pi}\log\frac{1+\rho}{1-\rho}.$$

Since $\partial\tilde{f}/\partial\theta = \rho(\partial f/\partial\rho)$, we have $f(\varphi) \le f(\rho,\varphi) + K\log 1/\rho$. Therefore (5.1) gives

$$\max f \le \frac{2\Delta(\tilde{f})}{\pi}\log\frac{2}{1-\rho} + K\log\frac{1}{\rho}.$$

Now choose $\rho = K\pi/(\Delta(\tilde{f})+K\pi)$. Since $\Delta(\tilde{f}) \le 2\pi K$, we obtain

$$\max f \le \frac{2\Delta(\tilde{f})}{\pi}\log\frac{2\pi K}{\Delta(\tilde{f})} + \left(K + \frac{\Delta(\tilde{f})}{\pi}\right)\log\left(1+\frac{\Delta(\tilde{f})}{\pi K}\right)$$
$$\le \frac{\Delta(\tilde{f})}{\pi}\left(3 + 2\log\frac{2\sqrt{3}\pi K}{\Delta(\tilde{f})}\right). \qquad \square$$

We end this section with a further remark concerning harmonic functions.

PROPOSITION 5.1. — *Let f be an harmonic function on $\mathbf{D}$ and assume that $f+i\tilde{f} \in H_1$. Then, for $0 \le \rho < 1$ and $\varphi \in R$,*

$$\text{(i)}\qquad -\frac{1+\rho}{1-\rho}\times\frac{1}{2\pi}\int_{-\pi}^{\pi} f^-(\theta)\,d\theta \le f(\rho,\varphi) \le \frac{1+\rho}{1-\rho}\times\frac{1}{2\pi}\int_{-\pi}^{\pi} f^+(\theta)\,d\theta.$$

Moreover, if $f(\theta) \le 0$, then

$$\text{(ii)}\qquad f(\rho,\varphi) \le \frac{1-\rho}{1+\rho}\times\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta.$$

Proof. — By Poisson's formula (2.3), for any $\rho \in (0,1)$ and for any $\varphi \in \mathbf{R}$ we have

$$f(\rho,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{K}(\rho,\theta-\varphi)\,f(\theta)\,d\theta.$$

Thus, by (2.6)

$$-\frac{1+\rho}{1-\rho}\times\frac{1}{2\pi}\int_{-\pi}^{\pi} f^-(\theta)\,d\theta \le f(\rho,\varphi) \le \frac{1+\rho}{1-\rho}\times\frac{1}{2\pi}\int_{-\pi}^{\pi} f^+(\theta)\,d\theta,$$

which proves (i).

Assume now $f(\theta) \leq 0$. Then

$$-f(\rho,\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{K}(\rho,\theta-\varphi)\,(-f(\theta))\,d\theta \geq \frac{1-\rho}{1+\rho} \times \frac{1}{2\pi}\int_{-\pi}^{\pi}(-f(\theta))\,d\theta,$$

which leads to (ii). □

COROLLARY 5.1. — *Let P be a polynomial with no zeros for $|z|<1$. Then, for $0 \leq \rho < 1$ and $\varphi \in \mathbf{R}$,*

$$-\frac{1+\rho}{1-\rho}\big(\tilde{h}(P) + \log \mathrm{M}(P)\big) \leq \log\big|P(\rho e^{i\varphi})\big| \leq \frac{1+\rho}{1-\rho} \times \tilde{h}(P).$$

Proof. — Use (i) and the relation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|P| = \log \mathrm{M}(P) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^{+}|P| - \frac{1}{2\pi}\int_{-\pi}^{\pi}\log^{-}|P|. \quad \square$$

COROLLARY 5.2. — *Let P be a polynomial with no zeros for $|z|<1$. Then, for $0 \leq r < 1$,*

$$|P|_r \leq |P|^{\frac{2r}{1+r}} \mathrm{M}(P)^{\frac{1-r}{1+r}}.$$

Proof. — This is an easy consequence of (ii). □

6. An extremal example.

Let x be a positive real number and consider the set Λ_x of polynomials $P(z) = a_n z^n + \cdots + a_1 z + a_0$ such that $a_0 a_n \neq 0$ and

$$\log \frac{|P|}{\sqrt{|a_0 a_n|}} \leq x \cdot n.$$

Let

$$f(x) = \sup_{P \in \Lambda_x} D_P.$$

Then, f is a non-decreasing function and Erdös-Turán's theorem implies the inequality

$$f(x) \leq c\sqrt{x}, \qquad c = \sqrt{2\pi/k}.$$

The aim of this section is to prove that this inequality is essentially sharp.

THEOREM 6.1. — *For any $x \in (0, 1/2)$ we have $f(x) \geq \sqrt{2x}$.*

Let n, r two positive integers with $r < n$. By the results of [ET], §14,

$$P(z) = \frac{r\binom{n+r}{r}}{(1+z)^r} \int_{-1}^{z} (z-t)^{r-1}(1+t)^r t^{n-r}\, dt \tag{6.1}$$

is a monic polynomial of degree n vanishing at -1 with multiplicity r such that

$$\log \|P\| = \frac{1}{2} \sum_{\nu=n-r+1}^{n} \log\left(1+\frac{r}{\nu}\right) \le \frac{r^2}{2(n-r)}, \tag{6.2}$$

where $\|P\|$ is the euclidean norm of the polynomial P, i.e. the quadratic mean of the moduli of the coefficients of P. Moreover, by (6.1)

$$a_0 = P(0) = (-1)^{n-r} r \binom{n+r}{r} \int_0^1 (1-s)^r s^{n-1}\, ds = (-1)^{n-r}\frac{r}{n}. \tag{6.3}$$

Since P has a root at -1 of multiplicity $\ge r$ we have $D_P \ge \dfrac{r}{n}$. On the other hand, by (6.2) and (6.3) we obtain

$$\log \frac{|P|}{\sqrt{|a_0 a_n|}} \le \log \frac{\sqrt{n}\,\|P\|}{\sqrt{|a_0|}} \le \frac{r^2}{2(n-r)} + \frac{1}{2}\log\frac{n^2}{r} \le \frac{r^2}{2(n-r)} + \log n.$$

Hence

$$\frac{r}{n} \le D_P \le f\left(\frac{r^2}{2n(n-r)} + \frac{\log n}{n}\right).$$

Let now $x \in (0, 1/2)$ and choose a sequence (n_k, r_k) such that $n_k \to +\infty$ and

$$\frac{r_k^2}{2n_k(n_k-r_k)} + \frac{\log n_k}{n_k}$$

increases to x as $k \to +\infty$. Then we have $r_k/n_k \le f(x)$ and, when $k \to +\infty$,

$$\sqrt{2x} \le f(x). \qquad \square$$

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