Irrationality measures for cubic irrationals whose conjugates lie on a curve

DOI:10.1007/s00209-021-02732-8

IRRATIONALITY MEASURES FOR CUBIC IRRATIONALS WHOSE CONJUGATES LIE ON A CURVE

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1. Introduction

This paper deals with irrationality measure of algebraic irrational numbers. There is a vast bibliography on this subject, the interested reader can refer for instance to [8] for further references. Let us recall only some basic facts. We say that μ is an irrationality measure for $\xi \in \mathbb{R}\backslash\mathbb{Q}$ if for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all p, $q \in \mathbb{Z}$ with $q > 0$ we have

$$\left|\xi - \frac{p}{q}\right| > \frac{C_\varepsilon}{q^{\mu+\varepsilon}}.$$

We define the irrationality measure $\mu(\xi)$ as the infimum of the irrationality measures for ξ. By Dirichlet's Theorem, or using the theory of continued fractions, $\mu(\xi) \geq 2$ and moreover for almost all real numbers equality holds. Nevertheless, "few" examples of numbers with (provable) irrationality measure 2 are known.

For real irrational *algebraic* numbers the situation is different: by a celebrated result of Roth, which extends the Thue-Gelfond-Dyson-Siegel method from 2 to n variables, every such number has irrationality measure 2, but this result is not effective, in the sense that the above C_ε cannot be effectively computed (at least for small $\varepsilon > 0$).

Let us say that μ is an *effective* irrationality measure for ξ, if the real number C_ε above is effectively computable for any $\varepsilon > 0$, and define $\mu_{\text{eff}}(\xi)$ as the infimum of the effective irrationality measures for ξ. Then Liouville's Theorem provides the easy bound $\mu_{\text{eff}}(\xi) \leq d$ for a real algebraic number of degree $d > 1$.

Feldman [12], using Baker's theory of linear forms in logarithms, proves that $\mu_{\text{eff}}(\xi) < d$ for a real algebraic number of degree $d \geq 3$. Bombieri [6] gives another proof of this result, using the original Thue-Dyson-Siegel method in two variables.

In some special case, essentially when ξ is "close" to 1, the theory of Padé approximant can give much better results. For instance for $\xi := \sqrt[3]{2}$, Baker [3] obtains $\mu_{\text{eff}}(\xi) \leq 2.955$, using $\frac{4}{5}\sqrt[3]{2} \approx 1.008$.

The aim of this short paper is to describe a new method (at any rate, we have no knowledge of appearance of this method in the literature) which provides *effective* irrationality measures for certain algebraic irrational numbers. The classical method needs a *family* of good approximations of ξ, which are in most cases obtained specializing Padé approximants. Our approach needs *only one* good approximation such that the point defined by its r

Date: February 20, 2022.

algebraic conjugates lies on a fixed curve $\mathcal{C} \subset \mathbb{P}_{r-1}$. We then take powers and eventually eliminate suitable coefficients. The height of the point must be sufficiently large with respect to the curve (*i. e.* with respect to the field of definition, the height and the degree of the equations defining $\mathcal{C}$). This constraint is effective, and moreover explicit if $\mathcal{C}$ is a line.

In section 2 we shall first consider the special case, but still interesting in its own, of cubic roots $\xi_l = \sqrt[3]{l^3+1}$ where l is a large natural number. The classical method deals with Padé approximants to the function $z \mapsto (1-z)^{1-1/3}$. The main ingredient of our approach is a bounded height estimate of Beukers and Schlickewei [4] for the solutions of the system of equations

$$\begin{aligned} A + B &= 1; \\ aA^n + bB^n &= 1. \end{aligned}$$

in the unknown A, $B \in \overline{\mathbb{Q}}^*$, with coefficients a, $b \in \overline{\mathbb{Q}}^*$.

In section 4 we generalize this approach to other cubic irrationals whose algebraic conjugates are coordinates of a point lying on a fixed line of $\mathbb{P}_2$. In some cases (see for instance exemples 4.3 1. and the discussion at the end of section 2), our results are close to the results obtained with classical methods. In other cases they are new (see for instance exemples 4.3 2.).

To deal with cubic irrationals whose algebraic conjugates are coordinates of a point lying on a fixed curve of $\mathbb{P}_2$, not necessarily a line, the results of [4] are no longer sufficient. We use instead, in section 5, a (very special) case of a recent bounded height result [2] obtained by Masser and the authors of this article. This allows us to prove (section 5) the following theorem, which we state here only for algebraic integers, in order to avoid technical definitions on heights (section 3). See Theorem 5.2 for the more general statement.

Theorem 1.1. *Let $\mathbb{K} \subseteq \mathbb{C}$ be a real cubic number field; denote by $\sigma_1 = \mathrm{Id}, \sigma_2, \sigma_3$ the immersions $\mathbb{K} \hookrightarrow \mathbb{C}$. Let $\mathcal{C} \subset \mathbb{P}_2$ be a projective curve defined over $\overline{\mathbb{Q}}$. Let $\hbar > 0$, $\lambda > 1/2$ and $\varepsilon \in (0,1)$ such that*

$$\hbar \geq c^2 \min\Big(1, \frac{(\lambda-1/2)^2}{8(\lambda+1)}\varepsilon\Big)^{-2}$$

where $c = c(\mathcal{C})$ is an effective constant. Let $\theta \in \mathcal{O}_{\mathbb{K}}$ with conjugates $\theta_1 = \theta$, θ_2, θ_3. We assume:

$$(\theta_1 : \theta_2 : \theta_3) \in \mathcal{C}(\overline{\mathbb{Q}})$$

and

$$\max(|\theta_2|, |\theta_3|) \leq \hbar, \qquad \log|\theta| \leq -\lambda\hbar.$$

Then

$$\mu_{\mathrm{eff}}(\xi) \leq \frac{\lambda+1}{\lambda-1/2}$$

for every generator ξ of $\mathbb{K}$.

As pointed out by the referee, it would be interesting to work out the dependence of the constant $c(\mathcal{C})$ appearing in this theorem on the degree, height and field of definition of the curve. This could be done, however at the price of a heavy technical work on the proofs of [2]. Some exemples (see the discussion after the statement of Theorem 5.1) seem to suggest that $c(\mathcal{C})$ grows polynomially with the degree and the height of the curve.

As a corollary of Theorem 1.1, we recover a (not so well known) result of Chudnwski [9] on irrationality measures for the values of a cubic algebraic function holomorphic at 0 (see Corollary 5.5).

In principle our method, could be further generalized to irrational numbers of higher degree, at the price of a technical extension of the relevant result of [2].

Acknowledgements. The authors would like to express their gratitude to Yann Bugeaud, Sinnou David, Michel Waldschmidt and Wadim Zudilin for helpful conversations. We kindly thank Pietro Corvaja for reading a preliminary version of this paper and David Masser for his suggestions on the behavior of $c(\mathcal{C})$. We also thank the anonymous referee for carefully reading and many helpful suggestions.

2. An Example

Let l be a positive (large) integer. We look for an irrationality measure of the cubic irrational $\xi_l = \sqrt[3]{l^3+1}$. To illustrate our method, we want to show that $\mu_{\rm eff}(\xi_l) \to 2$ when $l \to \infty$ (which is of course a well-known result). Put

$$\theta = \xi_l - l, \qquad \zeta = \exp(2\pi i/3).$$

We denote by $\theta_1 = \theta$, $\theta_2 = \zeta\xi_l - l$ and $\theta_3 = \zeta^2\xi_l - l$ the algebraic conjugates of θ and we write $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$. Let

$$\theta^n = \lambda_{n,0} + \lambda_{n,1}\theta + \lambda_{n,2}\theta^2 \tag{2.1}$$

with $\boldsymbol{\lambda}_n = (\lambda_{n,0}, \lambda_{n,1}, \lambda_{n,2}) \in \mathbb{Z}^3$. Thus $\boldsymbol{\theta}^n := (\theta_1^n, \theta_2^n, \theta_3^n) = \Theta\lambda_n$, where

$$\Theta = \begin{pmatrix} 1 & \theta_1 & \theta_1^2 \\ 1 & \theta_2 & \theta_2^2 \\ 1 & \theta_3 & \theta_3^2 \end{pmatrix}.$$

Solving in $\boldsymbol{\lambda}_n$ the system, we find $\boldsymbol{\lambda}_n = \Theta^{-1}\boldsymbol{\theta}^n$. We have $\theta_1\theta_2\theta_3 = 1$ and $|\theta_2| = |\theta_3| = O(l)$ as $l \to +\infty$. Thus

$$|\theta| = O(l^{-2}), \qquad |\boldsymbol{\lambda}_n|^{1/n} \ll l \tag{2.2}$$

and the projective (non logarithmic) Weil height of $\boldsymbol{\theta} \in \mathbb{P}_2(\overline{\mathbb{Q}})$ satisfies

$$H(\boldsymbol{\theta}) = |\theta_2| = |\theta_3| = O(l). \tag{2.3}$$

The implicit constants are absolute, for n sufficiently large with respect to l. In order to apply a standard irrationality criterium, we search "small" vectors $\mathbf{u}_n = (u_{n,0}, u_{n,1}, u_{n,2}) \in \mathbb{Z}^3$ such that $\alpha_n = u_{n,0} + u_{n,1}\theta + u_{n,2}\theta^2$ satisfies[1]

$$\alpha_n\theta^n \in \mathbb{Z} + \mathbb{Z}\theta.$$

By (2.1), this holds if and only if $\mathbf{u}_n = (u_{n,0}, u_{n,1}, u_{n,2})$ is orthogonal to the vector

$$\mathbf{v}_n = (\lambda_{n,2}, \lambda_{n+1,2}, \lambda_{n+2,2}).$$

of norm $|\mathbf{v}_n|^{1/n} \ll l$. By Minkowski's theorem we can find two *linearly independent* vectors $\mathbf{u}_n$, $\mathbf{u}'_n \in \mathbf{v}_n^\perp$ such that $|\mathbf{u}_n| \le |\mathbf{u}'_n|$ and

$$|\mathbf{u}_n|^{1/n} \cdot |\mathbf{u}'_n|^{1/n} \ll |\mathbf{v}_n|^{1/n} \ll l. \tag{2.4}$$

Thus $|\mathbf{u}_n|^{1/n} \ll l^{1/2}$. This gives a sequence $\{p_n/q_n\}_n$ of good rational approximations to θ, defined by $(u_{n,0} + u_{n,1}\theta + u_{n,2}\theta^2)\theta^n = q_n\theta - p_n$. In fact, by (2.1) and (2.2) we have

$$|q_n|^{1/n} \ll |\mathbf{u}_n|^{1/n}|\boldsymbol{\lambda}_n|^{1/n} \ll l^{3/2};$$
$$|q_n\theta - p_n|^{1/n} \ll |\mathbf{u}_n|^{1/n}|\theta| \ll l^{-3/2}.$$

Let us assume that $|\mathbf{u}'_n|$ satisfies a similar upper bound as $|\mathbf{u}_n|$, that is $|\mathbf{u}'_n|^{1/n} \ll l^{1/2}$. By (2.4) this holds if (and only if) the previous upper bound for $|\mathbf{u}_n|$ is essentially optimal, *i. e.*

$$|\mathbf{u}_n|^{1/n} \gg l^{1/2}. \tag{2.5}$$

[1]A similar requirement was worked by Bombieri and Van der Poorten for a different pourpose.

Note that in general this is what one would expect. Then, corresponding to u'_n, one has another sequence $\{p'_n/q'_n\}_n$ of good rational approximations such that $p_n q'_n \neq p'_n q_n$ for all n. This would allow to conclude (by a standard irrationality criterion) that

$$\mu_{\text{eff}}(\xi_l) \to 2 \text{ as } l \to +\infty$$

Hence the success of this approach is reduced to producing a *lower bound* for $|\mathbf{u}_n|^{1/n}$. Replacing n by $2n$, we can assume n even. For $j = 1, 2, 3$ write $\alpha_{n,j} = u_{n,0} + u_{n,1}\theta_j + u_{n,2}\theta_j^2$. Since the height of $\boldsymbol{\alpha}_n \in \mathbb{P}_2(\overline{\mathbb{Q}})$ satisfies

$$H(\boldsymbol{\alpha}_n)^{1/n} \ll |\mathbf{u}_n|^{1/n}, \tag{2.6}$$

it is enough to get a lower bound for this height.

Remark 2.1. *For $j = 1, 2, 3$ let $\sigma_j : \mathbb{Q}(\theta) \hookrightarrow \mathbb{C}$ be the immersions given by $\sigma_j\theta = \theta_j$. Then for $\gamma \in \mathbb{Z}[\theta]$ we have $\gamma \in \mathbb{Z} + \mathbb{Z}\theta$ if and only if $\sigma_1\gamma + \zeta\sigma_2\gamma + \zeta^2\sigma_3\gamma = 0$.*

Since both θ and $\alpha_{n,1}\theta^n$ are in $\mathbb{Z} + \mathbb{Z}\theta$, we have the two equations:

$$\begin{aligned} \theta_1 + \zeta\theta_2 + \zeta^2\theta_3 &= 0; \\ \alpha_{n,1}\theta_1^n + \zeta\alpha_{n,2}\theta_2^n + \zeta^2\alpha_{n,3}\theta_3^n &= 0. \end{aligned} \tag{2.7}$$

We now use the following result of Beukers and Schlickewei ([4], lemme 2.3), which follows from an explicit construction ([5], lemma 6) of Padé approximants to the function $z \mapsto (1 - z)^n$.

Lemma 2.2 (Beukers-Schlickewei)**.** *Let a, b, A, $B \in \overline{\mathbb{Q}}^*$ such that*

$$\begin{aligned} A + B &= 1; \\ aA^n + bB^n &= 1 \end{aligned} \tag{2.8}$$

for some even integer $n \in \mathbb{N}$. Then, $H(1 : A : B) \leq c\, 2^{2/n} H(1 : a : b)^{2/n}$, where $c = 6\sqrt{3}$.

The system (2.7) transforms into (2.8) by the change of variables:

$$A = -\frac{\zeta\theta_2}{\theta_1}, \quad B = -\frac{\zeta^2\theta_3}{\theta_1}, \quad a = -\frac{\zeta\alpha_{n,2}}{\zeta^n\alpha_{n,1}}, \quad b = -\frac{\zeta^2\alpha_{n,3}}{\zeta^{2n}\alpha_{n,1}}. \tag{2.9}$$

We have

$$H(1 : A : B) = H(\boldsymbol{\theta}) \quad \text{and} \quad H(1 : a : b) = H(\boldsymbol{\alpha}_n).$$

By the quoted result of Beukers and Schlickewei, by (2.3) and by (2.6),

$$l \ll H(\boldsymbol{\theta}) \leq c\, 2^{2/n} H(\boldsymbol{\alpha}_n)^{2/n} \ll |\mathbf{u}_n|^{2/n}.$$

Thus (2.5) holds and

$$\mu_{\text{eff}}(\xi_l) \to 2$$

as claimed.

Of course we can quantify all of this to see how near we go to effective irrationality measure 2. Assume $l \geq 3$. A more precise computation (see the first example 4.3) shows that ξ_l has irrationality measure

$$\begin{aligned}\mu_{\mathrm{eff}}(\xi_l) &\leq 1+\frac{-\log(\sqrt[3]{l^3+1}-l)+\frac{2}{3}\log(6\sqrt{3})}{-\log(\sqrt[3]{l^3+1}-l)-\frac{2}{3}\log(6\sqrt{3})}\\ &= 2+\frac{2\log(6\sqrt{3})}{3\log l}+O((\log l)^{-2}) \text{ for } l\to+\infty.\end{aligned}$$

Our method rests essentially on an explicit Padé construction (via hypergeometric function) of the Padé approximants to the function $z \mapsto (1-z)^n$, which is the main tool of the proof of Lemma 2.2. Let us compare our results with (a special case) of a result of Alladi and Robinson [1] which uses Padé approximants to the function $z \mapsto (1-z)^{1-1/3}$. Theorem 2 of *op.cit.* (with $k=3$, $f(3)=9/4$, $r=1$, $s=l^3$) gives, again for $l \geq 3$,

$$\begin{aligned}\mu_{\mathrm{eff}}(\xi_l) &\leq 1+\frac{-\log(\sqrt{l^3+1}-\sqrt{l^3})+9/4}{-\log(\sqrt{l^3+1}-\sqrt{l^3})-9/4}\\ &= 2+\frac{3}{2\log l}+O((\log l)^{-2}) \text{ for } l\to+\infty.\end{aligned}$$

Since $2\log(6\sqrt{3})/3 = 1.567... > 3/2$, the result of Alladi and Robisons is slightly better for large l. For small values of l we obtain:

$l=2$	$\mu_{\mathrm{eff}}=5.2381$	[5.5281]
$l=3$	$\mu_{\mathrm{eff}}=3.7865$	[3.8365]
$l=4$	$\mu_{\mathrm{eff}}=3.3480$	[3.3625]
$l=5$	$\mu_{\mathrm{eff}}=3.1312$	[3.1340]
$l=6$	$\mu_{\mathrm{eff}}=2.9996$	[2.9970]
$l=7$	$\mu_{\mathrm{eff}}=2.9099$	[2.9045]
$l=8$	$\mu_{\mathrm{eff}}=2.8443$	[2.8372]
$l=9$	$\mu_{\mathrm{eff}}=2.7937$	[2.7856]

where we have reproduced the corresponding results of [1] in brackets. Note that our method gives something better for $l \leq 5$. But in these cases our effective irrationality measure is worse than the bound $\mu_{\mathrm{eff}}(\xi_l) \leq 3$ predicted by Liouville's Theorem.

3. Heights and absolute values.

To go further, we need some more notations. As we have seen in section 2, given a cubic irrational θ the natural height associated with our method is not the Weil height of θ but instead the height of the projective point defined by its conjugates. Moreover, if θ is not a unit (as it was in section 2), for its absolute value we have to take into account the contribution at the finite places. We develop these remarks in the more general setting of algebraic numbers of degree r in view of further applications.

We consider algebraic numbers as complex numbers, on choosing an immersion $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Given $\theta_1, \ldots, \theta_r \in \overline{\mathbb{Q}}$ we denote by $H(\boldsymbol{\theta})$ the normalized, non-logarithmic Weil's height of $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_r)$, which we identify with the corresponding point in $\mathbb{P}_{r-1}(\overline{\mathbb{Q}})$. We put $h(\boldsymbol{\theta}) = \log H(\boldsymbol{\theta})$ for the corresponding logarithmic height. For an algebraic number θ we set $H(\theta) = H(1 : \theta)$ and $h(\theta) = \log H(\theta)$.

Let $\theta \in \mathbb{C}$ be an algebraic number. We choose a number field $\mathbb{K} \subseteq \mathbb{C}$ of degree say r, which contains θ, we let $\mathbb{L}$ be any number field containing its Galois closure, and we denote $\sigma_1, \ldots, \sigma_r$ the immersions $\mathbb{K} \hookrightarrow \mathbb{C}$. We set

$$H_0(\theta) = H(\sigma_1\theta : \cdots : \sigma_r\theta), \quad h_0(\theta) = \log H_0(\theta),$$

for the Weil height of the projective point $(\sigma_1\theta : \cdots : \sigma_r\theta)$, and we let

$$\|\theta\| = |\theta| \cdot H_{\text{finite}}(\theta)$$

where

$$H_{\text{finite}}(\theta) := \prod_{\substack{v \in \mathcal{M}_{\mathbb{L}} \\ v \nmid \infty}} \max(|\sigma_1\theta|_v, \ldots, |\sigma_r\theta|_v)^{[\mathbb{L}_v:\mathbb{Q}_v]/[\mathbb{L}:\mathbb{Q}]}.$$

It is clear that these definitions do not depend on the choice of $\mathbb{L}$ and $\mathbb{K}$ (but the definition of $\|\theta\|$ depends on the immersion $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$). We also remark that $H_{\text{finite}}(\theta) \leq 1$ if θ is an algebraic integer, with equality if θ is a unit.

Given the minimal polynomial of θ, say

$$x^r - a_1 x^{r-1} + \cdots + (-1)^r a_r \in \mathbb{Q}[x],$$

it is easy to compute $H_0(\cdot)$ and $\|\cdot\|$. First

$$\prod_{\substack{v \in \mathcal{M}_{\mathbb{L}} \\ v \mid \infty}} \max(|\sigma_1\theta|_v, \ldots, |\sigma_r\theta|_v)^{[\mathbb{L}_v:\mathbb{Q}_v]/[\mathbb{L}:\mathbb{Q}]} = \max(|\sigma_1\theta|, \ldots, |\sigma_r\theta|).$$

Thus

$$H_0(\theta) = \max(|\sigma_1\theta|, \ldots, |\sigma_r\theta|) H_{\text{finite}}(\theta), \qquad \|\theta\| = |\theta| H_{\text{finite}}(\theta).$$

The computation of $H_{\text{finite}}(\theta)$ is a little bit more involved.

Lemma 3.1. *Let $v \in \mathcal{M}_{\mathbb{L}}$, $v \nmid \infty$. Then*

$$\max(|\sigma_1\theta|_v, \ldots, |\sigma_r\theta|_v) = \max(|a_1|_v, |a_2|_v^{1/2}, \ldots, |a_r|_v^{1/r}).$$

Proof. Let for short $\theta_j := \sigma_j\theta$ and $\Theta = \max(|\theta_1|_v, \dots, |\theta_r|_v)$. After renumbering we can assume

$$|\theta_1|_v = \cdots = |\theta_k|_v > |\theta_{k+1}|_v \geq \cdots \geq |\theta_r|_v \tag{3.1}$$

for some k with $1 \leq k \leq r$. Then obviously

$$|a_j|_v = \left| \sum_{1 \leq i_1 < \cdots < i_j \leq r} \theta_{i_1} \cdots \theta_{i_j} \right|_v \leq \Theta^j$$

for $j = 1, \dots, r$. Moreover $|a_k|_v = \Theta^k$. Indeed, let $i_1, \dots, i_k$ with $1 \leq i_1 < \cdots < i_k \leq r$. Then, by (3.1)

$$|\theta_{i_1} \cdots \theta_{i_k}|_v < |\theta_1 \cdots \theta_k|_v = \Theta^k$$

if $(i_1, \dots, i_k) \neq (1, \dots, k)$.

□

Thus

$$H_{\text{finite}}(\theta) = \prod_p \max(|a_1|_p, |a_2|_p^{1/2}, \dots, |a_r|_p^{1/r}),$$

where the product is over the rational primes.

Even more explicitly, assume $r = 3$ (which is the relevant case for this article) and let

$$x^3 + f_1x^2 + f_2x + f_3 \in \mathbb{Q}[x]$$

with root θ. Then

$$H_{\text{finite}}(\theta) = \operatorname{lcm}(\operatorname{den}(f_1)^6, \operatorname{den}(f_2)^3, \operatorname{den}(f_3)^2)^{1/6}. \tag{3.2}$$

We also remark that (by Remark 3.2 1) below) we may assume $f_1 = 1$ replacing θ by $f_1\theta$.

We now state some obvious properties of $H_0(\cdot)$ and $\|\cdot\|$.

Remark 3.2.

1) *$H_0(\cdot)$ and $\|\cdot\|$ are functions $\overline{\mathbb{Q}} \to \mathbb{R}^+$ invariant by multiplication by non-zero rational numbers.*
2) *For $\theta \in \overline{\mathbb{Q}}$ of degree r we have $\|\theta\| \leq H_0(\theta) \leq H(\theta)^r$ and $\|\theta\| \leq |\theta| \cdot H(\theta)^r$.*
3) *For θ, $\theta' \in \overline{\mathbb{Q}}$ we have $H_0(\theta \cdot \theta') \leq H_0(\theta) \cdot H_0(\theta')$ and $\|\theta \cdot \theta'\| \leq \|\theta\| \cdot \|\theta'\|$.*

We state below a variant of Liouville's inequality:

Remark 3.3. *Let θ be a non zero algebraic number of degree r. Then*

$$\|\theta\| \geq H_0(\theta)^{-(r-1)}.$$

We also need a variant of a classical lemma on irrationality measures. We start with

Lemma 3.4. *Let $\xi \in \overline{\mathbb{Q}}$ and $q_0, \dots, q_{r-1} \in \mathbb{Q}$. Put*

$$\theta = q_0 + q_1\xi + \cdots + q_{r-1}\xi^{r-1}.$$

Then

$$H_0(\theta) \leq rH(\xi)^{r(r-1)} \cdot H(\mathbf{q}) \quad \textit{and} \quad \|\theta\| \leq H(\xi)^{r(r-1)} d(\mathbf{q}) \cdot |\theta| \tag{3.3}$$

with $d(\mathbf{q}) = \prod_{p \text{ prime}} \max(|q_0|_p, \ldots, |q_{r-1}|_p)$*. Moreover, if* ξ *has degree* r*,*

$$H(\mathbf{q}) \leq c_1 H_0(\theta) \quad \textit{and} \quad d(\mathbf{q}) \cdot |\theta| \leq c_2 \|\theta\| \tag{3.4}$$

for some positive constants c_1 *and* c_2 *depending only on* ξ*.*

Proof. We keep all the notations above and we let $\xi_j = \sigma_j \xi$, $\theta_j = \sigma_j \theta$. Let v be a place of $\mathbb{L}$. Put $\varepsilon_v = 1$ if $v \nmid \infty$ and $\varepsilon_v = 0$ otherwise. Then

$$|\theta_j|_v \leq r^{\varepsilon_\nu} \max(|q_0|_v, \ldots, |q_{r-1}|_v) \max(1, |\xi_j|_v)^{r-1}$$

Thus

$$\begin{aligned} \max(|\theta_1|_v, \ldots, |\theta_r|_v) &\leq r^{\varepsilon_\nu} \max(|q_0|_v, \ldots, |q_{r-1}|_v) \max(1, |\xi_1|_v, \ldots, |\xi_r|_v)^{r-1} \\ &\leq r^{\varepsilon_\nu} \max(|q_0|_v, \ldots, |q_{r-1}|_v) \prod_{j=1}^{r} \max(1, |\xi_j|_v)^{r-1}. \end{aligned}$$

Inequality (3.3) follows. To prove (3.4), we solve the system

$$q_0 + q_1 \xi_j + \cdots + q_{r-1} \xi_j^{r-1} = \theta_j, \qquad j = 1, \ldots, r$$

in the unknowns $q_0, \ldots, q_{r-1}$. We find

$$(q_0, \ldots, q_{r-1}) = \Xi^{-1} (\theta_1, \ldots, \theta_r)^T$$

where ^{T} denote transposition and where Ξ is the $r \times r$ matrix ξ_i^{j-1} of determinant $\sqrt{|\mathrm{disc}(\xi)|}$. Thus

$$\max(|q_0|_v, \ldots, |q_{r-1}|_v) \leq c_v(\xi) \max(|\theta_1|_v, \ldots, |\theta_r|_v)$$

which gives (3.4). □

We can now state our lemma on irrationality measures.

Lemma 3.5. *Let* ξ *be an algebraic number of degree* r*. Let also* s *be a positive integer* $\leq r$ *and* $\theta_n^{(i)}$ $(i = 1,, \ldots, s)$ *be sequences in*

$$\mathbb{Q} + \mathbb{Q}\xi + \cdots + \mathbb{Q}\xi^{s-1}$$

such that $\theta_n^{(1)}, \ldots, \theta_n^{(s)}$ *are* $\mathbb{Q}$*-linearly independent for large* n*. Let us assume that there exist* $H > 1$*,* $\omega > 0$ *and* $n_0 \in \mathbb{N}$ *such that*

$$H_0(\theta_n^{(i)}) \leq H^n, \qquad \|\theta_n^{(i)}\| \leq H^{-\omega n},$$

for $i = 1, \ldots, s$ *and for* $n \geq n_0$*. Then* ξ *has an effective irrationality measure* $\leq (s-1)(1 + 1/\omega)$*.*

Proof. We fix an index i. Multiplying $\theta_n^{(i)}$ by suitable rationals numbers, we may assume (by 1) of Remark 3.2)

$$\theta_n^{(i)} = p_{n,0}^{(i)} + p_{n,1}^{(i)} \xi + \cdots + p_{n,s-1}^{(i)} \xi^{s-1}$$

with $p_{n,0}^{(i)}, \ldots, p_{n,s-1}^{(i)}$ coprime integers. Since $\theta_n^{(1)}, \ldots, \theta_n^{(s)}$ are $\mathbb{Q}$-linearly independent for large n, the vectors $(p_{n,0}^{(i)}, \ldots, p_{n,s-1}^{(i)})$ are also $\mathbb{Q}$-linearly independent. Moreover, by (3.4) of Lemma 3.4,

$$\max(|p_{n,0}^{(i)}|, |p_{n,1}^{(i)}|, \ldots, |p_{n,s-1}^{(i)}|) \leq C H^n$$

and

$$|p_{n,0}^{(i)} + p_{n,1}^{(i)} \xi + \cdots + p_{n,s-1}^{(i)} \xi^{s-1}| \leq C H^{-\omega n}$$

for some $C > 0$ depending only on ξ. We now apply a standard irrationality criterium.

$\square$

4. Conjugates lying on a fixed line.

The computations in section 2 can be easily generalized to a (real) cubic irrational θ with algebraic conjugates θ_1, θ_2, θ_3 defining a projective point in a fixed projective line. Moreover, the method gives an upper bound for the effective irrationality measure of an arbitrary generator of $\mathbb{Q}(\theta)$.

Let L be a projective line in $\mathbb{P}_2$ defined over $\overline{\mathbb{Q}}$ by

$$\beta_1^{-1}x_1 + \beta_2^{-1}x_2 + \beta_3^{-1}x_3 = 0.$$

Put $c(L) = (6\sqrt{3})^{1/2}\, H(\boldsymbol{\beta})^{3/2}$.

Theorem 4.1. *Let $\mathbb{K} \subseteq \mathbb{C}$ be a real cubic number field; denote by $\sigma_1 = \mathrm{Id}, \sigma_2, \sigma_3$ the immersions $\mathbb{K} \hookrightarrow \mathbb{C}$. Let $\theta \in \mathbb{K}$ such that*

$$(\sigma_1\theta : \sigma_2\theta : \sigma_3\theta) \in L(\overline{\mathbb{Q}}).$$

We assume

$$\|\theta\| < c(L)^{-1} H_0(\theta)^{-1/2}. \tag{4.1}$$

Then every generator of $\mathbb{K}$ has effective irrationality measure

$$\leq 1 + \frac{\frac{3}{2}h_0(\theta) + \log c(L)}{\log(1/\|\theta\|) - \frac{1}{2}h_0(\theta) - \log c(L)}.$$

Proof. We first remark that θ is irrational by Remark 3.3 since $\|\theta\| < 1$, and thus $\mathbb{K} = \mathbb{Q}(\theta)$. Let $\theta_j = \sigma_j\theta$. We then proceed as in section 2. For the reader's convenience we make all details explicit. Let ξ be a generator of $\mathbb{Q}(\theta)$ with algebraic conjugates $\xi_1 = \xi$, ξ_2, ξ_3. For $n \in \mathbb{N}$ write

$$\theta^n = \lambda_{n,0} + \lambda_{n,1}\xi + \lambda_{n,2}\xi^2$$

with $\lambda_{n,j} \in \mathbb{Q}$. We look for "small" vectors $\mathbf{u}_n = (u_{n,0}, u_{n,1}, u_{n,2}) \in \mathbb{Z}^3$ such that

$$(u_{n,0} + u_{n,1}\theta + u_{n,2}\theta^2)\theta^n \in \mathbb{Q} + \mathbb{Q}\xi.$$

This holds if and only if $\mathbf{u}_n = (u_{n,0}, u_{n,1}, u_{n,2})$ is orthogonal to the vector $\mathbf{v}_n = (\lambda_{n,2}, \lambda_{n+1,2}, \lambda_{n+2,2})$. Solving the 9×9 system

$$\lambda_{i,0} + \lambda_{i,1}\theta_j + \lambda_{i,2}\theta_j^2 = \theta_j^i, \qquad i = n, n+1, n+2;\ j = 1, 2, 3,$$

we see that

$$H(\mathbf{v}_n) \leq H(\lambda_{n,0} : \lambda_{n,1} : \cdots : \lambda_{n+2,2}) \ll H(\theta_1^n : \theta_2^n : \cdots : \theta_3^{n+2}),$$

where from now on the implicit constants in $\ll$ may depend on $\boldsymbol{\xi}$ and on $\boldsymbol{\theta}$, but not on n. By a standard computation on heights,

$$H(\theta_1^n : \theta_2^n : \cdots : \theta_3^{n+2}) \leq H(\theta_1 : \theta_2 : \theta_3)^n H(\theta_1)^2 H(\theta_2)^2 H(\theta_3)^2.$$

Thus

$$H(\mathbf{v}_n) \ll H_0(\theta)^n.$$

By Minkowski's theorem we can find two linearly independent vectors $\mathbf{u}_n$, $\mathbf{u}'_n \in \mathbf{v}_n^{\perp}$ with rational entries such that

$$H(\mathbf{u}_n)H(\mathbf{u}'_n) \ll H(\mathbf{v}_n) \ll H_0(\theta)^n. \tag{4.2}$$

Write

$$\alpha_n = u_{n,0} + u_{n,1}\theta + u_{n,2}\theta^2, \qquad \alpha'_n = u'_{n,0} + u'_{n,1}\theta + u'_{n,2}\theta^2.$$

Thus $\alpha_n\theta_n$, $\alpha'_n\theta_n$ are $\mathbb{Q}$-linearly independent algebraic numbers in $\mathbb{Q}+\mathbb{Q}\xi$. To get an irrationality measure for θ *via* Lemma 3.5 we need an upper bound for $H_0(\alpha_n\theta_n)$, $\|\alpha_n\theta_n\|$ and for the corresponding dashed quantities. By symmetry, it is enough to consider the last ones; we can also assume n even. By 2) and 3) of Remark 3.2 we have

$$H_0(\alpha'_n\theta^n) \le H_0(\alpha'_n)H(\theta)^n, \qquad \|\alpha'_n\theta^n\| \le H_0(\alpha'_n).$$

By the first inequality in (3.3) of Lemma 3.4, $H_0(\alpha_n) \ll H(\mathbf{u}_n)$ and $H_0(\alpha'_n) \ll H(\mathbf{u}'_n)$. Thus by (4.2)

$$H_0(\alpha_n)H_0(\alpha'_n) \ll H_0(\theta)^n$$

and

$$\begin{aligned} H_0(\alpha'_n\theta^n) &\ll H_0(\alpha_n)^{-1}H_0(\theta)^{2n} \\ \|\alpha'_n\theta^n\| &\ll H_0(\alpha_n)^{-1}H_0(\theta)^n\|\theta\|^n. \end{aligned} \tag{4.3}$$

As in section 2, we need a lower bound for $H_0(\alpha_n)$ (note that this represents the crucial issue). Put

$$\alpha_{n,j} := u_{n,0} + u_{n,1}\theta_j + u_{n,2}\theta_j^2 \quad \text{for } j = 1,2,3.$$

We have $\boldsymbol{\theta} \in L$; thus

$$\beta_1^{-1}\theta_1 + \beta_2^{-1}\theta_2 + \beta_3^{-1}\theta_3 = 0. \tag{4.4}$$

Moreover $\alpha_n\theta^n \in \mathbb{Q}+\mathbb{Q}\xi$ and thus

$$(\alpha_{n,1}\theta_1^n : \alpha_{n,2}\theta_2^n : \alpha_{n,3}\theta_3^n) \in L',$$

where L' is the line in $\mathbb{P}_2$ through the points $\mathbf{1}$ and $\boldsymbol{\xi}$, of equation

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ 1 & 1 & 1 \\ \xi_1 & \xi_2 & \xi_3 \end{vmatrix} = \xi'_1x_1 + \xi'_2x_2 + \xi'_3x_3 = 0$$

with $\xi' = (\xi_3-\xi_2, \xi_1-\xi_3, \xi_2-\xi_1)$. Thus

$$\xi'_1\alpha_{n,1}\theta_1^n + \xi'_2\alpha_{n,2}\theta_2^n + \xi'_3\alpha_{n,3}\theta_3^n = 0. \tag{4.5}$$

Let

$$A = -\beta_2^{-1}\theta_2/\beta_1^{-1}\theta_1, \quad B = -\beta_3^{-1}\theta_3/\beta_1^{-1}\theta_1$$

and

$$a = -\xi'_2\alpha_{n,2}\beta_2^n/\xi'_1\alpha_{n,1}\beta_1^n, \quad b = -\xi'_3\alpha_{n,3}\beta_3^n/\xi'_1\alpha_{n,1}\beta_1^n.$$

Recall that we assume n even. By (4.4) and (4.5) we have

$$A+B=1, \qquad aA^{2\rho}+bB^{2\rho}=1$$

with $\rho = n/2 \in \mathbb{N}$. By the already quoted [4], Lemma 2.3,

$$H(1:A:B) \le c\,2^{2/n}H(1:a:b)^{2/n}.$$

with $c = 6\sqrt{3}$. We have

$$\begin{aligned} H(1:A:B) &= H(\beta_1^{-1}\theta_1 : \beta_2^{-1}\theta_2 : \beta_3^{-1}\theta_3) \\ &\geq H(\boldsymbol{\beta})^{-1}H(\boldsymbol{\theta}) = H(\boldsymbol{\beta})^{-1}H_0(\theta) \end{aligned}$$

and

$$\begin{aligned} H(1:a:b) &= H(\xi_1'\alpha_{n,1}\beta_1^n : \xi_2'\alpha_{n,2}\beta_2^n : \xi_3'\alpha_{n,3}\beta_3^n) \\ &\leq H(\boldsymbol{\xi}')H(\boldsymbol{\beta})^n H(\boldsymbol{\alpha}_n) = H(\boldsymbol{\xi}')H(\boldsymbol{\beta})^n H_0(\alpha_n). \end{aligned}$$

Thus

$$H(\boldsymbol{\beta})^{-n/2}H_0(\theta)^{n/2} \leq 2c^{n/2}H(\boldsymbol{\xi}')H(\boldsymbol{\beta})^n H_0(\alpha_n) \ll c^{n/2}H(\boldsymbol{\beta})^n H_0(\alpha_n)$$

and finally

$$H_0(\alpha_n)^{-1} \ll c^{n/2}H(\boldsymbol{\beta})^{3n/2}H_0(\theta)^{-n/2} = c(L)^n H_0(\theta)^{-n/2}.$$

By (4.3),

$$H_0(\alpha_n'\theta^n) \ll H_0(\alpha_n)^{-1}H_0(\theta)^{2n} \ll \big(c(L)H_0(\theta)^{3/2}\big)^n$$

and

$$\begin{aligned} \|\alpha_n'\theta^n\| &\ll H_0(\alpha_n)^{-1}H_0(\theta)^n\|\theta\|^n \ll \big(c(L)H_0(\theta)^{1/2}\|\theta\|\big)^n \\ &= \big(c(L)H_0(\theta)^{3/2}\big)^{-\omega n} \end{aligned}$$

with

$$\omega = \frac{\log(1/\|\theta\|) - \frac{1}{2}h_0(\theta) - \log c(L)}{\frac{3}{2}h_0(\theta) + \log c(L)}.$$

Since $\omega > 0$ by (4.1), Lemma 3.5 (with $r = 3$ and $s = 2$) shows that ξ has irrationality measure $\leq 1 + 1/\omega$.

□

Remark 4.2. *We get a non-trivial*[2] *effective irrationality measure, if*

$$\|\theta\| < c(L)^{-3/2}H_0(\theta)^{-5/4}.$$

We get effective irrationality measures ≤ 2 *if*

$$\|\theta\| \leq c(L)^{-2}H_0(\theta)^{-2}.$$

Example 4.3.
1. In the situation of section 2, $\xi_l = \sqrt[3]{l^3+1}$, $\theta = \xi_l - l \sim \frac{1}{3}l^{-2}$ *and* L *is the projective line of equation*

$$x_1 + \zeta x_2 + \zeta^2 x_3 = 0$$

The point $\boldsymbol{\beta} = (1 : \zeta^{-1} : \zeta^{-2})$ *has height* 1 *and thus* $c(L) = (6\sqrt{3})^{1/2}$. *We have* $\theta_1\theta_2\theta_3 = 1$ *and* $|\theta_2| = |\theta_3|$. *Thus* $\|\theta\| = \theta$ *and* $H_0(\theta) = \theta^{-1/2}$.

[2] *i. e.* better than the bound $\mu_{\rm eff}(\xi) \leq 3$ given by Liouville's Theorem

Inequality (4.1) *becomes* $\theta < (6\sqrt{3})^{-2/3} \approx 0.209...$, *which is satisfied for* $l \geq 2$. *In this case, Theorem 4.1 gives*

$$\begin{aligned}\mu_{\text{eff}}(\xi_l) &\leq 1 + \frac{\frac{3}{2}h_0(\theta) + \log c(L)}{\log(1/\|\theta\|) - \frac{1}{2}h_0(\theta) - \log c(L)} \\ &= 1 + \frac{-\frac{3}{4}\log(\xi_l - l) + \frac{1}{2}\log(6\sqrt{3})}{-\frac{3}{4}\log(\xi_l - l) - \frac{1}{2}\log(6\sqrt{3})} \\ &= 2 + \frac{2\log(6\sqrt{3})}{3\log l} + O((\log l)^{-2}) \text{ for } l \to +\infty.\end{aligned}$$

2. Another example is given by irrational cubic numbers of trace 0, *e.g.* $\theta = \eta_b \sim b^{-1}$ *is the real root of* $x^3 + bx - 1 = 0$ *for some integer* $b \geq 5$. *Now* L *is the projective line of equation*

$$x_1 + x_2 + x_3 = 0.$$

The same computation shows that

$$\begin{aligned}\mu_{\text{eff}}(\theta) &= 1 + \frac{-\frac{3}{4}\log(\eta_b) + \frac{1}{2}\log(6\sqrt{3})}{-\frac{3}{4}\log(\eta_b) - \frac{1}{2}\log(6\sqrt{3})} \\ &= 2 + \frac{4\log(6\sqrt{3})}{3\log l} + O((\log l)^{-2}) \text{ for } b \to +\infty.\end{aligned}$$

5. Conjugates lying on a fixed curve.

We want to generalize Theorem 4.1 replacing L by any fixed projective curve $\mathcal{C}$. To do that, in place of the Beukers-Schlickewei estimate in [4] used above, we need a recent result of Masser and the authors of this article.

Theorem 5.1. *Let* $\mathcal{C} \subset \mathbb{P}_{r-1}$ *be a projective curve defined over* $\overline{\mathbb{Q}}$. *There exists a* $c = c(\mathcal{C}) > 1$ *effectively depending on* $\mathcal{C}$ *such that the following holds. Let* $\boldsymbol{\alpha} \in \mathbb{P}_{r-1}(\overline{\mathbb{Q}})$, $\boldsymbol{\theta} \in \mathcal{C}(\overline{\mathbb{Q}})$ *and choose* $K \geq c$ *and a natural number* $n \geq K$. *Then, if*

$$\alpha_1\theta_1^n + \cdots + \alpha_r\theta_r^n = 0$$

and if there are no proper vanishing subsums,

$$\frac{h(\boldsymbol{\alpha})}{n} \geq \frac{h(\boldsymbol{\theta})}{r-1} - c\left(\frac{1}{K}h(\boldsymbol{\theta}) + h(\boldsymbol{\theta})^{1/2} + K\right). \tag{5.1}$$

Proof. Let $f_1, \ldots, f_r \in \overline{\mathbb{Q}}(\mathcal{C})$ be the coordinate functions on $\mathcal{C}$. Then f_i/f_j is non-constant for some $i \neq j$. We apply [2, Theorem 4.1], taking into account [2, Remark 4.2 i) and ii)]. □

The explicit computation of $c(\mathcal{C})$ is an open question. However, the following couple of exemples may suggest a polynomial dependence on the degree and on the height of the polynomial defining $\mathcal{C}$.

For simplicity we assume $r = 3$ (which is the relevant case for this article) and $\alpha_1 = \alpha_2 = \alpha_3 = 1$. A short computation shows that Theorem 5.1 then gives $h(\boldsymbol{\theta}) \leq K^2$ provided that $n \geq K := 144c^2$.

Take first for $\mathcal{C}$ the affine curve $x+y = \gamma$, where γ is an algebraic number, and let $(\theta_1, \theta_2) \in \mathcal{C}$ such that $\theta_1^n + \theta_2^n = 1$, where for simplicity $n \in \mathbb{N}$ is even. Then, by [14, Exercices 17.3 and 17.4] (apply Beukers-Schlickewei

Lemma 2.2 with $A = \theta_1/\gamma$, $B = \theta_2/\gamma$, and $a = b = \gamma^n$), we easily see that $h(1 : \theta_1 : \theta_2) \leq c_1 + c_2 h(\gamma)$ with c_1, $c_2 > 0$ absolute.

Concerning the dependence on the degree, take now for $\mathcal{C}$ the affine curve defined by $P = 0$, where

$$P(x, y) := \prod_{\omega^d=1} \prod_{\eta^d=1} (\omega x^{1/d} + \eta y^{1/d} - 1) \in \mathbb{Z}[x, y]$$

has degree d, and let $(\theta_1, \theta_2) \in \mathcal{C}$ such that $\theta_1^n + \theta_2^n = 1$, where for simplicity n is now a positive multiple of $2d$. Then we again easily deduce from Beukers-Schlickewei Lemma 2.2 that $h(1 : \theta_1 : \theta_2) \leq c_3 + c_4 d$, with c_3, $c_4 > 0$ absolute.

Replacing in our construction [4], Lemma 2.3, by Theorem 5.1 with $r = 3$ we get the following statement. Theorem 1.1 announced in the introduction immediately follows from it.

Theorem 5.2. *Let $\mathbb{K} \subseteq \mathbb{C}$ be a real cubic number; denote by $\sigma_1 = \mathrm{Id}, \sigma_2, \sigma_3$ the immersions $\mathbb{K} \hookrightarrow \mathbb{C}$. Let $\mathcal{C} \subset \mathbb{P}_2$ be a projective curve defined over $\overline{\mathbb{Q}}$ and let $\theta \in \mathbb{K}$ such that*

$$(\sigma_1\theta : \sigma_2\theta : \sigma_3\theta) \in \mathcal{C}(\overline{\mathbb{Q}}).$$

Let $c = c(\mathcal{C})$ be the constant in Theorem 5.1, and $\hbar > 0$, $\lambda > 1/2$ and $\varepsilon \in (0, 1)$ be parameters such that

$$\hbar \geq c^2 \min\left(1, \frac{(\lambda - 1/2)^2}{8(\lambda + 1)}\varepsilon\right)^{-2}. \tag{5.2}$$

We assume

$$h_0(\theta) \leq \hbar, \qquad \log \|\theta\| \leq -\lambda\hbar. \tag{5.3}$$

Then

$$\mu_{\text{eff}}(\xi) \leq \frac{\lambda + 1}{\lambda - 1/2} + \varepsilon$$

for every generator ξ of $\mathbb{K}$.

Proof. We argue as in the proof of Theorem 4.1, keeping the same notations. Recall equation (4.5),

$$\xi_1'\alpha_{n,1}\theta_1^n + \xi_2'\alpha_{n,2}\theta_2^n + \xi_3'\alpha_{n,3}\theta_3^n = 0.$$

We apply Theorem 5.1 with $r = 3$ and with $\boldsymbol{\alpha}$ replaced by $\tilde{\boldsymbol{\alpha}}_n := (\xi_1'\alpha_{n,1} : \xi_2'\alpha_{n,2} : \xi_3'\alpha_{n,3})$. Remark that the condition on no proper vanishing subsums is trivially satisfied since $r = 3$ (indeed ξ_j', $\alpha_{n,j}$ and θ_j are non zero). By this theorem, for any $K \geq c$ and for any integer $n \geq K$,

$$\begin{aligned} \frac{1}{n}h(\tilde{\boldsymbol{\alpha}}_n) &\geq \frac{1}{2}h_0(\theta) - c\left(\frac{1}{K}h_0(\theta) + h_0(\theta)^{1/2} + K\right) \\ &\geq \frac{1}{2}h_0(\theta) - c\left(\frac{1}{K} + \frac{1}{\hbar^{1/2}} + \frac{K}{\hbar}\right)\hbar \end{aligned}$$

with $c = c(\mathcal{C})$ the constant appearing in that theorem. Let $\varepsilon' \in (0, 1)$, which will be fixed later, and choose $K = c/\varepsilon'$. Then for $\hbar \geq (c/\varepsilon')^2$ and $n \geq c/\varepsilon'$,

$$\frac{1}{n}h(\tilde{\boldsymbol{\alpha}}_n) \geq \frac{1}{2}h_0(\theta) - 3\varepsilon'\hbar.$$

By equation (4.3) and since $h(\tilde{\boldsymbol{\alpha}}_n) \leq h(\boldsymbol{\xi}') + h_0(\alpha_n)$, we have

$$\begin{aligned} \frac{1}{n} h_0(\alpha_n'\theta_n) &\leq \frac{h(\boldsymbol{\xi}')}{n} - \frac{1}{n} h(\tilde{\boldsymbol{\alpha}}_n) + 2h_0(\theta) \\ \frac{1}{n}\log\|\alpha_n'\theta_n\| &\leq \frac{h(\boldsymbol{\xi}')}{n} - \frac{1}{n} h(\tilde{\boldsymbol{\alpha}}_n) + h_0(\theta) + \log\|\theta\|. \end{aligned} \tag{5.4}$$

Thus, if in addition $n \geq \frac{h(\boldsymbol{\xi}')}{\varepsilon'\hbar}$, we obtain from (5.4) and (5.3)

$$\frac{1}{n} h_0(\alpha_n'\theta_n) \leq \varepsilon'\hbar - \frac{1}{2} h_0(\theta) + 3\varepsilon'\hbar + 2h_0(\theta) \leq \left(\frac{3}{2} + 4\varepsilon'\right)\hbar$$

and

$$\begin{aligned} \frac{1}{n}\log\|\alpha_n'\theta_n\| &\leq \varepsilon'\hbar - \frac{1}{2}h_0(\theta) + 3\varepsilon'\hbar + h_0(\theta) - \lambda\hbar \\ &\leq \left(\frac{1}{2} - \lambda + 4\varepsilon'\right)\hbar = -\omega\left(\frac{3}{2} + 4\varepsilon'\right)\hbar \end{aligned}$$

with

$$\omega = \frac{\lambda - 1/2 - 4\varepsilon'}{3/2 + 4\varepsilon'}.$$

Note that $\omega > 0$ if $4\varepsilon' < \lambda - 1/2$. Assume further $8\varepsilon' < \lambda - 1/2$. Lemma 3.5 (with $r = 3$ and $s = 2$) then shows that ξ has irrationality measure

$$\begin{aligned} \leq 1 + 1/\omega &= \frac{\lambda+1}{\lambda - 1/2} + \frac{4(\lambda+1)\varepsilon'}{(\lambda-1/2)(\lambda - 1/2 - 4\varepsilon')} \\ &\leq \frac{\lambda+1}{\lambda-1/2} + \frac{8(\lambda+1)\varepsilon'}{(\lambda-1/2)^2}. \end{aligned}$$

The result follows on choosing $\varepsilon' := \min\left(1, \frac{(\lambda-1/2)^2}{8(\lambda+1)}\varepsilon\right) \leq \frac{1}{8}(\lambda - 1/2)$.

□

Example 5.3. *As an easy example, let $l > 1$ be an integer and consider the cubic polynomial $x^3 + x^2 + \frac{1}{l}x + \frac{1}{l^6}$, which is easily seen to be an irreducible polynomial over the rationals. Take for θ_l the root of $x^3 + x^2 + \frac{1}{l}x + \frac{1}{l^6}$ closest to* 0*. The projective point $P_l \in \mathbb{P}_2$ defined by the conjugates of θ_l lies on the projective curve $(x_1x_2 + x_1x_3 + x_2x_3)^6 - (x_1 + x_2 + x_3)^9 x_1x_2x_3 = 0$, which is absolutely irreducible and of genus* 10*, according to the computer algebra system* [11]*. Moreover, an easy computation shows that the above conjugates have all absolute value ≤ 1 and that $\log|\theta_l| \sim -5\log l$ as $l \to +\infty$. By* (3.2)*, $H_{\text{finite}}(\theta_l) = l^2$. Thus*

$$h_0(\theta_l) \leq (2 + \varepsilon_1(l))\log l, \qquad \text{and} \qquad \log\|\theta_l\| = -(3 + \varepsilon_2(l))\log l$$

for some $\varepsilon_1(l), \varepsilon_2(l) \to 0$ as $l \to +\infty$ (and it is possible to show that the first inequality is indeed an equality). We fix $\varepsilon > 0$ and we apply Theorem 5.2 choosing

$$\mathbb{K} = \mathbb{K}_l = \mathbb{Q}(\theta_l), \quad \hbar = \hbar_l = (2 + \varepsilon_1(l))\log l \quad \text{and} \quad \lambda = \lambda_l = \frac{3 + \varepsilon_2(l)}{2 + \varepsilon_1(l)}.$$

Let us assume l sufficiently large with respect to $1/\varepsilon$. Since $\lambda_l \to \frac{3}{2}$ as $l \to +\infty$, assertions (5.2) *and* (5.3) *are satisfied and we get*

$$\mu_{\text{eff}}(\theta_l) \leq \frac{\lambda_l + 1}{\lambda_l - 1/2} + \varepsilon \leq \frac{5}{2} + 2\varepsilon.$$

Remark 5.4. *Let E be an elliptic curve (of rank > 0) and $u, v\colon E \dashrightarrow \mathbb{P}_1$ be non constant morphisms. It would be tempting to apply Theorem 5.2 to the family of cubic equations $x^3 + x^2 + u(P)x + v(P) = 0$ parametrized by $P \in E(Q)$, but unfortunately this does not work, as we briefly show. Let θ_P be the root of $x^3 + x^2 + u(P)x + v(P) = 0$ closest to 0. Then, using "Weil's Height Machine" and a conjecture of Lang proved by David and Hirata-Kohno* [10, Conjecture 1.2]*, it is possible to show that $\log \|\theta_P\| \sim \frac{d}{6}\hat{h}(P) \to +\infty$ with $d = -\deg \operatorname{div}(1, u^3, v^2)$, preventing any application of Theorem 5.2.*

Back to cubic equations parametrized by rational points on a curve of genus 0. The example above can be generalized to find an effective upper bound for the irrationality measure of the values of a cubic algebraic function holomorphic at 0. More precisely, let g be a power series in the variable t, with rational coefficients, representing a cubic algebraic function, i.e. g has degree 3 over $\overline{\mathbb{Q}}(t)$. We assume that the conjugates $g_1 = g$, g_2, g_3 of g over $\overline{\mathbb{Q}}(t)$, are represented by analytic functions in the closed disk $|t| \leq e^{-c_0}$, for some $c_0 > 0$. We have:

Corollary 5.5. *Let $\varepsilon \in (0, 1/12)$. There exists $C(\varepsilon, g) \geq c_0 \geq 1$ such that the following holds. Let $a \in \mathbb{Z}$, $b \in \mathbb{N}$ coprime, with*

$$\log |a| \leq \varepsilon \log b \tag{5.5}$$

and

$$\log b \geq C(\varepsilon, g). \tag{5.6}$$

Then $g(a/b)$ is irrational with effective irrationality measure $\leq 2 + 13\varepsilon$.

Proof. Let $t_0 = a/b$. Then, by (5.5), (5.6) and since $\varepsilon < 1/2$,

$$\log |t_0| = \log |a| - \log b \leq -(1 - \varepsilon) \log b \leq -(\log b)/2 \leq -c_0$$

if $C(\varepsilon, g) \geq c_0^2$. Observe also that $H(t_0) = b$, since a, b are coprime and $|t_0| < 1$. We denote by $c_1, \ldots, c_5$ positive constants depending only on g and on ε.

By the functorial properties of the height, setting $\xi = g(t_0)$, we have

$$H(\xi) \leq e^{c_1} b^d, \tag{5.7}$$

where $d \geq 1$ is the geometric degree of the algebraic function g (*i. e.* the degree of its polar divisor).

Let $N = [6d/\varepsilon]$. We consider a Padé approximant at 0 of order N for $(1, g, g^2)$, *i. e.* a non-zero vector of polynomials $(Q_0, Q_1, Q_2) \in \mathbb{Q}[t]$ of degree at most N such that

$$f = Q_0 + Q_1 g + Q_2 g^2$$

vanishes at 0 with multiplicity $\geq [3(N + 1)] \geq 3N$. This implies

$$\begin{aligned} H(Q_0(t_0) : Q_1(t_0) : Q_2(t_0)) &\leq e^{c_2} b^N \\ |f(t_0)| &\leq e^{c_2} |t_0|^{3N}, \end{aligned} \tag{5.8}$$

where c_2 depends only on the Q_i. Moreover $f \neq 0$ since g has degree 3 over $\mathbb{Q}(t)$. Since f is analytic in $D(0, e^{-c_0})$, there exists $c_3 \geq c_0$ such that f does not vanish for $0 < |t| \leq e^{-c_3}$. Thus, if $C(\varepsilon, g) \geq c_3$ then $\theta := f(t_0)$ is a non zero algebraic number of degree ≤ 3. By (5.7), (5.8) and by (3.3) of Lemma 3.4 we have

$$\begin{aligned} H_0(\theta) &\leq 3H(\xi)^6 H(Q_0(t_0) : Q_1(t_0) : Q_2(t_0)) \leq e^{c_4} b^{(1+\varepsilon)N}, \\ \|\theta\| &\leq H(\xi)^6 H(Q_0(t_0) : Q_1(t_0) : Q_2(t_0)) |\theta| \leq e^{c_4} b^{(1+\varepsilon)N} |t_0|^{3N}. \end{aligned} \tag{5.9}$$

For $j = 1, 2, 3$ let $f_j = Q_0 + Q_1 g_j + Q_2 g_j^2$ and $\theta_j = f_j(t_0)$. We want to apply Theorem 5.2 with $\mathcal{C} \subseteq \mathbb{P}_2$ be the projective curve parametrized by $(f_1 : f_2 : f_3)$. Let $c > 0$ be the constant appearing in that theorem. Set $c_5 := \max(c_3, c_4/N, c)$ and

$$\kappa := \frac{\log(b/|a|)}{(1+\varepsilon)\log b + c_5}.$$

Set also

$$\hbar := ((1+\varepsilon)\log b + c_5)N > 0, \quad \lambda := 3\kappa - 1$$

and choose $C(\varepsilon, g) := (\frac{32c_5}{\varepsilon})^2$. Remark that

$$1 - \kappa = \frac{\varepsilon \log b + \log|a| + c_5}{(1+\varepsilon)\log b + c_5} \leq 3\varepsilon. \tag{5.10}$$

by (5.5) and (5.6). In particular

$$\kappa \geq 1 - 3\varepsilon \geq 3/4 \tag{5.11}$$

since $\varepsilon \leq 1/12$. This implies

$$\frac{(\lambda - 1/2)^2}{8(\lambda+1)}\varepsilon = \frac{3(\kappa - 1/2)^2}{8\kappa}\varepsilon \geq \frac{\varepsilon}{32}.$$

Thus

$$c^2 \min\Big(1, \frac{(\lambda - 1/2)^2}{8(\lambda+1)}\varepsilon\Big)^{-2} \leq \Big(\frac{32c_5}{\varepsilon}\Big)^2 = C(\varepsilon, g) \leq \log b \leq \hbar$$

by (5.6). Assumption (5.2) is satisfied. Moreover, by (5.9) we have $h_0(\theta) \leq \hbar$ and

$$-\frac{\log\|\theta\|}{\hbar} \geq \frac{3N\log(b/|a|) - (1+\varepsilon)N\log b - c_5 N}{((1+\varepsilon)\log b + c_5)N} = 3\kappa - 1 = \lambda.$$

Thus (5.3) is also satisfied.

By Theorem 5.2, θ is a cubic irrational and every generator of $\mathbb{Q}(\theta)$ has irrationality measure

$$\leq \frac{\lambda+1}{\lambda - 1/2} + \varepsilon = \frac{\kappa}{\kappa - 1/2} + \varepsilon = 2 + \frac{1-\kappa}{\kappa - 1/2} + \varepsilon \leq 2 + 13\varepsilon$$

where we have plugged in (5.10) and (5.11) into the computation.

We finally remark that ξ is a generator of $\mathbb{Q}(\theta)$, since $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(\xi)$ and $[\mathbb{Q}(\xi) : \mathbb{Q}] \leq 3 = [\mathbb{Q}(\theta) : \mathbb{Q}]$.

□

Let g be an arbitrary algebraic function (of degree possibly > 3). In [9, Theorem 1], Chudnovski announced the upper bound $\mu_{\rm eff}(g(a/b)) \leq 2+\varepsilon$ for every $\varepsilon > 0$ provided that $b^\varepsilon \geq c_1(\varepsilon, g)|a|^{2+\varepsilon}$. The Author presents a complete proof only in the special case of a cubic algebraic function; his proof rests on the well-known fact that cubic algebraic functions satisfies a Riccati equation. A standard computation, starting from the choice of the parameter N at the end of the proof of his Theorem 1, shows that his method is indeed able to provide a more explicit result:

$$(5.12) \quad \mu_{\rm eff}(g(a/b)) \leq 2 + \frac{2c(g)}{\varepsilon\sqrt{\log b}} + \frac{2}{\varepsilon}\cdot\frac{\log|a|}{\log b}, \qquad \text{if } |a| \leq b^{1/2-\varepsilon} \text{ and } \log b \geq 4c(g)^2/\varepsilon^2.$$

Our method (at the price of an explicit computation of the Padé approximation f and of a more precise zero's lemma, see the Appendix for details) can provide an estimate of a similar shape:

$$(5.13) \quad \mu_{\rm eff}(g(a/b)) \leq 2 + \frac{9}{\varepsilon}\sqrt{\frac{c(g)}{\log b}} + \frac{2}{\varepsilon}\cdot\frac{\log|a|}{\log b} \qquad \text{if } |a| \leq b^{1/2-\varepsilon} \text{ and } \log b \geq 2^{10}c(g)^3/\varepsilon^4.$$

In this context, it is not out of place to mention that Zudilin [15] obtained a general statement on the values of G-functions, and thus in particular applying to algebraic functions. A very special case of his result gives

$$\mu_{\rm eff}(g(a/b)) \leq 2 + c(g)\left(\frac{1}{(\log b)^{1/3}} + \frac{\log|a|}{\log b}\right),$$

for an algebraic function g (of arbitrary large degree) and a rational a/b, provided that $\log|a| < \frac{1}{2}\log b - c(g)(\log b)^{2/3}$ and $\log b > c(g)$. This is a weaker version of (5.12) (essentially $(\log b)^{1/3}$ instead of $\sqrt{\log b}$), which is not surprising, since his estimate is a specialization of a more general result.

6. Appendix

The aim of this appendix is to provide a complete proof of the bound (5.13) announced in the last section.

Let g be a cubic algebraic function, regular at 0 with rational Taylor coefficients. We choose a determination of the conjugates $g_1 = g$, g_2, g_3 of g over $\overline{\mathbb{Q}}(t)$, which we assume defined (and analytic) in the closed disk $|t| \leq e^{-c_0}$, for some $c_0 \geq 1$.

Proposition 6.1. *There exists $C = C(g) \geq c_0 \geq 1$ such that the following holds. Let ε_1, ε_2, ε_3, $\varepsilon_4 \in (0,1)$ be such that $\varepsilon_2+\varepsilon_3+\varepsilon_4 < 1/2$. Let $a \in \mathbb{Z}$,*

$b \in \mathbb{N}$ *coprime, with*

$$\log b \geq \begin{cases} \log|a| + \dfrac{C}{\varepsilon_2\varepsilon_3}, & (a) \\ \dfrac{\log|a| + C/\varepsilon_3}{1/2 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4}, & (b) \\ \dfrac{\varepsilon_2 C^2}{\varepsilon_1^2\varepsilon_4^4}. & (c) \end{cases} \tag{6.1}$$

Let also

$$\kappa = \frac{(1-\varepsilon_3)\log(b/|a|)}{(1+\varepsilon_2)\log b + C/\varepsilon_3}. \tag{6.2}$$

Then $g(a/b)$ is irrational with effective irrationality measure

$$\leq \frac{\kappa}{\kappa - 1/2} + \varepsilon_1 \leq 2 + \varepsilon_1 + \frac{1}{\varepsilon_4}\left(\varepsilon_2 + \varepsilon_3 + \frac{\log|a| + C/\varepsilon_3}{\log b}\right).$$

Proof. We first make some elementary remarks on the parameters. By definition (6.2) of κ we have

$$\begin{aligned} 1 - \kappa &= \frac{(\varepsilon_2+\varepsilon_3)\log b + (1-\varepsilon_3)\log|a| + C/\varepsilon_3}{(1+\varepsilon_2)\log b + C/\varepsilon_3} \\ &\leq \varepsilon_2 + \varepsilon_3 + \frac{\log|a| + C/\varepsilon_3}{\log b}. \end{aligned} \tag{6.3}$$

Thus $1-\kappa > 0$ and $1 - \kappa \leq \frac{1}{2} - \varepsilon_4$ by (6.1b), *i. e.*

$$1/2 + \varepsilon_4 \leq \kappa < 1 \tag{6.4}$$

Let now $t_0 = a/b$. Then, by (6.1a), $\log b \geq \log|a| + C$ and

$$|t_0| \leq \frac{1}{2}e^{-c_0}, \tag{6.5}$$

assuming $C \geq c_0 + \log 2$. Remark also that $H(t_0) = b$. We denote by c_1, ..., c_9 positive constants depending only on g. By the functorial properties of the height,

$$H(\xi) \leq e^{c_1} b^d \tag{6.6}$$

where $\xi = g(t_0)$ and where where $d \geq 1$ is the geometric degree of the algebraic function g (*i. e.* the degree of its polar divisor).

Let N be a positive integer and $\delta \in [1/(N+1), 3)$. We consider, in the terminology of [7], a (N,δ)-Padé approximant at 0 for $(1, g, g^2)$, *i. e.* a non-zero vector of polynomials $(Q_0, Q_1, Q_2) \in \mathbb{Q}[t]$ of degree at most N such that

$$f = Q_0 + Q_1 g + Q_2 g^2$$

vanishes at 0 with multiplicity $M \geq [(3-\delta)(N+1)]$. By Theorem 2 of [7] we can find a such vector of polynomials satisfying moreover

$$\max_j h(Q_j) \leq \frac{c_2(3-\delta)^2(N+1)}{\delta}.$$

We choose

$$N := [6d/\varepsilon_2] + 1 \quad \text{and} \quad \delta := \frac{3(\varepsilon_3 N + 1) - 1}{N+1}.$$

Since $\varepsilon_3 \leq 1$ we have $\delta \in [1/(N+1), 3)$ as needed. Moreover

$$M \geq (3-\delta)(N+1) - 1 = 3(1-\varepsilon_3)N \tag{6.7}$$

and

$$\delta \geq \frac{2(\varepsilon_3 N + 1) - 1}{N+1} = \frac{2\varepsilon_3 N + 1}{N+1} \geq \varepsilon_3.$$

Thus

$$\max_j h(Q_j) \leq \frac{c_3 N}{\varepsilon_3}. \tag{6.8}$$

Remark that this implies

$$|f|_R \leq e^{c_4 N/\varepsilon_3} \text{ on the disk } |t| \leq R := e^{-c_0} \leq 1. \tag{6.9}$$

Let $\theta := f(t_0)$. We want first show that θ is non-zero. Let us assume the contrary. Let $f(t) = \sum_{k=0}^{\infty} f_k t^k$ and $g(t)^j = \sum_{k=0}^{\infty} a_{j,k} t^k$ $(j = 1, 2)$ be the expansions of f and of g, g^2 around $t = 0$. Let also $a_{0,k} = 1$ if $k = 0$ and $a_{0,k} = 0$ otherwise. Writing $Q_j(t) = \sum_{l=0}^{N} q_{j,l} t^l$ for $j = 0, 1, 2$ we have

$$f_M = \sum_{j=0}^{2} \sum_{l=0}^{N} q_{jl} a_{j,M-l} \neq 0$$

since f vanishes at 0 with multiplicity exactly M. Let v be a place of $\mathbb{Q}$. The Local Eisenstein Theorem ([7, Corollary p.161]) gives $|a_{j,k}|_v \leq c'(v) R_v^{2k-1}$ for some $c'(v)$, R_v depending on v and on g, with moreover $c'(v) = 1$ if v is finite. Thus, by (6.8)

$$h(f_M) \leq c_5 M + \max_j h(Q_j) \leq c_5 M + c_3 N/\varepsilon_3$$

and, by Liouville's inequality,

$$\log |f_M| \geq -c_5 M - c_3 N/\varepsilon_3.$$

On the other hand, $f_M t_0^M = -\sum_{j>M} f_j t_0^j$ since we are assuming $f(t_0) = 0$. Let as before $R := e^{-c_0} \leq 1$. Using Cauchy's estimates $|f_j| \leq |f|_R R^{-j}$, we get

$$\begin{aligned} |f_M| &\leq |t_0|^{-M} \sum_{j=M+1}^{\infty} |f|_R (|t_0|/R)^j = |t_0|^{-M} |f|_R \frac{(|t_0|/R)^{M+1}}{1 - |t_0|/R} \\ &\leq 2(1/R)^{M+1} e^{c_4 N/\varepsilon_3} |t_0| \end{aligned}$$

by (6.9) and since $|t_0| \leq |R|/2$ by (6.5). Comparing the lower and the upper bound for $|f_M|$ we find

$$\log(1/|t_0|) \leq \log(2/R) + (\log(1/R) + c_5)M + (c_3 + c_4)\frac{N}{\varepsilon_3}.$$

We still need an (elementary) zero's estimate to bound M. Since the Q_j's are polynomials of degree $\leq N$, the polar divisor of f has degree $\leq c_6 N$.

Thus[3] $M \le c_6 N$. Inserting this bound in the last displayed formula we get

$$\log(1/|t_0|) \le \log(2/R) + (\log(1/R) + c_5)c_6 N + (c_3 + c_4)\frac{N}{\varepsilon_3} < \frac{c_7}{\varepsilon_2\varepsilon_3}$$

since $N \le 6d/\varepsilon_2 + 1$ by our choice. This contradicts (6.1a) provided that $C \ge c_7$. The proof of $\theta \ne 0$ is concluded.

We now prove an upper bound for $H_0(\theta)$ and for $\|\theta\|$. Inequality (6.8) implies

$$H(Q_0(t_0) : Q_1(t_0) : Q_2(t_0)) \le e^{c_8 N/\varepsilon_3} b^N. \tag{6.10}$$

Using Schwarz lemma in the disk $|t| \le R := e^{-c_0} \le 1$ and inequalities (6.9), (6.7), we get

$$|\theta| \le |f|_R (|t_0|/R)^M \le e^{c_4 N/\varepsilon_3} (|t_0|/R)^{3(1-\varepsilon_3)N}. \tag{6.11}$$

By (3.3) of Lemma 3.4, by (6.6), (6.10), (6.11), and since $N \ge 6d\varepsilon_2$, we have

$$H_0(\theta) \le 3H(\xi)^6 H(Q_0(t_0) : Q_1(t_0) : Q_2(t_0)) \le e^{c_9 N/\varepsilon_3} b^{(1+\varepsilon_2)N}, \tag{6.12}$$

and

$$\begin{aligned} \|\theta\| &\le H(\xi)^6 H(Q_0(t_0) : Q_1(t_0) : Q_2(t_0))|\theta| \\ &\le e^{c_9 N/\varepsilon_3} b^{(1+\varepsilon_2)N} |t_0|^{3(1-\varepsilon_3)N}. \end{aligned} \tag{6.13}$$

We are quite in position to apply Theorem 5.2. Assuming $C \ge c_9$ and taking into account (6.13), (6.2), (6.4), we have

$$\begin{aligned} -\frac{1}{N}\log\|\theta\| &\ge 3(1-\varepsilon_3)\log(b/|a|) - (1+\varepsilon_2)\log b - C/\varepsilon_3 \\ &= ((1+\varepsilon_2)\log b + C/\varepsilon_3)(3\kappa - 1) > 0. \end{aligned}$$

Liouville's inequality (Lemma 3.3) implies that θ is irrational and thus ξ is irrational as well, since $\theta \in \mathbb{Q}(\xi)$. If ξ is a quadratic irrational our result is trivial, since its effective irrationality measure is 2 which is $\le \frac{\kappa}{\kappa - 1/2}$ by (6.4). Thus we may assume that ξ is a cubic irrational. Let $\mathbb{K} = \mathbb{Q}(\xi)$, denote by $\sigma_1 = \mathrm{Id}, \sigma_2, \sigma_3$ the immersions $\mathbb{K} \hookrightarrow \mathbb{C}$ and put $\theta_j = f_j(t_0)$.

We apply Theorem 5.2 with $\mathcal{C} \subseteq \mathbb{P}_2$ be the projective curve parametrized by $(f_1 : f_2 : f_3)$, and $\varepsilon = \varepsilon_1$. Let $c = c(\mathcal{C})$ be the constant appearing in that theorem. Set $C := \max(c_0 + \log 2, c_7, c_9, 4c)$,

$$\hbar := ((1+\varepsilon_2)\log b + C/\varepsilon_3)N, \qquad \lambda := 3\kappa - 1.$$

Inequality (6.4) implies

$$\frac{(\lambda - 1/2)^2}{8(\lambda+1)} = \frac{3(\kappa - 1/2)^2}{8\kappa} \ge \frac{\varepsilon_4^2}{4}.$$

Thus, by (6.1c),

$$c^2 \min\Big(1, \frac{(\lambda-1/2)^2}{8(\lambda+1)}\varepsilon_1\Big)^{-2} \le \frac{C^2}{\varepsilon_1^2\varepsilon_4^4} \le \frac{\log b}{\varepsilon_2} \le N\log b \le \hbar.$$

[3]Equivalently, as suggested by Waldschmidt, we can argue on the non zero resultant $\mathrm{Res}_x(P, Q) \in \mathbb{C}[t]$ between $Q(t, x) = Q_0 + Q_1 x + Q_2 x^2$ and the minimal polynomial P of g over $\mathbb{C}[t]$.

Assumption (5.2) is satisfied. Moreover, by (6.12) we have $h_0(\theta) \le \hbar$, and, by (6.13) and (6.2),

$$-\frac{\log\|\theta\|}{\hbar} \ge \frac{3(1-\varepsilon_3)\log(b/|a|)-(1+\varepsilon_2)\log b - C/\varepsilon_3}{(1+\varepsilon_2)\log b + C/\varepsilon_3} = 3\kappa - 1 = \lambda.$$

Thus (5.3) is also satisfied. By Theorem 5.2, ξ has irrationality measure

$$\le \frac{\lambda+1}{\lambda-1/2} + \varepsilon_1 = \frac{\kappa}{\kappa-1/2} + \varepsilon_1.$$

We finally remark that, by (6.4) and (6.3),

$$\frac{\kappa}{\kappa-1/2} + \varepsilon_1 = 2 + \frac{1-\kappa}{\kappa-1/2} + \varepsilon_1 \le 2 + \varepsilon_1 + \varepsilon_4^{-1}\left(\varepsilon_2 + \varepsilon_3 + \frac{\log|a| + C/\varepsilon_3}{\log b}\right).$$

□

From proposition (6.1), we easily deduce the bound (5.13) for the irrationality measure of the values of a cubic algebraic function.

Corollary 6.2. *Let g be a cubic algebraic function, regular at 0 with rational Taylor coefficients. Then there exists $C = C(g) \ge 1$ such that the following holds. Let $a \in \mathbb{Z}$, $b \in \mathbb{N}$ coprime and $\varepsilon \in (0, 4/7)$ such that*

$$|a| \le b^{1/2-\varepsilon} \tag{6.14}$$

and

$$\log b \ge 2^{10}C^3/\varepsilon^4. \tag{6.15}$$

Then $g(a/b)$ is irrational with effective irrationality measure

$$\le 2 + \frac{1}{\varepsilon}\left(9\sqrt{\frac{C}{\log b}} + 2\frac{\log|a|}{\log b}\right).$$

Proof. Let $C = C(g)$ be the constant appearing in proposition (6.1). We let for short

$$\varepsilon' = \sqrt{\frac{C}{\log b}}.$$

By (6.15) we have

$$\varepsilon' \le \sqrt{\frac{C}{2^{10}C^3/\varepsilon^4}} = \frac{\varepsilon^2}{32C} \le \frac{\varepsilon}{8}. \tag{6.16}$$

We apply the proposition 6.1 choosing

$$\varepsilon_1 = \varepsilon'/\varepsilon, \quad \varepsilon_2 = 2\varepsilon', \quad \varepsilon_3 = \varepsilon', \quad \varepsilon_4 = \varepsilon/2$$

which is an admissible choice since $\varepsilon_1 \le 1/8 < 1$ by (6.16) and

$$\varepsilon_2 + \varepsilon_3 + \varepsilon_4 = 3\varepsilon' + \varepsilon/2 \le 7\varepsilon'/8 < 1/2$$

again by (6.16) and by the assumption $\varepsilon < 4/7$.

By the choice of ε' and by (6.14) we have

$$\log|a| + \frac{C}{\varepsilon_2\varepsilon_3} = \log|a| + \frac{C}{2\varepsilon'^2} = \log|a| + \frac{1}{2}\log b \le \log b.$$

Moreover, by (6.14) and since $1/2 - \varepsilon + \varepsilon' \leq 1/2 - 3\varepsilon' - \varepsilon/2$ by (6.16),

$$\frac{1}{\log b} \cdot \frac{\log|a| + C/\varepsilon_3}{1/2 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4} = \frac{\frac{\log|a|}{\log b} + \varepsilon'}{1/2 - 3\varepsilon' - \varepsilon/2} \leq \frac{1/2 - \varepsilon + \varepsilon'}{1/2 - 3\varepsilon' - \varepsilon/2} \leq 1.$$

Finally, by the choice of ε' and by (6.15),

$$\frac{1}{\log b} \cdot \frac{\varepsilon_2 C^2}{\varepsilon_1^2 \varepsilon_4^4} = \frac{1}{\log b} \cdot \frac{2\varepsilon' C^2}{(\varepsilon'/\varepsilon)^2 (\varepsilon/2)^4} = \frac{32C^2}{\varepsilon' \varepsilon^2 \log b} = \frac{32C^{3/2}}{\varepsilon^2 \sqrt{\log b}} \leq 1.$$

The last three displayed lines show that (6.1) is satisfied. Proposition 6.1 asserts that $g(a/b)$ is irrational with effective irrationality measure

$$\begin{aligned} &\leq 2 + \varepsilon_1 + \frac{1}{\varepsilon_4}\left(\varepsilon_2 + \varepsilon_3 + \frac{\log|a| + C/\varepsilon_3}{\log b}\right) \\ &= 2 + \frac{\varepsilon'}{\varepsilon} + \frac{2}{\varepsilon}\left(4\varepsilon' + \frac{\log|a|}{\log b}\right) \\ &= 2 + 9\frac{\varepsilon'}{\varepsilon} + \frac{2}{\varepsilon}\frac{\log|a|}{\log b} \\ &= 2 + \frac{1}{\varepsilon}\left(9\sqrt{\frac{C}{\log b}} + 2\frac{\log|a|}{\log b}\right). \end{aligned}$$

□

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