On a conjecture of C. Berenstein and A. Yger

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Abstract.

Let $\mathbf{K}$ be a field and let $\mathbf{a} \subset \mathbf{K}[x_1, \ldots, x_n]$ be an ideal generated by polynomials $f_1, \ldots, f_m$ of degree $\leq d$ $(d \geq 3)$. Put

$$e = 3^{\min\{n,m\}}, \qquad \delta = d^{\min\{n,m\}} + d.$$

We prove that for any polynomial $f \in \mathbf{a}$ of degree d_f there exist polynomials

$$a_1, \ldots, a_m \in \mathbf{K}[x_1, \ldots, x_n]$$

of degree $\leq e \cdot d_f + \delta$ such that

$$f^e = a_1 f_1 + \cdots + a_m f_m.$$

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§1 - Introduction.

Let $d \geq 5$ and $k \geq 1$ be two integers and let $n = 10k+1$. A well-known example of E.Mayr and A.Meyer (see [MM]) shows that there are n polynomials $f_1, \ldots, f_n \in \mathbf{C}[x_1, \ldots, x_n]$ of degree $\leq d$ such that x_1 belongs to the ideal generated by $f_1, \ldots, f_n$ and each solution $a_1, \ldots, a_n$ of the equation

$$x_1 = a_1 f_1 + \cdots + a_n f_n, \qquad a_1, \ldots, a_n \in \mathbf{C}[x_1, \ldots, x_n]$$

satisfies $\max \deg\, a_i > (d-2)^{2^{k-1}}$. In other words, the growth of the degrees of the polynomial coefficients in the representation problem for an ideal $\mathbf{a} \subseteq \mathbf{C}[x_1, \ldots, x_n]$ is, in general, double-exponential.

Given an ideal $\mathbf{a} \subset \mathbf{K}[x_1, \ldots, x_n] = \mathcal{R}$ and a positive integer d, we define $\phi_{\mathbf{a}}(d)$ as the minimum integer D such that for all systems of generators $\{f_1, \ldots, f_m\}$ of $\mathbf{a}$ with $\deg\, f_i \leq d$ and for all $f \in \mathbf{a}$, we can find a representation

$$f = a_1 f_1 + \cdots + a_m f_m$$

with

$$\max_i \deg\, a_i \leq \deg\, f + D.$$

A classical result of Hermann (see [H]) shows that $\phi_{\mathbf{a}}(d) \leq 2(2d)^{2^{n-1}}$ for all ideals $\mathbf{a}$. In 1991 Krick and Logar (see [KL]), using effective linear algebra tecnics, improved the

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previous bound to $\phi_{\mathbf{a}}(d) \leq d^{O(n^2 3^n)}$ where r is the dimension of $\mathbf{a}$. On the other hand, the quoted result of Mayr-Meyer gives an ideal $\mathbf{a}$ for which

$$\phi_{\mathbf{a}}(d) > (d-2)^{2^{(n-11)/10}}.$$

Therefore, it seems to be interesting to give algebraic conditions on $\mathbf{a}$ to avoid the double exponential growth of $\phi_{\mathbf{a}}$. Several authors have written papers on this subject, working with different technics: elimination theory, complex analysis, algebraic geometry, Gröbner basis theory, algebraic complexity theory. The principle results we know are the following ones.

First, for $\mathbf{a} = \mathcal{R}$ the problem is reduced to the effective Nullstellensatz. In 1987 Brownawell (see [B]), using the theory of Chow forms and a theorem of Skoda, found $\phi_R(d) \leq 3n^2 d^n$ provided that car $\mathbf{K} = 0$. In 1988 Caniglia-Galligo-Heintz (see [CGH]), with simple algebraic arguments, proved the weaker result $\phi_R(d) \leq d^{n(n+3)/2}$ without any assumption on $\mathbf{K}$. The same year Kollár (see [K]), using algebraic geometry, proved $\phi_R(d) \leq d^n$ without any assumption on $\mathbf{K}$ but under the technical condition $d \geq 3$. A similar result was found by Philippon (see [P1]), who used the homology of Koszul complex instead of local cohomology, and by Fitchas-Galligo (see [FG]).

Also for zero-dimensional ideals and for complete intersection ideals, $\phi_{\mathbf{a}}$ grows exponentially. In 1990 Berenstein-Yger (see [BY1]) proved with analytic methods $\phi_{\mathbf{a}}(d) \leq 3(n+1)d^n$ for zero-dimensional ideals and $\phi_{\mathbf{a}}(d) \leq 6(n+1)^2(2k^{k+1}+n)d^k$ for complete intersection ideals of codimension k, again under the assumption car $\mathbf{K} = 0$. In 1991 Dickenstein-Fitchas-Giusti-Sessa (see [DFGS]) obtained from Gröbner basis theory $\phi_{\mathbf{a}}(d) \leq nd^{2n} + d^n + d$ (if dim $\mathbf{a} = 0$) and $\phi_{\mathbf{a}}(d) \leq d^k$ (if $\mathbf{a}$ is a complete intersection of codimension k). In 1989 the Author (see [A1] and also [A2] for an errata-corrige), using Kollár-Philippon's method, showed that $\phi_{\mathbf{a}}(d) \leq d^n + d - 1$ provided that $d \geq 3$ for all zero-dimensional ideals, and $\phi_{\mathbf{a}}(d) \leq d^{k(n-k+1)} + d^k + 1$

($d \geq 3$) for unmixed, locally complete intersection ideals of codimension k.

A problem which is related but not equivalent to the previous one is the membership problem: for a given ideal $\mathbf{a} \subset \mathcal{R}$, decide whether f belongs to $\mathbf{a}$. Of course a solution for the representation problem gives a solution for the membership problem, but the converse is not true. A strong result of Dickenstein-Fitchas-Giusti-Sessa (see [DFGS]) shows that the membership problem for unmixed ideals is solvable in polynomial time.

Last but not least, the analogous diophantine problem. We assume $f_1, \ldots, f_m \in \mathcal{R}' = \mathbf{Z}[x_1, \ldots, x_n]$ and we try to find a representation

$$\lambda f = a_1 f_1 + \cdots + a_m f_m, \qquad \lambda \in \mathbf{Z} \backslash \{0\},\ a_1, \ldots, a_m \in \mathcal{R}'$$

for an arbitrary $f \in (f_1, \ldots, f_m)$, with good bounds not only for deg a_i but also for λ and for $H(a_i)$ (=max |coefficients(a_i)|). In 1991 Berenstein-Yger (see [BY2] and [BY3]), combining analytic methods with a theorem of Philippon (see [P2]), solved this problem when $f_1, \ldots, f_m$ don't have common zeros in $\mathbf{C}^n$. Recently, Elkadi (see [E]) and Krick-Pardo (see [KP1] and [KP2]) independently have also found good bounds when $f_1, \ldots, f_m$ is a regular sequence.

Instead of looking at a representation of $f \in \mathbf{a}$ it seems also interesting to try a representation of "small" powers of f. Given two positive integers e and d, let us define $\phi_{\mathbf{a}}(e, d)$ as the minimum integer D such that for all systems of generators $\{f_1, \ldots, f_m\}$ of $\mathbf{a}$ with deg $f_i \leq d$ and for all $f \in \mathbf{a}$, we can find a representation

$$f^e = a_1 f_1 + \cdots + a_m f_m$$

with

$$\max_i \deg\ a_i \leq e \deg\ f + D.$$

Obviously, $\phi_{\mathbf{a}}(e,d) \leq \phi_I(d)$. Moreover, the effective Nullstelensatz implies $\phi_{\mathbf{a}}(e,d) \leq d^n$, provided that $d \geq 3$ and $e \geq d^n$. The study of the applications of integral representation formulas (such as Weil formula or weighted Bochner Martinelli formula) suggests that $\phi_{\mathbf{a}}(n,d)$ is bounded essentially by d^n for all ideals $\mathbf{a}$ (see [BY4]). Recently (see [A2]) the Author has proved this conjecture for 1-dimensional ideals (not necessarly unmixed): $\phi_{\mathbf{a}}(n,d) \leq d^n + d - 1$ for $d \geq 3$. The proof combined Kollár-Philippon's method with the theory of reduction of ideals (developed by Nortchott and Rees), a theorem of Lipman-Tessier (see [LT]) on the integral closure of ideals, Bertini's theorem and a theorem on Gruson-Lazarsfeld-Peskine (see [GPL]) on the regularity of the Hilbert's function of a reduced ideal of dimension 1. In this paper we generalise this result, obtaining $\phi_{\mathbf{a}}(3^n,d) \leq d^n + d$ $(d \geq 3)$ for all ideals $\mathbf{a}$.

More precisely, let us define for an ideal $\mathbf{a}$ in a ring $\mathcal{R}$ the integral closure of $\mathbf{a}$ as the set $\overline{\mathbf{a}}$ of elements $g \in \mathcal{R}$ for which $g^n \in \mathbf{a}(\mathbf{a},g)^{n-1}$ (notice that this set is an ideal of $\mathcal{R}$). We have the following result.

Theorem.

Let $\mathbf{K}$ be a field of arbitrary characteristic, $\mathbf{a} \subseteq \mathbf{K}[x_1,\ldots,x_n]$ an ideal generated by polynomials $f_1,\ldots,f_m$ of degrees $d_1 \geq d_2 \geq \cdots \geq d_m \geq 3$ and set

$$\eta = \frac{3}{8}(3^m - 1) + \frac{m^2}{4}, \qquad \gamma = d_1 \cdots d_m, \qquad \text{if } m \leq n-1,$$
$$\eta = \frac{3}{8}(3^{n-1} - 1) + \frac{(n-1)^2}{4} + 1, \qquad \gamma = d_1 \cdots d_n + d_1, \qquad \text{if } m \geq n.$$

Then, for any $f \in (\overline{\mathbf{a}})^\eta$, we can find polynomials $a_1,\ldots,a_m \in \mathbf{K}[x_1,\ldots,x_n]$ with

$$\max \deg\ (a_i f_i) \leq \deg\ f + \gamma$$

such that

$$f = a_1 f_1 + \cdots + a_m f_m.$$

§2 Superficial ideals.

As in the proofs of the explicit Nullstellensatz (see [B] and [K]), the first step in order to prove our main theorem consists in replacing the sequence $f_1, \ldots, f_m$ of generators of $\mathbf{a}$ by $m' \leq \min\{m, n+1\}$ suitable linear combinations $g_1, \ldots, g_{m'} \in \mathbf{K}f_1 + \cdots + \mathbf{K}f_m$, in such a way that some "regularity" assumptions are satisfied. However, in our case, the polynomials g_i must be chosen more carefully. For this, we need some definitions coming from local algebra, namely the notions of superficial element and reduction of an ideal. A complete discussion about this theory for a local ring can be found in [ZS] and [NR], but for our purposes we must work on a finitely generated K-algebra and we also need some additional results. For the above reasons and also for completeness, we prefer to develop the results we need independently, even though there will be some overlap with the quoted papers.

We start from the following definitions. Let $\mathcal{R}$ be a noetherian ring and let $\mathbf{a} \subseteq \mathcal{R}$ be an ideal. An element $x \in \mathbf{a}$ is called a superficial element (for $\mathbf{a}$) if there exists a natural number c such that $\big(\mathbf{a}^n : (x)\big) \cap \mathbf{a}^c = \mathbf{a}^{n-1}$ for any sufficiently large n. An ideal $\mathbf{b} \subseteq \mathbf{a}$ is called superficial (for $\mathbf{a}$) if $\mathbf{a}^n \cap \mathbf{b} = \mathbf{b}\mathbf{a}^{n-1}$ for any sufficiently large n. Finally, a reduction of $\mathbf{a}$ is an ideal $\mathbf{b} \subseteq \mathbf{a}$ such that $\mathbf{a}^n = \mathbf{b}\mathbf{a}^{n-1}$ for some (and then for any sufficiently large) n.

Remarks.

(1) An ideal $\mathbf{b} \subseteq \mathbf{a}$ is superficial if and only if there exists a natural number c such that $\mathbf{a}^n \cap \mathbf{b}\mathbf{a}^c = \mathbf{b}\mathbf{a}^{n-1}$ for any sufficiently large n. Indeed, by a lemma of E. Artin and D.G. Rees ([ZS] Theorem $4'$, p.254), there exists $k \in \mathbf{N}$ such that $\mathbf{a}^n \cap \mathbf{b} = \mathbf{a}^{n-k}(\mathbf{a}^k \cap \mathbf{b})$ for any $n \geq k$. Therefore, if $n \geq k + c$, we get $\mathbf{a}^n \cap \mathbf{b} \subseteq \mathbf{a}^n \cap \mathbf{b}\mathbf{a}^c$.

(2) Let $\mathbf{b} = (x) \subseteq \mathbf{a}$ be a principal ideal. If x is superficial, the ideal $\mathbf{b}$ is also superficial (apply the last remark) and the converse is true if x is a non zero-divisor.

(3) Let $(\mathcal{R}, \mathbf{m})$ be a local ring and let $\mathbf{b} \subseteq \mathbf{a}$ be two open ideals (i.e. $\sqrt{\mathbf{a}} =$

$\sqrt{\mathbf{b}} = \mathbf{m}$). Then $\mathbf{b}$ is a reduction of $\mathbf{a}$ if and only if $\mathbf{b}$ is superficial (indeed we have $\mathbf{a}^n \subseteq \mathbf{b}$ for any sufficiently large n).

Superficial ideals will be constructed inductively using the following lemma:

Lemma 1.

Let $\mathbf{b} \subseteq \mathbf{a} \subseteq \mathcal{R}$ be two ideals with $\mathbf{b}$ superficial, and let x be an element of $\mathbf{a}$ such that its image $\overline{x}$ in $\mathbf{a}/\mathbf{b}$ is a superficial element. Then $(\mathbf{b}, x)$ is superficial for $\mathbf{a}$.

Proof.

If $\overline{x} \in \mathbf{a}/\mathbf{b}$ is a superficial element, by definition there exists $c \in \mathbf{N}$ such that $\big((\mathbf{a}^n + \mathbf{b}) : (x)\big) \cap (\mathbf{a}^c + \mathbf{b}) = \mathbf{a}^{n-1} + \mathbf{b}$ for any $n \geq n_1$. Therefore, if $y \in \mathbf{a}^n \cap (\mathbf{b}, x)\mathbf{a}^c$ and n is sufficiently large, we have $y = \alpha x + \beta$ where $\alpha \in \mathbf{a}^{n-1}$ and $\beta \in \mathbf{b}\mathbf{a}^c$. Hence $\beta \in \mathbf{a}^n \cap \mathbf{b}\mathbf{a}^c = \mathbf{b}\mathbf{a}^{n-1}$, since $\mathbf{b}$ is superficial for $\mathbf{a}$, and so $y \in (\mathbf{b}, x)\mathbf{a}^{n-1}$. From the remark (1) above, we deduce that $(\mathbf{b}, x)$ is superficial for $\mathbf{a}$.

Q.E.D.

If the ring $\mathcal{R}$ is a finitely generated $\mathbf{K}$-algebra ($\mathbf{K}$ being an infinite field), we can always find superficial elements $x \in \mathbf{a}$. More precisely, we have the following result (see also [ZS] p.286-287).

Lemma 2.

Let $\mathcal{R}$ be as above and let $\mathbf{a} = (x_1, \ldots, x_m) \subseteq \mathcal{R}$ be an ideal. Then there exists a finite number of proper linear subspaces $\mathbf{V}_1, \ldots, \mathbf{V}_u \subset \mathbf{K}^m$ such that for any vector $\lambda \in \mathbf{K}^m \setminus \bigcup_{i=1}^{u} \mathbf{V}_i$ [(1)] the element $x = \lambda_1 x_1 + \cdots + \lambda_m x_m \in \mathbf{a}$ is superficial for $\mathbf{a}$.

Proof.

Let $G(\mathcal{R}) = \sum_{h=0}^{\infty} \mathbf{a}^h/\mathbf{a}^{h+1}$ be the graded associated ring and let $\mathbf{X} = \sum_{h=1}^{\infty} \mathbf{a}^h/\mathbf{a}^{h+1}$ be the ideal of the elements of positive degree. We must find $x \in \mathcal{R}$ such that

(1) This last set is non-empty because $\mathbf{K}$ is infinite.

$\overline{x} \in \mathbf{a}/\mathbf{a}^2 \subseteq G(\mathcal{R})$ satisfies

$$\overline{x}y = 0, \quad y \in \mathbf{X}^c \Rightarrow y = 0$$

for some natural number c. Let $\wp_1, \ldots, \wp_s$ be the minimal primes of $G(\mathcal{R})$ and assume $\wp_1, \ldots, \wp_u \not\supseteq \mathbf{a}/\mathbf{a}^2$ and $\wp_{u+1}, \ldots, \wp_s \supseteq \mathbf{a}/\mathbf{a}^2$. Then, for $i = 1, \ldots, u$, the set

$$\mathbf{V}_i = \{\lambda \in \mathbf{K}^m \mid \lambda_1\overline{x}_1 + \cdots \lambda_m\overline{x}_m \in \wp_i\}$$

is a proper subspace of $\mathbf{K}^m$; choose $\lambda \in \mathbf{K}^m \setminus \bigcup_{i=1}^{u} \mathbf{V}_i$ and set $x = \lambda_1 x_1 + \cdots \lambda_m x_m \in \mathbf{a}$. We have

$$(0) = Q_1 \cap \cdots \cap Q_u \cap Q_{u+1} \cap \cdots \cap Q_s$$

where the Q_i's are $\wp_i$-primary. For some c, $\mathbf{X}^c \subseteq Q_{u+1} \cap \cdots \cap Q_s$; therefore, if $y \in \mathbf{X}^c$ and $\overline{x}y = 0$, we have $y \in Q_{u+1} \cap \cdots \cap Q_s$; moreover $y \in Q_1 \cap \cdots \cap Q_u$, since $\overline{x} \notin \wp_i$ for $i = 1, \ldots, u$, and so $y = 0$.

Q.E.D.

§3 **Reduction to an $\mathcal{U}$-regular sequence.**

Let $\mathbf{a} \subseteq \mathcal{R}$ be an ideal and let us consider the Zariski open set $\mathcal{U} = \mathcal{U}_{\mathbf{a}} = \{\wp \in \mathrm{Spec}\mathcal{R},\ \wp \not\supseteq \mathbf{a}\}$. A sequence $x_1, \ldots, x_k$ is called $\mathcal{U}$-regular if it is a regular sequence in the ring $\mathcal{R}_{\mathcal{U}} = \bigcap_{\wp \in \mathcal{U}} \mathcal{R}_{\wp}$. In other words, $x_1, \ldots, x_k$ is $\mathcal{U}$-regular if $(x_1, \ldots, x_k) \colon \langle \mathbf{a} \rangle = \bigcup_{r \in \mathbf{N}} (x_1, \ldots, x_k) \colon \mathbf{a}^r$ is a proper ideal and if x_i is a non zero-divisor in $\mathcal{R}/(x_1, \ldots, x_{i-1}) \colon \langle \mathbf{a} \rangle$ for $i = 1, \ldots, k$. If $\mathcal{R}$ is a Cohen-Maculay ring, $\mathcal{R}_{\mathcal{U}}$ is also Cohen-Maculay and the purity theorem holds; therefore, if $\mathbf{b} \subseteq \mathcal{R}$ is an ideal generated by an $\mathcal{U}$-regular sequence of length k, then $\mathbf{b} \colon \langle \mathbf{a} \rangle$ is unmixed of rank k.

We also need the following corollary of Bertini's theorem:

Lemma 3.

Let $\mathcal{R}$ be a finitely generated algebra over an algebraically closed field $\mathbf{K}$ and let $\mathbf{a} = (x_1, \ldots, x_m) \subseteq \mathcal{R}$ be an ideal. Then there exists a non-empty Zariski open set $\mathbf{V} \subset \mathbf{K}^m$ such that for any $\lambda \in \mathbf{K}^m \backslash \mathbf{V}$ the ideal $(\lambda_1 x_1 + \cdots + \lambda_m x_m)\colon \langle \mathbf{a} \rangle$ is radical (i.e. it coincides with its radical).

Proof.

Apply [J], corollary 6.7, to $X = \mathrm{Spec}\mathcal{R}_{\mathcal{U}}$ and to the morphism $f\colon X \to \mathrm{Aff}^m_{\mathbf{K}}$ defined by $f = (x_1, \ldots, x_m)$.

Q.E.D.

Proposition 1.

Let $f_1, \ldots, f_m$ be a system of generators of the ideal $\mathbf{a} \subseteq \mathcal{R} = K'$, where $\mathbf{K}$ is an algebraically closed field. Then we can find an integer $m' \leq \min\{m, n+1\}$ and m' linear combinations (with coefficients in $\mathbf{K}$) of $f_1, \ldots, f_m$, say $g_1, \ldots, g_{m'}$, such that $\sqrt{\mathbf{a}} = \sqrt{(g_1, \ldots, g_{m'})}$ and the following assertions hold for $i = 1, \ldots, m'$:

i) The ideal $\mathbf{b}_i = (g_1, \ldots, g_i)$ is superficial for $\mathbf{a}$;

ii) The ideal $\mathbf{b}'_i = \mathbf{b}_i\colon \langle \mathbf{a} \rangle$ is unmixed of rank i and g_i is a non zero-divisor in $R/\mathbf{b}'_{i-1}$;

iii) $\mathbf{b}'_i$ is radical.

Proof.

We choose $g_1, \ldots, g_{m'}$ by induction. Let us assume that assertions i), ii) and iii) are satisfied for $i = 1, \ldots h$. If $\sqrt{(g_1, \ldots, g_h)} = \sqrt{\mathbf{a}}$ we put $m' = h$; otherwise $\mathbf{b}_h\colon \langle \mathbf{a} \rangle$ is a proper ideal and so (by ii)) $g_1, \ldots, g_h$ is a $\mathcal{U}$-regular sequence and $\mathbf{b}_h\colon \langle \mathbf{a} \rangle$ is unmixed of rank h. Let $\wp_1, \ldots, \wp_r$ be its associated primes and let us consider the proper subspaces $\mathbf{W}_i = \{\lambda \in \mathbf{K}^m$ such that $\lambda_1 f_1 + \cdots \lambda_m f_m \notin \wp_i\}$, $i = 1, \ldots, r$. We also denote by $\mathbf{V}_i$ $(i = 1, \ldots, s)$ and by $\mathbf{V}$ the subspaces and the Zariski open set of $\mathbf{K}^m$ obtained respectively applying lemma 2 and lemma 3 to the $\mathbf{K}$-algebra $\mathcal{R}/\mathbf{b}_h$ and to the ideal $\mathbf{a}/\mathbf{b}_h$ generated by the images of $f_1, \ldots, f_m$. Then, if $\lambda \in \mathbf{K}^m$ lies outside $\mathbf{W}_i$, $\mathbf{V}_i$ and $\mathbf{V}$, the polynomial $g_{h+1} = \lambda_1 f_1 + \cdots + \lambda_m f_m$ has the required properties.

Q.E.D.

Remark.

It is easy to see that we can choose g_i as a linear combination of $f_i, \ldots, f_m$.

Let us define for an ideal $\mathbf{a}$ in a ring $\mathcal{R}$ its integral closure $\overline{\mathbf{a}}$ as the set of elements $x \in \mathcal{R}$ for which $x^n \in \mathbf{a}(\mathbf{a}, x)^{n-1}$; notice that this set is an ideal of $\mathcal{R}$. We have

Corollary 1.

Let g_i, $\mathbf{b}_i$ and $\mathbf{b}'_i$ be as in the previous proposition. Then the ideals $\mathbf{b}'_{i-1} + (g_i)$ and $\mathbf{b}'_{i-1} + \mathbf{a}$ have the same integral closure in $\mathcal{R}_\wp$, for any prime ideal $\wp \supseteq \mathbf{b}'_{i-1} + \mathbf{a}$ of rank i.

Proof.

Let $\wp \supseteq \mathbf{b}'_{i-1} + \mathbf{a}$ be a prime ideal of rank i. The ideal $(\mathbf{b}'_{i-1}, g_i)\mathcal{R}_\wp$ is open (use ii) of the proposition above), hence there exists $c_1 \in \mathbf{N}$ such that $\mathbf{a}^{c_1}\mathcal{R}_\wp \subseteq (\mathbf{b}'_{i-1}, g_i)\mathcal{R}_\wp$. Moreover, there exists another constant $c_2 \in \mathbf{N}$ such that $\mathbf{a}^{c_2}\mathbf{b}'_{i-1} \subseteq \mathbf{b}_{i-1} \subseteq \mathbf{b}'_{i-1}$. Therefore, for any $n \geq c_1 + c_2$,

$$\mathbf{a}^n\mathcal{R}_\wp \subseteq \mathbf{a}^{c_1+c_2}\mathcal{R}_\wp \subseteq \mathbf{a}^{c_2}(\mathbf{b}'_{i-1}, g_i)\mathcal{R}_\wp \subseteq (\mathbf{b}_{i-1}, g_i)\mathcal{R}_\wp = \mathbf{b}_i\mathcal{R}_\wp.$$

Hence, for any sufficiently large n,

$$\begin{aligned}(\mathbf{b}'_{i-1} + \mathbf{a})^n\mathcal{R}_\wp &= \mathbf{b}'_{i-1}(\mathbf{b}'_{i-1} + \mathbf{a})^{n-1}\mathcal{R}_\wp + \mathbf{a}^n\mathcal{R}_\wp \\ &= \mathbf{b}'_{i-1}(\mathbf{b}'_{i-1} + \mathbf{a})^{n-1}\mathcal{R}_\wp + (\mathbf{a}^n \cap \mathbf{b}_i)\mathcal{R}_\wp \\ &= \mathbf{b}'_{i-1}(\mathbf{b}'_{i-1} + \mathbf{a})^{n-1}\mathcal{R}_\wp + \mathbf{b}_i\mathbf{a}^{n-1}\mathcal{R}_\wp \\ &\subseteq (\mathbf{b}'_{i-1}, g_i)(\mathbf{b}'_{i-1} + \mathbf{a})^{n-1}\mathcal{R}_p\end{aligned}$$

since $\mathbf{b}_i$ is superficial for $\mathbf{a}$. Hence $(\mathbf{b}'_{i-1}, g_i)\mathcal{R}_\wp$ is a reduction of $(\mathbf{b}'_{i-1} + \mathbf{a})\mathcal{R}_\wp$. To conclude the proof, we use the following lemma (see [NR], corollary of theorem 1, p.155)

Lemma 4.

Let $\mathbf{b} \subseteq \mathbf{a}$ *two ideals in a noetherian ring* $\mathcal{R}$ *and let us assume that* $\mathbf{a}$ *is not entirely composed by zero-divisors. Then, if* $\mathbf{b}$ *is a reduction of* $\mathbf{a}$*, these two ideals have the same integral closure.*

Proof.

Let $x \in \mathbf{a}$. From $\mathbf{a}\mathbf{a}^{n-1} = \mathbf{b}\mathbf{a}^{n-1}$ we see from a determinant argument that there exists $\phi \in x^n + \mathbf{b}(\mathbf{b}, x)^{n-1}$ such that $\phi\mathbf{a}^{n-1} = 0$. Hence $\phi = 0$ and $x^n \in \mathbf{b}(\mathbf{b}, x)^{n-1}$.

Q.E.D.

Now we quote the following theorem of J.Lipman and B.Tessier:

Theorem. (see [LT])

Let $\mathcal{R}$ *be a regular local ring of dimension* d *and let* $\mathbf{a} \subseteq \mathcal{R}$ *be an open ideal. Then* $\overline{\mathbf{a}}^d \subseteq \mathbf{a}$.

From this theorem and from the corollary above, we deduce that

$$\overline{\mathbf{a}}^i \subseteq \big(\mathbf{b}'_{i-1} + (g_i)\big)\mathcal{R}_\wp \cap \mathcal{R} \tag{1}$$

for any prime $\wp \supseteq b'_{i-1} + \mathbf{a}$ of rank i. The previous statement is one of the crucial points in the proof of our main result.

§4 Proof of the main theorem.

We shall work over the homogeneous ring $\mathcal{A} = \mathbf{K}[x_0, \ldots, x_n]$; for $i = 1, \ldots, m'$ let $G_i = {}^h g_i$ be the homogenization of the polynomials g_i, given by proposition 1, and let $J_i = {}^h b'_i$ be the homogenization of the ideal b'_i; we also put $J_0 = (0)$. We denote by I_{i+1} the intersection of the isolated components of (J_i, G_{i+1}) whose radicals contain $I = {}^h\mathbf{a}$ but not x_0. Similarly, let K_{i+1} be the intersection of the isolated components whose radicals contain x_0 and let L_{i+1} be the intersection of the embedded ones. Then

$$(J_i, G_i) = J_{i+1} \cap I_{i+1} \cap K_{i+1} \cap L_{i+1}$$

and $x_0 \cdot I \subseteq \sqrt{L_{i+1}}$, since $G_1, \ldots, G_{m'}$ is an $\mathcal{U}$-regular sequence in the Zariski open set $\mathcal{U} = \{\wp \in \mathrm{Spec}\mathcal{A}$ such that $x_0 \cdot I \not\subseteq \wp\}$.

We want to give explicitly two integers, γ_{i+1} and η_{i+1} such that

$$x_0^{\gamma_{i+1}} \overline{I}^{\eta_{i+1}} J_{i+1} \subseteq (J_i, G_{i+1}).$$

To this end, let $\delta_{i+1} = \deg \mathcal{A}/K_{i+1}$, which is easily estimated by Bezout's theorem. From (1), we find that

$$x_0^{\delta_{i+1}} \overline{I}^{i+1} J_{i+1} \subseteq J_{i+1} \cap I_{i+1} \cap K_{i+1}. \tag{2}$$

Thus, we must only control the embedded components with a suitable modification of Kollar's method in Patrice Philippon's algebraic version (see [P1] and [P2]).

Definition.

Let $J \subseteq \mathcal{A}$ be a homegeneous ideal. We say that J has type (ε, ρ) if for any

$$\alpha \in \mathcal{F} = \{(\alpha_1, \ldots, \alpha_s) \subseteq \mathcal{A}^s \mid x_0 \cdot I \subseteq \sqrt{(\alpha_1, \ldots, \alpha_s)}\}$$

and for any

$$\sigma < \dim \mathcal{A}/J - \dim \mathcal{A}/(\alpha)$$

we have $x_0^{\varepsilon} \cdot I^{\rho} \cdot H_{s-\sigma}(\alpha|\mathcal{A}/J) = 0$ [(2)]

Lemma 5.

Let us assume that J_i has type (ε_i, ρ_i). Then

$$x_0^{\varepsilon_i + \delta i+1} \overline{I}^{\rho_i + i+1} J_{i+1} \subseteq (J_i, G_{i+1})$$

---

(2) The $H_{s-\sigma}(\alpha|M)$ are the homological modules associated with the Koszul complex $K(\alpha|\mathcal{A}|M)$ (see [N] §8.2).

Proof.

Let $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathcal{F}$ be a family of generators of L_{i+1} and put

$$M = (J_{i+1} \cap I_{i+1} \cap K_{i+1})/(J_i, G_{i+1}).$$

From the inclusion $M \hookrightarrow \mathcal{A}/(J_i, G_{i+1})$ we get

$$M = H_s(\alpha|M) \hookrightarrow H_s(\alpha|\mathcal{A}/(J_i, G_{i+1})).$$

On the other hand, the exact sequence

$$0 \longrightarrow \mathcal{A}/J_i \overset{\times G_{i+1}}{\longrightarrow} \mathcal{A}/J_i \longrightarrow \mathcal{A}/(J_i, G_{i+1}) \longrightarrow 0 \qquad (3)$$

gives

$$H_s(\alpha|\mathcal{A}/(J_i, G_{i+1})) \hookrightarrow H_{s-1}(\alpha|\mathcal{A}/J_i)$$

(since any associated prime of J_i contains neither x_0 nor I we have $H_s(\alpha|\mathcal{A}/J_i) = 0{:}_{\mathcal{A}/J_i} L_{i+1} = 0$). Therefore, composing the two last inclusions we get

$$M \hookrightarrow H_{s-1}(\alpha|\mathcal{A}/J_i).$$

Now, taking into account that $\dim \mathcal{A}/J_i - \dim \mathcal{A}/L_{i+1} > 1$ and our definition of type, the last line implies $x_0^{\varepsilon_i} \overline{I}^{\rho_i} M = 0$. From this and (2) we obtain our claim.

Q.E.D.

Lemma 6.

Let us assume that J_i has type (ε_i, ρ_i). Then J_{i+1} has type $(3\varepsilon_i + \delta_{i+1}, 3\rho_i + i + 1)$.

Proof.

Let $\alpha \in \mathcal{F}$ and let $\sigma < \dim \mathcal{A}/J_{i+1} - \dim \mathcal{A}/(\alpha) < \dim \mathcal{A}/J_i - \dim \mathcal{A}/(\alpha)$. From (3) we get

$$x_0^{2\varepsilon_i} \overline{I}^{2\rho_i} \cdot H_{s-\sigma}(\alpha|\mathcal{A}/(J_i, G_{i+1})) = 0. \qquad (4)$$

On the other hand, the exact sequence

$$0 \longrightarrow J_{i+1}/(J_i, G_{i+1}) \longrightarrow \mathcal{A}/(J_i, G_{i+1}) \longrightarrow \mathcal{A}/J_{i+1} \longrightarrow 0$$

gives rise to

$$H_{s-\sigma}(\alpha|\mathcal{A}/(J_i, G_{i+1})) \to H_{s-\sigma}(\alpha|\mathcal{A}/J_{i+1}) \to H_{s-\sigma-1}(\alpha|J_{i+1}/(J_i, G_{i+1})).$$

From lemma 5 we know that $x_0^{\varepsilon_i+\delta i+1}\overline{I}^{\rho_i+i+1}$ kills $J_{i+1}/(J_i, G_{i+1})$ hence also $H_{s-\sigma-1}(\alpha|J_{i+1}/(J_i, G_{i+1}))$. Taking into account (4), we find

$$x_0^{3\varepsilon_i+\delta i+1}\overline{I}^{3\rho_i+i+1} \cdot H_{s-\sigma}(\alpha|\mathcal{A}/J_{i+1})) = 0.$$

Q.E.D.

The ideal J_0 has type $(0, 0)$, hence we easily obtain from the lemma above that the ideal J_i has type (ε_i, ρ_i), where

$$\varepsilon_i = \sum_{h=1}^{i} 3^{i-h}\delta_h, \qquad \rho_i = \sum_{h=1}^{i} 3^{i-h}h.$$

Now repeated applications of lemma 6 give

$$x_0^{\gamma_i}\overline{I}^{\eta_i}J_i \subseteq (G_1, \ldots, G_i) \tag{5}$$

where

$$\begin{aligned} \eta_i &= \sum_{h=1}^{i} h + \sum_{h=1}^{i-1} \rho_h = \frac{3}{8}(3^i - 1) + \frac{i^2}{4} \\ \gamma_i &= \sum_{h=1}^{i} \delta_h + \sum_{h=1}^{i-1} \varepsilon_{h-1} \leq \sum_{h=1}^{i} 3^{i-h}\delta_h \\ &\leq \sum_{h=1}^{i} d_{k+1}\cdots d_i\delta_k \leq d_1\cdots d_i - \deg\, \mathcal{A}/J_i, \end{aligned} \tag{6}$$

the last inequality coming from Bezout's theorem. We distinguish two cases.

- **First case** $m' \leq n-1$.

  We take $i = m' \leq \min\{m, n-1\}$ in (5) and (6).

- **Second case** $m' \geq n$ (and so $m \geq n$)

The ideal J_{n-1} is a homogeneous radical ideal of rank $n-1$ (use iii) of proposition 1) and we apply the following two lemmas (for the proofs, see [A2] lemma 5 and lemma 4 respectively):

Lemma 7.

Let $c_1 = d_n \deg \mathcal{A}/J_{n-1} - \deg \mathcal{A}/J_n \cap I_n$ *and* $c_2 = \deg \mathcal{A}/J_n \cap I_n + 1$. *Then*

$$x_0^{c_1}(J_n \cap I_n)_\nu \subseteq (J_{n-1}, G_n)$$

for any integer $\nu \geq c_2$.

Lemma 8.

Let $F_i = {}^h f_i$, $i = 1, \dots, m$. *Then,*

$$I_\nu \subseteq (J_n \cap I_n, F_1, \dots, F_m)$$

for $\nu \geq c_3 = \deg \mathcal{A}/J_n \cap I_n + d_1$.

Combining these two results, we get $x_0^{c_1 + \max\{c_2, c_3\}} I \subseteq (J_{n-1}, G_n, F_1, \dots, F_m)$. Taking into account (5) with $i = n-1$, this gives

$$x_0^{\gamma'_n} \overline{I}^{\eta'_n} J_{n-1} \subseteq (G_1, \dots, G_{n-1}, G_n, F_1, \dots, F_m).$$

where

$$\gamma'_n = \gamma_{n-1} + d_n \deg \mathcal{A}/J_{n-1} + d_1 \leq d_1 \cdots d_n + d_1$$
$$\eta'_n = \eta_{n-1} + 1 = \frac{3}{8}(3^{n-1} - 1) + \frac{(n-1)^2}{4} + 1.$$

Our theorem follows.

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