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Research Paper

Progressive iterative Schoenberg-Marsden variation diminishing operator and related quadratures

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ABSTRACT

In this paper we propose an approximation method based on the classical Schoenberg-Marsden variation diminishing operator with applications to the construction of new quadrature rules. We compare the new operator with the multilevel one studied in [12] in order to characterize both of them with respect to the well known classical one. We discuss convergence properties and present numerical experiments.

1. Introduction

Recently two kinds of approximation techniques have been introduced and studied in the literature. As far as we know, both seminal papers [18,19], one dealing with the so called *progressive iterative approximation* (PIA) [9,11,16,17,20,28,29] and the other one with the so called *multilevel approximation* (MA) [6,12,13,15,24] were published in 2004.

They are two different ways of triggering an iterative procedure, acting on some kind of remainder (or error) in the chosen base method.

Induced by these ideas, in [12] the authors present and study MA, providing some new results on polynomial reproduction and convergence, when applied to the well known variation-diminishing Schoenberg-Marsden operator, also in order to define new quadrature formulas, based on it.

Similarly now we are interested in investigating PIA and comparing it with MA in a sort of a twin paper. So we are going to use similar notation with application to the same operator as in [12].

The reason to use the simplest among quasi-interpolating (QI) operators [3] is to make evident such techniques. For MA a second reason is explained in [6,13], where, even if in 2D, the best performance results are obtained right by this operator. The same is done in [24]. Indeed the simplicity of such an operator definition limits a possible error propagation. For PIA a second reason is taken from its very definition, born in the parametric setting. We remark that PIA has been widely used in the parametric setting for CAGD purposes, but, as far as we know, no studies have been carried out in the functional setting for approximation theory scope.

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The aim of using such a spline quasi-interpolating (QI) operator [22,25] is to exploit the good properties of B-spline functions, as locality, shape smoothness and approximation accuracy [2].

The aim of using PIA techniques is to improve the performances of the base QI spline results, by taking into account that in the classical PIA method the limit of the generated sequence of QI spline functions is an interpolating spline at certain fixed points of the approximation interval $[a,b]$, while in an evolution of the classical PIA technique the limit function is the least square function of the original data to be approximated [11,20].

Such a property of asymptotic interpolation is interesting, since, as we know, interpolation has a well-developed theory and it is a very powerful tool for function approximation, but, since it needs to solve, often large, systems of linear equations with possibly bad condition, the weaker form of asymptotic interpolation of QI operators becomes a very good alternative. In fact they can directly yield approximations and do not require solution of any linear system. For this reason QI operators are very important in the study of approximation theory and its applications, e.g. in solving partial differential equations and integral equations, curve and surface fitting, integration, differentiation, approximation of zeros, and so on, and they have been widely studied in recent years [1,7,14,21,23,26,27].

Therefore in this paper we continue the study of spline QI operators on bounded domains, restricted to the C^1 quadratic variation-diminishing Schoenberg-Marsden operator, but suitably modified by PIA technique in order to bypass the problem of solving a linear system of equations.

The paper is organized as follows. In Section 2 we recall the definition and the properties of the classical Schoenberg-Marsden operator. In Section 3 we modify such an operator by PIA technique and study polynomial reproduction and convergence. Then the new improved operator is used in Section 4 to approximate integrals, while in Section 5 we propose another approach of PIA. Finally in Section 6 some numerical results and comparisons between the classical and the improved operators are presented, showing the new operators provide better performances than the QI classical one.

2. The variation-diminishing Schoenberg-Marsden operator

Let $I=[a,b]\subset\mathbb{R}$ and Λ_n be a uniform partition, dividing I in n subintervals, whose associated extended partitions are

$$x_{-2}<x_{-1}<x_0=a<x_1<\cdots<x_{n-1}<x_n=b<x_{n+1}<x_{n+2}, \tag{1}$$

and

$$x_{-2}=x_{-1}=x_0=a<x_1<\cdots<x_{n-1}<x_n=b=x_{n+1}=x_{n+2} \tag{2}$$

with $x_i=a+ih$, $i=-2,\dots,n+2$ and $h=\frac{b-a}{n}$, except at multiple knots, where it is zero.

Let also $S_2^1(\Lambda_n)$ be the space of spline functions $s\in C^1(I)$ on Λ_n whose restriction on any subinterval is a polynomial in $\mathbb{P}_2$, space of polynomials of degree at most 2. The set of $n+2$ quadratic B-spline functions $\mathcal{B}_n=\{B_i:=B_{i,2}\,:\,i=0,\dots,n+1\}$ is a basis of $S_2^1(\Lambda_n)$. In case of partitions (1) they can be obtained as scaled translated of the quadratic C^1 B-spline B centered at $x=0$ with support 'radius' $\frac{3}{2}$, i.e. $B_{i,2}(x)=B(nx-i+\frac{1}{2})$ (see also [5] for a similar definition in the bivariate setting on triangulations), while for partitions (2) the multiplicity at the interval extrema is to be taken into account. Such basis functions satisfy the usual good properties of non negativity, partition of unity, minimal compact support. In particular the support of B_i is $[x_{i-2},x_{i+1}]$. Moreover they can be generated by the well known Cox-de Boor recurrence relation [2]

$$B_{i,d}(x)=\frac{x-x_{i-d}}{x_i-x_{i-d}}B_{i-1,d-1}(x)+\frac{x_{i+1}-x}{x_{i+1}-x_{i-d+1}}B_{i,d-1}(x),\ i=0,\dots,n+d-1 \tag{3}$$

with

$$B_{i,0}(x)=\begin{cases}1 & \text{if } x\in[x_i,x_{i+1}),\\ 0 & \text{otherwise}\end{cases}$$

for B-splines of degree d.

Let us consider the quadratic version of the variation-diminishing Schoenberg-Marsden operator
$S:C(I^*)\to S_2^1(\Lambda_n)$, defined as

$$Sf=\sum_{i=0}^{n+1}f(s_i)B_i, \tag{4}$$

where I^* is an open interval containing I and where

$$s_i=\frac{x_{i-1}+x_i}{2}=a+\frac{(2i-1)}{2}h,\quad i=0,\dots,n+1$$

for partitions (1) and

$$s_0=x_0,\quad s_i=\frac{x_{i-1}+x_i}{2},\ \text{ con }\ i=1,\dots,n,\quad s_{n+1}=x_n$$

for partitions (2) are the so-called *Greville sites* [2]. Moreover S reproduces $\mathbb{P}_1$ and $\|S\|=1$.

Theorem 2.1. *[12] Let $f \in C(\overline{I})$, where $\overline{I}$ is the closure of I^*. Then*

$$\|f - Sf\|_I \le \omega\left(f, \frac{3}{2}h\right) \tag{5}$$

for sufficiently small h. If $f \in C^1(I)$, then

$$\|f - Sf\|_I \le h\omega\left(f', \frac{h}{2}\right)$$

and, if $f \in C^2(I)$, then

$$\|f - Sf\|_I \le \frac{1}{4}h^2\|f''\|,$$

where $\|\cdot\|_I$ denotes the maximum norm on I and $\omega(f,\delta) = \max\{|f(x) - f(y)|, x, y \in \overline{I} : \|x - y\| \le \delta\}$ is the classical modulus of continuity of f on $\overline{I}$.

3. The progressive iterative variation-diminishing Schoenberg-Marsden operator

The aim of modifying the operator (4) is to reach better performances by defining a progressive iterative operator starting from the corresponding base one.

In literature progressive iterative approximation (PIA) is located in the parametric setting and it is meant to be an approximation technique where, starting from initial data points $\{P_i\}_{i=0}^{n+1}$, at each iteration they are slightly modified till when some suitable conditions are satisfied. In the case of classical PIA such conditions provide asymptotic interpolation of initial data points. The close relationship between the value of a spline and the nearby B-spline coefficient has led to the definition and use of *control points*, term that comes from Computer Aided Geometric Design, where spline curves, i.e. vector-valued spline, rather than spline functions, i.e. scalar-valued spline, are used. However the graph of a spline function defines a spline curve $(x, Sf(x))$. Then, since by a B-spline property $x = \sum_i s_i B_i(x)$, then we can call $\{P_i := (s_i, f(s_i))\}_{i=0}^{n+1}$ the control point sequence of the spline function Sf [2]. Therefore here with PIA we mean the above iterative technique, where, starting from initial function data $\{(s_i, f(s_i))\}$, at each iteration they are slightly modified, so that the resulting spline sequence tends to interpolate them.

So we rewrite Sf, given in (4), as

$$S^{0I}f(x) = \sum_{i=0}^{n+1} f_0(s_i)B_i(x) \tag{6}$$

with $f_0(s_i) = f(s_i)$. Then the iterative procedure is thus triggered by defining

$$\Delta_0 f_i = f_0(s_i) - S^{0I}f(s_i), \quad i = 0, 1, \dots, n+1, \tag{7}$$

and, setting

$$f_1(s_i) = f_0(s_i) + \Delta_0 f_i \;\; i = 0, 1, \dots, n+1, \tag{8}$$

at the first iteration we get the new operator S^{1I} defined as

$$S^{1I}f(x) = \sum_{i=0}^{n+1} f_1(s_i)B_i(x). \tag{9}$$

In general we define the operator S^{pI} after p iterations as

$$S^{pI}f(x) = \sum_{i=0}^{n+1} f_p(s_i)B_i(x), \tag{10}$$

with

$$f_p(s_i) = f_{p-1}(s_i) + \Delta_{p-1} f_i \;\text{ and }\; \Delta_{p-1} f_i = f_0(s_i) - S^{(p-1)I}f(s_i). \tag{11}$$

If

$$\lim_{p\to\infty} S^{pI}f(s_i) = f(s_i), \quad i = 0, 1, \dots, n+1, \tag{12}$$

holds, then the initial operator S satisfies the property of progressive iterative approximation (PIA).

From (11) and (12) we note that

$$\lim_{p\to\infty} \Delta_{p-1} f_i = f_0(s_i) - \lim_{p\to\infty} S^{(p-1)I}f(s_i) = f_0(s_i) - f_0(s_i) = 0, \quad i = 0, 1, \dots, n+1. \tag{13}$$

Moreover, from (7) and (8) we get

$$f_1(s_i) = 2f_0(s_i) - \sum_{j=0}^{n+1} f_0(s_j)B_j(s_i), i = 0, 1, \dots, n+1. \tag{14}$$

Iterating this process, we get

$$f_p(s_i) = (p+1)f_0(s_i) - \sum_{j=0}^{n+1}\sum_{r=0}^{p-1} f_r(s_j)B_j(s_i). \tag{15}$$

Indeed, starting from (14), by induction we suppose (15) hold for $p-1$:

$$f_{p-1}(s_i) = pf_0(s_i) - \sum_{j=0}^{n+1}\sum_{r=0}^{p-2} f_r(s_j)B_j(s_i)$$

and prove it for p. From (11) we write

$$\begin{aligned}
f_p(s_i) &= f_{p-1}(s_i) + \Delta_{p-1} f_i \\
&= pf_0(s_i) - \sum_{j=0}^{n+1}\sum_{r=0}^{p-2} f_r(s_j)B_j(s_i) + f_0(s_i) - S^{(p-1)I} f(s_i) \\
&= (p+1)f_0(s_i) - \sum_{j=0}^{n+1}\sum_{r=0}^{p-2} f_r(s_j)B_j(s_i) - \sum_{j=0}^{n+1} f_{p-1}(s_j)B_j(s_i) \\
&= (p+1)f_0(s_i) - \sum_{j=0}^{n+1}\sum_{r=0}^{p-1} f_r(s_j)B_j(s_i).
\end{aligned}$$

So, if PIA property holds, the coefficients in (15) for all i and for increasing p asymptotically tend to the unique coefficients obtained by solving the linear system coming from the interpolation conditions, i.e., if $If = \sum_i \alpha_i B_i$ is the unique spline such that $If(s_j) = f(s_j)$, then $\lim_{p\to\infty} S^{pI} = If$, since $\lim_{p\to\infty} f_p(s_i) = \alpha_i$ for all i, j.

Now, similarly to multilevel approximation in [12], we study the operator S^{pI} on both partitions (1) and (2).

3.1. PIA operator S^{pI} on simple knot partitions

In case of partitions (1), from $\Delta_{p-1} f_j$ in (11) let us analyse

$$S^{(p-1)I} f(s_j) = \sum_{i=0}^{n+1} f_{p-1}(s_i)B_i(s_j),$$

when $j = 0, \dots, n+1$. If $j = 0$, $s_0 \in (x_{-1}, x_0)$ and the B-splines containing such a point are B_0 and B_1. In particular

$$B_0(s_0) = \frac{6}{8}, \quad B_1(s_0) = \frac{1}{8}.$$

Then

$$S^{(p-1)I} f(s_0) = \sum_{i=0}^{n+1} f_{p-1}(s_i)B_i(s_0) = \frac{6}{8} f_{p-1}(s_0) + \frac{1}{8} f_{p-1}(s_1).$$

If $j = 1, \dots, n$, $s_j \in (x_{j-1}, x_j)$ and the B-splines containing such a point are $B_{j-1}(x)$, $B_j(x)$ and $B_{j+1}(x)$. In particular

$$B_{j-1}(s_j) = \frac{1}{8}, \quad B_j(s_j) = \frac{6}{8}, \quad B_{j+1}(s_j) = \frac{1}{8}.$$

Then

$$S^{(p-1)I} f(s_j) = \sum_{i=0}^{n+1} f_{p-1}(s_i)B_i(s_j) = \frac{1}{8} f_{p-1}(s_{j-1}) + \frac{6}{8} f_{p-1}(s_j) + \frac{1}{8} f_{p-1}(s_{j+1}).$$

If $j = n+1$, $s_{n+1} \in (x_n, x_{n+1})$ and the B-splines containing such a point are $B_n(x)$ and $B_{n+1}(x)$. In particular

$$B_n(s_{n+1}) = \frac{1}{8}, \quad B_{n+1}(s_{n+1}) = \frac{6}{8}.$$

Then

$$S^{(p-1)I} f(s_{n+1}) = \sum_{i=0}^{n+1} f_{p-1}(s_i)B_i(s_{n+1}) = \frac{1}{8} f_{p-1}(s_n) + \frac{6}{8} f_{p-1}(s_{n+1}).$$

Therefore

$$\begin{aligned}
&\Delta_{p-1} f_0 = f_0(s_0) - \frac{6}{8} f_{p-1}(s_0) - \frac{1}{8} f_{p-1}(s_1);\\
&\Delta_{p-1} f_i = f_0(s_i) - \frac{1}{8} f_{p-1}(s_{i-1}) - \frac{6}{8} f_{p-1}(s_i) - \frac{1}{8} f_{p-1}(s_{i+1}),\ i = 1, \dots, n;\\
&\Delta_{p-1} f_{n+1} = f_0(s_{n+1}) - \frac{1}{8} f_{p-1}(s_n) - \frac{6}{8} f_{p-1}(s_{n+1}).
\end{aligned} \tag{16}$$

3.2. PIA operator S^{pI} on boundary triple knot partitions

Now let us consider the case of partitions (2). The procedure is similar, but here the B-splines have different supports near a and b, so let us study

$$S^{(p-1)L} f(s_j) = \sum_{i=0}^{n+1} f_{p-1}(s_i) B_i(s_j),\ j = 0, \dots, n+1,$$

to get the correct value of $\Delta_{p-1} f_j$ in (11).

If $j = 0$, $s_0 = x_0$ and the B-splines containing such a point are $B_0(x), B_1(x)$ e $B_2(x)$. In particular

$$B_0(x_0) = 1,\quad B_1(x_0) = 0,\quad B_2(x_0) = 0.$$

Then

$$S^{(p-1)L} f(s_0) = \sum_{i=0}^{n+1} f_{p-1}(s_i) B_i(s_0) = f_{p-1}(s_0).$$

If $j = 1$, $s_1 \in (x_0, x_1)$ and the B-splines containing such a point are $B_0(x), B_1(x)$ e $B_2(x)$. In particular

$$B_0(s_1) = \frac{1}{4},\quad B_1(s_1) = \frac{5}{8},\quad B_2(s_1) = \frac{1}{8}.$$

Then

$$S^{(p-1)L} f(s_1) = \sum_{i=0}^{n+1} f_{p-1}(s_i) B_i(s_1) = \frac{1}{4} f_{p-1}(s_0) + \frac{5}{8} f_{p-1}(s_1) + \frac{1}{8} f_{p-1}(s_2).$$

If $j = 2, \dots, n-1$, $s_j \in (x_{j-1}, x_j)$ and the B-splines containing such a point are $B_{j-1}(x), B_j(x)$ e $B_{j+1}(x)$. In particular

$$B_{j-1}(s_j) = \frac{1}{8},\quad B_j(s_j) = \frac{6}{8},\quad B_{j+1}(s_j) = \frac{1}{8}.$$

Then

$$S^{(p-1)L} f(s_j) = \sum_{i=0}^{n+1} f_{p-1}(s_i) B_i(s_j) = \frac{1}{8} f_{p-1}(s_{j-1}) + \frac{6}{8} f_{p-1}(s_j) + \frac{1}{8} f_{p-1}(s_{j+1}).$$

If $j = n$, $s_n \in (x_{n-1}, x_n)$ and the B-splines containing such a point are $B_{n-1}(x), B_n(x)$ e $B_{n+1}(x)$. In particular

$$B_{n-1}(s_n) = \frac{1}{8},\quad B_n(s_n) = \frac{5}{8},\quad B_{n+1}(s_n) = \frac{1}{4}.$$

Then

$$S^{(p-1)L} f(s_n) = \sum_{i=0}^{n+1} f_{p-1}(s_i) B_i(s_n) = \frac{1}{8} f_{p-1}(s_{n-1}) + \frac{5}{8} f_{p-1}(s_n) + \frac{1}{4} f_{p-1}(s_{n+1}).$$

If $j = n+1$, $s_{n+1} = x_n$ and the B-splines containing such a point are $B_{n-1}(x), B_n(x)$ e $B_{n+1}(x)$. In particular

$$B_{n-1}(x_n) = 0,\quad B_n(x_n) = 0,\quad B_{n+1}(x_n) = 1.$$

Then

$$S^{(p-1)L} f(s_{n+1}) = \sum_{i=0}^{n+1} f_{p-1}(s_i) B_i(x_{n+1}) = f_{p-1}(x_{n+1}).$$

Therefore

$$\begin{aligned}
&\Delta_{p-1} f_0 = f_0(s_0) - f_{p-1}(s_0);\\
&\Delta_{p-1} f_1 = f_0(s_1) - \frac{1}{4} f_{p-1}(s_0) - \frac{5}{8} f_{p-1}(s_1) - \frac{1}{8} f_{p-1}(s_2);
\end{aligned}$$

$$\Delta_{p-1} f_i = f_0(s_i) - \frac{1}{8} f_{p-1}(s_{i-1}) - \frac{6}{8} f_{p-1}(s_i) - \frac{1}{8} f_{p-1}(s_{i+1}), \quad \text{con} \quad i = 2, \dots, n-1; \tag{17}$$
$$\Delta_{p-1} f_n = f_0(s_n) - \frac{1}{8} f_{p-1}(s_{n-1}) - \frac{5}{8} f_{p-1}(s_n) - \frac{1}{4} f_{p-1}(s_{n+1});$$
$$\Delta_{p-1} f_{n+1} = f_0(s_{n+1}) - f_{p-1}(s_{n+1}).$$

We remark that $\Delta_{p-1} f_0 = \Delta_{p-1} f_{n+1} = 0$, since

$$\begin{aligned}
\Delta_{p-1} f_0 =& f_0(s_0) - f_{p-1}(s_0) \\
=& f_0(s_0) - (f_{p-2}(s_0) + \Delta_{p-2} f_0) \\
=& f_0(s_0) - (f_{p-2}(s_0) + f_0(s_0) - f_{p-2}(s_0)) \\
=& f_0(s_0) - f_{p-2}(s_0) - f_0(s_0) + f_{p-2}(s_0) \\
=& 0.
\end{aligned}$$

Similarly it can be carried out for $\Delta_{p-1} f_{n+1} = 0$.

3.3. Polynomial reproduction

Now we present a result on polynomial reproduction, that holds for partitions (2), but not for partitions (1).

Theorem 3.1. *In case of partitions* (2) *the operator S^{pI} reproduces linear polynomials, as in the base case of operator S, i.e.*

$$S^{pI} f(x) = f(x) \text{ if } f \in \mathbb{P}_1.$$

Proof. If $f \in \mathbb{P}_1$, then all $\Delta_{p-1} f_i$ in (17) are zero. We show it by induction. The case $i = 0$ has already been shown and $\Delta_{p-1} f_0 = 0$ holds for all f, so $f_p(s_0) = f_{p-1}(s_0) = f_0(s_0)$. Similarly in the case $i = n+1$ where $\Delta_{p-1} f_{n+1} = 0$ for all f, so $f_p(s_{n+1}) = f_{p-1}(s_{n+1}) = f_0(s_{n+1})$.

Let be $i = 1$ and set $p = 1$ as base condition. In particular from (17)

$$\begin{aligned}
\Delta_0 f_1 =& f_0(s_1) - \frac{1}{4} f_0(s_0) - \frac{5}{8} f_0(s_1) - \frac{1}{8} f_0(s_2) \\
=& \frac{3}{8} f_0(s_1) - \frac{1}{4} f_0(s_0) - \frac{1}{8} f_0(s_2).
\end{aligned}$$

Setting $f(x) = x$, by definition of s_i in partitions (2) we get

$$\begin{aligned}
\Delta_0 f_1 =& \frac{3}{8}\left(a + \frac{1}{2}h\right) - \frac{1}{4}a - \frac{1}{8}\left(a + \frac{3}{2}h\right) \\
=& \frac{3}{8}a + \frac{3}{16}h - \frac{1}{4}a - \frac{1}{8}a - \frac{3}{16}h \\
=& 0.
\end{aligned}$$

From (11) $f_1(s_1) = f_0(s_1)$ holds.

Now let us assume $f_{p-1}(s_1) = f_{p-2}(s_1)$, that by recursion coincides with $f_0(s_1)$, hold and prove it holds for a general p:

$$\begin{aligned}
\Delta_{p-1} f_1 =& f_{p-1}(s_1) - \frac{1}{4} f_{p-1}(s_0) - \frac{5}{8} f_{p-1}(s_1) - \frac{1}{8} f_{p-1}(s_2) \\
=& \frac{3}{8} f_{p-1}(s_1) - \frac{1}{4} f_{p-1}(s_0) - \frac{1}{8} f_{p-1}(s_2) \\
=& \frac{3}{8} f_0(s_1) - \frac{1}{4} f_0(s_0) - \frac{1}{8} f_0(s_2).
\end{aligned}$$

For $f(x) = x$ by definition of s_i for partitions (2) we have

$$\begin{aligned}
\Delta_{p-1} f_1 =& \frac{3}{8}\left(a + \frac{1}{2}h\right) - \frac{1}{4}a - \frac{1}{8}\left(a + \frac{3}{2}h\right) \\
=& \frac{3}{8}a + \frac{3}{16}h - \frac{1}{4}a - \frac{1}{8}a - \frac{3}{16}h \\
=& 0.
\end{aligned}$$

From (11) $f_p(s_1) = f_{p-1}(s_1) = f_0(s_1)$ follows.

Let be $i = 2, \dots, n-1$ and set $p = 1$ as base condition. In particular from (17)

$$\begin{aligned}
\Delta_0 f_i =& f_0(s_i) - \frac{1}{8} f_0(s_{i-1}) - \frac{6}{8} f_0(s_i) - \frac{1}{8} f_0(s_{i+1}) \\
=& \frac{2}{8} f_0(s_i) - \frac{1}{8} f_0(s_{i-1}) - \frac{1}{8} f_0(s_{i+1}).
\end{aligned}$$

Setting $f(x) = x$, by definition of s_i in partitions (2) we get

$$\begin{aligned}\Delta_0 f_i =& \frac{2}{8}\left(a + \frac{2i-1}{2}h\right) - \frac{1}{8}\left(a + \frac{2i-3}{2}h\right) - \frac{1}{8}\left(a + \frac{2i+1}{2}h\right)\\ =& \frac{2}{8}a + \frac{2i-1}{8}h - \frac{1}{8}a - \frac{2i-3}{16}h - \frac{1}{8}a - \frac{2i+1}{16}h\\ =& 0.\end{aligned}$$

From (11) $f_1(s_i) = f_0(s_i)$ holds.

Now let us assume $f_{p-1}(s_i) = f_{p-2}(s_i)$, that by recursion coincides with $f_0(s_i)$, hold and prove it holds for a general p:

$$\begin{aligned}\Delta_{p-1} f_i =& f_{p-1}(s_i) - \frac{1}{8}f_{p-1}(s_{i-1}) - \frac{6}{8}f_{p-1}(s_i) - \frac{1}{8}f_{p-1}(s_{i+1})\\ =& \frac{2}{8}f_{p-1}(s_i) - \frac{1}{8}f_{p-1}(s_{i-1}) - \frac{1}{8}f_{p-1}(s_{i+1})\\ =& \frac{2}{8}f_0(s_i) - \frac{1}{8}f_0(s_{i-1}) - \frac{1}{8}f_0(s_{i+1}).\end{aligned}$$

For $f(x) = x$ by definition of s_i for partitions (2) we have

$$\begin{aligned}\Delta_{p-1} f_i =& \frac{2}{8}\left(a + \frac{2i-1}{2}h\right) - \frac{1}{8}\left(a + \frac{2i-3}{2}h\right) - \frac{1}{8}\left(a + \frac{2i+1}{2}h\right)\\ =& \frac{2}{8}a + \frac{2i-1}{8}h - \frac{1}{8}a - \frac{2i-3}{16}h - \frac{1}{8}a - \frac{2i+1}{16}h\\ =& 0.\end{aligned}$$

From (11) $f_p(s_i) = f_{p-1}(s_i) = f_0(s_i)$ follows.

Finally let be $i = n$ and set $p = 1$ as base condition. In particular from (17)

$$\begin{aligned}\Delta_0 f_n =& f_0(s_n) - \frac{1}{8}f_0(s_{n-1}) - \frac{5}{8}f_0(s_n) - \frac{1}{4}f_0(s_{n+1})\\ =& \frac{3}{8}f_0(s_n) - \frac{1}{8}f_0(s_{n-1}) - \frac{1}{4}f_0(s_{n+1}).\end{aligned}$$

Setting $f(x) = x$, by definition of s_i in partitions (2) we get

$$\begin{aligned}\Delta_0 f_n =& \frac{3}{8}\left(b - \frac{1}{2}h\right) - \frac{1}{8}\left(b - \frac{3}{2}h\right) - \frac{1}{4}b\\ =& \frac{3}{8}b - \frac{3}{16}h - \frac{1}{8}b + \frac{3}{16}h - \frac{1}{4}b\\ =& 0.\end{aligned}$$

From (11) $f_1(s_n) = f_0(s_n)$ holds.

Now let us assume $f_{p-1}(s_n) = f_{p-2}(s_n)$, that by recursion coincides with $f_0(s_n)$, hold and prove it holds for a general p:

$$\begin{aligned}\Delta_{p-1} f_n =& f_{p-1}(s_n) - \frac{1}{8}f_{p-1}(s_{n-1}) - \frac{5}{8}f_{p-1}(s_n) - \frac{1}{4}f_{p-1}(s_{n+1})\\ =& \frac{3}{8}f_{p-1}(s_n) - \frac{1}{8}f_{p-1}(s_{n-1}) - \frac{1}{4}f_{p-1}(s_{n+1})\\ =& \frac{3}{8}f_0(s_n) - \frac{1}{8}f_0(s_{n-1}) - \frac{1}{4}f_0(s_{n+1}).\end{aligned}$$

For $f(x) = x$ by definition of s_i for partitions (2) we have

$$\begin{aligned}\Delta_{p-1} f_n =& \frac{3}{8}\left(b - \frac{1}{2}h\right) - \frac{1}{8}\left(b - \frac{3}{2}h\right) - \frac{1}{4}b\\ =& \frac{3}{8}b - \frac{3}{16}h - \frac{1}{8}b + \frac{3}{16}h - \frac{1}{4}b\\ =& 0.\end{aligned}$$

From (11) $f_p(s_n) = f_{p-1}(s_n) = f_0(s_n)$ holds.

From all these facts $f_p(s_i) = f_{p-1}(s_i)$ follows for all p and $i = 0, \dots, n+1$. Then

$$S^{pI} f(x) = \sum_{i=0}^{n+1} f_0(s_i) B_i(x).$$

Therefore, since S reproduces $\mathbb{P}_1$, we get the thesis. $\square$

Remark 3.1. Note that the above theorem does not hold for partitions (1). This is due to the definition of $\Delta_{p-1} f_i$ in (16), when $i = 0, n+1$, while in cases $i = 1, \dots, n$ the behaviour is similar to what happens for partitions (2). Indeed, if $i = 0$ and $p = 1$, from (16) it results

$$\begin{aligned}\Delta_0 f_0 =& f_0(s_0) - \frac{6}{8} f_0(s_0) - \frac{1}{8} f_0(s_1)\\ =& \frac{2}{8} f_0(s_0) - \frac{1}{8} f_0(s_1).\end{aligned}$$

For $f(x) = x$ by definition of s_i for partitions (1) we have

$$\begin{aligned}\Delta_0 f_0 =& \frac{2}{8}\left(a - \frac{1}{2}h\right) - \frac{1}{8}\left(a + \frac{1}{2}h\right)\\ =& \frac{2}{8}a - \frac{2}{16}h - \frac{1}{8}a - \frac{1}{16}h\\ =& \frac{1}{8}a - \frac{3}{16}h.\end{aligned}$$

Going on this way, increasing p, in general we never get $\Delta_{p-1} f_0 = 0$.
A similar argument holds for $i = n+1$. Indeed, for $p = 1$, from (16)

$$\begin{aligned}\Delta_0 f_{n+1} =& f_0(s_{n+1}) - \frac{1}{8} f_0(s_n) - \frac{6}{8} f_0(s_{n+1})\\ =& \frac{2}{8} f_0(s_{n+1}) - \frac{1}{8} f_0(s_n).\end{aligned}$$

For $f(x) = x$ by definition of s_i for partitions (1) we have

$$\begin{aligned}\Delta_0 f_{n+1} =& \frac{2}{8}\left(b + \frac{1}{2}h\right) - \frac{1}{8}\left(b - \frac{1}{2}h\right)\\ =& \frac{2}{8}b + \frac{2}{16}h - \frac{1}{8}b + \frac{1}{16}h\\ =& \frac{1}{8}b + \frac{3}{16}h.\end{aligned}$$

Going on this way, increasing p, in general we never get $\Delta_{p-1} f_{n+1} = 0$.
This fact holds for any n.

3.4. Convergence

In order to provide results on convergence, we recall some properties presented in [4,9,10,17].

Theorem 3.2. *[17] Let $U = (u_0, \dots, u_n)$ be a blending basis. Then, if the basis U is totally positive (TP) and its collocation matrix M at $t_0 < t_1 < \dots < t_n$ is nonsingular, the curve $\gamma^0(t) = \sum_{i=0}^{n} P_i^0 u_i(t)$ satisfies the PIA property, where t_i is the assigned parameter value of the point P_i^0, $i = 0, \dots, n$.*

Thus such a sufficient condition ensures the convergence of PIA procedure. However we are interested in bases with fastest convergence rates in a given space with normalized totally positive (NTP) bases.

Definition 3.1. [4,9] Let $U = (u_0, \dots, u_n)$ be a TP basis of a vector space of functions $\mathcal{U}$. Then we say that $(u_0, \dots, u_n)$ is a B-basis if for any other TP basis $(v_0, \dots, v_n)$ of $\mathcal{U}$ the matrix K of change of basis $(v_0, \dots, v_n) = (u_0, \dots, u_n)K$ is TP.

Let us recall that, if a vector space of functions has a TP (resp., NTP) basis, then it has a B-basis (resp., normalized B-basis). It has been proved [4] that in a vector space of functions with NTP bases its normalized B-basis has optimal shape preserving properties among all the NTP bases of the space.

Theorem 3.3. *[10] Given a space of functions $\mathcal{U}$ with an NTP basis, the normalized B-basis provides a PIA with the fastest convergence rate among all NTP bases of $\mathcal{U}$.*

A necessary and sufficient condition for PIA convergence is $\rho(I - M) = 1 - \min_{0 \le i \le n} \lambda_i < 1$, where $\rho(B)$ is the spectral radius of matrix B and $\{\lambda_k\}_{k=0}^{n}$ are the $n+1$ eigenvalues of M. So the higher $\min_{0 \le i \le n} \lambda_i$ is, the faster is the convergence of PIA procedure [9].
An example of B-basis for polynomial spaces is the Bernstein basis. For such a basis the minimal eigenvalue of the corresponding collocation matrix decreases quickly as the degree increases, so in this case PIA method is practically not interesting for its slow convergence. In contrast to the case of polynomials, when applying PIA with a B-spline basis, that is a B-basis and so the optimal NTP basis of the spline space [4], the convergence rate keeps high, even for high dimensional spaces, if degree is kept low. Indeed

the minimal eigenvalue of the collocation matrix M of the B-spline basis of degree 2 keeps almost constant (from 0.45 to 0.49) in spite of the dimension $n = 10, \dots, 100$, while in case of B-spline of degree 3 the convergence is slower because of the behaviour of the minimal eigenvalue of the corresponding collocation matrices, when n increases, since it is smaller [9]. So this is a good reason to use quadratic splines.

4. Numerical integration based on S^{pI}

Approximating the integral $\mathcal{I}(f) = \int_a^b f$ of a function f on an interval $[a,b]$ by quadrature formulas based on QI splines is a well known topic in literature (see for example the wide bibliography in [8]). For the operator S we have

$$\mathcal{I}_S(f) = \sum_{i=0}^{n+1} f(s_i) \int_a^b B_i.$$

Since $\omega_i = \int_a^b B_i = \frac{x_{i+1} - x_{i-d}}{d+1}$, $i = 0, \dots, n+1$, here for degree $d = 2$ and symmetric partitions, for which $\omega_{n-i+1} = \omega_i$, $i = 0, \dots, n+1$, we have $\omega_0 = \omega_{n+1} = \frac{h}{3}$, $\omega_1 = \omega_n = 2\frac{h}{3}$, $\omega_j = h$, $j = 2, \dots, n-1$. Then

$$\mathcal{I}_S(f) = \sum_{i=0}^{n+1} \omega_i f(s_i) = \frac{h}{3}\left[f(s_0) + f(s_{n+1})\right] + \frac{2h}{3}\left[f(s_1) + f(s_n)\right] + h\sum_{i=2}^{n-1} f(s_i). \tag{18}$$

A quadrature formula $\mathcal{I}_{S^{pI}}(f)$ associated to S^{pI} in (10) is consequently obtained by substituting f with f_p in (18), in both cases of partitions (1) and (2), only taking into account the definition of the corresponding s_i's.

5. Another approach to PIA property

Following [9], now we approach PIA property by considering the progressive iteration on the blending functions, instead of on the functional coefficients, in order to keep the initial data unchanged.

So, starting from $S^{0I} f$ in (6), in order to compute the second function of the sequence $\{S^{pI} f\}_{p=0}^{\infty}$, we modify the basis instead of modifying the coefficients. Then we have (9), rewritten as

$$S^{1I} f(x) = \sum_{i=0}^{n+1} f_1(s_i) B_i^0(x)$$

and, taking into account (8), rewritten as

$$f_1(s_i) = f_0(s_i) + \Delta_0 f_i = 2f_0(s_i) - \sum_{j=0}^{n+1} f_0(s_j) B_j^0(s_i),$$

we have

$$S^{1I} f(x) = \sum_{i=0}^{n+1} f_0(s_i) 2B_i^0(x) - \sum_{i=0}^{n+1}\sum_{j=0}^{n+1} f_0(s_j) B_j^0(s_i) B_i^0(x). \tag{19}$$

Changing the indices i and j in the second sum on the right hand side of (19), we can write

$$S^{1I} f(x) = \sum_{i=0}^{n+1} f_0(s_i)\left[2B_i^0(x) - \sum_{j=0}^{n+1} B_i^0(s_j) B_j^0(x)\right].$$

Denoting $B_i^1(x) := 2B_i^0(x) - \sum_{j=0}^{n+1} B_i^0(s_j) B_j^0(x)$, $i = 0, 1, \dots, n+1$, it results $S^{1I} f(x) = \sum_{i=0}^{n+1} f_0(s_i) B_i^1(x)$. Iterating this process, we get $S^{pI} f(x) = \sum_{i=0}^{n+1} f_0(s_i) B_i^p(x)$, where, similarly to (15), but this time acting on the B-splines and not on the coefficients, we get the new blending functions

$$B_i^p(x) := (p+1)B_i^0(x) - \sum_{j=0}^{n+1}\sum_{r=0}^{p-1} B_i^r(s_j) B_j^0(x), i = 0, 1, \dots, n+1, p \geq 0. \tag{20}$$

In [9] the example of an NTP basis of $\mathbb{P}_n$ is considered, where by Theorem 3.2 the authors can state that the PIA approximant so defined tends to the Lagrange interpolant of degree n as p tends to infinity. In particular the i-th polynomial of the NTP basis tends to the i-th Lagrange interpolation polynomial, while the functional coefficients do not change with iterations. So, if we choose the Bernstein basis of degree n, that is the normalized B-basis of $\mathbb{P}_n$, by Theorem 3.3 we get the fastest convergence to the Lagrange interpolation polynomials.

Now, if in the B-spline case the iterations modify the coefficients, and not the blending functions, as we have shown in Section 3, by Theorem 3.2 such coefficients asymptotically tend to the unique coefficients obtained by solving the linear system coming from the

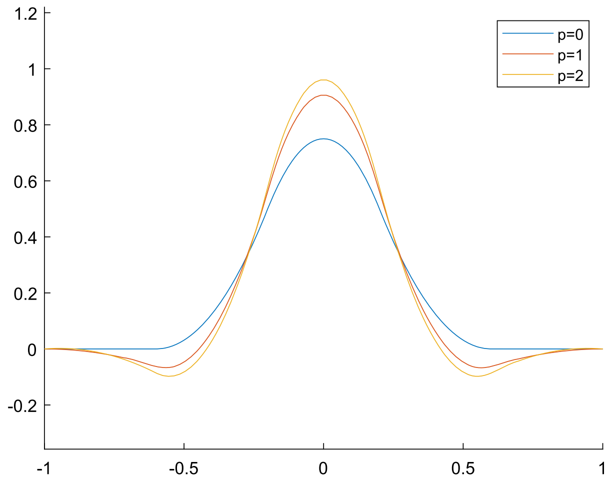

Fig. 1. $B_4^p(x)$, $p=0,1,2$, $x\in[-1,1]$, defined on a partition of type (2) with $n=5$. (For interpretation of the colours in the figure, the reader is referred to the web version of this article.)

interpolation conditions. However, if we proceed as in the above polynomial case, i.e. by keeping the coefficients fixed and modifying the B-spline basis, from (12) we can write:

$$f(s_i)=\lim_{p\to\infty} S^{pI}f(s_i)=\sum_{j=0}^{n+1} f_0(s_j)\lim_{p\to\infty} B_j^p(s_i), i=0,\dots,n+1.$$

By PIA property, while p increases, we know that $S^{pI}f$ tends to the unique spline interpolating f at the Greville sites, so

$$\lim_{p\to\infty} B_j^p(x), j=0,\dots,n+1,$$

comes out to be the unique j-th asymptotic nodal blending piecewise polynomial, where with *nodal* we mean a behaviour as Lagrange interpolation polynomials, i.e.

$$\lim_{p\to\infty} B_j^p(s_i)=\begin{cases}1 & \text{if } j=i\\ 0 & \text{if } j\neq i\end{cases}, i,j=0,\dots,n+1.$$

In Fig. 1 we show how B_4^p modifies its shape when $p=0,1,2$.

Consequently we can also immediately rewrite the corresponding quadrature formula, given in Section 4, taking into account the new approach above described, as

$$\mathcal{I}_{S^{pI}}(f)=\sum_i f(s_i)w_i$$

with

$$w_i=\int_a^b B_i^p=\left[(p+1)\omega_i-\sum_j\sum_r B_i^r(s_j)\omega_j\right].$$

6. Numerical results

Now we present some numerical results, obtained both on approximation and numerical integration and based on the operators studied throughout this paper, applied to the following test functions:

- $\varphi_1(x)=2x, \quad x\in[0,1];$
- $\varphi_2(x)=\frac{1}{1+16x^5}, \quad x\in[0,1];$
- $\varphi_3(x)=2x^2-5x+4, \quad x\in[2,3.5];$
- $\varphi_4(x)=\frac{1}{3}e^{(-\frac{81}{16}(x-\frac{1}{2})^2)}, \quad x\in[1.5,6];$
- $\varphi_5(x)=\sin(4.5x), \quad x\in[1.5,3];$
- $\varphi_6(x)=\arctan(100(x-0.3)), \quad x\in[2,4];$
- $\varphi_7(x)=e^{-x}\sin(5\pi x), \quad x\in[2.8,5].$

Table 1
Maximum norm of $\Delta_{p-1} f_i$, $i = 0, \ldots, n+1$, $n = 30$, in (16) and in (17) for increasing values of p and for the two test functions φ_2 and φ_7 in case of partitions (1) and (2), respectively.

	$\varphi_2(x)$, $x \in [0,1]$		$\varphi_7(x)$, $x \in [2.8,5]$	
p	partitions (1)	partitions (2)	partitions (1)	partitions (2)
8	1.19(−04)	1.80(−07)	1.22(−05)	1.69(−06)
16	2.31(−07)	4.71(−10)	2.28(−08)	4.10(−09)
32	1.77(−12)	4.94(−15)	1.71(−13)	4.20(−14)
40	5.44(−15)	5.55(−17)	5.27(−16)	1.46(−16)
48	5.55(−17)	5.55(−17)	7.59(−18)	5.20(−18)
64	5.55(−17)	5.55(−17)	3.47(−18)	5.20(−18)
96	5.55(−17)	5.55(−17)	3.47(−18)	5.20(−18)

Table 2
Maximum norm of the error in the approximation of the test function φ_1 for increasing values of n: on the left for partitions (1) and on the right for partitions (2).

n	E_S	$E_{S^{1I}}$	$E_{S^{2I}}$	E_S	$E_{S^{1I}}$	$E_{S^{2I}}$
12	4.44(-16)	1.41(-01)	1.58(-01)	4.44(-16)	6.66(-16)	6.66(-16)
28	6.66(-16)	1.32(-01)	1.48(-01)	4.44(-16)	4.44(-16)	4.44(-16)
56	4.44(-16)	1.28(-01)	1.44(-01)	4.44(-16)	4.44(-16)	4.44(-16)
112	4.44(-16)	1.27(-01)	1.43(-01)	4.44(-16)	4.44(-16)	4.44(-16)
224	4.44(-16)	1.25(-01)	1.42(-01)	4.44(-16)	4.44(-16)	4.44(-16)

6.1. Numerical results on approximation

First of all, in order to verify (13), i.e. the convergence from approximation to interpolation, two examples of decreasing $\Delta_{p-1} f_i$, while increasing p with $n = 30$, are shown in Table 1 for partitions (1) and (2), respectively. From Table 1 we can see that for increasing p PIA operator tends to interpolate the given functions at Greville sites, as in the parametric case it tends to interpolate the control points, iteration after iteration.

Now for $Q = S, S^{1I}, S^{2I}$ we set $E_Q(f) = \|f - Q(f)\|_\infty$, evaluated on 500 equispaced points in $[a,b]$. Moreover we define the numerical (or observed) approximation order as follows:

$$r_Q = \log_2 \frac{E_Q|_{n=7\cdot 2^{l-1}}}{E_Q|_{n=7\cdot 2^l}} \quad \text{for} \quad Q = S, S^{1I}, S^{2I} \quad \text{and} \quad l = 3,4,5,$$

with $E_Q := E_Q(f)$, when it is not necessary to point out f.

Unfortunately results in case of partitions (1) are comparable or worse with respect to the ones of the base operator S, so we do not report them here, except the ones in Table 2, where $\mathbb{P}_1$ reproduction for PIA operators on partitions (2) is confirmed.

Conversely, as we can see in Tables 3-4, for increasing n in general PIA operators improve the results obtained with the corresponding base operator S on partitions (2), already at the first iteration of p, i.e. $p = 1$.

In Tables 5-6 the maximum norm of the error and the corresponding numerical approximation order are presented for the test function φ_3 on partitions (2), confirming the asymptotic interpolation, as p tends to infinity. However, since here the maximum norm on $[a,b]$ and not at the Greville points is reported, one could even conjecture the 'asymptotic reproduction' of $\mathbb{P}_2$.

6.2. Numerical results on integration

For numerical integration we compare results obtained by quadratures $\mathcal{I}_S(f)$ and $\mathcal{I}_{S^{1I}}(f)$ to evaluate the integral of some functions whose exact value is known.

Let us set $E_Q(f) = \mathcal{I}(f) - \mathcal{I}_Q(f)$ and

$$r_Q = \log_2 \frac{|E_Q|_{n=2^{l-1}}}{|E_Q|_{n=2^l}} \quad \text{for} \quad Q = S, S^{1I}, S^{2I} \quad \text{with} \quad l = 8,9,10.$$

In Tables 7-8 we approximate

$$\mathcal{I}(g_1) = \int_{-1}^{1} \frac{1}{1+16x^2}\,dx = \frac{1}{4}\Big[\arctan(4) - \arctan(-4)\Big] \approx 0.662908831834016,$$

Table 3
Maximum norm of the error in the approximation of test function φ_2-φ_7 for increasing values of n on partitions (2).

	n	E_S	$E_{S^{1I}}$	$E_{S^{2I}}$		E_S	$E_{S^{1I}}$	$E_{S^{2I}}$
φ_2	12	8.17(-03)	1.21(-03)	6.92(-04)	φ_5	3.93(-02)	5.39(-03)	2.31(-03)
	28	1.57(-03)	5.61(-05)	3.49(-05)		7.23(-03)	9.49(-04)	3.08(-04)
	56	3.95(-04)	9.80(-06)	4.40(-06)		1.82(-03)	2.37(-04)	7.92(-05)
	112	9.91(-05)	1.35(-06)	5.56(-07)		4.54(-04)	3.56(-05)	6.99(-06)
	224	2.48(-05)	6.34(-08)	6.32(-08)		1.13(-04)	2.18(-07)	6.37(-07)
φ_3	12	7.81(-03)	1.29(-03)	4.40(-04)	φ_6	1.07(-05)	2.12(-06)	8.11(-07)
	28	1.43(-03)	2.34(-04)	8.05(-05)		2.28(-06)	4.12(-07)	1.38(-07)
	56	3.59(-04)	5.84(-05)	2.01(-05)		6.07(-07)	1.04(-07)	3.55(-08)
	112	8.97(-05)	8.46(-06)	1.45(-06)		1.57(-07)	1.55(-08)	2.88(-09)
	224	2.24(-05)	6.33(-15)	1.90(-07)		3.92(-08)	3.74(-11)	3.07(-10)
φ_4	12	5.12(-04)	3.39(-04)	2.78(-04)	φ_7	2.59(-02)	1.83(-02)	1.72(-02)
	28	2.04(-04)	8.16(-05)	5.21(-05)		1.01(-02)	2.84(-03)	1.56(-03)
	56	8.03(-05)	2.08(-05)	6.33(-06)		2.60(-03)	1.72(-04)	8.06(-05)
	112	2.77(-05)	3.78(-06)	9.67(-07)		6.59(-04)	1.05(-05)	1.09(-05)
	224	6.86(-06)	5.21(-08)	4.00(-08)		1.66(-04)	1.31(-06)	1.58(-06)

Table 4
Numerical approximation order of test function φ_2-φ_7 for increasing values of n on partitions (2).

	n	r_S	$r_{S^{1I}}$	$r_{S^{2I}}$		r_S	$r_{S^{1I}}$	$r_{S^{2I}}$
φ_2	28	-	-	-	φ_5	-	-	-
	56	1.99	2.51	2.93		1.99	2.00	1.96
	112	1.99	2.86	2.99		2.00	3.73	3.50
	224	1.99	4.41	3.14		2.00	3.46	3.46
φ_3	28	-	-	-	φ_6	-	-	-
	56	2.00	2.00	2.00		1.99	1.99	1.96
	112	2.00	2.79	3.79		1.95	2.75	3.62
	224	1.99	30.00	1.57		2.00	8.70	3.23
φ_4	28	-	-	-	φ_7	-	-	-
	56	2.64	2.00	3.04		1.99	2.81	4.27
	112	1.54	1.54	3.04		1.98	2.09	2.88
	224	2.00	2.02	3.57		1.99	9.87	2.79

Table 5
Maximum norm $E_{S^{pI}}(\varphi_3)$ in $[2, 3.5]$ for increasing values of p in case of partitions (2).

	n	$E_{S^{2I}}$	$E_{S^{10I}}$	$E_{S^{20I}}$	$E_{S^{30I}}$	$E_{S^{35I}}$	$E_{S^{40I}}$
φ_3	12	4.39(-04)	7.00(-07)	4.77(-10)	3.73(-13)	1.15(-14)	3.55(-15)
	28	8.05(-05)	1.29(-07)	8.69(-11)	6.88(-14)	5.33(-15)	5.33(-15)
	56	2.01(-05)	3.21(-08)	2.17(-11)	1.78(-14)	5.33(-15)	5.33(-15)
	112	1.45(-06)	2.25(-09)	2.67(-12)	3.99(-15)	3.55(-15)	3.55(-15)
	224	1.90(-07)	5.62(-10)	6.68(-13)	3.99(-15)	3.55(-15)	3.55(-15)

Table 6
Numerical approximation order $r_{S^{pI}}(\varphi_3)$ in $[2, 3.5]$ for increasing values of p in case of partitions (2).

	n	$r_{S^{2I}}$	$r_{S^{10I}}$	$r_{S^{20I}}$	$r_{S^{30I}}$	$r_{S^{35I}}$	$r_{S^{40I}}$
φ_3	28	-	-	-	-	-	-
	56	2.00	2.00	2.00	1.95	1.11	-
	112	3.79	3.84	3.03	2.16	-	-
	224	2.93	2.00	1.99	-	-	-

Table 7
Absolute errors and numerical approximation order for $\mathcal{I}(g_1)$ and $\mathcal{I}(g_2)$ on partitions (2) for $p = 1$ and increasing values of n.

	$\mathcal{I}(g_1)$				$\mathcal{I}(g_2)$			
n	E_S	r_S	$E_{S^{1I}}$	$r_{S^{1I}}$	E_S	r_S	$E_{S^{1I}}$	$r_{S^{1I}}$
128	6.86(-06)		5.11(-08)		1.65(-04)		6.67(-07)	
256	1.70(-06)	2.00	6.23(-09)	3.03	4.13(-05)	1.99	8.40(-08)	2.99
512	4.24(-07)	2.00	7.70(-10)	3.01	1.03(-05)	2.00	1.05(-08)	3.00
1024	1.06(-07)	2.00	9.56(-11)	3.00	2.59(-06)	2.00	1.32(-09)	3.00

Table 8
Absolute errors and numerical approximation order for $\mathcal{I}(g_3)$ and $\mathcal{I}(g_4)$ on partitions (2) for $p = 1$ and increasing values of n.

	$\mathcal{I}(g_3)$				$\mathcal{I}(g_4)$			
n	E_S	r_S	$E_{S^{1I}}$	$r_{S^{1I}}$	E_S	r_S	$E_{S^{1I}}$	$r_{S^{1I}}$
128	2.03(-05)		5.09(-06)		1.63(-04)		5.43(-07)	
256	5.09(-06)	1.99	1.27(-06)	2.00	4.09(-05)	1.99	4.64(-08)	3.55
512	1.27(-06)	2.00	3.18(-07)	2.00	1.02(-05)	2.00	4.44(-09)	3.39
1024	3.18(-07)	2.00	7.95(-08)	2.00	2.56(-06)	2.00	4.70(-10)	3.24

$$\mathcal{I}(g_2) = \int_{-1}^{1} xe^x dx = \frac{2}{e} \approx 0.735758882342885,$$

$$\mathcal{I}(g_3) = \int_{0}^{1} |x^2 - 0.25| dx = \frac{1}{4},$$

$$\mathcal{I}(g_4) = \int_{0}^{1} e^{-x} \sin(5\pi x) dx = \frac{5\pi(e+1)}{e(25\pi^2+1)} \approx 0.086730404755780.$$

7. Conclusions

In conclusion, while for partitions (1) the results of base operator e PIA operators are comparable, so that we do not report them here, we can see a good improvement in case of partitions (2).

We remark that partitions (1) and (2) tend to identify while n increases, but not while p increases, so, if on one hand for both partitions convergence to interpolation holds while p increases, on the other hand it is reasonable that by increasing n we can get improvements on the error maximum norm. However, while convergence to identification of both partitions with increasing n is confirmed for MA approximation, here numerical evidence shows us that it seems not to be the same for PIA, for which partitions (2) give always better results. Moreover better results on the speed of convergence are obtained with PIA quadratic than cubic splines.

Another good reason for presenting PIA in the functional setting is to have a tool for approximation and asymptotic interpolation, without managing systems of linear equations.

Finally, for sake of clearness and completeness, in Table 9 we highlight similarities and differences of PIA versus MA. In general we can remark that PIA on partitions (2) and MA operators improve the results obtained by the corresponding base method and the numerical convergence orders confirm this fact. This means that for partitions (1) MA provides better results than PIA ones, while for partitions (2) the results obtained by PIA and MA are comparable. Moreover all polynomial reproduction results are confirmed.

CRediT authorship contribution statement

Elena Fornaca: Software, Investigation. **Paola Lamberti:** Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

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Appendix A. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.apnum.2024.08.014.

Table 9
A comparison between PIA and MA.

PIA	MA [12]
$Sf = S^{0I}f = \sum_{i=0}^{n+1} f(s_i)B_i$, with $s_i = \frac{x_{i-1}+x_i}{2}$, $i = 0, \dots, n+1$.	$Sf = S^{(0)}f = \sum_{i=0}^{n+1} f(s_i)B_i$, with $s_i = \frac{x_{i-1}+x_i}{2}$, $i = 0, \dots, n+1$.
$S^{1I}f = \sum_{i=0}^{n+1} f_1(s_i)B_i$, with $f_1(s_i) = f_0(s_i) + \Delta_0 f_i$, $\Delta_0 f_i = f_0(s_i) - Sf(s_i)$ and $f_0(s_i) = f(s_i)$.	$S^{1L}f = S^{(1)}f + S^{(0)}\Delta_1 f$, with $S^{(1)}f = \sum_{i=-1}^{\frac{n}{2}+2} f(s_i^{(1)})B_i^{(1)}$ and $\Delta_1 f = f - S^{(1)}f$.
$S^{2I}f = \sum_{i=0}^{n+1} f_2(s_i)B_i$, with $f_2(s_i) = f_1(s_i) + \Delta_1 f_i$ and $\Delta_1 f_i = f_0(s_i) - S^{1I}f(s_i)$.	$S^{2L}f = S^{(2)}f + S^{(1)}\Delta_2 f + S^{(0)}\Delta_1^2 f$, with $S^{(2)}f = \sum_{i=-1}^{\frac{n}{2}+2} f(s_i^{(2)})B_i^{(2)}$, $\Delta_2 f = f - S^{(2)}f$ and $\Delta_1^2 f = \Delta_2 f - S^{(1)}\Delta_2 f$.
$S^{pI}f = \sum_{i=0}^{n+1} f_p(s_i)B_i$, $S^{pI}f(x) = \sum_{i=0}^{n+1} f(s_i)B_i^p(x)$, with $f_p(s_i) = f_{p-1}(s_i) + \Delta_{p-1} f_i$, $\Delta_{p-1} f_i = f_0(s_i) - S^{(p-1)I}f(s_i)$. $B_i^p(x) := (p+1)B_i^0(x) - \sum_{j=0}^{n+1}\sum_{r=0}^{p-1} B_i^r(s_j)B_j^0(x)$, $i = 0, 1, \dots, n+1, p \geq 0$	$S^{pL}f = S^{(p)}f + S^{(p-1)}\Delta_p f(x) + \dots$ $\dots + S^{(1)}\Delta_2^{p-1} f + S^{(0)}\Delta_1^p f$, with $S^{(p)}f = \sum_{i=-1}^{\frac{n}{2^p}+2} f(s_i^{(p)})B_i^{(p)}$, $\Delta_p f = f - S^{(p)}f, \dots$ $\dots, \Delta_2^{p-1} f = \Delta_3^{p-2} f - S^{(2)}\Delta_3^{p-2} f$ and $\Delta_1^p f = \Delta_2^{p-1} f - S^{(1)}\Delta_2^{p-1} f$.
On partitions (1): no polynomial reproduction. On partitions (2): S^{pI}, $p \geq 0$ reproduce $\mathbb{P}_1$ and numerical evidence shows an asymptotic reproduction of $\mathbb{P}_2$.	S^{pL}, $p \geq 0$ reproduce $\mathbb{P}_1$. S^{pL}, $p \geq 1$ reproduce $\mathbb{P}_2$ on partitions (1). S^{pL}, $p \geq 1$ asymptotically reproduce $\mathbb{P}_2$ for increasing n on partitions (2).

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