EHNANCING REFLECTION ON THE CRITICAL ATTRIBUTES OF THE FIGURES: THE HEIGHT CHALLENGE GAME

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Geometrical concepts are characterized by the intertwining of theoretical and perceptual aspects, where examples play a crucial role from an educational perspective. We focus on the concept of height of triangles and propose an inquiry-game activity within GeoGebra, with the goal of prompting students to consider non-prototypical cases of triangles and enhancing their reflection on the critical attributes of this concept. After outlining a theoretical framework on the role of examples in geometrical reasoning, we present the Height Challenge Game and report some excerpt from a grade 7 classroom discussion. An emerging model for the role of different kind of figures at play in the game is proposed.

INTRODUCTION AND THEORETICAL FRAMEWORK

In most Italian primary school textbooks, the height of the triangles is exemplified mainly on equilateral or isosceles triangles. The choice to use these types of triangles can lead students to mistakenly think that the height divides the angle at the vertex in half and that its foot coincides with the midpoint of the base. In addition, in textbooks the base-height configuration is often represented following the horizontal-vertical orientation of the page, which may lead students to think that the base and the height of the triangle (and of other figures) must always be horizontal or vertical. In other words, elements not contained within the definition of height because they do not characterize the concept, are assumed as *critical attributes*, namely properties that an example of a concept must have in order to be considered as such (Hershkowitz & Vinner, 1983; Hershkowitz, 1987). The student's unexperienced eye is therefore deceived by accidental elements observable in the examples.

This kind of errors can be linked to spontaneous processes of categorization of things, people or events implemented in everyday life. In the field of cognitive psychology, the studies of Eleonor Rosch (1999) have highlighted the presence of two dimensions within the structure of the categorical system. In a *vertical dimension* the relationships between categories are established hierarchically on the basis of a three-level taxonomy: *superordinate*, *basic*, *subordinate*. For example, polygons - triangles - isosceles triangles represent the superordinate, basic and subordinate levels respectively. The *horizontal dimension* provides for an organization within the same category, which revolves around prototypes, namely "the clearest cases of category membership defined operationally by people's judgments of goodness of membership in a category" (Rosch, 1999, p.196). The relationship between formal-theorical-conceptual aspects and figural-diagrammatic-material ones has been at the core of

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research in geometry education. Fishbein (1993) introduced the expression of *figural concept* to indicate that the objects of geometrical reasoning "reflect spatial properties (shape, position, size), and at the same time possess conceptual qualities - such as ideality, abstractness, generality, perfection" (p.143). Building a figural concept means pursuing "the integration of conceptual and figural properties into a unitary mental structure, with the predominance of conceptual constraints over figural ones". (Fischbein 1993, p.156).

Referring more in general to mathematics, Tall and Vinner distinguished between a student's *personal conceptual image*, that is "[the] cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes […] built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (Tall & Vinner, 1981, p. 152) and the *formal definition of the concept*, which is established by the community of mathematicians within a certain theory. Hershkowitz (1987) spoke of *misconception* to indicate a conceptual image that is either partial or contains elements in conflict with the formal definition of the concept. In the intertwine between the conceptual and the figural dimensions, the use of prototypes represents a delicate element for learning geometry, since "popular examples" are often taken as prototypes of a concept (ibid.).

Applying this conceptual framework, we can hypothesize that when the height prototype for a student coincides with the height of the isosceles triangle, i.e. with an example of height belonging to a subordinate level, the student will show difficulties in recognizing and/or producing heights of geometric figures that belong to basic or higher level. It is possible that initially students will be able to operate correctly with this prototype at a subordinate level (therefore on isosceles and equilateral triangles), but later, when they will encounter examples of the same concept belonging to superordinate or basic levels (non-isosceles or non-equilateral triangles, quadrilaterals), the prototype at their disposal will lead them to make mistakes. Hershkowitz (1987) uses the expression *prototype effect* to characterize students' productions in which it is recognizable the selection of non-critical attributes and their extension to all the examples of a given mathematical object. The examples that come to mind to students depend strongly on the situations they encountered, so it is important that students meet different typologies of examples of a given concept. Referring to this variation, Watson and Mason (2005) stress the importance of developing students' rich *examples spaces*, indicating a metaphorical space that contains some examples and excludes others, in which different examples play different roles in structuring students' sense-making. It is of particular relevance that students experience, besides prototypical examples, also non-prototypical ones. In the next paragraph we will present an activity that we purposefully designed to this aim.

THE HEIGHT CHALLENGE GAME

In order to foster the enlargement of students' example space related to triangles and enhance their reflection on critical attributes of the height concept, we designed the

Height Challenge Game (HCG). The HCG is an inquiring-game activity in Geogebra, based on challenges between two players. The HCG may be found at https://www.geogebra.org/m/xnjuu2zc. In the challenge, students act on a given geometrical figure and each student plays a role: falsifier or verifier. The falsifier is the first to play: he "messes up" the triangle by dragging its vertices and chooses a vertex from which the verifier must position the height. This choice is made by clicking on one of "Vertex" check boxes: as a result, a segment is added to the figure as a candidate for the height of the triangle that starts from the chosen vertex. For example, in Figure 1a the falsifier has chosen vertex A, and the segment AE has appeared. E is the candidate foot of the height and can be dragged on the line passing through the base BC. Basing on visual perception, the verifier must drag the point E on the straight line in order to place the height in the correct position. Once the move is finished, in order to check if the height has been positioned correctly, students click on the "Test" check box, which displays the height segment built with the GeoGebra commands (in Fig. 1b, Test A is selected). If the two segments visually coincide, the verifier wins a point, otherwise the falsifier wins a point.

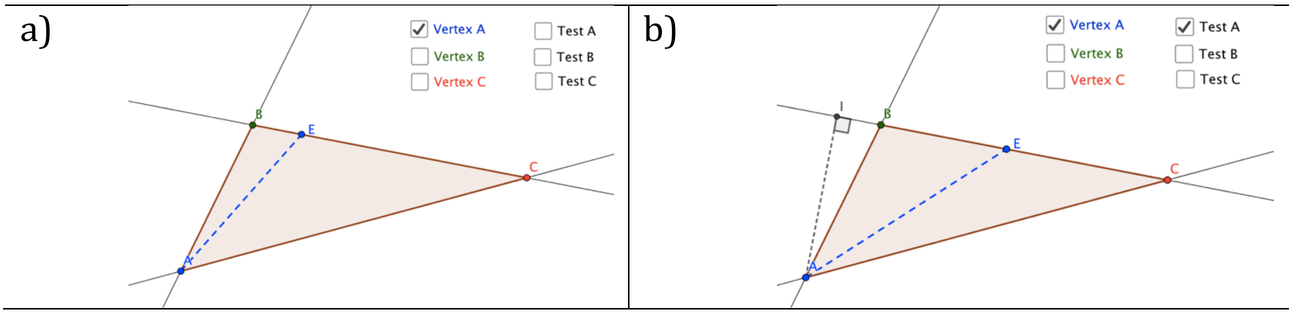

Figure 1: a) Effect of the "Vertex A" button. b) Effect of the "Test A" button

Inquiring-game activities are based on Hintikka's game theoretical semantic (1998). They have been used in previous research at secondary school level to investigate the effect of inquiring-game approach to geometry learning (Soldano & Arzarello, 2017, Soldano & Sabena, 2019). Differently from the previous inquiring-game activities, in the HCG an instrumented feedback is introduced (by means of the Test check box), to support students to visualize the correctness of the verifier's moves. We experimented the HCG in three 7th grade classes, where the height concept had been presented to students by their teachers in the previous year. In the experimented design, students are asked to play at least 4 times, exchanging their roles. At the end of the game, the students answer some questions contained in a written sheet. The first question (*When you were playing the verifier role, what did you pay attention to in order to place the height in the correct position?*) focuses students' attention on critical attributes of the height. The second one (*When you were playing the falsifier role, what geometric characteristics did the most difficult figures proposed to the verifier have?*) focuses on the falsifier's move and the possible use of non-prototypical configurations in the game. Finally, the third question (*When you played as a verifier, did you always manage to reach the goal? If yes, explain how you did it, otherwise explain why you* *did not succeed*) intends to investigate the formative role of the activity and see if, thanks to the feedback of the computer, students become aware of any errors and misconceptions. These questions were the ground of a collective discussion, in which we were present as participant observers, in collaboration with the classroom teacher.

EXCERPTS FROM THE CLASS DISCUSSION

R1 and R2 introduce the activity, videotape a couple of students while playing the game and the class discussion. The focuses of questions contained in the written sheet were proposed in the class discussion, of which we will analyse some extracts. We will consider words, gestures and written signs produced in the discussion as semiotic productions through which mathematical thinking evolves in the learning path of students, according to a multimodal perspective (Arzarello, 2006). Words, gestures and representations in GeoGebra will be analysed in order to investigate the critical attributes identified by students for the height concept and to have access to the students' personal conceptual image.

The discussion opens with a challenge on the IWB between two volunteer students. After the challenge, researcher R1 prompts students to reflect on their own processes:

1 R1: What did you watch in order to reach the goal as verifiers?

2 S1: Making the angles of 90 degree, that is, in order to make the height of a triangle you have to look at the angles of 90°

3 R1: Ok, come and show us which angles you were looking at. Can you point them out on the IWB?

4 S1: I was looking at this, the blue point, and that point ... [*he is indicating first a vertex and then one of the two angles that the height forms with the base, Figure 2a-b*]

5 T: Who wants to say something?

6 S2: I was looking… I was looking at the side on which to place the height point [*while walking towards the IWB, he raises the right arm horizontally, Figure 2c*]. I look at the side below and I look that the height is 90° [*he is raising his left forearm perpendicularly, Figure 2d*]

7 R1: So, can you tell us exactly where [you look]?

8 S2: I look at this here, this side below [*indicating the base, Figure 2e*] and I see that the height line is 90° [*placing his hand vertically, Figure 2f*]

In answering R1's question, S1 recalls "angles of 90°" (line 2) as a critical attribute for the height concept. He then indicates the vertex from which to trace the height and one of the right angles that the height forms with the base (line 4, Fig. 2a-b). The teacher gives the floor to another student, S2, who expresses the critical attributes in words and through gestures produced while walking towards the IWB.

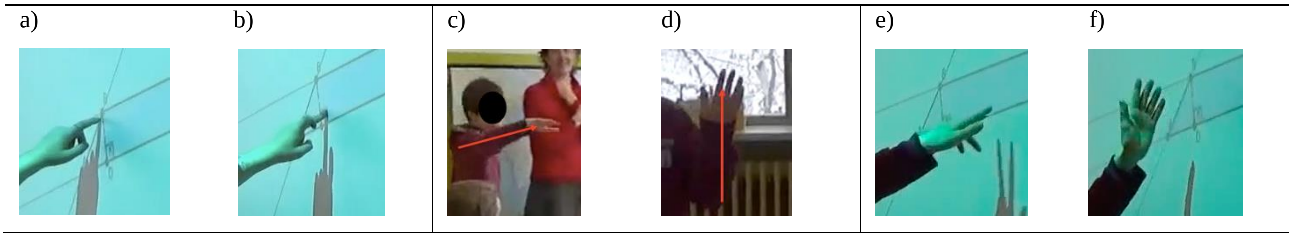

Figure 2: Students' gestures during the discussion.

Whereas S1 mentioned first the vertex of the triangle and then the right angle, S2 is first referring to the base of the triangle. As a matter of fact, he is first lifting the right arm horizontally and then, while lifting the left forearm vertically, he says that he looks for the height to be at 90° (line 6). The combination of the vertical and horizontal gestures makes the property of perpendicularity between base and height—which is actually a critical attribute for the height concept—visible. The discussion continues:

9 R2: And how can you conclude whether "it's 90" or "it is not"?

10 S2: You can see it

11 S1: You have to see if the two ... the height and the base are more or less at 90° ... if they are ... that ... [*he traces the base and height in the air with the two arms and then places his left hand perpendicular to the surface of the desk, Figure 3a-b*]

12 S3: Well yes, but we see that it is like this

13 R2: What do you mean? What is it that you see?

14 S3: You can see that it is an angle of 90° because it is like this [*he places his palms perpendicular to each other, Figure 3c-d-e*]

15 S4: It's L-shaped

16 S2: If I have in mind what a 90° angle does look like [*gestures with the two hands, as Figure 3f-g*]

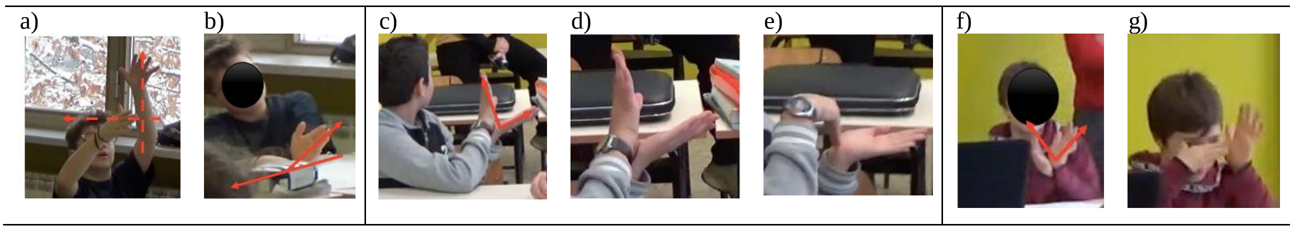

Figure 3: Students' gestures relating to a 90° angle.

To explain how to visually establish whether an angle measures 90°, the students produce gestural configurations with two hands/arms representing the perpendicular sides of angles. If we observe closely the pictures reported in Figure 3, we may identify cases in which hands or arms are placed in the horizontal-vertical orientations (Fig. 3 a, b, e, g), according to the prototypical stance, and cases in which hands or arms are arranged in a rotated position (Fig. 3 c, d, f)—let us remember that the triangle shown at the IBW has no horizontal or vertical sides. S2 and S3 shift fluently between one case and the other, indicating that their conceptual

images are not anchored to a prototypical representation and that their examples space include both prototypical and non-prototypical examples of height.
Then R2 direct students' attention to reflect on mistakes made while playing:

17 R2: During the game, did anyone find a case where they got it wrong? Where you say, I was really wrong, I put the height just somewhere else. [*S5 raises his hand*]. Please, can you [*S5*] show us what it happened? What kind of mistake [you made]? And then what did you do?

18 S5: [*He gets up and walks towards the IWB*] I put it in this way [*he moves F very close to B, Fig. 4a*]

19 S1: No, he did place it in the center [*S5 drags F to the midpoint of AB, Fig. 4b*]

a) b)

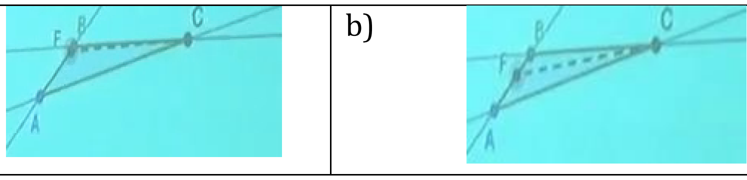

Figure 4: IWB figure proposed by S5 to discuss his mistake

S5's mistake consisted in attributing to the height a non-critical attribute, namely to have an extreme in the midpoint of a triangle side: this is true for isosceles cases, which are so often used in Italian school textbooks (and by many teachers) even if they belong to the subordinate category of triangles and not to the basic category. Such a mistake may be interpreted with reference to the prototype effect, in which a non-critical attribute for the concept of height has been erroneously attributed to it. During the game, the feedback from the computer allowed S5 to notice the inconsistency between where he expected the height to be (basing on his personal conceptual image), and where it was placed by the software. When prompted to reflect on his own mistake, the student brings it back to the attention of the whole class. Possibly, he has not yet realized *what* kind of mistake he did, as we may see in Fig. 4a, but his fellow S1 (challenger in the game) remembers precisely the mistake and is able to offer a description in words (line 19) which helps S5 to reproduce it (Fig. 4b) and may contribute to make him to reflect further.

The inconsistency between a student's incorrect personal conceptual image and the correct concept image can be perceived not only thanks to the feedback of the computer but also to the careful observation of the moves made by the challenger in the game:

20 R1: And this maybe happened at the beginning of the challenges? Or…

21 S1: Yes at the beginning, the second challenge

22 S3: I was always wrong!

23 R2: Who says that he was always wrong, then what did he do? Did they stop then making mistakes?

24 S3: I lost both challenges and then later, that is when we finished everything, when answering the questions on the sheet I understood how to do it

25 T: Did you then understand what was wrong?

26 S3: Yes, because I saw how he made them and then I understood how to put them

29 R2: But he also explained his trick to you or not ... you looked at

30 S3: No, I just looked at his moves

During the game, the careful observation of the moves made by a more experienced opponent takes on a formative role in understanding how to put the height and therefore can evolve the personal conceptual image of the student. The challenge partner is an opponent in the game but an ally in learning.

CONCLUSION

Identifying the critical attributes for the height concept and applying it in different examples (including non-prototypical ones) is crucial for winning the Height Challenge Game. From our observation, the students' engagement in the game and in reflecting on their thinking processes by means of the worksheet questions and during the discussion allows the teacher (and the students themselves) getting precious information on their personal conceptual images of the height of triangles. In addition, the game may enact an evolution of students' conceptual images, by means of the dialectical relationships between different kind of figures, as schematized in Figure 5.

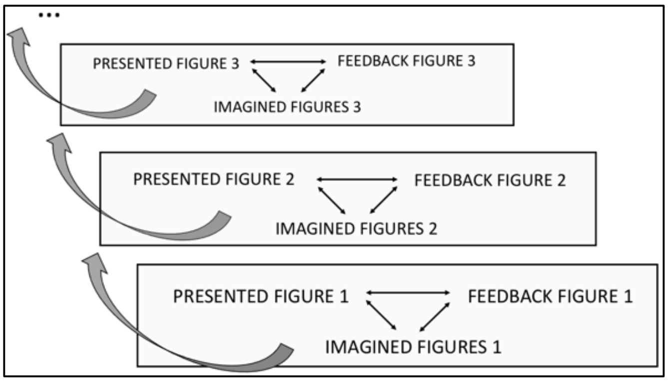

Figure 5: Evolution of the dialectical relationships between the different figures.

At the beginning of the game, the students have some personal conceptual images of height, which include the figures they imagine (Imagined Figures). Playing the verifier role, the figure imagined by the verifier is made visible through his move (Presented Figure). This move may show—to an expert's eye—possible students' misconceptions relative to the critical attributes of the geometric concept at stake. Thanks to the computer feedback, students are helped to notice these possible inconsistencies between the Imagined, Presented and Feedback Figure. Observing the inconsistencies offers the opportunity for a first evolution of the personal conceptual image of height. If such an evolution has occurred, then this will have effects in subsequent challenges where new Imagined, Presented and Feedback figures will be produced. The cycle repeats itself while playing. As the game is played by two students and the role are exchanged, it is likely that Imagined Figures of different kinds are proposed in the game, thus contributing to enlarging the students' example space. For the students who already have the correct conceptual image of height, the game plays a consolidating role. As part and parcel of our design and theoretical stance, the teacher's role is crucial for guiding the students in becoming aware of their personal concept image and for comparing it with the formal concept definition. From our observations, the HCG may offer rich material from which such a discussion may be orchestrated. In this paper we reported some brief excepts from the first experimentation, and further data will help us to validate the proposed model.

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