Entanglements of Mathematics Education Research and Large-Scale Assessment: Rethinking Formulas as Relational

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Entanglements of mathematics education research and large-scale assessment: Rethinking formulas as relational

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Abstract

In this article, we examine eighth graders' incorrect responses to a specific task on a national standardized assessment of mathematics. The task asked students to write the formula for the perimeter of a given figure as a function of a variable. We focus on incorrect responses to better understand students' difficulties with the algebraic thinking demanded by the task, especially with the formula and the variable. We show how these responses identify a variety of approaches to the solution of the task, which we name *routes*. We use these data to conclude that, at the end of middle school in our country, there still appears to be a lack of a relational view of formula, pointing to a need to reconceptualize formulas as relations rather than procedures. The strength of large-scale data in fueling mathematics education research is also discussed.

Keywords: formula; algebraic thinking; route; relational; assessment

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1. The relevance of formulas to mathematics

In this article, we investigate middle grade students' difficulties in solving a mathematical task that requires algebraic modelling. The task asked students to write the formula for the perimeter of a geometric figure as a function of a variable, providing the relevant information in a text and a figure. Solving the task involves algebraic thinking and the ability to grasp relationships from both text and figure. The geometric figure provides the initial context, but the target of the task is the function. Once relevant information is properly combined and translated into symbols, the perimeter of the figure is given by a simple linear function.

That the task focused on writing a formula is a central aspect of our study. At a large-scale, formulas are important for those STEM-oriented or business-related future studies and careers in which mathematical statements or investigations are required. The relevance of formulas to mathematics is traditionally associated with the fact that, to a great extent, they save time, space, and effort by reducing cognitive load and simplifying or facilitating mathematical practice. A formula directs attention, through symbols, to the relation or the operation expressed by it, often extricated by the length, bewilderment, or complexity of ordinary language.

In the elementary and middle grades, students use formulas as a murky translation of a mathematical statement from words, a furtive means to relate different domains (e.g., geometry and algebra) when modeling a situation, and a vivid site of encounter between arithmetic and algebra when used to measure areas and perimeters. Working with formulas in mathematics therefore evokes problematic pairs, such as conceptual and procedural, or relational and operational (see, for example, Arcavi, 1994; Sfard, 1991). Each pair foregrounds a tension between the meaning of formulas (conceptual, relational) and their application as rote procedures (procedural, operational).

Because we are interested in the activities of students in the classroom, at a closer look, we can see this tension in terms of the dynamic of explorative and ritual, two kinds of routines "at the extremes" (see Lavie et al., 2019 for a detailed discussion). While rituals are seen as inevitable in the initial stages of learning, de-ritualization must occur to gradually turn them into explorations. One of the problems we have in mathematics education—stress Lavie and colleagues—is that many of the routines taught in school are learned as rituals and there is insufficient de-ritualization. In this sense, a formula, like the area formula $A = b \times h$ for a rectangle with the base b and the height h, is a ritual when its use is limited to applying it as a static rule, like when it is used to calculate the value of A by inserting the known values of b and h. It must be the responsibility of the teacher to draw students' attention to things that should be related, especially if they are conceptually important.

A formula requires variable notation (using letters). We know that students often possess static images of variables as unknown values, and thus of formulas, with difficulty in conceptualizing quantities in a problem context, how they can be related, and how they vary together (e.g., Carlson et al., 2015; Thompson & Carlson, 2017). A formula also explicitly implicates the meaning of the equals sign, which is itself a source of misconceptions, particularly when interpreted under an operational view as indicating the need to compute an answer, rather than in a relational manner, in terms of equivalence (e.g., Carpenter et al., 2003; Kieran, 2006; Knuth et al., 2011; Molina & Ambrose, 2008).

Conceptually, we see that the formula (e.g., $A = b\times h$) involves an algebraic relationship between quantities (e.g., area, base and height), or a function when it expresses one quantity as dependent on another (e.g., $A = 2b\times b$ for a rectangle with one dimension twice the size of the other). In many instances, this is crucial to algebraic thinking (Blanton et al., 2011; Carraher et al., 2008). Researchers stress that "the growing body of evidence that K–grade 8 students *can* successfully reason algebraically about functional relationships opens the possibility that the difficulties exhibited by older students might stem from a lack of experiences with functional thinking in the elementary grades" (Stephens et al., 2017, pp. 397–398, *emphasis in the original*). With this focus on formulas, we ask what happens if we turn our attention to the challenge of reasoning functionally about algebraic relationships. We see formulas as being relations by nature, but what will students do when they solve the mathematical task of writing a formula and writing it as a function?

We use this introductory discourse as a point of departure for our article's focus on ways of interpreting formulas in grade 8, when students should be ready to progress to the more formal algebra of high school. Despite their role in algebraic or functional reasoning, formulas have not received extensive attention in mathematics education research, especially in relation to students' performance. While we want to draw attention to a notion that is central to mathematics teaching and learning, we are concerned about the culture of mathematics education in our country being often too much about procedural practices and rituals. We therefore focus on the cognitive challenges students encounter when experiencing conceptual tasks with formulas. Our main interest in this article is in how formulas are conceptualized at the end of middle school in our country and in the difficulty in solving the problem of writing a formula as a function in a given situation (we call this problem the "formula task"). To address this interest, we consider a sample of Italian students, who were given the formula task during a national standardized assessment of mathematics and analyze the incorrect answers to the task to better understand students' underlying view of the formula. In this way, we hope to contribute to a line of research that draws specific attention to formulas in mathematics teaching and learning and their relevance as algebraic tools, not only as arithmetic procedures (the way they are usually practiced and framed). We argue that a reframing of formulas is a promising lever for shifting traditional mathematics education towards more conceptual, explorative sense making practices.

2. Theoretical Framework

In this section, we focus our attention on the cognitive challenges students encounter when engaging in conceptual tasks with formulas. We situate the study described in this article within the field of algebraic thinking in elementary and middle grades (Kieran et al., 2016), because it involves the elementary symbolic treatment of variables. Milestone studies have increased our knowledge of algebraic thinking processes and their nature. Kieran (2004) pointed out that algebraic thinking in the lower grades may involve the development of ways of thinking within activities for which "the letter-symbolic" could be used as a tool, or it may refer, for example, to finding relationships among quantities and noticing structure without resorting to letters or symbols. For Arcavi (1994), even students who can handle algebraic techniques (or rituals) successfully do not necessarily use algebra as a tool to understand, express, and communicate generalizations, to reveal structure, or to establish connections. Briefly speaking, they do not possess "symbol sense," the algebraic component of mathematical sense-making. Following Arcavi, symbol sense comprises aspects such as: "understanding how and when symbols can and should be used in order to display relationships" and "awareness that one can successfully engineer symbolic relationships which express the verbal or graphical information needed to make progress in a problem, and the ability to engineer those expressions" (p. 31). These two aspects are especially relevant to the task analyzed in our study, which requires the symbolic grasp of the relationships provided in figural and textual form before writing a formula that connects them. In Arcavi's vision, algebraic reasoning is versatile and transformative, which is to say, different from procedural, ritual reasoning that often obscures conceptual understanding.

2.1. Variables and letter usage

The versatile and transformative character of algebraic reasoning seems challenging and difficult to achieve, especially in the lower grades, as it speaks directly to letter usage and understanding of the roles and multiple meanings of variables (Küchemann, 1978; Philipp, 1992; see also Bush & Karp, 2013 for an extensive review on middle school students' misconceptions). Both Philipp (1992) and Küchemann (1978) noted that variables can be seen and used in many ways. While Philipp focused on explaining the difficulty students encounter with variables as related to an inability to recognize the role of the variable, Küchemann studied how certain ways of using letters are typically perceived as less demanding than others, thus proposing a hierarchical structure of variables. According to Philipp, we can think of variables as labels, constants, unknowns, generalized numbers, varying quantities, parameters, or abstract symbols. Drawing on Küchemann, letters can be evaluated or ignored, and treated as objects, specific unknowns, generalized numbers or variables. For him, however, students understand the meaning of using symbols in algebra only when they can work with letters as variables. These two visions are entwined with each other. For example, letters standing as names or labels, such as p for the perimeter and A for the area of a geometric figure, are used as objects. Letters standing for unknown quantities that cannot be evaluated are used as specific unknowns, like in the following propositions: "Stefania is 2 cm taller than Francesca. Francesca's height is denoted by *a*. Therefore, Stefania's height is a+2" (adapted from Bush & Karp, 2013). Usiskin (1988) pointed out similar misconceptions where variables are viewed as simple labels or where students fail to understand a variable as a varying quantity rather than as a missing value. Usiskin also made diverse meanings of variables correspond with diverse conceptions of algebra: algebra as generalized arithmetic, when the variable represents any number; algebra as a tool to solve problems, when the variable is seen as an unknown; algebra as the study of relationships among quantities, when variables are varying quantities. These conceptions effectively reflect Küchemann's rising degrees of difficulty in letter usage.

2.2. Working with variables in context

Working with variables in context is a primary aspect of algebraic thinking, which implicates looking at numbers from a more structural perspective (Warren et al., 2016) and brings forth discontinuities with arithmetical thinking (Kieran, 2006; Malara & Navarra, 2003; Stacey & MacGregor, 2000), while supporting generalization processes and actions (e.g., Kaput et al., 2008; Radford, 2006). For example, Warren and colleagues made the case that prior arithmetical use of letters in formulas and as labels can negatively impact students' understanding of the variable. This aspect is particularly influenced by algebraic reasoning routines in the classroom. Typically, consistent use of the first letter of a word as the unknown is one of those habitual, context-based practices that confuse the difference between labels and variables (MacGregor & Stacey, 1997). The use of *p* and *A* for perimeter and area, as mentioned above, provides two instances related to measurement practices. The letter *n* used for a natural number gives another instance related to early algebraic practices.

In line with these studies, Álvarez and colleagues (2015) suggested that the complexity of variable as a concept exists because its meaning varies depending on the context, a factor advanced by Kieran (2007) as a matter of concern for algebraic reasoning and as an obstacle for learners when confronted with problems involving this notion. These researchers contend that for students to acquire a vision of variable as "a multifarious entity" or "a global entity with several sides" (Álvarez et al., 2015, p. 1512), students must work with each usage separately and develop flexibility to change from one use to another. Álvarez and colleagues described successful algebra problem solvers as being able to see variables as changing entities, specifically by recognizing the correspondence between related variables independent of the representation used (e.g., tables, graphs, word problems, or analytic expressions), and symbolizing a functional relation based on the analysis of the data of a problem. In the case of problems involving unknowns, instead, one is primarily required to recognize and identify the presence of something unknown that can be determined by considering the restrictions of the problem based on the situation.

2.3. Working with formulas

Working with formulas involves variables in context. Take again the case of $b \times h$ for the area of a rectangle, where b and h typically refer to the base and the height of the rectangle. Rather than seeing it as an algebraic expression, which does not depend on chosen lengths or specific letters, students easily see it as an arithmetic rule or operation to obtain the area of the figure using known numbers for base and height. The formula $A = b \times h$ embeds both procedural and conceptual aspects as soon as we contextualize it as a measurement or problem-solving task which requires the use of letters or symbols. The interplay of procedural and conceptual is mastered through problem solving. In fact, the formula $A = b \times h$ embeds a purely operational meaning—the area measurement for a given rectangle, and a more structural meaning—an open bundle of possibilities for the family of all rectangles as the area changes together with b and h. Usiskin (1988) would say that the formula describes a relationship among three quantities and that there is not the feel of an unknown, because we do not solve for anything. Additionally, the feel of formulas is different from the feel of generalizations, even though a formula is a special kind of generalization.

Stacey and MacGregor (2000) argued that difficulty in formulating or solving algebra word problems often has to do with a compulsion to calculate, which stems from arithmetic ways of solving problems and accounts for many of the misconceptions students experience in algebra learning. Interestingly, these researchers observed a variety of "routes" from problem to solution followed by grade 10 learners when solving simple problems. They identified non-algebraic routes where students used arithmetic reasoning or trial and error without attempting to use algebra; superficially algebraic routes where students wrote equations to solve the problem as formulas describing sequences of calculations to work out the answer from known information but did not use an algebraic approach; and algebraic routes where students engaged in equation writing and solving. Capraro and Joffrion (2006) also noted that writing equations from word problems is difficult for middle grades students due to both misconceptions or literal translation. Literal translation occurs when directly changing from natural language to symbolic expression, simply moving from left to right. So, a sentence like "Seven less than a number" is interpreted by many students as "$7–n$" instead of "$n–7$", in relation to the "less than" following the 7 in the sentence. For Capraro and Joffrion, students' inclination to engage in direct translation may be reinforced with procedural approaches, sometimes resulting in an inability to apply problem solving methods. Direct translation is also commonly associated with reversal errors (Bush & Karp, 2013).

Much of the research highlighted above does not focus specifically on the understanding of formulas. Consequently, we turn to research on the teaching and learning of measurement. A number of studies have considered conceptual difficulties that students face in understanding formulas. Many students learn and apply formulas for the area of simple figures without understanding why they work, making them prone to errors (Lehrer et al., 2003; Tan, 1998; Zacharos, 2006). Additionally, the problem of confusing area and perimeter, especially for rectangles, is widely documented and indicates weakness in grasping both formulas (Smith & Barrett, 2017). The challenge of distinguishing area from the length of the region's boundary extends into middle school for many students (Chappell & Thompson, 1999; Tan-Sisman & Aksu, 2012). These studies are particularly significant to this article since the formula task analyzed below concerns the perimeter of a simple trapezoid.

2.4. Formulas as relational

Recall that Stacey and MacGregor (2000) identified algebraic and non-algebraic routes that learners followed to solve problems. They identified the superficially algebraic route in accordance with Janvier's (1996) view of formulas, where formulas are conceptualized as defined procedures for computing or rules telling one what to do, distinct from equations. Janvier stressed that an equation as a statement about equality differs from an equation as a formula, which does not require any knowledge of algebra. In contrast, rather than conceptualizing the formula as a rule, which only relies upon the procedural, we conceptualize it as an algebraic statement, valuing the interplay of procedural and conceptual (operational and relational) aspects as the very essence of the formula (more in line with the intertwined nature of procedural and conceptual in mathematics knowledge embraced in the study by Capraro and Joffrion, 2006). From this perspective, solving problems that require formulas mobilizes symbol sense.

Returning to $A = b \times h$ (or any other formula), only the flexibility of moving between the rule and a view of the formula as an equality, where variables are varying quantities and the equals sign establishes relationships, can feed a *relational* understanding of the formula as an algebraic statement. It is in these circumstances that the formula can become for the students "a tool to establish connections" and to transform two (or more) variables. In our example, the changing b and h are transformed into a new variable, the changing area. This dynamic, *transformative* nature allows an individual to conceptualize the formula algebraically, in ways that involve structural relationships or functional reasoning depending on the context. In our study, the view of formulas as relational is important since the task at hand is to write a formula as a function of a variable.

2.5. Rationale and aim

The formula task considered in this study was administered in the context of a national computer-based assessment for grade 8. In our investigation, we are interested primarily in the "routes" followed by the students who solved the task, and in the particular and unstable nature of these routes in relation to problem solving. To that end, we focus on the incorrect answers that elude stability. In fact, the incorrect answers and errors students may include are associated with the struggles with mathematics they experience that are critical to meaning making and understanding (e.g., Borasi, 1987; Hiebert & Grouws, 2007; Granberg, 2016; Pozio, 2011). Taking the full sample of students from the national assessment, we can have a realistic glimpse of how and to what extent middle school students struggle with understanding formulas as relations in our country. We ask: What are the routes to an incorrect solution of the formula task?

What is the relationship between the routes and the students' knowledge concerning formulas and the use of variables?

Ultimately, we aim to contribute to the discussion on algebraic thinking in young learners by shedding light on how formulas are interpreted by grade 8 students. We believe this can help address the challenges of mathematics education research relative to the nature and processes of algebraic thinking, which has mostly disregarded formulas as an object of study, while largely considering mathematical structure and relationships as central to the practice of algebra (Blanton et al., 2011; Kaput, 2008; Kieran, 2007; Stephens et al., 2017).

3. Research Context and Methods

We first provide an overview of the research context, including the participants and the kind of research data that were collected. We then describe the mathematical task. Finally, we articulate our data analysis process.

3.1. Participants and data

The data for this article were collected from a national standardized computer-based assessment of mathematics conducted with Italian students at the conclusion of grade 8, as part of the evaluation of their middle school-based education. The assessment was administered by the National Institute for the Evaluation of the System of Education and Training (INVALSI), which tests students' knowledge and skills (proficiency) in mathematics, reading, and English in different grades. The computer-based testing (CBT) aligns with the trends of international surveys, such as PISA 2015 and eTIMSS 2019 (see Mullis & Martin; 2017; OECD, 2017).

The test assesses the entire student population but estimates of overall proficiency are obtained for a sample of students from the total number of participating classes. For the sample selection, schools are nationally sampled from all eligible schools, through size, geographical position, etc., and two classes are randomly sampled for each selected school. This process guarantees reliability and quality of the collected data, and the sample is considered representative of the entire population of learners, thereby providing information to establish national and regional standards.

The test reports student progress by means of empirically-based levels of knowledge and skills, defined as levels of mathematical proficiency, and estimated by the student performance on the test. Based on Rasch analysis, the assessment also provides estimates of the empirical difficulty of each item in the test (level of difficulty), using the same scale, depending on the proportion of participants who solve the item correctly. Briefly speaking, item difficulty can be seen as the location along the ability continuum at which a person is just as likely to answer the item correctly or not. The mathematics scale is divided into five proficiency levels, 1 to 5, lowest to highest, with items scaled according to their difficulty. Students' positions on the scale help indicate their proficiency based on the kind of tasks they are able to solve.

The subjects of our study took part in the national CBT of mathematics that was administered in grade 8 in April 2018. They spent 90 minutes completing a 38-item test. A total of 574506 students (student population) completed the assessment test, out of which 31300 constituted the sample. Of these, 4543 students received the item (the formula task), which is the focus of this article, as part of two comparable test forms.

The test produces simple quantitative information about the respondents' proficiency levels and item difficulty. We collected all the students' written responses to the item, made available by the central coding of the test. For our study's purpose, we have conducted the analysis on the unsuccessful responses, which goes beyond identifying proficiency levels.

3.2. The formula task

The test item we consider in this article is shown in Figure 1. The three numbered lines contain the following statements:

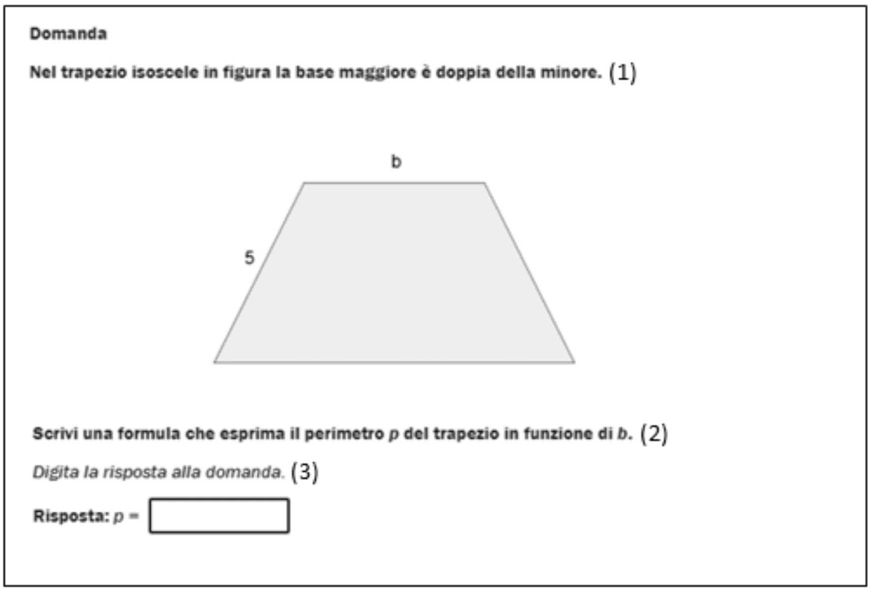
(1) In the isosceles trapezoid in the picture, the long base is twice the short base.

(2) Write a formula that expresses the trapezoid's perimeter *p* as a function of *b*.

(3) Type your answer to the question.

Figure 1

The formula task



The item includes both text and a figure. It shows an isosceles trapezoid whose long base is twice the short base, and this relational information is presented with an initial textual statement (1). The figure also provides the lengths of the leg and the short base respectively with a number and a letter. The second textual statement (2) asks the students to write a formula to express the trapezoid's perimeter "as a function of *b*". The task requires connecting the letter *b* to the length of the short base and considering all the given relationships, in addition to applying knowledge of the calculation of the perimeter. Therefore, it focuses on reasoning functionally about the algebraic relations captured by the formula and implies coordination among the different semiotic registers. The final textual statement (3) directs to type the answer inside the box.

This open constructed-response item involves the mathematical domain of relations and functions, and its intent is for students to write a formula that expresses a relationship in a geometric context. The intent aligns with the specific learning objective of the content domain: to interpret, construct, and transform formulas that contain letters to express relations and properties in a general form, and the grade 8 goal of using and interpreting mathematical language (formulas, equations, etc.).

The item requires relating different elements (the figure, the text, a number and a letter) to each other. The solution to the task is obtained from the sum of the four sides of the quadrilateral, after having considered that the length of the long base, which is missing in the figure, corresponds to $2b$. The formula is: $p = 3b+10$, but the correct answer can be written in many ways, no matter if condensed into a single expression or not. For example, $p = b+b+b+5+5$, $p = b+2b+10$, $p = b+2b+5\times2$, $p = 3b+5\times2$ are all acceptable answers.

The seemingly simple task requires various knowledge and skills to be applied effectively. The student must put together what double means, what the perimeter of a quadrilateral is, what an isosceles trapezoid is, and what “a function of” means. Thus, the student must know how to calculate the perimeter, that for the given figure the two legs are equal (and the other leg’s length is also 5), the relationship between short and long base, and how to express that relationship using the unique letter b so that the long base depends on the short base. The student who completes the task successfully must coordinate control over these steps and maintain coordination between the different relationships at play in the situation. Therefore, the complexity of the task resides in the ability to establish relations among the various elements and connect information.

3.3. Method and process

We situate our investigation process within the larger context of qualitative research methods. According to Denzin and Lincoln (2011), "qualitative research is a situated activity that locates the observer in the world" and "consists of a set of interpretive, material practices that make the world visible. These practices transform the world" (p. x). The world we strive to make visible in this article is that of 8th grade students' views of the formula and corresponding algebraic proficiency when they are required to write a formula in context (given by the task above), with specific attention to their difficulties.

Qualitative research is also "attentive to those aspects of data about individual or group behaviour or sites of inquiry that often get lost when we turn too quickly to coding and quantifying that which is under study" (de Freitas et al., 2017, p. 160). There is always some sort of flow or quality in the empirical data which is always in excess of any simplistic code. Instead of inferring overall algebraic ability from the correct, incorrect and omitted answer percentages (which we can easily calculate), we are more interested in how the students write their (incorrect) answers, which we use as a way to examine their approach to solving the task and the nature of their mathematical conceptualizations. By focusing on this aspect, we see errors in the incorrect responses as positive events which produce reasoning and disrupt solving routines to show what the students bring to the task in relation to their view of the formula.

We see our method as one which relies on forms of observation at a distance. We consider it as based on interpretive empirical practice, which seeks to better understand questions about what is happening in particular places, ways that people and material objects are organized in various settings, patterns of interaction, and relationships among settings (de Freitas et al., 2017). Our "place," as nonparticipant observers, is the wide context of the national computer-based test. With software easily performing searches on large data corpus, we additionally characterize the methodology we use as mainly pursuing a qualitative, exploratory analysis of the students' incorrect answers while also considering limited quantitative information about their percentage and the proficiency levels of the students. Since the respondents are part of the sampled group, their answers represent those of the national population of grade 8 learners and the analysis provides insight into the situation across the country.

After taking the set of incorrect answers, made accessible to us thanks to the central coding of the item, our data analysis proceeded by identifying common elements in the answers that were more or less related to the mathematical thinking provoked by the task. Types of commonalities included the following: prevalent reference to a single semiotic register occurred when incorrect responses were written using only symbols or only numbers, or when other responses were expressed only in words. Attention to specific relationships embraced not only the given relationship between the bases of the trapezoid but also new relationships between them, or between the short base and the leg (expressed for example by associating the same length to the two sides). Specific letters were used to name specific sides (like *l* for the leg).

We viewed and discussed the incorrect answers in the data set, attending to instances of commonality as described above. This analysis enabled us to divide the responses into different categories and sub-categories, which emerged as a responsive and inventive way of reading the data. The categories were established when we could discern a main characterizing commonality and then other common characteristics within it. For this article, we focus on the main series of incorrect answers that we considered significant with respect to the nature of the mathematical task, as mentioned above. Rather than seeing these in terms of errors to the task (thinking products or failing outcomes), we see them as traces of the ways in which letters are used and the formula is viewed in the specific context (dynamic processes). We also pay attention to the extent to which they are generative of differences regarding the correct solution. Other incorrect responses occurred in the test, such as purely numerical values or expressions, and word answers which were often out of context, but in this article, we report on these only briefly.

By so doing, we offer a qualitative interpretation of the students' incorrect responses, borrowing from Stacey and MacGregor (2000) to rethink the categories as *routes* not only to the solution, but also to algebraic or functional reasoning about the formula. This helps us better understand ways that middle school students may still struggle with formulas.

4. Students' routes and incorrect responses

As an overview of our data, the overall performance on the formula task indicated a very difficult item (level 5). Some 2147 students out of our 4543 respondents gave an incorrect answer (47,2%) and 835 students omitted the answer (18,4%), while only 1561 students answered the task correctly (34,4%). As shown in Table 1, among the successful students, about 42% were at level 5 of proficiency and 32% at level 4. We also observe that around 71% of students with low proficiency (levels 1 and 2) did not even attempt an answer. As mentioned already, we focus specifically on the main group of incorrect answers, without numerical values or expressions and irrelevant word answers, which consisted of a total of 1759 responses out of 2147.

Table 1

Percentages of students at a given proficiency level in relation to response type

	Number	% level 5 students	% level 4 students	% level 3 students	% level 2 students	% level 1 students
Incorrect responses	2147	6,9	14,4	26,9	29,6	22,1
Missing responses	835	1,4	7,8	20	35,4	35,3
Correct responses	1561	41,6	31,6	17,2	7,9	1,6
Total number of students	4543	17,9	19,1	22,3	23,3	17,5

Analyzing the kinds of incorrect answers, we identified the major groups of responses which we see as approaches students used to solve the task: those that tried to write a formula for the perimeter of the trapezoid making explicit reference to the relationship between the bases (first route); those that referred only partially to the given elements to write the perimeter formula (second route); and those that wrote an area formula instead (third route). An additional approach included answers that presented letters but seemed to escape a decipherable orientation (fourth route).

The four routes are outlined in Table 2, which provides a short description of the nature of each approach and percentages of related answers (with respect to the total of 2147 incorrect answers). We name these routes: *relational* (first route described above), *partial* (second route), *habitual* (third route), and *murky* (fourth route). The following sections introduce and discuss each route.

Table 2

Students' main routes to an incorrect solution of the formula task

1. The relational route	2. The partial route	3. The habitual route	4. The murky route

Written expression with *explicit reference to the relationship between the bases*	Written expression with *non-complete reference to all the elements*, but partial focus	*Writing an area formula instead of the perimeter formula*, albeit using letters	Using letters with *a generally unclear connection with the formula*
10,6% (228)	31,4% (676)	13,6% (293)	26,2% (562)

4.1. The relational route

The answers in this route (228 in total) appear to be closest of the four routes to solving the task correctly and provided examples of how students who understood the task could complete it unsuccessfully. The answers all exhibited a perimeter formula (using operations) and explicitly referred to the relationship between the bases of the trapezoid. This route has a "relational" character not only because of this explicit reference, but also because we interpret the students who undertook it as essentially capable of orienting themselves to the information presented across the different semiotic registers and to the various relationships in the task. In fact, we observed that in addition to writing a relationship between the two bases, these students knew how to find the perimeter of an isosceles trapezoid and took into consideration all its elements, obtaining information from both the text and the figure.

Table 3 shows examples of the answers in this route and how they distribute across five sub-groups. Each sub-group is characterized by some specific choice regarding letter use, the role of the variable, or the view of the formula, which makes the answers differ. Each sub-group demonstrates an approach to the specific task, linking to the algebraic thinking that was activated in the solving process. We use "*R*" to denote the route and "*Rx*", with *x* changing, to denote its sub-groups, while the total number of students undertaking that approach is in parentheses.

Table 3 reports the students' answers as written (words have been translated from the original language, while the multiplication sign is expressed by the symbol * or the letter *x*, and the use of letter pairs, like *lb* and *sb* (or *SB*), refers to the long base and the short base, respectively).

Table 3

Examples of students' answers in the relational route

The relational route (R)				
R1 (62)	**R2** (32)	**R3** (4)	**R4** (66)	**R5** (64)
Using a letter in place of number 5	Using only numbers for the formula	Using mainly words for the formula	Reversing the relationship	Using wrong typing for operation signs
bx3+2l	[6+(6*2)]+(5*2)	You have to multiply b by 2 to find the long base then add up 5 plus 5 and finally add up all numbers to find the perimeter	B+1/2B+(5*2)	5*2+b+b*b
b+(b*2)+(d*2)	5+5+5+(5*2)	Short base plus short base times two plus ten	(B : 2) + B + (5+5)	b + b x b + 5 x 2
b+2b+a+a	3+(3x2)+(5+5)		(5x2)+(B/2)+B	(b*bx2)*(5*5)
b+2b+2l	8+4+5+5=22	Short base+(short base*2)+(leg*2)	(l*2)+lb+lb/2	(5x5)+(bxbx2)
b+(bx2)+(hx2)	5*3+5*2	P=2times5+SB+ 2timesB	long base+(5x2)+ long base:2	5+5+b+:2b
b+(b*2)+(l*2)	10 cm + 15 cm = 25 cm		10+B+1/2B	b+(2+b)+(2+5)

4.1.1. Sub-routes of the relational route

*R*1 depicts ways in which students wrote the perimeter formula without using the known value 5 for the length of the leg while introducing a new letter to refer to it. Despite this substitution, the writing of the formula is appropriate and, usually, appears as the sum of three pieces, associated with the length of the short base, the length of the long base, and the total length of the two legs. Particularly, the relationship between the bases is correct and is expressed through the letter *b*. The three pieces indicate the successive identifications of the length of the short base as a quantity captured by the variable *b*, of the length of the long base as dependent on *b* according to doubling, and of the equal lengths of the two legs. Therefore, various relationships are considered before writing the perimeter as a sum of these essential elements. In Table 3, the only exception is given by the expression *bx*3+2*l*, which considers the single piece *bx*3 as the sum of the bases, still keeping the multiplication sign for the operation “three times *b*”.

We see the answers of R1 (including the one just mentioned) as a trace of a peculiar vision of the formula as an entity that generally evokes a procedure in which only letters can be used, but not numbers. We might interpret this approach as emerging from working habits with formulas in contexts in which the formula is ritualized as a direct rule (like in our example of $A = b \times h$) to calculate a measure (of the area or the perimeter) once known values are substituted for letters. The typical approach, then, is that of using letters for all the sides that have a role in the formula, which for these students implicates the use of two letters, the given *b* for the short base and the long base, and another letter for the equal legs.

From the examples, we observe that the choice of the new letter varies, from *l* to *a* to *h* or *d*. Interestingly, the majority of the 62 students who approached the task in this manner used the letter *l* as a variable for the leg, that is, exactly the first letter of the Italian word for side (“lato”). This is the most evident example of the misconception related to the role of the variable as a label. Some students used *a* as the second letter, along with *b*, and this usually occurs with figures (such as rectangles) where only two letters are needed, and the letters *a* and *b* are chosen (the first two letters of the alphabet). This approach reinforces the ritual view of the formula as a procedure which also relies on the use of the letter as an object to name a side.

*R*2 and *R*3 are two approaches of the relational route that we see as similar due to their focus on a single semiotic register. Both *R*2 and *R*3 offer examples of answers to the task where the relationship between the bases is clearly stated, but the answers contain only operations with numbers (*R*2) or are mainly expressed in words (*R*3). What is relevant in these kinds of responses is again the way that the answer was written. The 32 answers for *R*2 provided the perimeter as a sum of numerical terms where the bases were given by two numbers, one double the other, and the equality of the two legs were expressed through the number 5 added up twice. Therefore, these students assigned a known value to the short base, and then they doubled it to indicate the long base. Most of them chose 5 as the value for the length of the short base, which was the same value given for the leg, perhaps because they paid special attention to the figure, which may have influenced these students to establish new relationships between the lengths of the two sides, which visually appear almost the same. Others chose 4 or 6, which were significant in the same respect.

For students following this approach, the formula seems to recall once again the rote application of a rule, as it requires operations and a numerical result. We might interpret this approach as related to ritual uses of the formula to calculate a value, the measure of the length of the perimeter. It might also be related to how students find algebraic expressions challenging in problem solving. The responses in *R2* can be connected with Stacey and MacGregor's (1997) non-algebraic routes based on arithmetic reasoning. Stacey and MacGregor indeed found that not only do students not conceptually understand what variables mean, but they also struggle with answers to a problem that are different from numbers (as in the case of a+2 for Stefania's height, which is discussed in the theoretical highlights). Similar troubles have to do with the idea that literal symbols and numbers should not be combined in a single equation (Brizuela et al., 2015).

Only 4 answers belong to *R*3, but it is interesting to find a textual approach to the task. We see how the procedure to find the trapezoid's perimeter was described in words, regardless of some use of addition and multiplication signs. We might interpret such an approach as one that tracks how students struggle to manage symbols, while noticing that the procedure or its steps were correctly expressed and that the relationship between the bases was made explicit. But we might also interpret it as a response to the specific request to "write" a formula, as if this request asked students to explain in words how to find the trapezoid's perimeter.

*R*4 presents a backward interpretation of the given relationship between the bases of the trapezoid (activated by 66 students, the largest group in *R*). All the answers in *R*4 in fact expressed the inverse relationship, that is, that the short base is half of the long base, and assumed to have the latter. They generally contained the letter *B* (the usual name given to the long base in Italian) and the term $B/2$ or $B{:}2$. Most of these answers were correct conceptually, since the leg's length is associated with the number 5 instead of a letter (*l* when present). An example illustrating this interpretation is $(B : 2) + B + (5+5)$. The writing of the inverse relationship might easily be interpreted as evidence of a reversal error. This error in fact arises every time the students exchange the role of the variables used to translate from the written to the symbolic, as in the case of: "There are 6 times as many students as professors" translated into $6S = P$ instead of $6P = S$ (see Bush & Karp, 2013). But if we consider how the formula was written and how the information was given, it may have been that the students of this group were very influenced by the first textual statement they encountered. This conjecture gains strength when we realize that in the initial text, the long base is the subject of the phrase and that the short base has no associated value (given the introduction of the letter *b* in the figure). We might also interpret the approach here as implying the conceptual understanding that the doubling can always be seen in reverse as halving, in which case it makes no difference whether to consider either base as a starting point because that relationship can be written one way or another, and the two are equivalent. The choice of the capital *B*, commonly attributed to the length of the long base (the biggest one), might be interpreted once again in relation to the letter used as a label. But we can also see how the perimeter is mainly offered using a single variable. What is missing from this approach is the focus on using the given letter *b*. It may be that paying close attention to the initial text meant that these students lost sight of the figure and the information in it or forgot the "function of *b*" part of the task.

The approaches of *R1* and *R4* can be considered close to the algebraic routes of Stacey and MacGregor (1997). In both approaches, the written formula expresses in symbolic form the relationship between the bases of the trapezoid (and in the case of R4 it is expressed as a function). However, the use of letters as labels and the initial textual statement seem to interfere with a correct solution.

Finally, 64 incorrect answers contained wrong signs for some of the operations (*R*5). For example, for some students writing a multiplication of *b* by 2 was the same as writing the multiplication of *b* by itself, so they presented the term *b***b* instead of 2**b*. Others used the multiplication sign instead of the addition sign, or vice versa. Errors of the first kind are conceptual, habitually conveyed by those particular operations whose results are the same (e.g., 2*2 and 2^2). Examples are the expressions 5*2+*b*+*b***b* (the last term refers to the long base) and (*b*+*b*^2)+5*2 (the second term in brackets refers to the long base; this example is not shown in Table 3). Students in this group showed that they connected different semiotic registers, wrote a formula, grasped the relationship between the bases, calculated the perimeter, and considered leg equivalence, but surprisingly, they confused the double and the square of a number.

Errors of the second kind (some are shown in Table 3) were more related to the use of the computer keyboard, where a single key has, for example, both the addition sign (+) and the multiplication sign (*). This was the case with the expressions 10*3*b*, *b*+(2+*b*)+(2+5) and (*b***b*x2)*(5*5). The first expression uses * in place of +, the second contains 2+*b* instead of 2**b*, and in the third expression, the three signs would all have to be + to make sense. A similar example is the expression (*b**2)+*b*+(5*“), since “ and 2 are on the same key.

4.2. The partial route

About three times the number of answers (676 in total) fell into the partial route than in the relational route. Furthermore, the answers in the partial route appear to be further from a correct solution than answers in *R*. The students who fell into the partial route all showed a similar approach to solving the task, an approach that was essentially captured by knowing the formula for the perimeter of the isosceles trapezoid. However, students seemed to not fully comply with the writing of the formula by using the letter *b* for the bases and the known value for the leg. It was as if they had put particular focus on writing the perimeter formula while forgetting some elements of the item. We interpret this approach as having a “partial” character because of the explicit attention paid to the request for the formula and the only partial focus on the various pieces of information provided. The students who undertook this route understood the request to write a perimeter formula but did not apply the formula to the whole context, which requires consideration of the different relationships. In particular, most students seemed to not pay attention to the first textual information provided about the relationship between the bases.

Table 4 provides examples of answers in the partial route and how they are divided into four sub-groups, each of which is again characterized by different choices regarding the use of letters, the role of the variable, or the view of the formula. Each sub-group indicates an approach to solving the task related to the specific information used in the solving process. The route is labeled "*P*" and the sub-groups "*Px*", with *x* changing, and the corresponding number of responses in parentheses. The table shows the answers as written, with translation from the original language.

Table 4

Examples of students' answers in the partial route

The partial route (*P*)			
***P*1** (229)	***P*2** (276)	***P*3** (145)	***P*4** (26)
Using the letter *B* for the long base	Multiplying *b* by 2 instead of 3	Seeing the perimeter as sum of any 4 sides	Writing the procedure in words
B+*b*+5+5	(5+5)+*b**2	*B*+*b*+*l*+*l*	Long base + short base + 2*5
b+*B*+5*2	(5+5) + (*b*+*b*)	AB+BC+CD+DE	Long base + short base + leg+leg
(5*2)+(*b*+*B*)	2*b*+10	*b*1+*b*2+2*l*	Short base plus long base plus side times two
5+5+short*b*+long*b*	(*b**2)+(5*2)	*L*+*L*+(*b*+*b*)	You add up all the sides, but first you have to find '*b*' and then add up the sides
*b*1+*b*2+(5+5)	(5+*b*)2	*L*1+*L*2+*L*3+*L*4	all sides added up

10+b+B	(b*2)+10=2b+10= 12b	B+b+lx2	2sides+short base+long base

4.2.1. Sub-routes of the partial route

*P*1 gives examples of ways in which 229 students wrote the perimeter formula of the isosceles trapezoid by adding a short base, a long base, and two legs with length 5. The two bases are always written using the letters *b* and *B*. Even if the writing of the formula was appropriate, these students focused on the trapezoid to extract the length of the legs but not on the initial text. They showed that they knew what the perimeter is and how to find it for an isosceles trapezoid, as evidenced when they doubled the length of the leg. However, they only considered part of the information provided because they did not explicitly write any relationship between the bases while introducing the letter *B* to refer to the length of the long base. These students incompletely grasped the task because they satisfied the writing of a formula that expresses the trapezoid's perimeter but neglected to write it "as a function of *b*". For their answers to be correct, a single step was missing. However, this step concerned the most demanding part of the task, namely being able to express a quantity as a function of another quantity, or to relate the two quantities to each other. We might interpret this approach as derived from a vision of the figure as providing everything necessary to write the formula (a ritual of working with geometric figures in the context of measurement), which implies only one element missing from the sides of the trapezoid, exactly the long base. But we might also interpret it as supported by the usual definition of the perimeter of a quadrilateral as the sum of its four sides. In both cases, the letter *B* was introduced as the name for the long base (again, the classical label for the long base of a trapezoid), not as a variable quantity that depends on *b*.

A different approach was followed by the 276 students whose answers belong to *P*2. These students wrote the perimeter formula by adding only double the short base and double the leg. Students considered both text and figure information to solve the task, but the formula did not include the short base. Students showed that they built the relationship between short base and long base, and that they focused on and understood the initial textual statement. However, they failed to write the correct expression for the perimeter since they precisely forgot the short base. They may have translated the initial text into symbols ("the long base is twice the short base"), moved to the figure, figured out the length of the leg and doubled it, and then omitted one side. We see this approach as strictly related to the specific focus on writing the formula as a function of *b*, where the long base is the only element that can be expressed in function of *b*. It is as if these students had interpreted the request to write the perimeter formula in terms of what is to be entered in the formula as a function of *b*, excluding *b*, since the latter was already given in the task. The initial text told what to do to write the long base as a function of *b*, then the figure added visual and numerical information to relate the legs (their equivalence, beyond value 5). The result was a partial writing of the formula which missed the term present in the figure, even if the answer was written in function of *b*, which was therefore used as a variable. The perimeter, in fact, is always expressed as the sum of two pieces, a numerical part referring to the legs and a symbolic part referring to the long base. In short, our interpretation is that the students did not feel the need to consider another *b* because the task already provided it. A different interpretation might be that students confused the trapezoid with a rectangle, taking the leg as one dimension and the two bases as equal to write the perimeter as double of *b* plus double of 5.

145 students fell into *P*3. They showed a procedural approach to solving the task, which rests on writing the formula using only letters and which we might see as emerging from habits related to defining the perimeter of a quadrilateral. The perimeter was offered as the sum of four addends captured by letters, most commonly the letter *l* to name the leg (as already seen for *R*) and the letter *b* to refer to one or both bases. We cannot know for sure whether the letter *b* was used for the base since this letter is given in the figure or simply because it is generally used to indicate a base. Nevertheless, for the overall approach of the answers, we may interpret that the students used letters mainly as labels, often linked to habitual ways of using letters with geometric figures. For example, we find again the long base called *B*, but we also observe the use of AB, BC, CD, DE for the four sides, which usually occurs to name the sides of a given quadrilateral in sequence. An exception in Table 4 is the expression $b1+b2+2l$ in which the two bases are distinguished by numbers and the legs have been joined. Interestingly, almost half of the answers in *P*3 exhibit the multiplication of the leg's length by 2, while in the rest of the answers all four sides appear, mostly denoted by *B*, *b* and *l*, as in the expression $B+b+l+l$. Only 8 students wrote the perimeter using four different terms, as in the case of the aforementioned AB+BC+CD+DE, or in the case of $L1+L2+L3+L4$. There were also students who introduced the letter *h* for the leg. Since *h* is primarily related to height, especially when working with areas, we might interpret that these students showed a confusion between the leg and the height of the trapezoid. It is not likely that students were thinking of an area formula, as their answers were like the others except that we find *h* in place of *l*. But we might also see the use of *h* as related to the formula for the perimeter of a rectangle, where base and height are often used. The vision of letters as labels leads us to think that the students who fell into this group ignored the information given by the figure and the initial text. They did not show they

were able to write a relationship between quantities or extract elements from the given context, although they had an idea of the perimeter of a trapezoid as the sum of four sides. The answers also indicate a vision of the formula as a rule, in this case strictly connected with the definition of the perimeter of a figure as the sum of its sides (often not stated in textbooks but independent of the figure).

Finally, even in the partial route a few students (26, the smallest percentage, like in the relational route) provided written descriptions on how to find the trapezoid's perimeter. While they were all appropriate with respect to the procedure, we interpret these responses in the same way as the answers in *R*3, as essentially focused on the specific request to "write" a formula, thus seeing it in terms of description rather than in terms of the algebraic expression of the formula in the given context.

In most of the answers in *P*, the view of the perimeter formula appears to be procedural as it involves sequences of operations and letters used as names. Stacey and MacGregor (2000) would likely have seen these kinds of solutions as part of a superficially algebraic route, mainly centered on working out the answer.

4.3. The habitual route

This route comprises answers (293 in total) that we consider far from solving the task correctly, because they appear to refer to writing an area formula rather than a perimeter formula. We can see this in principle from the use of the letter *h*, which usually occurs for the height of a geometric figure, multiplied by a base or by the sum of two bases in many answers. The answers all showed a similar approach to solving the task, one that we interpret as reducing the focus to the presence of a simple geometric figure and the subsequent request to write a formula. The use of the letter *h* is peculiar not only because it is the typical letter for naming height in mathematical practice, but also because of the specific role it plays in the area of geometric figures. In other words, the case where we find the letter h in a formula is usually that of area formulas. It is principally this feature that pushed us to interpret the route as having a "habitual" character, mainly linked to working habits with the perimeter and area formulas in geometry. In fact, extensive work on the perimeter is unusual, because the perimeter has the same generic meaning of the sum of the lengths of the sides, for all geometric figures. Textbooks rarely mention perimeter formulas. If anything, they only provide the general idea of perimeter. Conversely, students often learn to measure the area of specific figures by rote procedures involving different ways of proceeding. In addition, work on perimeter is often taken for granted in middle grades, and in elementary grades it is generally limited to squares, rectangles, and triangles. The students who followed this route may have immediately associated the trapezoid and the formula with finding the trapezoid's area. Some seemed to consider the relationship between the bases. However, many showed only a vague awareness of a proper area formula.

Table 5 offers examples of answers in the habitual route and how they are distributed across five sub-groups, each characterized by choices on the use of letters, the role of the variable, or the view of the formula. We label the route "*H*" and the sub-groups "*Hx*" (*x* changing). The parentheses contain the total number of students for each group, and the answers are reported as written and translated if necessary.

Table 5

Examples of students' answers in the habitual route

The habitual route (*H*)				
***H*1** (112)	***H*2** (35)	***H*3** (31)	***H*4** (28)	***H*5** (87)
Using the letter *B* for the long base	Taking the relationship between the bases	Writing the area of a triangle	Using words to write an area formula	Using the letter *h* in some way
(*b*+*B*)*xh*/2	[*b*+(*b**2)]**h*/2	*b***h*/2	*b* plus *B* times 5 all divided by 2	[(*B*+*b*)**h*]

B+*b* times h divided by 2	(*b**3**h*:2	*b* * *h* : 2	Long base plus short base times height divided by two	(*B*+*b*)/*h*
B+*bxh*:2	3*b***h*/2	*bxh*:2	Base times height	*b*+*b***h*
*b*1 + *b*2 times h :2	(*b*+*b* times 2)times *h* :2	(*B***H*)/2	Long base plus short base times height	*Bxb*:*h*
(*b*+*b*2)**h*/2	(*b*+*b**2)*5/2	2*b***h*/2	LONG BASE +SHORT BASE* HEIGHT/2	*b***h*+5*2
(*Lb*+*sb*)**h*/2	[(*b**2)+*b**5]/2	*b* times *h* : 2	*B* TIMES *A* DIVIDED BY 2	[*B*+(*B*/2)]**h*

4.3.1. Sub-routes of the habitual route

Students' answers that were closely related to a formula for the area of the isosceles trapezoid belong to the sub-routes *H*1 and *H*2. *H*1 provides examples of ways 112 students wrote the area formula using the rule learned in school or found in textbooks (the sum of the bases multiplied by the height and divided by 2). However, they used the letter *B* for the long base, or the letters *b*1 and *b*2 for the two bases (like in the expression *b*1 + *b*2 times *h* :2). These students used letters as names, without considering the relationship between the bases, as if they neglected the initial text.

By contrast, the 35 answers in *H*2 exhibited a formula for the isosceles trapezoid that was conceptually consistent with the information given, except that it captured the area rather than the perimeter. The relationship between the bases was clearly expressed by the term 2*b* or *b**2 used for the long base, or by the presence of the term 3*b* or *b**3 involving some manipulation. The students who undertook this approach knew how to write the long base in function of *b*, although most of them used the letter *h* to denote the height of the trapezoid. Five students used the known value of the leg instead of *h*. This choice may highlight a confusion between leg and height, but we might also interpret it as a way for these students to offer an understanding of "as a function of *b*". In this way, they advanced a formula written as a function of *b*. In either case, the intent of the original test question was surprisingly met with an incorrect answer.

The students falling into *H*1 and *H*2 may have connected trapezoid and formula with the ritual of learning area formulas in the geometric context. The two approaches may also indicate confusion and a weak grasp of perimeter and area formulas, which occurs even with simple two-dimensional shapes. Research demonstrates, for example, that students are more likely to report the perimeter of rectangles as their area when numerical information about the lengths of the sides is present (Miller, 2013), while vague descriptive language, especially for perimeter, can perpetuate such confusion (Clements & Sarama, 2009). This could be the case with the formula task, where the leg length is given by a number and the relationship between the bases is given in words.

The 31 answers in *H*3 show a half product of a base and a height expressing the area of a triangle (one of the first areas encountered in school). The students who gave these answers seemed not to distinguish the type of figure given, nor to consider the text or any other element provided. In *H*4, there are 28 answers that verbally expressed a procedure that recalls the calculation of the area but is often described incorrectly.

The last group, *H*5, is made up of 87 answers, which combined letters pertinent to the trapezoid (e.g., *b* for the short base, sometimes *B* for the long base, *h* for the height) but inconsistently. For example, there were cases of multiplying one base by another, as in the expression *B*x*b*:*h*. Almost a third of the answers contained the sum of the two bases multiplied or divided by the height. We might interpret this as an attempt by the students to construct a formula, or as an effort to dig into memory to find rules (rituals) learned without exploration.

Summarizing, route *H* collects answers that manifested difficulties not only in the way to treat formulas as functions or relations, but especially with the concepts of area and perimeter, or possibly with the request to "write" a perimeter formula, which some students may have seen as far from their experience with trapezoids.

4.4. The murky route

We collected the rest of the incorrect responses using letters (562) in this route. The responses showed a similar approach to solving the task that consisted of the use of letters but did not capture transparent reasoning on the perimeter formula, except in the case of a few examples. We interpret the route as having a "murky" character because of the somewhat indecipherable approach of the answers to the formula. We see that these answers considered partial elements, in many cases excluding the legs' length or the relationship between the bases. About two-thirds of the answers appeared to show little or no idea of which elements to use to find the perimeter of the trapezoid. Alternatively, it may be that the request to write the perimeter as a function of *b* had involved specific reasoning about how to use the letter *b*, and other letters, in a formula, or a specific focus on one or the other base.

Table 6 provides examples of answers and how they were divided into three sub-groups, each characterized by the use of letters, essentially confined to one or two letters. The route is labeled "*M*" and the sub-groups "*Mx*" (with *x* changing and the indication of the number of associated answers). The answers are reported as written.

Table 6

Examples of students' answers in the murky route

The murky route (*M*)

***M*1** (202)	***M*2** (180)	***M*3** (180)
Taking essentially *b* and 5	Taking essentially *b* and 2	Taking *b* and other letters
5*2+2**b*+*B*-*b*	*b**2+*b**2+*b*+*b*	*l*+*l*=*b*+*B*
b+5+5+10	[*b*+(*bx*2)]:2	*b**(*C**3)-(*C*:2)
(*b x* 2) +5 *x* 3 : 2	*b*+*b*+*b*+2*b*	*b*+*B x* (*l*+*l*)
{3*b**[5+(*b*:2)]:2}	*B*=2 times *b*	*pXb*
b+5+5+7	*b*+*b*+*b**2/2	(*b*+*b*)+(*B*+*B*)
5+5-*b*=*B*	*Lbase* + *Lbase*:2	*b*+*B*

4.4.1. Sub-routes of the murky route

The three sub-routes all contain nearly the same number of answers. *M*1 (202) shows answers from students who clearly focused on the information given in the figure. In fact, they mostly contained the number 5 and the letter *b*, and in few cases other letters such as *B* or *h*, or specific names (labels) for the bases. Interestingly, the elements were sometimes combined in a form that resembled a formula for the perimeter, as in the expressions *b*+5+5+10 and *b*+5+5+7 which would give the perimeter of the trapezoid by assigning a value of 10 or 7 to the length of the long base. Therefore, in these instances, the students may have inferred the length of the long base by relating it to the length of the leg. We can also note that both the example expressions above are written in function of *b*. Another interesting answer was given as 5*2+2**b*+*B*-*b*, which in the end might be seen as capturing the perimeter of the trapezoid, despite using *B* for the long base (in fact, 2**b* diminished by one *b* gives one *b*). The vision of the formula is here typically procedural. Other times, the elements in the figure were combined without apparent relationships, such as when only the multiplication of 5 by *b* appears.

*M*2 contains 180 answers, which primarily included multiplication or division by 2, often of *b* or a different label used for either base. There is no trace of the leg in these responses, and the students merely considered the bases to obtain the perimeter. Some students summed the bases in some way, even contemplating the relationship given in the initial textual statement. For example, the expression *Lbase* + *Lbase*:2 (where *Lbase* stands for the length of the long base) exactly provides the sum of the two bases, written in a way that exploits the inverse relationship again. Expressions like *b*+*b*+*b**2/2 and [*b*+(*bx*2)]:2 seem to refer to a formula that involves division by 2, which may be that of a half perimeter or area. Expressions of the kind of *b**2+*b**2+*b*+*b* might suggest the reference to a rectangle, whose two dimensions are identified with the two bases, or to a trapezoid whose leg is perceived as long as the short base. Other students simply wrote the long base as the double of the short base, translating the relationship between the bases without going further to construct the formula. Answers in *M*2 are the only ones of the murky route that focused attention to relating the bases using a single letter or label (*b*, *Lbase*).

Finally, in most of the 180 answers in *M*3, only the letter for the short base was taken from the task. In many cases, the letter was added up to or multiplied by the long base, while in the rest of the cases, it was used together with other letters like *l* or *p*, respectively introduced for the leg or perimeter. Also in this group, we find procedural visions of the formula, which we might hypothesize as referring to a rectangle, for example in the case of the expression (*b*+*b*)+(*B*+*B*). Answers such as *b*+*B x* (*l*+*l*) or

$l+l=b+B$ might also demonstrate sums of the four sides of the trapezoid, showing probable typing errors.

It cannot be said that the answers in this route were given by chance. They seem to reveal a type of reasoning, albeit a rough one. Although the approach may seem illogical on the surface, traces of attention to relevant elements of the task make us better see it as a murky approach in which students strove to unravel the knot but were far from being able to solve the task correctly.

4.5. Single numbers or numerical expressions

Beyond the four routes, there is another sizable group of incorrect answers (348), which corresponded to single numbers or numerical expressions. We do not analyze these answers in detail, but it is worth mentioning that the most common number response was 25, which can be obtained by assigning a value of 5 to the length of the short base, and then using the relationship between the bases to express the length of the long base. In this group, we also find other values, such as 19 and 22, which are possible results of numerical answers collected in $R1$ and based on the use of specific numbers to write the perimeter formula (3+6+5+5 and 4+8+5+5). We might infer that they are the results of a similar approach to solving the task. We might interpret them as conveyed by a ritualized view of the formula, implying a numerical result for the perimeter through a precise calculation. The students who gave these answers may have concentrated solely on the figure and the relationships between the lengths involved, particularly those of the short base and the leg, which could be perceived as being very close to one another, exactly as in the case of $R1$. So, they may have estimated the lengths of the bases. The difference is that no procedure was explicitly offered to make such estimates visible. Although the three answers (19, 22 and 25) are all sensible estimates for the perimeter and reasonable ways to look at the information in this context, taken together they make up only a low percentage of all the incorrect responses in the group. The remainder of the responses consisted of out-of-context numerical responses. Similarly, 40 additional out-of-context word responses were not examined. Therefore, no further discussion about these responses has been included as part of significant and generalizable results.

5. Final discussion

In this article, we have examined the incorrect answers provided to the formula task by a sample of eighth graders during a national assessment test of mathematics. We focused on how these answers are written in order to investigate the ways in which middle school students solved the task and the nature of their mathematical conceptualizations, especially of formulas. In doing so, we see errors as positive events that generate reasoning and help us understand the students’ difficulties, as well as what the students brought to the task in relation to their view of the formula.

By analyzing all the incorrect answers, we identified various *routes*, which show similarities and differences in the conceptual understandings of the formula. We contend that these routes are different approaches that students followed to solve the task, depending on the elements they focused on. We use them to interpret how students view formulas and can reason about them functionally or algebraically.

The formula task is neither a simple algebra word problem nor an arithmetic-to-algebra problem, but a modelling task that specifically involves a relationship between a geometric formula and elementary algebra. The routes, therefore, also provide a way of examining the extent to which middle school students think of formulas as generalizations of geometric relationships. While the first three routes show the main attempts to write the formula (and ways of seeing it) in the given context, the fourth route completes the landscape with answers that still consider letters, albeit often without a clear orientation to the perimeter.

We found intersections between the routes. For example, we found textual approaches to the formula in the three main routes, as if the request to “write” a formula involved only the use of words. Often, we also found procedural or ritual views of the perimeter formula as a rule or calculation in which letters are mainly used as names or labels for the sides, or similarly, the perimeter ended up being conceptualized as a numerical value rather than a variable length. These intersections, as well as the specific choices that characterize the routes, seem to highlight overall difficulties encountered by middle school students in conceptualizing formulas in a versatile and transformative sense (according to Arcavi’s vision of algebraic thinking). A specific difficulty is seeing formulas as tools for establishing relationships between quantities, in this case between the trapezoid’s perimeter and the length of its sides, or directly between the perimeter and the length of the short base. We have suggested that these difficulties may arise from habitual, operational ways and ritual routines of using formulas to provide results or from struggling with answers to a problem that are different from numbers. Additionally, they may be based on arithmetic thinking applied to the task.

We have also found in the answers the problem of distinguishing perimeter from area, a problem that has been well-documented by the extant literature. Finally, some incorrect answers show attempts at formulas that nevertheless express the relationship between the bases of the trapezoid, or even a function of *b*, meeting the question’s original intent.

5.1. About mathematics education research and large-scale assessment

While the study we have presented in this article focused only on one item of the national assessment test, it analyzed incorrect answers from a sample of students that were representative of grade 8 students in our country. Because these students came from different classes and different schools throughout the country, the analysis leads to significant and generalizable results. In particular, it reveals two needs: first, to have students work more with formulas as a means of establishing algebraic relationships in different contexts; and second, to *rethink* formulas as *relational* rather than procedural, so as to reconcile the view of the formula with its multiple uses in mathematics and to shift attention to the transformative nature of algebraic thinking, of which the culture and practice of formulas should be a major part.

Identifying the *routes* as an analytical process to investigate students' ways of interpreting the formula provided us with a tool to explain students' struggles and difficulties. This approach may hold promise for further investigations on other tasks involving formulaic reasoning or other mathematical thinking. Studying the relationships between students' mathematical conceptualizations and their proficiency levels, for example, seems a promising line of inquiry.

We believe that the data from the national mathematics assessment, although raw, are invaluable material for research in mathematics education. They allow for a better understanding of the problems in large-scale mathematics education and, ultimately, an understanding of what is missing, what is needed, and what interventions can be made to improve mathematics teaching and learning. We also believe that mathematics education research and large-scale assessment can learn from each other. We hope that other researchers will take up the challenge of examining this fascinating and unexplored connection in favor of new lines of flight.

The overall results of our collaboration did not end in this article. Ongoing emergent visions of the relevance of incorrect answers continue and herald new discussions and questions each time we return to investigating them. While it might be difficult for researchers to appreciate incorrectness in school mathematics or mathematical performance, the unanticipated richness unfolded by the responses to the formula task offers opportunities for further exploration.

Disclosure statement

No potential conflict of interest was reported by the authors.

Data availability statement

The data that support the findings of this study are available on request from the corresponding author, FF. The data are not publicly available due to ethical restrictions.

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