Cassirer in the Context of the Philosophy of Mathematics

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1 Introduction

Mathematical topics from the late nineteenth century play a central role in Cassirer's philosophy. He provided an interpretation of Marburg neo-Kantian epistemology based on the mathematical concept of function in his first major epistemological work, *Substance and Function* (1910). His account of mathematical objectivity in structural terms was one of the key motives of his philosophy of symbolic forms, in particular with regard to the third volume of Cassirer's work, *The Phenomenology of Knowledge* (1929). Several studies have pointed out the importance of these topics for the historical reconstruction of the development of Cassirer's thought.[1] But some of Cassirer's views on mathematical knowledge have been reconsidered also in more recent discussions on mathematical structuralism as well as in the broader context of the philosophy of science.

This chapter will offer a general introduction to Cassirer's views and their development throughout his intellectual career. Subsequently, it will be outlined how these views have been connected to two main directions of research in contemporary philosophy of mathematics: (1) as providing an interesting variant of mathematical philosophy taking inspiration from the methodology of the exact sciences at the turn of the nineteenth century; (2) as offering an interesting philosophical framework for an account of structural practices that emerged in nineteenth and early twentieth-century mathematical works.[2] Whereas the first direction of research moves from Cassirer's methodological analyses to reconsider the potential of his philosophical project as a whole, the second direction uses some insights from Cassirer's philosophy to improve our current understanding of key mathematical concepts and practices.

I would like to suggest that these directions of research emphasize different (but in Cassirer's original view complementary) aspects of his thought, and offer various examples of how his philosophy continues to show its potential in the philosophy of mathematics.

2 From the mathematical concept of function to the philosophy of symbolic forms

2.1 The mathematical concept of function in Cassirer's neo-Kantian epistemology

As Natorp put it in an important paper of 1912, what Marburg neo-Kantians shared was not a particular body of doctrine, but a commitment to the "transcendental method", that is, to rely on the "firm basis of the given and historically documented facts of science, ethical life, art, religion" in an attempt to discover the reason for the "validity" of these facts.[3]

[1] See esp. Ferrari (1996/2003), Ihmig (1997).

[2] The first direction has been emphasized especially by Richardson (1997), Friedman (2005), Heis (2010). The second direction has taken off from the parallel between Cassirer's account of function-concepts and Reck's (2003) interpretation of Dedekind, and has been explored further by Mormann (2008), Yap (2017), Schiemer (2018), Cantù (2018).

[3] Natorp (1912), p.196f.

By the time Natorp was writing, several internal debates divided the key figures of the School. These divisions can be traced back as early as the 1880s, when Natorp, dissatisfied with Cohen's interpretation of the infinitesimal calculus against the background of early-modern philosophy, undertook his own attempt at a transcendental account of the "fact" of modern mathematics.[4] The declining phase of Marburg neo-Kantianism, so to speak, begun with the lively debate on the tenability of Cohen's system following the publication of his *Logik der reinen Erkenntnis* (*Logic of Pure Knowledge*).[5] The disagreements notwithstanding, Natorp continued to identify a common ground in the twofold direction of inquiry that emerged from Cohen's interpretation of Kant[6]: from the fact of science (and generally of culture) to the presuppositions for the lawfulness of this fact, and from the laws to the variety of their possible instantiations.

Cassirer's education at Marburg led him to elaborate on original interpretation of the transcendental method as grounded in the mathematical concept of function.[7] This interpretation is foreshadowed in his reconstruction of the early modern roots of critical idealism, especially of Leibniz's mathematical work in Cassirer (1902). He expressed the main ideas for his interpretation of the transcendental inquiry in his important paper of 1907, "Kant und die moderne Mathematik" (Kant and Modern Mathematics), by saying that with the foundation of mathematics in a general logic of relations (and thereby ultimately in the idea of "functionality"), modern mathematical logic offered "more fruitful points of connection with epistemological problems and a safer guide than that possessed by Kant in the traditional logic of his time".[8]

Cassirer articulated his view in *Substance and Function* by contrasting two different models of concept formation associated with the mathematical concept of function, on the one side, and with the predicates of the Aristotelian syllogistic logic, on the other. The latter were understood as concepts of "things" or "substances" that are supposed to exist in themselves and to become recognisable as the bearers of a series of characteristics. Typically, this model of concept formation offers a categorization of all objects into genus and species, where the more general concepts are also the less specific, and individual differences escape all conceptualization. By contrast, the relevant notion of function for Cassirer's inquiry is that of a "universal law, which, by virtue of the successive values which the variable can assume, contains within itself all the particular cases for which it holds".[9] The idea is that scientific objects, unlike Aristotelian substances, are supposed to

[4]On the discussion of Cohen's interpretation of the calculus within the Marburg School as well as in the broader context of late nineteenth-century philosophies informed by the exact sciences, see Giovanelli (2011, 2016).

[5]For evidence and further references on the disagreements between Cohen and Natorp, see Holzhey (1986). On the declining phase of Marburg neo-Kantianism in the literature, see esp. Sieg (1994).

[6]See esp. Cohen (1877).

[7]For a thorough reconstruction of Cassirer's education at Marburg and of his relations to Cohen and Natorp, see Ferrari (1988).

[8]Cassirer (1907), p.8. All translations from original German texts are my own, unless otherwise indicated.

[9]Cassirer (1910/1923), p.21. This notion, which is taken from Drobisch (1875, p.22), leaves out arbitrary functions, in which there is no such law. Although Drobisch's notion does not include all possible cases of mathematical functions, it is understood by Cassirer as *the* mathematical notion that lies at the basis of the new transcendental account of objectivity in the exact sciences.

exist only insofar as they can be determined by a network of mutual relations expressed in functional terms.

Mathematical constructions from late nineteenth-century function theory and geometry offer the paradigmatic examples for Cassirer's account of concept formation, and the starting point for his account of objectivity, generally speaking. Notably, Cassirer highlighted the logic of the mathematical concept of function with reference to the way in which Dedekind generated the series of natural numbers as the chain N of the initial element 1 under the successor function $\phi(N)$.[10]

Cassirer referred to Dedekind also for the idea of using functions to coordinate various such series in a higher-oder system or structure. The main reference here is the way in which Dedekind (1872) introduced the axiom of continuity of the real numbers. Dedekind's starting point was the fact that the ordering relations, "being lower/higher than" holding between the rational numbers and "laying at the left/right of" holding between the points of a straight line, fulfill the same formal characteristics (*i.e.*, transitivity and the existence of infinitely many intermediate elements between any two given elements). It follows that every rational number can be mapped onto one and only one point; however, the inverse does not hold true. There are points for which there is no corresponding number in the original domain. So, Dedekind's idea was to introduce or, as he said, "create"[11] the irrational numbers that make the coordination univocal. His axioms of continuity justifies this way of proceeding by stating the existence of all elements that produce a division of the rational numbers into two separates classes O_1, O_2, where all the elements of O_1 are less than all elements of O_2. The element producing the division can be taken as the highest of O_1 or the least of O_2, and it can be either one of the rational numbers already in the system or an irrational number.

According to Cassirer, Dedekind's procedure showed that such "things" as numbers are actually "terms of relations", that can never be "given" in isolation "but only in ideal community with each other"; but it also offered a starting point for a novel account of knowledge, not as the reproduction of outer impressions in thought, but as "the intellectual coordination (*Zuordnung*) by which we bind otherwise totally diverse elements in a systematic unity".[12] Further examples analyzed by Cassirer include the use of transfer principles in analytic and projective geometry by mathematicians such as Julius Plücker, Otto Hesse and Felix Klein, as well as the group-theoretical view of geometry proposed by Klein in 1872 and articulated further by Sophus Lie and Henri Poincaré. All these examples of how structural procedures were extended from numerical and algebraic to geometrical domains showed for Cassirer an "inclusion of the spatial concepts in the schema of the pure serial concepts".[13] The aim of his work was to show how with the law of the conservation of energy and other nineteenth-century reformulations of the principles of physics, the schema of the mathematical concept of function extended over into empirical domains as well, providing "the general schema and model according to which the modern

[10] A brief presentation of Dedekind's construction is given later in the chapter, in Section 3.2.
[11] Dedekind (1872/1901), p.15.
[12] Cassirer (1910/1923), p.36.
[13] *Ibid.*, p.87.

concept of nature has been molded in its progressive historical development".[14]

The thesis that mathematical and physical knowledge are of the same kind (also known in the literature as *the sameness thesis*[15]) allowed Cassirer to provide a new formulation of the task of the transcendental inquiry in the wake of Marburg neo-Kantianism. In the place of Kant's articulation of the a priori apparatus into categories of the understanding and forms of intuitions, Cassirer articulated the conditions of knowledge into a hierarchy of syntheses culminating with the construction of abstract mathematical concepts. As he put it already in a pregnant passage from "Kant and Modern Mathematics", the task of critical philosophy is to show that "the same fundamental syntheses, which lie at the foundation of logic and mathematics, rule over the scientific articulation of empirical knowledge", notably: "that only these syntheses enable us to establish a lawful order of appearances, and therefore their objective meaning".[16]

Cassirer's interpretation differs from Kant's transcendental inquiry, insofar as the system outlined by Cassirer is continuously put to the test by scientific developments. Insofar as the highest principles themselves are subject to change with regard to their content, Cassirer has been acknowledged as one of the first to advocate a historicized and relativized conception of the Kantian a priori.[17] Cassirer pointed out this possibility in *Substance and Function* by saying that the goal of isolating the ultimate common elements of all possible forms of scientific experience can never be perfectly achieved.[18] Subsequently, Cassirer himself reconsidered many of his assumptions substantially in the light of Einstein's general relativity, quantum physics and other classical cases of scientific change in the first half of the twentieth century.

Before turning to the contemporary reception of Cassirer's philosophy of mathematics, the next paragraph will outline how he returned to the problem of characterizing mathematical objectivity after the foundational crisis of mathematics in the 1920s.

2.2 Mathematical objectivity in the philosophy of symbolic forms

We have seen that the mathematical concept of function offered the model for Cassirer's interpretation of the neo-Kantian epistemology as a back and forth between analysis and justification of the preconditions for scientific inquiries. Subsequently, Cassirer elaborated on the dynamical aspect of his epistemology in a further and more fundamental way in the context of his philosophy of symbolic forms. The starting point of this perspective is the expressive power of symbols underlying the idea of functionality in the exact sciences,

[14] *Ibid.*, p.21.

[15] See Mormann (2008).

[16] Cassirer (1907), p.45.

[17] See, e.g., Richardson (1997), Ryckman (2005). A different interpretation has been given by Friedman (2001), who takes Cassirer, and Marburg neo-Kantians in general, to advocate a purely regulative conception of a priori cognition as an ideal towards which all actual systems of knowledge tend to converge. Cf., however, Ferrari (2012) for evidence of the fact that the transcendental method in the sense of Marburg neo-Kantianism offered a theory of the constitution of scientific objectivity. Without discussing these different interpretations here, I will limit myself to point out that both readings recognize the key role of the mathematical and structural procedures in shaping Cassirer's dynamical understanding of a priori cognition.

[18] Cassirer (1910/1923), p.269.

as well as the whole variety of ways in which the humans' understanding of the world is articulated, from myth to religion and art. Cassirer beginning in 1921 called these "symbolic forms", explaining that the task of systematic philosophy is to go beyond epistemology and "grasp the whole system of symbolic forms, the application of which produces for us the concept of an ordered reality".[19]

The philosophy of symbolic forms introduces a new perspective on mathematics, insofar as all constitution of objectivity appears to be specific to the different symbolic forms in the system. There arose the problem of accounting for what counts as objectivity within mathematics, regardless of the way in which mathematical concepts are applied in other disciplines.

Cassirer in the late 1920s had to engage, furthermore, with the new agenda of the different attempts to overcome the paradoxes of set theory in the foundation of mathematics. Therefore, a large section of the third volume of the *Philosophy of the Symbolic Forms* is dedicated to the discussion of the main foundational perspectives of Cassirer's time, namely, logicism, intuitionism, and formalism.[20]

Cassirer sought to identify a common ground between these different frameworks in the intended mathematical procedures. Notably, Cassirer relied on Hilbert's axiomatics, insofar as this helped highlighting the structural character of modern mathematical theories as complexes of propositions and deductive inferences that satisfy the consistency requirement and can be applied to wholly different domains. By the same token, Cassirer distanced himself from all attempts to set limits to mathematical developments based on some further assumptions about what mathematical objects should be in themselves. According to Cassirer, such an attempt showed the problematic aspect of Brouwer's intuitionism. But a similar objection can be raised also against logicism, insofar as logical objects considered in their individuality are supposed to be something different from the structures containing them. To mention one of the previous examples, Russell distanced himself from Dedekind's foundation of arithmetics by requesting that numbers, if they are anything at all, must be "intrinsically something" besides the form of a progression that can be instantiated by the most diverse objects.[21] Already in *Substance and Function*, Cassirer defended Dedekind's approach against Russell's criticism, pointing out that numbers as the objects of arithmetics can be completely determined only in their mutual relations of position, that is, as a system.

Without undermining the novelty of Cassirer's perspective of 1929, it is important to notice that he still relied on the idea of functionality as constitutive of mathematical objectivity. In his account, Hilbert's approach offered a way to study systematically the structural characteristics that had emerged in the mathematical developments already considered in *Substance and Function* (in particular projective and group-theoretical geometry). That such a study is possible at all, according to Cassirer, confirms the idea that mathematical objectivity, understood in terms of functionality, offers a paradigmatic example for scientific concept formation. The formation of mathematical concepts illustrates what Cassirer called the "symbolic pregnance" of experience, namely, the fact that the

[19] Cassirer (1921/1923), p.447

[20] *Ibid.*, Ch.4.

[21] Russell (1903), p.249.

categorization of the phenomena into some "orders", as Cassirer put it, "detracts in no way from its concrete abundance; but it does provide a guarantee that this abundance will not simply dissipate itself, but it will round itself into a stable, self-contained form".[22] There is a clear parallel between the notion of symbolic pregnancy and the above caracterization of the mathematical concept of function as a law that is the richer in content the more it is universal.

But Cassirer in 1929 addressed the symbolic function of mathematical constructions also from a "bottom up" perspective, so to speak, that is, considering the phenomenological basis for mathematical concept formation. Cassirer maintained, for example, that even such abstract concepts as that of transformation groups find their roots in the way in which a variety of phenomena in spatial perception (*e.g.*, the single optical images) are grouped together as representations of one and the same object.[23] All sense perceptions thereby are arranged around some "centers" or elements that have to remain constant in the perception of something in space. A change of perspective (*e.g.*, an optical inversion), however, can show that even these centers can be varied under specific circumstances, whereby the change in our "way of seeing" determines a change of the object itself.[24] The fact that humans learn how to perform such changes with increasing degrees of "freedom", according to Cassirer, introduces the symbolic function of spatiality that found its clearest expression in the late nineteenth-century way to characterize geometrical properties as the relative invariants of the underlying transformation group. On the level the formation of mathematical concept, the choice of the group is completely free, which is why the group-theoretical view of geometry can be made fruitful to a variety of purposes.

With regard to this broadening of perspective, Heis (2008, 2015) emphasizes that the third volume of the *Philosophy of Symbolic Forms* inaugurated a new phase in Cassirer's philosophy of mathematics. Whereas the constructions of mathematics in Cassirer's earlier account (culminating with *Substance and Function*) gained an objective meaning in their physical applications, the main issue addressed by him in the philosophy of symbolic forms concerned the objectivity of mathematics considered in its own right. According to Heis, Cassirer had to abandon the sameness thesis (according to which mathematical syntheses occupy a fundamental place within a comprehensive system of the precondition of scientific knowledge) in favor of an attempt to show that mathematics builds a unitary whole despite the structural turn of late nineteenth-century and early twentieth-century mathematical disciplines, and despite the various directions of twentieth-century foundational inquiries. Insofar as it is still possible to identify a unitary motive in the relevant structural procedures, mathematics can be said to keep its own standards of objectivity in the changed context of the twentieth century.

As Mormann (2008) points out, evidence from Cassirer's later epistemological works suggests that he continued to subscribe to the sameness thesis throughout the development

[22]Cassirer (1929/1957), p.204.

[23]Cassirer (1929/1957), Ch.3. Cassirer elaborated further on this idea in an important paper of 1944 published in English with the title "The Concept of group and the theory of perception". For a discussion of Cassirer's argument, along with further references, see Biagioli (2018).

[24]Cassirer relied for the characterization of optical inversions on the work of "*Gestalt*" psychologists, in particular Wolfgang Köhler's and Erich Moritz von Hornbostel's.

of his thought. He reproduced entire sections of *Substance and Function* in *Determinism and Indeterminism in Modern Physics* to emphasize the fact that his original account of objectivity in terms of functionality seemed to receive a surprising confirmation throughout the scientific revolutions of early twentieth-century physics.[25] Cassirer continued, furthermore, to suggest that such developments had been made possible by a continuous back and forth between mathematics and physics dating back to the nineteenth century. The fourth volume of *The Problem of Knowledge*, written in 1940, opens with Cassirer's appreciation of the "revolutionary" stance taken by Riemann in identifying the principles of geometry as "hypotheses": "Where absolute and self-evident propositions had been envisioned he sees 'hypothetical' truths that are dependent upon the validity of certain assumptions, and no longer expects a decision on this validity from logic or mathematics but from physics."[26]

I have argued elsewhere[27] that the connection between the different directions of Cassirer's philosophy of mathematics depends on a functional account of mathematical reasoning that shows the structure of a transcendental argument, according to which structural and mathematical constructions ground the possibility of knowledge by extending the same procedures to natural processes.

The following section will discuss some of the examples of how Cassirer's theses have been reconsidered in contemporary philosophy of mathematics.

3 Cassirer and contemporary philosophy of mathematics

3.1 A place for neo-Kantianism in contemporary philosophy of mathematics

It is well known that Cassirer was one of the first neo-Kantians to engage with early twentieth-century mathematical logic or the so-called "logistics" in a critical but also constructive way.[28] Cassirer in "Kant and Modern Mathematics" drew a clear distinction between the logicists' task of a logical derivation of mathematical theories, on the one hand, and the critical philosophers' task of investigating the role of mathematics in the acquisition of knowledge or what Cassirer also called the "logic of objective knowledge", on the other.[29] At the same time, he relied on some of the results of the modern logic of relations (in particular Russell's) to account for various instantiations of the mathematical concept of function. Cassirer seems to depart from Russell mainly for the epistemological implications of his account.

More recent scholarship initiated by Heis(2010)[30] has reconsidered Cassirer's twofold

[25] Cassirer (1936), pp.137f.

[26] Cassirer (1950), p.21.

[27] Biagioli (2020a).

[28] See Pulkkinen (2001), Heis (2010).

[29] Cassirer (1907), pp.44f. This language allowed Cassirer to draw a distinction between mathematical logic or logistics and the received notion of logic as a philosophical discipline. What most philosophers called "logic" throughout the nineteenth-century and until the first decades of the the twentieth century was a series of investigations dealing with what is now currently known as "epistemology" or a theory of justified belief.

[30] See also Heis (2011), as well as several contributions contained in Edgar & Patton (2018)

attitude towards mathematical logic, emphasizing how it differs in interesting ways from Russell's approach also when it comes to account for purely mathematical procedures. Whereas Russell's deductions are supposed to show that the subject matter of mathematical disciplines is constituted by logical objects, Cassirer highlighted the modern understanding of mathematical objects in terms of structures as an implication of the new methodology at work in nineteenth-century mathematics. The fact that Cassirer – in the wake of the Marburg neo-Kantian interpretation of the transcendental method – relied primarily on methodological considerations to draw the relevant epistemological and ontological implications, allowed him to appreciate aspects of early mathematical structuralism that simply escaped the more standard reconstruction in set-theoretic terms. This includes a variety of idealizations in use in mathematics, such as the way in which Dedekind introduced irrational numbers by a sort of expansion of the original system or the rationals. As Heis put it: "Cassirer, because he shared with other Neo-Kantians an appreciation of the *developmental* and *historical* nature of mathematics, was attentive to questions in the philosophy of mathematics to which Frege, Russell, and the logical empiricists paid scant attention".[31]

The resurgence of interest in Cassirer's philosophy of mathematics in recent years is due also to the fact that these kinds of questions are again in the agenda of the philosophy of mathematical practice.[32] Without calling into question the achievements of twentieth-century foundational inquiries, the philosophy of mathematical practice has considerably broadened the scope of the philosophical reflections on mathematics to epistemological questions including those concerning conceptual changes, growth and development of mathematical disciplines, the relation of mathematical disciplines among themselves, as well as the use of mathematics in other disciplines. Not only did Cassirer address the same questions in ways that are still relevant, but he advocated a similar approach by presenting his neo-Kantian epistemology as a way to integrate the formal approaches of his time with an inquiry into the preconditions of mathematical knowledge starting from the "fact" of mathematics considered in its historical development. This led Cassirer to address the philosophical implications of a variety of mathematical and scientific practices, whereby Cassirer, unlike contemporary practice-oriented approaches in the philosophy of science, sought to provide a comprehensive account of the system of knowledge.

As a matter of fact, we have seen that a certain tension between Cassirer's methodological standpoint and his theorizing about the system of knowledge emerged in the development of his thought itself. Mathematical syntheses in the original sense of the sameness thesis play a fundamental role in the system as preconditions for the lawfulness of nature. With regard to the system of the symbolic forms, however, there arises the question whether mathematical objectivity deserves an account in its own right, regardless of whether the sameness thesis can be reformulated even starting from the more abstract standpoint of

[31] Heis (2011), p.761.

[32] See Mancosu *et al.* (2008) for a comprehensive introduction to the motivations of this direction of research. Although the authors do not refer to Cassirer or other long-term influences for their approach, Cassirer's reading of Dedekind has received increasing attention in connection with the new literature on the development of structural practices in nineteenth-century and early twentieth-century mathematics. The content of some of these studies will be presented in more details in the next section.

twentieth-century mathematics or not.

Besides this interpretative issue, there is the question of how the said tension ought to be solved by someone who is willing to use the main isights of Cassirer's philosophy in the context of contemporary philosophy of mathematics. In emphasizing the novelty of the philosophy of symbolic forms, Heis suggests that Cassirer himself saw the potential of his philosophy to provide a unitary account of the various directions of purely mathematical research. In Heis's view, Cassirer's commitment to the unity of mathematics allowed him to account for mathematical objectivity; however, this must come at the price of the sameness thesis. By contrast, Mormann emphasizes that the sameness thesis continued to lie at the core of Cassirer's philosophy and shed light on his relation to Russell. In Mormann's reading, the sameness thesis was directed initially against Russell (1903), insofar as this work seemed to set apart the realm of logic and mathematics from empirical domains. Mormann points put that, nevertheless, there are interesting affinities between Cassirer's thesis and Russells later "exterior world program" of a logicist reconstruction including mathematics as well as physics and psychology.[33]

Starting from such a reconstruction, Mormann aims to show that the sameness thesis can be retained even in the case of incompatible idealizations. This includes, for example, the different solutions that can be given to the task of completing a domain in set theory and in category theory. The fact that there are different solutions in alternative frameworks, according to Mormann, does not call into question Cassirer's essential point that new mathematical elements can be introduced as systematic unfolding of the old; however, as Mormann puts it, it shows that "there is no reason to expect that there are not various ways of undfolding".[34] In other words, contrary to Heis's suggestion, Mormann's proposal is to retain the main insights expressed by Cassirer in the sameness thesis while dropping his original commitment to the unity of mathematics.

In order to show how the sameness thesis can be retained from a pluralist perspective on mathematical idealization, Mormann focuses on another example from twentieth-century mathematics, namely, Stone's representation theorem for Boolean algebras. The proof of this theorem makes use of continuous ideal completions that can be shown to be a generalization of Dedekind's idealizing completion of the rational numbers.[35] Not only do these mathematical results provide further evidence that idealizing completions have played a crucial role in the conceptual evolution of twentieth-century mathematics, but they offer what Mormann considers to be a compelling argument for the sameness thesis, insofar as the same idealizing procedure can be generalized further to the constitution of the points of physical space.

The potential of Cassirer's view, according to Mormann, lies not so much in Cassirer's attempt at a single all-encompassing account of idealization as in the fact that idealizing procedures continued to have a significant role in twentieth-century mathematics and physics. Mormann's suggestion is to solve the said tension in favor of a closer connection of Cassirer's neo-Kantian epistemology to scientific methodologies and their developments. This allowed Mormann to take into account the possibility of incompatible idealizations,

[33]See Russell (1914).
[34]Mormann (2008), p.171.
[35]See Gierz & Keimel (1981).

disregarded by Cassirer, as well as to reformulate the sameness thesis about topological and physical spaces regardless of the fact that neither Cassirer nor other philosophers (including Russell) appreciated the importance of Stone's results in 1936.

The next section will offer a brief account of how Cassirer had a more direct influence on some recent literature about the philosophical implications of early mathematical structuralism.

3.2 Cassirer's influences in the literature on early mathematical structuralism

We have seen in the previous sections that Cassirer modelled his notion of function on structural procedures from nineteenth-century mathematics such as Dedekind's completion of the system of rational numbers by the introduction of newly created irrational numbers. This section will sketch how more recent discussions of Dedekind's mathematical structuralism benefitted from Cassirer's interpretation. This discussion offers an interesting example of how Cassirer's neo-Kantian epistemology can still offer important resources in the philosophy of mathematics.

Dedekind's foundation of arithmetic has been widely acknowledged to have foreshadowed the view of mathematics as concerned with the investigation of abstract structures independently of the nature of individual objects making up those structures. This view became known as one the main directions of contemporary philosophy of mathematics under the label of "mathematical structuralism".[36] However, it is has been debated whether Dedekind's view can be brought into the contemporary debate, and on what basis. According to Dummett (1991), Dedekind's talk about numbers as "free creations of the human mind" revealed a psychological notion of abstract objects widely shared by his contemporaries (with the relevant exception of Frege). Not only would be such a notion incompatible with the modern understanding of structure, but much of twentieth-century philosophy of mathematics departed from a psychologistic standpoint in emphasizing the logical status of mathematical objects.

Contrary to Dummett's reading, Tait (1996), Ferreiròs (1999), Reck (2003) have emphasized that Dedekind himself implies a purely logical procedure of abstraction from all spatio-temporal as well as psychological determinations in the characterization of numerical structures. This way of proceeding emerged most clearly in Dedekind's (1888), which includes what in current model-theoretic terminology is called a characterization of the natural numbers up to isomorphism or the the proof of the categoricity of second oder Peano axioms of arithmetic.[37]

Reck (2003) offered important insights in this debate by pointing out that the set-theoretical reconstruction of Dedekind's way of proceeding accounts only partially for his understanding of abstraction as leading to the "creation" of numbers. Dedekind's numbers differ from mere positions within patterns, insofar as Dedekind identified them as new

[36] See Reck & Schiemer (2019) for a comprehensive introduction to the main issues of mathematical structuralism, and its different variants in contemporary philosophy of mathematics. I will refer mainly to Reck (2003) for the issue of locating Dedekind's position in the contemporary taxonomy.

[37] Dedekind (1888), §X, shows that all systems satisfying his characterization of natural numbers are also similar to (*i.e.*, can be mapped one-to-one onto) the natural numbers.

objects, whose introduction is justified logically by a structural procedure. We have seen that Cassirer himself emphasized this point with regard to the way in which Dedekind in 1872 introduced irrational numbers in the place of all univocal divisions of rationals that are not produced by a rational number. In order to highlight Reck's point, let us examine somewhat more closely the main steps of Dedekind's characterization of natural numbers of 1888.[38]

Dedekind started with the construction of a *simply infinite* system, where an infinite system is characterized by the fact that it can be mapped one-to-one onto a proper subset of itself. Dedekind called an infinite system N *simple* when it can be generated by an ordering function ϕ and a base-element 1 not contained in $\phi(N)$. While the symbols 1 and N already indicate that the natural numbers are the intended system, it is important to notice that Dedekind formulated his conditions for simply infinite systems in general. In Dedekind's symbolism:

(a) $\phi(N) \subset N$

(b) $N = 1_0$

(c) The element 1 is not contained in $\phi(N)$

(d) The function ϕ is similar.[39]

He characterized the natural numbers by saying that: "If in the consideration of a simply infinite system N set in order by a transformation ϕ we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation ϕ, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-element* of the *number-series N*. With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind".[40]

This passage indicates that numbers in Dedekind's sense are not the same as the set-theoretical construction outlined above, but they are abstracted from such a construction by retaining only the structure of a simply infinite system and neglecting every other content otherwise associated with numerical concepts or counted objects. As Reck pointed out, the natural numbers obtained in this way form a system of *sui generis* objects, differing both from physical objects and from objects in other simple infinities in mathematics, such as those constructed in set theory. Insofar as the identity of all the natural numbers is determined together, Reck recognizes that Dedekind's view can be identified as a form of structuralism; however, his construction differs from the later variants of mathematical structuralism on account of the fundamental role of his ordering function ranging over a domain that is not fixed but is generated in part by the function itself. The construction

[38] The following account focuses on the aspects of Dedekind's characterization that are more relevant for Cassirer's reading.

[39] Dedekind 1888, §VI. Dedekind's conditions on a system N contain the first formulation of what are known today as the Dedekind-Peano axioms of arithmetic.

[40] Dedekind (1888/1901), p.68.

provides the necessary and sufficient conditions for what Dedekind takes to be a purely logical determination of numbers, and thus the only basis for assuming that they exist objectively. With regard to the logical status thus granted to numbers, Reck proposed to call Dedekind's view "logical structuralism".

After developing his interpretation independently of Cassirer's, Reck pointed out some important points of agreement in Reck (2003) and (2020). Notably, Reck recognized that Cassirer was the first to defend the possibility of a logical rather than psychological understanding of the abstraction at work in Dedekind's construction. Elaborating on this point, Yap (2017) argued that the appropriate philosophical background for a logical understanding of numbers in Dedekind's sense can be found in Cassirer rather than in Dedekind's remarks as such. Yap relies on Cassirer to highlight the claim that the essential relations between elements are completely determined. Expressed in Cassirer's neo-Kantian language: "Here [in Dedekind's definition of natural numbers] abstraction means logical concentration on the relational connection as such with rejection of all psychological circumstances, that may force themselves into the subjective stream of presentations, but which form no actual constitutive aspect of this connection".[41] As Yap points out, the notion of constitution of objectivity offers a suitable explanation for Dedekind's claim that we can forget about the special character of the elements in a simply infinite system. Yap shows, furthermore, how Cassirer's claim can be articulated further in terms of the categoricity of the natural numbers, even though, as Yap points out, Cassirer himself did not mention categoricity, arguably because other of his examples of mathematical concepts (in particular transformation groups) are defined by non-categorical sets of axioms.

Heis (2012), Schiemer (2018) and myself[42] emphasized how Cassirer's account of mathematical concept formation in terms of the concepts of function offers no less important philosophical insights about structuralist techniques from nineteenth-century geometry, from the foundation of the geometry of position to the theory of transformation groups. The geometry of position offered numerous examples of idealizing completions outside numerical domains, whereby principles such as duality as well as continuity determine a functionalization of spatial concepts in analogy with the concept of number. Such a process in Cassirer's account culminates in the group-theoretical view of geometry, according to which what counts as geometrical properties can be determined only relative to a given group of transformations (in the sense of one-to-one mappings of space onto itself). Cassirer emphasized that hereby there is a "logical priority" of structure over the appearance of the single figures, such that "the intuition of our Euclidean three-dimensional space only gains in clear comprehension when, in modern geometry, we ascend to the 'higher' forms of space; for in this way the total axiomatic structure of our space is first revealed in full distinctness".[43]

To put it in current structuralist terminology, transformation groups offer a clear example of how an abstract concept can be instantiated in different ways. Not only did higher geometry highlight the logical status of Euclidean geometry (as one of such instantiations), but it informed a broader epistemological discussion on the presuppositions of measurement

[41] Cassirer (1910/1923), p.39.
[42] See esp. Biagioli (2018), (2020b).
[43] Cassirer (1910/1923), p.20.

and the form of physical space. Besides being one of the first philosophers to appreciate the structural turn of nineteenth-century mathematics, Cassirer was one of the protagonists of the early twentieth-century debate on the geometrical foundations of modern physics together with Henri Poincaré, Hermann Weyl, Moritz Schlick, Hans Reichenbach and Albert Einstein.[44]

Insofar as Cassirer recognized that Euclidean space is only one of the possible instantiations of the abstract theory of the forms of space, he clearly distanced himself from the Kantian view of geometry as grounded in pure spatial intuition. It is important to notice that, nevertheless, Cassirer believed that an important aspect of the Marburg interpretation of Kant was retained insofar as his epistemological inquiries confirmed that there is a logical priority of conceptual structures over the individual contents of perception: "The procedures of mathematics here points to the analogous procedure of theoretical natural science, for which it contains the key and the justification".[45] As argued in Biagioli (2020a), there is a clear parallel between the articulation of Kant's transcendental deduction and the way in which Cassirer justifies the extensibility of the mathematical concept of function from numerical to spatial domains, and further on from mathematical to spatio-temporal manifolds.

Without addressing these broader issues here, I would like to suggest that Cassirer's account of Dedekindian abstraction offers an interesting example of how his reflection on early mathematical structuralism can still offer illuminating insights both in the different context of a practice-oriented philosophy of mathematics and when considered in connection with fundamental theses of his philosophy.

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[44]See esp. Ryckman (2005) for a thorough reconstruction that sheds light on Cassirer's role in this debate.

[45]Cassirer (1910/1923), p.94.

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