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Quantization and Path Integrals: a Time-Frequency Analysis Approach

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Sommario (Italian)

In questa tesi sono affrontati alcuni problemi relativi a due aree alla frontiera tra l'analisi e la fisica matematica (quantizzazione e integrali sui cammini di Feynman) per cui l'impiego di tecniche e strategie dell'analisi tempo-frequenza risulta essere particolarmente utile.

Per quanto riguarda il problema della quantizzazione, ci occupiamo di operatori pseudodifferenziali parametrizzati da matrici come

$$\sigma^T f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i(x-y)\cdot\xi} \sigma((I-T)x + Ty, \xi) f(y) dy d\xi = \langle \sigma, W_T(g, f) \rangle,$$

dove il simbolo è in generale una distribuzione temperata ($\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$) e $T \in \mathbb{R}^{d\times d}$ è una matrice arbitraria; abbiamo introdotto la distribuzione tempo-frequenza di tipo Wigner associata

$$W_T(g, f)(x, \xi) = \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi} g(x + Ty) \overline{f(x - (I-T)y)} dy.$$

Nella tesi determiniamo in che misura i ben noti risultati che caratterizzano il calcolo di Weyl e la trasformata di Wigner (corrispondenti al caso $T = I/2$) possano essere estesi in questo contesto più generale, con particolare riguardo ai risultati di continuità su spazi di modulazione e Wiener amalgam con simboli nelle stesse classi. Sebbene regole di quantizzazione ancora più generali possano essere prese in considerazione, la scelta precedente è particolarmente rilevante perché, in un certo senso, esaurisce la classe delle ragionevoli modificazioni lineari delle trasformate di Weyl/Wigner.

Motivati poi dalle applicazioni dell'analisi di Gabor allo studio delle equazioni differenziali dispersive, ci occupiamo dell'analisi di operatori metaplettici tramite pacchetti d'onda; ricade in questa classe di operatori il propagatore di Schrödinger associato a una Hamiltoniana di tipo quadratico. Dimostriamo, nella fattispecie, alcune stime per le corrispondenti rappresentazioni nello spazio delle fasi in cui gli attesi fenomeni di dispersione, sparpagliamento e sparsità dei pacchetti d'onda

sono tutti rappresentati. Si considerano infine applicazioni alla propagazione di singolarità nello spazio delle fasi.

Un altro problema affrontato è quello delle stime di continuità su spazi di modulazione e Wiener amalgam per le soluzioni dell'equazione di Dirac

$$i\partial_t\psi(t,x) = (\mathcal{D}_m + V)\psi(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \quad \psi(t,x) \in \mathbb{C}^n,$$

sotto ipotesi di bassa regolarità per il potenziale V. Si studia inoltre, nel caso libero, la buona positura per nonlinearità reali di tipo analitico a valori vettoriali; in questo scenario ricade il noto modello di Thirring. Lo studio dell'equazione di Dirac con tecniche di analisi tempo-frequenza è ancora limitato: il nostro contributo sul tema è solo il secondo che si registra nella letteratura.

Per quanto riguarda gli integrali sui cammini di Feynman, grazie a tecniche dell'analisi di Gabor è stato possibile risolvere il problema della convergenza puntuale dei nuclei integrali nell'approccio sequenziale alla Nelson (formulazione di Feynman-Trotter) per una vasta classe di Hamiltoniane quadratiche con perturbazioni di bassa regolarità. Questo problema risale alla formulazione originale proposta da Feynman ed è rimasto a lungo aperto. Il nostro risultato di convergenza vale per quasi ogni istante fissato (anche tutti, ammettendo una formulazione più debole) e generalizza risultati precedenti ottenuti per altri schemi di approssimazione sotto forti ipotesi di regolarità per i potenziali.

Abbiamo inoltre introdotto nuovi propagatori approssimati del tipo operatore integrale oscillante (*à la* Fujiwara), la cui ispirazione proviene dalla pratica corrente in fisica e chimica, dove è uso introdurre versioni approssimate del funzionale d'azione. Nonostante queste approssimazioni e le ipotesi di bassa regolarità sui potenziali, la nostra formulazione permette di garantire la stessa velocità di convergenza che caratterizza costruzioni più sofisticate.

Abstract

This dissertation deals with several aspects of two broad research areas which lie at the interface between analysis and physics, namely quantization and path integrals, where the techniques of Gabor analysis naturally play a significant role.

We consider the class matrix-parametrized pseudodifferential operators such as

$$\sigma^T f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i(x-y)\cdot\xi}\sigma((I-T)x+Ty,\xi)f(y)dyd\xi = \langle \sigma, W_T(g,f)\rangle,$$

where the symbol is in general a temperate distribution ($\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$) and $T \in \mathbb{R}^{d\times d}$ is an arbitrary matrix. We introduced the associated phase-space distributions of Wigner type

$$W_T(g,f)(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi} g(x+Ty)\overline{f(x-(I-T)y)}dy.$$

We then investigate whether the well-known results for Weyl quantization and the Wigner distribution ($T=I/2$) extend to this more general context, with a focus on boundedness results on modulation and Wiener amalgam spaces for symbols in the same classes. While more general quantization rules of this type can be taken into account, we characterize the relevance of our choice - which, in a sense, exhausts the family of appropriate linear modifications of Weyl/Wigner transforms.

Motivated by the applications of Gabor analysis to dispersive equations, we deal with the wave packet analysis of metaplectic operators - which include Schrödinger propagators with quadratic Hamiltonians. We prove refined estimates for their phase-space representations where dispersive, spreading and sparsity phenomena for Gabor wave packets are simultaneously represented, with applications to propagation of singularities on the phase space.

We also provide a broad set of continuity estimates on modulation and Wiener amalgam spaces for the solutions of the Dirac equation

$$i\partial_t \psi(t,x) = (\mathcal{D}_m + V)\psi(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \quad \psi(t,x) \in \mathbb{C}^n,$$

under several choices of potentials V of low regularity. We also study the local well-posedness in the free case for vector-valued real-analytic nonlinearities, including the Thirring model. The study of the Dirac equation from a time-frequency analysis perspective can be further broadened - our contribution is only the second one on the topic appeared in the literature.

As far as Feynman path integrals are concerned, thanks to the machinery of Gabor analysis we solve the problem of pointwise convergence of integral kernels in Nelson's sequential approach (also known as the Feynman-Trotter formulation), for a large class of quadratic Hamiltonians with rough potential perturbations. This issue has been conjectured by Feynman and its solution has been open for a long time; our result is almost global in time (even global, if a weaker formulation is allowed) and generalizes known results for different time-slicing approximation schemes and well-behaved potentials.

Finally, we consider a new family of approximate propagators in the form of oscillatory integral operators (*à la* Fujiwara). The inspiration comes from the custom in physics and chemistry, where approximations of the action functional are usually considered. In spite of these approximations and the low regularity of potentials, the same rate of convergence of more sophisticated parametrices is guaranteed.

Acknowledgements

I would like to take this occasion to express my heartfelt gratitude to my mentors, Elena Cordero and Fabio Nicola. I had the good fortune and privilege of having both of them to guide me during this training path, as I could take part into an active research team since my very first day. This condition gave me the opportunity to confront with intriguing problems, but most of all with manifold perspectives on mathematical issues and aspects of professional life. I can safely claim that they taught me all the important things that I know, as well as many others that I have not been able to learn.

I am specially grateful to Elena for setting me on the way of time-frequency analysis, when she accepted to supervise my Master thesis. In spite of her countless commitments, she always found time and enthusiasm to advise me over these years and continuously provided me with precious suggestions on mathematics and career. I learned from her the way of concreteness and the audacity to take all the opportunities that may show up on the road.

At the beginning of my Ph.D. training I was given a desk at Politecnico, where Fabio accepted to guide me along the way. In many occasions I stressed his generous availability beyond the reasonable borders of acceptability; still, there is not a single time when he did not reply to my (long!) messages or give an answer to my questions, even the silly ones, always with peace and clarity. I am especially thankful for his honest advice, even in moments of discouragement. We had so many stimulating discussions on several topics, ranging from puzzling mathematical problems to non-elementary aspects of elementary physics, but also on good reads and teaching. I truly hope that I will be able to make all the time I borrowed from him bear fruit.

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I am glad to take this occasion to publicly thank Professor Luigi Picasso and to let him know once again how his legacy really set me on the road when I was moving the first steps of my education.

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Above all, I wholeheartedly thank my family for endless support and encouragement, as well as for their constant example. I hope that my achievements will repay their efforts with pride.

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Outline

We briefly describe the organization of the material in this dissertation.

First of all, in **Chapter 1** the reader can find a detailed introduction to the problems that have been taken into account, together with an exposition of the main results.
The rest of the thesis is organized in three parts.

- **Part I** collects the background material on time-frequency analysis. In an attempt of providing an essentially self-contained presentation, in **Chapter 2** we fix the notation and recall the general notions of analysis that are needed below, whereas **Chapter 3** and **Chapter 4** are devoted to the basic results of Gabor analysis of functions/distributions and operators respectively.
- In **Part II** we collect the results concerning the problem of quantization and its applications. More in detail:
    - In **Chapter 5** we deal with matrix-parametrized quantization rules and the related time-frequency distributions. The results come from the papers [BCGT20; CDT19; CNT19b; CT20].
    - **Chapter 6** is devoted to the wave packet analysis of metaplectic operators, in the spirit of the paper [CNT20].
    - The time-frequency analysis of the Dirac equation conducted in the paper [Tra20] is the main content of **Chapter 7**.
- **Part III** contains the results concerning Feynman path integrals. In particular:
    - The problem of pointwise convergence of integral kernels is the focus of **Chapter 8**. The original results can be found in the papers [FNT20; NT20].
    - In **Chapter 9** we report on the results proved in [NT19] on the convergence of suitable sequences of operators.

Chapter 1

Introduction and Discussion of the Results

This dissertation deals with several aspects of two broad research areas which lie at the interface between analysis and physics, namely quantization and path integrals, where recourse to techniques of phase space analysis is particularly well suited.

1.1 The elements of Gabor analysis

To be precise, the ensemble of techniques used in this thesis should be referred to as *time-frequency analysis* or, even better, *Gabor analysis.* The origin of this fascinating branch of modern harmonic analysis dates back to D. Gabor's article *Theory of Communication* of 1946 [Gab46], where the author suggested that a family of functions obtained by translation and modulation of a single Gaussian signal may provide a collection of elementary building blocks (usually known as atoms, wave packets or coherent states depending on the context) for any square-integrable signal, meaning that for any $f \in L^2(\mathbb{R})$ there exist $c_{mn} \in \mathbb{C}$, $m, n \in \mathbb{Z}$, such that

$$f(x) = \sum_{m,n\in\mathbb{Z}} c_{mn} e^{2\pi i n x} g(x-m), \quad g(x) = e^{-\pi x^2}. \tag{1.1}$$

The same problem can be considered in the general d-dimensional Euclidean setting and also with different atoms than Gaussian functions, provided that suitable decay and smoothness conditions are guaranteed - for instance, one may assume $g \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz class of rapidly decreasing functions.

Interestingly enough, Gaussian wave packets were already well known in physics since the early work of E. Schrödinger on minimum uncertainty states [Sch82] and also in connection with coherent states of the Weyl-Heisenberg group [Gla63;

Per86] and von Neumann lattices [Neu18]. In fact, it was J. von Neumann the one who claimed (without proof) in [Neu18] that the family of Gaussians $\mathcal{G} = \{e^{2\pi inx}e^{-\pi(x-m)^2}\}_{m,n\in\mathbb{Z}}$ spans a dense subset of $L^2(\mathbb{R})$.

Both the claims by Gabor and von Neumann turned out to be true, but proofs were given only in the 1970s [BBGK71; Per71]. Nevertheless, the expansion (1.1) is unstable in many aspects: for instance, even for $f \in \mathcal{S}(\mathbb{R})$ the series in (1.1) converges only in the sense of distributions [Jan81]; moreover, the sequence of coefficients (c_{mn}) is not uniquely determined and does not precisely characterize the signal f in terms of its energy, meaning that $\|c_{mn}\|_{\ell^2(\mathbb{Z}^2)}$ is not comparable in general with $\|f\|_{L^2(\mathbb{R})}$. More precisely, this means that $\mathcal{G}$ is not a *frame* nor a *Riesz basis* for $L^2(\mathbb{R})$ [Chr16; Hei11]. In fact, the main obstruction here resides in the choice of a critical time-frequency density: it is now well known that the family $\{e^{2\pi i\beta nx}e^{-\pi(x-\alpha m)^2}\}$ is a frame for $L^2(\mathbb{R})$ if and only if $\alpha\beta < 1$ (overcritical sampling) - see in this connection the celebrated series of papers by Kristian Seip [Sei92; SW92] and also [ALS09; LS99]. The mathematical literature on Gabor expansions and their applications has astonishingly grown in sophistication in the last forty years; we recommend the classic monograph [Grö01] as a point of departure.

On the other hand, expansions like those in (1.1) unravel only the *discrete* facet of time-frequency analysis. Let us first elaborate more on the notion of wave packet, that is a function which does possess good localization in phase space. To be more concrete, recall that good energy concentration of a non-trivial function $g \in \mathcal{S}(\mathbb{R}^d)$ on a measurable set $X \subset \mathbb{R}^d$ is achieved if there exists $0 \le \delta_X \le 1/2$ such that

$$\left(\int_{\mathbb{R}^d\setminus X} |g(y)|^2 dy\right)^{1/2} \le \delta_X \|g\|_{L^2}.$$

The spectral content of g is well concentrated on a set $\Xi \subset \mathbb{R}^d$ if the analogous estimate is satisfied by its Fourier transform $\hat{g}$ for some small $\delta_\Xi \ge 0$. Therefore g is concentrated on the cell (or "logon" [Gab46]) $X \times \Xi$ in the phase space and the Donoho-Stark uncertainty principle prescribes a lower bound for the measure of such cell in terms of δ_X and δ_Ξ, namely $|X||\Xi| \ge (1-\delta_X-\delta_\Xi)^2$ [DS89].

The essential phase-space support of g can be moved to $(x+X)\times(\xi+\Xi)$ for any choice of $(x,\xi) \in \mathbb{R}^{2d}$ by applying a so-called *time-frequency shift* $\pi(x,\xi) = M_\xi T_x$ to g, namely as a result of the joint action of the modulation operator M_ξ and the translation operator T_x, respectively defined by

$$M_\xi g(y) = e^{2\pi iy\cdot\xi}g(y), \qquad T_x g(y) = g(y-x), \quad y \in \mathbb{R}^d.$$

Functions of the type $\pi(z)g$ for some fixed $z \in \mathbb{R}^{2d}$ and $g \in \mathcal{S}(\mathbb{R}^d)$ are called *Gabor wave packets* or *atoms*. We retrieve Gaussian wave packets for the choice

$g(y) = e^{-\pi|y|^2}$; note in particular that the Gabor expansion in (1.1) coincides with $\sum_{m,n\in\mathbb{Z}} c_{mn}\pi(m,n)g$.

In according with the program of modern harmonic analysis, the dictionary of atomic elements provided by Gabor wave packets can be used to decompose functions and operators into elementary pieces - that is the so-called *analysis* step. The focus is then shifted to the level of phase space, where one is lead to investigate how Gabor atoms interact or how they evolve under operators. Finally, one collects all these results and tries to read the overall effect on the original objects (*synthesis*), hence coming back to the primary domain hopefully with new information.

1.1.1 The Gabor analysis of functions and distributions

Let us briefly discuss how this program is carried out for functions and distributions. As far as the analysis step is concerned, we can design a phase-space representation of a signal $f \in L^2(\mathbb{R}^d)$ by means of a decomposition along the uniform boxes in phase space occupied by the Gabor atoms $\pi(z)g$, $z \in \mathbb{R}^{2d}$, for some fixed $g \in L^2(\mathbb{R}^d) \setminus \{0\}$. This is the continuous analogue of the expansion (1.1) and is called the *Gabor transform* of f with *window function* g - also known as *short-time Fourier transform* (STFT) or *sliding/windowed Fourier transform*:

$$V_g f(x,\xi) \coloneqq \langle f, \pi(x,\xi)g\rangle = \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi} f(y)\,\overline{g(y-x)}\,dy, \quad (x,\xi) \in \mathbb{R}^{2d}, \tag{1.2}$$

where the bracket $\langle\cdot,\cdot\rangle$ denote the inner product in L^2 - or its extension to the duality $\mathcal{S}' - \mathcal{S}$ in the case where f is a temperate distribution and $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. In order to understand the heuristics behind this expression, it is worth noting that it can be equivalently recast as follows:

$$V_g f(x,\xi) = \mathcal{F}(f \cdot \overline{T_x g})(\xi) = e^{-2\pi i x\cdot\xi}(f * M_\xi g^*)(x),$$

where we set $g^*(y) = \overline{g(-y)}$. Now, fix $x \in \mathbb{R}^d$ and assume for simplicity that g is a real smooth function with compact support; then $f \cdot T_x g$ is just a slice of the original signal f near the "instant" x and $V_g f(x,\xi)$ provides the spectral content of this piece of the signal. It is clear the role of $T_x g$ as sliding cut-off function as x varies on $\mathbb{R}^d$, but due to overlaps we have that $V_g f$ is a highly redundant representation; the design of the window function is in fact a major problem for obtaining a satisfactory resolution. For this and other problems of interest for applications we suggest the comprehensive references [Boa15; HA08].

The idea behind the STFT encodes the paradigm of local Fourier analysis, which first appeared in the signal processing community in an attempt to overcome the practical limitations of the Fourier transform - see the pioneering papers by Jean Ville [Vil48] and the classic monographs [Coh95; Fla99]. It is indeed well-known that the standard Fourier analysis suffers from several limitations for the purposes of signal processing. Just to mention some of them, note that the computation of a single frequency value of the Fourier transform requires the knowledge of the entire history of the signal; conversely, the inversion of the Fourier transform shows that the value of a signal at one instant comes from superposition of everlasting monochromatic waves $t_\xi(y) = e^{2\pi i \xi\cdot y}$, which are global in nature. As a result, the Fourier transform is not stable under local perturbations in the time (or frequency) domain.

Another concrete experience of the drawbacks of the Fourier transform is provided by listening to music; in a sense, this just amounts to the knowledge of a signal in the time domain, where the transition between notes can be perceived but the latter cannot be identified. Conversely, on the spectral side we may easily derive a statistics on the abundance of single notes forming the piece, at the cost of little information on when (and for how long) they are in play. The solution here is provided by the musical score, which is ultimately the prototype of a joint time-frequency representation of a signal. The Gabor transform (1.2) can be thought of as a mathematical analogue of the musical score (or, in more evocative terms, a rough mathematical model of hearing). We stress that the analysing function $t_\xi(y) = e^{2\pi i \xi\cdot y}$ of the Fourier transform (formally $\mathcal{F}(f)(\xi) = \langle f, t_\xi\rangle$) is replaced here with the phase-space localized Gabor atom $\pi(z)g$ - note that the latter belong to $L^2(\mathbb{R}^d)$ while the former do not.

The short-time Fourier transform is a rich source of intriguing mathematical problems, including invertibility/reconstruction of a signal from the knowledge of its STFT and the connection with discrete samples and Gabor expansions in the spirit of (1.1) (with $c_{mn} = \langle f, \pi(m,n)g\rangle$), see again [Grö01] for a comprehensive account. Moreover, the Gabor transform can be used to perform fine-tuning of the phase-space properties of distributions and thus introduce new function spaces. For instance, the so-called *modulation spaces* were designed by H.G. Feichtinger in the 1980s [Fei03; Fei81; Fei83] and can be thought of as the parallel of Besov-Triebel-Lizorkin spaces for wavelets, with uniform rather than dyadic geometry. This can be equivalently obtained by constraining the summability/decay of signals on phase space, that is the global behaviour of their time-frequency representations. To be precise, given a continuous and positive function m on $\mathbb{R}^{2d}$ with at most polynomial growth and $1 \le p, q \le \infty$ the modulation space $M^{p,q}_m(\mathbb{R}^d)$ is the subset of all the distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M^{p,q}_m} \coloneqq \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x,\xi)|^p m(x,\xi)^p dx\right)^{q/p} d\xi\right)^{1/q} < \infty, \tag{1.3}$$

for some (in fact, any) non-trivial $g \in \mathcal{S}(\mathbb{R}^d)$, with obvious modifications in the case where $p = \infty$ or $q = \infty$. We omit the subscript in the unweighted case $m = 1$ and also write $M^p(\mathbb{R}^d)$ when $p = q$. Even if general weight functions will be taken into account below, we are usually concerned with weights of polynomial type: for $s \in \mathbb{R}$ we set $v_s(x) := (1+|x|)^s$, $x \in \mathbb{R}^d$. We then consider amalgamated weights of type $m(z) = v_s(z)$, $z = (x, \xi) \in \mathbb{R}^{2d}$, or splitted weights like $m(x, \xi) = v_r(x)v_s(\xi)$ for $r, s \in \mathbb{R}$ - in this case we write $M^{p,q}_{r,s}(\mathbb{R}^d)$ for clarity. As a rule of thumb, the decay and smoothness of $f \in M^{p,q}_{r,s}(\mathbb{R}^d)$ are related to the weighted L^p and L^q summability of $V_g f(x, \xi)$ with respect to x and ξ respectively.

In addition, reversing the order of integration in (1.3), namely

$$\|f\|_{W^{p,q}_{r,s}} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\xi)|^p (1+|\xi|)^{rp} d\xi \right)^{q/p} (1+|x|)^{sq} dx \right)^{1/q},$$

gives rise to a norm that characterizes the so-called *Wiener amalgam spaces* $W^{p,q}_{r,s}(\mathbb{R}^d)$. In fact, they are strictly related to modulation spaces via the Fourier transform, since $W^{p,q}_{r,s}(\mathbb{R}^d) = \mathcal{F}M^{p,q}_{r,s}(\mathbb{R}^d)$.

These families of Banach spaces enjoy a large number of nice properties and connections with other well-known spaces of harmonic analysis - notably $M^2(\mathbb{R}^d) = W^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$; we refer to Section 3.2 for an account of the main features of these spaces. Modulation and Wiener amalgam spaces provide an optimal framework for the the problems of Gabor analysis, but in the last twenty years they had a non-negligible impact on the study of pseudodifferential operators and nonlinear partial differential equations. We cannot hope to frame all the relevant literature here; we just mention the classic monographs [FS02; FS98; Grö01] and the more recent ones [BO20; CR20; WHHG11] for a wide perspective and further references. We also recommend the survey [RSW12] for applications to PDEs.

While the Gabor transform is a well-defined continuous mapping $V_g : M^{p,q}_m(\mathbb{R}^d) \to L^{p,q}_m(\mathbb{R}^{2d})$ that performs the analysis of a function in terms of Gabor wave packets, the inverse problem of synthesis/reconstruction is encoded by the so-called *adjoint Gabor transform.* Precisely, fix a non-trivial atom $\gamma \in \mathcal{S}(\mathbb{R}^d)$; for any measurable function $F : \mathbb{R}^{2d} \to \mathbb{C}$ on phase space that grows at most polynomially (i.e., $|F(z)| = O(|z|^N)$ for some positive integer N), the adjoint Gabor transform is the distribution-valued integral defined by

$$V_\gamma^* F := \int_{\mathbb{R}^{2d}} F(z)\pi(z)\gamma \, dz \in \mathcal{S}'(\mathbb{R}^d).$$

The choice of the name is justified by the identity

$$\langle V_\gamma^* F, f\rangle = \langle F, V_\gamma f\rangle, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

It can be proved that the mapping V_γ^* is continuous from $L_m^{p,q}(\mathbb{R}^{2d})$ to $M_m^{p,q}(\mathbb{R}^d)$, and the following crucial *inversion formula* holds if $\langle \gamma, g\rangle \neq 0$:

$$f = \frac{1}{\langle \gamma, g\rangle} V_\gamma^* V_g f = \frac{1}{\langle \gamma, g\rangle} \int_{\mathbb{R}^{2d}} V_g f(z)\pi(z)\gamma \, dz, \quad f \in M_m^{p,q}(\mathbb{R}^d). \tag{1.4}$$

The role of the adjoint Gabor transform in the synthesis step is thus clarified.

1.1.2 The analysis of operators via Gabor wave packets

The machinery developed so far can also be used to perform the Gabor wave packet analysis of operators. Given a linear continuous operator $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, if we fix two windows $g, \gamma \in \mathcal{S}(\mathbb{R}^d)$ such that $\|g\|_{L^2} = \|\gamma\|_{L^2} = 1$ and apply the inversion formula (1.4) twice we obtain

$$A = V_\gamma^* V_\gamma A V_g^* V_g = V_\gamma^* \widetilde{A} V_g,$$

where we set $\widetilde{A} = V_\gamma A V_g^*$. It turns out that $\widetilde{A}$ corresponds to a representation of the operator A on phase space; precisely, a straightforward computation shows that $\widetilde{A}$ is an integral operator satisfying

$$\widetilde{A}(V_g f)(w) = V_\gamma(Af)(w) = \int_{\mathbb{R}^{2d}} K_A(w,z) V_g f(z) dz, \quad w \in \mathbb{R}^{2d},$$

where the integral kernel corresponds to the so-called *Gabor matrix* of A with respect to analysis and synthesis windows g and γ respectively:

$$K_A(w,z) \coloneqq \langle A\pi(z)g, \pi(w)\gamma\rangle, \quad w,z \in \mathbb{R}^{2d}.$$

Indeed, the Gabor matrix K_A can be thought of as an infinite-dimensional matrix encoding the whole information on the phase-space features of A, since it does precisely characterize how wave packets evolve and interact under the action of A.

The phase-space analysis of an operator thus corresponds to a detailed investigation of the properties of the corresponding Gabor matrix. In particular, it is clear that some form of sparsity of K_A is a highly desirable property, for both theoretical and numerical purposes.

The techniques of Gabor analysis are the backbone of all the results contained in this dissertation, hence in the first part - **Chapters 3 and 4** - we provide a hopefully extensive review of the background material with pointers to the literature and proofs of new technical results needed below.

1.2 The problem of quantization

The first class of problems that we are concerned with may be labelled "quantization and related issues" for the sake of conciseness. More precisely, this amounts to the Gabor analysis of pseudodifferential and related operators, with applications to the analysis of PDEs.

We already mentioned that Gabor analysis has been largely influenced by problems arising in quantum physics; in fact, several notions have usually undergone parallel and independent developments, such as Gabor expansions (they were introduced by J. von Neumann in [Neu18] within a rigorous theory of the measurement process). Another example is the origin of the *Wigner transform*: recall that for $f, g \in L^2(\mathbb{R}^d)$ it is defined by

$$W(f,g)(x,\xi) \coloneqq \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot y} f\Big(x + \frac{y}{2}\Big) \overline{g\Big(x - \frac{y}{2}\Big)} dy.$$

If $f = g$ we write Wf. Even if its appearance is not much revealing, this sort of Fourier transform of the two-point cross-correlation between f and g was mysteriously introduced by E. Wigner in a celebrated paper of 1932 [Wig32] as a quasi-probability distribution on phase space in order to derive quantum corrections to classical statistical mechanics, where the relevant terms are functions of jointly position and momentum. It was later rediscovered in the context of signal analysis by J. Ville [Vil48] and eventually became a popular tool in this community because it enjoys several properties desired from a good time-frequency representation [Jan97; MH97]. In fact, some of such features are shared with the STFT since

$$W(f,g)(x,\xi) = 2^d e^{4\pi i x \cdot \xi} V_{g^\vee} f(2x, 2\xi),$$

where we set $g^\vee(y) \coloneqq g(-y)$. Nevertheless, there is an intrinsic difference: the Wigner transform Wf is a *quadratic* time-frequency representation - in the sense that $W(cf) = |c|^2 Wf$ for any $c \in \mathbb{C}$, while the Gabor transform (with fixed window g) is linear. Heuristically, $Wf(x,\xi)$ is interpreted as a measure of the energy content of the signal f in a "tight" spectral band around ξ during a "short" time interval near x. See Section 3.1.2 for further details.

The Wigner transform plays a key role in the problem of *quantization*. In a nutshell, this requires to associate in a "robust" way a classical observable σ (i.e., a suitable function on the phase space) with a quantum observable $\mathrm{op}(\sigma)$ - which is represented by a self-adjoint operator on the Hilbert space of the system in the canonical Schrödinger picture of quantum mechanics. A naive way to perform quantization is provided by the following recipe: given an observable $\sigma : \mathbb{R}^{2d} \to \mathbb{R}$, then $\mathrm{op}(\sigma)$ is obtained by formally replacing x_j with the position operator X_j and ξ_j with the momentum operator D_j, where

$$X_j f(x) = x_j f(x), \quad D_j f(x) = \frac{h}{2\pi i}\frac{\partial}{\partial x_j} f(x), \quad f \in \mathcal{S}(\mathbb{R}^d), \quad j = 1, \ldots, d,$$

where $h > 0$ is a small real parameter (the analogue of the Planck constant). The well-posedness issues related to such a functional calculus of operators are of primary concern. It should be also emphasized that the operators X_j and D_j may be defined in a different way as long as they satisfy the canonical commutation relations

$$[X_j, X_k] = 0, \quad [D_j, D_k] = 0, \quad 2\pi[X_j, D_k] = ih\delta_{jk} I, \quad j, k = 1, \ldots, d,$$

where $[A, B] = AB - BA$ is the commutator of the operators A and B. Moreover, the correspondence $\sigma \mapsto \mathrm{op}(\sigma)$ should depend on the parameter h in such a way that the classical observable σ can be recovered by taking the "classical limit" $\lim_{h\to 0} \mathrm{op}_h(\sigma)$ in a suitable sense (this is known as the *correspondence principle* [Boh76]). For simplicity we temporarily ignore this aspect and fix $h = 1$ in this discussion, in line with the custom in harmonic analysis. Note that it is customary to introduce the reduced parameter $\hbar := h/2\pi$, hence we have $\hbar = 1/2\pi$ for the moment.

There is plenty of quantization schemes in the literature, each with its own strengths and weaknesses. Let us commence our discussion with the case of monomial observables for concreteness, namely $\sigma(x, \xi) = x_j^m \xi_j^n$ for some $m, n \in \mathbb{N}$ and $j = 1, \ldots, d$. We immediately remark that a clear ordering problem occurs due to the non-commutativity of X_j and D_j, which is an unavoidable source of ambiguity in the definition of $\mathrm{op}(\sigma)$. In this respect, providing a quantization rule corresponds to fixing an order for the quantization of monomials. Two options are quite natural: the *normal* and *antinormal orderings*, also called left (resp. right) or $q - p$ (resp. $p - q$) quantization, the name being clear from the very definition:

$$x_j^m \xi_j^n \xrightarrow{\text{left}} X_j^m D_j^n, \quad x_j^m \xi_j^n \xrightarrow{\text{right}} D_j^n X_j^m.$$

A compromise between these prescriptions which favours symmetry is provided by the *Weyl quantization*:

$$\mathrm{op_w}(x_j^m \xi_j^n) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{b}{k} D_j^{n-k} X_j^n D_j^k.$$

The correspondence introduced by H. Weyl in the late 1920s [Wey] is usually acknowledged as the one having optimal properties. In fact, we highlight that M.

de Gosson has recently made a strong case for the Born-Jordan quantization as the optimal quantization rule, cf. [Gos16]; this is an equally weighted average of the operator orderings, namely

$$\mathrm{op_{BJ}}(x_j^m \xi_j^n) = \frac{1}{m+1} \sum_{k=0}^{m} X_j^{m-k} D_j^n X_j^k = \frac{1}{n+1} \sum_{k=0}^{n} D_j^{n-k} X_j^m D_j^k.$$

Note that the Weyl and Born-Jordan orderings coincide for $m = n = 1$ and both yield the operator $(X_j D_j + D_j X_j)/2$.

The quantization of polynomial observables ultimately reduces to the previous rules. The coverage of more general functions crucially relies on a simple though powerful remark. Let $P(X,D) = \sum_{|\alpha| \le m} X^\alpha D_x^\alpha$ be a linear partial differential operator of order $m \in \mathbb{N}$ - we use the multi-index notation; note that $P(X,D)$ corresponds to the normal quantization of the polynomial $P(x,\xi) = \sum_{|\alpha| \le m} x^\alpha \xi^\alpha$. The inversion formula for the Fourier transform yields

$$P(X,D)f = \int_{\mathbb{R}^{2d}} e^{2\pi i (x-y)\cdot\xi} P(x,\xi) f(y) dy d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

It is quite tempting to take this integral representation as a definition of the normal quantization of a general function σ of both position and momentum beyond the polynomial case, at least formally; the hard work of providing a rigorous and consistent framework for the study of these operators is left to the theory of *pseudodifferential operators* [BS72; Fol89; Hör85; Kg81; NR10; Shu87; Tay11; Won98], with the aim to relate the properties of the operator $\sigma(x,D)$ (e.g., invertibility, composition, etc.) to those enjoyed by the corresponding symbol $\sigma(x,\xi)$ at the algebraic level.

Operators of the form

$$\mathrm{op_{KN}}(\sigma) f = \int_{\mathbb{R}^{2d}} e^{2\pi (x-y)\cdot\xi} \sigma(x,\xi) f(y) dy d\xi \tag{1.5}$$

are usually known as *classical* (or *Kohn-Nirenberg*) *pseudodifferential operators*. In the case of the polynomial symbol we retrieve the normal ordering discussed above. Similarly, given a generalized phase-space *symbol* $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$, the *Weyl quantization* prescribes that the operator $\sigma^{\mathrm{w}} = \mathrm{op_w}(\sigma) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is (formally) defined by

$$\mathrm{op_w}(\sigma) f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i (x-y)\cdot\xi} \sigma\left(\frac{x+y}{2}, \xi\right) f(y) dy d\xi. \tag{1.6}$$

The rigorous handling of the integral expressions in (1.5) and (1.6) is non-trivial and easily becomes a quite technical issue - one is required to precisely adjust the regularity and decay assumptions on the symbol under which they can be meaningfully interpreted. Nevertheless, a straightforward computation reveals the role of the Wigner transform in the Weyl quantization:

$$\langle \mathrm{op_w}(\sigma) f, g\rangle = \langle \sigma, W(g,f)\rangle, \quad f,g \in \mathcal{S}(\mathbb{R}^d). \tag{1.7}$$

This weak formulation is certainly easier to handle and allows one to cover distributional symbols without further effort. More importantly, this is also the bridge to Gabor analysis, since modulation and Wiener amalgam spaces may be used both as symbol classes as well as background where to study boundedness of pseudodifferential operators; the basic results in this connection may be found in [Grö01; Grö06a], while [CR20] is devoted to more advanced outcomes, also including applications to PDEs.

1.2.1 Linear perturbations of the Wigner transform and the Weyl quantization

Let us focus on the Wigner distribution, in order to understand the role of time-frequency representations in quantization rules. It is well known that not all properties which are desired from a time-frequency representation are compatible. For instance, the Wigner transform is real-valued, but it may take negative values; this is a serious obstruction to the interpretation of the Wigner transform as a probability distribution or as an energy density of a signal. A key result in the problem of the positivity of the Wigner distribution is Hudson's theorem (cf. Proposition 3.1.3), stating that generalized Gaussian functions have positive Wigner transforms [Jan84]. The question of zeros of the Wigner distribution is a highly non-trivial problem which requires the contribution of several branches of analysis, see the recent paper [GJM20].

In order to obtain time-frequency representations that are positive for all functions but still retaining the nice properties of the Wigner distribution (marginal densities, orthogonality relations, etc.), one is lead to take local averages of the Wigner transform in the hope of taming sign oscillations. This is usually done by convolving Wf with a suitable kernel θ and such a procedure yields a general class of quadratic time-frequency representations, which is called *Cohen's class* after L. Cohen [Coh66].

Time-frequency representations in Cohen's class are parametrized by a kernel $\theta \in \mathcal{S}'(\mathbb{R}^{2d})$, in the sense of the following definition:

$$Q_\theta(f,g) \coloneqq W(f,g) * \theta, \qquad f,g \in \mathcal{S}(\mathbb{R}^d). \tag{1.8}$$

Most of the time-frequency representations proposed so far belong to Cohen's class [Coh95; HA08], and the correspondence between properties of θ and Q_θ is well understood [HA08]. In many respects Q_θ can be interpreted as a perturbation of the Wigner distribution and the Cohen class provides a unifying framework for the study of several time-frequency representations appearing in signal processing - see for instance [Coh89; Coh95; HA08; HB92].

For every time-frequency representation in Cohen's class one can naturally introduce a quantization rule in analogy to the Weyl quantization (1.7), namely,

$$\langle \mathrm{op}_\theta(\sigma) f, g\rangle = \langle \sigma, Q_\theta(g,f)\rangle = \langle \sigma * \theta^*, W(g,f)\rangle, \quad f,g \in \mathcal{S}(\mathbb{R}^d)\,, \tag{1.9}$$

whenever the expressions make sense [Coh13; Grö01].

Although the new operator $\mathrm{op}_\theta(\sigma)$ is just a Weyl operator with the modified symbol $\sigma * \theta^*$ (whenever defined in $\mathcal{S}(\mathbb{R}^{2d})$), the variety of pseudodifferential calculi given by definition (1.9) adds flexibility and a new flavour to the description and analysis of operators. For example, a first important variation of the Wigner transform are the *τ-Wigner transforms* [BDDO10; BDDOC10], which are parametrized by a real number τ and defined by

$$W_\tau(f,g)(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi} f(x+\tau y)\overline{g(x-(1-\tau)y)}\,dy, \quad f,g \in \mathcal{S}(\mathbb{R}^d)\,. \tag{1.10}$$

Such distributions belong to Cohen's class, their kernel $\theta_\tau \in \mathcal{S}'(\mathbb{R}^{2d})$ being given (on the spectral side) by

$$\widehat{\theta_\tau}(x,\xi) = e^{-2\pi i(\tau-1/2)x\cdot\xi}, \quad (x,\xi)\in\mathbb{R}^{2d}. \tag{1.11}$$

The corresponding pseudodifferential calculi are known as Shubin's τ-pseudodifferential operators [Shu87]; according to formula (1.9) these are explicitly given by

$$\mathrm{op}_\tau(\sigma) f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi(x-y)\cdot\xi}\sigma((1-\tau)x+\tau y,\xi) f(y) dy d\xi. \tag{1.12}$$

In some sense, the dependence on x and y of the symbol is now amalgamated in an affine combination of the variables. Note that for the parameter $\tau = 1/2$ this is just the Weyl transform, while for $\tau = 0$ we recover the standard Kohn-Nirenberg quantization. We highlight that the already mentioned Born-Jordan quantization rule [BJ25] also belongs to the Cohen class and in fact can be obtained as an integral average over $\tau \in [0,1]$, see [BDDO10; CGN17a] and the monograph [Gos16].

While the distributions in the Cohen class are definitely more general, τ-distributions provide a precise control over the deviation from the Wigner distribution thanks to the parameter τ - or equivalently $\mu = \tau - 1/2$. Accordingly, it is interesting to investigate whether some of the most relevant properties of the Wigner distributions and the Weyl operators “survive” the perturbation, that is if they extend (possibly in a weaker form) to τ distributions and the corresponding quantizations. This is the motivation for the papers [CDT19] and [CNT19b], which are now briefly described; expanded accounts are given in Sections 5.5.3, 5.7.3, 5.8 and 5.9.

The main purpose of the paper [CDT19], which is a joint work with E. Cordero and L. D'Elia, is to derive uniform upper bounds with respect to $\tau \in (0,1)$ for the operator norm of $\mathrm{op}_\tau(\sigma)$ on modulation and Wiener amalgam spaces for several symbol classes of the same type, cf. Theorem 5.7.7. This is achieved by duality, in the sense that we first derived uniform continuity estimates for τ-Wigner distributions - see Theorem 5.5.9. We stress that the endpoint quantizations $\tau = 0$ and $\tau = 1$ are more delicate to handle, being unbounded in general - see Proposition 5.7.8 below.

In the article [CNT19b] (joint work with E. Cordero and F. Nicola) the focus is on Weyl operators with symbols in the modulation space $M^{\infty,1}(\mathbb{R}^d)$. In fact, this space of rough symbols (essentially, bounded continuous functions which locally coincide with the Fourier transform of an integrable function) was independently discovered by J. Sjöstrand in 1994 [Sjö94] in an attempt to extend the well-behaved Hörmander class $S^0_{0,0}$. He showed that this exotic symbol class still yield pseudodifferential operators which are bounded on $L^2(\mathbb{R}^d)$. In addition, it is a Banach algebra under the *Weyl product*, namely if $\rho, \sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ then $\rho^{\mathrm{w}} \circ \sigma^{\mathrm{w}}$ is again a pseudodifferential operator with Weyl symbol in $M^{\infty,1}(\mathbb{R}^{2d})$ - see Section 4.2 for further details.

Later on, K. Gröchenig exploited the full power of time-frequency analysis to further study the features of this symbol class [Grö06c] and proved an important characterization that could be condensed in the following statement: the Gabor matrix of σ^{w} is approximately diagonalized by Gabor wave packets. More precisely (see Theorem 5.8.1 for further details): $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ if and only if there exists $H \in L^1(\mathbb{R}^{2d})$ such that, for all $g \in M^1(\mathbb{R}^d)$,

$$|\langle \sigma^{\mathrm{w}} \pi(z) g, \pi(w) g\rangle| \le H(w-z), \quad w, z \in \mathbb{R}^{2d}. \tag{1.13}$$

This sparsity estimate is a rich source of consequences as far as boundedness results for Weyl operators are concerned. For instance, it can be used to easily prove that if $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ then σ^{w} is bounded on every modulation space[1] $M^{p,q}(\mathbb{R}^d)$, $1 \le p, q \le \infty$. Indeed, recall that lifting the continuity problem to the level of phase space by means of the Gabor transform amounts to study the continuity

[1] This result can be interpreted as a generalization of the classical result by Caldéron and Vaillancourt [CV71], since the space $C^k_{\mathrm{b}}(\mathbb{R}^{2d})$ of k-times continuously differentiable functions with bounded derivatives up to k-th order is embedded in $M^{\infty,1}(\mathbb{R}^{2d})$ for $k > 2d$ [Grö01, Theorem 14.5.3].

of a phase-space representation of σ^{w}, that is an integral operator with integral kernel given by the Gabor matrix $K_{\sigma^{\mathrm{w}}}$. Using the estimate above we see that the action of σ^{w} on phase space is essentially that of a convolution operator with kernel $H \in L^1(\mathbb{R}^{2d})$, that is continuous on $L^{p,q}(\mathbb{R}^{2d})$, $1 \le p, q \le \infty$ (cf. [Grö01, Proposition 11.3(a)]). We refer to Theorem 5.8.2 for further details and related results.

In [CNT19b] we extended this result to τ-operators and used it to prove new boundedness results on Wiener amalgam spaces. In some sense, this family of results is stable under the perturbation represented by τ; this is expected for many reasons, including the fact that the change-of-quantization map

$$\sigma_1^{\tau_1} = \sigma_2^{\tau_2} \quad \Longleftrightarrow \quad \widehat{\sigma_2}(\xi,\eta) = e^{-2\pi i(\tau_2-\tau_1)\xi\cdot\eta}\widehat{\sigma_1}(\xi,\eta),$$

is well behaved on modulation spaces (cf. [Tof04a, Proposition 1.2 and Remark 1.5]). The same is not true for Wiener amalgam spaces [CN10, Proposition 6.4], therefore the results for Weyl operators with symbols in some $W^{p,q}$ do not extend in general to other quantizations; for instance, Weyl operators with symbols in $L^1(\mathbb{R}^{2d}) \subset W^{\infty,1}(\mathbb{R}^{2d})$ are bounded on $L^2(\mathbb{R}^d)$ (cf. [WHHG11]) but neither Kohn–Nirenberg operators ($\tau = 0$) nor anti-Kohn-Nirenberg operators ($\tau = 1$) are [Bou95; Bou97]. See also [DT18] and Proposition 5.7.8 in this connection.

It is then interesting to investigate whether almost-diagonalization results in the same spirit also hold for operators with symbols in the Fourier-Sjöstrand class $W^{\infty,1}(\mathbb{R}^{2d}) = \mathcal{F}M^{\infty,1}$, which contains rougher functions and distributions - as an example, the Dirac delta belongs to $W^{\infty,1}$. As expected, the path is somehow more involved due to the peculiar way τ comes across; nevertheless, this has been done in [CNT19b, Theorem 4.3], leading to several boundedness and algebraic results on modulation and Wiener amalgam spaces for τ-operators with $\tau \in (0,1)$ - weaker results hold also for endpoint quantizations. We highlight that some properties of the Weyl quantization are destroyed by τ-perturbations, like the discrete version of (1.13).

In view of this discussion, it is therefore tempting to stress the resistance of the Weyl quantization under more general perturbations in the Cohen class, still with some loose control on the perturbation. One could try to fill the gap between generality and controllability by allowing general parametrizations $\tau : \mathbb{R}^d \to \mathbb{R}^d$ in (1.12). For instance, in the recent paper [ER20] the authors consider smooth quantizing functions with bounded or unbounded derivatives. While in the nonlinear scenario there is still too much freedom in choosing the function τ, the linear case can be characterized in full generality. This just amounts to replace the scalar parameter τ with a matrix parameter $T \in \mathbb{R}^{d\times d}$.

The resulting family of T-*pseudodifferential operators* under our attention is

$$\sigma^T f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i(x-y)\cdot\xi}\sigma((I-T)x+Ty,\xi)f(y)dyd\xi = \langle\sigma, W_T(g,f)\rangle, \quad (1.14)$$

where $I = I_d \in \mathbb{R}^{d\times d}$ is the identity matrix and we introduced the *matrix-Wigner distribution*

$$W_T(g,f)(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi} g(x+Ty)\overline{f(x-(I-T)y)}dy. \quad (1.15)$$

These are members of the Cohen class in (1.8) with a kernel θ_T given by

$$\theta_T = \mathcal{F}^{-1}\Theta_T \in \mathcal{S}'(\mathbb{R}^{2d}), \quad \Theta_T(u,v) = e^{-2\pi i\xi\cdot(T-I/2)\eta}.$$

An even more general definition in the spirit of (1.15) uses an arbitrary linear mapping of the pair $(x,y)\in\mathbb{R}^{2d}$. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix} \in \mathbb{R}^{2d\times 2d}$ be an invertible, real-valued $2d\times 2d$-matrix. We define the bilinear time-frequency transform $\mathcal{B}_A$ of two functions f,g by

$$\mathcal{B}_A(f,g)(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi} f(A_{11}x + A_{12}y)\overline{g(A_{21}x+A_{22}y)}dy. \quad (1.16)$$

Note that this general framework also encompasses the Gabor transform (1.2). Clearly, W_T in (1.15) is a special case by choosing

$$A = A_T = \begin{bmatrix} I & T \\ I & -(I-T)\end{bmatrix}.$$

Once again, every matrix-Wigner transform $\mathcal{B}_A$ is associated with a quantization rule: given an invertible $2d\times 2d$-matrix A and a symbol $\sigma\in\mathcal{S}'(\mathbb{R}^{2d})$, we define the operator σ^A by

$$\langle\sigma^A f, g\rangle = \langle\sigma, \mathcal{B}_A(g,f)\rangle, \qquad f,g\in\mathcal{S}(\mathbb{R}^d)\,. \quad (1.17)$$

The class of matrix-Wigner transforms has already a sizeable history. To the best of our knowledge they were first introduced in [FM89] in dimension $d=1$ for a different purpose, but the original contribution to the subject went unnoticed. The first thorough investigation of matrix-Wigner transforms $\mathcal{B}_A$ is contained in the unpublished Ph.D. thesis [Bay10] of D. Bayer, who studied the general properties of this class of time-frequency representations and the associated pseudodifferential operators. Independently, in [BCO11] the authors introduced and studied "Wigner representations associated with linear transformations of the time-frequency plane" in dimension $d = 1$. In [GG13] matrix-Wigner transforms were used for a signal estimation problem. Recently, in [Tof17] the author discusses "matrix parametrized pseudo-differential calculi on modulation spaces", which precisely correspond to T-operators in (1.14) in the context of modulation space, while in [CT17] they are studied in the Gelfand-Shilov regularity setting.

Finally, E. Cordero and the author in [CT20] reworked and streamlined several results of [Bay10] and showed that matrix-Wigner distributions as in (1.15) are all and only the members of the general family (1.16) that belong to Cohen's class. Several results concerning such distributions, including boundedness results on modulation and Wiener amalgam spaces, are given.

In the joint contribution [BCGT20] with D. Bayer, E. Cordero and K. Gröchenig, we used the accumulated knowledge on the topic to provide several results for T-pseudodifferential operators, including broad sets of norm estimates in the spirit of [CDT19] and almost-diagonalization properties in the spirit of [CNT19b]. As expected, the results involving modulation spaces are somehow stable under matrix perturbations, while those in the framework of Wiener amalgam spaces are less easy to extend. Moreover, the Wigner distribution and the Weyl calculus have stood out for their remarkable properties.

All these results are presented in **Chapter 5**. A detailed outline of the findings can be found at the beginning of that chapter. In some sense, our effort can be seen as a complete time-frequency characterization of pseudodifferential operators and associated distributions for the general linear case of quantizing parametrization $\tau : \mathbb{R}^d \to \mathbb{R}^d$ in (1.10).

1.3 Wave packet analysis of metaplectic operators and applications

We have already mentioned that the analysis of a linear continuous operator $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ in terms of Gabor wave packets comes through a detailed study of the corresponding Gabor matrix K_A with respect to fixed $g, \gamma \in \mathcal{S}(\mathbb{R}^d)\setminus\{0\}$, namely

$$K_A(w, z) \coloneqq \langle A\pi(z)g, \pi(w)\gamma\rangle, \quad w, z \in \mathbb{R}^{2d}.$$

In the previous section we have seen this principle in action in the case where A is a Weyl operator with symbol in the Sjöstrand class $M^{\infty,1}(\mathbb{R}^d)$. Results in the same spirit have been appearing in the literature for several families of operators, including pseudodifferential operators [CNT19b; GR08; Grö06c; RT98; Tat04], Fourier integral operators [CGNR13; CGNR14; CNR10; Tat04] and propagators associated with Cauchy problems for Schrödinger-type evolution equations

[CNR15a; CNR15b; CNR15c; KT05; MMT08; Tat04]. We also recommend the recent monograph [CR20] for a more systematic account.

We stress that wave packets should be tailored in order to best fit the geometry of the problem. For instance, the Gabor matrix of Fourier integral operators arising as propagators for strictly hyperbolic equations does not display a sparse behaviour, while analogous representations involving curvelet atoms do enjoy super-polynomial decay, cf. [CD05; CF78]. See also [GL07; ST05; Tat04] for other applications of wave packet analysis.

For the sake of concreteness let us focus on the Schrödinger propagator for the free particle $U(t) = e^{i(t/2\pi)\Delta}$, $t \in \mathbb{R}$, and fix $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. It has been proved that the corresponding Gabor matrix is well organized [CNR09; CNR10; CNR12]; precisely, for any $t \in \mathbb{R}$ and $N \in \mathbb{N}$ there exists a constant $C = C(t,N) > 0$ such that the following decay estimate for the Gabor matrix elements of $U(t)$ holds:

$$|\langle e^{i(t/2\pi)\Delta}\pi(z)g, \pi(w)g\rangle| \le C(1+|w-S_t z|)^{-N}, \quad w,z \in \mathbb{R}^{2d}, \tag{1.18}$$

where $S_t \in \mathbb{R}^{2d\times 2d}$ is the block matrix

$$S_t = \begin{bmatrix} I & 2tI \\ O & I \end{bmatrix}, \tag{1.19}$$

where $O \in \mathbb{R}^{d\times d}$ is the null matrix. We remark that $t \mapsto S_t$ coincides with the Hamiltonian flow for the free particle in phase space; precisely, the classical equations of motion with Hamiltonian $H(x,\xi) = |\xi|^2$ and initial datum $(x_0,\xi_0) \in \mathbb{R}^{2d}$ are solved by $(x(t),\xi(t)) = S_t(x_0,\xi_0)$. Hence the wave packet analysis (1.18) shows that the time evolution of wave packets under $U(t)$ approximately follows the classical flow, in according with the correspondence principle of quantum mechanics mentioned before.

Nevertheless, a distinctive feature of wave propagation dynamics is the unavoidable effect of diffraction. In the situation under our attention it does manifest itself as the well-known phenomenon of the *spreading* of wave packets [Dir78, Section 31]. Moreover, a straightforward consequence of the *dispersive estimates* for the Schrödinger propagator [Tao06] is that there exists $C > 0$ such that

$$|\langle e^{i(t/2\pi)\Delta}\pi(z)g, \pi(w)g\rangle| \le C(1+|t|)^{-d/2}, \quad w,z \in \mathbb{R}^{2d}. \tag{1.20}$$

It may therefore appear quite unsatisfactory that there is no trace of such issues in quasi-diagonalization estimates as (1.18).

This question has been addressed in the article [CNT20], which is a joint work with E. Cordero and F. Nicola; the results therein are presented in **Chapter 6** below. Our purpose was exactly to prove refined estimates for the Gabor matrix of $U(t)$ where sparsity, spreading and dispersive phenomena are simultaneously represented. To the best of our knowledge, we are not aware of results in this spirit for pseudodifferential or evolution operators.

Our quest is in fact motivated by the more general situation where $U(t)$ is the Schrödinger propagator corresponding to the Hamiltonian $H = Q^{\mathrm{w}}$, where Q is a real homogeneous quadratic polynomial on $\mathbb{R}^{2d}$ and Q^{w} denotes its Weyl quantization as in (1.6). To be precise, consider

$$Q(x,\xi) = \frac{1}{2}A\xi\cdot\xi + Bx\cdot\xi + \frac{1}{2}Cx\cdot x, \tag{1.21}$$

where $A, B, C \in \mathbb{R}^{d\times d}$ and A, C are symmetric matrices. It is not difficult to compute the corresponding Weyl quantization, that is

$$Q^{\mathrm{w}} = -\frac{1}{8\pi^2}\sum_{j,k=1}^{d} A_{j,k}\partial_j\partial_k - \frac{i}{2\pi}\sum_{j,k=1}^{d} B_{j,k}x_j\partial_k + \frac{1}{2}\sum_{j,k=1}^{d} C_{j,k}x_jx_k - \frac{i}{4\pi}\mathrm{Tr}(B). \tag{1.22}$$

The Schrödinger propagator $U(t) = e^{-2\pi i t Q^{\mathrm{w}}}$, $t \in \mathbb{R}$, is in turn an instance of a *metaplectic operator*. In short, the metaplectic representation is a machinery which associates a symplectic matrix $S \in \mathrm{Sp}(d,\mathbb{R})$ with a member of the metaplectic group $\mu(S) \in \mathrm{Mp}(d,\mathbb{R})$, that is a unitary operator on $L^2(\mathbb{R}^d)$ defined up to the sign. If $\mathbb{R} \ni t \mapsto S_t \in \mathrm{Sp}(d,\mathbb{R})$ denotes the classical flow on phase space associated with the quadratic Hamiltonian $H(x,\xi) = Q(x,\xi)$ then $\mu(S_t) = e^{-2\pi i t Q^{\mathrm{w}}}$ - see (1.19) for the free particle case. We refer to Section 4.3 for further details.

It is therefore convenient to focus on metaplectic operators as primary objects of our investigation. The spreading of wave packets under $\mu(S)$ is now connected with the *singular values* of $S \in \mathrm{Sp}(d,\mathbb{R})$ [CNT19a], which occur in couples (σ, σ^{-1}) of positive real numbers. We fix the ordering by labelling the largest d singular values in such a way that $\sigma_1 \ge \ldots \ge \sigma_d \ge 1$; moreover we set $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_d)$ and introduce the matrices

$$D = \begin{bmatrix} \Sigma & O \\ O & \Sigma^{-1} \end{bmatrix}, \quad D' = \begin{bmatrix} \Sigma^{-1} & O \\ O & I \end{bmatrix}, \quad D'' = \begin{bmatrix} I & O \\ O & \Sigma^{-1} \end{bmatrix}. \tag{1.23}$$

The singular value decomposition of $S \in \mathrm{Sp}(d,\mathbb{R})$ (also known as the *Euler decomposition* in this setting) has a peculiar form due to the symplectic condition, namely there exist (non-unique) orthogonal and symplectic matrices U, V such that $S = U^\top DV$, cf. Proposition 4.3.2 and Section 4.3.1 below. Such factorization is identified by the triple (U, V, Σ). In the following for a given $S \in \mathrm{Sp}(d,\mathbb{R})$ we will denote by (U, V, Σ) an Euler decomposition of S and by D, D', D'' the above defined related matrices.

We are now in the position to state our first result, concerning rapidly decaying Gabor wave packets.

Theorem 1.3.1. *For any $g, \gamma \in \mathcal{S}(\mathbb{R}^d)$ and $N > 0$ there exists $C > 0$ such that, for every $S \in \mathrm{Sp}(d,\mathbb{R})$ and any Euler decomposition (U, V, Σ) of S,*

$$|\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle| \le C(\det\Sigma)^{-1/2}(1+|D'U(w-Sz)|)^{-N}, \quad z,w\in\mathbb{R}^{2d}. \tag{1.24}$$

We see that the simultaneous occurrence of sparsity, spreading and dispersive phenomena is represented by the quasi-diagonal structure along S, the dilation by $D'U$ and the factor $(\det\Sigma)^{-1/2}$ respectively. An equivalent form of the previous estimate where the spreading phenomenon is somehow more distributed follows by noticing that $D'U(w-Sz) = D'Uw - D''Vz$.

The special case of the free particle propagator is treated in detail in Section 6.3. We just mention here that, for any fixed $t \in \mathbb{R}$ and any Euler decomposition (U_t, V_t, Σ_t) of S_t, the estimate (1.24) reads

$$\left|\langle e^{i(t/2\pi)\Delta}\pi(z)g, \pi(w)\gamma\rangle\right| \le C(1+|t|)^{-d/2}(1+|D_t'U_t(w-S_tz)|)^{-N}, \quad w,z\in\mathbb{R}^{2d}.$$

We see that the features of both (1.18) and (1.20) are now represented, whereas the spreading phenomenon manifests itself as a dilation by the matrix $D_t'U_t$, the nature of which is investigated in Section 6.3.

We also provide results in the same spirit of Theorem 1.3.1 for wave packets associated with less regular atoms; in particular we assume that g and γ belong to suitable modulation spaces. The latter provide an optimal environment where to investigate the behaviour of the Gabor matrix of a metaplectic operator, as evidenced by the following result.

Theorem 1.3.2. *(i) Let $1 \le p, q, r \le \infty$ satisfy $1/p + 1/q = 1 + 1/r$. For any $g \in M^p(\mathbb{R}^d)$, $\gamma \in M^q(\mathbb{R}^d)$, $S \in \mathrm{Sp}(d,\mathbb{R})$ and any Euler decomposition (U, V, Σ) of S, there exists $H \in L^r(\mathbb{R}^{2d})$ such that, for any $z, w \in \mathbb{R}^{2d}$,*

$$|\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle| \le H(D'U(w-Sz)), \tag{1.25}$$

with

$$\|H\|_{L^r} \le (\det\Sigma)^{1/2-1/r}\|g\|_{M^p}\|\gamma\|_{M^q}. \tag{1.26}$$

(ii) Let $s > 2d$. For any $g, \gamma \in M^\infty_{v_s}(\mathbb{R}^d)$ there exists $H \in L^\infty_{v_{s-2d}}(\mathbb{R}^{2d})$ such that (1.25) holds, with

$$\|H\|_{L^\infty_{v_{s-2d}}} \le (\det\Sigma)^{-1/2}\|g\|_{M^\infty_{v_s}(\mathbb{R}^d)}\|\gamma\|_{M^\infty_{v_s}(\mathbb{R}^d)}.$$

Here we used the notation $\|H\|_{L^\infty_{v_s}} = \|Hv_s\|_{L^\infty}$. We remark that the best decay in (1.26) is achieved in the case where $p = q = r = 1$, namely for Gabor atoms belonging to the Feichtinger algebra $M^1(\mathbb{R}^d)$. We also highlight the inclusion $M^\infty_{v_s}(\mathbb{R}^d) \subset M^1(\mathbb{R}^d)$ for $s > 2d$, which follows directly from the definition.

In Theorem 1.3.3 we prove an estimate in the same spirit of Theorem 1.3.2 for the Gabor matrix of the so-called *generalized metaplectic operators*. This family of operators characterized by the sparsity of their phase-space representation has been introduced and studied in [CGNR13; CGNR14] in connection with inverse-closed algebras of Fourier integral operators. In short, a linear operator $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is in the class $FIO(S)$ if there exists $H \in L^1(\mathbb{R}^{2d})$ such that

$$|\langle A\pi(z)g, \pi(w)g\rangle| \le H(w - Sz), \quad w, z \in \mathbb{R}^{2d}.$$

The main properties are recalled in Section 4.3.5. In particular, it can be proved that $A \in FIO(S)$ if and only if A is the composition of a metaplectic operator $\mu(S)$ and a Weyl operator σ^{w} for a suitable symbol $\sigma \in M^{\infty,1}$ - cf. Theorem 4.3.10.

Theorem 1.3.3. *Let* $1 \le p, q, r \le \infty$ *satisfy* $1/p + 1/q = 1 + 1/r$. *Consider* $S \in \mathrm{Sp}(d, \mathbb{R})$ *with an Euler decomposition* (U, V, Σ), $a \in M^{\infty,1}(\mathbb{R}^{2d})$ *so that* $A := a^{\mathrm{w}}\mu(S) \in FIO(S)$, *cf. Theorem 4.3.10. For any* $g \in M^p(\mathbb{R}^d)$, $\gamma \in M^q(\mathbb{R}^d)$ *there exists* $H \in L^r(\mathbb{R}^{2d})$ *such that, for any* $z, w \in \mathbb{R}^{2d}$,

$$|\langle A\pi(z)g, \pi(w)\gamma\rangle| \le H(D'U(w - Sz)),$$

with

$$\|H\|_{L^r} \le (\det \Sigma)^{1/2-1/r} \|a\|_{M^{\infty,1}} \|g\|_{M^p} \|\gamma\|_{M^q}.$$

Finally, we provide an application of the enhanced estimates for the Gabor matrix to the propagation of singularities for the Schrödinger equation. The fruitful interplay between time-frequency and microlocal analysis lead to new notions of *global* wave front sets after Hörmander [Hör91]. Several notions of global wave front set have been introduced to detect (the lack of) regularity at modulation space level, see [RW14] for a more detailed discussion and [PSRW18; Wah18] for further applications.

Given an open cone Γ in $\mathbb{R}^{2d}$ and $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ we define the space $M^1_{(g)}(\Gamma)$ of M^1-regular distributions on the cone Γ with respect to g as the set of all the distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M^1_{(g)}(\Gamma)} := \int_\Gamma |V_g f(z)| dz < \infty. \tag{1.27}$$

The next result shows that M^1-regularity of a function f on a conic subset of the phase space is preserved by the action of $\mu(S)$ provided that a slightly smaller cone, evolved under S, is taken into account. We set $\mathbb{S}^{2d-1}$ for the sphere in $\mathbb{R}^{2d}$.

Theorem 1.3.4. *Let $S \in \mathrm{Sp}(d,\mathbb{R})$, $g,\gamma \in \mathcal{S}(\mathbb{R}^d)\setminus\{0\}$ and $\Gamma,\Gamma' \subset \mathbb{R}^{2d}$ be open cones such that $\overline{\Gamma'\cap\mathbb{S}^{2d-1}} \subset \Gamma\cap\mathbb{S}^{2d-1}$. If $f \in \mathcal{S}'(\mathbb{R}^d)$ is M^1-regular on Γ with respect to g then $\mu(S)f$ is M^1-regular on $S(\Gamma')$ with respect to γ.*

Precisely, given $r \geq 0$ there exists $C > 0$ such that, for any $f \in M^1_{v_{-r}}(\mathbb{R}^d) \cap M^1_{(g)}(\Gamma)$ (cf. (3.10)) and $S \in \mathrm{Sp}(d,\mathbb{R})$ the following estimate holds:

$$\|\mu(S)f\|_{M^1_{(\gamma)}(S(\Gamma'))} \leq C(\det\Sigma)^{1/2}\Big(\|f\|_{M^1_{(g)}(\Gamma)} + (\det\Sigma)^r\|f\|_{M^1_{v_{-r}}(\mathbb{R}^d)}\Big).$$

If we specialize the previous result to the free particle propagator we get

$$\big\|e^{i(t/2\pi)\Delta}f\big\|_{M^1_{(\gamma)}(S_t(\Gamma'))} \leq C\Big((1+|t|)^{d/2}\|f\|_{M^1_{(g)}(\Gamma)} + (1+|t|)^{d(1/2+r)}\|f\|_{M^1_{v_{-r}}(\mathbb{R}^d)}\Big),$$

where S_t is the classical flow in (1.19). The latter can be regarded as a microlocal refinement of known estimates, cf. for instance [WHHG11, Proposition 6.6] and Corollary 6.2.3 below.

1.4 Time-frequency analysis of the Dirac equation

As already evidenced in the previous sections, the study of dispersive equations may certainly take advantage from the techniques of modern harmonic analysis. In the last decades we have witnessed an increasing interest in the application to PDEs of strategies and function spaces arising in time-frequency analysis. Even if it is impossible to offer a comprehensive list of results, we suggest the papers [BGOR07; BO09; CF12; CN08b; CN09; CN14; CNR15b; KKI12; KKI14; WH07; ZCG14] and the monographs [CR20; WHHG11] as examples of the manifold aspects one can handle from this perspective.

The relevance of modulation and Wiener amalgam spaces to the study of dispersive PDEs is closely related to the evolution of the phase-space concentration under the corresponding propagators. As an example, while the Schrödinger propagator $e^{it\Delta}$ is not bounded on $L^p(\mathbb{R}^d)$ except for $p = 2$, it is a bounded unimodular Fourier multiplier on any modulation space $M^{p,q}(\mathbb{R}^d)$ [BGOR07]. Many results of this type, including improved dispersive and Strichartz estimates, are also known for the wave equation and the Klein-Gordon equation (see the list of papers above).

In spite of this established trend, little is known concerning the Gabor analysis of the *Dirac equation*. Recall that the Cauchy problem for the n-dimensional Dirac equation with a potential V reads

$$\begin{cases} i\partial_t \psi(t,x) = (\mathcal{D}_m + V)\psi(t,x), \\ \psi(0,x) = \psi_0(x), \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d. \tag{1.28}$$

Here $\psi(t,x) = (\psi_1(t,x), \dots, \psi_n(t,x)) \in \mathbb{C}^n$ is a vector-valued complex wavefunction and the Dirac operator $\mathcal{D}_m$ is defined by

$$\mathcal{D}_m = 2\pi m \alpha_0 - i \sum_{j=1}^{d} \alpha_j \partial_j, \tag{1.29}$$

where $m \geq 0$ (mass) and $\alpha_0, \alpha_1, \dots, \alpha_d \in \mathbb{C}^{n\times n}$ is a set of *Dirac matrices*, i.e. $n \times n$ Hermitian matrices satisfying the identities

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} I_n, \quad \forall\, 0 \leq i,j \leq d. \tag{1.30}$$

For $d = 3$ and $n = 4$ the standard choice for such matrices is the so-called Dirac representation:

$$\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}, \quad i = 1,2,3, \qquad \alpha_0 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix},$$

where we introduced the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In general, for any d there exist several iterative schemes to obtain a set of Dirac matrices and in general the dependence of the (even) dimension $n = n(d)$ on d is a consequence of the chosen construction [KY01].

To the best of our knowledge, the only contribution on the Dirac equation in the perspective discussed above is the recent paper [KN19] by K. Kato and I. Naumkin. The authors proved estimates for the solutions of the Dirac equation (1.28) in the free case (Theorem 1.1) and also for quadratic and subquadratic time-dependent smooth potentials (Theorem 1.2); the latter setting also includes an electromagnetic potential with linear growth.

Broadly speaking, the main difficulty in dealing with (1.28) lies in that it is a system of coupled equations, hence a strategy for disentangling the components is needed. For instance, this can be done approximately at the level of phase space (see [KN19, (3.17)]) or by projection onto the spectrum of the Dirac operators (see the proof of the dispersive estimate [KN19, (1.8)]). Another standard procedure consists of exploiting the connection with the wave and Klein-Gordon equations when $m = 0$ and $m > 0$ respectively. Nevertheless, when a non-zero potential V is taken into account most of these procedures loose their usefulness and new ideas are required (cf. for instance [CD13; CF17; DF07; EGG19]).

Some aspects of the Gabor analysis of the Dirac equation have been considered by the author in [Tra20] - proofs and further details are to be found in **Chapter 7**. The first aim of this contribution is to offer a different point of view that does not require an explicit decoupling technique nor any preliminary knowledge about the Klein-Gordon equation. A naive look at (1.28) would suggest to treat it like a Schrödinger-type equation with matrix-valued Hamiltonian $\mathcal{H} = \mathcal{D}_m + V$. For the free case ($V = 0$) the corresponding propagator $U(t) = e^{-it\mathcal{D}_m}$ can be formally viewed as a Fourier multiplier with matrix symbol

$$\mu_t(\xi) = \exp\left[-2\pi it\left(m\alpha_0 + \sum_{j=1}^{d} \alpha_j\xi_j\right)\right]. \tag{1.31}$$

This perspective naturally leads to consider estimates on *vector-valued modulation and Wiener amalgam spaces* by studying the regularity of μ_t and extending the ordinary boundedness results for Fourier multipliers and more general pseudodifferential operators. Vector-valued modulation spaces were first considered by J. Toft [Tof04b] and then extensively studied by P. Wahlberg [Wah07]. Roughly speaking, the E-valued modulation space $M^{p,q}_{r,s}(\mathbb{R}^d, E)$, E being in general a Banach space, is defined by above with $|\cdot|$ replaced by the Banach space norm $|\cdot|_E$. We address the reader to Chapter 3 for a rigorous discussion.

The study of the Dirac equation would only require to consider finite-dimensional vector spaces such as $\mathbb{C}^n$ and $\mathbb{C}^{n\times n}$, so that the subtleties connected with infinite-dimensional target spaces are not relevant here and most of the proofs reduce to componentwise computation. Nevertheless, we decided to embrace a wider perspective and thus in the preparatory Section 3.2 we extended several results of scalar-valued time-frequency analysis to the vector-valued context; this framework allows us to derive very natural and compact proofs for the main results on the Dirac equation. In passing, we remark that the core of results concerning *vector-valued time-frequency analysis* is in fact of independent interest and falls into the larger area of infinite-dimensional harmonic analysis. We plan to devote future investigations to the topic.

In that spirit, we are then able to prove the following estimates for the free Dirac propagator.

Theorem 1.4.1. *Let $1 \le p, q \le \infty$ and $r, s \in \mathbb{R}$; denote by X any of the spaces $M^{p,q}_{r,s}(\mathbb{R}^d, \mathbb{C}^n)$ or $W^{p,q}_{r,s}(\mathbb{R}^d, \mathbb{C}^n)$. Let $\psi(t, x)$ be the solution of* (1.28) *with $V = 0$.*

For any $t \in \mathbb{R}$ there exists a constant $C_X(t) > 0$ such that

$$\|\psi(t,\cdot)\|_X \le C_X(t)\|\psi_0\|_X.$$

In particular, if $X = M^{p,q}_{0,s}(\mathbb{R}^d, \mathbb{C}^n)$ there exists a constant $C' > 0$ such that

$$C_X(t) \le C'(1 + |t|)^{d|1/2-1/p|}. \tag{1.32}$$

While the results are not unexpected in themselves in view of the discussion above on the connection with the Klein-Gordon propagator, we stress that our method improves the known estimates in two aspects. First, we are able to cover weighted modulation and Wiener amalgam spaces with no extra effort, resulting in a more precise description of the action of the propagator (no loss of derivatives in Theorem 1.4.1 or asymptotic smoothing in Theorem 7.1.1 below). On the other hand, at least for modulation spaces we are able to explicitly characterize the time-dependence of the constant $C(t)$ in (1.32) in a straightforward way, essentially by inspecting the symbol (1.31).

The second purpose of our contribution is to provide boundedness results on modulation and Wiener amalgam spaces for suitable potentials V in (1.28). We relax the regularity assumptions in [KN19] in two aspects. First, we replace the multiplication operator by V with a genuine matrix pseudodifferential operator σ^{w} in the Weyl form, where the matrix symbol σ belongs to the Sjöstrand class $M^{\infty,1}(\mathbb{R}^d, \mathbb{C}^{n\times n})$. In addition, while the dependence on the time of the potential V is assumed to be smooth in [KN19], we require here a milder condition, namely *continuity for the narrow convergence* - cf. Definition 3.2.21 below. In the following claim we use the spaces $\mathcal{M}^{p,q}_{r,s}$ and $\mathcal{W}^{p,q}_{r,s}$, defined by the closure of the Schwartz class in the corresponding modulation and Wiener amalgam spaces respectively; they coincide with the standard spaces except for $p = \infty$ or $q = \infty$. This is necessary in order to avoid technical difficulties arising in those cases.

Theorem 1.4.2. *Let $1 \le p, q \le \infty$, $\gamma \ge 0$ and $r, s \in \mathbb{R}$ be such that $|r| + |s| \le \gamma$; denote by X any of the spaces $\mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d, \mathbb{C}^n)$ or $\mathcal{W}^{p,q}_{r,s}(\mathbb{R}^d, \mathbb{C}^n)$. Let $T > 0$ be fixed and assume the map $[0,T] \ni t \mapsto \sigma(t,\cdot) \in M^{\infty,1}_{0,\gamma}(\mathbb{R}^d, \mathbb{C}^{n\times n})$, to be continuous for the narrow convergence. For any $\psi_0 \in X$ there exists a unique solution $\psi \in C([0,T], X)$ to (1.28) with $V = \sigma(t,\cdot)^{\mathrm{w}}$. The corresponding propagator is bounded on X.*

Note that the result stated here contains a slight improvement of the regularity of the symbol compared to [Tra20, Theorem 1.2].

We then consider the case of potentials with quadratic and sub-quadratic growth as in [KN19]. As a consequence of a useful splitting lemma, namely Proposition 3.2.20, we are able to prove a generalized rough counterpart of the smooth scenario considered in [KN19, Theorem 1.2]. In particular, the potential contains non-smooth functions with a certain number of derivatives in the Sjöstrand class plus a pseudodifferential perturbation in the Weyl form.

Theorem 1.4.3. *Let $1 \le p \le \infty$ and $\psi_0 \in \mathcal{M}^p(\mathbb{C}^n)$. Consider the Cauchy problem* (1.28) *with potential*

$$V = QI_n + L + \sigma^{\mathrm{w}}, \tag{1.33}$$

where

- *$Q : \mathbb{R}^d \to \mathbb{C}$ is such that $\partial^\alpha Q \in M^{\infty,1}(\mathbb{R}^d)$ for $\alpha \in \mathbb{N}^d$, $|\alpha| = 2$,*
- *$L : \mathbb{R}^d \to \mathbb{C}^{n\times n}$ is such that $\partial^\alpha L \in M^{\infty,1}(\mathbb{R}^d, \mathbb{C}^{n\times n})$ for $\alpha \in \mathbb{N}^d$, $|\alpha| = 1$, and*
- *$\sigma \in M^{\infty,1}(\mathbb{R}^d, \mathbb{C}^{n\times n})$.*

For any $t \in \mathbb{R}$ there exists a constant $C(t) > 0$ such that the solution ψ of (1.28) *satisfies*

$$\|\psi(t,\cdot)\|_{\mathcal{M}^p} \le C(t)\|\psi_0\|_{\mathcal{M}^p}.$$

Furthermore, if V is as in (1.33) *and $Q = 0$, then for any $1 \le p, q \le \infty$ and $t \in \mathbb{R}$ there exists a constant $C(t) > 0$ such that the solution ψ of* (1.28) *satisfies*

$$\|\psi(t,\cdot)\|_{\mathcal{M}^{p,q}} \le C(t)\|\psi_0\|_{\mathcal{M}^{p,q}}.$$

Finally, we study the local well-posedness for the nonlinear setting, namely

$$\begin{cases} i\partial_t\psi(t,x) = \mathcal{D}_m\psi(t,x) + F(\psi(t,x)), \\ \psi(0,x) = \psi_0(x), \end{cases} \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \tag{1.34}$$

where the nonlinear term F considered below comes in the form of a vector-valued real-analytic entire function $F : \mathbb{C}^n \to \mathbb{C}^n$ such that $F(0) = 0$, i.e.

$$F_j(z) = \sum_{\alpha,\beta\in\mathbb{N}^n} c^j_{\alpha,\beta} z^\alpha \bar{z}^\beta, \quad c^j_{\alpha,\beta} \in \mathbb{C}, \quad j = 1,\dots,n, \tag{1.35}$$

with absolute convergence for any $z \in \mathbb{C}$. We remark that this general choice includes nonlinearities of power type, such as

$$F(\psi) = |\psi|^{2k}\psi, \qquad k \in \mathbb{N}; \tag{1.36}$$

and the cubic nonlinearity known as the Thirring model, namely

$$F(\psi) = (\alpha_0\psi, \psi)\alpha_0\psi;$$

The choice of even powers in (1.36) and entire functions as in (1.35) are standard in the context of modulation and amalgam spaces, because of the Banach algebra property enjoyed by certain spaces of these families [STW11]. On the other hand, the nonlinear spinor field appearing in the Thirring model has been largely investigated; cf. for instance [BH16; Huh11; MNO03; Nau16], also in view of its physical relevance - it is a model for self-interacting Dirac fermions in quantum field theory [Sol70; Thi58].

The main result in this respect reads as follows.

Theorem 1.4.4. *Let* $1 \le p \le \infty$ *and* $r, s \ge 0$*; denote by* X *any of the spaces* $M^{p,1}_{0,s}(\mathbb{R}^d, \mathbb{C}^n)$ *or* $W^{1,p}_{r,s}(\mathbb{R}^d, \mathbb{C}^n)$*. If* $\psi_0 \in X$ *then there exists* $T = T(\|\psi_0\|_X)$ *such that the Cauchy problem* (1.34) *with* F *as in* (1.35) *has a unique solution* $\psi \in C^0([0,T], X)$*.*

We conclude this discussion by emphasizing a few aspects that may be further developed in the context of modulation spaces, such as Strichartz estimates and perturbations due to a magnetic field, i.e. the Dirac operator in (1.29) becomes $\mathcal{D}_{m,A} = 2\pi m\alpha_0 - i\sum_{j=1}^{d} \alpha_j(\partial_j - iA_j)$, where $A(x) = (A_1(x), \dots, A_d(x))$, $x \in \mathbb{R}^d$, is a static magnetic potential. We also point out that more general nonlinear terms could be considered, for instance as in the Soler model [Sol70] and other interactions arising in condensed matter; cf. [Pel11] for the state of the art in 1+1 dimensions.

1.5 Gabor analysis meets Feynman path integrals

There has been plenty of opportunities to discuss the fruitful interplay between Gabor analysis and physics for what concerns motivations as well as applications; the problem of quantization is perhaps the most striking example in this respect. In fact, another problem arising in mathematical physics has recently benefited from the techniques of Gabor analysis, that is the rigorous formulation of *Feynman path integrals*. From the mathematical point of view, while the quantization problem deals with the characterization of an operator A in terms of its symbol a and the correspondence $a \mapsto A$, path integrals are basically a way to provide explicit representation formulae for the evolution operator e^{-itA} (and also e^{-tA} in general).

The path integral formulation of non-relativistic quantum mechanics is a paramount contribution by R. Feynman (Nobel Prize in Physics, 1965) to modern theoretical physics. The origin of this approach goes back to Feynman's Ph.D. thesis of 1942 at Princeton University (recently reprinted, cf. [Bro05]) but was first published in the form of research paper in 1948 [Fey48]; see also [Sau08] for some historical hints. In rough terms we could say that this approach provides a quantum counterpart to Lagrangian mechanics, while the standard framework for canonical quantization as developed by Dirac relies on the Hamiltonian formulation of classical mechanics. Path integrals and Feynman's deep physical intuition were the main ingredients of the celebrated diagrams, introduced in the 1949 paper [Fey49], which gave a whole new outlook on quantum field theory.

For a first-hand pedagogical introduction we recommend the textbook [FH10], where it is clarified how the physical intuition of path integrals comes from a deep understanding of the lesson given by the two-slit experiment. We briefly outline below the main features of Feynman's approach.

Recall that the state of a non-relativistic particle in the Euclidean space $\mathbb{R}^d$ at time t is represented by the wave function $\psi(t,x)$, $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, such that $\psi(t,\cdot) \in L^2(\mathbb{R}^d)$. The time-evolution of a state $f(x)$ at $t = 0$ is governed by the Cauchy problem for the Schrödinger equation:

$$\begin{cases} i\hbar\partial_t\psi = (H_0 + V(t,x))\psi \\ \psi(0,x) = f(x), \end{cases} \tag{1.37}$$

where $0 < \hbar \leq 1$ is a parameter[2], $H_0 = -\hbar^2\Delta/2$ is the standard Hamiltonian for a free particle and V is a real-valued potential; we conveniently set $m = 1$ for the mass of the particle. The map $U(t,s) : \psi(s,\cdot) \mapsto \psi(t,\cdot)$, $t, s \in \mathbb{R}$, is a unitary operator on $L^2(\mathbb{R}^d)$ and is known as *propagator*[3] or *evolution operator*; we set $U(t)$ for $U(t,0)$. Since $U(t)$ is a linear operator we can formally represent it as an integral operator, namely

$$\psi(t,x) = \int_{\mathbb{R}^d} u_t(x,y) f(y) dy,$$

where the kernel $u_t(x,y)$ is interpreted as the transition amplitude from the position y at time 0 to the position x at time t. In a nutshell, Feynman's prescription is a recipe for this kernel, the main ingredients being all the possible paths from y to x that the particle could follow. The contribution of each interfering alternative path

[2] This is the reduced Planck constant $\hbar = h/2\pi$. We temporarily keep track of its presence in view of semiclassical arguments below, but then we restore the harmonic analysis convention $h = 1$. We stress that h denotes a dimensionless parameter which should not be systematically identified with the true Planck constant - the latter being just a physical motivation for this mathematical scenario.

[3] We remark that in the physics literature the term "propagator" is usually reserved to the integral kernel u_t of $U(t)$, see below. This may possibly lead to confusion since it is in conflict with the traditional nomenclature adopted in the analysis of PDEs.

to the total probability amplitude is a phase factor involving the *action functional* evaluated on the path, that is

$$S[\gamma] = \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau), \tau) d\tau, \tag{1.38}$$

where L is the Lagrangian of the underlying classical system, namely

$$L(x, v, \tau) = \frac{1}{2}|v|^2 - V(x, \tau).$$

Therefore, the kernel should be formally represented as

$$u_t(x, y) = \int e^{\frac{i}{\hbar}S[\gamma]} \mathcal{D}\gamma, \tag{1.39}$$

that is a sort of integral over the infinite-dimensional space of paths satisfying the conditions above. This intriguing picture is further reinforced by the following remark: a formal application of the stationary phase method shows that the semiclassical limit $\hbar \to 0$ selects the classical trajectory, hence we recover the principle of stationary action of classical mechanics.

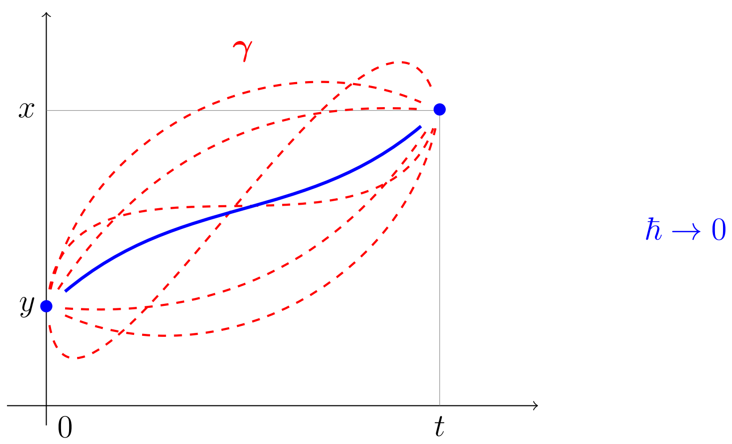

Figure 1.1: Some of all the possible trajectories with given endpoints $(0, x_0)$ and (t, x) that a particle could follow (in red). In blue: the classical path, which is recovered in the semiclassical limit $\hbar \to 0$.

It is well known after R. Cameron [Cam60] that $\mathcal{D}\gamma$ cannot be a Lebesgue-type measure on the space of paths, neither it can be constructed as a Wiener measure with complex variance - it would have infinite total variation [Maz09]. The literature concerning the problem of putting formula (1.39) on firm mathematical ground is huge; the interested reader could benefit from the monographs [AHKM08; Fuj17; Maz09] as points of departure. We will describe below only two of the several schemes which have been manufactured in order to give a rigorous meaning to (1.39); they both rely on operator-theoretic strategies and are called the *sequential approach* and the *time slicing approach*. Basically, one is lead to study sequences of operators on $L^2(\mathbb{R}^d)$ which converge to the exact propagator $U(t)$ in a sense to be specified, the strength of convergence competing against the regularity of the potential V.

This is the point where time-frequency analysis enters the scene: the accumulated knowledge on the wave packet analysis of operators will be crucial in the study of approximate propagators arising in the theory of path integrals. Moreover, function spaces of time-frequency analysis enjoy a fruitful balance between nice properties (Banach algebra structures, embeddings, decomposition, etc.) and regularity of their members, so that they can be used as reservoirs of potentials.

We now briefly outline the main features of a pair of mathematical schemes which are in fact two faces of the same philosophy: the closest approach to Feynman's original intuition (cf. [FH10]) requires to interpret formula (1.39) by means of a limiting procedure involving suitably designed finite-dimensional approximations. While in the literature one may easily notice that different names are interchangeably used for them, we consider the classification below for the sake of clarity.

1.5.1 The sequential approach

The so-called *sequential approach* to path integrals was first introduced by E. Nelson in [Nel64] and relies on two basic results. First, recall that the free evolution operator for the Schrödinger equation $U_0(t) = e^{-\frac{i}{\hbar}tH_0}$, $H_0 = -\hbar^2\Delta/2$, is a Fourier multiplier; routine computation yields the following integral representation [RS75, Section IX.7]:

$$e^{-\frac{i}{\hbar}tH_0}f(x) = \frac{1}{(2\pi i t\hbar)^{d/2}}\int_{\mathbb{R}^d}\exp\left(\frac{i}{\hbar}\frac{|x-y|^2}{2t}\right)f(y)dy, \qquad f\in\mathcal{S}(\mathbb{R}^d). \tag{1.40}$$

Notice that the phase factor in the integral actually coincides with the action functional evaluated along the line $\gamma_{\mathrm{cl}}(\tau) = y + (x-y)\tau/t$, namely the classical trajectory of a free particle moving from position y at time $\tau = 0$ to position x at time $\tau = t$ in the absence of external forces.

Next, we need a result from the theory of operator semigroups. Provided that suitable conditions on the domain of H_0 and on the potential V are satisfied (see below), the *Trotter product formula* holds for the propagator $U(t) = e^{-\frac{i}{\hbar}t(H_0+V)}$

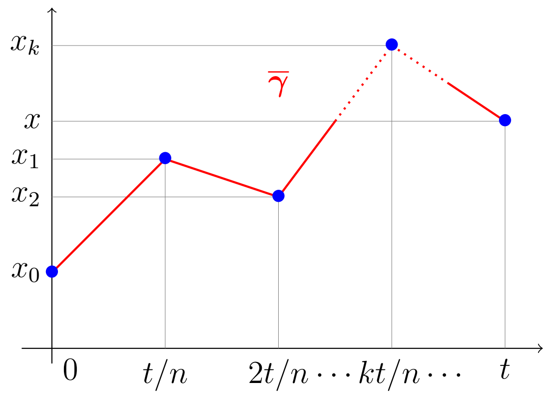

Figure 1.2: The broken line approximation $\overline{\gamma}$ introduced in (1.43).

generated by $H = H_0 + V$:

$$U(t)f = \lim_{n\to\infty} E_n(t)f, \quad f \in L^2(\mathbb{R}^d), \qquad E_n(t) \coloneqq \left(e^{-\frac{i}{\hbar}\frac{t}{n}H_0}e^{-\frac{i}{\hbar}\frac{t}{n}V}\right)^n, \tag{1.41}$$

hence we have convergence of the approximate propagators $E_n(t)$ (also called *Feynman-Trotter parametrices*) in the strong topology of operators in $L^2(\mathbb{R}^d)$ to the exact propagator $U(t)$. Combining these two ingredients yields the following representation of the complete propagator $e^{-\frac{i}{\hbar}tH}$ as limit of integral operators [RS75, Theorem X.66]:

$$e^{-\frac{i}{\hbar}t(H_0+V)}f(x) = \lim_{n\to\infty}\left(2\pi\hbar i\frac{t}{n}\right)^{-\frac{nd}{2}}\int_{\mathbb{R}^{nd}} e^{\frac{i}{\hbar}S_n(t;x_0,\dots,x_{n-1},x)}f(x_0)dx_0\dots dx_{n-1}, \tag{1.42}$$

where we set

$$S_n(t;x_0,\dots,x_{n-1},x) = \sum_{k=1}^{n}\frac{t}{n}\left[\frac{1}{2}\left(\frac{|x_k-x_{k-1}|}{t/n}\right)^2 - V(x_k)\right], \quad x_n = x.$$

With the aim of understanding the role of $S_n(t;x_0,\dots,x_n)$, consider the following argument. Given the points $x_0,\dots,x_{n-1},x\in\mathbb{R}^d$, let $\overline{\gamma}$ be the polygonal path (broken line) through the vertices $x_k = \overline{\gamma}(kt/n)$, $k=0,\dots,n$, $x_n = x$, parametrized as

$$\overline{\gamma}(\tau) = x_k + \frac{x_{k+1}-x_k}{t/n}\left(\tau - k\frac{t}{n}\right), \qquad \tau\in\left[k\frac{t}{n},(k+1)\frac{t}{n}\right], \qquad k=0,\dots,n-1. \tag{1.43}$$

Hence $\overline{\gamma}$ prescribes a classical motion with constant velocity along each segment. The action for this path is thus given by

$$S[\overline{\gamma}] = \sum_{k=1}^{n} \frac{1}{2}\frac{t}{n}\left(\frac{|x_k - x_{k-1}|}{t/n}\right)^2 - \int_0^t V(\overline{\gamma}(\tau))d\tau.$$

According to Feynman's interpretation formula (1.42) can be thought of as an integral over all polygonal paths, where $S_n(t; x_0, \dots, x_n)$ is a finite-dimensional approximation of the action functional evaluated on them. The limiting behaviour for $n \to \infty$ is now intuitively clear: the set of polygonal paths becomes the set of all paths and in some sense we recover (1.39). We remark that the custom in the physics community after Feynman is exactly to employ the suggestive formula (1.39) as a placeholder for (1.42) and the related arguments - see for instance [GS98; Kle09].

For what concerns the assumptions on the potential perturbation V under which the Trotter product formula holds, a standard result shows that it is enough to choose V in such a way that $H_0 + V$ is essentially self-adjoint on $D = D(H_0) \cap D(V)$ in $L^2(\mathbb{R}^d)$, cf. for instance [RS72, Theorem VIII.31]. The power of Nelson's approach is that one can cover wide classes of wild perturbations, such as *Kato potentials*, including finite sums of real-valued functions in $L^p(\mathbb{R}^d)$ with $2p > d$ and $p \geq 2$ [Nel64, Theorem 8].

There exist several generalizations of the Trotter formula for semigroups on Banach spaces, see for instance [EN06, Corollary 2.7]. The following simpler variant will be enough for our purposes.

Theorem 1.5.1 ([EN06, Exercise 2.9])**.** *Let H_0 be a self-adjoint operator on the domain $D(H_0) \subset L^2(\mathbb{R}^d)$ and let $V \in \mathcal{L}(L^2(\mathbb{R}^d))$ be a bounded perturbation. The Trotter product formula for the propagator $U(t) = e^{-\frac{i}{\hbar}t(H_0+V)}$ holds: for any $t \in \mathbb{R}$,*

$$U(t)f = \lim_{n\to\infty} E_n(t)f \quad \forall f \in L^2(\mathbb{R}^d), \qquad E_n(t) \coloneqq \left(e^{-\frac{i}{\hbar}\frac{t}{n}H_0} e^{-\frac{i}{\hbar}\frac{t}{n}V}\right)^n.$$

1.5.2 The time-slicing approximation

We now consider another scheme that could be informally called "the Japanese way" to rigorous path integrals, in honour of the leading players in its construction: D. Fujiwara and N. Kumano-go, with further developments by W. Ichinose and T. Tsuchida. The main references for this approach are the papers [FKg05; Fuj79; Fuj80; Ich03; Ich97; Kg04; Kg95] and the monograph [Fuj17], to which the reader is referred for further details.

Let us briefly reconsider equation (1.42) and its interpretation in terms of finite-dimensional approximations along broken lines; a similar result can be achieved without recourse to the Trotter formula as detailed below. First, let us specify the class of potentials involved in this approach.

Assumption (A). *The potential* $V : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *satisfies* $\partial_x^\alpha V \in C^0(\mathbb{R} \times \mathbb{R}^d)$ *for any* $\alpha \in \mathbb{N}^d$ *and*

$$|\partial_x^\alpha V(t,x)| \le C_\alpha, \quad |\alpha| \ge 2, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d$$

for suitable constants $C_\alpha > 0$.

In particular, $V(t,x)$ is smooth in x and has at most quadratic growth. Consider then the Hamiltonian

$$H(t,x,\xi) = \frac{1}{2}|\xi|^2 + V(t,x).$$

We denote by $(x(t,s,y,\eta), \xi(t,s,y,\eta))$ $(s,t \in \mathbb{R},\ y,\eta \in \mathbb{R}^d)$, the solution of the corresponding classical equations of motion

$$\begin{cases} \dot{x} = \xi \\ \dot{\xi} = -\nabla_x V(t,x) \end{cases}$$

with initial condition at time $t = s$ given by $x(s,s,y,\eta) = y$, $\xi(s,s,y,\eta) = \eta$. The flow

$$(x(t,s,y,\eta), \xi(t,s,y,\eta)) = \chi(t,s)(y,\eta)$$

defines a smooth canonical transformation $\chi(t,s) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ satisfying for every $T_0 > 0$ the estimates

$$|\partial_y^\alpha \partial_\eta^\beta x(t,s,y,\eta)| + |\partial_y^\alpha \partial_\eta^\beta \xi(t,s,y,\eta)| \le C_{\alpha,\beta}(T_0), \quad y,\eta \in \mathbb{R}^d$$

for some constant $C_{\alpha,\beta}(T_0) > 0$, if $|t-s| \le T_0$ and $|\alpha| + |\beta| \ge 1$ (see [Fuj80, Proposition 1.1]). In particular the flow is globally Lipschitz and the same holds for its inverse.

Moreover, there exists $\delta > 0$ such that for $0 < |t-s| \le \delta$ and every $x,y \in \mathbb{R}^d$, there exists only one classical path γ such that $\gamma(s) = y$, $\gamma(t) = x$. By computing the action functional along this path γ, as in (1.38), we define the generating function

$$S(t,s,x,y) = S[\gamma] = \int_s^t L(\gamma(\tau), \dot{\gamma}(\tau), \tau)\, d\tau,$$

for $0 < |t-s| \le \delta$.

Then Fujiwara showed [Fuj80] that the propagator $U(t,s)$ is an *oscillatory integral operator* (for short, OIO) provided that $|t-s|$ is small enough, that is

$$U(t,s)f(x) = \frac{1}{(2\pi i\hbar(t-s))^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}S(t,s,x,y)} a(\hbar,t,s)(x,y) f(y)\, dy, \tag{1.44}$$

for an amplitude function $a(\hbar, t, s) \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^{2d})$ - the space of smooth functions with bounded derivatives of any order, also known as the Hörmander class $S^0_{0,0}$ in microlocal analysis. Moreover, $a(\hbar, t, s)$ is such that $\partial_x^\alpha \partial_y^\beta a(\hbar, t, s)$ is of class C^1 in t, s and satisfies

$$\|a(\hbar, t, s)\|_{C_{\mathrm{b}}^m} := \sup_{|\alpha| \leq m} \|\partial^\alpha a(\hbar, t, s)\|_{L^\infty} \leq C_m,$$

for $0 < t - s \leq \delta$, $0 < \hbar \leq 1$, $m \in \mathbb{N}$.

In concrete situations, except for a few cases, there is no hope to compute the exact propagator in an explicit, closed form. Due to this difficulty and inspired by the free particle operator (1.40), one is lead to consider approximate propagators (*parametrices*), such as

$$E^{(0)}(t, s)f(x) = \frac{1}{(2\pi i\hbar(t - s))^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}S(t,s,x,y)} f(y) dy. \tag{1.45}$$

As suggested indeed by the case of the free particle, this operator is supposed to provide a good approximation of the $U(t, s)$ for $|t - s|$ small enough. The case of a large interval $|t - s|$ can be treated by means of composition of such operators in the spirit of the time slicing method proposed by Feynman: given a subdivision $\Omega = t_0, \ldots, t_L$ of the interval $[s, t]$ such that $s = t_0 < t_1 < \ldots < t_L = t$ and $t_j - t_{j-1} \leq \delta$, we define the operator

$$E^{(0)}(\Omega, t, s) = E^{(0)}(t_L, t_{L-1})E^{(0)}(t_{L-1}, t_{L-2}) \cdots E^{(0)}(t_1, t_0),$$

whose integral kernel $e^{(0)}(\Omega, t, s)(x, y)$ can be explicitly computed from (1.45), namely

$$e^{(0)}(\Omega, t, s)(x, y) = \prod_{j=1}^{L} \frac{1}{(2\pi i(t_j - t_{j-1})\hbar)^{d/2}} \int_{\mathbb{R}^{d(L-1)}} \exp\Big(\frac{i}{\hbar} \sum_{j=1}^{L} S(t_j, t_{j-1}, x_j, x_{j-1})\Big) \prod_{j=1}^{L-1} dx_j,$$

with $x = x_L$ and $y = x_0$. A detailed analysis can be found in [Fuj17, Chapter 2].

The parametrix $E^{(0)}(\Omega, t, s)$ is then expected to converge (in some sense) to the actual propagator $U(t, s)$ in the limit

$$\omega(\Omega) := \max\{t_j - t_{j-1},\ j = 1, \ldots, L\} \to 0.$$

Note that this scheme is definitely more sophisticate than Nelson's one discussed before, also because of the fact that the broken line approximation is now replaced with more refined piecewise classical paths. This issue will be relevant in view of

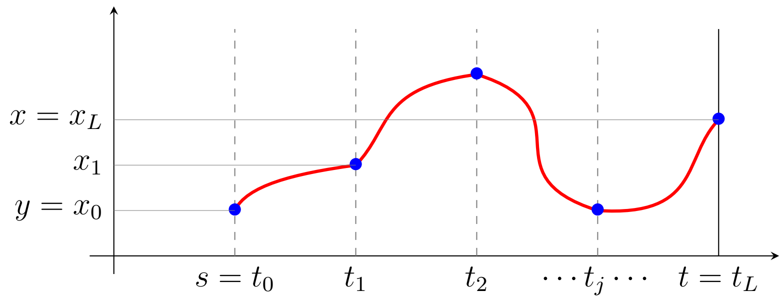

Figure 1.3: A piecewise classical path in spacetime.

the semiclassical limit, as will be shown in a moment. However, a quite complete theory of path integration for approximations on straight lines in this spirit has been developed by Kumano-go [Kg04].

Among the large number of results proved in the framework discussed so far we mention two milestones from forerunner papers by Fujiwara. In [Fuj79] he proved convergence of $E^{(0)}(\Omega, t, s)$ to $U(t, s)$ in the norm operator topology in $\mathcal{L}(L^2(\mathbb{R}^d))$. Under the same hypotheses convergence at the level of integral kernels in a very strong topology was proved in [Fuj80]. It should be emphasized that the aforementioned results are also given for the higher order parametrices $E^{(N)}(t, s)$, $N = 1, 2, \ldots$, also known as *Birkhoff-Maslov parametrices* [Bir33; Mas70] and defined by

$$E^{(N)}(t, s)f(x) = \frac{1}{(2\pi i\hbar(t - s))^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}S(t,s,x,y)} a^{(N)}(\hbar, t, s)(x, y)f(y)dy, \quad (1.46)$$

where $a^{(N)}(\hbar, t, s)(x, y) = \sum_{j=1}^{N} (\frac{i}{\hbar})^{1-j} a_j(t, s)(x, y)$ and the functions $a_j(t, s) \in C_b^\infty(\mathbb{R}^{2d})$ uniformly with respect to $0 < \hbar \le 1$, $0 < |t - s| \le \delta$.

As before, the case of larger $|t - s|$ can be treated by means of composition over a sufficiently fine subdivision $\Omega = \{t_0, \ldots, t_L\}$ of the interval $[s, t]$ such that $s = t_0 < t_1 < \ldots < t_L = t$, namely

$$E^{(N)}(\Omega, t, s) = E^{(N)}(t_L, t_{L-1})E^{(N)}(t_{L-1}, t_{L-2}) \cdots E^{(N)}(t_1, t_0). \quad (1.47)$$

The core results of the L^2 theory for the time slicing approximation read as follows.

Theorem 1.5.2. *Let the potential V satisfy Assumption (A) and fix $T_0 > 0$. For $0 < t - s \le T_0$ and any subdivision Ω of the interval $[s, t]$ such that $\omega(\Omega) \le \delta$, the following claims hold.*

(i) There exists a constant $C = C(N, T_0) > 0$ such that

$$\left\| E^{(N)}(\Omega, t, s) - U(t, s) \right\|_{L^2 \to L^2} \le C\hbar^N \omega(\Omega)^{N+1}(t - s), \quad N \in \mathbb{N}. \quad (1.48)$$

(ii) There exists $C = C(m, N, T_0) > 0$ such that

$$\left\| a(\hbar, t, s) - a^{(N)}(\Omega, \hbar, t, s) \right\|_{C_b^m} \leq C\hbar^N \omega(\Omega)^{N+1}(t-s), \quad m, N \in \mathbb{N}$$

cf. (1.44). *In particular,*

$$\lim_{\omega(\Omega)\to 0} a^{(N)}(\Omega, \hbar, t, s) = a(\hbar, t, s) \quad \textit{in } C_b^\infty(\mathbb{R}^{2d}).$$

The proof of these results ultimately relies on fine analysis of OIOs. The underlying overall strategy can be condensed as follows:

1. prove that "time slicing approximation is an oscillatory integral" (cf. [Fuj17]), i.e., that the operators arising from (1.46) are well-defined OIOs under suitable assumptions;
2. derive precise estimates for the operator norm of such OIOs;
3. deduce corresponding results for the compositions in (1.47).

The last step is extremely delicate because estimates uniform in L, the number of points in the partition Ω, are required. Moreover, the composition of OIOs results in an OIO only for short times, due to the occurrence of caustics.

For the sake of completeness we also mention that F. Nicola showed in [Nic19] how parts of the conclusions in Theorem 1.5.2 still hold under weaker regularity assumptions for the potential. Assumption (A) is now replaced by the following one.

Assumption (A'). *The potential $V : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^d)$ and for almost every $t \in \mathbb{R}$ and $|\alpha| \leq 2$ the derivatives $\partial_x^\alpha V(t, x)$ exist and are continuous with respect to x. Furthermore*

$$\partial_x^\alpha V(t, x) \in L^\infty(\mathbb{R}; H^{d+1}_{\mathrm{ul}}(\mathbb{R}^d)), \quad |\alpha| = 2,$$

where $H^n_{\mathrm{ul}}(\mathbb{R}^d)$ is the Kato-Sobolev space (also known as uniformly local Sobolev space) of functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfying $\|f\|_{H^n_{\mathrm{ul}}} = \sup_B \|f\|_{H^n(B)} < \infty$, the supremum being computed on all open balls $B \subset \mathbb{R}^d$ of radius 1.

Theorem 1.5.3 ([Nic19, Theorem 1.1])**.** *Let the potential V satisfy Assumption (A'). For any $T > 0$ there exists $C = C(T) > 0$ such that for any $0 < t - s \leq T$ and any subdivision Ω of the interval $[s, t]$ with $\omega(\Omega) \leq \delta$ and $0 < \hbar \leq 1$,*

$$\left\| E^{(0)}(\Omega, t, s) - U(t, s) \right\|_{L^2 \to L^2} \leq C\omega(\Omega)(t-s).$$

It is natural to wonder whether there exists an L^p analogue of Theorem 1.5.2 with $p \neq 2$. We cannot expect a naive transposition of the claim for several reasons. First of all, notice that the Schrödinger propagator is not even bounded on $L^p(\mathbb{R}^d)$ for $p \neq 2$. The parabolic geometry of its characteristic manifold implies that a peculiar loss of derivative, ultimately due to dispersion, occurs [BTW75; Miy80]:

$$\left\| e^{i\hbar\Delta} f \right\|_{L^p} \le C \left\| (1-\hbar\Delta)^{k/2} f \right\|_{L^p}, \quad k = 2d|1/2 - 1/p|, \quad 1 < p < \infty.$$

On the basis of this observation one is lead to consider the following scale of semiclassical L^p-based Sobolev spaces: for $1 < p < \infty$ and $k \in \mathbb{R}$ define

$$\tilde{L}^p_k(\mathbb{R}^d) = \{ f \in S'(\mathbb{R}^d) \,:\, \|f\|_{\tilde{L}^p_k} = \left\| (1-\hbar\Delta)^{k/2} f \right\|_{L^p} < \infty \}.$$

This is indeed a suitable setting for the analysis Fourier integral operators arising as Schrödinger propagators associated with quadratic Hamiltonians, cf. [DN16].

In addition, one is also confronted with another issue: the space of bounded operators $\tilde{L}^p_k \to L^p$ (or viceversa) is clearly not an algebra under composition. This is a major obstacle for a proficient time slicing approximation, having in mind the construction of the parametrices $E^{(N)}(\Omega, t, s)$ in (1.47) and the role of this feature in the L^2 setting.

A significant breakthrough in this respect comes from Gabor analysis, since all these issues become manageable as soon as one transfers the problem to the phase space setting. The first key results in this context are due to Nicola [Nic16] and read as follows.

Theorem 1.5.4 ([Nic16, Theorem 1.1])**.** *Assume the condition in Assumption (A) and let $1 < p < \infty$, $k = 2d|1/2 - 1/p|$.*

1. *For any $T > 0$ there exists a constant $C = C(T) > 0$ such that for all $f \in \mathcal{S}(\mathbb{R}^d)$, $|t-s| \le T$ and $0 < \hbar \le 1$:*

$$\|U(t,s)f\|_{L^p} \le C\|f\|_{\tilde{L}^p_k}, \quad 1 < p \le 2,$$

$$\|U(t,s)f\|_{\tilde{L}^p_{-k}} \le C\|f\|_{L^p}, \quad 2 \le p < \infty.$$

2. *For any $T > 0$ and $N \in \mathbb{N}$ there exists a constant $C = C(T) > 0$ such that for $0 < t-s \le T$ and any subdivision Ω of the interval $[s,t]$ with $\omega(\Omega) \le \delta$, $f \in \mathcal{S}(\mathbb{R}^d)$ and $0 < \hbar \le 1$:*

$$\left\| \left(E^{(N)}(\Omega,t,s) - U(t,s) \right) f \right\|_{L^p} \le C\hbar^N \omega(\Omega)^{N+1}(t-s)\|f\|_{\tilde{L}^p_k}, \quad 1 < p \le 2,$$

$$\left\| \left(E^{(N)}(\Omega,t,s) - U(t,s) \right) f \right\|_{\tilde{L}^p_{-k}} \le C\hbar^N \omega(\Omega)^{N+1}(t-s)\|f\|_{L^p}, \quad 2 \le p < \infty.$$

A discussion of this result is beyond the purposes of this introduction. In short, the strategy goes as follows: one lifts the analysis to the phase space level by means of non-trivial embeddings relating modulation and Sobolev spaces [KS11], then proves that the approximate propagators in the form of OIOs belong to a family of operators characterized by the sparsity of their Gabor matrix - hence well behaved on modulation spaces. One should also keep track of $\hbar$ by means of suitable dilations (semiclassical modulation and Sobolev spaces).

1.6 Pointwise convergence of integral kernels in the Feynman-Trotter formula

A concise way to resume the philosophy behind the operator-theoretic approaches to rigorous path integral discussed in Sections 1.5.1 and 1.5.2 could be the following one: design suitable sequences of approximation operators and prove that they are bounded together with their compositions, where the latter should converge to the exact propagator in a suitable topology on $\mathcal{L}(L^2(\mathbb{R}^d))$. There are good reasons for not being completely satisfied with this state of affairs. First of all, looking back at Feynman's original paper [Fey48] and the textbook [FH10] one immediately notices that the entire process of defining path integrals can be read in terms of a sequence of integral operators (finite-dimensional approximation operators as in (1.42) or (1.46)); in particular, Feynman's insight calls for the *pointwise* convergence of their integral kernels to the kernel u_t of the propagator. This remark strongly motivates a focus shift from the operators to their kernels, which may appear as an unaffordable problem in general: approximation operators should be first explicitly characterized as integral operators, at least in the sense of distributions by some version of the Schwartz kernel theorem, then one should determine if the kernels are in fact functions and finally hope for convergence to the integral kernel u_t of the propagator $U(t)$. Both the approximation schemes discussed insofar are well suited for this purpose and a clue in this direction, already mentioned at the beginning of the previous section, is that the regularity assumptions in Theorem 1.5.2 imply convergence in a finer topology at the level of integral kernels.

In this connection, some new results in the framework of the sequential approach as presented in Section 1.5.1 have been recently obtained by the author and F. Nicola in the paper [NT20], where techniques of time-frequency analysis of functions and operators are heavily used. The results are extensively discussed in **Chapter 8** below.

In order to state the main results in full generality we need some preparation.

First of all, the Schrödinger equation under our attention (with $\hbar = 1/2\pi$) is

$$\begin{cases} i\partial_t \psi = 2\pi H_0 \psi \\ \psi(0,x) = f(x) \end{cases},$$

where $H_0 = Q^{\mathrm{w}}$ is the Weyl quantization a real-valued, time-independent, quadratic homogeneous polynomial Q on $\mathbb{R}^{2d}$, cf. (1.21) and (1.22) above. Note that a linear magnetic potential or a quadratic electric potential are allowed and included in H_0. Moreover, notice that the factor 2π can be harmlessly embedded in Q (or V below) so that we can formally reduce to the case $\hbar = 1$ in order to have lighter formulae - semiclassical aspects are not taken into account here.

We already discussed above that the associated propagator is a *metaplectic operator*, that is $U_0(t) = e^{-itH_0} = \mu(S_t)$, where the mapping

$$\mathbb{R} \ni t \mapsto S_t = \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \in \mathrm{Sp}(d,\mathbb{R}) \tag{1.49}$$

is the phase-space flow determined by the Hamilton equations for the corresponding classical model with Hamiltonian $Q(x,\xi)$; we refer to Section 4.3.3 for an extensive account.

It is a standard result of harmonic analysis in phase space (see Theorem 4.3.6 below) that if S_t is a *free symplectic matrix*, namely the upper-right block B_t of S_t is invertible, then the corresponding metaplectic operator coincides with a *quadratic Fourier transform* (up to a phase factor), namely

$$U_0(t)f(x) = c(t)|\det B_t|^{-1/2} \int_{\mathbb{R}^d} e^{2\pi i \Phi_t(x,y)} f(y)dy, \tag{1.50}$$

where $c(t)$ is a variable phase factor ($c(t) \in \mathbb{C}$, $|c(t)| = 1$) and we introduced the quadratic form (also known as generating function of S_t, cf. (4.3) below)

$$\Phi_t(x,y) = \frac{1}{2} D_t B_t^{-1} x \cdot x - B_t^{-1} x \cdot y + \frac{1}{2} B_t^{-1} A_t y \cdot y. \tag{1.51}$$

This representation of $\mu(S_t)$ is a main ingredient of our results, hence we stress that it does hold for any $t \in \mathbb{R} \setminus \mathfrak{E}$, where we define the set of *exceptional times* to be

$$\mathfrak{E} = \{t \in \mathbb{R} \,:\, \det B_t = 0\}.$$

Some of the properties of this set can be immediately deduced from the fact that it is indeed the zero set of an analytic function: apart from the case $\mathfrak{E} = \mathbb{R}$ (which trivially happens when $H_0 = 0$), $\mathfrak{E}$ is a discrete (hence at most countable) subset of $\mathbb{R}$ which always includes $t = 0$ - in particular $\mathfrak{E} = \{0\}$ in the case of the free Schrödinger equation.

It is known that $H_0 = Q^{\mathrm{w}}$ is a self-adjoint operator on the maximal domain (see [Hör95])

$$D(H_0) = \{f \in L^2(\mathbb{R}^d) \, : \, H_0 f \in L^2(\mathbb{R}^d)\}.$$

We can thus consider the perturbed problem

$$\begin{cases} i\partial_t \psi = (H_0 + V)\psi \\ \psi(0, x) = f(x), \end{cases} \tag{1.52}$$

where we included the potential perturbation $V \in \mathcal{L}(L^2(\mathbb{R}^d))$. We are in the position to use the Trotter product formula in the form of Theorem 1.5.1: if $U(t) = e^{-it(H_0+V)}$ denotes the evolution operator associated with (1.52), then

$$U(t)f = \lim_{n\to\infty} E_n(t)f \quad \forall f \in L^2(\mathbb{R}^d),$$

where the Feynman-Trotter approximate propagators $E_n(t)$ are defined by

$$E_n(t) := \left(e^{-i\frac{t}{n}H_0} e^{-i\frac{t}{n}V}\right)^n, \quad n \in \mathbb{N},\, n \geq 1. \tag{1.53}$$

We denote by $e_{n,t}(x, y)$ the distribution kernel of $E_n(t)$ and by $u_t(x, y)$ that of $U(t) = e^{-it(H_0+V)}$. We study the problem of the convergence of $e_{n,t}(x, y)$ to $u_t(x, y)$ as $n \to +\infty$.

Let us first discuss the case where the potential perturbation is just the pointwise multiplication by a function $V \in L^\infty(\mathbb{R}^d)$. There is some room left for tuning the regularity of potentials and we have indeed available from time-frequency analysis a scale of decreasing regularity spaces.

1. The best option for our purposes is given by the Hörmander class $C_{\mathrm{b}}^\infty(\mathbb{R}^d)$, the space of smooth bounded functions on $\mathbb{R}^d$ with bounded derivatives of any order.

2. At an intermediate level we have the (scale of) modulation spaces $M_{0,s}^\infty(\mathbb{R}^d)$, $s > 2d$, consisting of distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that for any $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$

$$|V_g f(x, \xi)| \leq C(1 + |\xi|)^{-s}, \quad (x, \xi) \in \mathbb{R}^{2d},$$

for some $C > 0$. $M_{0,s}^\infty(\mathbb{R}^d)$ contain bounded continuous functions, becoming less regular as $s \searrow 2d$ - the parameter s can be thought of as a measure of (fractional) differentiability.

3. We finally consider the Sjöstrand class $M^{\infty,1}(\mathbb{R}^d)$ as a maximal space, where the partial regularity of the previous level is completely lost. Recall that $f \in M^{\infty,1}(\mathbb{R}^d)$ if for any $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$

$$\|f\|_{M^{\infty,1}} = \int_{\mathbb{R}^d} \sup_{x\in\mathbb{R}^d} |V_g f(x,\xi)| d\xi < \infty.$$

   It is still a space of bounded continuous functions which locally enjoy the mild regularity of the Fourier transform of a L^1 function - cf. Section 3.2.4 for further details.

We have indeed the following chain of strict inclusions for $s > d$ (cf. Proposition 3.2.16):

$$C_{\mathrm{b}}^\infty(\mathbb{R}^d) \subset M_{0,s}^\infty(\mathbb{R}^d) \subset M^{\infty,1}(\mathbb{R}^d) \subset (\mathcal{F}L^1)_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \subset C_{\mathrm{b}}(\mathbb{R}^d).$$

It seems worthwhile to highlight that results on the convergence of path integrals are already known for special elements of the Sjöstrand class; for instance, a class of potentials widely investigated by means of different approaches in the papers of S. Albeverio and co-authors [AB93; ABHK82; AHK77; AM16] and K. Itô [Itô61; Itô67] is $\mathcal{FM}(\mathbb{R}^d)$, namely the space of Fourier transforms of (finite) complex measures on $\mathbb{R}^d$. In fact, we have $\mathcal{FM}(\mathbb{R}^d) \subset M^{\infty,1}(\mathbb{R}^d)$, cf. Proposition 3.2.16, and the above inclusion is strict; for instance, $f(x) = \cos|x|$, $x \in \mathbb{R}^d$, clearly belongs to $C_{\mathrm{b}}^\infty(\mathbb{R}^d)$, but it is easy to realize that $f \notin \mathcal{FM}(\mathbb{R}^d)$ as soon as $d > 1$, by the known formula for the fundamental solution of the wave equation [Eva10].

We are able to cover a more general family of potential perturbations in the form of Weyl operators, namely we assume that $V = \sigma^{\mathrm{w}}$ where the symbol σ belongs to any of the spaces $C_{\mathrm{b}}^\infty(\mathbb{R}^{2d})$, $M_{0,s}^\infty(\mathbb{R}^{2d})$ with $s > 2d$, $M^{\infty,1}(\mathbb{R}^{2d})$. Notice indeed that the multiplication by a function V on $\mathbb{R}^d$ is an easy example of Weyl operator with symbol $\sigma_V(x,\xi) = V(x) = (V \otimes 1)(x,\xi)$, and the correspondence $V \mapsto \sigma$ is continuous from $M^{\infty,1}(\mathbb{R}^d)$ to $M^{\infty,1}(\mathbb{R}^{2d})$ and similarly for the other spaces mentioned above, cf. Remark 4.2.1 below.

Let us first state our main result on the pointwise convergence *at the level of integral kernels*, at the intermediate regularity encoded by $M_{0,s}^\infty$.

Theorem 1.6.1. *Consider the problem* (1.52) *with* $H_0 = Q^{\mathrm{w}}$ *as discussed above and* $V = \sigma^{\mathrm{w}}$ *with* $\sigma \in M_{0,s}^\infty(\mathbb{R}^{2d})$ *and* $s > 2d$. *Let* S_t *denote the classical flow associated with* H_0 *as in* (1.49). *For any* $t \in \mathbb{R} \setminus \mathfrak{E}$*:*

1. *the distributions* $e^{-2\pi i \Phi_t} e_{n,t}$, $n \geq 1$, *and* $e^{-2\pi i \Phi_t} u_t$ *belong to a bounded subset of* $M_{0,s}^\infty(\mathbb{R}^{2d})$*;*

2. $e_{n,t} \to u_t$ *in* $(\mathcal{F}L^1_r)_{\mathrm{loc}}(\mathbb{R}^{2d})$ *for any* $0 < r < s - 2d$*, hence uniformly on compact subsets.*

The first part of the claim assures that the kernel convergence problem is well posed in this case - the amplitudes are bounded continuous functions. The second part precisely characterizes the regularity at which convergence occurs, hence the desired pointwise convergence.

We expect to improve the convergence result in the smooth scenario in view of the characterizations

$$C^\infty_{\mathrm{b}}(\mathbb{R}^{2d}) = \bigcap_{s\ge 0} M^\infty_{0,s}(\mathbb{R}^{2d}), \quad C^\infty(\mathbb{R}^{2d}) = \bigcap_{r>0} \left(\mathcal{F}L^1_r\right)_{\mathrm{loc}}(\mathbb{R}^{2d})$$

(cf. Propositions 3.2.16 and (8.6) below).

Corollary 1.6.2. *Consider the problem* (1.52) *with* $H_0 = Q^{\mathrm{w}}$ *as discussed above and* $V = \sigma^{\mathrm{w}}$ *with* $\sigma \in C^\infty_{\mathrm{b}}(\mathbb{R}^{2d})$*. Let* S_t *denote the classical flow associated with* H_0 *as in* (1.49)*. For any* $t \in \mathbb{R} \setminus \mathfrak{E}$*:*

1. *the distributions* $e^{-2\pi i \Phi_t} e_{n,t}$*,* $n \ge 1$*, and* $e^{-2\pi i \Phi_t} u_t$ *belong to a bounded subset of* $C^\infty_{\mathrm{b}}(\mathbb{R}^{2d})$*;*

2. $e_{n,t} \to u_t$ *in* $C^\infty(\mathbb{R}^{2d})$*, hence uniformly on compact subsets together with any derivatives.*

This result should be compared with the second claim in Theorem 1.5.2 by Fujiwara, which motivated our quest. In spite of the different assumptions and approximation schemes, we stress that our result is almost global in time - more details on exceptional times are given below.

We conclude with the analogous convergence result for potentials in the Sjöstrand class.

Theorem 1.6.3. *Consider the problem* (1.52) *with* $H_0 = Q^{\mathrm{w}}$ *as discussed above and* $V = \sigma^{\mathrm{w}}$ *with* $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$*. Let* S_t *denote the classical flow associated with* H_0 *as in* (1.49)*. For any* $t \in \mathbb{R} \setminus \mathfrak{E}$*:*

1. *the distributions* $e^{-2\pi i \Phi_t} e_{n,t}$*,* $n \ge 1$*, and* $e^{-2\pi i \Phi_t} u_t$ *belong to a bounded subset of* $M^{\infty,1}(\mathbb{R}^{2d})$*;*

2. $e_{n,t} \to u_t$ *in* $(\mathcal{F}L^1)_{\mathrm{loc}}(\mathbb{R}^{2d})$*, hence uniformly on compact subsets.*

The occurrence of a set of exceptional times in Theorems 1.6.1 and 1.6.3 comes not as a surprise from a mathematical point of view. For instance, the correspondence between free symplectic matrices and quadratic Fourier transforms as in (1.50) (cf. Proposition 4.3.4 for a precise formulation) can be used to determine the abundance of free matrices in $\mathrm{Sp}(d,\mathbb{R})$: not being free is an exceptional feature of a symplectic matrix, in the sense that follows.

Proposition 1.6.4 ([Gos17, Proposition 171])**.** *The subset* $\mathrm{Sp}_0(d,\mathbb{R})$ *of free symplectic matrices has codimension* 1 *in* $\mathrm{Sp}(d,\mathbb{R})$*.*

Moreover, it may very well happen that the integral kernel of the evolution operator degenerates into a distribution. A standard example of this phenomenon is provided by the harmonic oscillator, namely

$$\frac{i}{2\pi}\partial_t \psi = \left(-\frac{1}{8\pi^2}\Delta + \frac{1}{2}|x|^2\right)\psi.$$

The integral kernel of the corresponding evolution operator is known as the *Mehler kernel* and can be explicitly characterized [Gos11; KRY97]: for $k \in \mathbb{Z}$,

$$u_t(x,y) = \begin{cases} c(k)|\sin t|^{-d/2}\exp\Big(\pi i \frac{x^2+y^2}{\tan t} - 2\pi i \frac{x\cdot y}{\sin t}\Big) & (\pi k < t < \pi(k+1)) \\ c'(k)\delta((-1)^k x - y) & (t = k\pi) \end{cases},$$

for suitable phase factors $c(k), c'(k) \in \mathbb{C}$. This shows the expected degenerate behaviour at integer multiples of π, which is consistent with the fact that the associated classical flow S_t is given by

$$S_t = \begin{bmatrix} (\cos t)I & (\sin t)I \\ -(\sin t)I & (\cos t)I \end{bmatrix},$$

and we retrieve $\mathfrak{E} = \{t \in \mathbb{R} : \sin t = 0\} = \{k\pi : k \in \mathbb{Z}\}$.

It is natural to wonder whether convergence of integral kernels still occurs in some distributional sense, hopefully better than the broadest one (that is $\mathcal{S}'(\mathbb{R}^{2d})$). This question has been settled in the paper [FNT20], joint work with H. Feichtinger and F. Nicola, where we show that a suitable framework is offered by the *Banach-Gelfand triple* (M^1, L^2, M^∞) of modulation spaces, which has better properties than the standard triple $(\mathcal{S}, L^2, \mathcal{S}')$ of real harmonic analysis. This is further discussed in Section 3.2; see also [FLC08] in this connection.

Our main convergence result at exceptional times reads as follows.

Theorem 1.6.5. *Assume* $V = \sigma^{\mathrm{w}}$ *for some* $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$*. Let* $\{E_n(t)\}$ *be the sequence of Feynman-Trotter parametrices defined in* (1.53) *and* $U(t)$ *be the*

Schrödinger evolution operator $U(t)$ *associated with the Cauchy problem* (1.52). *For any fixed* $t \in \mathbb{R}$ *we have*

$$\lim_{n\to\infty} E_n(t) = U(t), \quad \lim_{n\to\infty} E_n(t)^* = U(t)^*$$

in the strong topology of operators acting on $M^1(\mathbb{R}^d)$. *In particular, for all* $t \in \mathbb{R}$ *and* $f \in M^1(\mathbb{R}^d)$, *the functions*

$$\langle e_{n,t}(x,\cdot), f\rangle, \quad \langle e_{n,t}(\cdot,y), f\rangle, \quad \langle u_t(x,\cdot), f\rangle, \quad \langle u_t(\cdot,y), f\rangle$$

belong to $M^1(\mathbb{R}^d)$, *and*

$$\langle e_{n,t}(x,\cdot), f\rangle \to \langle u_t(x,\cdot), f\rangle, \quad \langle e_{n,t}(\cdot,y), f\rangle \to \langle u_t(\cdot,y), f\rangle$$

in $M^1(\mathbb{R}^d)$, *hence in* $L^p(\mathbb{R}^d)$ *for every* $1 \le p \le \infty$.

Let us conclude this presentation with a few words on the techniques employed for the proofs. The main idea is to exploit the very rich structure enjoyed by the modulation spaces $M^\infty_{0,s}(\mathbb{R}^{2d})$ (with $s > 2d$) and $M^{\infty,1}(\mathbb{R}^{2d})$: they are Banach algebras for both pointwise multiplication and Weyl product of symbols (cf. Remark 4.2.4) and the corresponding families of Weyl operators are inverse-closed Banach subalgebras of $\mathcal{L}(L^2(\mathbb{R}^d))$ (cf. Theorem 4.2.5 below).

There is a certain number of questions which seem worthy of further consideration. For example, Theorem 1.6.1 and Corollary 1.6.2 should hopefully extend to Hamiltonians H_0 given by the Weyl quantization of a smooth real-valued function with derivatives of order ≥ 2 bounded, using techniques from [Nic16; Nic19]. We observe that the strategies introduced here could hopefully be useful to study similar convergence problems of the integral kernels for other approximation formulas arising in semigroup theory; cf. [EN06].

1.7 Approximation of Feynman path integrals with non-smooth potentials

Let us reconsider the time-slicing approximation for Feynman path integrals discussed in Section 1.5.2, and in particular Theorem (1.5.2) by Fujiwara, in order to make some important remarks. First, the occurrence of convergence results at two different levels, a coarser one (parametrices in $\mathcal{L}(L^2(\mathbb{R}^d))$) and a finer one (OIO amplitudes in $C^\infty_{\mathrm{b}}(\mathbb{R}^{2d})$), suggests that the assumptions may be relaxed in order to preserve convergence in operator norm. A first step in this direction is the aforementioned paper [Nic19], where a delicate analysis of low-regular potentials leads to the desired result. We are now going to consider another class of non-smooth potentials.

Assumption ($\tilde{\mathbf{A}}$). *$V(t,x)$ is a real-valued function of $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ and there exists $N \in \mathbb{N}$, $N \geq 1$, such that*[4]

$$\partial_t^k \partial_x^\alpha V \in C_{\mathrm{b}}(\mathbb{R}, M^{\infty,1}(\mathbb{R}^d)),$$

for any $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$ satisfying $2k + |\alpha| \leq 2N$.

Roughly speaking, potentials satisfying Assumption ($\tilde{\mathrm{A}}$) are bounded continuous functions together with a certain number of derivatives. Assumptions in the same spirit, or even stronger (e.g., smooth potentials with compact support), are quite popular in scattering theory [Mel95].

In the second place, the estimate (1.48) reveals other interesting aspects of the parametrices $E^{(N)}$. In particular, notice that while the approximation power increases with N from the point of view of semiclassical analysis (positive powers of $\hbar$), the rate of convergence with respect to the length of the time interval does not enjoy any improvement. Moreover, sophisticate parametrices like those introduced in (1.46) have limited applicability to concrete situations and computational problems since the knowledge of the exact action functional is required, the latter being an intractable problem except for a number of simple systems. These remarks lead one to consider short-time approximations for the action by means of the so-called *midpoint rules* [Sch81]. In short, given the action functional corresponding to the standard Hamiltonian $H(x,\xi) = |\xi|^2/2 + V(x)$, that is

$$S(t,s,x,y) = \frac{|x-y|^2}{2(t-s)} - \mathcal{V}(t,s,x,y), \quad \mathcal{V} = \int_s^t V(\gamma(\tau))d\tau,$$

the latter integral involving paths with $\gamma(s) = y$ and $\gamma(t) = x$, $\mathcal{V}$ is replaced with approximate expressions such as

$$\mathcal{V}_1 = \frac{V(x)+V(y)}{2}(t-s), \quad \text{or} \quad \mathcal{V}_2 = V\left(\frac{x+y}{2}\right)(t-s).$$

A simple test in the case of known models reveals that, in spite of their popularity within the physics literature, these procedures are not sufficiently accurate. For the harmonic oscillator and the corresponding approximate actions S_1, S_2 one has indeed

$$S(t,s,x,y) - S_j(t,s,x,y) = O(t-s), \quad j = 1,2.$$

[4] $C_{\mathrm{b}}(\mathbb{R}, E)$ is the space of bounded continuous functions $f : \mathbb{R} \to E$, see Chapter 2.

The quest for a correct short-time approximation was initiated by N. Makri and W. Miller [Mak91; MM88a; MM88b], leading to the integral average

$$\overline{\mathcal{V}}(s,x,y) = \int_0^1 V(\tau x + (1-\tau)y, s) d\tau.$$

This procedure satisfies a correct first-order approximation, i.e. $S(t,s,x,y) - \overline{S}(t,s,x,y) = O((t-s)^2)$ for small $t-s$. We refer the interested reader to the aforementioned papers and the recent one [Gos18] by de Gosson.

Chapter 9 of this dissertation is based on the joint article [NT19] with F. Nicola. Inspired by this discussion and the current practice in physics and chemistry, we considered different time slicing approximation operators than (1.46), namely

$$\widetilde{E}^{(N)}(t,s)f(x) = \frac{1}{(2\pi i\hbar(t-s))^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}S^{(N)}(t,s,x,y)} f(y) dy, \tag{1.54}$$

where the approximate action $S^{(N)}$ ultimately is a Taylor-like expansion of the exact action S at $t=s$:

$$S^{(N)}(t,s,x,y) = \frac{|x-y|^2}{2(t-s)} + \sum_{k=1}^{N} W_k(s,x,y)(t-s)^k. \tag{1.55}$$

The coefficients $W_k(s,x,y)$ are recursively constructed after careful analysis of power series solutions for the modified Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2}|\nabla_x S|^2 + V(t,x) + \frac{i\hbar d}{2(t-s)} - \frac{i\hbar}{2}\Delta_x S = 0.$$

The last two terms are tailored to enhance the approximating power of $\widetilde{E}^{(N)}$ as parametrix. Nevertheless, the "counterterm" is first order in $\hbar$ and identically vanishes in the free particle case ($V=0$). Plus, we remark that $W_1(s,x,y) = \overline{\mathcal{V}}(s,x,y)$ as expected. All these aspects are discussed in detail in Section 9.1, which is devoted to a rigorous short-time analysis of the action functional.

Given a subdivision $\Omega = \{t_0, \dots, t_L\}$ of the interval $[s,t]$ such that $s = t_0 < t_1 < \ldots < t_L = t$, we introduce the long-time composition

$$\widetilde{E}^{(N)}(\Omega,t,s) = \widetilde{E}^{(N)}(t_L,t_{L-1})\widetilde{E}^{(N)}(t_{L-1},t_{L-2}) \cdots \widetilde{E}^{(N)}(t_1,t_0), \tag{1.56}$$

which has integral kernel

$$K^{(N)}(\Omega,t,s,x,y) = \prod_{j=1}^{L} \frac{1}{(2\pi i(t_j - t_{j-1})\hbar)^{d/2}} \times \int_{\mathbb{R}^{d(L-1)}} \exp\Big(\frac{i}{\hbar}\sum_{j=1}^{L} S^{(N)}(t_j,t_{j-1},x_j,x_{j-1})\Big) \prod_{j=1}^{L-1} dx_j,$$

with $x = x_L$ and $y = x_0$. It is reasonable to believe that the operators $\widetilde{E}^{(N)}(\Omega, t, s)$ converge to the actual propagator as $\omega(\Omega) = \sup\{t_j - t_{j-1} : j = 1, \dots, L\} \to 0$, in line with Feynman's insight. The main result in [NT19] reads as follows.

Theorem 1.7.1 ([NT19, Theorem 1]). *Let V satisfy Assumption* $(\tilde{\mathrm{A}})$ *above. For any $T > 0$ there exists a constant $C = C(T) > 0$ such that, for $0 < t - s \le T\hbar$, $0 < \hbar \le 1$, and any sufficiently fine subdivision Ω of the interval $[s, t]$, we have*

$$\|\widetilde{E}^{(N)}(\Omega, t, s) - U(t, s)\|_{L^2 \to L^2} \le C\omega(\Omega)^N.$$

We remark that the increasing semiclassical approximation power of Birkhoff-Maslov parametrices (1.46) is lost when one considers rougher parametrices as those in (1.54), where the balance weights in favour of accelerated rate of convergence with respect to time. A cursory glance at the estimates for the operators $\widetilde{E}^{(N)}$ reveals that negative powers of $\hbar$ are involved, making them completely unfit for semiclassical arguments. Nevertheless, one can also notice that all the estimates are uniform in $\hbar$ as soon as time is measured in units of $\hbar$, which is a particularly interesting feature.

Part I

Background Material

Chapter 2

Basic Facts of Real, Functional and Fourier Analysis

The purpose of this chapter is to fix the notation used in the manuscript, as well as to collect the basic facts and main results of real, functional and harmonic analysis that are needed below.

2.1 General notation

We denote the set of positive integer numbers by $\mathbb{N} = \{0, 1, 2, \ldots\}$, while $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ are the usual sets of integers, real and complex numbers respectively. In particular, i denotes the imaginary unit and $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$. We also set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

If f is a function from some set A with values in some set B we write $f : A \to B$. We assume to deal with complex-valued functions, i.e. $B = \mathbb{C}$, if not otherwise specified. The *characteristic function* of a set A is denoted with χ_A, hence

$$\chi_A(a) = \begin{cases} 0 & (a \notin A) \\ 1 & (a \in A) \end{cases} .$$

Given two real numbers $x, y \in \mathbb{R}$, the symbol $x \lesssim y$ means that the underlying inequality holds up to a universal positive constant factor, namely

$$x \lesssim y \quad \Longrightarrow \quad \exists C > 0 : x \leq Cy.$$

If the constant $C = C(a) > 0$ depends on some "allowable" parameter $a \in A$ we write $x \lesssim_a y$. Moreover, $x \asymp y$ means that x and y are *equivalent quantities*, that is both $x \lesssim y$ and $x \lesssim y$ hold.

We will be mainly concerned with the d-dimensional real Euclidean space $\mathbb{R}^d$. The standard inner product on $\mathbb{R}^d$ and the induced Euclidean norm are denoted by

$$x \cdot y \coloneqq \sum_{j=1}^{d} x_j y_j, \quad |x| \coloneqq \sqrt{x \cdot x} = (x_1^2 + \ldots x_d^2)^{1/2}.$$

We write x^2 in place of $|x|^2 = x \cdot x$. Notice that $|x|$ is the absolute value of x in the case where $d = 1$. Examples of equivalent norms on $\mathbb{R}^d$ are

$$|x|_1 \coloneqq |x_1| + \ldots |x_d|, \quad |x|_\infty \coloneqq \max\{|x_1|, \ldots, |x_d|\}.$$

The open ball of radius $R > 0$ and center $x_0 \in \mathbb{R}^d$ is the set

$$B_R(x_0) \coloneqq \{x \in \mathbb{R}^d : |x - x_0| < R\}.$$

We use several symbols for partial differential operators on $\mathbb{R}^d$:

$$\partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j}, \quad D_j \coloneqq \frac{1}{2\pi i}\partial_j, \quad j = 1, \ldots, d.$$

We employ the *multi-index notation*: given $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$, we write

$$|\alpha| = |\alpha|_1 = \alpha_1 + \ldots + \alpha_d, \quad \alpha! = \alpha_1! \cdots \alpha_d!, \quad x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}.$$

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}, \quad D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_d}^{\alpha_d}.$$

The following relations between multi-indices $\alpha, \beta \in \mathbb{N}^d$ are defined:

$$\alpha \le \beta \Leftrightarrow \alpha_j \le \beta_j,\ j = 1, \ldots, d, \qquad \alpha < \beta \Leftrightarrow \alpha \ne \beta \text{ and } \alpha \le \beta,$$

$$\binom{\alpha}{\beta} = \prod_{j=1}^{d} \binom{\alpha_j}{\beta_j} = \frac{\alpha!}{\beta!(\alpha - \beta)!} \quad (\text{if } \beta \le \alpha).$$

If $x \in \mathbb{R}$ and $\alpha \in \mathbb{N}^d$ we write x^α in place of $x^{|\alpha|}$ to lighten the notation.

We repeatedly make use of the *Japanese brackets* to denote the inhomogeneous magnitude of $x \in \mathbb{R}^d$, namely $\langle x \rangle \coloneqq (1 + x^2)^{1/2}$. Given $m > 0$ we similarly define $\langle x \rangle_m \coloneqq (m^2 + x^2)^{1/2}$; note that $\langle x \rangle = \langle x \rangle_1$. It is useful to remark that the so-called *Peetre inequality* holds:

$$\langle x + y \rangle^s \lesssim_s \langle x \rangle^s \langle y \rangle^{|s|}, \quad x, y \in \mathbb{R}^d,\ s \in \mathbb{R}. \tag{2.1}$$

The set of real matrices of dimension $d \times d$ is denoted by $\mathbb{R}^{d \times d}$. In particular, $I = I_d \in \mathbb{R}^{d \times d}$ is the identity matrix and $O = O_d \in \mathbb{R}^{d \times d}$ is the null matrix.

Recall that the canonical symplectic matrix $J \in \mathbb{R}^{2d\times 2d}$ is the nonsingular and skew-symmetric block matrix

$$J = \begin{bmatrix} O & I \\ -I & O \end{bmatrix}.$$

The direct sum $A \oplus B$ of $A, B \in \mathbb{R}^{d\times d}$ is the $2d \times 2d$ matrix defined by

$$A \oplus B = \begin{bmatrix} A & O \\ O & B \end{bmatrix}.$$

Given a list $a_1, \dots, a_d$ of real numbers, we define the matrix $\mathrm{diag}(a_1, \dots, a_d) \in \mathbb{R}^{d\times d}$ by

$$[\mathrm{diag}(a_1, \dots, a_d)]_{ij} := \begin{cases} a_j & (i = j) \\ 0 & (i \neq j) \end{cases},$$

where $[M]_{ij}$ is the element on the i-th row and the j-th column of the matrix $M \in \mathbb{R}^{d\times d}$, $1 \le i, j \le d$. Accordingly, we extend the notation to block matrices by setting $\mathrm{diag}(A, B) := A \oplus B$.

We will thoroughly work with invertible matrices, namely elements of the group

$$\mathrm{GL}(2d, \mathbb{R}) = \{M \in \mathbb{R}^{2d\times 2d} \mid \det M \neq 0\}.$$

We employ the following symbol to denote the transpose of an inverse matrix:

$$M^{\#} = (M^{-1})^{\top} = (M^{\top})^{-1}, \qquad M \in \mathrm{GL}(2d, \mathbb{R}).$$

2.1.1 Function spaces

In general we denote by E a complex Banach space with norm $|\cdot|_E$, whereas the symbol H is usually reserved for a complex separable Hilbert space. The topological dual space of E is denoted by E'. The brackets $(\cdot, \cdot)$ are used for the duality between E' and E and in particular for the inner product in H - we assume $(\cdot, \cdot)$ to be conjugate-linear in the second argument.

Consider the space $\mathcal{L}(X, Y)$ of all the *continuous linear mappings* between two Hausdorff topological vector spaces X and Y (we write $\mathcal{L}(X)$ if $Y = X$). It can be endowed with different topologies [Tre67], in which cases we write:

(i) $\mathcal{L}_{\mathrm{b}}(X, Y)$, if equipped with the topology of *bounded convergence*, that is uniform convergence on bounded subsets of X;

(ii) $\mathcal{L}_{\mathrm{c}}(X, Y)$, if equipped with the topology of *compact convergence*, that is uniform convergence on compact subsets of X;

(iii) $\mathcal{L}_{\mathrm{s}}(X,Y)$, if equipped with the topology of *pointwise convergence*, that is uniform convergence on finite subsets of X.

Notice that if $Y = \mathbb{C}$, $\mathcal{L}_{\mathrm{b}}(X,Y) = X'_{\mathrm{b}}$ (the strong dual of X), while $\mathcal{L}_{\mathrm{s}}(X,Y) = X'_{\mathrm{s}}$ (the weak dual of X). It is tacitly assumed that $\mathcal{L}(X,Y) = \mathcal{L}_{\mathrm{b}}(X,Y)$ unless otherwise specified.

The symbol $E_1 \simeq E_2$ means that E_1 and E_2 are isomorphic Banach spaces, namely there exists a bijective linear operator $T : E_1 \to E_2$ such that $T \in \mathcal{L}(E_1, E_2)$ and $T^{-1} \in \mathcal{L}(E_2, E_1)$.

Recall that the *singular values* $s_j(T)$ of a compact operator $T \in \mathcal{L}(H)$ are the positive eigenvalues of $|T| := \sqrt{T^*T}$, where T^* denotes the *adjoint operator* of T; we arrange them in a sequence $s(T) = (s_j(T))_{j\geq 1}$ in such a way that $s_1 \geq s_2 \ldots \ldots > 0$. The p-th order Schatten class $\mathfrak{S}^p(H)$, $1 \leq p < \infty$, is the subset of all compact operators T on H such that $\|T\|_{\mathfrak{S}^p(H)} := \|s(T)\|_{\ell^p(\mathbb{N})} < \infty$. In particular, operators in $\mathfrak{S}^1(H)$ are said to be in the *trace class*, while $\mathfrak{S}^2(H)$ is known as the set of *Hilbert-Schmidt operators*. We also recall that an equivalent norm for $\mathfrak{S}^2(H)$ is given by $\|\|T\psi_j\|_H\|_{\ell^2(\mathbb{N})}$, for any orthonormal basis $\{\psi_j\}_{j\in\mathbb{N}}$ of H. Similarly, for $T \in \mathfrak{S}^1(H)$ we have $\|T\|_{\mathfrak{S}^1} = \mathrm{Tr}|T|$, where the *trace* of an operator $A \in \mathfrak{S}^1(H)$ is

$$\mathrm{Tr}(A) := \sum_{j\in\mathbb{N}} (A\psi_j, \psi_j)_H,$$

for any orthonormal basis $\{\psi_j\}$ of H. We also recall that if $S \in \mathcal{L}(H)$ and $T \in \mathfrak{S}^1(H)$ then $ST, TS \in \mathfrak{S}^1(H)$ and $\mathrm{Tr}(ST) = \mathrm{Tr}(TS)$.

We will occasionally make use of the *Dirac notation* for projection operators: given $\phi, \psi \in H$, we define

$$|\psi\rangle\langle\phi| \in \mathfrak{S}^1(H), \quad |\psi\rangle\langle\phi|(w) := (w,\phi)\psi, \quad w \in H.$$

Given a triple E_1, E_2 and E_3 of complex Banach spaces, we say that the map

$$\bullet : E_1 \times E_2 \to E_3, \quad (x_1, x_2) \mapsto x_3 = x_1 \bullet x_2$$

is a *multiplication* [Ama19] if it is a continuous bilinear operator such that $\|\bullet\|_{\mathcal{L}(E_1\times E_2, E_3)} \leq 1$. The following are common examples of multiplications that will be used below:

(i) multiplication with scalars: $\mathbb{C} \times E \to E$, $(\lambda, x) \mapsto \lambda x$;

(ii) the duality pairing: $E' \times E \to \mathbb{C}$, $(u, x) \mapsto u(x)$;

(iii) the evaluation map: $\mathcal{L}(E_1, E_2) \times E_1 \to E_2$, $(T, x) \mapsto Tx$;

(iv) multiplication in a Banach algebra.

Remark 2.1.1. *To unambiguously fix the notation: whenever concerned with a product of elements $a_1, \ldots, a_N$ in a Banach algebra $(A, \star)$, we write*

$$\prod_{k=1}^{N} a_k \coloneqq a_1 \star a_2 \star \ldots \star a_N.$$

This relation is meant to hold even when $(A, \star)$ is a non-commutative algebra, provided that the symbol on the LHS exactly designates the ordered product on the RHS. Moreover, if A is a unital algebra, it is a well known general fact that one can provide an equivalent norm on A for which the identity element has norm equal to 1 *(cf. [Rud91, Theorem 10.2]). From now on, we assume to work with such equivalent norm whenever concerned with a Banach algebra.*

2.2 Function spaces

We collect below some fundamental results of real analysis and fix the related notation once for all. We also devote a part of this section to the presentation of such results in the broadest setting of vector-valued analysis. The reason behind this choice is that some tools are directly needed below in order to extend the results of scalar time-frequency analysis. In fact, the main parts of this dissertation would only require to consider the standard case $E = \mathbb{C}$ or at most finite-dimensional vector spaces such as $E = \mathbb{C}^n$ and $E = \mathbb{C}^{n\times n}$, so that the subtleties related to infinite-dimensional target spaces are not relevant here and most of the proofs ultimately reduce to componentwise arguments. Nevertheless, we decided to embrace this wider perspective because developing these tools in full generality does not require an excessive effort. Moreover, this approach is repaid by a unifying and powerful framework which provides very natural and compact proofs also in the cases $E = \mathbb{C}^n$ or $E = \mathbb{C}^{n\times n}$.

The notation and the basic results of analysis on infinite-dimensional spaces are rather standard. All the proofs and further details on the results mentioned below may be found for instance in [Ama19; GW03; HNVW16].

In what follows we always consider functions $f : \mathbb{R}^d \to E$, where $\mathbb{R}^d$ is provided with the Lebesgue measure and E is a Banach space.

2.2.1 Weight functions

We will constantly employ *weight functions*, that is families of non-negative functions, to precisely tune the decay or the regularity of other functions. Several types of weight functions are used in harmonic and time-frequency analysis depending on the need. We collect below some basic definitions and facts, possibly under slightly more restrictive assumptions than usual. We refer the reader to [Grö07] for generalizations and further details.

Definition 2.2.1. *(i) We say that $v : \mathbb{R}^d \to (0, +\infty)$ is a weight function if it is a continuous and symmetric function in each coordinate:*

$$v(\pm x_1, \dots, \pm x_d) = v(x_1, \dots, x_d), \quad x \in \mathbb{R}^d.$$

(ii) v is called submultiplicative if

$$v(x + y) \le v(x)v(y), \quad x, y \in \mathbb{R}^d.$$

(iii) Given a submultiplicative weight v, we say that a weight function m is v-moderate if

$$m(x + y) \lesssim v(x)m(y), \quad x, y \in \mathbb{R}^d.$$

In general, we say that m is moderate if it is v-moderate for some submultiplicative weight v.

(iv) A weight v satisfies the Gelfand-Raikov-Shilov (GRS) condition if

$$\lim_{n\to\infty} v(nx)^{1/n} = 1, \quad x \in \mathbb{R}^d.$$

A standard family of weights is

$$m_{a,b,s,t}(x) \coloneqq e^{a|x|^b}(1 + |x|)^s \log^t(e + |x|), \quad a, b, s, t \in \mathbb{R}.$$

Note that tuning the parameters allows us to control polynomial, (sub)logarithmic and (sub)exponential rates of decay/growth. In particular,

(i) if $a, s, t \ge 0$ and $0 \le b \le 1$ then $m_{a,b,s,t}$ is submultiplicative;

(ii) if $a, s, t \in \mathbb{R}$ and $|b| \le 1$ then $m_{a,b,s,t}$ is moderate;

(iii) if $a, s, t \ge 0$ and $0 \le b < 1$ then $m_{a,b,s,t}$ satisfies the GRS condition.

Moreover, a convenient characterization of the GRS property is as follows: if v is a submultiplicative weight, then v is a GRS weight if and only if $v(x) \lesssim e^{-\epsilon x}$ for any $\epsilon > 0$ [FGT14].

Weights of particular relevance for our purposes are those of polynomial type, namely

$$v_s(x) \coloneqq (1 + |x|)^s, \quad s \in \mathbb{R},\, x \in \mathbb{R}^d.$$

Note that

$$v_s(x) \asymp \langle x\rangle^s \asymp (1+|x_1|+\ldots+|x_d|)^s, \quad x\in\mathbb{R}^d,$$

hence we will tacitly switch from one form to another one whenever convenient. In particular, for a polynomial weight v_s on $\mathbb{R}^{2d}$ we have

$$v_s(z) \asymp \langle z\rangle^s \asymp (1+|x|+|\xi|)^s, \quad z=(x,\xi)\in\mathbb{R}^{2d}.$$

We collect below some elementary properties of weight functions that will be used below.

Lemma 2.2.2. *Let v be a submultiplicative weight and m be a v-moderate weight on $\mathbb{R}^d$.*

(i) The weight $1/v$ is v-moderate.

(ii) v grows at most exponentially, namely there exist $C>0$ and $a\ge 0$ such that $v(x)\le Ce^{a|x|}$, $x\in\mathbb{R}^d$.

(iii) For any $x,y\in\mathbb{R}^d$,

$$\frac{m(x)}{v(y)} \lesssim m(x-y) \lesssim m(x)v(y).$$

In particular, $1/v\lesssim m\lesssim v$ and $1/v\lesssim 1/m\lesssim v$.

(iv) If $s\ge 0$ then v_s is submultiplicative. Moreover, if $0\le |r|\le s$, then both v_s and v_{-s} are v_r-moderate.

Important. *In order to avoid the technical difficulties related with the exponential growth of general submultiplicative weights, in the rest of the dissertation **all** the weight functions are always assumed to grow at most polynomially. We thus denote by $\mathcal{M}_v(\mathbb{R}^{2d})$ the space of all the weight functions on $\mathbb{R}^{2d}$ which are moderate with respect to an **admissible** weight function v, that is a submultiplicative weight satisfying the GRS condition and of temperate growth (that is, $v\lesssim v_s$ for some $s\in\mathbb{R}$); the same applies when we say that m is a moderate weight.*

2.2.2 Lebesgue spaces

Fix $1\le p<\infty$ and let m be a moderate weight on $\mathbb{R}^d$. Consider the set $\mathcal{L}^p_m$ of measurable functions $f:\mathbb{R}^d\to\mathbb{C}$ such that

$$\|f\|_{L^p_m} := \left(\int_{\mathbb{R}^d} |f(x)|^p m(x)^p\right)^{1/p} < \infty.$$

The Lebesgue space $L^p_m(\mathbb{R}^d)$ is the quotient set of $\mathcal{L}^p_m$ by the equivalence relation

$$f \sim g \Longleftrightarrow \|f-g\|_{L^p_m} = 0.$$

The space $L^\infty_m(\mathbb{R}^d)$ is defined similarly, where

$$\|f\|_{L^\infty_m} := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} |f(x)|m(x).$$

As usual, we write $f \in L^p_m(\mathbb{R}^d)$ instead of $[f] \in L^p_m(\mathbb{R}^d)$, where $[f]$ is the equivalence class of functions which coincide with $f \in \mathcal{L}^p_m$ almost everywhere on $\mathbb{R}^d$.

The family of *Lebesgue-Bochner spaces* is the natural analogue of Lebesgue spaces for vector-valued functions $\mathbb{R}^d \to E$. They are defined by above with $|\cdot|$ replaced by $|\cdot|_E$. When there is no risk of confusion, we will write L^p for $L^p(\mathbb{R}^d)$ and $L^p_m(E)$ for $L^p_m(\mathbb{R}^d, E)$; we omit the subscript for the weight when $m = 1$ and write $L^p(E)$. In the case where $m = v_s$, $s \in \mathbb{R}$, we write $L^p_s(E)$.

Most of the usual properties from the scalar-valued case extend in a natural way (with the remarkable exception of duality [HNVW16]). We list those that are needed below.

Proposition 2.2.3. *(i) For any $1 \le p \le \infty$ and $m \in \mathcal{M}_v(\mathbb{R}^d)$, $L^p_m(E)$ is a Banach space with the norm $\|f\|_{L^p_m(E)} = \||f(\cdot)|_E\|_{L^p_m}$.*

(ii) $L^2(H)$ is a Hilbert space with inner product given by

$$\langle f, g\rangle_{L^2(H)} = \int_{\mathbb{R}^d} (f(t), g(t))_H dt.$$

(iii) (Hölder inequality) Given a multiplication $\bullet : E_1 \times E_2 \to E_3$, $s_1, s_2 \in \mathbb{R}$ and $1 \le p_1, p_2, p \le \infty$ such that $1/p_1 + 1/p_2 = 1/p$, if $f \in L^{p_1}_{s_1}(E_1)$ and $g \in L^{p_2}_{s_2}(E_2)$ then $f \bullet g \in L^p_{s_1+s_2}(E_3)$ and $\|f \bullet g\|_{L^p_{s_1+s_2}(E_3)} \le \|f\|_{L^{p_1}_{s_1}(E_1)}\|g\|_{L^{p_2}_{s_2}(E_2)}$.

(iv) (Duality) For $1 \le p \le \infty$ define the conjugate index p' in such a way that $1/p + 1/p' = 1$ - with the understanding that $p' = 1$ if $p = \infty$. If $m \in \mathcal{M}_v$ and E is reflexive then $(L^p_m(E))' \simeq L^{p'}_{1/m}(E')$ for $1 \le p < \infty$, the duality being given by

$$\langle f, g\rangle = \int_{\mathbb{R}^d} (f(t), g(t))dt,$$

for $f \in L^p_m(E)$, $g \in L^{p'}_{1/m}(E')$.

(v) Fix $s \in \mathbb{R}$ and $1 \le p \le \infty$. Then $f = (f_1, \ldots, f_n) \in L^p_s(\mathbb{C}^n)$ if and only if $f_j \in L^p_s$ for any $j = 1, \ldots, n$.

For $1 \le p, q \le \infty$ and a moderate weight m on $\mathbb{R}^{2d}$, we introduce weighted *mixed-norm Lebesgue-Bochner spaces* $L^{p,q}_m(\mathbb{R}^d, E)$ as above, the norm being

$$\|f\|_{L^{p,q}_m(E)} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x,\xi)|_E^p m(x,\xi)^p dx \right)^{q/p} d\xi \right)^{1/q},$$

with trivial modifications for the cases $p = \infty$ or $q = \infty$. In the case where $m = v_r \otimes v_s$ for some $r, s \in \mathbb{R}$ we write $L^{p,q}_{r,s}(E)$.

Most of mixed-norm Lebesgue spaces coincide with vector-valued Lebesgue spaces as above. Precisely:

Proposition 2.2.4 ([Wah07, Lemma 3.8])**.** *If $1 \le p, q < \infty$ and m is a moderate weight on $\mathbb{R}^{2d}$ then $L^{p,q}_m(\mathbb{R}^d, E) = L^q(\mathbb{R}^d, L^p_m(\mathbb{R}^d, E))$ with equal norms. If $p = \infty$ or $q = \infty$ then $L^q(\mathbb{R}^d, L^p_m(\mathbb{R}^d, E)) \subset L^{p,q}_m(\mathbb{R}^d, E)$ with equal norms and possibly strict inclusion.*

As a result, such spaces enjoy most of the expected properties from the scalar-valued case, cf. [Grö01, Lemma 11.1.2] and [Wah07, Lemma 3.5-8].

2.2.3 Differentiable functions and distributions

Recall that a function $f : \mathbb{R}^d \to E$ is said to be differentiable at $x_0 \in \mathbb{R}^d$ if there exist $v_1, \dots, v_d \in E$ such that

$$\lim_{|x - x_0| \to 0} \frac{f(x) - f(x_0) - \sum_{j=1}^d (x_j - (x_0)_j) v_j}{|x - x_0|} = 0.$$

The vectors $v_1, \dots, v_d$ are called first partial derivatives of f at x_0, that is we set $\partial_j f(x_0) := v_j$. A differentiable function at x_0 is clearly continuous there; a function f is said to be differentiable (on $\mathbb{R}^d$) if it is differentiable at every $x \in \mathbb{R}^d$.

Let $C(\mathbb{R}^d, E) = C^0(\mathbb{R}^d, E)$ denote the space of continuous maps $\mathbb{R}^d \to E$. We also write $C_{\mathrm{u}}(\mathbb{R}^d, E)$ for the set of uniformly continuous maps, $C_{\mathrm{b}}(\mathbb{R}^d, E)$ [resp. $C_{\mathrm{ub}}(\mathbb{R}^d, E)$] for the set of [resp. uniformly] bounded continuous functions and $C_0(\mathbb{R}^d, E)$ for the set of continuous functions vanishing at infinity, namely such that $|f(x)|_E \to 0$ as $|x| \to +\infty$.

Given $k \in \mathbb{N}$, $k \ge 1$, we recursively define the vector space $C^k(\mathbb{R}^d, E)$ as the space of differentiable functions $f : \mathbb{R}^d \to E$ such that $\partial_j f \in C^{k-1}(\mathbb{R}^d, E)$, $j = 1, \dots, d$. We similarly define $C^k_{\mathrm{b}}(\mathbb{R}^d, E)$ as the space of k times continuously differentiable E-valued bounded functions on $\mathbb{R}^d$ with bounded derivatives up to k-th order; it is a Banach space with norm

$$\|f\|_{C^k_{\mathrm{b}}(E)} := \max_{|\alpha| \le k} \sup_{x \in \mathbb{R}^d} |\partial^\alpha f(x)|_E.$$

We denote by $C^\infty(\mathbb{R}^d, E) := \bigcap_{k\geq 0} C^k(\mathbb{R}^d, E)$ the space of smooth functions. We also introduce the space $C^\infty_{\geq k}(\mathbb{R}^d, E)$ of smooth functions with bounded derivatives of any order larger than $k \in \mathbb{N}$, namely

$$C^\infty_{\geq k}(\mathbb{R}^d, E) := \big\{ f \in C^\infty(\mathbb{R}^d, E) \,:\, |\partial^\alpha f(x)| \leq C_\alpha \quad \forall x \in \mathbb{R}^d,\, \alpha \in \mathbb{N}^d,\, |\alpha| \geq k \big\}.$$

Notice that $C^\infty_{\mathrm{b}}(\mathbb{R}^d) := C^\infty_{\geq 0}(\mathbb{R}^d) = \cap_{k\geq 0} C^k_{\mathrm{b}}(\mathbb{R}^d)$ coincides with the well-known Hörmander class $S^0_{0,0}(\mathbb{R}^d)$ [GR08; Hör85]. Recall that the latter is a Fréchet space under the family of seminorms $\{\|\cdot\|_{C^k_{\mathrm{b}}(E)}\}_{k\in\mathbb{N}}$. We emphasize that the class $C^\infty_{\geq k}$ is also known as $S^{(k)}_{0,0}$ in microlocal analysis [Tat04].

The set $C^k_{\mathrm{c}}(\mathbb{R}^d, E)$, $k \in \mathbb{N} \cup \{\infty\}$, is the subspace of functions in $C^k(\mathbb{R}^d, E)$ with compact support.

Recall that the Schwartz class of E-valued rapidly decreasing functions $\mathcal{S}(\mathbb{R}^d, E)$ is the subset of $C^\infty(\mathbb{R}^d, E)$ such that

$$p_{m,E}(f) := \max_{|\alpha|+|\beta|\leq m} \sup_{x\in\mathbb{R}^d} \big| x^\alpha \partial^\beta f(x) \big|_E < \infty, \quad \forall m \in \mathbb{N}.$$

It is a Fréchet space with the topology induced by the family of seminorms $\{p_{m,E}\}_{m\in\mathbb{N}}$ and is a dense subset of $L^p(\mathbb{R}^d, E)$ for any $1 \leq p < \infty$.

The space of E-valued *temperate distributions* $\mathcal{S}'(\mathbb{R}^d, E)$ consists of bounded (complex-)linear maps from $\mathcal{S}(\mathbb{R}^d)$ to E, that is $\mathcal{S}'(\mathbb{R}^d, E) = \mathcal{L}(\mathcal{S}(\mathbb{R}^d), E)$, and we set

$$\langle f, g \rangle = f(\overline{\phi}) \in E, \quad f \in \mathcal{S}'(\mathbb{R}^d, E),\, \phi \in \mathcal{S}(\mathbb{R}^d).$$

Example 2.2.5. *1. For $1 \leq p \leq \infty$ any p-integrable E-valued function f can be identified with a E-valued temperate distribution as usual:*

$$\langle f, \phi \rangle = \int_{\mathbb{R}^d} f(t)\overline{\phi(t)} dt, \qquad g \in \mathcal{S}(\mathbb{R}^d).$$

Notice that this is a further meaning for the brackets $\langle \cdot, \cdot \rangle$.

2. Recall that a vector measure μ on $\mathbb{R}^d$ is a E-valued σ-additive map on the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^d}$ of $\mathbb{R}^d$ satisfying $\mu(\emptyset) = 0$. The total variation $|\mu| : \mathcal{B}_{\mathbb{R}^d} \to [0, +\infty]$ of a vector measure μ is defined by

$$|\mu|(B) := \sup_{\pi(B)} \sum_{A\in\pi(B)} |\mu(A)|_E,$$

where the supremum is taken over all the partitions $\pi(B)$ of B into a finite number of pairwise disjoint Borel subsets. The space $\mathcal{M}(\mathbb{R}^d, E)$ of all the E-valued measures on $\mathbb{R}^d$ such that $|\mu|(\mathbb{R}^d) < \infty$ (bounded variation) is provided *with the norm* $\|\mu\|_{\mathcal{M}} \coloneqq |\mu|(\mathbb{R}^d) < \infty$. *We have that* $\mathcal{M}(\mathbb{R}^d, E) \subset \mathcal{S}'(\mathbb{R}^d, E)$ *(cf. [Ama19, Appendix 2]):*

$$\langle \mu, \phi \rangle = \int_{\mathbb{R}^d} \overline{\phi(t)} d\mu(t), \quad \phi \in \mathcal{S}(\mathbb{R}^d).$$

We will use the following results concerning approximation of distributions and extension of functionals, which are slight modifications of [Ama19, Theorems 1.3.3 and 1.7.2] - cf. [Wah07].

Theorem 2.2.6. 1. *Fix* $u \in \mathcal{S}'(\mathbb{R}^d, E)$. *There exists a sequence* $u_n \in \mathcal{S}(\mathbb{R}^d, E)$ *such that* $u_n \to u$ *in* $\mathcal{S}'(\mathbb{R}^d, E)$.

2. *Let* $\bullet : E_1 \times E_2 \to E_3$ *a multiplication on Banach spaces and consider the map* $F : \mathcal{S}(\mathbb{R}^d, E_1) \times \mathcal{S}(\mathbb{R}^d, E_2) \to E_3$ *defined by* $F(u, v) = \int_{\mathbb{R}^d} u(x) \bullet v(x) dx$. *There exists a unique continuous bilinear extension of* F *to* $\mathcal{S}'(\mathbb{R}^d, E_1) \times \mathcal{S}(\mathbb{R}^d, E_2)$.

We restrict now to the case $E = \mathbb{C}$ for future reference. Given a normed linear space of distributions $X \subset \mathcal{S}'(\mathbb{R}^d)$, we set

$$X_{\text{comp}} \coloneqq \{u \in X \,:\, \text{supp}(u) \text{ is a compact subset of } \mathbb{R}^d\},$$

$$X_{\text{loc}} \coloneqq \{u \in \mathcal{S}'(\mathbb{R}^d) \,:\, \phi u \in X \ \forall \phi \in C_c^\infty(\mathbb{R}^d)\}.$$

2.2.4 Basic operations on functions and distributions

Consider a function $f : \mathbb{R}^d \to E$ and let $x, \xi \in \mathbb{R}^d$. The translation and modulation operators T_x and M_ξ are respectively defined by

$$T_x f(t) \coloneqq f(t - x), \quad M_\xi f(t) \coloneqq e^{2\pi i t \cdot \xi} f(t), \quad t \in \mathbb{R}^d.$$

Note that such operators do not commute unless $x \cdot \xi \in \mathbb{Z}$, since

$$T_x M_\xi f(t) = e^{-2\pi i x \cdot \xi} M_\xi T_x f(t).$$

Composition of translation and modulation operators are of primary relevance in time-frequency analysis. We fix the ordering once for all and define the *time-frequency shift operator* along $z = (x, \xi) \in \mathbb{R}^{2d}$ as $\pi(z) \coloneqq M_\xi T_x$.

The reflection operator $\mathcal{I}$ acts on a function $f : \mathbb{R}^d \to E$ as $\mathcal{I} f(t) \coloneqq f(-t)$, $t \in \mathbb{R}^d$. We usually write $f^\vee$ in place of $\mathcal{I} f$ for the sake of readability. We also define the involution f^* of f as $f^*(t) \coloneqq \overline{f(-t)}$, $t \in \mathbb{R}^d$.

Recall that the *tensor product* of two functions $f, g : \mathbb{R}^d \to \mathbb{C}$ is defined by

$$f \otimes g : \mathbb{R}^{2d} \to \mathbb{C} \,:\, (x, y) \mapsto (f \otimes g)(x, y) = f(x)g(y).$$

The tensor product $\otimes$ maps $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^{2d})$. The tensor product of two temperate distributions $f, g \in \mathcal{S}'(\mathbb{R}^d)$ is the distribution $f \otimes g \in \mathcal{S}'(\mathbb{R}^{2d})$ acting on any $\Phi \in \mathcal{S}(\mathbb{R}^{2d}_{(x,y)})$ as

$$\langle f \otimes g, \Phi\rangle = \langle f, \overline{\langle g, \Phi(x, y)\rangle_y}\rangle_x,$$

meaning that g acts on the section $\Phi(x, \cdot)$ and then f acts on $\langle g, \Phi(x, \cdot)\rangle \in \mathcal{S}(\mathbb{R}^d_x)$. In particular, $f \otimes g$ is the unique distribution such that

$$\langle f \otimes g, \phi_1 \otimes \phi_2\rangle = \langle f, \phi_1\rangle\langle g, \phi_2\rangle, \quad \forall \phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d).$$

In conclusion, recall that the complex conjugate of a temperate distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ is denoted by $\overline{f} \in \mathcal{S}'(\mathbb{R}^d)$ and defined by the rule

$$\langle \overline{f}, \phi\rangle = \overline{\langle f, \overline{\phi}\rangle}, \qquad \phi \in \mathcal{S}(\mathbb{R}^d).$$

Given a matrix $A \in \mathbb{R}^{d\times d}$, the *dilation operator* D_A acts on a function $f : \mathbb{R}^d \to E$ as $D_A f(t) \coloneqq f(At)$. If $A = A_\lambda = \lambda I$ for some $\lambda \in \mathbb{R}$ we write D_λ for D_{A_λ}. In Section 5.3 we will use the notation $\mathfrak{T}_A$ for D_A as it is customary in that context. We also introduce the unitary dilation $U_A f(t) \coloneqq |\det A|^{1/2} f(At)$; if $A = A_\lambda = \lambda I$ for some $\lambda \in \mathbb{R}$ we write U_λ for U_{A_λ}.

2.3 The Fourier transform

The Fourier transform can be initially defined by a Bochner integral for $f \in L^1(\mathbb{R}^d, E)$. Precisely, it is the operator $\mathcal{F} : L^1(\mathbb{R}^d, E) \to L^\infty(\mathbb{R}^d, E)$ defined by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) \coloneqq \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)dx, \quad \xi \in \mathbb{R}^d.$$

As a rule of thumb, the results which holds in the L^1 scalar-valued case generally extend to the vector-valued case, but this principle does not hold in the L^2 framework. For instance, we have the Riemann-Lebesgue lemma (that is $\widehat{f} \in C_0(\mathbb{R}^d, E)$ and $\|\widehat{f}\|_{L^\infty(E)} \le \|f\|_{L^1(E)}$) and the inversion formula, namely if $f, \hat{f} \in L^1(\mathbb{R}^d, E)$ then

$$f(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \hat{f}(\xi)d\xi, \quad \text{for a.e. } x \in \mathbb{R}^d,$$

but the Hausdorff-Young inequality does not hold in general [HNVW16]. In particular, it is a deep result by Kwapień [Kwa72] that the Plancherel theorem holds only for Hilbert spaces. To be precise, $\mathcal{F} : L^1(\mathbb{R}^d, E) \cap L^2(\mathbb{R}^d, E) \to L^2(\mathbb{R}^d, E)$ extends to a unitary operator on $L^2(\mathbb{R}^d, E)$ if and only if E is isomorphic to a Hilbert space, namely $E \simeq H$. In that case we have $\|\hat{f}\|_{L^2(H)} = \|f\|_{L^2(H)}$.

The restriction of $\mathcal{F}$ to $\mathcal{S}(\mathbb{R}^d, E)$ yields a continuous automorphism that enjoys the usual properties, in particular the inversion formula $\mathcal{F}^{-1} = \mathcal{I}\mathcal{F}$. Moreover, the Fourier transform extends by duality to an isomorphism on $\mathcal{S}'(\mathbb{R}^d, E)$ as follows:

$$\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle, \qquad f \in \mathcal{S}'(\mathbb{R}^d, E),\, g \in \mathcal{S}(\mathbb{R}^d).$$

For future reference we also define the (Bochner-)Fourier-Lebesgue spaces $\mathcal{F}L^q_s(\mathbb{R}^d, E)$, where $1 \le q \le \infty$ and $s \in \mathbb{R}$, consisting of distributions $f \in \mathcal{S}'(\mathbb{R}^d, E)$ such that

$$\|f\|_{\mathcal{F}L^q_s(E)} := \left\|\mathcal{F}^{-1} f\right\|_{L^q_s(E)} < \infty.$$

The following Bernstein-type lemma can be proved just as in the scalar-valued case, cf. [WHHG11, Proposition 1.11].

Lemma 2.3.1. *Let $N > d/2$ be an integer and $\partial_j^k f \in L^2(\mathbb{R}^d, H)$ for any $j = 1, \dots, d$ and $0 \le k \le N$. Then*

$$\|f\|_{\mathcal{F}L^1(H)} \lesssim \|f\|_{L^2(H)}^{1-d/2N} \left(\sum_{j=1}^{d} \left\|\partial_j^N f\right\|_{L^2(H)} \right)^{d/2N}. \tag{2.2}$$

The *symplectic Fourier transform* $\mathcal{F}_\sigma$ of a function $F \in L^1(\mathbb{R}^{2d}, E)$ is defined by

$$\mathcal{F}_\sigma F(x, \xi) := \mathcal{F}F(J(x, \xi)) = \mathcal{F}F(\xi, -x), \quad (x, \xi) \in \mathbb{R}^{2d}.$$

Note that this is an involution, that is $\mathcal{F}_\sigma(\mathcal{F}_\sigma F) = F$.

2.3.1 Convolution and Fourier multipliers

The convolution of vector-valued functions can be meaningfully defined as long as the target spaces are provided with a multiplication structure. We consider some easy examples and refer to [Ama19; HNVW16] for extensive accounts on the topic.

The convolution of two functions $f \in L^p(\mathbb{R}^d, E)$, $1 \le p \le \infty$, and $g \in L^1(\mathbb{R}^d)$, is defined by

$$f * g(x) := \int_{\mathbb{R}^d} f(x - y) g(y) dy.$$

It is a well-defined Bochner integral for a.e. $x \in \mathbb{R}^d$ and satisfies the Young inequality $\|f * g\|_{L^p(E)} \leq \|f\|_{L^p(E)}\|g\|_{L^1}$.

The convolution of $f \in \mathcal{S}'(\mathbb{R}^d, E)$ with a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$ is defined by the distribution $f * g \in \mathcal{S}'(\mathbb{R}^d, E)$ such that

$$\langle f * g, \phi \rangle = \langle f, g^* * \phi \rangle, \qquad \forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

In fact, $f * g \in C^\infty(\mathbb{R}^d, E)$ is a function of polynomial growth together with all its derivatives.

In general, the $\bullet$-convolution $f_1 *_\bullet f_2$ of $f_1 \in \mathcal{S}(\mathbb{R}^d, E_1)$ and $f_2 \in \mathcal{S}'(\mathbb{R}^d, E_2)$ can be similarly defined by a smooth E_3-valued function for any multiplication $\bullet : E_1 \times E_2 \to E_3$ [Ama19, Theorem 1.9.1]. We state some results that will be used below; the proofs of more general versions of these facts can be found in [Ama19, Sec. 1.9] and [Ker83].

Proposition 2.3.2. *(i) (Young inequality) Let $1 \leq p, q, r \leq \infty$ satisfy $1/p + 1/q = 1 + 1/r$ and $s_1, s_2, s_3 \in \mathbb{R}$ satisfy*

$$s_1 + s_3 \geq 0, \quad s_2 + s_3 \geq 0, \quad s_1 + s_2 \geq 0.$$

If $f \in L^p_{s_1}(\mathbb{R}^d, E_1)$ and $g \in L^q_{s_2}(\mathbb{R}^d, E_2)$, then $f *_\bullet g \in L^r_{-s_3}(\mathbb{R}^d, E_3)$, with

$$\|f *_\bullet g\|_{L^r_{-s_3}(E_3)} \lesssim \|f\|_{L^p_{s_1}(E_1)}\|g\|_{L^q_{s_2}(E_2)}.$$

(ii) For any $f \in \mathcal{S}'(\mathbb{R}^d, E_1)$ and $g \in \mathcal{S}(\mathbb{R}^d, E_2)$:

$$\mathcal{F}(f *_\bullet g) = \hat{f} \bullet \hat{g}.$$

In the standard scalar-valued setting we have the following version of Young inequality for mixed-norm spaces, cf. [BP61].

Proposition 2.3.3. *Let $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ be a v-moderate weight on $\mathbb{R}^{2d}$, and $1 \leq p_i, q_i, p, q \leq \infty$, $i = 1, 2$. If $F \in L^{p_1,q_1}_v(\mathbb{R}^{2d})$ and $G \in L^{p_2,q_2}_m(\mathbb{R}^{2d})$ then $F * G \in L^{p,q}_m(\mathbb{R}^{2d})$, with $1/p_1 + 1/p_2 = 1 + 1/p$, $1/q_1 + 1/q_2 = 1 + 1/q$ and*

$$\|F * G\|_{L^{p,q}_m} \leq \|F\|_{L^{p_1,q_1}_v}\|G\|_{L^{p_2,q_2}_m}. \tag{2.3}$$

We also introduce for future convenience the *twisted convolution* [Grö01] of $F, G \in L^1(\mathbb{R}^{2d})$ to be

$$(F \natural G)(x, \xi) := \int_{\mathbb{R}^{2d}} e^{\pi i (x,\xi)\cdot J(x',\xi')} F(x', \xi') G(x - x', \xi - \xi')\, dx' d\xi'. \tag{2.4}$$

We finally define the *Fourier multiplier* with symbol $\mu \in \mathcal{S}'(\mathbb{R}^d, E_1)$ to be the linear map

$$\mu(D)f := \mathcal{F}^{-1}(\mu \bullet \hat{f}) = \mathcal{F}^{-1}\mu *_\bullet f \in \mathcal{S}'(\mathbb{R}^d, E_3),$$

the domain consisting of all $f \in \mathcal{S}'(\mathbb{R}^d, E_2)$ such that the latter convolution is well defined.

Chapter 3

Preliminaries of Time-Frequency Analysis

In this chapter we provide an exposition of the main results of time-frequency analysis of functions and distributions. We do not always provide pointers to the literature for each result, not even proofs and heuristic comments, since our presentation is largely inspired to the reference monographs [BO20; CR20; Grö01] and the papers [Tof04a; Tof04b; Wah07]. The reader is invited to consult these references for further details.

3.1 Time-frequency representations

3.1.1 The short-time Fourier transform

The first example of a phase-space representation of a signal $f \in L^2(\mathbb{R}^d)$ is provided by the *short-time Fourier transform* (STFT), also known as windowed/sliding Fourier transform or Gabor transform. As already mentioned in the introductory Chapter 1, it does ultimately amount to a decomposition of f along the uniform boxes in phase space occupied by the Gabor atoms $\pi(z)g$, $z \in \mathbb{R}^{2d}$, for some fixed window function $g \in L^2(\mathbb{R}^d) \setminus \{0\}$. Precisely, it is defined by

$$V_g f(x,\xi) \coloneqq \langle f, \pi(x,\xi) g \rangle = \int_{\mathbb{R}^d} e^{-2\pi i y \cdot \xi} f(y) \, \overline{g(y-x)} \, dy, \quad (x,\xi) \in \mathbb{R}^{2d}.$$

Recall the following equivalent representations:

$$V_g f(x,\xi) = \mathcal{F}(f \cdot \overline{T_x g})(\xi) = e^{-2\pi i x \cdot \xi} (f * M_\xi g^*)(x). \tag{3.1}$$

Note that the while we assumed $g \neq 0$ in the definition of the STFT, that is natural in view of the related motivational discussions, is not essential from the mathematical point of view. The case $g = 0$ is clearly of little significance, but still admissible, and the same holds for most of the properties given below. For this reason we usually do not explicitly require the condition $g \neq 0$ to hold, unless strictly necessary.

We stress that if $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, and $g \in L^{p'}(\mathbb{R}^d)$, then $V_g f$ is defined pointwise on $\mathbb{R}^{2d}$. In general, $V_g f$ is defined pointwise by duality if $f \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$, and also if $f \in X'$ and $g \in X$ for any Banach space X such that $\mathcal{S}(\mathbb{R}^d) \hookrightarrow X$ with dense inclusion (so that $X' \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$) and such that X is invariant under time-frequency shifts. It can also be defined by a temperate distribution in the case where $f, g \in \mathcal{S}'(\mathbb{R}^d)$, see Section 5.3 below. Note that the definition also extends to the infinite-dimensional case for $f \in L^p(\mathbb{R}^d, E)$ and $g \in L^{p'}(\mathbb{R}^d)$ or $f \in \mathcal{S}'(\mathbb{R}^d, E)$ and $g \in \mathcal{S}(\mathbb{R}^d)$.

In any case where the STFT is defined in some sense, notice that the map $(f, g) \mapsto V_g f$ is a sesquilinear mapping - namely linear in f and conjugate-linear in g. In particular, for a fixed window g, the map $V_g : f \mapsto V_g f$ is linear.

As far as the regularity of the STFT is concerned, we have the following result.

Proposition 3.1.1. *1. If $f \in L^p(\mathbb{R}^d)$, $1 < p < \infty$, and $g \in L^{p'}(\mathbb{R}^d)$ then $V_g f \in C_0(\mathbb{R}^{2d})$ and*

$$\|V_g f\|_\infty \leq \|f\|_{L^p} \|g\|_{L^{p'}}.$$

2. If $f \in \mathcal{S}'(\mathbb{R}^d, E)$ and $g \in \mathcal{S}(\mathbb{R}^d)$ then $V_g f \in C^\infty(\mathbb{R}^{2d})$ and there exist $N \in \mathbb{N}$ and $C > 0$ such that

$$|V_g f(z)|_E \leq C(1 + |z|)^N, \quad z \in \mathbb{R}^{2d}.$$

Moreover $V_g : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^{2d})$ is continuous.

3. Fix $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. If $f \in \mathcal{S}'(\mathbb{R}^d)$ then $f \in \mathcal{S}(\mathbb{R}^d)$ if and only if $V_g f \in \mathcal{S}(\mathbb{R}^{2d})$, or equivalently: for all $N \geq 0$ there exists $C_N > 0$ such that

$$|V_g f(z)| \leq C_N (1 + |z|)^{-N}, \quad z \in \mathbb{R}^{2d}.$$

Moreover $V_g : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^{2d})$ is continuous.

The following fundamental properties of the STFT will be used below.

Proposition 3.1.2. *1. (The fundamental STFT identity)*

$$V_g f(x, \xi) = e^{-2\pi i x \cdot \xi} V_{\widehat{g}} \widehat{f}(\xi, -x),$$

for all $f, g \in L^2$ and $(x, \xi) \in \mathbb{R}^{2d}$.

2. *(Switching identity) If* $f, g \in \mathcal{S}(\mathbb{R}^d)$ *then* $V_g f(x,\xi) = e^{-2\pi i x\cdot\xi}\overline{V_f g(-x,-\xi)}$, $(x,\xi) \in \mathbb{R}^{2d}$.

3. *(Covariance property) Fix* $(u,v) \in \mathbb{R}^{2d}$. *Then*
$$V_g(\pi(u,v)f)(x,\xi) = e^{-2\pi i u\cdot(\xi-v)} V_g f(x-u, \xi-v),$$
for all $f, g \in L^2$ *and* $(x,\xi) \in \mathbb{R}^{2d}$.

4. *(Orthogonality relations) If* $f_1, f_2 \in L^2(H)$ *and* $g_1, g_2 \in L^2$, *then* $V_{g_i} f_i \in L^2(H)$, $i = 1, 2$, *and*
$$\langle V_{g_1} f_1, V_{g_2} f_2\rangle_{L^2(H)} = \langle f_1, f_2\rangle_{L^2(H)} \overline{\langle g_1, g_2\rangle_{L^2}}.$$

5. *(Adjoint STFT) The adjoint STFT with window* g *is the operator* V_g^* *which maps a measurable function* $F : \mathbb{R}^{2d} \to E$ *of at most polynomial growth (i.e.,* $|F(z)|_E = O(|z|^N)$) *for some* $N \in \mathbb{N}$) *into the temperate distribution*
$$V_g^* F = \int_{\mathbb{R}^{2d}} F(z)\pi(z)g\, dz \in \mathcal{S}'(\mathbb{R}^d, E)$$
which satisfies $\langle V_g^* F, \phi\rangle = \langle F, V_g \phi\rangle$, $\phi \in \mathcal{S}(\mathbb{R}^d)$. *If* $F \in \mathcal{S}(\mathbb{R}^{2d})$ *then* $V_g^* F \in \mathcal{S}(\mathbb{R}^d)$.

6. *(Inversion formula) Let* $f \in \mathcal{S}'(\mathbb{R}^d, E)$ *and* $g, \gamma \in \mathcal{S}(\mathbb{R}^d)$ *be such that* $\langle g, \gamma\rangle \neq 0$. *Then,*
$$f = \frac{1}{\langle \gamma, g\rangle} V_\gamma^* V_g f = \frac{1}{\langle \gamma, g\rangle} \int_{\mathbb{R}^{2d}} V_g f(z)\pi(z)\gamma dz. \tag{3.2}$$
In particular, the STFT is injective on $\mathcal{S}'(\mathbb{R}^d, E)$.

7. *(Change-of-window identity) Let* $g, h, \gamma \in \mathcal{S}(\mathbb{R}^d)$ *be such that* $\langle h, \gamma\rangle \neq 0$. *Then, for any* $f \in \mathcal{S}'(\mathbb{R}^d, E)$,
$$|V_g f(z)|_E \leq \frac{1}{|\langle h, \gamma\rangle|}(|V_h f|_E * |V_g \gamma|)(z), \quad z \in \mathbb{R}^{2d}. \tag{3.3}$$

8. *(Tensor product) Let* $f_1, f_2 \in \mathcal{S}'(\mathbb{R}^d)$ *and* $g \in \mathcal{S}(\mathbb{R}^{2d})$ *be such that* $g = g_1 \otimes g_2$ *for some* $g_1, g_2 \in \mathcal{S}(\mathbb{R}^d)$. *Then*
$$V_{g_1\otimes g_2}(f_1 \otimes f_2)(z,\zeta) = V_{g_1} f_1(z_1,\zeta_1) V_{g_2} f_2(z_2,\zeta_2),$$
for any $z = (z_1, z_2) \in \mathbb{R}^{2d}$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$.

It is easy to see that the definition of $V_g^* F$, $g \in \mathcal{S}(\mathbb{R}^d)$, extends naturally to $F \in \mathcal{S}'(\mathbb{R}^{2d})$, and defines continuous mappings $V_g^* : \mathcal{S}'(\mathbb{R}^{2d}) \to \mathcal{S}'(\mathbb{R}^d)$ and $V_g^* : \mathcal{S}(\mathbb{R}^{2d}) \to \mathcal{S}(\mathbb{R}^d)$.

3.1.2 Quadratic representations

It is useful in several settings to produce time-frequency representations that depend quadratically on the signal. For instance, similar functions are interpreted as phase-space energy densities in signal analysis or as quasi-probability distributions in quantum physics.

A standard way to obtain a quadratic representation given a sesquilinear one, say $L(f,g)$, is to consider either $Q(f) = |L(f,g)|^2$ for fixed g or $Q(f) = L(f,f)$ - possibly up to complex factors. In the latter case, for any $\alpha_1, \alpha_2 \in \mathbb{C}$ and suitable functions f_1, f_2 we have

$$Q(\alpha_1 f_1 + \alpha_2 f_2) = |\alpha_1|^2 Q(f_1) + \alpha_1 \overline{\alpha_2} L(f_1, f_2) + \overline{\alpha_1} \alpha_2 L(f_2, f_1) + |\alpha_2|^2 Q(f_2).$$

The non-linear behaviour due to the occurrence of interference cross-terms is distinctive of quadratic representations. It is a major drawback both for applications and theoretical purposes, but it is somehow compensated by other nice features as showed below.

The most common quadratic representations (also called quadratic functions, distributions or transforms) in signal analysis are the *spectrogram* and the *radar ambiguity distribution*, which are derived from the short-time Fourier transform as described above. In particular, the spectrogram of the signal $f \in L^2(\mathbb{R}^d)$ with respect to a window $g \in L^2(\mathbb{R}^d)$ such that $\|g\|_{L^2} = 1$ is defined by $S_g(f)(z) := |V_g f(z)|^2$, $z \in \mathbb{R}^{2d}$, while the ambiguity function of the signal $f \in L^2(\mathbb{R}^d)$ is defined by

$$\mathrm{Amb} f(x,\xi) := \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi} f(x+y/2)\overline{f(x-y/2)}dy, \quad (x,\xi) \in \mathbb{R}^{2d}.$$

In general, the cross-ambiguity function of the signals $f, g \in L^2(\mathbb{R}^d)$ is

$$\mathrm{Amb}(f,g)(x,\xi) := \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi} f(x+y/2)\overline{g(x-y/2)}dy, \quad (x,\xi) \in \mathbb{R}^{2d}.$$

A trivial substitution in the previous definitions reveal that

$$\mathrm{Amb} f(x,\xi) = e^{\pi i x\cdot\xi} V_f f(x,\xi), \quad \mathrm{Amb}(f,g)(x,\xi) = e^{\pi i x\cdot\xi} V_g f(x,\xi).$$

It is then clear that most of the properties of the short-time Fourier transform are inherited by the spectrogram and the ambiguity function, hence we omit an explicit description here; we also mention that further properties are established in Section 5.3 below.

The *Wigner distribution* (WD) of a signal $f \in L^2(\mathbb{R}^d)$ is defined to be

$$Wf(x,\xi) := \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi} f(y+x/2)\overline{f(y-x/2)}dy, \quad (x,\xi) \in \mathbb{R}^{2d},$$

while its polarized version (also known as the cross-Wigner distribution) for signals $f, g \in L^2(\mathbb{R}^d)$ is given by

$$W(f,g)(x,\xi) := \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi} f(y + x/2)\overline{g(y - x/2)}dy, \quad (x,\xi) \in \mathbb{R}^{2d}.$$

It is not difficult to show that the Wigner transform is related to the STFT as follows:

$$W(f,g)(x,\xi) = 2^d e^{4\pi i x\cdot\xi} V_{g^\vee} f(2x, 2\xi), \quad (x,\xi) \in \mathbb{R}^{2d}. \tag{3.4}$$

Moreover, it is related to the ambiguity function by means of the symplectic Fourier transform:

$$W(f,g) = \mathcal{F}_\sigma \mathrm{Amb}(f,g), \quad \mathrm{Amb}(f,g) = \mathcal{F}_\sigma W(f,g).$$

As a result of these connections, most of the properties of the STFT in Propositions 3.1.1 and 3.1.2 extend to the Wigner distribution. Nevertheless, there are some properties which are quite distinctive of this representation; we list below those which are used in the rest of this dissertation.

Proposition 3.1.3. *Let $g, g_1, g_2 \in L^2(\mathbb{R}^d) \setminus \{0\}$ and $f, f_1, f_2 \in L^2(\mathbb{R}^d)$. For any $z = (x,\xi) \in \mathbb{R}^{2d}$:*

1. *We have*

   $$W(f,g) = \mathcal{F}_2 \mathfrak{T}_s (f \otimes \overline{g}),$$

   where $\mathcal{F}_2$ is the partial Fourier transform with respect to the second variable on $\mathbb{R}^{2d}$ and $\mathfrak{T}_s$ acts on $F : \mathbb{R}^{2d} \to \mathbb{C}$ as

   $$\mathfrak{T}_s F(x,y) = F\left(\frac{x+y}{2}, x-y\right).$$

   These operators are automorphisms of $\mathcal{S}(\mathbb{R}^{2d})$, hence extend to $\mathcal{S}'(\mathbb{R}^{2d})$ and allow one to define $W(f,g)$ for $f, g \in \mathcal{S}'(\mathbb{R}^d)$ (see also Chapter 5 below).

2. *(Schwartz regularity) If $f, g \in \mathcal{S}(\mathbb{R}^d)$ then $W(f,g) \in \mathcal{S}(\mathbb{R}^{2d})$. In particular, the correspondence $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \ni (f,g) \mapsto W(f,g) \in \mathcal{S}(\mathbb{R}^{2d})$ is continuous.*

3. *(The fundamental WD identity) $W(f,g)(x,\xi) = W(\widehat{f}, \widehat{g})(\xi, -x)$.*

4. *(Real-valuedness) $W(f,g)(z) = \overline{W(g,f)(z)}$. In particular, $Wf : \mathbb{R}^{2d} \to \mathbb{R}$.*

5. *(Covariance property) Fix $u, v \in \mathbb{R}^{2d}$. Then*

   $$W(\pi(u)f, \pi(v)g)(z) = e^{\pi i (v_1 + u_1)\cdot(v_2 - u_2)} M_{J(u-v)} T_{\frac{u+v}{2}} W(f,g)(z). \tag{3.5}$$

   In particular,

   $$Wf(\pi(u)f)(z) = Wf(z-u).$$

6. *(Moyal's formula)* $W(f_i, g_i) \in L^2(\mathbb{R}^d)$, $i = 1, 2$, *and*

$$\langle W(f_1, g_1), W(f_2, g_2)\rangle = \langle f_1, f_2\rangle \overline{\langle g_1, g_2\rangle}. \tag{3.6}$$

In particular, $\|Wf\|_{L^2} = \|f\|_{L^2}^2$.

7. *(Marginal densities) If* $f, g \in \mathcal{S}(\mathbb{R}^d)$ *then*

$$\int_{\mathbb{R}^d} W(f,g)(x,\xi)d\xi = f(x)\overline{g(x)}, \quad \int_{\mathbb{R}^d} W(f,g)(x,\xi)dx = \widehat{f}(\xi)\overline{\widehat{g}(\xi)}.$$

8. *(Projective identification) If* $Wf = Wg$ *then there exists* $c \in \mathbb{C}$, $|c| = 1$, *such that* $f = cg$.

9. *(The "magic formula" [Grö06a]) Let* $\phi \in \mathcal{S}(\mathbb{R}^d)$ *and set* $\Phi = W\phi \in \mathcal{S}(\mathbb{R}^{2d})$. *For all* $z = (z_1, z_2), \zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$,

$$V_\Phi W(g,f)(z,\zeta) = e^{-2\pi i z_2 \cdot \zeta_2}\overline{V_\phi f(z_1 + \zeta_2/2, z_2 - \zeta_1/2)} V_\phi g(z_1 - \zeta_2/2, z_2 + \zeta_1/2). \tag{3.7}$$

10. *(Hudson's theorem)* $W(f,g)(z) \geq 0$ *for all* $z \in \mathbb{R}^{2d}$ *if and only if* $f = cg$ *for some* $c \geq 0$ *and* g *is a generalized Gaussian function, namely* $g(t) = e^{Q(t)}$ *where* $Q : \mathbb{R}^d \to \mathbb{C}$ *is a quadratic polynomial such that* $\operatorname{Re} Q(x) \to +\infty$ *as* $|x| \to \infty$.

We stress once again that finding a remedy to the lack of positivity of the Wigner distribution is a central issue in signal analysis and mathematical physics. Inspired by heuristic arguments on the uncertainty principle, the following class of phase-space representations were introduced by L. Cohen [Coh95]. The following definition is widely accepted in the mathematical literature.

Definition 3.1.4 ([Grö01])**.** *A sesquilinear form* $Q : \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^{2d})$ *belongs to the Cohen class if there exists* $\theta \in \mathcal{S}'(\mathbb{R}^{2d})$ *(the Cohen kernel) such that*

$$Q(f,g) = Q_\theta(f,g) = W(f,g) * \theta, \qquad \forall f, g \in \mathcal{S}(\mathbb{R}^d).$$

More details on this topic may be found in Chapter 5 below.

We conclude this section with a brief description of the Wigner distribution in the infinite-dimensional setting (in addition to the obvious considerations related to (3.4)). Given $f, g \in L^2(\mathbb{R}^d, H)$, the Wigner distribution $W(f,g)(x,\xi) \in \mathcal{L}(H)$, $x, \xi \in \mathbb{R}^d$, is defined by follows:

$$W(f,g)(x,\xi) = [\mathcal{F}\mathfrak{T}_p P(f,g)(x,\cdot)](\xi),$$

where we introduced the projector-valued function

$$P(f,g) : \mathbb{R}^{2d} \to \mathfrak{S}^1(H), \qquad P(f,g)(x,y) \coloneqq |f(x)\rangle\langle g(y)|,$$

and $\mathfrak{T}_p$ acts on $F : \mathbb{R}^{2d} \to H$ as $\mathfrak{T}_p F(x,y) = F(x+y/2, x-y/2)$. It is therefore clear that $W(f,g) : \mathbb{R}^{2d} \to \mathfrak{S}^1(H)$ and in particular [Fol89; Wah07]

$$(W(f,g)(x,\xi)u,v)_H = \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi}(f(x+y/2),v)_H \overline{(g(x-y/2),u)_H} dy,$$

for any $u,v \in H$. More concisely, we have $(W(f,g)(x,\xi)u,v)_H = W(\widetilde{f_v},\widetilde{g_u})(x,\xi)$, where on the right-hand side we have the ordinary Wigner distribution of the functions $\widetilde{f_v}(t) = (f(t),v)_H$ and $\widetilde{g_u}(t) = (g(t),u)_H$.

The following properties of the Wigner distributions are similar to those listed in Proposition 3.1.3 and can be easily derived in the vector-valued context.

Proposition 3.1.5. *For any $f,g \in \mathcal{S}(\mathbb{R}^d, H)$ and $x,\xi \in \mathbb{R}^d$:*

(i) $W(f,g) \in \mathcal{S}(\mathbb{R}^{2d}, \mathfrak{S}^1(H))$.

(ii) $W(f,g)(x,\xi) = W(\hat{f},\hat{g})(\xi,-x)$.

(iii) $\int_{\mathbb{R}^d} W(f,g)(x,\xi)dx = |\hat{f}(\xi)\rangle\langle\hat{g}(\xi)|$.

(iv) $\int_{\mathbb{R}^d} W(f,g)(x,\xi)d\xi = |f(x)\rangle\langle g(x)|$.

The Wigner transform can be extended to $f,g \in \mathcal{S}'(\mathbb{R}^d, H)$ as follows [Wah07]. Let $\Phi = W(\phi_1,\phi_2)$ for $\phi_1,\phi_2 \in \mathcal{S}(\mathbb{R}^d)$; then $W(f,g) \in \mathcal{S}'(\mathbb{R}^{2d}, \mathfrak{S}^1(H))$ is the distribution satisfying

$$(\langle W(f,g),\Phi\rangle u,v)_H = (\langle f,\phi_1\rangle,v)_H \overline{(\langle g,\phi_2\rangle,u)_H}, \quad u,v \in H.$$

The following result is crucial for the results on pseudodifferential operators. It is the parallel of the magic formula (3.7).

Proposition 3.1.6 ([Wah07, Lemma 4.5])**.** *Let $f,g \in \mathcal{S}'(\mathbb{R}^d, H)$, $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $\Phi = W\phi \in \mathcal{S}(\mathbb{R}^{2d})$. For all $z = (z_1,z_2), \zeta = (\zeta_1,\zeta_2) \in \mathbb{R}^{2d}$,*

$$\|V_\Phi W(g,f)(z,\zeta)\|_{\mathfrak{S}^1(H)} = \|V_\phi f(z_1+\zeta_2/2, z_2-\zeta_1/2)\|_H \|V_\phi g(z_1-\zeta_2/2, z_2+\zeta_1/2)\|_H. \tag{3.8}$$

3.2 Modulation spaces

Modulation spaces were introduced by H. Feichtinger in the early 1980s [Fei03; Fei81]. According to the inventor [Fei06], the original motivation in defining modulation spaces is rooted in the theory of harmonic analysis over locally compact Abelian groups; in particular, the need of a whole family of Banach spaces closed under duality and complex interpolation lead to the definition of modulation spaces in terms of uniform decompositions on the spectral side of their members. Therefore, in the first instance they can be thought of as Besov spaces with isometric boxes in the frequency domain instead of dyadic annuli. A much more insightful definition is given in terms of the global decay/summability of the phase-space concentration of a function or a distribution; this is in fact the so-called *coorbit representation* of modulation spaces, which falls in the perspective of the general coorbit theory developed by Feichtinger and Gröchenig around 1990 [FG88; FG89a; FG89b].

Definition 3.2.1. *Let $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ be a moderate weight and $1 \le p, q \le \infty$, and fix $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. The modulation space $M^{p,q}_m(\mathbb{R}^d)$ is the set of all temperate distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that*

$$\|f\|_{M^{p,q}_m} := \|V_g f\|_{L^{p,q}_m} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\xi)|^p m(x,\xi)^p dx \right)^{q/p} d\xi \right)^{1/q} < \infty, \quad (3.9)$$

with suitable modifications in the cases $p = \infty$ or $q = \infty$. If $p = q$ we write $M^p_m(\mathbb{R}^d)$; we omit the subscript for the weight if $m = 1$ and write $M^p(\mathbb{R}^d)$. If $m(x,\xi) = v_r(x)v_s(\xi)$, $r, s \in \mathbb{R}$, we write $M^{p,q}_{r,s}(\mathbb{R}^d)$.

Remark 3.2.2. *1. The name "modulation space" comes from noticing that $|V_g f(x,\xi)| = |f * M_\xi g^*(x)|$. The test function g is thus deformed by means of modulation, while in context of Besov spaces one is ultimately concerned with the Lebesgue regularity of $f * D_\xi g$, where $D_\xi g$ is a suitable dilation. Such characterizations were used by Peetre and Triebel and lead to develop the theory of modulation spaces by paralleling that of the Besov spaces, cf. [Pee76; Tri78].*

2. *The definition of modulation spaces can be given in more general settings. A definition of $M^{p,q}$ in the case $0 < p, q \le \infty$ (quasi-Banach setting) was provided by Galperin and Samarah in [GS04], while mixed modulation spaces were considered in [CN19].*

3. *Recall that we always restrict to weights of temperate growth in order to frame the theory of modulation spaces in the context of temperate distributions. More general weights can be taken into account, at the price of enlarging the universe space to ultra-distributions [Teo06].*

We collect in the following result the main properties of modulation spaces that will be used throughout the dissertation.

Proposition 3.2.3. *Let $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ be a moderate weight.*

1. *(Banach space property) For any $1 \le p,q \le \infty$, the modulation space $M_m^{p,q}(\mathbb{R}^d)$ is a Banach space with the norm (3.9) which is invariant under time-frequency shifts. The definition does not depend on the window function g in (3.9), in the sense that $\|V_\phi f\|_{L^{p,q}_m}$ is an equivalent norm for $M^{p,q}_m$ for any choice of $\phi \in M^1_v(\mathbb{R}^d)$.*

2. *(Density of $\mathcal{S}$) The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is a subset of $M^{p,q}_m(\mathbb{R}^d)$ for any $1 \le p,q \le \infty$, in particular a dense subset if $p,q \neq \infty$. Moreover, the following characterization holds for any $1 \le p,q \le \infty$:*

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{s\ge 0} M^{p,q}_{v_s}(\mathbb{R}^d), \quad \mathcal{S}'(\mathbb{R}^d) = \bigcup_{s\ge 0} M^{p,q}_{v_s}(\mathbb{R}^d). \tag{3.10}$$

3. *(Reconstruction formula) If $g,\gamma \in M^1_v(\mathbb{R}^d)\setminus\{0\}$, then $V_g : M^{p,q}_m(\mathbb{R}^d) \to L^{p,q}_m(\mathbb{R}^{2d})$ and $V^*_\gamma : L^{p,q}_m(\mathbb{R}^d) \to M^{p,q}_m(\mathbb{R}^{2d})$ are bounded operators. Moreover, if $\langle g,\gamma\rangle \neq 0$ then the inversion formula (3.2) holds for any $f \in M^{p,q}_m(\mathbb{R}^d)$; in short, $\mathrm{Id}_{M^{p,q}_m(\mathbb{R}^d)} = \langle \gamma, g\rangle^{-1} V^*_\gamma V_g$.*

4. *(Duality) If $1 \le p,q < \infty$ then $(M^{p,q}_m(\mathbb{R}^d))' \simeq M^{p',q'}_{1/m}(\mathbb{R}^d)$ and the duality pairing is given by*

$$\langle f,h\rangle = \int_{\mathbb{R}^{2d}} V_g f(z)\overline{V_g h(z)}dz,$$

for $f \in M^{p,q}_m(\mathbb{R}^d)$, $h \in M^{p',q'}_{1/m}(\mathbb{R}^d)$ and $g \in M^1_v(\mathbb{R}^d)\setminus\{0\}$. As a consequence, for any $1 < p,q \le \infty$,

$$\|f\|_{M^{p,q}_m} = \sup_{h\in M^{p',q'}_{1/m}} |\langle f,h\rangle|.$$

5. *(Inclusions) If $p_1 \le p_2$, $q_1 \le q_2$ and $m_1 \gtrsim m_2$ then $M^{p_1,q_1}_{m_1}(\mathbb{R}^d) \subset M^{p_2,q_2}_{m_2}(\mathbb{R}^d)$. In particular, for any $1 \le p,q \le \infty$,*

$$M^1_v(\mathbb{R}^d) \subset M^{p,q}_m(\mathbb{R}^d) \subset M^\infty_{1/v}(\mathbb{R}^d).$$

6. *(Local properties) For any $1 \le p,q \le \infty$,*

$$(M^{p,q}(\mathbb{R}^d))_{\mathrm{loc}} = (\mathcal{F}L^q(\mathbb{R}^d))_{\mathrm{loc}}, \quad (M^{p,q}(\mathbb{R}^d))_{\mathrm{comp}} = (\mathcal{F}L^q(\mathbb{R}^d))_{\mathrm{comp}}.$$

7. *(Complex interpolation) Let* $0 < \theta < 1$, $1 \le p_1, p_2, q_1, q_2 \le \infty$ *and* $m_1, m_2 \in \mathcal{M}_v(\mathbb{R}^{2d})$. *Then* $[M^{p_1,q_1}_{m_1}(\mathbb{R}^d), M^{p_2,q_2}_{m_2}(\mathbb{R}^d)]_{[\theta]} = M^{p,q}_m(\mathbb{R}^d)$, *where*
$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad m = m_1^{1-\theta} m_2^{\theta}.$$

The boundedness of some operations on modulation spaces is established in the following result.

Proposition 3.2.4. 1. *(Convolution) Let* $1 \le p, q, p_1, q_1, p_2, q_2 \le \infty$. *Then*
$$\|f * g\|_{M^{p,q}} \lesssim \|f\|_{M^{p_1,q_1}} \|g\|_{M^{p_2,q_2}}$$
if and only if
$$\frac{1}{p_1} + \frac{1}{p_2} \ge 1 + \frac{1}{p}, \quad \frac{1}{q_1} + \frac{1}{q_2} \ge \frac{1}{q}.$$

2. *(Multiplication) Let* $1 \le p, q, p_1, q_1, p_2, q_2 \le \infty$. *Then*
$$\|f \cdot g\|_{M^{p,q}} \lesssim \|f\|_{M^{p_1,q_1}} \|g\|_{M^{p_2,q_2}}$$
if and only if
$$\frac{1}{p_1} + \frac{1}{p_2} \ge \frac{1}{p}, \quad \frac{1}{q_1} + \frac{1}{q_2} \ge 1 + \frac{1}{q}.$$

3. *(Dilation) Let* $A \in \mathrm{GL}(d, \mathbb{R})$ *and* $1 \le p, q \le \infty$. *For any* $f \in M^{p,q}(\mathbb{R}^d)$,
$$\|D_A f\|_{M^{p,q}} \lesssim C_{p,q}(A) \|f\|_{M^{p,q}},$$
where $C_{p,q}(A) = |\det A|^{-(1/p+1/q')} (\det(I + A^\top A))^{1/2}$.

4. *(Tensor product) Let* $m_i \in \mathcal{M}_{v_i}(\mathbb{R}^{2d})$ *and* $f_i \in M^{p,q}_{m_i}(\mathbb{R}^d)$, $i = 1, 2$. *Then*
$$\|f_1 \otimes f_2\|_{M^{p,q}_m} \lesssim \|f_1\|_{M^{p,q}_{m_1}} \|f_2\|_{M^{p,q}_{m_2}}, \tag{3.11}$$
where $m(z, \zeta) \coloneqq m_1(z_1, \zeta_1) m_2(z_2, \zeta_2)$ *for* $z = (z_1, z_2) \in \mathbb{R}^{2d}$ *and* $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$.

In conclusion, we mention that many common function spaces are embedded in modulation spaces.

Proposition 3.2.5. *(i) If* $m \in \mathcal{M}_v(\mathbb{R}^d)$, *then* $M^2_{m\otimes 1}(\mathbb{R}^d)$ *coincides with* $L^2_m(\mathbb{R}^d)$, *while* $M^2_{1\otimes m(\mathbb{R}^d)}$ *coincides with* $\mathcal{F}L^2_m(\mathbb{R}^d)$. *In particular,* $M^2_{0,s}(\mathbb{R}^d)$ *coincides with the usual* L^2*-based Sobolev space* $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$.

(ii) The following continuous embeddings with Lebesgue spaces hold:
$$M^{p,q_1}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d) \subset M^{p,q_2}(\mathbb{R}^d), \quad q_1 \le \min\{p, p'\}, \quad q_2 \ge \max\{p, p'\}.$$
Similarly,
$$M^{q_1,p}(\mathbb{R}^d) \subset \mathcal{F}L^p(\mathbb{R}^d) \subset M^{q_2,q}(\mathbb{R}^d), \quad q_1 \le \min\{2, p'\}, \quad q_2 \ge \max\{2, p'\}.$$

3.2.1 Vector-valued modulation spaces

The generalization of modulation spaces in the infinite-dimensional setting is quite natural.

Definition 3.2.6. *Let $1 \leq p, q \leq \infty$ and $m \in \mathcal{M}_v(\mathbb{R}^{2d})$. The E-valued modulation space $M_m^{p,q}(\mathbb{R}^d, E)$ consists of distributions $f \in \mathcal{S}'(\mathbb{R}^d, E)$ such that*

$$\|f\|_{M_m^{p,q}(E)} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\xi)|_E^p m(x,\xi)^p dx \right)^{q/p} d\xi \right)^{1/q} < \infty, \tag{3.12}$$

for some $g \in \mathcal{S}(\mathbb{R}^d)$, with suitable modification for $p = \infty$ or $q = \infty$.

We will mainly focus on the case where $m = v_r \otimes v_s$ for some $r, s \in \mathbb{R}$. In this case we write $M_{r,s}^{p,q}(\mathbb{R}^d, E)$. Furthermore, when there is no risk of confusion we write $M_m^{p,q}$ in the ordinary setting $E = \mathbb{C}$ and $M_m^{p,q}(E)$ in general.

Most of the ordinary theory extends to the vector-valued context by simply substituting $|\cdot|$ with $|\cdot|_E$ in the proofs. For our purposes, it is enough to mention the following properties.

Proposition 3.2.7. *Let $1 \leq p, q \leq \infty$ and $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ be a moderate weight.*

(i) *$M_m^{p,q}(E)$ is a Banach space with the norm (3.12), which is independent of the window function $g \in M_v^1$ (i.e., different windows yield equivalent norms).*

(ii) *If $p, q < \infty$ the Schwartz class $\mathcal{S}(\mathbb{R}^d, E)$ is dense in $M_m^{p,q}(E)$.*

(iii) *If $p_1 \leq p_2$, $q_1 \leq q_2$ and $m_2 \gtrsim m_1$, then $M_{m_1}^{p_1,q_1}(E) \subset M_{m_2}^{p_2,q_2}(E)$.*

(iv) *If $f_1 \in \mathcal{S}(\mathbb{R}^d, E')$, $f_2 \in \mathcal{S}(\mathbb{R}^d, E)$ and $g_1, g_2 \in \mathcal{S}(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^{2d}} \langle V_{g_1} f_1(z), V_{g_2} f_2(z) \rangle dz = \overline{\langle g_1, g_2 \rangle} \int_{\mathbb{R}^d} \langle f_1(y), f_2(y) \rangle dy.$$

(v) *If $g, \gamma \in M_v^1(\mathbb{R}^d) \setminus \{0\}$, then $V_g : M_m^{p,q}(\mathbb{R}^d, E) \to L_m^{p,q}(\mathbb{R}^{2d}, E)$ and $V_\gamma^* : L_m^{p,q}(\mathbb{R}^{2d}, E) \to M_m^{p,q}(\mathbb{R}^d, E)$ are bounded operators. Moreover, if $\langle g, \gamma \rangle \neq 0$ then the inversion formula (3.2) holds for any $f \in M_m^{p,q}(E)$; in short, $\mathrm{Id}_{M_m^{p,q}(E)} = \langle \gamma, g \rangle^{-1} V_\gamma^* V_g$.*

(vi) *If $E = \mathbb{C}^{a \times b}$, then $f \in M_m^{p,q}(\mathbb{C}^{a \times b})$ if and only if $f_{ij} \in M_m^{p,q}$ for any $i = 1, \dots, a$, $j = 1, \dots, b$.*

Remark 3.2.8. *In contrast to the aforementioned properties, duality is a quite subtle question (cf. [Wah07]). In order to avoid related issues, which usually occur when $p, q \in \{1, \infty\}$, it is convenient to introduce the space $\mathcal{M}^{p,q}_m(E)$, namely the closure of $\mathcal{S}(E)$ with respect to the $M^{p,q}_m$ norm. In particular we have $\mathcal{M}^{p,q}_m(E) = M^{p,q}_m(E)$ for $1 \le p, q < \infty$.*

3.2.2 Modulation spaces as Wiener amalgams on the Fourier side

Let us consider the action of the Fourier transform on $f \in M^{p,q}_m$, where $1 \le p, q \le \infty$ and $m \in \mathcal{M}_v(\mathbb{R}^{2d})$. Recall from Proposition 3.1.2 that, for $g \in \mathcal{S}$,

$$|V_g f(z)| = |V_{\hat{g}} \hat{f}(Jz)|,$$

hence

$$\|f\|_{M^{p,q}_m} = \left(\int_{\mathbb{R}^d} \left(|V_{\hat{g}} \hat{f}(x,\xi)|^p m(-\xi, x)^p d\xi \right)^{q/p} dx \right)^{1/q}.$$

Therefore, the space $\mathcal{F}M^{p,q}_m$ comes with a Banach norm given by

$$\|h\|_{\mathcal{F}M^{p,q}_m} = \left(\int_{\mathbb{R}^d} (|V_g h(x,\xi)|^p m(\xi, -x)^p d\xi)^{q/p} dx \right)^{1/q}.$$

We see that this norm is similar to that of the corresponding modulation space but with reversed order of integration over time and frequency and a swap of variables in the weight function.

We define in general the space $W^{p,q}_m(\mathbb{R}^d, E)$ as $W^{p,q}_m(\mathbb{R}^d, E) \coloneqq \mathcal{F}M^{p,q}_{D_J m}(\mathbb{R}^d, E)$. The related notation is derived from that of modulation spaces, in particular we write $W^{p,q}_m$ when $E = \mathbb{C}$, $W^{p,q}_{r,s}(E)$ in the case where $m = u \otimes w$, $u = v_r$ and $w = v_s$ for some $r, s \in \mathbb{R}$, and $W^p(E)$ for $p = q$ and $m = 1$.

It is worth mentioning that if $m = u \otimes w$, where u, v are (even) moderate weights on $\mathbb{R}^d$, then for $f \in M^{p,q}_{u \otimes v}(E)$ we have

$$\|\hat{f}\|_{W^{p,q}_{u\otimes v}(E)} = \| \| \hat{f} \cdot \overline{T_x \hat{g}}(\xi) \|_{\mathcal{F}L^p_u(\mathbb{R}^d_\xi, E)} \|_{L^q_w(\mathbb{R}^d_x)} = \|f\|_{M^{p,q}_{u\otimes v}(E)}.$$

Note that the norm of this space is defined by imposing a global L^q-regularity condition on a "sliding" local $\mathcal{F}L^p$-regularity measure. This structure in fact characterizes the norm of Wiener amalgam spaces, where global and local features of functions are "amalgamated". The seminal papers on the topic date back to the work of N. Wiener [Wie26; Wie32; Wie88]; for a systematic review of the subject we address to the references [BS81; Fei90; FS85; Hei03; Hol75].

The formulation of amalgam spaces can be given in greater generality, we restrict to those spaces appearing in the dissertation.

Definition 3.2.9. *Let* $(B, \|\cdot\|_B)$ *any of the spaces* $L^p_u(\mathbb{R}^d, E)$, $\mathcal{F}L^p_u(\mathbb{R}^d, E)$ *or* $L^{p,q}_m(\mathbb{R}^{2d}, E)$ *for some* $1 \le p, q \le \infty$ *and moderate weights* u, m. *Let* $(C, \|\cdot\|_C)$ *be any of the spaces* $L^p_u(\mathbb{R}^d, E)$ *or* $L^{p,q}_m(\mathbb{R}^{2d}, E)$ *for some* $1 \le p, q \le \infty$ *and* v_s*-moderate weights* u, m *for some* $s \ge 0$. *The Wiener amalgam space* $W(B, C)$ *with local component* B *and global component* C *is defined by*

$$W(B, C)(\mathbb{R}^d) = \{f \in B_{\mathrm{loc}} \,:\, F_g(f) \in C\},$$

where $g \in C^\infty_{\mathrm{c}}(\mathbb{R}^d) \setminus \{0\}$ *is a fixed window function and* F_g *is the associated control function:* $F_g(f)(x) := \|f \cdot T_x g\|_B$. *The natural norm on* $W(B, C)$ *is* $\|f\|_{W(B,C)} := \|F_g\|_C$.

It turns out that $W^{p,q}_{u\otimes v}(\mathbb{R}^d) = W(\mathcal{F}L^p_u, L^q_v)(\mathbb{R}^d)$. This should not come as a surprise, since Wiener amalgam spaces were introduced before modulation spaces and in fact Feichtinger admitted that the latter were originally designed as amalgam spaces on the Fourier side [Fei06]. Note that $W^p(E) = M^p(E)$ for any $1 \le p \le \infty$, in particular $M^2(E) = W^2(E) = L^2(E)$.

We now state the more important properties of Wiener amalgam spaces.

Proposition 3.2.10. *Let* B, B_i *be local component spaces and* C, C_i *be global component spaces,* $(i = 1, 2, 3)$.

1. *(Banach space property)* $(W(B, C)(\mathbb{R}^d), \|\cdot\|_{W(B,C)})$ *is a Banach space continuously embedded into* $\mathcal{S}'(\mathbb{R}^d)$*; moreover, different window functions provide equivalent norms.*
2. *(Embeddings) If* $B_1 \hookrightarrow B_2$ *and* $C_1 \hookrightarrow C_2$, *then* $W(B_1, C_1) \hookrightarrow W(B_2, C_2)$.
3. *(Complex interpolation) For* $0 < \theta < 1$ *we have*

$$[W(B_1, C_1), W(B_2, C_2)]_{[\theta]} = W([B_1, B_2]_{[\theta]}, [C_1, C_2]_{[\theta]})$$

   if C_1 *or* C_2 *has absolutely continuous norm*[1].
4. *(Duality) If* B', C' *are the topological dual spaces of* B, C *respectively, and the space of test functions* C^∞_{c} *is dense in both* B *and* C, *then* $W(B, C)' = W(B', C')$.

[1] Recall that a Banach space $(Y, \| \, \|_Y)$ of functions $\mathbb{R}^n \to \mathbb{C}$ has an absolutely continuous norm if the Banach dual Y' coincides with $Y* := \{g : \mathbb{R}^n \to \mathbb{C} \text{ measurable } : gf \in L^1(\mathbb{R}^d), \forall f \in Y\}$. For instance, $L^p(\mathbb{R}^d)$ has an absolutely continuous norm for $1 \le p < \infty$. See [BS88] for further details.

5. *(Admissible windows) If $u_i \in \mathcal{M}_{v_i}(\mathbb{R}^d)$, $i = 1, 2$, and $1 \leq p, q \leq \infty$, the class of admissible windows for the norm in the definition of $W^{p,q}_{u_1 \otimes u_2}(E)$ can be extended from C_c^∞ to $W^1_{v_1 \otimes v_2}$.*

Note that this general framework allows one to use the tools of decomposition spaces to further study the properties of such spaces. In order to exploit this connection for future purposes we introduce a useful equivalent discrete norm for the amalgam spaces $W^{p,q}_{r,s}(\mathbb{R}^d, E)$; the general case is covered in [Fei83]. Recall that a bounded uniform partition of function (BUPU) $(\{\psi_i\}_{i\in I}, (x_i)_{i\in I}, U)$ consists of a family of non-negative functions in $\mathcal{F}L^1_{|r|}(\mathbb{R}^d)$ $\{\psi_i\}_{i\in I}$ such that the following conditions are satisfied:

1. $\sum_{i\in I} \psi_i(x) = 1$, for any $x \in \mathbb{R}^d$;
2. $\sup_{i\in I} \|\psi_i\|_{\mathcal{F}L^1_{|r|}} < \infty$;
3. there exist a discrete family $(x_i)_{i\in I}$ in $\mathbb{R}^d$ and a relatively compact set $U \subset \mathbb{R}^d$ such that $\operatorname{supp}(\psi_i) \subset x_i + U$ for any $i \in I$, and
4. $\sup_{i\in I} \#\{j \,:\, x_i + U \cap x_j + U \neq \emptyset\} < \infty$.

A general result in the theory of amalgam spaces is the following norm equivalence in the spirit of decomposition spaces:

$$\|f\|_{W^{p,q}_{r,s}(E)} \asymp \left(\sum_{i\in I} \|f\, \psi_i\|^q_{\mathcal{F}L^p_r(E)} \langle x_i \rangle^{sq} \right)^{1/q}. \tag{3.13}$$

A similar characterization holds for modulation spaces [Fei03; WZG06], providing a norm comparable to that of Besov spaces:

$$\|f\|_{M^{p,q}_{r,s}(E)} \asymp \left(\sum_{i\in I} \|\Box_i f\|^q_{L^p_r(E)} \langle x_i \rangle^{sq} \right)^{1/q} \tag{3.14}$$

where we introduced the frequency-uniform decomposition operators

$$\Box_i := \mathcal{F}^{-1} \psi_i \mathcal{F}, \qquad i \in I.$$

In particular, a Young type result can be obtained after a suitable modification of the proof of [Fei83, Thmeorem 3].

Theorem 3.2.11. *Let $\bullet : E_1 \times E_2 \to E_3$ be a multiplication for the triple of Banach spaces (E_1, E_2, E_3). For any $1 \le p_1, p_2, p_3, q_1, q_2, q_3 \le \infty$ and $r_1, r_2, r_3, s_1, s_2, s_3 \in \mathbb{R}$ such that*

$$\mathcal{F}L^{p_1}_{r_1}(\mathbb{R}^d, E_1) *_\bullet \mathcal{F}L^{p_2}_{r_2}(\mathbb{R}^d, E_2) \hookrightarrow \mathcal{F}L^{p_3}_{r_3}(\mathbb{R}^d, E_3),$$
$$L^{q_1}_{s_1}(\mathbb{R}^d) * L^{q_2}_{s_2}(\mathbb{R}^d) \hookrightarrow L^{q_3}_{s_3}(\mathbb{R}^d),$$

the following inclusion holds:

$$W^{p_1,q_1}_{r_1,s_1}(\mathbb{R}^d, E_1) *_\bullet W^{p_2,q_2}_{r_2,s_2}(\mathbb{R}^d, E_2) \hookrightarrow W^{p_3,q_3}_{r_3,s_3}(\mathbb{R}^d, E_3).$$

Proof. For the benefit of the reader we sketch here a short proof in the spirit of [Hei03, Theorem 11.8.3]. We consider as a BUPU for $W^{p,q}_{r,s}(\mathbb{R}^d, E)$ the family $\{\psi_k\}_{k\in\mathbb{Z}^d} \subset C^\infty_c(\mathbb{R}^d) \subset \mathcal{F}L^1_{|r|}(\mathbb{R}^d)$ defined by

$$\psi_k(t) = \frac{\phi(t-k)}{\sum_{k\in\mathbb{Z}^d} \phi(t-k)}, \quad t \in \mathbb{R}^d,$$

for a fixed $\phi \in C^\infty_c(\mathbb{R}^d)$, $\phi \ge 0$, such that $\phi(t) = 1$ for $t \in [0,1]^d$ and $\phi(t) = 0$ for $t \in \mathbb{R}^d \setminus [-1,2]^d$. After introducing the control functions

$$\Psi_{f,p,r,E}(k) := \|f\,\psi_k\|_{\mathcal{F}L^p_r(\mathbb{R}^d,E)}, \quad k \in \mathbb{Z}^d,$$

the equivalent norm (3.13) becomes

$$\|f\|_{W^{p,q}_{r,s}(E)} \asymp \left(\sum_{k\in\mathbb{Z}^d} \|f\,\psi_k\|^q_{\mathcal{F}L^p_r(\mathbb{R}^d,E)} \langle k\rangle^{qs} \right)^{1/q} \asymp \|\Psi_{f,p,r,E}\|_{\ell^q_s(\mathbb{Z}^d)}.$$

For $f \in W^{p_1,q_1}_{r_1,s_1}(\mathbb{R}^d, E_1)$ and $g \in W^{p_2,q_2}_{r_2,s_2}(\mathbb{R}^d, E_2)$ set $f_m = f\psi_m$, $g_n = g\psi_n$ for $m, n \in \mathbb{Z}^d$. In view of the support property [Ama19, Remark 1.9.6(f)] and the properties of BUPUs, we have

$$\operatorname{supp}(f_m *_\bullet g_n) \subset \operatorname{supp}(f_m) + \operatorname{supp}(g_n) = m + n + 2\operatorname{supp}\psi.$$

It is then clear that the cardinality of the set $J_k := \{(m,n) \in \mathbb{Z}^{2d} : \operatorname{supp}((f_m *_\bullet g_n)\psi_k) \ne \emptyset\}$ is finite for any $k \in \mathbb{Z}^d$ and is uniformly bounded with respect to m, n, k. In fact, notice that

$$J_k = \{(m,n) \in \mathbb{Z}^{2d} : m = k - n + \alpha,\ |\alpha| \le N(d)\},$$

for a fixed constant $N(d) \in \mathbb{N}$ depending only on the dimension d. Therefore, an easy computation yields

$$\Psi_{f*_\bullet g,p_3,r_3,E_3}(k) = \sum_{|\alpha|\le N(d)} \Psi_{f,p_1,r_1,E_1} * \Psi_{g,p_2,r_2,E_2}(k+\alpha),$$

and hence
$$\|f *_\bullet g\|_{W^{p_3,q_3}_{r_3,s_3}(E_3)} \lesssim \|f\|_{W^{p_1,q_1}_{r_1,s_1}(E_1)} \|g\|_{W^{p_2,q_2}_{r_2,s_2}(E_2)},$$
that is the claim. □

Remark 3.2.12. *In view of the relation with modulation spaces and Young inequality for convolution, under the same assumptions of the previous theorem we also have*
$$M^{p_1,q_1}_{r_1,s_1}(\mathbb{R}^d, E_1) \bullet M^{p_2,q_2}_{r_2,s_2}(\mathbb{R}^d, E_2) \hookrightarrow M^{p_3,q_3}_{r_3,s_3}(\mathbb{R}^d, E_3).$$

An interesting relation between modulation and Wiener amalgam spaces is given by the following set of generalized Hausdorff-Young inequalities, which are a direct consequence of Minkowski's integral inequality:

$$M^{p,q}_{r,s}(E) \hookrightarrow W^{q,p}_{s,r}(E), \quad 1 \le q \le p \le \infty,\ r,s \in \mathbb{R}. \tag{3.15}$$
$$W^{p,q}_{r,s}(E) \hookrightarrow M^{p,q}_{s,r}(E), \quad 1 \le p \le q \le \infty,\ r,s \ge 0.$$

3.2.3 A Banach-Gelfand triple of modulation spaces

The modulation space $M^1(\mathbb{R}^d)$ has a rather special role in the theory of modulation spaces. It is in fact the first kind of modulation space introduced by Feichtinger as a special, new Segal algebra [Fei81]. It is known since then as the Feichtinger algebra; we address the reader to the recent paper [Jak18] for a comprehensive survey on the topic.

Note that $M^1 \subset M^{p,q} \subset M^\infty$ for any $1 \le p,q \le \infty$, and in particular $M^1(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset M^\infty(\mathbb{R}^d)$. There has been an increasing interest for the triple (M^1, M^2, M^∞) as a replacement of the standard Gelfand triple $(\mathcal{S}, L^2, \mathcal{S}')$ of real harmonic analysis [FLC08]. Recall that a *Gelfand triple* (also known as rigged Hilbert space in quantum mechanics) consists of a separable Hilbert space H and a topological vector space Q such that the inclusion $i : Q \to H$ is an injective bounded operator with dense image [Ant98]; note that the adjoint map $i^* : H \simeq H' \to Q'$ is injective too and the inner product on H extends to the duality pairing $Q - Q'$ in a natural way.

In particular, Feichtinger's algebra is a good substitute of the Schwartz class of test functions for the purposes of Gabor analysis - except for applications to PDEs or to situations where a control on the regularity is required [Fei19; FJ20]. We list below the main properties of M^1 as a function space - it may be helpful to recall that $M^1 = W^1$.

Proposition 3.2.13. *1. $f \in M^1$ if and only if $f \in L^2$ and $V_g f \in L^1$ for all $g \in M^1$. In particular $f \in M^1 \iff \mathrm{Amb} f \in L^1$.*

2. *If $f \in M^1$, then $f, \hat{f} \in L^1$ and f is continuous.*

3. *M^1 is a time-frequency homogeneous Banach space: for any $z \in \mathbb{R}^{2d}$, $f \in M^1(\mathbb{R}^d)$, one has $\pi(z)f \in M^1(\mathbb{R}^d)$ and $\|\pi(z)f\|_{M^1} = \|f\|_{M^1}$. In particular, it is the smallest time-frequency homogeneous Banach space containing the Gaussian function.*

4. *The Schwartz class $\mathcal{S}$ is a dense subset of M^1 and L^2 is the completion of M^1 with respect to the L^2 norm.*

5. *M^1 is invariant under the Fourier transform, i.e. for any $f \in M^1(\mathbb{R}^d)$ one has $\hat{f} \in M^1(\mathbb{R}^d)$ and $\|\hat{f}\|_{M^1} = \|f\|_{M^1}$.*

6. *M^1 is a Banach algebra under both convolution and pointwise multiplication.*

The Feichtinger algebra is also well behaved under tensor products and we list a few results in this respect.

Proposition 3.2.14. *(i) The tensor product $\otimes : M^1(\mathbb{R}^d) \times M^1(\mathbb{R}^d) \to M^1(\mathbb{R}^{2d})$ is a bilinear bounded operator.*

(ii) M^1 enjoys the projective tensor factorization property:

$$M^1(\mathbb{R}^{2d}) \simeq M^1(\mathbb{R}^d) \widehat{\otimes} M^1(\mathbb{R}^d),$$

namely the space $M^1(\mathbb{R}^{2d})$ consists of all functions of the form

$$f = \sum_{n \in \mathbb{N}} g_n \otimes h_n,$$

where $\{g_n\}, \{h_n\}$ are (sequences of) functions in $M^1(\mathbb{R}^d)$ such that

$$\sum_{n \in \mathbb{N}} \|g_n\|_{M^1} \|h_n\|_{M^1} < \infty.$$

(iii) The tensor product is well defined on M^∞: for any $f, g \in M^\infty(\mathbb{R}^d)$, $f \otimes g$ is the unique element of $M^\infty(\mathbb{R}^{2d})$ such that

$$\langle f \otimes g, \phi_1 \otimes \phi_2 \rangle = \langle f, \phi_1 \rangle \langle g, \phi_2 \rangle, \quad \forall \phi_1, \phi_2 \in M^1(\mathbb{R}^d).$$

The parallel with the standard triple $(\mathcal{S}, L^2, \mathcal{S}')$ is further reinforced by a fundamental kernel theorem; just as the (temperate) distributions are related to the Schwartz kernel theorem, the *Feichtinger kernel theorem* [FG97] characterizes operators $M^1 \to M^\infty$ as follows - see also [CN19].

Theorem 3.2.15. *(i) Every distribution $k \in M^\infty(\mathbb{R}^{2d})$ defines a bounded linear operator $T_k : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ according to the rule*

$$\langle T_k f, g\rangle = \langle k, g \otimes \overline{f}\rangle, \quad \forall f, g \in M^1(\mathbb{R}^d),$$

with $\|T_k\|_{M^1 \to M^\infty} \le \|k\|_{M^\infty}$.

(ii) For any bounded operator $T : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ there exists a unique kernel $k_T \in M^\infty(\mathbb{R}^{2d})$ such that

$$\langle Tf, g\rangle = \langle k_T, g \otimes \overline{f}\rangle, \quad \forall f, g \in M^1(\mathbb{R}^d).$$

3.2.4 The Sjöstrand class and related spaces

Another special member of the family of modulation spaces is $M^{\infty,1}$. This is also known as the Sjöstrand class after the seminal paper [Sjö94], where it was introduced as an exotic symbol class still yielding bounded pseudodifferential operators on L^2. It was later recognized that operators with symbols in this space enjoy several other properties, see Section 5.8 for a thorough account.

Pseudodifferential operators with Sjöstrand symbols have also been used as potential perturbations for the Schrödinger equation, since they are particularly well suited to the Gabor analysis of the corresponding propagator - cf. Chapter 4 below.

For the moment we highlight some important properties that will be used below.

Proposition 3.2.16. *1. $M^{\infty,1}(E) \subset (\mathcal{F}L^1(E))_{\mathrm{loc}} \cap L^\infty(E) \subset C_{\mathrm{b}}(E)$.*

2. $(M^{\infty,1}(E))_{\mathrm{loc}} = (\mathcal{F}L^1(E))_{\mathrm{loc}}$.

3. If $k \in \mathbb{N}$ and $k > d$ then $C^k_{\mathrm{b}}(E) \subset M^{\infty,1}(E)$. Moreover, $C^\infty_{\mathrm{b}}(E) = \bigcap_{s\ge 0} M^\infty_{0,s}(E) = \bigcap_{s\ge 0} M^{\infty,1}_{0,s}(E)$.

4. $\mathcal{F}\mathcal{M}(E) \subset M^{\infty,1}(E)$, where $\mathcal{M}(E)$ is the space of E-valued vector measures of bounded variation on $\mathbb{R}^d$.

Proof. 1. It is a direct consequence of the definition.

2. See Proposition 3.2.3 - for the proof, see [BO20, Proposition 2.9].

3. The proof in the scalar-valued case can be found in [Grö01, Theorem 14.5.3] and [GR08, Lemma 6.1]. The proof in the vector-valued case goes exactly as the previous one, with $|\cdot|$ replaced by $|\cdot|_E$.

4. We prove equivalently that $\mathcal{M}(E) \subset W^{\infty,1}(E)$. Recall that $\mathcal{M}(E) \subset \mathcal{S}'(E)$, hence for any non-zero window $g \in \mathcal{S}$ we can explicitly write the STFT of $\mu \in \mathcal{M}(E)$:

$$V_g\mu(x,\xi) = \langle \mu, M_\xi T_x g\rangle = \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi}\overline{g(y-x)}d\mu(y).$$

Therefore,

$$\begin{aligned}
\|\mu\|_{W^{\infty,1}(E)} &= \int_{\mathbb{R}^d} \sup_{\xi\in\mathbb{R}^d} |V_g\mu(x,\xi)|_E dx \\
&\le \int_{\mathbb{R}^d} \sup_{\xi\in\mathbb{R}^d} \int_{\mathbb{R}^d} |e^{-2\pi i y\cdot\xi}\overline{g(y-x)}|d|\mu|(y)\, dx \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |g(y-x)|d|\mu|(y)\, dx \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |g(y-x)|dx\, d|\mu|(y) \\
&= \|g\|_{L^1}|\mu|(\mathbb{R}^d) < \infty.
\end{aligned}$$

□

We mention that the (scalar-valued) Sjöstrand class is provided with a Banach algebra structure with respect to the pointwise multiplication, as a result of the following characterization - in fact, it also enjoys a non-commutative algebra structure under Weyl product of symbols, see Section 4.2 below.

Proposition 3.2.17 ([RS16, Theorem 3.5 and Corollary 2.10])**.** *Let $1 \le p,q \le \infty$ and $s \in \mathbb{R}$. The following facts are equivalent.*
(i) $M^{p,q}_{0,s}$ is a Banach algebra for pointwise multiplication.
(ii) $M^{p,q}_{0,s} \hookrightarrow L^\infty$.
(iii) Either $s = 0$ and $q = 1$ or $s > d/q'$.

Remark 3.2.18. *We clarify once for all that the preceding results concern the conditions under which the embedding $M^{p,q}_{0,s} \cdot M^{p,q}_{0,s} \hookrightarrow M^{p,q}_{0,s}$ is continuous, hence there exists a constant $C > 0$ such that*

$$\|fg\|_{M^{p,q}_{0,s}} \le C\|f\|_{M^{p,q}_{0,s}}\|g\|_{M^{p,q}_{0,s}}, \qquad \forall f,g \in M^{p,q}_{0,s}.$$

Thus, the algebra property holds up to a constant. Recall that we always assume to normalize the norm in such a way that $C = 1$, in according with Remark 2.1.1.

Functions in $M^{\infty,1}$ enjoy nice low-pass decompositions, in the sense of the following results.

Proposition 3.2.19. *For any $\epsilon > 0$ and $f \in M^{\infty,1}(E)$, there exist $f_1 \in C_{\mathrm{b}}^{\infty}(E)$ and $f_2 \in M^{\infty,1}(E)$ such that*

$$f = f_1 + f_2, \qquad \|f_2\|_{M^{\infty,1}(E)} \leq \epsilon.$$

Proof. Fix $g \in \mathcal{S}$ with $\|g\|_{L^2} = 1$, and set

$$f_1(y) = V_g^*(V_g f \cdot 1_{A_R})(y) = \int_{A_R} V_g f(x,\xi) e^{2\pi i y\cdot\xi} g(y-x) dx d\xi, \tag{3.16}$$

in the sense of distributions, where $A_R = \{(x,\xi) \in \mathbb{R}^{2d} : |\xi| \leq R\}$, and $R > 0$ will be chosen later, depending on ϵ.

The integral in (3.16) actually converges for every y and defines a bounded function. Indeed, setting $S(\xi) = \sup_{x\in\mathbb{R}^d} |V_g f(x,\xi)|_E$, we have $S \in L^1$ by the assumption $f \in M^{\infty,1}(E)$, and for any $y \in \mathbb{R}^d$,

$$\begin{aligned} |f_1(y)|_E &\leq \int_{A_R} |V_g f(x,\xi)|_E |g(y-x)| dx d\xi \\ &\leq \|g\|_{L^1} \|S\|_{L^1}. \end{aligned}$$

Similarly one shows that all the derivatives $\partial^\alpha f_1$ are bounded, using that $\xi^\alpha S(\xi)$ is integrable on $|\xi| \leq R$. Differentiation under the integral sign is permitted because for y in a neighbourhood of any fixed $y_0 \in \mathbb{R}^d$ and every N,

$$|V_g f(x,\xi) \partial_y^\alpha [e^{2\pi i y\cdot\xi} g(y-x)]|_E \leq C_N (1+|\xi|)^{|\alpha|} S(\xi)(1+|y_0 - x|)^{-N},$$

which is integrable in A_R. Hence $f_1 \in C_{\mathrm{b}}^{\infty}(E)$.

Now, let $f_2 = f - f_1 = V_g^*(V_g f \cdot 1_{A_R^c})$, where in the second equality we used the inversion formula for the STFT (3.2). The continuity of $V_g^* : L^{\infty,1}(\mathbb{R}^{2d}, E) \to M^{\infty,1}(\mathbb{R}^d, E)$ yields

$$\begin{aligned} \|f_2\|_{M^{\infty,1}(E)} &= \|V_g^*(V_g f \cdot 1_{A_R^c})\|_{M^{\infty,1}(E)} \\ &\lesssim \|V_g f \cdot 1_{A_R^c}\|_{L^{\infty,1}(E)} \\ &= \int_{|\xi|>R} S(\xi) d\xi \leq \epsilon \end{aligned}$$

provided that $R = R_\epsilon$ is large enough. ∎

Proposition 3.2.20. *Let $f : \mathbb{R}^d \to E$ be such that $\partial^\alpha f \in M^{\infty,1}(E)$ for any $\alpha \in \mathbb{N}^d$ such that $|\alpha| = k$ for some $k \in \mathbb{N}$. Then there exist $f_1 \in C_{\geq k}^{\infty}(E)$ and $f_2 \in M^{\infty,1}(E)$ such that $f = f_1 + f_2$.*

Proof. Fix a smooth cut-off function $\chi \in C_c^\infty(\mathbb{R}^d)$ supported in a neighbourhood of the origin and such that $\chi = 1$ near zero, then consider the Fourier multiplier $\chi(D)$ with symbol χ. Set $f_1 = \chi(D)f$ and $f_2 = (I - \chi(D))f$. Clearly $f = f_1 + f_2$ and we argue that f_1 and f_2 satisfy the claimed properties.

Indeed, $f_1 \in C^\infty(E)$ and for any $\alpha \in \mathbb{N}^d$, $|\alpha| = k$, we have

$$\partial^\alpha f_1 = \partial^\alpha(\chi(D)f) = \chi(D)(\partial^\alpha f) \in M^{\infty,1}(E),$$

since $\partial^\alpha \chi(D)$ is a Fourier multiplier with symbol $(2\pi i\xi)^\alpha \chi(\xi) \in C_c^\infty(\mathbb{R}^d)$, hence $\partial^\alpha \chi(D) = \chi(D)\partial^\alpha$ and $\chi(D)$ is continuous on $M^{\infty,1}(E)$ by Proposition 4.1.1 below. Furthermore, similar arguments imply that for any $\alpha \in \mathbb{N}^d$, $|\alpha| \geq k$,

$$\partial^\alpha f_1 = \partial^{\alpha-\beta}\partial^\beta(\chi(D)f) = (\partial^{\alpha-\beta}\chi(D))(\partial^\beta f) \in M^{\infty,1}(E).$$

where $\beta \in \mathbb{N}^d$ satisfies $|\beta| = k$. In order to prove the claim for f_2 consider the finite smooth partition of unity $\{\varphi_j\}_{j=1}^N$ of the unit sphere $S^{d-1} \subset \mathbb{R}^d$ subordinated to the open cover $\{U_j\}_{j=1}^d$, where $U_j = \{x \in S^{d-1} \,:\, x_j \neq 0\}$. Then we extend each function φ_j on $\mathbb{R}^d \setminus \{0\}$ by zero-degree homogeneity, namely

$$\sum_{j=1}^d \varphi_j(x) = 1, \qquad \varphi_j(\alpha x) = \varphi_j(x), \qquad \forall x \in S^{d-1},\ \alpha > 0.$$

This procedure gives a finite partition of unity $\{\varphi_j\}_{k=1}^d$ on $\mathbb{R}^d \setminus \{0\}$. Then

$$\begin{aligned} f_2(x) &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}(1-\chi(\xi))\hat{f}(\xi)d\xi \\ &= \sum_{j=1}^d \left[\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\left(\frac{1-\chi(\xi)}{(2\pi i\xi_j)^k}\varphi_j(\xi)\right)\widehat{\partial_j^k f}(\xi)d\xi \right] \\ &= \sum_{j=1}^d \widetilde{\chi_j}(D)(\partial_j^k f)(x) \end{aligned}$$

and thus $f_2 \in M^{\infty,1}(E)$ since each $\widetilde{\chi_j}(D)$ is a Fourier multiplier with symbol $(1-\chi(\xi))\varphi_j(\xi)/(2\pi i\xi_j)^k \in C^\infty_{\geq 0}(\mathbb{R}^d)$, hence bounded on $M^{\infty,1}(E)$. □

Narrow convergence. Convergence in $M^{\infty,1}$ norm is a quite strong requirement. For instance, it is well known that C_{c}^∞ is not dense in $M^{\infty,1}$ with the norm topology [Sjö94]; this fact inhibits the standard approximation arguments and leads to restrict to subspaces such as $\mathcal{M}^{\infty,1}$. Another way to cope with this problem consists in weakening the notion of convergence as follows [CNR15b; Sjö94].

Definition 3.2.21. *Let* $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ *and let* Ω *be a subset of some Euclidean space and* $s \in \mathbb{R}$*. The map* $\Omega \ni \nu \mapsto \sigma_\nu \in M^{\infty,1}_m(\mathbb{R}^d, E)$ *is said to be continuous for the narrow convergence if:*

1. *it is a continuous map in* $\mathcal{S}'(\mathbb{R}^d, E)$ *(weakly), and*

2. *there exists a function* $h \in L^1(\mathbb{R}^d)$ *such that for some (hence any) nonzero window* $g \in \mathcal{S}(\mathbb{R}^d)$ *one has* $\sup_{x\in\mathbb{R}^d} |V_g\sigma_\nu(x,\xi)m(x,\xi)|_E \le h(\xi)$ *for any* $\nu \in \Omega$ *and a.e.* $\xi \in \mathbb{R}^d$*.*

The advantages of narrow convergence will be used below in connection with the Weyl quantization. For the moment, we just stress that $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{\infty,1}$ with respect to the narrow convergence [Sjö94].

3.3 Gabor frames

We already stressed that the STFT $V_g f$ may be heuristically interpreted as a continuous expansion of the function f with respect to the system $\{\pi(z)g : z = (x,\xi) \in \mathbb{R}^{2d}\}$ of highly localized wave packets on phase space. The theory of Gabor frames may be used to give a precise meaning to this suggestion in terms of discrete samples and expansions.

To be precise, given a non-zero window function $g \in L^2(\mathbb{R}^d)$ and a *full-rank lattice* $\Lambda \subset \mathbb{R}^{2d}$ (namely, a countable and discrete additive subgroup of $\mathbb{R}^{2d}$ with compact quotient group $\mathbb{R}^{2d}/\Lambda$), the *Gabor system* $\mathcal{G}(g,\Lambda)$ is the collection of time-frequency shifts of g along Λ, namely

$$\mathcal{G}(g,\Lambda) = \{\pi(z)g \,:\, z \in \Lambda\}.$$

Standard examples of lattices are $\Lambda = M\mathbb{Z}^{2d}$ where $M \in \mathrm{GL}(2d,\mathbb{R})$, in particular separable lattices such as

$$\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z} = \{(\alpha k, \beta n) \,:\, k,n \in \mathbb{N}\},$$

for suitable lattice parameters $\alpha, \beta > 0$; we write $\mathcal{G}(g,\alpha,\beta)$ for the corresponding Gabor system.

Recall that a *frame* for $L^2(\mathbb{R}^d)$ is a sequence $\{\phi_j\}_{j\in J} \subset L^2(\mathbb{R}^d)$, J being a countable index set, such that for all $f \in L^2(\mathbb{R}^d)$

$$A\|f\|^2_{L^2} \le \sum_{j\in J} |\langle f, \phi_j\rangle|^2 \le B\|f\|^2_{L^2},$$

for some positive constants $A, B > 0$ (frame bounds). We say that a frame is tight if $A = B$ and Parseval if $A = B = 1$.

One may naturally define some bounded linear operators related to a given frame:

- The *analysis operator* $C : L^2(\mathbb{R}^d) \to \ell^2(J)$, $Cf = \{\langle f, \phi_j\rangle\}_{j\in J}$.
- The *synthesis operator* $D : \ell^2(J) \to L^2(\mathbb{R}^d)$, $Da = \sum_{j\in J} a_j\phi_j$.
- The *frame operator* $S : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $Sf = DCf = \sum_{j\in J}\langle f, \phi_j\rangle\phi_j$.

If a Gabor system $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$ it is called *Gabor frame.* Notice that the Gabor frame operator reads

$$Sf = \sum_{\lambda\in\Lambda} V_g f(\lambda)\pi(\lambda)g,$$

and is a positive, bounded invertible operator on $L^2(\mathbb{R}^d)$. A remarkable result of frame theory is that a function can be reconstructed from its Gabor coefficients, in the sense that

$$f = \sum_{\lambda\in\Lambda} V_g f(\lambda)\pi(\lambda)\gamma, \tag{3.17}$$

where $\gamma = S^{-1}g$ is the *canonical dual window* and the sum is unconditionally convergent in L^2.

Discrete Gabor analysis can be extended to modulation spaces under suitable assumptions (cf. [Grö01, Corollary 12.2.6 and Corollary 12.2.8]); for instance, if $g \in M^1(\mathbb{R}^d)$ and $\mathcal{G}(g, \alpha, \beta)$ is a tight frame for $L^2(\mathbb{R}^d)$ then for any $1 \le p \le \infty$,

$$\|f\|_{M^{p,q}_m} = \frac{1}{A}\left(\sum_{n\in\mathbb{Z}^d}\left(\sum_{k\in\mathbb{Z}^d} |V_g f(\alpha k, \beta n)|^p m(\alpha k, \beta n)^p|\right)^{q/p}\right)^{1/q},$$

and the Gabor expansion (3.17) holds with $\gamma = g/A$ and unconditional convergence if $1 \le p < \infty$ (weak*-convergence if $p = \infty$).

Chapter 4

The Gabor Analysis of Operators

The inversion formula for the short-time Fourier transform given in (3.2) enables an efficient phase-space analysis of operators. Consider a bounded linear operator $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ and $g, \gamma \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$; it is not restrictive to assume $\|g\|_{L^2} = \|\gamma\|_{L^2} = 1$. Therefore, have that

$$A = V_\gamma^* V_\gamma A V_g^* V_g = V_\gamma^* \widetilde{A} V_g, \tag{4.1}$$

where $\widetilde{A} \coloneqq V_\gamma A V_g^*$ is an integral operator in $\mathbb{R}^{2d}$ with integral kernel given by the *Gabor matrix* K_A, that is

$$\widetilde{A}F(w) = \int_{\mathbb{R}^{2d}} K_A(w,z)F(z)dz, \quad K_A(w,z) = \langle A\pi(z)g, \pi(w)\gamma\rangle, \quad w \in \mathbb{R}^{2d}. \tag{4.2}$$

This decomposition will be of primary concern for the time frequency-analysis of operators. We consider below some special classes of operators and derive fundamental boundedness estimates that will be repeatedly used to prove the main results.

4.1 Fourier multipliers

We provide sufficient conditions on the symbol of a Fourier multiplier in order for it to be bounded on modulation and Wiener amalgam spaces.

Proposition 4.1.1. *Let $\bullet : E_0 \times E_1 \to E_2$ be a multiplication and $\mu \in W^{1,\infty}_{|r|,\delta}(E_0)$ for some $r, \delta \in \mathbb{R}$. The Fourier multiplier $\mu(D)$ is bounded from $M^{p,q}_{r,s}(E_1)$ to $M^{p,q}_{r,s+\delta}(E_2)$ for any $1 \le p, q \le \infty$ and $s \in \mathbb{R}$. In particular,*

$$\|\mu(D)f\|_{M^{p,q}_{r,s+\delta}(E_2)} \lesssim \|\mu\|_{W^{1,\infty}_{|r|,\delta}(E_0)} \|f\|_{M^{p,q}_{r,s}(E_1)}, \quad f \in M^{p,q}_{r,s}(E_1).$$

Proof. The proof is a straightforward generalization of the argument used in the scalar-valued case; see for instance [BGOR07, Lemma 8]; we report it here for the sake of completeness. Choose as window a function $g \in \mathcal{S}$ that factors as $g = g_0 * g_0$ for some $g_0 \in \mathcal{S}$ - consider for instance $g_0(t) = e^{-\pi t^2}$. It is then easy to prove that $M_\xi g^* = M_\xi g_0^* * M_\xi g_0^*$. Thanks to (3.1), Proposition 3.2.11 and the associativity of convolutions [Ama19, Remark 1.9.6(c)] we have

$$\begin{aligned}
\|\mu(D)f\|_{M^{p,q}_{r,s+\delta}(E_2)} &= \left(\int_{\mathbb{R}^d} \|\mu(D)f * M_\xi g^*\|^q_{L^p_r(E_2)} \langle\xi\rangle^{s+\delta} d\xi\right)^{1/q} \\
&= \left(\int_{\mathbb{R}^d} \|\check{\mu} *_\bullet f * M_\xi g_0^* * M_\xi g_0^*\|^q_{L^p_r(E_2)} \langle\xi\rangle^{s+\delta} d\xi\right)^{1/q} \\
&= \left(\int_{\mathbb{R}^d} \|\check{\mu} * M_\xi g_0^* *_\bullet f * M_\xi g_0^*\|^q_{L^p_r(E_2)} \langle\xi\rangle^{s+\delta} d\xi\right)^{1/q} \\
&\leq \left(\int_{\mathbb{R}^d} \|\check{\mu} * M_\xi g_0^*\|^q_{L^1_{|r|}(E_0)} \|f * M_\xi g_0^*\|^q_{L^p_r(E_1)} \langle\xi\rangle^{s+\delta} d\xi\right)^{1/q} \\
&\leq \left(\sup_{\xi\in\mathbb{R}^d} \|\check{\mu} * M_\xi g_0^*\|_{L^1_{|r|}(E_0)} \langle\xi\rangle^{\delta}\right)\left(\int_{\mathbb{R}^d} \|f * M_\xi g_0^*\|^q_{L^p_r} \langle\xi\rangle^{s} d\xi\right)^{1/q} \\
&= \|\mu\|_{W^{1,\infty}_{|r|,\delta}(E_0)} \|f\|_{M^{p,q}_{r,s}(E_1)}.
\end{aligned}$$

The cases where $p = \infty$ or $q = \infty$ can be handled after slight modifications. □

A similar result holds for Fourier multipliers on Wiener amalgam spaces.

Proposition 4.1.2. *Let $\bullet : E_0 \times E_1 \to E_2$ be a multiplication and $\mu \in M^{\infty,1}_{\delta,|s|}(\mathbb{R}^d, E_0)$ for some $s, \delta \in \mathbb{R}$. The Fourier multiplier with symbol μ is bounded from $W^{p,q}_{r,s}(\mathbb{R}^d, E_1)$ to $W^{p,q}_{r+\delta,s}(\mathbb{R}^d, E_2)$ for any $1 \leq p, q \leq \infty$ and $r \in \mathbb{R}$. In particular,*

$$\|\mu(D)f\|_{W^{p,q}_{r+\delta,s}(E_2)} \lesssim \|\mu\|_{M^{\infty,1}_{\delta,|s|}(E_0)} \|f\|_{W^{p,q}_{r,s}(E_1)}, \quad f \in W^{p,q}_{r,s}(E_1).$$

Proof. Recall that $W^{p,q}_{r,s}(\mathbb{R}^d, E) = W(\mathcal{F}L^p_r(\mathbb{R}^d, E), L^q_s(\mathbb{R}^d))$. Theorem 3.2.11 and the relation $\mathcal{F}M^{p,q}_{r,s} = W^{p,q}_{r,s}$ thus yield

$$\begin{aligned}
\|\mu(D)f\|_{W^{p,q}_{r+\delta,s}(E_2)} &= \left\|\mathcal{F}^{-1}\mu *_\bullet f\right\|_{W^{p,q}_{r+\delta,s}(E_2)} \\
&\lesssim \left\|\mathcal{F}^{-1}\mu\right\|_{W^{\infty,1}_{\delta,|s|}(E_0)} \|f\|_{W^{p,q}_{r,s}(E_1)} \\
&\lesssim \|\mu\|_{M^{\infty,1}_{\delta,|s|}(E_0)} \|f\|_{W^{p,q}_{r,s}(E_1)}.
\end{aligned}$$

□

4.2 The Weyl quantization

In this section we provide some basic results on the Weyl transform from the perspective of time-frequency analysis. The topic of quantization will be thoroughly developed in Chapter 5, hence here we confine ourselves to provide the essential notions.

The standard definition of the Weyl transform $\sigma^{\mathrm{w}} = \mathrm{op}_{\mathrm{w}}(\sigma)$ of the symbol $\sigma : \mathbb{R}^{2d} \to \mathbb{C}$ is

$$\sigma^{\mathrm{w}} f(x) := \int_{\mathbb{R}^{2d}} e^{2\pi i (x-y)\cdot\xi} \sigma\left(\frac{x+y}{2}, \xi\right) f(y) dy d\xi.$$

The meaning of this formal integral operator heavily relies on the function space to which the symbol σ belongs; the classical symbol classes investigated within the long tradition of pseudodifferential calculus are usually defined by means of decay/smoothness conditions (such as the general Hörmander classes $S^m_{\rho,\delta}(\mathbb{R}^{2d})$ in [Hör85]).

We adopt below the perspective of time-frequency analysis [Grö06a] and define the Weyl quantization of a rough symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ via duality as follows:

$$\sigma^{\mathrm{w}} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d), \qquad \langle \sigma^{\mathrm{w}} f, g \rangle = \langle \sigma, W(g,f) \rangle, \qquad \forall f, g \in \mathcal{S}(\mathbb{R}^d).$$

Equivalently, σ^{w} can be defined in terms of the so-called spreading representation:

$$\sigma^{\mathrm{w}} f := \int_{\mathbb{R}^{2d}} \widehat{\sigma}(\xi, u) e^{\pi i \xi \cdot u} \pi(-u, \xi) f \, du d\xi.$$

Remark 4.2.1. *The multiplication by a function V is a special example of Weyl operator with symbol*

$$\sigma_V(x,\xi) = V(x) = (V \otimes 1)(x,\xi), \quad (x,\xi) \in \mathbb{R}^{2d}.$$

It is not difficult to prove that the correspondence $V \mapsto \sigma_V$ is continuous from $M^{\infty,q}_{0,s}(\mathbb{R}^d)$ to $M^{\infty,q}_{0,s}(\mathbb{R}^{2d})$ for any $1 \le q \le \infty$ and $s \in \mathbb{R}$, cf. (3.11)*. This identification shall be implicitly assumed whenever needed below; by a slight abuse of notation, we will write V also for σ_V^{w} for the sake of legibility.*

One may similarly prove that a Fourier multiplier with symbol $\mu(\xi)$ is a Weyl operator with symbol $\widetilde{\sigma}_\mu(x,\xi) = (1 \otimes \mu)(x,\xi)$.

Modulation spaces can be used both as symbol classes as well as target spaces for pseudodifferential operator. For instance, we mention that symbols in the Sjöstrand class yield Weyl operators which are bounded on any modulation space.

Theorem 4.2.2 ([Grö01, Theorem 14.5.2])**.** *If $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ then σ^{w} is bounded on $M^{p,q}(\mathbb{R}^d)$ for all $1 \le p, q \le \infty$, with*

$$\|\sigma^{\mathrm{w}}\|_{M^{p,q}\to M^{p,q}} \le \|\sigma\|_{M^{\infty,1}}.$$

The symbolic calculus relies on the composition of Weyl transforms, which provides a bilinear form on symbols known as the *Weyl* (or *twisted*) *product*:

$$\sigma^{\mathrm{w}} \circ \rho^{\mathrm{w}} = (\sigma\#\rho)^{\mathrm{w}}, \quad \sigma\#\rho = \mathcal{F}^{-1}(\hat{\sigma}\natural\hat{\rho}),$$

where the twisted convolution is defined in (2.4). Although explicit formulas for the twisted product of symbols can be derived (cf. [Won98]), we will not need them hereafter. Nevertheless, this is a fundamental notion in order to establish an algebra structure on symbol spaces; the problem has been studied in several papers (cf. for instance [CTW14; Grö06b; HTW07]). For the sake of completeness we provide here sufficient conditions in a simplified form.

Theorem 4.2.3 ([HTW07, Theorem 0.3'])**.** *Let $r, s \in \mathbb{R}$ be such that $0 \le r \le 2s$. Let $1 \le p_j, q_j \le \infty$, $j = 0, 1, 2$, satisfy*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{q_1} + \frac{1}{q_2} = 1 + \frac{1}{p_0} + \frac{1}{q_0}, \quad q_1, q_2 \le q_0,$$

$$0 \le \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_0} \le \frac{1}{p_j}, \frac{1}{q_j} \le \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q_0}, \quad j = 0, 1, 2.$$

The map $\mathcal{S}(\mathbb{R}^{2d}) \times \mathcal{S}(\mathbb{R}^{2d}) \ni (\sigma, \rho) \mapsto \sigma\#\rho \in \mathcal{S}(\mathbb{R}^{2d})$ uniquely extends to a continuous map $\mathcal{M}^{p_1,q_1}_{r,s}(\mathbb{R}^{2d}) \times \mathcal{M}^{p_2,q_2}_{r,s}(\mathbb{R}^{2d}) \to \mathcal{M}^{p_0,q_0}_{r,s}(\mathbb{R}^{2d})$ and there exists $C > 0$ such that

$$\|\sigma\#\rho\|_{\mathcal{M}^{p_0,q_0}_{r,s}} \le C\|\sigma\|_{\mathcal{M}^{p_1,q_1}_{r,s}}\|\rho\|_{\mathcal{M}^{p_2,q_2}_{r,s}}.$$

Remark 4.2.4. *It is a quite distinctive property of $M^{\infty,1}(\mathbb{R}^{2d})$ as well as the scale of spaces $M^{\infty}_{0,s}(\mathbb{R}^{2d})$ with $s > 2d$, to enjoy a double Banach algebra structure:*

- *a commutative one with respect to the pointwise multiplication as detailed in Proposition 3.2.17;*
- *a non-commutative one with respect to the twisted product of symbols [GR08; Sjö94].*

In fact, much more is true, as detailed in the following statement.

Theorem 4.2.5 ([GR08; Sjö94])**.** *Let X be any of the spaces $C^{\infty}_{\mathrm{b}}(\mathbb{R}^{2d})$, $M^{\infty}_{0,s}(\mathbb{R}^{2d})$ with $s > 2d$ or $M^{\infty,1}(\mathbb{R}^{2d})$. The corresponding family of Weyl operators $\mathrm{op}_{\mathrm{w}}(X)$ is a Wiener subalgebra of $\mathcal{L}(L^2(\mathbb{R}^d))$ under composition, that is:*

(i) (Boundedness) If $\sigma \in X$, then σ^{w} is a bounded operator on $L^2(\mathbb{R}^d)$.

(ii) (Algebra property) If $\sigma_1, \sigma_2 \in X$ then $\sigma_1^{\mathrm{w}} \circ \sigma_2^{\mathrm{w}}$ is a Weyl operator with symbol $\sigma_1 \# \sigma_2 \in X$.

(iii) (Wiener property) If $\sigma \in X$ and σ^{w} is invertible on $L^2(\mathbb{R}^d)$, then there exists $\rho \in X$ such that $(\sigma^{\mathrm{w}})^{-1} = \rho^{\mathrm{w}}$.

New proofs of this and more general results (see Theorem 5.8.1 below) in the spirit of time-frequency analysis were provided by Gröchenig [Grö06c], unravelling a deep and fascinating analogy between Weyl operators with symbols in the Sjöstrand's class and Fourier series with ℓ^1 coefficients. Similarities of this kind come under the multifaceted problem of spectral invariance, a topic thoroughly explored by Gröchenig in [Grö10]; see also the related papers [CGNR13; GR08].

We stress that the latter algebra structure has been deeply investigated from a time-frequency analysis perspective. Indeed, it is subtly related to a characterizing property satisfied by pseudodifferential operators with symbols in those spaces, namely *almost diagonalization* of the Gabor matrix by time-frequency shifts. We address the reader to [CGNR13; CGNR14; GR08; Grö06c] for further discussions on these aspects and to Section 5.8 below for generalizations.

Theorem 4.2.6. *Fix $g, \gamma \in \mathcal{S}(\mathbb{R}^d)$ and consider $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$.*

1. $\sigma \in M^\infty_{0,s}(\mathbb{R}^{2d})$ if and only if

$$|\langle \sigma^{\mathrm{w}} \pi(z) g, \pi(w)\gamma\rangle| \le C(1+|w-z|)^{-s}, \quad z, w \in \mathbb{R}^{2d}.$$

2. $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ if and only if there exists a function $H \in L^1(\mathbb{R}^{2d})$ such that

$$|\langle \sigma^{\mathrm{w}} \pi(z) g, \pi(w)\gamma\rangle| \le H(w-z), \quad z, w \in \mathbb{R}^{2d}.$$

The controlling function H can be chosen as

$$H(w) = \sup_{z \in \mathbb{R}^{2d}} |V_\Phi \sigma(z,w)|, \quad \Phi = W(\gamma, g),$$

hence $\|H\|_{L^1} \asymp \|\sigma\|_{M^{\infty,1}}$.

4.2.1 Vector-valued Weyl transform

Let H be a separable Hilbert space and consider an operator-valued Schwartz function $\sigma \in \mathcal{S}(\mathbb{R}^{2d}, \mathcal{L}(H))$. The Weyl transform σ^{w} of $f \in \mathcal{S}(\mathbb{R}^d, H)$ is well defined by a Bochner integral by

$$\sigma^{\mathrm{w}} f(x) := \int_{\mathbb{R}^{2d}} e^{2\pi i(x-y)\cdot\xi} \sigma\left(\frac{x+y}{2}, \xi\right) f(y) dy d\xi \in H.$$

As in the scalar-valued setting, we wish to extend this definition in order to cover more general symbols $\sigma \in \mathcal{S}'(\mathbb{R}^{2d}, \mathcal{L}(H))$ and to make a connection with the vector-valued Wigner function. In this respect we note that, under the previous assumptions, a straightforward computation yields

$$\langle \sigma^{\mathrm{w}} f, g \rangle = \int_{\mathbb{R}^{2d}} \mathrm{Tr}[\sigma(x,\xi) W(f,g)(x,\xi)] dx d\xi, \quad f, g \in \mathcal{S}(\mathbb{R}^d, H).$$

Motivated by this identity, let us first consider $\sigma \in \mathcal{S}'(\mathbb{R}^{2d}, \mathcal{L}(H))$ and fix $f \in \mathcal{S}(\mathbb{R}^d, H)$. Recall from Theorem 2.2.6 that there exists a sequence $\sigma_n \in \mathcal{S}(\mathbb{R}^{2d}, \mathcal{L}(H))$ such that $\sigma_n \to \sigma$ in $\mathcal{S}'$. We thus define the H-valued functional $\sigma^{\mathrm{w}} f$ in such a way that

$$\langle \sigma^{\mathrm{w}} f, \phi \rangle = \langle \sigma, W(\phi, f) \rangle, \quad \phi \in \mathcal{S}(\mathbb{R}^d),$$

where the multiplication on the RHS denotes the unique extension to $\mathcal{S}'(\mathbb{R}^{2d}, \mathcal{L}(H)) \times \mathcal{S}(\mathbb{R}^{2d}, H)$ of the underlying functional, originally defined on $\mathcal{S}(\mathbb{R}^{2d}, \mathcal{L}(H)) \times \mathcal{S}(\mathbb{R}^{2d}, H)$, in according with Theorem 2.2.6 - that is $\langle \sigma, W(\phi, f) \rangle = \lim_{n\to\infty} \langle \sigma_n, W(\phi, f) \rangle$. It is easy to prove that $\sigma^{\mathrm{w}} : \mathcal{S}(\mathbb{R}^d, H) \to \mathcal{S}'(\mathbb{R}^d, H)$.

Under the same assumptions and again in view of the extension Theorem 2.2.6 (with the multiplication by scalars $H \times \mathbb{C} \to H$) one also has that the sequence $\{\sigma_n^{\mathrm{w}} f\} \subset \mathcal{S}(\mathbb{R}^d, H)$ converges to $\sigma^{\mathrm{w}} f$ in $\mathcal{S}'(\mathbb{R}^d, H)$. Therefore, once again by the extension theorem 2.2.6 (with the multiplication $H \times H \to \mathbb{C}$ given by the inner product on H) we define

$$\langle \sigma^{\mathrm{w}} f, g \rangle = \lim_{n\to\infty} \langle \sigma_n^{\mathrm{w}} f, g \rangle, \quad g \in \mathcal{S}(\mathbb{R}^d, H).$$

Arguing as before one can prove that

$$\langle \sigma^{\mathrm{w}} f, g \rangle = \lim_{n\to\infty} \int_{\mathbb{R}^{2d}} \mathrm{Tr}[\sigma_n(x,\xi) W(f,g)(x,\xi)|_{H_0}] dx d\xi,$$

where H_0 is the largest closed separable subspace of H where $\sigma_n(x,\xi)W(f,g)(x,\xi) : H_0 \to H_0$ for all n and almost all $(x,\xi) \in \mathbb{R}^{2d}$ - see [Wah07] for more details.

Moreover, by enlarging H_0 and using the magic formula (3.8) and Proposition 3.2.7, for any $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$ with $\|\Phi\|_{L^2} = 1$ we can write

$$\langle \sigma^{\mathrm{w}} f, g \rangle = \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \mathrm{Tr}[V_\Phi \sigma(z,\zeta) V_\Phi W(f,g)(z,-\zeta)|_{H_0}] dz d\zeta.$$

This representation is the main ingredient for proving boundedness results for symbols in modulation and amalgam spaces. For instance, the classical result of Theorem 4.2.2 for symbols in the Sjöstrand class extends to the vector valued case as follows.

Theorem 4.2.7. *Let $1 \leq p, q \leq \infty$, $\gamma \geq 0$ and $r, s \in \mathbb{R}$ be such that $|r| + |s| \leq \gamma$; denote by X any of the spaces $\mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d, H)$ or $\mathcal{W}^{p,q}_{r,s}(\mathbb{R}^d, H)$. If $\sigma \in M^{\infty,1}_{0,2\gamma}(\mathbb{R}^{2d}, \mathcal{L}(H))$ then the Weyl operator σ^{w} is bounded on X.*

If $H = \mathbb{C}^n$ then, under the same assumptions, the same claim holds for $\sigma \in M^{\infty,1}_{0,\gamma}(\mathbb{R}^{2d}, \mathbb{C}^{n\times n})$.

Proof. The case $X = \mathcal{M}^{p,q}_{r,s}(H)$ is covered in [Wah07, Corollary 4.8], and it is stated here with small modifications in the spirit of [Grö01, Theorem 14.5.6] in order to take the weights into account. In the case where $H = \mathbb{C}$ the regularity of the symbol can be improved using the characterization in Theorem 5.8.1 and and convolution relations for weighted Lebesgue spaces, in particular $L^{p,q}_m(\mathbb{R}^{2d}) * L^1_v(\mathbb{R}^{2d}) \subset L^{p,q}_m(\mathbb{R}^{2d})$ (cf. Theorem 2.3.3). The case $H = \mathbb{C}^n$, $n > 1$, follows arguing as above for each component.

For the case $X = \mathcal{W}^{p,q}_{r,s}(H)$ we need a special case of the symplectic covariance of Weyl calculus, namely

$$\mathcal{F}\sigma^{\mathrm{w}} = \sigma^{\mathrm{w}}_{J^{-1}}\mathcal{F}, \qquad \sigma \in \mathcal{S}'(\mathbb{R}^{2d}, \mathcal{L}(H)),$$

where $\sigma_{J^{-1}} = \sigma \circ J^{-1}$; the proof is a straightforward application of Proposition 3.1.5 above. In view of this property, consider the following diagram:

$$\begin{array}{ccc} \mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d, H) & \xrightarrow{\sigma^{\mathrm{w}}_J} & \mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d, H) \\ \uparrow{\scriptstyle \mathcal{F}^{-1}} & & \downarrow{\scriptstyle \mathcal{F}} \\ \mathcal{W}^{p,q}_{r,s}(\mathbb{R}^d, H) & \xrightarrow{\sigma^{\mathrm{w}}} & \mathcal{W}^{p,q}_{r,s}(\mathbb{R}^d, H) \end{array}$$

It is easy to prove that if $\sigma \in M^{\infty,1}_{0,2\gamma}(\mathbb{R}^{2d}, \mathcal{L}(H))$ then $\sigma_J \in M^{\infty,1}_{0,2\gamma}(\mathbb{R}^{2d}, \mathcal{L}(H))$ too (cf. for instance the proof of Lemma 5.9.6 below in the scalar setting), hence the preceding case implies that σ^{w}_J is bounded on $\mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d, H)$ for any $1 \leq p, q \leq \infty$ and $r, s \in \mathbb{R}$ such that $|r| + |s| \leq \gamma$.

The case $H = \mathbb{C}^n$ is covered by identical arguments. $\square$

To conclude, we mention that the benefits of narrow continuity in the scalar-valued case carry over to the Hilbert-valued case. The following property will be used below.

Theorem 4.2.8. *For any $1 \le p, q \le \infty$ and $\gamma \ge 0$, $r, s \in \mathbb{R}$ such that $|r| + |s| \le \gamma$, let X denote either $\mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d, H)$ or $\mathcal{W}^{p,q}_{r,s}(\mathbb{R}^d, H)$. If $\Omega \ni \nu \mapsto \sigma_\nu \in M^{\infty,1}_{0,2\gamma}(\mathbb{R}^{2d}, \mathcal{L}(H))$ is continuous for the narrow convergence then the corresponding map of operators $\nu \mapsto \sigma_\nu^{\mathrm{w}}$ is strongly continuous on X.*

If $H = \mathbb{C}^n$ then, under the same assumptions, the same claim holds for $\sigma \in M^{\infty,1}_{0,\gamma}(\mathbb{R}^{2d}, \mathbb{C}^{n\times n})$.

Proof. The proof for $X = \mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d, H)$ is a suitable adaption of the one given in [CNR15b, Proposition 3]. For the strong continuity on $X = \mathcal{W}^{p,q}_{r,s}(\mathbb{R}^d, H)$ we reduce to the latter case by the same arguments in the proof of Proposition 4.2.7, which imply that $\sigma_\nu^{\mathrm{w}} u = \mathcal{F}(\sigma_\nu)^{\mathrm{w}}_J \mathcal{F}^{-1} u$ for $u \in \mathcal{W}^{p,q}_{r,s}(\mathbb{R}^d, H)$. The claimed result easily follows from the continuity of the map $\nu \mapsto (\sigma_\nu)^{\mathrm{w}}_J \mathcal{F}^{-1} u$ on $\mathcal{M}^{p,q}_{r,s}(\mathbb{R}^d, H)$. □

4.3 Metaplectic operators

4.3.1 Notable facts on symplectic matrices

We recall the definition and the main properties of symplectic matrices, cf. [Gos11] and the references therein for further details.

The canonical symplectic matrix $J \in \mathbb{R}^{2d\times 2d}$ is defined by

$$J = \begin{bmatrix} O & I \\ -I & O \end{bmatrix}.$$

Note that $J^\top = J^{-1} = -J$. An invertible matrix $S \in \mathrm{GL}(2d, \mathbb{R})$ is said to be symplectic if $S^\top J S = J$. In this case, the matrices $S^\top$ and S^{-1} are symplectic too; note that the product of two symplectic matrices is clearly symplectic. As a result, the set of all symplectic matrices is a (Lie) group: we define the real symplectic group $\mathrm{Sp}(d, \mathbb{R})$ as

$$\mathrm{Sp}(d, \mathbb{R}) = \{S \in \mathrm{GL}(2d, \mathbb{R}) \,:\, S^\top J S = J\}.$$

We list below some well-known properties of symplectic matrices.

Proposition 4.3.1. *1. The eigenvalues of a symplectic matrix $S \in \mathrm{Sp}(d, \mathbb{R})$ occur in quadruples, meaning that if $\lambda \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of S then so are $\overline{\lambda}$ and $1/\lambda$ - hence $1/\overline{\lambda}$.*

2. *If* $S \in \mathrm{Sp}(d,\mathbb{R})$ *then* $\det S = 1$.

3. *Let* $S \in \mathrm{GL}(2d,\mathbb{R})$ *have the following block structure:*

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Then $S \in \mathrm{Sp}(d,\mathbb{R})$ *if and only if any of the two sets of conditions are satisfied:*

$$A^\top C,\ B^\top D \ \text{ are symmetric, and } A^\top D - C^\top B = I;$$

$$AB^\top,\ CD^\top \ \text{ are symmetric, and } AD^\top - CB^\top = I.$$

Moreover, in that case the inverse matrix is explicitly given by

$$S^{-1} = \begin{bmatrix} D^\top & -B^\top \\ -C^\top & A^\top \end{bmatrix}.$$

4. *The complex unitary group* $U(d,\mathbb{C})$ *is isomorphic to the subgroup of symplectic rotations* $\mathrm{U}(2d,\mathbb{R})$ *of* $\mathrm{Sp}(d,\mathbb{R})$ *defined by* $\mathrm{U}(2d,\mathbb{R}) = \mathrm{Sp}(d,\mathbb{R}) \cap \mathrm{O}(2d,\mathbb{R})$. *In particular, the following characterization holds:*

$$\mathrm{U}(2d,\mathbb{R}) = \left\{ \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \in \mathbb{R}^{2d\times 2d} \,:\, AA^\top + BB^\top = I,\ AB^\top = B^\top A \right\}.$$

We recall a result on a SVD-like decomposition of symplectic matrices, also known as the *Euler decomposition* in the literature; see [Ser17, Appendix B.2] for details and proofs.

Proposition 4.3.2. *For any* $S \in \mathrm{Sp}(d,\mathbb{R})$ *there exist* $U, V \in \mathrm{U}(2d,\mathbb{R})$ *such that*

$$S = U^\top D V, \quad D = \Sigma \oplus \Sigma^{-1},$$

where $\Sigma = \mathrm{diag}(\sigma_1,\ldots,\sigma_d)$ *and* $\sigma_1 \ge \ldots \ge \sigma_d \ge \sigma_d^{-1} \ge \ldots \ge \sigma_1^{-1}$ *are the singular values of* S.

We stress that while Σ is uniquely determined for given S once the order of the singular values is fixed, the matrices U and V appearing in such factorization are not unique in general due to possible occurrence of degenerate singular values. We identify any Euler decomposition of S as $U^\top D V$ with the triple (U,V,Σ).

Let us introduce the notion of *free symplectic matrix.*

Definition 4.3.3. *Let* $S \in \mathrm{Sp}(d,\mathbb{R})$. *We say that* S *is a free symplectic matrix if any of the following equivalent conditions is satisfied:*

(i) If $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *then* $\det B \neq 0$.

(ii) Given $(x, y) \in \mathbb{R}^{2d}$, *there exists a unique* $(\xi, \eta) \in \mathbb{R}^{2d}$ *such that* $(x, \xi) = S(y, \eta)$.

(iii) Set $(x, \xi) = S(y, \eta)$, $(y, \eta) \in \mathbb{R}^{2d}$. *Then* $\det(\partial_\eta x) \neq 0$.

The subset of free symplectic matrices is denoted by $\mathrm{Sp}_0(d, \mathbb{R})$.

Free symplectic matrices are naturally associated with quadratic forms. These are also called *generating functions*, in connection with those of canonical transformations in classical mechanics [Gol80].

Proposition 4.3.4. *Let* $S \in \mathrm{Sp}_0(d, \mathbb{R})$ *and define the generating function*

$$\Phi_S(x, y) := \frac{1}{2} DB^{-1}x \cdot x - B^{-1}x \cdot y + \frac{1}{2} B^{-1}Ay \cdot y. \tag{4.3}$$

Therefore,

$$(x, \xi) = S(x', \xi') \iff \begin{cases} \xi = \partial_x \Phi_S(x, y) \\ \xi' = -\partial_y \Phi_S(x, y) \end{cases}.$$

Conversely, let $L, P, Q \in \mathbb{R}^{d \times d}$ *be such that* $P = P^\top$, $Q = Q^\top$ *and* $\det L \neq 0$, *and set*

$$\Phi(x, y) = \frac{1}{2} Px \cdot x - Lx \cdot y + \frac{1}{2} Qy \cdot y.$$

Then,

$$S_\Phi := \begin{bmatrix} L^{-1}Q & L^{-1} \\ PL^{-1}Q - L^\top & PL^{-1} \end{bmatrix} \in \mathrm{Sp}_0(d, \mathbb{R}).$$

We finally recall a factorization result in terms of free matrices and provide a special set of generators of symplectic matrices.

Proposition 4.3.5. *1. For any* $S \in \mathrm{Sp}(d, \mathbb{R})$ *there exist (non unique)* $S_1, S_2 \in \mathrm{Sp}_0(d, \mathbb{R})$ *such that* $S = S_1 S_2$.

2. Given $P, L \in \mathbb{R}^{d \times d}$ *with* $P = P^\top$ *and* $\det L \neq 0$, *define*

$$V_P := \begin{bmatrix} I & 0 \\ P & I \end{bmatrix}, \quad U_P := \begin{bmatrix} P & I \\ -I & 0 \end{bmatrix}, \quad M_L := \begin{bmatrix} L & 0 \\ 0 & L^\# \end{bmatrix}. \tag{4.4}$$

If $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_0(d, \mathbb{R})$ *then*

$$S = V_{DB^{-1}} M_B U_{B^{-1}A} = V_{DB^{-1}} M_B J V_{B^{-1}A}.$$

As a consequence, any of the sets $\{V_P, M_L, J \,:\, P = P^\top, \det L \neq 0\}$ *and* $\{U_P, M_L \,:\, P = P^\top, \det L \neq 0\}$ *generates* $\mathrm{Sp}(d,\mathbb{R})$.

4.3.2 Metaplectic operators: definitions and basic properties

The metaplectic group $\mathrm{Mp}(d,\mathbb{R})$ is the double cover of the symplectic group $\mathrm{Sp}(d,\mathbb{R})$ and the corresponding faithful, strongly continuous unitary representation in $L^2(\mathbb{R}^d)$ allows us to identify $\mathrm{Mp}(d,\mathbb{R})$ with its range, the subgroup of unitary operators on L^2 consisting of *metaplectic operators*. Comprehensive accounts on the topic can be found in [Fol89; Gos11]; below we prefer to pay some precision and omit the technicalities since the finer aspects of the construction of the metaplectic representation will not be relevant to the applications in this dissertation.

By a slight abuse of language we use the notation $\mu(S)$ to denote metaplectic operators defined up to sign, where $S = \rho^{\mathrm{Mp}}(\mu(S)) \in \mathrm{Sp}(d,\mathbb{R})$ and $\rho^{\mathrm{Mp}} : \mathrm{Mp}(d,\mathbb{R}) \to \mathrm{Sp}(d,\mathbb{R})$ is the group projection, hence

$$\mu(AB) = \pm\mu(A)\mu(B), \quad A, B \in \mathrm{Sp}(d,\mathbb{R}).$$

Depending on the need, we think of $\mu(S)$ both as a pair of operators that differ only in sign or as one of the members of this pair.

Note that an operator $\mu(S)$ satisfies the intertwining relation

$$\pi'(Sz) = \mu(S)\pi'(z)\mu(S)^{-1}, \quad z \in \mathbb{R}^{2d},$$

where we introduced the symmetric time-frequency shifts [Grö01] (also known as Weyl-Heisenberg operators [Gos11])

$$\pi'(x,\xi) \coloneqq M_{\xi/2}T_xM_{\xi/2} = e^{-\pi i x\cdot\xi}\pi(x,\xi), \quad (x,\xi) \in \mathbb{R}^{2d}.$$

To be concrete, we recall the following representation result for metaplectic operators associated with free symplectic matrices in terms of *quadratic Fourier transforms* [Gos11].

Theorem 4.3.6 ([Fol89, Theorems 4.51 and 4.53])**.** *Let* $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(d,\mathbb{R})$.

1. *If* $\det B \neq 0$ *then*

$$\mu(S)f(x) = c|\det B|^{-1/2}\int_{\mathbb{R}^d} e^{2\pi i\Phi_S(x,y)}f(y)dy, \qquad f \in \mathcal{S}(\mathbb{R}^d), \tag{4.5}$$

for some $c \in \mathbb{C}$ *with* $|c| = 1$*, where* Φ_S *is the generating function of* S *defined in* (4.3).

2. *If* $\det A \neq 0$ *then*

$$\mu(S)f(x) = c|\det A|^{-1/2}\int_{\mathbb{R}^d} e^{2\pi i\Phi_{SJ^{-1}}(x,y)}\hat{f}(y)dy, \qquad f\in\mathcal{S}(\mathbb{R}^d),$$

for some $c\in\mathbb{C}$ *with* $|c|=1$*, where*

$$\Phi_{SJ^{-1}} = \frac{1}{2}CA^{-1}x\cdot x + A^{-1}x\cdot y - \frac{1}{2}A^{-1}By\cdot y. \tag{4.6}$$

In fact, the metaplectic group can be equivalently designed as the subgroup of unitary operators on L^2 generated by quadratic Fourier transforms [Gos11].

Metaplectic operators have been thoroughly studied in the framework of phase-space analysis. We mention below two relevant results concerning the Gabor matrix of a metaplectic operator and the boundedness on modulation spaces.

Theorem 4.3.7 ([CGNR13]). *Consider* $\mu(S)\in \mathrm{Mp}(d,\mathbb{R})$ *and* $g,\gamma\in\mathcal{S}(\mathbb{R}^d)$*. For any* $N\geq 0$ *we have*

$$|\langle \mu(S)\pi(z)g,\pi(w)\gamma\rangle| \lesssim_{N,S} (1+|w-Sz|)^{-N}, \quad w,z\in\mathbb{R}^{2d}.$$

As a consequence, for any $1\leq p\leq\infty$ *and* $s\in\mathbb{R}$*, the operator* $\mu(S)$ *is bounded from* $M^p_{v_s}(\mathbb{R}^d)$ *into itself; the continuity of* $\mu(S)$ *on* $M^{p,q}(\mathbb{R}^d)$ *with* $p\neq q$ *fails in general, cf. [CNR10, Proposition 7.1].*

In view of Theorem 4.3.7 we observe that the Gabor matrix $\langle \mu(S)\pi(z)g,\pi(w)\gamma\rangle$ of a metaplectic operator $\mu(S)\in\mathrm{Mp}(d,\mathbb{R})$ is well defined in the case where

$$g\in M^p(\mathbb{R}^d),\ \gamma\in M^q(\mathbb{R}^d), \quad \frac{1}{p}+\frac{1}{q}\geq 1. \tag{4.7}$$

To be precise,

$$\|\mu(S)\pi(z)g\|_{M^p} \leq \|\mu(S)\|_{M^p\to M^p}\|\pi(z)g\|_{M^p} = \|\mu(S)\|_{M^p\to M^p}\|g\|_{M^p}, \quad z\in\mathbb{R}^d,$$

hence by Proposition 3.2.3 (iv)

$$\begin{aligned}|\langle \mu(S)\pi(z)g,\pi(w)\gamma\rangle| &\leq \|\mu(S)\|_{M^p\to M^p}\|g\|_{M^p}\|\pi(w)\gamma\|_{M^{p'}}\\ &\leq \|\mu(S)\|_{M^p\to M^p}\|g\|_{M^p}\|\gamma\|_{M^{p'}}\\ &\leq \|\mu(S)\|_{M^p\to M^p}\|g\|_{M^p}\|\gamma\|_{M^q},\end{aligned}$$

since from (4.7) we infer $q\leq p'$ and the inclusion $M^q(\mathbb{R}^d)\subset M^{p'}(\mathbb{R}^d)$ (Proposition 3.2.3 (v)) yields the last inequality.

4.3.3 Important examples of metaplectic operators

Using the results in Theorem 4.3.6 we provide some elementary examples of metaplectic operators which are associated with the special elements of $\mathrm{Sp}(d,\mathbb{R})$ highlighted in (4.4). In the sequel we let $c \in \mathbb{C}$ be a phase factor, namely $|c| = 1$, which can be different from time to time.

1. The Fourier transform is a metaplectic operator associated with the canonical symplectic matrix, that is $\mu(J)f = c\mathcal{F}(f)$. Notice in particular that $\mu(-J) = c\mathcal{F}^{-1}$.

2. Let $L \in \mathrm{GL}(d,\mathbb{R})$. The metaplectic operator $\mu(M_L)$ acts as a rescaling by L:
$$\mu(M_L)f(x) = c|\det L|^{-1/2} f(L^{-1}x).$$

3. Let $P \in \mathbb{R}^{d\times d}$ be a symmetric matrix. The metaplectic operator $\mu(V_P)$ is a chirp multiplication:
$$\mu(V_P)f(x) = ce^{\pi i x\cdot Px} f(x).$$

In fact, it turns out that $\{\mu(V_P), \mu(M_L), J \,:\, P = P^\top, \det L \neq 0\}$ is a set of generators of $\mathrm{Mp}(d,\mathbb{R})$, cf. [Gos11, Corollary 112].

An important example of metaplectic operator is provided by the Schrödinger propagator for the free particle $U(t) = e^{-i(t/2\pi)\Delta}$, $t \in \mathbb{R}$. This can be easily derived from the examples above since $U(t)$ is a Fourier multiplier with chirp symbol $m_t(\xi) = e^{2\pi i t|\xi|^2}$ on $\mathbb{R}^d$, hence
$$U(t) = \mathcal{F}^{-1} m_t \mathcal{F} = c(t)\mu(S_t), \quad S_t = \begin{bmatrix} I & 2tI \\ O & I \end{bmatrix} \in \mathrm{Sp}(d,\mathbb{R}), \quad t \in \mathbb{R}, \tag{4.8}$$
where $c(t) \in \mathbb{C}$ satisfies $|c(t)| = 1$. In general, let $Q : \mathbb{R}^{2d} \to \mathbb{R}$ be a homogeneous quadratic polynomial, namely
$$Q(x,\xi) = \frac{1}{2}A\xi\cdot\xi + Bx\cdot\xi + \frac{1}{2}Cx\cdot x,$$
for some $A, B, C \in \mathbb{R}^{d\times d}$ with $A = A^\top$ and $C = C^\top$. Let us consider the Schrödinger equation (with the normalization $\hbar = 1/2\pi$)
$$\begin{cases} i\partial_t f(t,x) = 2\pi Q^{\mathrm{w}} f(t,x) \\ f(0,x) = f_0(x) \end{cases}, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^{2d},$$

where the Hamiltonian is the Weyl quantization of Q, namely

$$Q^{\mathrm{w}} = -\frac{1}{8\pi^2}\sum_{j,k=1}^{d} A_{j,k}\partial_j\partial_k - \frac{i}{2\pi}\sum_{j,k=1}^{d} B_{j,k}x_j\partial_k - \frac{i}{4\pi}\mathrm{Tr}(B) + \frac{1}{2}\sum_{j,k=1}^{d} C_{j,k}x_jx_k.$$

It turns out that the propagator $U(t) = e^{-2\pi i t Q^{\mathrm{w}}}$ is an example of metaplectic operator [Gos11, Section 15.1.3]. In particular, the classical phase-space flow determined by the Hamilton equations (the factor 2π derives from the normalization of the Fourier transform adopted here)

$$2\pi\dot{z} = J\nabla_z Q(z) = \begin{bmatrix} B & A \\ -C & -B^\top \end{bmatrix} =: \mathbb{S}, \quad z = (x, y) \in \mathbb{R}^{2d},$$

defines a mapping $\mathbb{R} \ni t \mapsto S_t := e^{(t/2\pi)\mathbb{S}} \in \mathrm{Sp}(d,\mathbb{R})$. It follows from the general theory of covering manifolds that this path can be lifted in a unique way to a mapping $\mathbb{R} \ni t \mapsto M(t) \in \mathrm{Mp}(d,\mathbb{R})$, $M(0) = I$; hence $\rho^{\mathrm{Mp}}(M(t)) = S_t$ and we have $U(t) = e^{-2\pi i t Q^{\mathrm{w}}} = \mu(S_t)$.

4.3.4 Symplectic covariance of the Weyl calculus

The Weyl quantization satisfies a special intertwining property involving metaplectic operators, called *symplectic covariance*.

Proposition 4.3.8 ([Gos11, Theorem 215])**.** *For any $S \in \mathrm{Sp}(d,\mathbb{R})$ and $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$,*

$$(\sigma \circ S)^{\mathrm{w}} = \mu(S)^{-1}\sigma^{\mathrm{w}}\mu(S). \tag{4.9}$$

Symplectic covariance is in fact a distinctive property of the Weyl quantization among all possible quantization rules, as detailed below.

Theorem 4.3.9 ([Won98])**.** *Let $\mathrm{op} : \mathcal{S}'(\mathbb{R}^{2d}) \to \mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$ be a continuous linear operator such that:*

1. *if $\sigma(x,\xi) = m(x)$ and $m \in L^\infty(\mathbb{R}^d)$ then $\mathrm{op}(\sigma)f(x) = m(x)f(x)$;*
2. *for any $S \in \mathrm{Sp}(d,\mathbb{R})$ and $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$, $\mathrm{op}(\sigma \circ S) = \mu(S)^{-1}\mathrm{op}(\sigma)\mu(S)$.*

Then, $\mathrm{op}(\sigma) = \sigma^{\mathrm{w}}$.

4.3.5 Generalized metaplectic operators

General families of operators characterized by the sparsity of their phase-space representation were introduced in [CGNR13; CGNR14]. Given $S \in \mathrm{Sp}(d,\mathbb{R})$, $g \in \mathcal{S}(\mathbb{R}^d)$ and $s \geq 0$, we say that a linear operator $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is in the class $FIO(S, v_s)$ of *generalized metaplectic operators* if there exists $H \in L^1_s(\mathbb{R}^{2d})$ such that

$$|\langle A\pi(z)g, \pi(w)g\rangle| \leq H(w - Sz), \quad w, z \in \mathbb{R}^{2d}. \tag{4.10}$$

We write $FIO(S)$ in the case where $s = 0$. The definition of $FIO(S, v_s)$ does not depend on the choice of $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. In fact, careful inspection of the proof of [CGNR14, Proposition 3.1] reveals that for $s = 0$ the class of admissible windows for $FIO(S)$ may be extended to $M^1(\mathbb{R}^d)$, hence the estimate (4.10) is equivalent to its *polarized* version with two arbitrary windows $g, \gamma \in M^1(\mathbb{R}^d)$, that is,

$$|\langle A\pi(z)g, \pi(w)\gamma\rangle| \leq H(w - Sz), \quad w, z \in \mathbb{R}^{2d}.$$

Sparsity of the Gabor matrix of generalized metaplectic operators provides non-trivial algebraic properties for $FIO(S)$ in the spirit of Theorem 4.2.6, as detailed in the following result.

Theorem 4.3.10. *Let $S, S_1, S_2 \in \mathrm{Sp}(d,\mathbb{R})$ and $s \geq 0$.*

1. *An operator $T \in FIO(S, v_s)$ is bounded on $M^p_{v_s}(\mathbb{R}^d)$ for any $1 \leq p \leq \infty$.*
2. *If $T_1 \in FIO(S_1, v_s)$ and $T_2 \in FIO(S_2, v_s)$, then $T_1T_2 \in FIO(S_1S_2, v_s)$.*
3. *If $T \in FIO(S, v_s)$ is invertible on $L^2(\mathbb{R}^d)$ then $T^{-1} \in FIO(S^{-1}, v_s)$.*
4. *Let $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ be a linear continuous operator. $T \in FIO(S, v_s)$ if and only if there exist $\sigma_1, \sigma_2 \in M^{\infty,1}_{0,s}(\mathbb{R}^{2d})$ such that*

   $$T = \sigma_1^{\mathrm{w}}\mu(S) = \mu(S)\sigma_2^{\mathrm{w}}.$$

   In particular, $\sigma_2 = \sigma_1 \circ S$.
5. *Let $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(d,\mathbb{R})$ be such that $\det A \neq 0$ and $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ be a linear continuous operator. $T \in FIO(S, v_s)$ if and only if there exists $\sigma \in M^{\infty,1}_{0,s}(\mathbb{R}^{2d})$ such that it can be represented as a Fourier integral operator, namely*

   $$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi i\Phi_{SJ^{-1}}(x,y)}\sigma(x,y)\hat{f}(y)dy,$$

   where $\Phi_{SJ^{-1}}$ is given in (4.6).

In view of Theorem 4.3.10 and arguing as we did before for metaplectic operators, we observe that the Gabor matrix $\langle T\pi(z)g, \pi(w)\gamma\rangle$ of a generalized metaplectic operator $T \in FIO(S)$ is well defined in the case where

$$g \in M^p(\mathbb{R}^d),\ \gamma \in M^q(\mathbb{R}^d), \quad \frac{1}{p} + \frac{1}{q} \geq 1.$$

Part II

Time-Frequency Analysis of Operators and Applications

Chapter 5

Linear Perturbations of the Wigner Transform and the Weyl Quantization

5.1 Outline

In this chapter we provide a systematic survey of the accumulated knowledge about the matrix Wigner transforms and their pseudodifferential calculi introduced in Chapter 1. In the first part we discuss the general properties of the matrix Wigner transforms.

1. We state the main formulas for covariance, the behaviour with respect to the Fourier transform, the analogue of Moyal's formulas, and the inversion formula.

2. It is well known that, up to normalization, the ambiguity function and the short-time Fourier transform are just different versions and names for the Wigner transform. This is no longer true for the matrix Wigner transforms, so we give precise conditions on the parametrizing matrix A so that $\mathcal{B}_A$ can be expressed as a short-time Fourier transform, up to a phase factor and a change of coordinates.

3. Of special importance is the intersection of the class of matrix Wigner transforms with Cohen's class. The main result in this respect is that $\mathcal{B}_A$ belongs to Cohen's class, if and only if $A = \begin{bmatrix} I & T \\ I & -(I-T) \end{bmatrix}$ for some $T \in \mathbb{R}^{d\times d}$, namely is a matrix generalization of the τ-Wigner transform. A thorough

time-frequency analysis of the corresponding Cohen kernel and the related properties is carried out.

4. A further item is the boundedness of the bilinear mapping $(f, g) \to \mathcal{B}_A(f, g)$ on various function spaces. These results are quite useful in the analysis of the mapping properties of the pseudodifferential operators σ^A.

In the second part we study the pseudodifferential calculi defined by the rule (1.17).

1. We first show that every "reasonable" operator can be represented as a pseudodifferential operator σ^A. We remark that the map $(\sigma, A) \mapsto \sigma^A$ is highly non-injective and, given two matrices A and B and two symbols σ, ρ, we obtain formulas characterizing the condition $\sigma^A = \rho^B$.

2. A large section is devoted to the mapping properties of the pseudodifferential operator σ^A on various function spaces, in particular on L^p-spaces and on modulation and amalgam spaces. Sharp results for τ-operators derived in [CDT19] are highlighted.

3. Finally, we extend the boundedness results for symbols in the Sjöstrand class and related spaces to those pseudodifferential operators for which $\mathcal{B}_A$ is in Cohen's class, in the spirit of [CNT19b].

For most results we will include proofs, but we will omit those proofs that only require a formal computation. The organization of the chapter follows the structure of the paper [BCGT20].

5.2 Preliminary results

5.2.1 Bilinear coordinate transformations

Let us summarize the properties of bilinear coordinate transformations in the time-frequency plane. Given a matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{R}^{2d\times 2d}$ with $d \times d$ blocks $A_{ij} \in \mathbb{R}^{d\times d}$, $i, j = 1, 2$, we use the symbol $\mathfrak{T}_A$ to denote the corresponding dilation of $F : \mathbb{R}^{2d} \to \mathbb{C}$, namely

$$\mathfrak{T}_A F(x, y) := F(Ax) = F(A_{11}x + A_{12}y, A_{21}x + A_{22}y).$$

For $A \in \mathrm{GL}(d,\mathbb{R})$ and $u \in \mathcal{S}'(\mathbb{R}^d)$, the dilated distribution $\mathfrak{T}_A u \in \mathcal{S}'(\mathbb{R}^d)$ is defined by follows:

$$\langle \mathfrak{T}_A u, \phi \rangle := \left\langle u, |\det A|^{-1} \mathfrak{T}_{A^{-1}} \phi \right\rangle, \qquad \phi \in \mathcal{S}\big(\mathbb{R}^d\big).$$

The following lemma collects elementary facts on such transformations.

Lemma 5.2.1. *(i) For any $A, B \in \mathbb{R}^{2d\times 2d}$ we have $\mathfrak{T}_A \mathfrak{T}_B = \mathfrak{T}_{BA}$.*

(ii) If $A \in \mathrm{GL}(2d,\mathbb{R})$, the transformation $\mathfrak{T}_A$ is a topological isomorphism on $L^2(\mathbb{R}^{2d})$ with inverse $\mathfrak{T}_A^{-1} = \mathfrak{T}_{A^{-1}}$ and adjoint $\mathfrak{T}_A^* = |\det A|^{-1}\mathfrak{T}_{A^{-1}}$.

(iii) If $A \in \mathrm{GL}(2d,\mathbb{R})$, the transformation $\mathfrak{T}_A$ is an isomorphism on $M^1(\mathbb{R}^{2d})$.

(iv) For any $A \in \mathrm{GL}(d,\mathbb{R})$, $f \in L^2(\mathbb{R}^d)$ and $x, \xi \in \mathbb{R}^d$,

$$\mathfrak{T}_A T_x f = T_{A^{-1}x} \mathfrak{T}_A f, \qquad \mathfrak{T}_A M_\xi f = M_{A^\top \xi} \mathfrak{T}_A f.$$

(v) For any $A \in \mathrm{GL}(d,\mathbb{R})$ and $f \in L^2(\mathbb{R}^d)$,

$$\mathcal{F}\mathfrak{T}_A f = |\det A|^{-1} \mathfrak{T}_{A^\#} \mathcal{F} f.$$

(vi) For any $A \in \mathrm{GL}(d,\mathbb{R})$ and $u \in \mathcal{S}'(\mathbb{R}^d)$,

$$\mathfrak{T}_A \mathcal{F}^{-1} u = |\det A|^{-1} \mathcal{F}^{-1} \mathfrak{T}_{A^\#} u.$$

Proof. The only non-trivial issue is the continuity on M^1. By Proposition 3.2.4 we have

$$\|\mathfrak{T}_A F\|_{M^1} \le C |\det A|^{-1} \big(\det\big(I + A^\top A\big)\big)^{1/2} \|F\|_{M^1},$$

for some constant $C > 0$. □

Two special transformations deserve a separate notation. The first is the *flip* operator

$$\tilde{F}(x,y) = \mathfrak{T}_{\tilde{I}} F(x,y) = F(y,x), \qquad \tilde{I} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \in \mathrm{GL}(2d,\mathbb{R}),$$

while the other one is the *reflection* operator:

$$\mathcal{I}F(x,y) = \mathfrak{T}_{-I} F(x,y) = F(-x,-y).$$

Sometimes we will also write $\mathcal{I} = -I \in \mathrm{GL}(2d,\mathbb{R})$.

5.2.2 Partial Fourier transforms

Given $F \in L^1(\mathbb{R}^{2d})$, we use the symbols $\mathcal{F}_1$ and $\mathcal{F}_2$ to denote the partial Fourier transforms

$$\mathcal{F}_1 F(\xi, y) := \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot t} F(t, y) dt, \qquad \xi, y \in \mathbb{R}^d,$$

$$\mathcal{F}_2 F(x, \xi) := \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot t} F(x, t) dt, \qquad x, \xi \in \mathbb{R}^d.$$

The partial Fourier transforms of F are well defined pointwise as a consequence of Fubini's theorem. The Fourier transform $\mathcal{F}$ is related to the partial Fourier transforms as

$$\mathcal{F} = \mathcal{F}_1 \mathcal{F}_2 = \mathcal{F}_2 \mathcal{F}_1.$$

Using Plancherel's theorem and properties of modulation spaces (Proposition 3.2.13), the following extension of the partial Fourier transform is routine. For a different proof in a more general context see [FK98, Lemma 7.3.6].

Lemma 5.2.2. *(i) The partial Fourier transform $\mathcal{F}_2$ is a unitary operator on $L^2(\mathbb{R}^{2d})$. In particular, after setting $\mathcal{I}_2 = I \oplus (-I)$,*

$$\mathcal{F}_2^* F(x, y) = \mathcal{F}_2^{-1} F(x, y) = \mathcal{F}_2 F(x, -y) = \mathfrak{T}_{\mathcal{I}_2} \mathcal{F}_2 F(x, y).$$

(ii) The partial Fourier transform $\mathcal{F}_2$ is an isomorphism on $\mathcal{S}(\mathbb{R}^{2d})$ and $M^1(\mathbb{R}^{2d})$, hence on $\mathcal{S}'(\mathbb{R}^{2d})$ and $M^\infty(\mathbb{R}^{2d})$.

5.3 Matrix-Wigner distributions

Let us define the main characters of this chapter.

Definition 5.3.1. *Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathrm{GL}(2d, \mathbb{R})$. The time-frequency distribution of Wigner type for f and g associated with A (in short: matrix-Wigner distribution, MWD) is defined for suitable functions f, g as*

$$\mathcal{B}_A(f, g)(x, \xi) = \mathcal{F}_2 \mathfrak{T}_A (f \otimes \overline{g})(x, \xi). \tag{5.1}$$

When $g = f$, we write $\mathcal{B}_A f$ for $\mathcal{B}_A(f, f)$.

Explicitly, $\mathcal{B}_A$ is given by

$$\mathcal{B}_A(f, g)(x, \xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot y} f(A_{11} x + A_{12} y) \overline{g(A_{21} x + A_{22} y)} dy.$$

This definition is meaningful on many function spaces. We state a result for the Banach triple (M^1, L^2, M^∞), but it also holds if the latter is replaced by the standard $(\mathcal{S}, L^2, \mathcal{S}')$.

Proposition 5.3.2. *Assume $A \in \mathrm{GL}(2d, \mathbb{R})$.*

(i) If $f, g \in M^1(\mathbb{R}^d)$, then $\mathcal{B}_A(f,g)$ is defined pointwise and belongs to $M^1(\mathbb{R}^{2d})$. Moreover, the mapping $\mathcal{B}_A : M^1(\mathbb{R}^d) \times M^1(\mathbb{R}^d) \to M^1(\mathbb{R}^{2d})$ is continuous.

(ii) If $f, g \in L^2(\mathbb{R}^d)$, then $\mathcal{B}_A(f,g) \in L^2(\mathbb{R}^{2d})$ and the mapping $\mathcal{B}_A : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ is continuous. Furthermore, $\mathrm{span}\{\mathcal{B}_A(f,g) \mid f, g \in L^2(\mathbb{R}^d)\}$ is a dense subset of $L^2(\mathbb{R}^{2d})$.

(iii) If $f, g \in M^\infty(\mathbb{R}^d)$, then $\mathcal{B}_A(f,g) \in M^\infty(\mathbb{R}^{2d})$ and the mapping $\mathcal{B}_A : M^\infty(\mathbb{R}^d) \times M^\infty(\mathbb{R}^d) \to M^\infty(\mathbb{R}^{2d})$ is continuous.

The standard time-frequency representations covered within this framework include for instance:

- the short-time Fourier transform:

$$V_g f(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot y} f(y) \overline{g(y-x)} dy = \mathcal{B}_{A_{ST}}(f,g)(x,\xi),$$

  where

$$A_{ST} = \begin{bmatrix} 0 & I \\ -I & I \end{bmatrix};$$

- the cross-ambiguity function:

$$\mathrm{Amb}(f,g)(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot y} f\Big(y + \frac{x}{2}\Big) \overline{g\Big(y - \frac{x}{2}\Big)} dy = \mathcal{B}_{A_{\mathrm{Amb}}}(f,g)(x,\xi), \tag{5.2}$$

  where

$$A_{\mathrm{Amb}} = \begin{bmatrix} I/2 & I \\ -I/2 & I \end{bmatrix};$$

- the Wigner distribution:

$$W(f,g)(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot y} f\Big(x + \frac{y}{2}\Big) \overline{g\Big(x - \frac{y}{2}\Big)} dy;$$

- the Rihaczek distribution:

$$R(f,g)(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot y} f(x) \overline{g(x-y)} dy = e^{-2\pi i x \cdot \xi} f(x) \overline{\hat{g}(\xi)}.$$

The latter two distributions are special cases of the τ-Wigner distribution introduced in [BDDO10]. For any $\tau \in [0,1]$, we have

$$W_\tau(f,g)(x,\xi) = \mathcal{B}_{A_\tau}(f,g)(x,\xi),$$

where

$$A_\tau = \begin{bmatrix} I & \tau I \\ I & -(1-\tau)I \end{bmatrix}. \tag{5.3}$$

It is well known that the Wigner distribution and the ambiguity transform coincide with the STFT up to normalization. It is natural to wonder whether a similar relation holds for general matrix-Wigner distributions. Note that a characteristic which is common to the corresponding matrices is *right-regularity*: we say that a block matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{R}^{2d\times 2d}$ is left-regular (resp. right-regular) if the submatrices $A_{11}, A_{21} \in \mathbb{R}^{d\times d}$ (resp. $A_{12}, A_{22} \in \mathbb{R}^{d\times d}$) are invertible.

The right-regularity of A is an essential condition to express $\mathcal{B}_A(f,g)$ as a short-time Fourier transform. In fact, this characterization is very strong, as stated in the subsequent results. Beware that $(A^{\#})_{ij} \neq A^{\#}_{ij} = (A^{\top}_{ij})^{-1}$, $i,j = 1,2$.

Theorem 5.3.3 ([Bay10, Theorem 1.2.5])**.** *Assume that $A \in \mathrm{GL}(2d,\mathbb{R})$ is right-regular. For every $f,g \in M^1(\mathbb{R}^d)$ the following formula holds:*

$$\mathcal{B}_A(f,g)(x,\omega) = |\det A_{12}|^{-1} e^{2\pi i A^{\#}_{12}\omega\cdot A_{11}x} V_{\tilde{g}} f(c(x), d(\omega)), \quad x,\omega \in \mathbb{R}^d, \tag{5.4}$$

where $c(x) = \big(A_{11} - A_{12}A_{22}^{-1}A_{21}\big)x$, $d(\omega) = A^{\#}_{12}\omega$ and $\tilde{g}(t) = g\big(A_{22}A_{12}^{-1}t\big)$.

An equivalent and slightly more clear formulation is the following one.

Theorem 5.3.4. *Given matrices $M,N,P \in \mathbb{R}^{d\times d}$ and $Q,R \in \mathrm{GL}(\mathbb{R},d)$, set*

$$A = \begin{bmatrix} Q^{\#}N^{\top}M & Q^{\#} \\ R\big(Q^{\#}N^{\top}M - P\big) & RQ^{\#} \end{bmatrix}.$$

Then A is right-regular and for any $f,g \in M^1(\mathbb{R}^d)$ we have

$$\mathcal{B}_A(f,g)(x,\omega) = |\det Q| e^{2\pi i Mx\cdot N\omega} V_{g\circ R}(Px, Q\omega), \quad x,\omega \in \mathbb{R}^d.$$

The peculiar way the blocks of A are combined in $c(x) = (A_{11} - A_{12}A_{22}^{-1}A_{21})x$ is a well-known construction in linear algebra and is usually called *Schur complement*. The Schur complement comes up many times in our results, ultimately because of its distinctive role in the inversion of block matrices (cf. [LS02, Theorem 2.1]).

The right-regularity of A is not only a technical condition required for (5.4) to hold, but also has unexpected effects on the continuity of $\mathcal{B}_A$.

Theorem 5.3.5 ([Bay10, Theorem 1.2.9]). *Assume* $A \in \mathrm{GL}(2d, \mathbb{R})$ *such that* $\det A_{22} \neq 0$ *but* $\det A_{12} = 0$. *Then there exist* $f, g \in L^2(\mathbb{R}^d)$ *such that* $\mathcal{B}_A(f, g)$ *is not a continuous function on* $\mathbb{R}^{2d}$.

The study of the properties satisfied by general bilinear time-frequency distributions (including orthogonality relations, inversion and reconstruction formulae, algebraic identities, etc.) is mostly a matter of computation; it has been carried out in Bayer's Ph.D. thesis [Bay10] and slightly expanded in [CT20]. Moreover, a complete survey with all pointers can be found in [BCGT20]. We confine ourselves to mention two results that will be heavily used below. The first one is the behaviour under phase-space shifts, leading to the *covariance formula.*

Theorem 5.3.6 ([Bay10, Theorem 1.5.1]). *Let* $A \in \mathrm{GL}(2d, \mathbb{R})$. *For any* $f, g \in M^1(\mathbb{R}^d)$ *and* $a, b, \alpha, \beta \in \mathbb{R}^d$, *the following formula holds:*

$$\begin{aligned}\mathcal{B}_A(M_\alpha T_a f, M_\beta T_b g)(x, \xi) &= e^{2\pi i \sigma \cdot s} M_{(\rho, -s)} T_{(r, \sigma)} \mathcal{B}_A(f, g)(x, \xi) \\ &= e^{2\pi i \sigma \cdot s} e^{2\pi i (x \cdot \rho - \xi \cdot s)} \mathcal{B}_A(f, g)(x - r, \xi - \sigma),\end{aligned} \tag{5.5}$$

where

$$\begin{bmatrix} r \\ s \end{bmatrix} = A^{-1} \begin{bmatrix} a \\ b \end{bmatrix}, \qquad \begin{bmatrix} \rho \\ \sigma \end{bmatrix} = A^\top \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}.$$

This result clearly encompasses the covariance formula for the τ-Wigner distribution with $A = A_\tau$ as in (5.3), cf. [CNT19b, Proposition 3.3] and also for the STFT with $A = A_{ST}$, cf. 3.1.2.

We mention an amazing representation result for the STFT of a bilinear time-frequency distribution, sometimes called the short-time product formula or *magic formula* in similar contexts (cf. [Grö06a]).

Theorem 5.3.7 ([Bay10, Theorem 1.7.1]). *Assume* $A \in \mathrm{GL}(2d, \mathbb{R})$ *and* $f, g, \psi, \phi \in M^1(\mathbb{R}^d)$, *and set* $z = (z_1, z_2)$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$. *Then,*

$$V_{\mathcal{B}_A(\phi, \psi)} \mathcal{B}_A(f, g)(z, \zeta) = e^{-2\pi i z_2 \cdot \zeta_2} V_\phi f(a, \alpha) \overline{V_\psi g(b, \beta)}, \tag{5.6}$$

where

$$\begin{bmatrix} a \\ b \end{bmatrix} = A \mathcal{I}_2 \begin{bmatrix} z_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} A_{11} z_1 - A_{12} \zeta_2 \\ A_{21} z_1 - A_{22} \zeta_2 \end{bmatrix},$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \mathcal{I}_2 A^{\#} \begin{bmatrix} \zeta_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} (A^{\#})_{11} \zeta_1 + (A^{\#})_{12} z_2 \\ -(A^{\#})_{21} \zeta_1 - (A^{\#})_{22} z_2 \end{bmatrix}.$$

We also mention the following generalized Moyal formula, which extends the Parseval identity to matrix-Wigner distributions. In fact, the generalization of the orthogonality relations was one of the main motivations for introducing $\mathcal{B}_A$ in [Bay10].

Theorem 5.3.8 ([Bay10, Theorem 1.3.1])**.** *Let $A \in \mathrm{GL}(2d,\mathbb{R})$ and $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$. Then*

$$\langle \mathcal{B}_A(f_1,g_1), \mathcal{B}_A(f_2,g_2)\rangle_{L^2(\mathbb{R}^{2d})} = \frac{1}{|\det A|}\langle f_1, f_2\rangle_{L^2(\mathbb{R}^d)}\overline{\langle g_1, g_2\rangle_{L^2(\mathbb{R}^d)}}. \tag{5.7}$$

In particular,

$$\|\mathcal{B}_A(f,g)\|_{L^2(\mathbb{R}^{2d})} = \frac{1}{|\det A|^{1/2}}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.$$

Thus, the representation $\mathcal{B}_{A,g} : L^2(\mathbb{R}^d) \ni f \mapsto \mathcal{B}_A(f,g) \in L^2(\mathbb{R}^{2d})$ is a non-trivial constant multiple of an isometry whenever $g \neq 0$.

The proof follows directly from the definition in (5.1), since $\mathcal{F}_2$ is unitary and $\mathfrak{T}_A$ is a multiple of a unitary operator.

Let us underline that the benefits of linear algebra should be appreciated in view of the very short and simple proofs. This aspect should not be underestimated: the proof of similar results for certain special members has lead to quite cumbersome computations (cf. the proofs for the τ-Wigner distributions in [CNT19b]).

5.4 Cohen class members as perturbations of the Wigner transform

We already described the heuristics behind the Cohen class of distributions in Section 1.2.1. Although the Wigner distribution was the main inspiration for the bilinear distributions studied so far, the connection to the Cohen class is by no means clear. This question is the point of departure of the paper [CT20] and the following result completely characterizes the intersection between these families[1]

Theorem 5.4.1 ([CT20, Theorem 1.1])**.** *Let $A \in \mathrm{GL}(2d,\mathbb{R})$. The distribution $\mathcal{B}_A$ belongs to the Cohen class if and only if there exists $M \in \mathbb{R}^{d\times d}$ such that*

$$A = A_M = \begin{bmatrix} I & M + (1/2)I \\ I & M - (1/2)I \end{bmatrix},$$

[1] We take this opportunity to highlight a minor mistake in the published proof, namely the expression of A^{-1} is wrong. This issue does not affect the result, as showed in the corrected proof below. Moreover, the explicit formula for θ_M in [CT20, Eq. (7)] and [BCGT20, Eq. 31] has been mistakenly reported with M^{-1} in place of $M^{\#}$ in the exponent.

Furthermore, in this case we have

$$W_M(f,g) \coloneqq \mathcal{B}_{A_M}(f,g) = W(f,g) * \theta_M, \quad f,g \in \mathcal{S}(\mathbb{R}^d),$$

where the Cohen kernel $\theta_M \in \mathcal{S}'(\mathbb{R}^{2d})$ *is*

$$\theta_M = \mathcal{F}^{-1}\Theta_M, \quad \textit{with} \quad \Theta_M(u,v) = e^{-2\pi i u\cdot Mv}, \quad (u,v) \in \mathbb{R}^{2d}.$$

If M *is invertible, the kernel* θ_M *is explicitly given by*

$$\theta_M(x,\xi) = |\det M|^{-1} e^{2\pi i \xi\cdot M^{-1}x}, \quad (x,\xi) \in \mathbb{R}^{2d}. \tag{5.8}$$

We say that $A = A_M$ *is a Cohen-type matrix associated with* $M \in \mathbb{R}^{d\times d}$.

Proof. We prove necessity first. It follows directly from the definition that a member of Cohen's class must satisfy the covariance property:

$$Q(\pi(z)f) = T_z Qf, \quad z \in \mathbb{R}^{2d},\ f \in M^1(\mathbb{R}^d).$$

For fixed $z = (x,\xi) \in \mathbb{R}^d$, by Theorem 5.3.6 with $\alpha = \beta = \xi$, $a = b = x$ and $f = g$ we get

$$\begin{bmatrix} \rho \\ \sigma \end{bmatrix} = A^\top \begin{bmatrix} \xi \\ -\xi \end{bmatrix} = \begin{bmatrix} 0 \\ \xi \end{bmatrix}, \qquad \begin{bmatrix} r \\ s \end{bmatrix} = A^{-1} \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

Converting the relations for ρ and σ into conditions for the blocks of A yields

$$A_{11} = A_{21}, \qquad A_{12} - A_{22} = I.$$

Moreover, the conditions on r and s yield that $A_{11} = A_{21} = I$. In conclusion, if $\mathcal{B}_A$ belongs to the Cohen class, then A has the form

$$A = \begin{bmatrix} I & M + (1/2)I \\ I & M - (1/2)I \end{bmatrix},$$

where $M \in \mathbb{R}^{d\times d}$ is such that $A_{22} = M - (1/2)I$; other parametrizations are of course allowed, provided that $A_{12} - A_{22} = I$.

For what concerns sufficiency, assume that $A = A_M$ has this prescribed form. We shall show that $W_M = W * \theta_M$ for some $\theta_M \in \mathcal{S}'(\mathbb{R}^{2d})$. Applying the symplectic Fourier transform to both sides of the latter relation, this is equivalent to showing that, for any $f,g \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathcal{F}_\sigma W_M(f,g) = \mathcal{F}_\sigma W(f,g) \cdot \mathcal{F}_\sigma \theta_M = \mathrm{Amb}(f,g) \cdot \mathcal{F}_\sigma \theta_M,$$

where $\mathrm{Amb}(f,g)$ is defined in (5.2). A straightforward computation (cf. [CT20, Proposition 3.5] for the details) shows that

$$\mathcal{F} W_M(f,g)(u,v) = \mathcal{B}_{A_M J}(f,g)(v,u), \quad u,v \in \mathbb{R}^d,$$

where

$$A_M J = \begin{bmatrix} -(M+(1/2)I) & I \\ -(M-(1/2)I) & I \end{bmatrix}.$$

Therefore

$$\begin{aligned}\mathcal{F}_\sigma W_M(f,g)(u,v) &= \mathcal{F}W_M(f,g)(J(u,v)) \\ &= \mathcal{F}W_M(f,g)(v,-u)) \\ &= \mathcal{B}_{A_M J}(f,g)(-u,v) \\ &= \int_{\mathbb{R}^d} e^{-2\pi i v\cdot t} f\Big(t+\Big(M+\frac{1}{2}I\Big)u\Big)\overline{g\Big(t+\Big(M-\frac{1}{2}I\Big)u\Big)}dt.\end{aligned}$$

The substitution $t+(M-(1/2)I)u = z-u/2$ yields

$$\begin{aligned}\mathcal{F}_\sigma W_M(f,g)(u,v) &= e^{2\pi i v\cdot Mu}\int_{\mathbb{R}^d} e^{-2\pi i v\cdot z} f\Big(z+\frac{u}{2}\Big)\overline{g\Big(z-\frac{u}{2}\Big)}dz \\ &= e^{2\pi i v\cdot Mu}\cdot \mathrm{Amb}(f,g)(u,v),\end{aligned}$$

so that $\mathcal{F}_\sigma \theta_M(u,v) = e^{2\pi i v\cdot Mu}$. Defining $\chi_M(u,v) := e^{2\pi i v\cdot Mu}$ and using that $\mathcal{F}_\sigma^2 = I$, we finally obtain

$$\theta_M(u,v) = \mathcal{F}_\sigma \chi_M(u,v) = \mathcal{F}_\sigma\big[e^{2\pi i v\cdot Mu}\big] \in \mathcal{S}'(\mathbb{R}^{2d}).$$

We finally compute the expression of θ_M in the case where $M \in \mathrm{GL}(d,\mathbb{R})$. Let us denote by Θ_M the Fourier transform of the Cohen kernel, namely

$$\Theta_M(u,v) = \mathcal{F}\theta_M(u,v) = \chi_M(-v,u) = e^{-2\pi i u\cdot Mv}, \quad (u,v)\in\mathbb{R}^{2d}.$$

Note that $\Theta_M = \mathfrak{T}_{\tilde{M}}\Theta$, where we set $\Theta(u,v) = e^{2\pi i u\cdot v}$ and $\tilde{M} = (-I)\oplus M$. We deduce from Lemma 5.2.1 that

$$\theta_M = \mathcal{F}^{-1}\Theta_M = \mathcal{F}^{-1}\mathfrak{T}_{\tilde{M}}\Theta = |\det M|^{-1}\mathfrak{T}_{\tilde{M}^\#}\mathcal{F}^{-1}\Theta.$$

Notice that $\mathcal{F}^{-1}\Theta = \mathcal{I}\mathcal{F}\Theta$ and from [Fol89, Appendix A, Theorem 2]) we have $\mathcal{F}\Theta(u,v) = e^{-2\pi i u\cdot v}$, hence $\theta_M(x,\xi) = |\det M|^{-1}e^{2\pi i\xi\cdot M^{-1}x}$ as claimed. □

It is clear from the previous proof that a Cohen-type matrix A should be defined by the following conditions on the blocks:

$$A_{11} = A_{21} = I, \quad A_{12} - A_{22} = I. \tag{5.9}$$

The choice $A_{22} = M - (1/2)I$ with $M \in \mathbb{R}^{d\times d}$ is thus a suitable parametrization, but by no means the only possible one - and in fact neither the most natural one.

The reason underlying our choice is clarified by writing down the explicit formula for W_M, that is

$$W_M(f,g)(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot y} f\left(x + \left(M + \frac{1}{2}I\right)y\right)\overline{g\left(x + \left(M - \frac{1}{2}I\right)y\right)}dy,$$

which reveals the similarity with the Wigner distribution. A sort of symmetry with respect to the Wigner distribution (corresponding to $M = 0$) immediately stands out. We interpret these representations as a family of "linear perturbations" of the Wigner distribution and M as the control parameter, exactly as τ controls the degree of deviation of τ-Wigner distributions. For this reason, we will refer to $A = A_M$ as the *perturbative form* of a Cohen-type matrix.

The analogy with the τ-Wigner distributions naturally leads to another representation, hence another choice of A_{22} in (5.9). A closer inspection of the kernel (1.11) and also of (1.10) reveals that the role of perturbation parameter is not played by τ, rather by the deviation $\mu = \tau - 1/2$. In this analogy one chooses $A_{21} = T \in \mathbb{R}^{d\times d}$ and $A_{22} = -(I - T)$ and obtains

$$W_T(f,g)(x,\xi) = \mathcal{B}_{A_T}(f,g)(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot y} f(x + Ty)\overline{g(x - (I - T)y)}dy,$$

which should be compared to (1.10) (see also (5.3)). Occasionally, we refer to A_T as the *affine form* of the Cohen-type matrix A. It is clear that the two forms of a Cohen-type matrix are perfectly equivalent, the connection being

$$M = T - (1/2)I. \tag{5.10}$$

The notation W_M for the perturbative form and W_T for the affine form is in fact ambiguous but we prefer to keep the notation as light as possible; the correct alternative will be clear from the context.

Therefore, the choice of a form is just a matter of convenience: when studying the properties of $\mathcal{B}_A$ as a time-frequency representation, it seems better to explicitly see the effect of the perturbation M (which could be easily turned off setting $M = 0$) and use the perturbative form accordingly. As an example of this, the perturbed representation of a Gaussian signal is provided below.

Lemma 5.4.2 ([CT20, Lemma 4.1])**.** *Let $A = A_M \in \mathrm{GL}(2d,\mathbb{R})$ be a Cohen-type matrix and $\phi_\lambda(t) = e^{-\pi t^2/\lambda}$, $\lambda > 0$. Then,*

$$W_M\phi_\lambda(x,\xi) = (2\lambda)^{d/2}(\det S)^{-1/2}e^{-2\pi x^2/\lambda} \cdot e^{8\pi\left(M^\top x\cdot S^{-1}M^\top x\right)/\lambda}e^{8\pi i S^{-1}\xi\cdot M^\top x}e^{-2\pi\lambda\xi\cdot S^{-1}\xi}, \tag{5.11}$$

where $S = I + 4M^\top M \in \mathbb{R}^{d\times d}$.

5.4.1 Main properties of distributions in the Cohen class

The properties of a time-frequency distribution belonging to the Cohen class are intimately related to the structure of the Cohen kernel. There is an established list of correspondences between the kernel and the properties, which can be used to deduce the following results.

Proposition 5.4.3. *Assume that $\mathcal{B}_A$ belongs to the Cohen class, namely that $\mathcal{B}_A = W_M$. For any $f, g \in M^1(\mathbb{R}^d)$, the following properties are satisfied:*

(i) (Correct marginal densities)

$$\int_{\mathbb{R}^d} W_M f(x,\xi) d\xi = |f(x)|^2, \qquad \int_{\mathbb{R}^d} W_M f(x,\xi) dx = |\hat{f}(\xi)|^2, \quad x, \xi \in \mathbb{R}^d.$$

In particular, the energy is preserved:

$$\iint_{\mathbb{R}^{2d}} W_M f(x,\xi) dx d\xi = \|f\|^2_{L^2}.$$

(ii) (Moyal's identity)

$$\langle W_M f, W_M g \rangle_{L^2(\mathbb{R}^{2d})} = |\langle f, g\rangle|^2.$$

(iii) (Symmetry) For all $x, \xi \in \mathbb{R}^d$,

$$W_M(\mathcal{I}f)(x,\xi) = \mathcal{I} W_M f(x,\xi) = W_M f(-x,-\xi),$$

$$W_M\big(\overline{f}\big)(x,\xi) = \overline{\mathcal{I}_2 W_M f(x,\xi)} = \overline{W_M(x,-\xi)}.$$

(iv) (Convolution properties) For all $x, \xi \in \mathbb{R}^d$,

$$W_M(f * g)(x,\xi) = W_M f *_1 W_M g,$$

$$W_M(f \cdot g)(x,\xi) = W_M f *_2 W_M g.$$

Here $*_1$ (resp. $*_2$) denotes the convolution with respect to the first (resp. second) variable.

(v) (Scaling invariance) Setting $U_\lambda f(t) := |\lambda|^{d/2} f(\lambda t)$, $\lambda \in \mathbb{R} \setminus \{0\}$, $t \in \mathbb{R}^d$,

$$W_M(U_\lambda f)(x,\xi) = W_M f\big(\lambda x, \lambda^{-1}\xi\big).$$

Proof. The properties above are directly related to conditions satisfied by the Fourier transform Θ_M of the Cohen kernel, cf. for instance [Coh95; Jan84; Jan97] (for dimension $d = 1$ - the stated characterization easily extends to dimension $d > 1$), that is:

(i) $\Theta_M(0,\xi) = \Theta_M(x,0) = 1$ for any $x,\xi \in \mathbb{R}^d$ (in particular $\Theta_M(0,0) = 1$);

(ii) $|\Theta_M(x,\xi)| = 1$ for any $x,\xi \in \mathbb{R}^d$;

(iii) $\Theta_M(-x,-\xi) = \Theta_M(x,\xi)$ and $\overline{\Theta_M(x,\xi)} = \Theta_M(-x,\xi)$ respectively, for any $x,\xi \in \mathbb{R}^d$;

(iv) $\Theta_M(\cdot,\xi_1+\xi_2) = \Theta_M(\cdot,\xi_1)\Theta_M(\cdot,\xi_2)$ and $\Theta_M(x_1+x_2,\cdot) = \Theta_M(x_1,\cdot)\Theta_M(x_2,\cdot)$ respectively, for any $x_i,\xi_i \in \mathbb{R}^d$, $i = 1,2$.

(v) $\Theta_M(\lambda x, \lambda^{-1}\xi) = \Theta_M(x,\xi)$.

The kernel $\Theta_M(x,\xi) = e^{-2\pi i x\cdot M\xi}$ trivially satisfies conditions (i)-(v) above. □

We now give a few hints on several aspects of interests for both theoretical problems and applications; extensive discussions on these issues may be found in [Bay10; CT20].

Real-valuedness. An easy computation shows that $\mathcal{B}_A f$, $f \in L^2(\mathbb{R}^d)$, is a real-valued function if and only if $A_{11} = A_{21}$ and $A_{12} = -A_{22}$, cf. [CT20, Proposition 3.3]. Therefore, the Wigner distribution is the only real-valued member of the family W_M. This is one of the properties which does not survive the perturbation.

More on marginal densities. The marginal densities for a general distribution $\mathcal{B}_A$ can be easily computed, see [Bay10, Lemma 1.6.1] for details. For $f,g \in M^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \mathcal{B}_A f(x,\xi)d\xi = f(A_{11}x)\overline{f(A_{21}x)},$$
$$\int_{\mathbb{R}^d} \mathcal{B}_A f(x,\xi)dx = |\det A|^{-1}\hat{f}\big((A^\#)_{12}\xi\big)\overline{\hat{f}(-(A^\#)_{22}\xi)}.$$

The correct marginal densities are thus recovered if and only if $A_{11} = A_{21} = I$ and $(A^\#)_{12} = -(A^\#)_{22} = I$. These conditions force both $|\det A| = 1$ and the block structure of A as that of Cohen's type. This fact provides an equivalent characterization of the distributions $\mathcal{B}_A$ belonging to the Cohen class: these are exactly those satisfying the correct marginal densities.

Relation between two distributions. Let $A_1 = A_{M_1}$ and $A_2 = A_{M_2}$ be two Cohen-type matrices in perturbative form. The two distributions W_{M_1} and W_{M_2} are connected by a Fourier multiplier as follows. For $f,g \in M^1(\mathbb{R}^d)$,

$$\mathcal{F}W_{M_2}(f,g)(u,v) = e^{-2\pi i u\cdot(M_2-M_1)v}\mathcal{F}W_{M_1}(f,g)(u,v). \tag{5.12}$$

Furthermore, if $M_2 - M_1 \in \mathrm{GL}(d,\mathbb{R})$, we have explicitly

$$W_{M_2}(f,g)(x,\xi) = |\det(M_2 - M_1)|^{-1} e^{2\pi i \xi\cdot(M_2-M_1)^{-1}x} * W_{M_1}(f,g)(x,\xi).$$

The proofs follow at once from Theorem 5.4.1.

Covariance and magic formulae. For any $z = (z_1, z_2)$, $w = (w_1, w_2) \in \mathbb{R}^{2d}$, the covariance formula (5.5) now reads

$$W_M(\pi(z)f, \pi(w)g)(x,\xi) = e^{2\pi i\left[\frac{1}{2}(z_2+w_2)+M(z_2-w_2)\right]\cdot(z_1-w_1)} \times M_{J(z-w)} T_{\mathcal{T}_M(z,w)} W_M(f,g)(x,\xi), \quad (5.13)$$

where

$$\mathcal{T}_M(z,w) = \frac{1}{2}(z+w) + \begin{bmatrix} -M & 0 \\ 0 & M \end{bmatrix}(z-w).$$

Alternatively, adopting the affine representation of A (5.10), we introduce

$$P_T = \begin{bmatrix} -T & 0 \\ 0 & -(I-T) \end{bmatrix}, \quad I + P_T = \begin{bmatrix} I-T & 0 \\ 0 & T \end{bmatrix}, \quad (5.14)$$

and then we can write

$$\mathcal{T}_T(z,w) = \begin{bmatrix} (I-T)z_1 + Tw_1 \\ Tz_2 + (I-T)w_2 \end{bmatrix} = (I+P_T)z - P_T w. \quad (5.15)$$

For future use, let us also specialize the magic formula (5.6) to representations in the Cohen class: we have

$$V_{W_T(\phi,\psi)} W_T(f,g)(z,\zeta) = e^{-2\pi i z_2\cdot\zeta_2} V_\phi f(z + P_T J\zeta)\overline{V_\psi g(z + (I+P_T)J\zeta)}. \quad (5.16)$$

Support conservation. A desirable property for a time-frequency distribution is the preservation of the support of the original signal. A scale of precise mathematical conditions can be introduced in order to capture this heuristic feature. Following Folland's classic approach (see [Fol89, p. 59]), we define the support of a signal $f : \mathbb{R}^d \to \mathbb{C}$ as the smallest closed set outside of which $f = 0$ a.e., hence we may assume $f = 0$ everywhere outside $\mathrm{supp} f$.

Definition 5.4.4. *Let $Qf : \mathbb{R}^{2d}_{(x,\xi)} \to \mathbb{C}$ be the time-frequency distribution associated to the signal $f : \mathbb{R}^d_t \to \mathbb{C}$ in a suitable function space. Let $\pi_x : \mathbb{R}^{2d}_{(x,\xi)} \to \mathbb{R}^d_x$ and $\pi_\xi : \mathbb{R}^{2d}_{(x,\xi)} \to \mathbb{R}^d_\xi$ be the projections onto the first and second factors ($\mathbb{R}^{2d}_{(x,\xi)} \simeq \mathbb{R}^d_x \times \mathbb{R}^d_\xi$) and, for any $E \subset \mathbb{R}^d$, let $\mathcal{C}(E)$ denote the closed convex hull of E.*

We say that Q satisfies the time strong support property if $f(x) = 0 \Leftrightarrow Qf(x,\xi) = 0$ for all $\xi \in \mathbb{R}^d$. Similarly, Q satisfies the frequency strong support property if $\hat{f}(\xi) = 0 \Leftrightarrow Qf(x,\xi) = 0$ for all $x \in \mathbb{R}^d$.

Moreover, Q satisfies the time (resp. frequency) weak support property if $\pi_x(\mathrm{supp}Qf) \subset \mathcal{C}(\mathrm{supp}f)$ (resp. if $\pi_\xi(\mathrm{supp}Qf) \subset \mathcal{C}(\mathrm{supp}\hat{f})$).

We say that Q satisfies the strong (resp. weak) support property if both time and frequency strong (resp. weak) support properties hold.

We characterize the MWDs in the Cohen class satisfying the aforementioned properties, showing the optimality in this sense of τ-Wigner distributions.

Theorem 5.4.5. *The only MWDs in Cohen's class satisfying the strong correct support properties are Rihaczek and conjugate-Rihaczek distributions.*

Proof. This result can be inferred by directly inspecting the Fourier transform of Cohen's kernel. Indeed, by adapting the proof of Janssen (see [Jan97, Sec. 2.6.2]) to dimension $d > 1$ one can show that the only members of the Cohen class satisfying both time and frequency strong support properties are linear combinations of Rihackez and conjugate-Rihaczek distributions. This is equivalent to the following condition on the Fourier transform of the kernel Θ_M: for any $x, \xi \in \mathbb{R}^d$,

$$\Theta_M(x,\xi) = C_+ e^{\pi i x\cdot\xi} + C_- e^{-\pi i x\cdot\xi},$$

for some $C_+, C_- \in \mathbb{C}$. Since $\Theta_M(x,\xi) = e^{-2\pi i x\cdot M\xi}$, this can happen only if $M = \pm(1/2)I$ with $C_+ = 1, 0$ and $C_- = 0, 1$ respectively. □

Theorem 5.4.6. *Let $A = A_T \in \mathrm{GL}(2d,\mathbb{R})$ be a Cohen-type matrix. The only associated distributions satisfying the weak support property are the τ-Wigner distributions, namely $T = \tau I$ for $\tau \in [0,1]$.*

Proof. Assume $x \in \mathrm{supp}\mathcal{B}_A f(\cdot,\xi)$ for a fixed $\xi \in \mathbb{R}^d$. The only way for this to occur is $x + Ty$, $x - (I-T)y \in \mathrm{supp}f$. In order to have $x \in \mathcal{C}(\mathrm{supp}f)$, we require that

$$x = \lambda(x + Ty) + \mu(x - (I-T)y),$$

for some $\lambda, \mu \geq 0$ such that $\lambda + \mu = 1$. Rewriting this condition as $x = (\lambda + \mu)x + (\lambda+\mu)Ty - \mu y$ gives the constraints $\lambda + \mu = 1$ and $(T - \mu I) = 0$, and the claim follows. Similar arguments apply to the case of the frequency weak support property. □

Perturbation and interferences. The emergence of unwanted artefacts is a well-known drawback of any quadratic representation. The signal processing literature is full of strategies to mitigate these effects (see for instance [Coh95; HA08]). For what concerns the Cohen class, it is folklore that the severity of interferences is somewhat related to the decay of the Cohen kernel. In fact, a precise formulation of this principle is rather elusive and recent contributions unravelled further non-trivial fine points ([CGDN18, Proposition 4.4 and Theorem 4.6]). We remark that the chirp-like kernel $\Theta_M = \mathcal{F}\theta_M$ does not decay at all, and thus no smoothing effect should be expected for the perturbed representations. In order to experience this, we limit ourselves to dimension $d = 1$ and follow the geometrical approach employed in [BCO11]. As a toy model we consider signals consisting of pure frequencies confined in disjoint time intervals. It is well known that the Wigner transform displays "ghost frequencies" in between any couple of true frequencies of the signal. A similar phenomenon can be studied also in higher dimension considering Gaussian signals in the so-called "diamond configuration", see again [CGDN18].

We remark that for $d = 1$ the perturbation matrix M boils down to a scalar $m \in \mathbb{R}$. Let f be a signal with a frequency ω_1 appearing in the interval $I_1 = [x_1, x_1 + h_1]$ and ω_2 in $I_2 = [x_2, x_2 + h_2]$, with $h_2 \geq h_1 > 0$ such that $x_1 + h_1 < x_2$. We are then dealing with a τ-Wigner distribution (in perturbative form, namely $m = \tau - 1/2$), that is

$$\mathcal{B}_m f(x,\omega) = \int_{\mathbb{R}} e^{-2\pi i\omega y} f\left(x + \left(m + \frac{1}{2}\right)y\right)\overline{f\left(x + \left(m - \frac{1}{2}\right)y\right)}dy.$$

We see that $\mathcal{B}_m f$ is supported in the diamond-shaped regions D_i, $i = 1, ..., 4$, (see Figure 5.1) obtained by intersecting the following straight lines passing through the endpoints of the time intervals:

$$\begin{cases} x + \left(m \pm \frac{1}{2}\right)y = x_1 \\ x + \left(m \pm \frac{1}{2}\right)y = x_1 + h_1 \\ x + \left(m \pm \frac{1}{2}\right)y = x_2 \\ x + \left(m \pm \frac{1}{2}\right)y = x_2 + h_2. \end{cases}$$

With the notation of the figure, we see that D_1 and D_3 give account for the true frequencies of the signal, while D_2 and D_3 are non-zero interferences. A short computation shows that the coordinates of the two points V_1 and V_2 are

$$V_1 = \left(\frac{2m+1}{2}(x_2 + h_2) - \frac{2m-1}{2}x_1, x_1 - (x_2 + h_2)\right),$$

$$V_2 = \left(\frac{2m+1}{2}x_1 - \frac{2m-1}{2}(x_2 + h_2), (x_2 + h_2) - x_1\right),$$

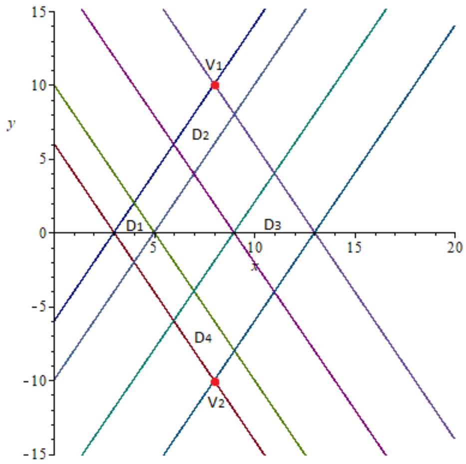

Figure 5.1: Support of $\mathcal{B}_m f$ with $m = 0$, $I_1 = [3, 5]$, $I_2 = [9, 13]$.

hence we see that the only effect of the perturbation parameter m is the horizontal translation of the diamond's corners, giving no room for damping. Moreover, as already seen before and also noticeable from the coordinates of V_1 and V_2, when $m \in \mathbb{R} \setminus \left[-\frac{1}{2}, \frac{1}{2}\right]$, the support of the signal is no longer conserved - neither in weak sense.

To sum up, the only effect of the perturbation consists of a distortion and relocation of interferences, but there is no damping. Following the engineering literature, we suggest that convolution with suitable decaying distributions may provide some improvement, probably at the price of loosing other nice properties.

5.4.2 Time-frequency analysis of the Cohen kernel

In this section we deepen the study of the Cohen kernel θ_M. Recall that $\Theta_M(u, v) = \mathcal{F}\theta_M(u, v) = e^{-2\pi i u \cdot M v}$, $(u, v) \in \mathbb{R}^{2d}$. At a first glance we notice that $\Theta_M \in C_b^\infty(\mathbb{R}^{2d})$ and $\Theta_M \in L^p_{\mathrm{loc}}(\mathbb{R}^{2d})$ for any $1 \le p \le \infty$. Hence, we are dealing with distributions whose Fourier transforms are well-behaved dilated chirps, and intuition suggests that the kernels themselves should belong to the same family. In the case where $M \in \mathrm{GL}(d, \mathbb{R})$ this outcome is clear from Theorem 5.4.1. Moreover, such a heuristic statement is enforced by the following result, already proved in [CGN18, Proposition 3.2 and Cor.3.4].

Lemma 5.4.7. *The function* $\Theta(x, \xi) = e^{2\pi i x \cdot \xi}$ *belongs to* $M^{1,\infty}(\mathbb{R}^{2d}) \cap W^{1,\infty}(\mathbb{R}^{2d})$.

Proposition 5.4.8. *Let $A = A_M \in \mathrm{GL}(2d,\mathbb{R})$ be a Cohen-type matrix with $M \in \mathrm{GL}(d,\mathbb{R})$. We have*

$$\theta_M \in M^{1,\infty}(\mathbb{R}^{2d}) \cap W^{1,\infty}(\mathbb{R}^{2d}).$$

Proof. Recall that $\Theta_M(u,v) = e^{-2\pi i u\cdot Mv} = \mathfrak{T}_{\tilde{M}}\Theta(u,v)$, where $\Theta(u,v) = e^{2\pi u\cdot v}$ and $\tilde{M} = (-I) \oplus M$. Note that $\tilde{M}$ is invertible since $M \in \mathrm{GL}(d,\mathbb{R})$ by assumption. Therefore, according to the dilation properties in [CN08a, Proposition 3.1 and Cor. 3.2], the results in [CGN18, Proposition 3.2] and Lemma 5.4.7 above, we have $\Theta_M \in M^{1,\infty}(\mathbb{R}^{2d}) \cap W^{1,\infty}(\mathbb{R}^{2d})$. Since $\Theta_M = \mathcal{F}\theta_M$ and $W^{1,\infty} = \mathcal{F}M^{1,\infty}$, we conclude that $\theta_M \in M^{1,\infty}(\mathbb{R}^{2d}) \cap W^{1,\infty}(\mathbb{R}^{2d})$. □

In the case of τ-Wigner distributions, that is $M = (\tau - 1/2)I$ with $\tau \in [0,1] \setminus \{1/2\}$, we recover a known result [BDDO10, Proposition 5.6]:

$$\theta_\tau(x,\xi) = 2^d|2\tau - 1|^{-d}e^{2\pi i\frac{2}{2\tau-1}x\cdot\xi}.$$

Notice that one cannot say much without assuming the invertibility of M. We do not explore this situation, apart from mentioning that for $M = 0$ most of these results do not hold: for instance, since $\theta_0 = \delta$, it is easy to verify that $\theta_0 \in M^{1,\infty}(\mathbb{R}^{2d}) \setminus W^{1,\infty}(\mathbb{R}^{2d})$, cf. [CGN17b].

To conclude this section we prove that, in according with heuristic expectations, the time-frequency regularity of a Wigner distribution survives linear perturbations. Similar results have been proved for the Born-Jordan distribution in [CGN18, Theorem 4.1] and its n-th order generalization in [CDGN18], although with a significant difference: no directional smoothing effects occur in our scenario, as already remarked above.

Theorem 5.4.9. *Let $A = A_M \in \mathrm{GL}(2d,\mathbb{R})$ be a matrix of Cohen's type with $M \in \mathrm{GL}(d,\mathbb{R})$ and $f \in \mathcal{S}'(\mathbb{R}^d)$ be a signal. Then, for any $1 \le p,q \le \infty$,*

$$Wf \in M^{p,q}(\mathbb{R}^{2d}) \Longleftrightarrow W_M f \in M^{p,q}(\mathbb{R}^{2d}).$$

Proof. Assume first $Wf \in M^{p,q}(\mathbb{R}^{2d})$, for some $1 \le p,q \le \infty$. Taking the symplectic Fourier transform, the claim is equivalent to showing that

$$\theta_M \cdot \mathrm{Amb}(f) \in W^{p,q}(\mathbb{R}^{2d}).$$

Since $\mathrm{Amb}(f) \in W^{p,q}(\mathbb{R}^{2d})$ because of the assumption on Wf, the desired result follows from Proposition 5.4.8 and the boundedness of Fourier multipliers with symbols in $W^{1,\infty}$ on $M^{p,q}$ (cf. [BGOR07, Lemma 8]), or equivalently $W^{1,\infty}(\mathbb{R}^{2d}) \cdot W^{p,q}(\mathbb{R}^{2d}) \subset W^{p,q}(\mathbb{R}^{2d})$ [Fei83, Theorem 1].

Conversely, assume $W_M f \in M^{p,q}(\mathbb{R}^{2d})$, for some $1 \leq p, q \leq \infty$. Taking the symplectic Fourier transform yields

$$\theta_M \cdot \mathrm{Amb}(f) = \mathcal{F}_\sigma W_M f \in W^{p,q}(\mathbb{R}^{2d}).$$

In view of (5.8) and Proposition 5.4.8 we have $\theta_M^{-1} \in W^{1,\infty}(\mathbb{R}^{2d})$, therefore

$$\mathrm{Amb}(f) = \theta_M^{-1} \cdot \mathcal{F}_\sigma W_M f \in W^{1,\infty}(\mathbb{R}^{2d}) \cdot W^{p,q}(\mathbb{R}^{2d}) \subset W^{p,q}(\mathbb{R}^{2d}).$$

□

5.5 Boundedness results for matrix-Wigner distributions

5.5.1 Boundedness on Lebesgue spaces

We characterize the boundedness of a matrix-Wigner distribution $\mathcal{B}_A(f,g)$ on Lebesgue spaces using that this is a completely established issue for the STFT, cf. [BDDO09].

Proposition 5.5.1 ([CT20, Proposition 3.9])**.** *Assume that $A \in \mathrm{GL}(2d,\mathbb{R})$ is right-regular. For any $1 \leq p \leq \infty$ and qge2 such that $q' \leq p \leq q$, $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p'}(\mathbb{R}^d)$, we have*

(i) *$\mathcal{B}_A(f,g) \in L^q(\mathbb{R}^{2d})$, with*

$$\|\mathcal{B}_A(f,g)\|_{L^q} \leq \frac{\|f\|_{L^p}\|g\|_{L^{p'}}}{|\det A|^{\frac{1}{q}}|\det A_{12}|^{\frac{1}{p}-\frac{1}{q}}|\det A_{22}|^{\frac{1}{p'}-\frac{1}{q}}}. \tag{5.17}$$

(ii) *If $1 < p < \infty$ then $\mathcal{B}_A(f,g) \in C_0(\mathbb{R}^{2d})$. In particular, $\mathcal{B}_A(f,g) \in L^\infty(\mathbb{R}^{2d})$.*

(iii) *If $1 \leq p, q \leq \infty$ such that $p < q'$ or $p > q$, the map $\mathcal{B}_A(f,g) : L^p(\mathbb{R}^d) \times L^{p'}(\mathbb{R}^d) \to L^q(\mathbb{R}^{2d})$ is not continuous.*

Proof. (i) Set $W = (A_{11} - A_{12}A_{22}^{-1}A_{21}) \oplus A_{12}^{\#}$ and note that $W \in \mathrm{GL}(2d,\mathbb{R})$, since

$$\det W = \det\bigl(A_{11} - A_{12}A_{22}^{-1}A_{21}\bigr) \cdot \frac{1}{\det A_{12}} = \frac{\det A}{\det A_{12}\det A_{22}} \neq 0.$$

We use Theorem 5.3.3 and the estimate [BDDO09, Proposition 3.1] for the L^q-norm of the STFT, that is

$$\begin{aligned}
\|\mathcal{B}_A(f,g)\|_{L^q} &= |\det A_{12}|^{-1}\|\mathfrak{T}_W V_{\tilde{g}}f\|_{L^q} \\
&= |\det A_{12}|^{-1}|\det W|^{-1/q}\|V_{\tilde{g}}f\|_{L^q} \\
&\le |\det A_{12}|^{-1}|\det W|^{-1/q}\|f\|_{L^p}\|\tilde{g}\|_{L^{p'}} \\
&= |\det A_{12}|^{-1}|\det W|^{-1/q}\|f\|_{L^p}\left(\frac{|\det A_{12}|^{1/p'}}{|\det A_{22}|^{1/p'}}\|g\|_{L^{p'}}\right) \\
&= \frac{\|f\|_{L^p}\|g\|_{L^{p'}}}{|\det W|^{1/q}|\det A_{12}|^{1/p}|\det A_{22}|^{1/p'}} \\
&\le \frac{\|f\|_{L^p}\|g\|_{L^{p'}}}{|\det A|^{1/q}|\det A_{12}|^{\frac{1}{p}-\frac{1}{q}}|\det A_{22}|^{\frac{1}{p'}-\frac{1}{q}}}.
\end{aligned}$$

(ii) Arguing by density, there exist sequences $\{f_n\},\{g_n\} \in \mathcal{S}(\mathbb{R}^d)$ such that $f_n \to f$ in L^p and $g_n \to g$ in $L^{p'}$. Therefore $\mathcal{B}_A(f_n,g_n) \in \mathcal{S}(\mathbb{R}^{2d}) \subset C_0(\mathbb{R}^{2d})$ and we have

$$\begin{aligned}
\|\mathcal{B}_A(f_n,g_n)-\mathcal{B}_A(f,g)\|_\infty &= \|\mathcal{B}_A(f_n,g_n)-\mathcal{B}_A(f_n,g)+\mathcal{B}_A(f_n,g)-\mathcal{B}_A(f,g)\|_\infty \\
&\le \|\mathcal{B}_A(f,g_n-g)\|_\infty + \|\mathcal{B}_A(f-f_n,g)\|_\infty \\
&\le \frac{\|f\|_{L^p}\|g_n-g\|_{L^{p'}}+\|f-f_n\|_{L^p}\|g\|_{L^{p'}}}{|\det A|^{\frac{1}{q}}|\det A_{12}|^{\frac{1}{p}-\frac{1}{q}}|\det A_{22}|^{\frac{1}{p'}-\frac{1}{q}}}.
\end{aligned}$$

Since the sequence $\{\|f_n\|_{L^p}\}$ is bounded, we conclude

$$\lim_{n\to\infty}\|\mathcal{B}_A(f_n,g_n)-\mathcal{B}_A(f,g)\|_\infty = 0.$$

This implies $\mathcal{B}_A(f,g) \in C_0(\mathbb{R}^{2d})$, as desired.

(iii) This is a direct application of [BDDO09, Proposition 3.2].

□

We can specialize the previous result to distributions of Cohen type as follows.

Theorem 5.5.2. *[CT20, Theorem 4.14] Let $A = A_T \in \mathrm{GL}(2d,\mathbb{R})$ be a right-regular Cohen-type matrix. For any $1 < p < \infty$ and $q \ge 2$ such that $q' \le p \le q$, $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p'}(\mathbb{R}^d)$, the following facts hold.*

(i) $\mathcal{B}_A(f,g) \in L^q\big(\mathbb{R}^{2d}\big)$, *with*

$$\|W_T(f,g)\|_{L^q} \leq \frac{\|f\|_{L^p}\|g\|_{L^{p'}}}{|\det(T)|^{\frac{1}{p}-\frac{1}{q}}|\det(I-T)|^{\frac{1}{p'}-\frac{1}{q}}}.$$

(ii) $W_T(f,g) \in C_0\big(\mathbb{R}^{2d}\big)$.

5.5.2 Boundedness on modulation and amalgam spaces

We provide here some results on the continuity of the distributions W_T on modulation and Wiener amalgam spaces. We would like to point out that results of this type were already proved for τ-Wigner distributions in [CDT19] and [CNT19b]. In fact, we prove generalized versions of [CDT19, Lemma 3.1 and Proposition 5.1].

Theorem 5.5.3. *Let* $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ *and* $A = A_T$ *be a Cohen-type matrix. Set* $v_J = \mathfrak{T}_J v$.

(i) *Let the indices* $1 \leq p_i, q_i, p, q \leq \infty$, $i = 1,2$, *satisfy the conditions*

$$p_i, q_i \leq q, \qquad i = 1,2, \tag{5.18}$$

and

$$\frac{1}{p_1}+\frac{1}{p_2} \geq \frac{1}{p}+\frac{1}{q}, \qquad \frac{1}{q_1}+\frac{1}{q_2} \geq \frac{1}{p}+\frac{1}{q}. \tag{5.19}$$

If $f \in M^{p_1,q_1}_m(\mathbb{R}^d)$ *and* $g \in M^{p_1,q_2}_{1/m}(\mathbb{R}^d)$ *then the* T*-Wigner distribution* $W_T(g,f)$ *belongs to* $M^{p,q}_{1\otimes 1/v_J}(\mathbb{R}^{2d})$, *with*

$$\|W_T(g,f)\|_{M^{p,q}_{1\otimes 1/v_J}} \lesssim_T \|f\|_{M^{p_1,q_1}_m}\|g\|_{M^{p_2,q_2}_{1/m}}. \tag{5.20}$$

(ii) *Let* $1 \leq p_1, p_2 \leq \infty$ *and assume that* A_T *is right-regular. If* $f \in M^{p_1,p_2}_m(\mathbb{R}^d)$ *and* $g \in M^{p_1',p_2'}_{1/m}(\mathbb{R}^d)$ *then the* T*-Wigner distribution* $W_T(g,f)$ *belongs to* $W^{1,\infty}_{(1/v_J)\otimes 1}(\mathbb{R}^{2d})$, *with*

$$\|W_T(g,f)\|_{W^{1,\infty}_{(1/v_J)\otimes 1}} \lesssim_T \alpha_{(p_1,p_2)}(T)\|f\|_{M^{p_1,p_2}_m}\|g\|_{M^{p_1',p_2'}_{1/m}},$$

where we set

$$\alpha_{(p_1,p_2)}(T) := (\det T)^{-(1/p_1'+1/p_2)}(\det(I-T))^{-(1/p_1+1/p_2')}. \tag{5.21}$$

(iii) *If* A_T *is right-regular,* $f \in M^2_m(\mathbb{R}^d)$ *and* $g \in M^2_{1/m}(\mathbb{R}^d)$, *then* $W_T(g,f) \in W^{2,2}_{(1/v_J)\otimes 1}(\mathbb{R}^{2d})$, *with*

$$\|W_T(g,f)\|_{W^{2,2}_{(1/v_J)\otimes 1}} \lesssim_T \|f\|_{M^2_m}\|g\|_{M^2_{1/m}}.$$

Proof. We start with the proof of (i) in the case where $p \leq q < \infty$. We fix $\phi \in \mathcal{S}(\mathbb{R}^d)\setminus\{0\}$ and compute the STFT of $W_T(g,f)$ with respect to the window function $\Phi_T = W_T\phi$ by means of the magic formula (5.16). Therefore, the substitution $\eta = z + (I + P_T)J\zeta$ yields

$$\begin{aligned}
&\|W_T(g,f)\|_{M^{p,q}_{1\otimes 1/v_J}} \\
&= \left(\int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^{2d}} |V_\phi g(z + P_TJ\zeta)|^p|V_\phi f(z+(I+P_T)J\zeta)|^p dz\right)^{\frac{q}{p}} \frac{1}{v^q(J\zeta)} d\zeta\right)^{\frac{1}{q}} \\
&= \left(\int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^{2d}} |V_\phi g(\eta - J\zeta)|^p|V_{\phi_2} f(\eta)|^p \frac{1}{v^p(J\zeta)} d\eta\right)^{\frac{q}{p}} d\zeta\right)^{\frac{1}{q}}.
\end{aligned}$$

Since m is a v-moderate weight and v is even, there exists $C > 0$, independent of T, such that

$$\frac{1}{v(J\zeta)} \leq C\frac{m(\eta)}{m(\eta - J\zeta)}.$$

Therefore,

$$\begin{aligned}
&\|W_T(g,f)\|_{M^{p,q}_{1\otimes 1/v_J}} \\
&\leq C\left(\int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^{2d}} |V_\phi g(\eta - J\zeta)|^p|V_\phi f(\eta)|^p \frac{m^p(\eta)}{m^p(\eta - J\zeta)} d\eta\right)^{\frac{q}{p}} d\zeta\right)^{\frac{1}{q}} \\
&= C\left(\int_{\mathbb{R}^{2d}}\left((|(V_\phi g)^\vee|^p m^{-p}) * (|V_\phi f|^p m^p)(J\zeta)\right)^{\frac{q}{p}} d\zeta\right)^{\frac{1}{q}} \\
&= C\|(|(V_\phi g)^\vee|^p m^{-p}) * (|V_{\phi_2} f|^p m^p)\|^{1/p}_{L^{q/p}}.
\end{aligned}$$

The rest of the proof goes exactly as in [CN18, Theorem 3.1] using the convolution inequalities in Proposition 2.3.3.

Let us consider the case $p = q = \infty$. A similar computation yields

$$\|W_T(g,f)\|_{M^\infty_{1\otimes 1/v_J}} \lesssim_T \|f\|_{M^\infty_m}\|g\|_{M^\infty_{1/m}},$$

and the claim follows by the inclusion properties of modulation spaces.

Finally, assume $p > q$. Again by the inclusion properties of modulation spaces we have

$$\|W_T(g,f)\|_{M^{p,q}_{1\otimes 1/v_J}} \leq \|W_T(g,f)\|_{M^{q,q}_{1\otimes 1/v_J}} \leq C\|f\|_{M^{p_1,q_1}_m}\|g\|_{M^{p_2,q_2}_{1/m}}.$$

The proof of *(ii)* goes along the same lines. In particular we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} |V_\Phi W_T(g,f)|(z,\zeta)\frac{1}{v(J\zeta)}d\zeta \\
&= \int_{\mathbb{R}^{2d}} |V_\phi g(z+P_TJ\zeta)||V_\phi f(z+(I+P_T)J\zeta)|\frac{1}{v(J\zeta)}d\zeta \\
&= C_T^{-1}\int_{\mathbb{R}^{2d}} |V_\phi g(z+P_T(I+P_T)^{-1}\eta)||V_\phi f(z+\eta)|\frac{1}{v((I+P_T)^{-1}\eta)}d\eta,
\end{aligned}$$

where we set $C_T=(\det T)(\det(I-T))$. Since m is a v-moderate weight and v is even, there exists $C>0$, independent of T, such that

$$\frac{1}{v((I+P_T)^{-1}\eta)} \le C\frac{m(z+\eta)}{m(z+P_T(I+P_T)^{-1}\eta)}.$$

Consequently,

$$\begin{aligned}
&\|W_T(g,f)\|_{W^{1,\infty}_{(1/v_J)\otimes 1}} \\
&\le CC_T^{-1}\sup_{z\in\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} |V_\phi g(z+P_T(I+P_T)^{-1}\eta)||V_\phi f(z+\eta)|\frac{m(z+\eta)}{m(z+P_T(I+P_T)^{-1}\eta)}d\eta \\
&\le CC_T^{-1}\|V_\phi f m\|_{L^{p_1,p_2}}\|\mathfrak{T}_{P_T(I+P_T)^{-1}}V_\phi g m^{-1}\|_{L^{p_1',p_2'}} \\
&\le C(\det T)^{-1}(\det(I-T))^{-1}\left(\frac{\det(I-T)}{\det T}\right)^{1/p_2-1/p_1}\|f\|_{M^{p_1,p_2}_m}\|g\|_{M^{p_1',p_2'}_{1/m}},
\end{aligned}$$

and the claim follows.

The proof of *(iii)* is a slight variation of the previous one. □

In the case of polynomial weights we can similarly extend a characterization for the Wigner distribution proved in [CN18, Theorem 3.1]. Note that this is not a special case of the previous result, in general.

Theorem 5.5.4 ([CT20, Theorem 4.12].)**.** *Let $A=A_T\in \mathrm{GL}(2d,\mathbb{R})$ be a Cohen-type matrix. Let $s\in\mathbb{R}$ and $1\le p_i,q_i,p,q\le\infty$, $i=1,2$, be such that conditions* (5.18) *and* (5.19) *are satisfied.*

(i) If $f_1\in M^{p_1,q_1}_{v_{|s|}}(\mathbb{R}^d)$ and $f_2\in M^{p_2,q_2}_{v_s}(\mathbb{R}^d)$, then $W_T(f_1,f_2)\in M^{p,q}_{0,s}(\mathbb{R}^{2d})$, and the following estimate holds:

$$\|W_T(f_1,f_2)\|_{M^{p,q}_{0,s}} \lesssim_T \|f_1\|_{M^{p_1,q_1}_{v_{|s|}}}\|f_2\|_{M^{p_2,q_2}_{v_s}}.$$

(ii) Assume further that both T and $I-T$ are invertible (equivalently: A_T is right-regular, or P_T is invertible, cf. (5.14)*), and define $\mathcal{B}_T := (I+P_T)^{-1}$ and $\mathcal{U}_T := (I+P_T^{-1})^{-1}$. Set $v_s^{\mathcal{B}_T} = \mathfrak{T}_{\mathcal{B}_T} v_s$, $v_s^{\mathcal{U}_T} = \mathfrak{T}_{\mathcal{U}_T} v_s$. If $f_1 \in M^{p_1,q_1}_{v_{|s|}}(\mathbb{R}^d)$ and $f_2 \in M^{p_2,q_2}_{v_s^{\mathcal{U}_T}}(\mathbb{R}^d)$, then $W_T(f_1,f_2) \in W^{p,q}_{1\otimes v_s^{\mathcal{B}_T}}(\mathbb{R}^{2d})$, and the following estimate holds:*

$$\|W_T(f_1,f_2)\|_{W^{p,q}_{1\otimes v_s^{\mathcal{B}_T}}} \lesssim_T (C_T)^{1/q-1/p} \|f_1\|_{M^{p_1,q_1}_{v_{|s|}}} \|f_2\|_{M^{p_2,q_2}_{v_s^{\mathcal{U}_T}}},$$

where $C_T = |\det T||\det(I-T)| > 0$.

Proof. Arguing as in the proof of Theorem 5.5.3 we get the estimate

$$\|W_T(f_1,f_2)\|_{M^{p,q}_{0,s}} \asymp \left\| |V_g f_2|^p * |(V_g f_1)^*|^p \right\|^{1/p}_{L^{q/p}_{v_{ps}}}.$$

From now on, the proof proceeds exactly as in [CN18, Theorem 3.1]. Similar arguments also apply in the case where $p=\infty$ or $q=\infty$.

For what concerns the boundedness on Wiener amalgam spaces, notice first that if A_T is right regular, $(I+P_T)$ and also $\left(I+P_T^{-1}\right)$ are invertible, with

$$(I+P_T^{-1})^{-1} = (I+P_T)^{-1}P_T = P_T(1+P_T)^{-1}.$$

Arguing as before, one similarly gets

$$\|W_T(f_1,f_2)\|_{W^{p,q}_{1\otimes v_s^{\mathcal{B}_T}}} \asymp_T C_T^{1/q-1/p} \| |V_g f_1|^p * |((V_g f_2)^*(\mathcal{U}_T \cdot))^*|^p \|^{1/p}_{L^{q/p}_{v_{ps}}},$$

where $C_T = |\det T||\det(I-T)|$. Again, the proof proceeds hereinafter as in [CN18, Theorem 3.1]. □

Remark 5.5.5. *1. We remark that the given estimates are not sharp, since we employed window functions depending on T in order to perform the computations and thus the hidden constants in the symbol $\lesssim_T$ may depend on T. However, the comments of [CN18, Rem. 3.2] are still valid here. In particular, the result holds for more general weight functions: for instance, sub-exponential weights or polynomial weights satisfying formula* (4.10) *in [Tof04b] are suitable choices. Notice that the proof of the theorem in fact reduces to the study of continuity estimates for convolutions on weighted Lebesgue mixed-norm spaces.*

2. The previous results can be easily specialized to τ-Wigner distributions. For instance, we recover [CNT19b, Lemma 4.2 (ii)] by noticing that $(I+P_T)^{-1} = \mathcal{B}_\tau$ and $(I+P_T^{-1})^{-1} = \mathcal{U}_\tau$ for $T=\tau I$, where the matrices $\mathcal{B}_\tau$ and $\mathcal{U}_\tau$ are defined in [CNT19b, (5) and (26)].

We will also need the following boundedness result involving more general weights; we omit the proof since it is just a slight variation of the previous ones. The details in the case of τ-Wigner distributions may be found in [CNT19b, Lemma 4.2].

Proposition 5.5.6. *Let $A = A_T \in \mathrm{GL}(2d,\mathbb{R})$ be a Cohen-type matrix and $m \in \mathcal{M}_v(\mathbb{R}^{2d})$.*

1. *Let $1 \le p \le \infty$ and set $v_{-J} = \mathfrak{T}_{J^{-1}}$. If $f_1 \in M^p_v(\mathbb{R}^d)$ and $f_2 \in M^1_v(\mathbb{R}^d)$ then $W_T(f_1,f_2) \in M^{1,p}_{1\otimes v_{-J}}(\mathbb{R}^{2d})$, with*
$$\|W_T(f_1,f_2)\|_{M^{1,p}_{1\otimes v_{-J}}} \lesssim_T \|f_1\|_{M^p_v}\|f_2\|_{M^1_v}.$$
2. *Let $1 \le p_i, q_i, p, q \le \infty$, $i = 1,2$, be such that conditions* (5.18) *and* (5.19) *are satisfied. Assume further that A_T is right-regular and define $\mathcal{B}_T := (I+P_T)^{-1}$ and $\mathcal{U}_T := (I+P_T^{-1})^{-1}$, where P_T is defined in* (5.14). *Set $m^{\mathcal{B}_T} = \mathfrak{T}_{\mathcal{B}_T} m$, $m^{\mathcal{U}_T} = \mathfrak{T}_{\mathcal{U}_T} m$. If $f_1 \in M^{p_1,q_1}_v(\mathbb{R}^d)$ and $f_2 \in M^{p_2,q_2}_{m^{\mathcal{B}_T}}(\mathbb{R}^d)$ then $W_T(f_1,f_2) \in W^{p,q}_{1\otimes m^{\mathcal{U}_T}}(\mathbb{R}^{2d})$, with*
$$\|W_T(g,f)\|_{W^{p,q}_{1\otimes m^{\mathcal{U}_T}}} \lesssim_T (C_T)^{1/q-1/p}\|f_1\|_{M^{p_1,q_1}_v}\|f_2\|_{M^{p_2,q_2}_{m^{\mathcal{B}_T}}},$$
where $C_T = |\det T||\det(I-T)| > 0$.

5.5.3 Sharp estimates for τ-Wigner distributions

We already pointed out that the results in Theorems 5.5.4 and 5.5.3 are not sharp with respect to the "parameter" T. In the slightly more manageable case of τ-Wigner distributions, namely for $T = \tau I$ with $\tau \in [0,1]$, sharp estimates were obtained in the paper [CDT19].

First, for $1 \le p_1, p_2 \le \infty$ let us define the function
$$\alpha_{(p_1,p_2)}(\tau) := \frac{1}{\tau^{d\left(\frac{1}{p_1'}+\frac{1}{p_2}\right)}(1-\tau)^{d\left(\frac{1}{p_1}+\frac{1}{p_2'}\right)}}, \quad \tau \in (0,1). \tag{5.22}$$

Note that the this function just coincides with $\alpha_{(p_1,p_2)}(T)$ defined in (5.21) in the case where $T = \tau I$. The function $\alpha_{(p_1,p_2)}(\tau)$ is unbounded on $(0,1)$ at least on one endpoint for any choice of p_1, p_2.

In order for the results of Theorem 5.5.3 to be made sharp, we provide two preliminary results. Given a moderate weight m on $\mathbb{R}^{2d}$, we say that a measurable function f on $\mathbb{R}^{2d}_z \times \mathbb{R}^{2d}_\zeta$ belongs to the space $L^\infty_z(L^1_{\zeta,m})(\mathbb{R}^{4d}_{z,\zeta})$ if
$$\|f\|_{L^\infty_z(L^1_{\zeta,m})} = \sup_{z\in\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |f(z,\zeta)|m(\zeta)d\zeta < \infty. \tag{5.23}$$

The following convolution inequality holds.

Lemma 5.5.7 ([CDT19, Lemma 2.11]). *If* $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, $f \in L^1_{1\otimes v}(\mathbb{R}^{4d})$ *and* $g \in L^\infty_z(L^1_{\zeta,m})(\mathbb{R}^{4d})$, *then* $f * g \in L^\infty_z(L^1_{\zeta,m})(\mathbb{R}^{4d})$, *with*

$$\|f * g\|_{L^\infty_z(L^1_{\zeta,m})} \le \|f\|_{L^1_{1\otimes v}} \|g\|_{L^\infty_z(L^1_{\zeta,m})}.$$

Proof. Using the definition of $L^\infty_z(L^1_{\zeta,m})$-norm in (5.23),

$$\begin{aligned}
I := \|f * g\|_{L^\infty_z(L^1_{\zeta,m})} &= \sup_{z\in\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |f * g|(z,\zeta) m(\zeta) d\zeta \\
&= \sup_{z\in\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \left| \int_{\mathbb{R}^{4d}} f(y,\eta) g(z-y,\zeta-\eta) dy d\eta \right| m(\zeta) d\zeta \\
&\le \sup_{z\in\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} |f|(y,\eta) |g|(z-y,\zeta-\eta) d\eta \right) m(\zeta) dy d\zeta \\
&= \sup_{z\in\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} (|f|(y,\cdot) * |g|(z-y,\cdot))(\zeta) m(\zeta) dy d\zeta.
\end{aligned}$$

By Young's inequality (2.3),

$$\begin{aligned}
I &= \sup_{z\in\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \||f|(y,\cdot)\|_{L^1_v} \||g|(z-y,\cdot)\|_{L^1_m} dy \\
&\le \int_{\mathbb{R}^{2d}} \||f|(y,\cdot)\|_{L^1_v} \sup_{z\in\mathbb{R}^d} \||g|(z-y,\cdot)\|_{L^1_m} dy \\
&= \|g\|_{L^\infty_z(L^1_{\zeta,m})} \|f\|_{L^1_{1\otimes v}},
\end{aligned}$$

as claimed. ☐

The second result is an estimate for the τ-Wigner distribution with a Gaussian window, uniform with respect to τ. This is an essential result for the purpose of obtaining sharp estimates but we just sketch the main steps of the proof, which ultimately amounts to a lengthy and painful computation.

Lemma 5.5.8 ([CDT19, Lemma 3.2]). *Let* $\tau \in [0,1]$ *and consider* $\Phi(x,\xi) = e^{-\pi(x^2+\xi^2)}$, $(x,\xi) \in \mathbb{R}^{2d}$. *Set* $\Phi_\tau = W_\tau(\phi,\phi)$, *where* $\phi(t) = e^{-\pi t^2}$, $t \in \mathbb{R}^d$. *Let* v *be an admissible weight and set* $v_J = \mathfrak{T}_J v$. *There exists a constant* $C > 0$ *such that*

$$\|\Phi_\tau\|_{M^1_{1\otimes v_J}} \asymp \|V_\Phi \Phi_\tau\|_{L^1_{1\otimes v_J}} \le C, \quad \forall \tau \in [0,1].$$

Sketch of the proof. Recall that a generalized Gaussian function is defined by

$$\phi_{a,b,c}(x,\xi) = e^{-\pi a x^2} e^{-\pi b \xi^2} e^{2\pi i c x\cdot\xi}, \quad (x,\xi) \in \mathbb{R}^{2d},$$

for arbitrary positive parameters $a, b, c > 0$. It is proved in [CN18, Proposition 2.2] that the STFT of such a function with window $\Phi(x,\xi) = e^{-\pi(x^2+\xi^2)}$ is given by

$$\begin{aligned} &V_\Phi \phi_{a,b,c}(z,\zeta) = C(a,b,c) \\ &\times \exp\left(-\pi \frac{[a(b+1)+c^2]z_1^2 + [(a+1)b+c^2]z_2^2 + (b+1)\zeta_1^2 + (a+1)\zeta_2^2 - 2c(z_1\cdot\zeta_2 + z_2\cdot\zeta_1)}{(a+1)(b+1)+c^2} \right) \\ &\qquad\qquad \times \exp\left(-\frac{2\pi i}{a+1}\left(z_1\cdot\zeta_1 + (cz_1 - (a+1)\zeta_2)\frac{c\zeta_1 + (a+1)z_2}{(a+1)(b+1)+c^2} \right)\right), \end{aligned}$$

for any $z = (z_1, z_2), \zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$, with $C(a,b,c) = [(a+1)(b+1)+c^2]^{-d/2}$. It is easy to see from (5.11) with $M = (\tau - 1/2)I$ and $\lambda = 1$ that

$$W_\tau \phi(x,\xi) = C(\tau) e^{-\pi a(\tau) x^2} e^{-\pi b(\tau)\xi^2} e^{2\pi i c(\tau) x\cdot\xi},$$

where

$$a(\tau) = b(\tau) = C(\tau)^{2/d} = \frac{1}{2\tau^2 - 2\tau + 1}, \quad c(\tau) = \frac{2\tau - 1}{2\tau^2 - 2\tau + 1}.$$

By combining the previous results we thus have

$$\begin{aligned} &|V_\Phi \Phi_\tau|(z,\zeta) = \frac{1}{(2\tau^2 - 2\tau + 5)^{d/2}} \\ &\quad \times \exp\left(-\pi \frac{3(z_1^2 + z_2^2) + (2\tau^2 - 2\tau + 2)(\zeta_1^2 + \zeta_2^2) + (2 - 4\tau)(z_1\cdot\zeta_2 + z_2\cdot\zeta_1)}{2\tau^2 - 2\tau + 5} \right). \end{aligned}$$

We now recall from Proposition 2.2.2 that a submultiplicative weight has at most exponential growth[2], namely $v(z) \lesssim e^{a|z|}$ for some $a \geq 0$. Note that for $\tau \in [0,1]$ the coefficient in front of the exponential is uniformly bounded, hence

$$\|V_\Phi \Phi_\tau\|_{L^1_{1\otimes v_J}} \leq C \int_{\mathbb{R}^{2d}} e^{-\pi \frac{3(z_1^2+z_2^2)}{2\tau^2 - 2\tau + 5}} I_1 dz_1 dz_2,$$

where

$$I_1 := \int_{\mathbb{R}^{2d}} e^{-\pi \frac{(2\tau^2 - 2\tau + 2)(\zeta_1^2 + \zeta_2^2) + (2-4\tau)(z_1\cdot\zeta_2 + z_2\cdot\zeta_1)}{2\tau^2 - 2\tau + 5}} e^{a|J\zeta|} d\zeta_1 d\zeta_2.$$

[2] In fact, admissible weights used in this dissertation grow at most polynomially, but it is better to use a rougher exponential estimate for the purposes of the proof.

The integral I_1 can be estimated as follows:

$$I_1 \leq \left(\int_{\mathbb{R}^d} e^{-\pi\frac{(2\tau^2-2\tau+2)\zeta_1^2+(2-4\tau)z_2\cdot\zeta_1}{2\tau^2-2\tau+5}} e^{a|\zeta_1|}d\zeta_1\right)\left(\int_{\mathbb{R}^d} e^{-\pi\frac{(2\tau^2-2\tau+2)\zeta_2^2+(2-4\tau)z_1\cdot\zeta_2}{2\tau^2-2\tau+5}} e^{a|\zeta_2|}d\zeta_2\right).$$

We compute the integral with respect to the variable ζ_1 (the other one is analogous):

$$\int_{\mathbb{R}^d} e^{-\pi\frac{(2\tau^2-2\tau+2)\zeta_1^2+(2-4\tau)z_2\cdot\zeta_1}{2\tau^2-2\tau+5}} e^{a|\zeta_1|}d\zeta_1 = e^{\pi\frac{(1-2\tau)^2 z_2^2}{(2\tau^2-2\tau+2)(2\tau^2-2\tau+5)}} I_2,$$

where we set

$$I_2 = \int_{\mathbb{R}^d} e^{-\pi\frac{((2\tau^2-2\tau+2)\zeta_1+(1-2\tau)z_2)^2}{(2\tau^2-2\tau+5)(2\tau^2-2\tau+2)}} e^{a|\zeta_1|}d\zeta_1.$$

After the substitution $(2\tau^2-2\tau+2)\zeta_1+(1-2\tau)z_2=\eta_1$ we have

$$\begin{aligned} I_2 &= \frac{1}{(2\tau^2-2\tau+2)^d}\int_{\mathbb{R}^d} e^{-\pi\frac{\eta_1^2}{(2\tau^2-2\tau+5)(2\tau^2-2\tau+2)}} e^{\frac{a}{2\tau^2-2\tau+2}|\eta_1-(1-2\tau)z_2|}d\eta_1 \\ &\leq C_1^d e^{\frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_2|}\int_{\mathbb{R}^d} e^{-\pi C_2\eta_1^2}e^{aC_1|\eta_1|}d\eta_1, \end{aligned}$$

where

$$C_1=\max_{\tau\in[0,1]}\frac{1}{(2\tau^2-2\tau+2)}=\frac{2}{3},\quad C_2=\min_{\tau\in[0,1]}\frac{1}{(2\tau^2-2\tau+5)(2\tau^2-2\tau+2)}=\frac{1}{10}.$$

Using $\lim_{|\eta_1|\to\infty} e^{-\pi\frac{C_1}{2}\eta_1^2}e^{aC_2|\eta_1|}=0$ we conclude that

$$I_2 \leq Ce^{\frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_2|},$$

hence

$$I_1 \leq Ce^{\pi\frac{(1-2\tau)^2 z_2^2}{(2\tau^2-2\tau+2)(2\tau^2-2\tau+5)}+\frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_2|}e^{\pi\frac{(1-2\tau)^2 z_1^2}{(2\tau^2-2\tau+2)(2\tau^2-2\tau+5)}+\frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_1|}.$$

Finally,

$$\begin{aligned} \|V_\Phi\Phi_\tau\|_{L^1_{1\otimes v_J}} \leq C'\int_{\mathbb{R}^d} & e^{-\pi\frac{3z_1^2}{2\tau^2-2\tau+5}} e^{\pi\frac{(1-2\tau)^2 z_1^2}{(2\tau^2-2\tau+2)(2\tau^2-2\tau+5)}+\frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_1|}dz_1 \\ &\times\int_{\mathbb{R}^d} e^{-\pi\frac{3z_2^2}{2\tau^2-2\tau+5}} e^{\pi\frac{(1-2\tau)^2 z_2^2}{(2\tau^2-2\tau+2)(2\tau^2-2\tau+5)}+\frac{a|1-2\tau|}{2\tau^2-2\tau+2}|z_2|}dz_2. \end{aligned}$$

The latter integrals can be uniformly estimated arguing as before for I_2, in particular

$$\|V_\Phi\Phi_\tau\|_{L^1_{1\otimes v_J}} \leq C'\left(\int_{\mathbb{R}^d} e^{-\pi z^2/2+a|z|/2}dz\right)^2<\infty,$$

where the constant $C'>0$ does not depend on τ. □

We are then ready to prove sharp version of Theorem 5.5.3 for τ-Wigner distributions.

Theorem 5.5.9 ([CDT19, Proposition 3.1-3.2-5.1]). *Under the assumptions of Proposition 5.5.3 with $T=\tau I$, there exist constants $C, C', C''>0$ independent of τ such that*

$$\|W_\tau(g,f)\|_{M^{p,q}_{1\otimes 1/v_J}} \lesssim C\|f\|_{M^{p_1,q_1}_m}\|g\|_{M^{p_2,q_2}_{1/m}}, \quad \tau\in[0,1].$$

$$\|W_\tau(g,f)\|_{W(\mathcal{F}L^1_{1/v_J},L^\infty)} \leq C'\alpha_{(p_1,p_2)}(\tau)\|f\|_{M^{p_1,p_2}_m}\|g\|_{M^{p_1',p_2'}_{1/m}}, \quad \tau\in(0,1),$$

$$\|W_\tau(g,f)\|_{W^{2,2}_{(1/v_J)\otimes 1}} \lesssim C''\|f\|_{M^2_m}\|g\|_{M^2_{1/m}}, \quad \tau\in(0,1). \tag{5.24}$$

Proof. Note that the estimate (5.20) in the case where $\tau=0$ or $\tau=1$ is already sharp; for instance, using the magic formula (5.16) in the case $T=0$ we have

$$\begin{aligned}
&\|W_0(g,f)\|_{M^{p,q}_{1\otimes 1/v_J}}\\
&=\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_\phi g(z_1,z_2+\zeta_1)|^p|V_\phi f(z_1+\zeta_2,z_2)|^p dz_1dz_2\right)^{q/p}\frac{1}{v_q(\zeta_2,-\zeta_1)}d\zeta_1d\zeta_2\right)^{1/q}\\
&=\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_\phi g(z_1-\zeta_2,z_2+\zeta_1)|^p|V_\phi f(z_1,z_2)|^p\frac{1}{v^p(\zeta_2,-\zeta_1)}dz_1dz_2\right)^{q/p}d\zeta_1d\zeta_2\right)^{1/q}\\
&\leq C\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_\phi g(z-J\zeta)|^p|V_\phi f(z)|^p\frac{m^p(z)}{m^p(z-J\zeta)}dz\right)^{q/p}d\zeta\right)^{1/q}\\
&=C\|(|(V_\phi g)^\vee|^p m^{-p}))*(|V_\phi f|^p m^p)\|_{L^{q/p}},
\end{aligned}$$

and similarly for $\tau=1$.

Assume then $\tau\in(0,1)$. We use the results of Theorem 5.5.3 with $T=\tau I$, the change-of-window lemma for the STFT in (3.3), Lemma 5.5.8 and Moyal's formula (3.6). Therefore, there exists a positive constant C independent of τ such that

$$\begin{aligned}
\|W_\tau(g,f)\|_{M^{p,q}_{1\otimes 1/v_J}} &\leq \frac{1}{|\langle\Phi_\tau,\Phi_\tau\rangle|}\||V_{\Phi_\tau}W_\tau(g,f)|*|V_\Phi\Phi_\tau|\|_{L^{p,q}_{1\otimes 1/v_J}}\\
&\leq \frac{1}{\|\phi\|^2_{L^2}\|\phi\|^2_{L^2}}\|W_\tau(g,f)\|_{M^{p,q}_{1\otimes 1/v_J}}\|\Phi_\tau\|_{M^1_{1\otimes v_J}}\\
&\leq C\|f\|_{M^{p_1,q_1}_m}\|g\|_{M^{p_2,q_2}_{1/m}}.
\end{aligned}$$

In a similar fashion, also by Lemma 5.5.7,

$$\begin{aligned}\|V_\Phi W_\tau(g,f)\|_{L^\infty_z(L^1_{\zeta,1/v_J})} &\le \frac{1}{|\langle\Phi_\tau,\Phi_\tau\rangle|}\||V_{\Phi_\tau}W_\tau(g,f)|*|V_\Phi\Phi_\tau|\|_{L^\infty_z(L^1_{\zeta,1/v_J})}\\ &\le \frac{1}{\|\phi\|^2_{L^2}\|\phi\|^2_{L^2}}\|V_{\Phi_\tau}W_\tau(g,f)\|_{L^\infty_z(L^1_{\zeta,1/v_J})}\|V_\Phi\Phi_\tau\|_{L^1_{1\otimes v_J}}\\ &\le C\alpha_{(p_1,p_2)}(\tau)\|f\|_{M^{p_1,p_2}_m}\|g\|_{M^{p_1',p_2'}_{1/m}}.\end{aligned}$$

The proof of (5.24) is similar. □

5.6 Pseudodifferential operators

In this section we discuss the formalism of pseudodifferential operators that is associated with every time-frequency representation $\mathcal{B}_A$.

Imitating the time-frequency analysis of Weyl operators, we introduce the following general pseudodifferential calculus.

Theorem 5.6.1 ([Bay10, Proposition 2.2.1])**.** *Let $A\in GL(2d,\mathbb{R})$ and $\sigma\in M^\infty\big(\mathbb{R}^{2d}\big)$. The mapping $\mathrm{op}_A(\sigma)=\sigma^A$ defined by duality as*

$$\left\langle\sigma^A f,g\right\rangle=\langle\sigma,\mathcal{B}_A(g,f)\rangle,\qquad f,g\in M^1(\mathbb{R}^d)$$

is a well-defined linear continuous map from $M^1(\mathbb{R}^d)$ to $M^\infty(\mathbb{R}^d)$ (and also from $\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}'(\mathbb{R}^d)$ if $\sigma\in\mathcal{S}'(\mathbb{R}^{2d})$.

The proof easily follows from the continuity of the distribution $\mathcal{B}_A: M^1(\mathbb{R}^d)\times M^1(\mathbb{R}^d)\to M^1(\mathbb{R}^{2d})$, from Proposition 5.3.2.

Definition 5.6.2. *Let $A\in GL(2d,\mathbb{R})$ and $\sigma\in M^\infty\big(\mathbb{R}^{2d}\big)$. The mapping defined in Theorem 5.6.1, namely*

$$\sigma^A: M^1(\mathbb{R}^d)\ni f\mapsto\sigma^A f\in M^\infty(\mathbb{R}^d):\left\langle\sigma^A f,g\right\rangle=\langle\sigma,\mathcal{B}_A(g,f)\rangle,\quad\forall g\in M^1(\mathbb{R}^d),$$

is called quantization rule with symbol σ associated with the matrix-Wigner distribution $\mathcal{B}_A$, or pseudodifferential operator with symbol σ associated with the matrix-Wigner distribution $\mathcal{B}_A$.

Sometimes we use the equivalent notation $\mathrm{op}_A(\sigma)$ for σ^A. Moreover, if $A=A_T$ is of Cohen type we write σ^T in place of σ^{A_T} with a slight abuse of notation. In particular, we retrieve τ-pseudodifferential operators $\mathrm{op}_\tau(\sigma)$ for $T=\tau I$, $\tau\in[0,1]$; we use the traditional notation σ^{w} for Weyl pseudodifferential operators ($\tau=1/2$). Using Feichtinger's kernel theorem (Theorem 3.2.15), we now provide a number of equivalent representations for $\sigma^A f$.

Theorem 5.6.3. *Let* $A \in \mathrm{GL}(2d, \mathbb{R})$*. Let* $T : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ *be a continuous linear operator and* $A \in \mathrm{GL}(2d, \mathbb{R})$*. There exist distributions* $k, \sigma, F \in M^\infty\big(\mathbb{R}^{2d}\big)$ *such that* T *admits the following representations:*

(i) as an integral operator with kernel k*:* $\langle Tf, g\rangle = \big\langle k, g \otimes \overline{f}\big\rangle$ *for any* $f, g \in M^1(\mathbb{R}^d)$*;*

(ii) as pseudodifferential operator with symbol σ *associated with* $\mathcal{B}_A$*:* $T = \sigma^A$*;*

(iii) as a superposition (in weak sense) of time-frequency shifts (also called spreading representation):

$$T = \iint_{\mathbb{R}^{2d}} F(x, \xi) T_x M_\xi \, dx d\xi.$$

The relations among k, σ, F *and* A *are the following:*

$$\sigma = |\det A| \mathcal{F}_2 \mathfrak{T}_A k, \qquad F = \mathcal{F}_2 \mathfrak{T}_{A_{ST}} k.$$

Proof. The first representation is exactly the claim of the kernel theorem. Now set $\sigma = |\det A| \mathcal{F}_2 \mathfrak{T}_A k \in M^\infty\big(\mathbb{R}^{2d}\big)$: this is a well-defined distribution, since $\mathcal{F}_2$ and $\mathfrak{T}_A$ are isomorphisms on $M^\infty\big(\mathbb{R}^{2d}\big)$. In particular, for any $f, g \in M^1(\mathbb{R}^d)$ we have

$$\begin{aligned}
\langle Tf, g\rangle &= \big\langle k, g \otimes \overline{f}\big\rangle \\
&= \big\langle |\det A|^{-1} \mathfrak{T}_A^{-1} \mathcal{F}_2^{-1} \sigma, g \otimes \overline{f}\big\rangle \\
&= \big\langle \sigma, \mathcal{F}_2 \mathfrak{T}_A\big(g \otimes \overline{f}\big)\big\rangle \\
&= \langle \sigma, \mathcal{B}_A(g, f)\rangle \\
&= \big\langle \sigma^A f, g\big\rangle.
\end{aligned}$$

This proves that $Tf = \sigma^A f$ in $M^\infty(\mathbb{R}^d)$. The relation between the kernel representation in item 1 and the spreading representation in item 3 is well-known, e.g. [Grö01]. It can also be deduced from item 2 from the special matrix $A_{ST} = \begin{bmatrix} 0 & I \\ -I & I \end{bmatrix}$ and

$$\langle Tf, g\rangle = \langle F, V_f g\rangle = \langle F, \mathcal{B}_{A_{ST}}(g, f)\rangle, \qquad f, g \in M^1(\mathbb{R}^d).$$

□

Remark 5.6.4. *Since* $k = |\det A|^{-1} \mathfrak{T}_{A^{-1}} \mathcal{F}_2^{-1} \sigma = |\det A|^{-1} \mathfrak{T}_{\mathcal{I}_2 A^{-1}} \mathcal{F}_1^{-1} \widehat{\sigma}$*, one can formally obtain another representation of the third type with a special spreading function:*

$$\sigma^A f(x) = \frac{1}{|\det A|} \int_{\mathbb{R}^{2d}} \widehat{\sigma}(\xi, -(A^{-1})_{21} x - (A^{-1})_{22} y) e^{2\pi i \xi \cdot [(A^{-1})_{11} x + (A^{-1})_{22} y]} f(y) \, d\xi dy.$$

Notice that the inverse of a Cohen-type matrix $A = A_T$ has the form

$$A_T^{-1} = \begin{bmatrix} -(I-T) & T \\ I & -I \end{bmatrix},$$

thus the previous formula becomes

$$\sigma^A f(x) = \int_{\mathbb{R}^{2d}} \widehat{\sigma}(\xi,u)e^{-2\pi i(I-T)u\cdot\xi}T_{-u}M_\xi f(x)d\xi du. \tag{5.25}$$

This should be compared with [Grö01, (14.14)] for the Weyl quantization and [CDT19, (20)] for τ-quantization.

We now study the relations among pseudodifferential operators associated with MWDs and the corresponding symbols.

Proposition 5.6.5. *Let $A, B \in \mathrm{GL}(2d,\mathbb{R})$ and $\sigma, \rho \in M^\infty(\mathbb{R}^{2d})$. Then,*

$$\sigma^A = \rho^B \quad \Longleftrightarrow \quad \sigma = \frac{|\det A|}{|\det B|}\mathcal{F}_2\mathfrak{T}_{B^{-1}A}\mathcal{F}_2^{-1}\rho$$

Proof. Assume that $T = \sigma^A = \rho^B$. According to Theorem 5.6.3, T has a distributional kernel k such that

$$\sigma = |\det A|\mathcal{F}_2\mathfrak{T}_A k, \qquad \rho = |\det B|\mathcal{F}_2\mathfrak{T}_B k.$$

Therefore,

$$\begin{aligned}\sigma &= |\det A|\mathcal{F}_2\mathfrak{T}_A k \\ &= \frac{|\det A|}{|\det B|}\mathcal{F}_2\mathfrak{T}_A\mathfrak{T}_B^{-1}\mathcal{F}_2^{-1}\rho \\ &= \frac{|\det A|}{|\det B|}\mathcal{F}_2\mathfrak{T}_{B^{-1}A}\mathcal{F}_2^{-1}\rho.\end{aligned}$$

On the other side, if $\sigma = |\det A||\det B|^{-1}\mathcal{F}_2\mathfrak{T}_{B^{-1}A}\mathcal{F}_2^{-1}\rho$, then for any $f, g \in M^1(\mathbb{R}^d)$

$$\begin{aligned}\langle \sigma^A f, g\rangle &= \langle \sigma, \mathcal{F}_2\mathfrak{T}_A(f\otimes\overline{g})\rangle \\ &= \left\langle |\det A||\det B|^{-1}\mathcal{F}_2\mathfrak{T}_A\mathfrak{T}_B^{-1}\mathcal{F}_2^{-1}\rho, \mathcal{F}_2\mathfrak{T}_A(f\otimes\overline{g})\right\rangle \\ &= \langle \rho, \mathcal{F}_2\mathfrak{T}_B(f\otimes\overline{g})\rangle \\ &= \langle \rho^B f, g\rangle.\end{aligned}$$

□

In the case of Cohen-type matrices we have a more explicit relation that covers the usual rule for τ-Shubin operators (cf. [Tof04a, Remark 1.5]). The proof is a straightforward application of (5.12).

Proposition 5.6.6. *Let $A_1 = A_{T_1}$, $A_2 = A_{T_2}$ be Cohen-type invertible matrices, and $\sigma, \rho \in M^\infty\big(\mathbb{R}^{2d}\big)$. Then,*

$$\sigma_1^{T_1} = \sigma_2^{T_2} \quad \Longleftrightarrow \quad \widehat{\sigma_2}(\xi,\eta) = e^{-2\pi i \xi\cdot(T_2-T_1)\eta}\widehat{\sigma_1}(\xi,\eta).$$

Moreover, if $T_2 - T_1 \in \mathrm{GL}(d,\mathbb{R})$ then

$$\sigma_1^{T_1} = \sigma_2^{T_2} \quad \Longleftrightarrow \quad \sigma_2(x,\xi) = |\det(T_2-T_1)|^{-1}e^{2\pi i\xi\cdot(T_2-T_1)^{-1}x}\sigma_1(x,\xi).$$

It is also interesting to characterize the matrices yielding self-adjoint operators.

Proposition 5.6.7 ([Bay10, Proposition 2.2.3])**.** *Let $A \in GL(2d,\mathbb{R})$ and $\sigma \in M^\infty\big(\mathbb{R}^{2d}\big)$. Then*

$$\big(\sigma^A\big)^* = \rho^B,$$

where

$$\rho = \overline{\sigma}, \qquad B = \tilde{I}A\mathcal{I}_2 = \begin{bmatrix} A_{21} & -A_{22} \\ A_{11} & -A_{12} \end{bmatrix}.$$

In particular, σ^A is self-adjoint if and only if $\sigma = \overline{\sigma}$ (real symbol) and $B = A$, hence

$$A_{21} = A_{11}, \qquad A_{12} = -A_{22}.$$

Remark 5.6.8. *The previous result shows that only matrices of the form $\begin{bmatrix} P & Q \\ P & -Q \end{bmatrix}$, with $P, Q \in GL(d,\mathbb{R})$, give rise to pseudodifferential operators which are self-adjoint for real symbols. This occurs only for Weyl calculus as far as the class of T-operators is concerned.*

5.7 Boundedness results for matrix-pseudodifferential operators

5.7.1 Boundedness on Lebesgue spaces

The boundedness of a pseudodifferential operator σ^A associated with $\mathcal{B}_A$ is intimately related to the boundedness of the distribution $\mathcal{B}_A$ on Lebesgue spaces, in view of the duality in the definition of σ^A and Theorem 5.5.1.

Theorem 5.7.1. *Let $A \in \mathrm{GL}(2d,\mathbb{R})$ be right-regular and $\sigma \in L^q\big(\mathbb{R}^{2d}\big)$. The quantization mapping*

$$\sigma \in L^q(\mathbb{R}^{2d}) \mapsto \sigma^A \in \mathcal{L}\big(L^p(\mathbb{R}^d)\big)$$

is continuous if and only if $q \le 2$ and $q \le p \le q'$, with norm estimate

$$\left\|\sigma^A\right\|_{L^p\to L^p} \le \frac{\|\sigma\|_{L^q}}{|\det A|^{\frac{1}{q'}}|\det A_{12}|^{\frac{1}{p}-\frac{1}{q'}}|\det A_{22}|^{\frac{1}{p'}-\frac{1}{q'}}}.$$

Proof. Assume $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p'}(\mathbb{R}^d)$, with $p \neq 1$ nor $p \neq \infty$. Therefore, by (5.17) (switch q and q') and Hölder inequality:

$$\begin{aligned}\left|\left\langle\sigma^A f, g\right\rangle\right| &= |\langle\sigma, \mathcal{B}_A(g,f)\rangle| \\ &\le \|\sigma\|_{L^q}\|\mathcal{B}_A(g,f)\|_{L^{q'}} \\ &\le \frac{\|\sigma\|_{L^q}}{|\det A|^{\frac{1}{q'}}|\det A_{12}|^{\frac{1}{p}-\frac{1}{q'}}|\det A_{22}|^{\frac{1}{p'}-\frac{1}{q'}}}\|f\|_{L^p}\|g\|_{L^{p'}}.\end{aligned}$$

□

Remark 5.7.2. *Note that the closed graph theorem implies the non-continuity of the quantization map. This means that there exists a symbol $\sigma \in L^q$ for which the operator σ^A is not bounded on $L^p(\mathbb{R}^{2d})$, cf. [BDDO09, Proposition 3.4].*

One can also study compactness and Schatten class properties for these operators. We confine ourselves to state a result for symbols in the modulation spaces M^1 and M^2.

Theorem 5.7.3 ([BCGT20, Theorem 16] and [Bay10, Theorem 2.2.9])**.** *Let $A \in \mathrm{GL}(2d,\mathbb{R})$.*

(i) If $\sigma \in M^1\big(\mathbb{R}^{2d}\big)$ then the operator $\sigma^A \in \mathcal{L}\big(L^2(\mathbb{R}^d)\big)$ belongs to the trace class $\mathfrak{S}^1\big(L^2(\mathbb{R}^d)\big)$, with

$$\left\|\sigma^A\right\|_{\mathfrak{S}^1} \lesssim |\det A|^{1/2}\|\sigma\|_{M^1}.$$

(ii) If $\sigma \in L^2\big(\mathbb{R}^{2d}\big)$ then $\sigma^A \in \mathcal{L}\big(L^2(\mathbb{R}^d)\big)$ is a Hilbert-Schmidt operator in $\mathfrak{S}^2\big(L^2(\mathbb{R}^d)\big)$, with

$$\left\|\sigma^A\right\|_{\mathfrak{S}^2} \lesssim |\det A|^{1/2}\|\sigma\|_{L^2} \asymp |\det A|^{1/2}\|\sigma\|_{M^2}.$$

Proof. We give a sketch of the proof of the first item, since the technique used is of independent interest. Proposition 5.7.1 immediately yields $\sigma^A \in \mathcal{L}\big(L^2(\mathbb{R}^d)\big)$, since $M^1\big(\mathbb{R}^{2d}\big) \subseteq L^1\big(\mathbb{R}^{2d}\big) \cap L^2\big(\mathbb{R}^{2d}\big)$. In line with the paradigm of time-frequency analysis of operators (cf. [Grö01, Section 14.5] and [Grö96, Theorem 3]), let us decompose the action of σ^A into elementary pseudodifferential operators with time-frequency shifts of a suitable function as symbols. The inversion formula for the STFT allows to write

$$\sigma = \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} V_\Phi \sigma(z, \zeta) M_\zeta T_z \Phi dz d\zeta,$$

for any window function $\Phi \in M^1\big(\mathbb{R}^{2d}\big)$ with $\|\Phi\|_{L^2} = 1$. Therefore, for any $f, g \in M^1(\mathbb{R}^d)$:

$$\begin{aligned}
\langle \sigma^A f, g\rangle &= \langle \sigma, \mathcal{B}_A(g, f)\rangle \\
&= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} V_\Phi \sigma(z, \zeta) \langle M_\zeta T_z \Phi, \mathcal{B}_A(g, f)\rangle dz d\zeta \\
&= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} V_\Phi \sigma(z, \zeta) \langle (M_\zeta T_z \Phi)^A f, g\rangle dz d\zeta.
\end{aligned}$$

This shows that σ^A acts (in an operator-valued sense on L^2) as a continuous weighted superposition of elementary operators:

$$\sigma^A = \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} V_\Phi \sigma(z, \zeta) (M_\zeta T_z \Phi)^A dz d\zeta. \tag{5.26}$$

The action of the building blocks $(M_\zeta T_z \Phi)^A$ can be unwrapped by means of the magic formula (5.3.7) provided one takes $\Phi = \mathcal{B}_A \varphi$ for some $\varphi \in M^1(\mathbb{R}^d)$ with $\|\varphi\|_{L^2} = |\det A|^{1/4}$ (see the orthogonality relations (5.7)):

$$\begin{aligned}
\langle (M_\zeta T_z \Phi)^A f, g\rangle &= \langle M_\zeta T_z \Phi, \mathcal{B}_A(g, f)\rangle \\
&= \overline{V_{\mathcal{B}_A \varphi} \mathcal{B}_A(g, f)(z, \zeta)} \\
&= e^{2\pi i z_2 \cdot \zeta_2} V_\varphi f(b, \beta) \overline{V_\varphi g(a, \alpha)},
\end{aligned}$$

where a, α, b, β are continuous functions of z and ζ. In particular, we have

$$(M_\zeta T_z \Phi)^A : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \quad : \quad f \mapsto e^{2\pi i z_2 \cdot \zeta_2} \langle f, M_\beta T_b \varphi\rangle M_\alpha T_a,$$

hence $(M_\zeta T_z \Phi)^A$ is a rank-one operator with trace class norm given by $\|(M_\zeta T_z \Phi)^A\|_{\mathfrak{S}^1} = \|\varphi\|_{L^2}^2 = |\det A|^{1/2}$, independent of z, ζ. To conclude, we reconstruct the operator σ^A and compute its norm by means of the estimates for the pieces:

$$\|\sigma^A\|_{\mathfrak{S}^1} \le \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_\Phi \sigma(z, \zeta)| \|(M_\zeta T_z \Phi)^A\|_{\mathfrak{S}^1} dz d\zeta \le C_A \|\sigma\|_{M^1}.$$

□

Remark 5.7.4. *The superposition formula* (5.26) *should be compared with the inversion formula* (4.1)*, which in this case reads*

$$\sigma^A f = \int_{\mathbb{R}^{2d}} V_g f(z)\sigma^A(\pi(z)g)dz.$$

5.7.2 Boundedness on modulation spaces

We now study the boundedness on modulation spaces of pseudodifferential operators associated with Cohen-type representations.

Theorem 5.7.5 (Symbols in $M^{p,q}$)**.** *Let $A = A_T \in \mathrm{GL}(2d,\mathbb{R})$ be a Cohen-type matrix and consider indices $1 \le p, p_1, p_2, q, q_1, q_2 \le \infty$, satisfying the following relations:*

$$p_1, p_2', q_1, q_2' \le q', \qquad \frac{1}{p_1} + \frac{1}{p_2'} \ge \frac{1}{p'} + \frac{1}{q'}, \qquad \frac{1}{q_1} + \frac{1}{q_2'} \ge \frac{1}{p'} + \frac{1}{q'}. \tag{5.27}$$

1. *Let $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ and set $v_J = \mathfrak{T}_J v$. If $\sigma \in M^{p,q}_{1\otimes v_J}(\mathbb{R}^{2d})$ then the pseudodifferential operator σ^T is bounded from $M^{p_1,q_1}_m(\mathbb{R}^d)$ to $M^{p_2,q_2}_{1/m}(\mathbb{R}^d)$, with*

$$\left\|\sigma^T\right\|_{M^{p_1,q_1}_m \to M^{p_2,q_2}_{1/m}} \lesssim_T \|\sigma\|_{M^{p,q}_{1\otimes v_J}}.$$

2. *Let $s \in \mathbb{R}$. If $\sigma \in M^{p,q}_{0,s}\big(\mathbb{R}^{2d}\big)$ then the pseudodifferential operator σ^T is bounded from $M^{p_1,q_1}_{-s}(\mathbb{R}^d)$ to $M^{p_2,q_2}_{-|s|}(\mathbb{R}^d)$, with*

$$\left\|\sigma^T\right\|_{M^{p_1,q_1}_{-s} \to M^{p_2,q_2}_{-|s|}} \lesssim_T \|\sigma\|_{M^{p,q}_{0,s}}.$$

Proof. Under the given assumptions on the indices, Theorem 5.5.3 (i) implies that $W_T(g,f) \in M^{p',q'}_{1\otimes 1/v_J}(\mathbb{R}^d)$ for any $f \in M^{p_1,q_1}_m(\mathbb{R}^d)$ and $g \in M^{p_2',q_2'}_{1/m}(\mathbb{R}^d)$. Therefore, by the duality of modulation spaces we obtain

$$\begin{aligned} \left|\left\langle \sigma^T f, g\right\rangle\right| &= |\langle \sigma, W_T(g,f)\rangle| \\ &\le \|\sigma\|_{M^{p,q}_{1\otimes v_J}} \|W_T(g,f)\|_{M^{p',q'}_{1\otimes 1/v_J}} \\ &\lesssim_A \|\sigma\|_{M^{p,q}_{1\otimes v_J}} \|f\|_{M^{p_1,q_1}_m} \|g\|_{M^{p_2',q_2'}_{1/m}}. \end{aligned}$$

The remaining claim follows by a similar argument using the results in Theorem 5.5.4 (i). ☐

We are also able to consider symbols in Wiener amalgam spaces, as detailed below.

Theorem 5.7.6 (Symbols in $W^{p,q}$)**.** *Let $A = A_T \in \mathrm{GL}(2d,\mathbb{R})$ be a right-regular Cohen-type matrix and m be a v-moderate weight on $\mathbb{R}^{2d}$; set $v_J = \mathfrak{T}_J v$. Consider indices $1 \le p, q, r_1, r_2 \le \infty$, satisfying the following relations:*

$$q \le p', \qquad r_1, r_1', r_2, r_2' \le p. \tag{5.28}$$

For any $\sigma \in W^{p,q}_{v_J \otimes 1}(\mathbb{R}^{2d})$ the pseudodifferential operator σ^T is bounded on $M^{r_1,r_2}_m(\mathbb{R}^d)$, with

$$\|\sigma^T\|_{M^{r_1,r_2}_m \to M^{r_1,r_2}_m} \lesssim_T C_T \alpha_{(r_1,r_2)}(T) \|\sigma\|_{W^{p,q}_{v_J\otimes 1}},$$

where $C_T = |\det T||\det(I-T)|$ and $\alpha_{(r_1,r_2)}(T)$ is defined in (5.21)*.*

Proof. We divide the proof into three parts.

Step 1. Let $\sigma \in W^{\infty,1}_{v_J\otimes 1}(\mathbb{R}^{2d})$. For any $1 \le p_1, p_2 \le \infty$, $f \in M^{p_1,p_2}_m(\mathbb{R}^d)$ and $g \in M^{p_1',p_2'}_{1/m}(\mathbb{R}^d)$, by Theorem 5.5.3 (*ii*) and duality of amalgam spaces we have

$$\begin{aligned}
\left|\langle \sigma^A f, g\rangle\right| &= |\langle \sigma, W_T(g,f)\rangle| \\
&\le \|\sigma\|_{W^{\infty,1}_{v_J\otimes 1}} \|W_T(g,f)\|_{W^{1,\infty}_{1/v_J\otimes 1}} \\
&\lesssim_T \alpha_{(p_1,p_2)}(T) \|\sigma\|_{W^{\infty,1}_{v_J\otimes 1}} \|f\|_{M^{p_1,p_2}} \|g\|_{M^{p_1',p_2'}}.
\end{aligned}$$

Step 2. Let $\sigma \in W^{2,2}_{v_J\otimes 1}(\mathbb{R}^{2d})$. If $f, g \in M^2_m(\mathbb{R}^d)$, by Theorem 5.5.3 (*iii*) and duality of amalgam spaces we get have, arguing as before,

$$\left\|\sigma^A\right\|_{M^2_m \to M^2_m} \lesssim_T \|\sigma\|_{W^{2,2}_{v_J\otimes 1}}.$$

Step 3. We proceed now by complex interpolation of the continuous mapping op_T on modulation spaces; in particular, we are dealing with

$$\mathrm{op}_T : W^{\infty,1}_{v_J\otimes 1}(\mathbb{R}^{2d}) \times M^{p_1,p_2}(\mathbb{R}^d) \to M^{p_1,p_2}(\mathbb{R}^d),$$

$$\mathrm{op}_T : W^{2,2}_{v_J\otimes 1} \times M^2(\mathbb{R}^d) \to M^2(\mathbb{R}^d).$$

For $\theta \in [0,1]$, we have

$$\left[W\left(\mathcal{F}L^\infty_{v_J}, L^1\right), W\left(\mathcal{F}L^2_{v_J}, L^2\right)\right]_\theta = W(\mathcal{F}L^p_{v_J}, L^{p'}), \qquad 2 \le p \le \infty,$$

$$\left[M^{p_1,p_2}_m, M^{2,2}_m\right]_\theta = M^{r_1,r_2}_m,$$

with

$$\frac{1}{r_i} = \frac{1-\theta}{p_i} + \frac{\theta}{2} = \frac{1-\theta}{p_i} + \frac{1}{p}, \qquad i = 1,2.$$

From these estimates we immediately derive the condition $r_1, r_1', r_2, r_2' \leq p$. The inclusion relations enjoyed by amalgam spaces allow us to extend the result to $W^{p,q}_{v_J\otimes 1}(\mathbb{R}^{2d})$ for any $q \leq p'$. The norm estimate is given by

$$\begin{aligned}\|\sigma^T\|_{\mathcal{L}(W^{p,q}_{v_J\otimes 1}\times M^{r_1,r_2}_m, M^{r_1,r_2}_m)} &\leq \|\sigma^T\|^{1-\theta}_{\mathcal{L}(W^{\infty,1}_{v_J\otimes 1}\times M^{p_1,p_2}_m, M^{p_1,p_2}_m)}\|\sigma^T\|^{\theta}_{\mathcal{L}(W^{2,2}_{v_J\otimes 1}\times M^{2}_m, M^{2}_m)}\\ &\lesssim_T (\alpha_{(p_1,p_2)}(T))^{1-\theta}\\ &\asymp_T \alpha_{(r_1,r_2)}(T) C_T^{\theta}\\ &\lesssim_T C_T\alpha_{(r_1,r_2)}(T).\end{aligned}$$

This concludes the proof. □

5.7.3 Sharp results for τ-pseudodifferential operators

The sharp results for τ-Wigner distributions obtained in [CDT19] and reported in Section 5.5.3 have a counterpart for boundedness results of the corresponding τ-operators. The proof follows the same pattern of those of Theorems 5.7.5 and 5.7.6, using the results of Theorem 5.5.9 instead of Theorem 5.5.3.

We highlight that the norm estimate in (5.29) below is slightly better than the one originally proved in [CDT19, Theorem 4.3], where we used that $C_T = C_\tau = (\tau(1-\tau))^d \leq 1$.

Theorem 5.7.7 ([CDT19, Theorem 4.3 and 5.1])**.** *Let m be a v-moderate weight on $\mathbb{R}^{2d}$ and set $v_J = \mathfrak{T}_J v$.*

1. *Consider indices $1 \leq p, p_1, p_2, q, q_1, q_2 \leq \infty$, satisfying the conditions (5.27). For any $\tau \in [0,1]$, if $\sigma \in M^{p,q}_{1\otimes v_J}(\mathbb{R}^{2d})$ then the pseudodifferential operator σ^τ is bounded from $M^{p_1,q_1}_m(\mathbb{R}^d)$ to $M^{p_2,q_2}_{1/m}(\mathbb{R}^d)$ and there exists a constant $C > 0$ independent of τ such that*

$$\|\sigma^\tau\|_{M^{p_1,q_1}_m\to M^{p_2,q_2}_{1/m}} \leq C\|\sigma\|_{M^{p,q}_{1\otimes v_J}}, \quad \forall \tau \in [0,1].$$

2. *Consider indices $1 \leq p, q, r_1, r_2 \leq \infty$, satisfying the conditions (5.28). For any $\tau \in (0,1)$, if $\sigma \in W^{p,q}_{v_J\otimes 1}(\mathbb{R}^{2d})$ then the pseudodifferential operator σ^τ is bounded on $M^{r_1,r_2}_m(\mathbb{R}^d)$ and there exists a constant $C > 0$ independent of τ such that*

$$\|\sigma^\tau\|_{M^{r_1,r_2}_m\to M^{r_1,r_2}_m} \leq C(\tau(1-\tau))^d\alpha_{(r_1,r_2)}(\tau)\|\sigma\|_{W^{p,q}_{v_J\otimes 1}} \quad \forall \tau \in (0,1), \quad (5.29)$$

where $\alpha_{(r_1,r_2)}(\tau)$ is defined in (5.22).

We finally consider the endpoints $\tau = 0$ and $\tau = 1$, for which the boundedness result of Theorem 5.7.7 (*ii*) does not hold in general. The following counterexample is inspired by an argument of Boulkhemair [Bou95].

Proposition 5.7.8 ([CDT19, Proposition 4.4])**.** *There exists a symbol* $\sigma \in W^{\infty,1}(\mathbb{R}^{2d})$ *such that the corresponding Kohn-Nirenberg* $\mathrm{op}_0(\sigma)$ *and anti-Kohn-Nirenberg* $\mathrm{op}_1(\sigma)$ *operators are not bounded on* $L^2(\mathbb{R}^d)$*.*

Proof. Consider the symbol

$$\sigma(x_1,\dots,x_d,\xi_1,\dots,\xi_d) = x_1^{-1/2}\dots x_d^{-1/2} 1_{(0,1]}(x_1)\dots 1_{(0,1]}(x_d) e^{-\pi\xi^2},$$

with $\xi^2 = \xi_1^2 + \cdots + \xi_d^2$. An easy computation shows that $\sigma \in L^1(\mathbb{R}^{2d}) = W(L^1,L^1)(\mathbb{R}^{2d}) \subset W^{\infty,1}(\mathbb{R}^{2d})$. We prove that the Kohn-Nirenberg operator $\mathrm{op}_0(a)$ is unbounded on $L^2(\mathbb{R}^d)$, in particular given the Gaussian function $f(t) = e^{\pi t^2} \in L^2(\mathbb{R}^d)$ we have that $\mathrm{op}_0(a)f \notin L^2(\mathbb{R}^d)$. Indeed, by a tensor product argument, we reduce to compute the following one-dimensional integral:

$$\int_{\mathbb{R}} e^{2\pi i x\cdot\xi} x^{-1/2} 1_{(0,1]}(x) e^{-\pi\xi^2} d\xi = (2x)^{-1/2} 1_{(0,1]}(x) e^{-\pi x^2/2},$$

resulting in a function that does not belong to $L^2(\mathbb{R})$.

The proof of the unboundedness of the anti-Kohn-Nirenberg operator $\mathrm{op}_1(\sigma)$ is an immediate consequence of the fact that $\mathrm{op}_1(\sigma)^* = \mathrm{op}_0(\sigma)$. □

5.8 Symbols in Sjöstrand's classes

We already mentioned in Section 4.2 that an important space of symbols is the (unweighted) Sjöstrand class. It has been introduced by Sjöstrand [Sjö94] to extend the well-behaved Hörmander class $S^0_{0,0} = C^\infty_{\mathrm{b}}(\mathbb{R}^{2d})$ and later recognized to coincide with the modulation space $M^{\infty,1}(\mathbb{R}^{2d})$. Accordingly, it consists of bounded symbols with low regularity in general, namely temperate distributions $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ such that

$$\int_{\mathbb{R}^{2d}} \sup_{z\in\mathbb{R}^{2d}} |\langle \sigma, \pi(z,\zeta)g\rangle| d\zeta < \infty.$$

As a rule of thumb, notice that a symbol in $M^{\infty,1}(\mathbb{R}^{2d})$ locally (i.e. for fixed $z \in \mathbb{R}^{2d}$) coincides with the Fourier transform of a function in L^1. Nevertheless, symbols in the Sjöstrand class lead to L^2-bounded pseudodifferential operators. Sjöstrand's results have been put into the context of time-frequency analysis by Gröchenig in [Grö06c] and were further generalized in [GR08; GS07]. In particular, weighted Sjöstrand's classes with weight functions of type $1 \otimes v$, where v is an admissible weight on $\mathbb{R}^{2d}$, have been taken into account. The key ingredient is a characterizing property of Weyl operators with Sjöstrand symbols, namely *approximate diagonalization by Gabor wave packets*; this means that the Gabor matrix of σ^{w} defined in (4.2) shows some kind of off-diagonal decay. In more evocative terms, we could say that Gabor wave packets are almost eigenvectors of σ^{w}. More precisely, the claim is the following.

Theorem 5.8.1. *Let v be an admissible weight on $\mathbb{R}^{2d}$ and fix an atom $g \in M^1_v(\mathbb{R}^d) \setminus \{0\}$ and a lattice $\Lambda \subset \mathbb{R}^{2d}$ in a way that $\mathcal{G}(g,\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. The following properties are equivalent:*

1. $\sigma \in M^{\infty,1}_{1\otimes v\circ J^{-1}}(\mathbb{R}^{2d})$.
2. $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ *and there exists a function $H \in L^1_v(\mathbb{R}^{2d})$ such that*
$$|\langle \sigma^{\mathrm{w}} \pi(z)g, \pi(w)g\rangle| \le H(w-z), \qquad w,z \in \mathbb{R}^{2d}.$$
3. $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ *and there exists a sequence $h \in \ell^1_v(\Lambda)$ such that*
$$|\langle \sigma^{\mathrm{w}} \pi(\mu)g, \pi(\lambda)g\rangle| \le h(\lambda-\mu), \qquad \lambda,\mu \in \Lambda.$$

The proof of this result heavily relies on a simple but crucial interplay between the Gabor matrix of σ^{w} and the short-time Fourier transform of σ, which will be discussed in complete generality below. We mention that in [Grö06c] Gröchenig established a strong link with matrix algebra, heading towards a more conceptual discussion of the almost diagonalization property; we give some hints on this connection below. The proofs of Sjöstrand's results provided by Gröchenig are to certain extent more natural, in view of this fresh new formulation, Furthermore, they extend the previous ones since weighted spaces are considered. We summarize the main outcomes in the following claim.

Theorem 5.8.2 ([Grö06c])**.** *Let $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ and set $v_{-J} = \mathfrak{T}_{J^{-1}} v$.*

1. *(Boundedness) If $\sigma \in M^{\infty,1}_{1\otimes v_{-J}}$ then σ^{w} is bounded on $M^{p,q}_m$ for any $1 \le p,q \le \infty$.*
2. *(Algebra property) $M^{\infty,1}_{1\otimes v}$ is a Banach $*$-algebra with respect to the Weyl product $\#$ and the involution $\sigma \mapsto \overline{\sigma}$.*
3. *(Wiener property) If (and only if) v is an admissible weight then $\mathrm{op_w}(M^{\infty,1}_{1\otimes v})$ is inverse-closed in $\mathcal{L}(L^2(\mathbb{R}^d))$: if $\sigma \in M^{\infty,1}_{1\otimes v}(\mathbb{R}^{2d})$ and σ^{w} is invertible on L^2 then there exists $\rho \in M^{\infty,1}_{1\otimes v}(\mathbb{R}^{2d})$ such that $(\sigma^{\mathrm{w}})^{-1} = \rho^{\mathrm{w}}$.*

4. *(Spectral invariance on modulation spaces) Let v be an admissible weight. If $\sigma \in M^{\infty,1}_{1\otimes v}(\mathbb{R}^{2d})$ and σ^{w} is invertible on $L^2(\mathbb{R}^d)$ then σ^{w} is simultaneously invertible on every modulation space $M^{p,q}_m(\mathbb{R}^d)$, for any $1 \le p, q \le \infty$.*

In this section we extend Theorem 5.8.2 to general T-pseudodifferential operators, in the spirit of the paper [CNT19b] where τ-operators were considered. Moreover, we explore the consequences of the approximate diagonalization in terms of boundedness results for the corresponding operators.

Let us first provide some general conditions on the matrices for which the associated pseudodifferential operators with symbols in $M^{\infty,1}(\mathbb{R}^{2d})$ are bounded on modulation spaces.

Theorem 5.8.3 ([BCGT20, Theorem 20]). *Let $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ and assume $A \in \mathrm{GL}(2d,\mathbb{R})$ is a left-regular matrix. The pseudodifferential operator σ^A is bounded on all modulation spaces $M^{p,q}(\mathbb{R}^d)$, $1 \le p, q \le \infty$, with*

$$\left\|\sigma^A\right\|_{M^{p,q}\to M^{p,q}} \lesssim_A \frac{1}{\left|\det A_{11}\right|^{1/p'}\left|\det A_{21}\right|^{1/p}} \cdot \frac{1}{\left|\det(A_{12})^{\#}\right|^{1/q'}\left|\det(A_{22})^{\#}\right|^{1/q}}\|\sigma\|_{M^{\infty,1}}. \tag{5.30}$$

Proof. Let $f, g \in M^1(\mathbb{R}^d)$ and $\Phi \in M^1\left(\mathbb{R}^{2d}\right)\setminus\{0\}$. Then,

$$\begin{aligned}\left|\left\langle \sigma^A f, g\right\rangle\right| &= \left|\langle \sigma, \mathcal{B}_A(g,f)\rangle\right| \\ &= \left|\langle V_\Phi\sigma, V_\Phi\mathcal{B}_A(g,f)\rangle\right| \\ &\le \|V_\Phi\sigma\|_{L^{\infty,1}}\|V_\Phi\mathcal{B}_A(g,f)\|_{L^{1,\infty}},\end{aligned}$$

where in the last line we used Hölder inequality for mixed-norm Lebesgue spaces. Let us choose for instance $\Phi_A = \mathcal{B}_A\phi$ for a fixed non-zero $\phi \in M^1(\mathbb{R}^d)$. We fix $\zeta \in \mathbb{R}^{2d}$ and introduce the affine transformations

$$P_\zeta(z_1,z_2) = P_1 z + P_2\zeta = \begin{bmatrix} A_{11} & 0 \\ 0 & (A_{12})^{\#}\end{bmatrix}\begin{bmatrix} z_1 \\ z_2\end{bmatrix} + \begin{bmatrix} 0 & -A_{12} \\ (A_{11})^{\#} & 0\end{bmatrix}\begin{bmatrix}\zeta_1 \\ \zeta_2\end{bmatrix},$$

$$Q_\zeta(z_1,z_2) = Q_1 z + Q_2\zeta = \begin{bmatrix} A_{21} & 0 \\ 0 & -(A_{22})^{\#}\end{bmatrix}\begin{bmatrix} z_1 \\ z_2\end{bmatrix} + \begin{bmatrix} 0 & -A_{22} \\ -(A_{21})^{\#} & 0\end{bmatrix}\begin{bmatrix}\zeta_1 \\ \zeta_2\end{bmatrix},$$

in according with the magic formula (5.6), and using again Hölder's inequality we get

$$\begin{aligned}
\|V_{\Phi_A}\mathcal{B}_A(g,f)\|_{L^{1,\infty}} &= \sup_{(\zeta_1,\zeta_2)\in\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_\phi g(P_\zeta(z_1,z_2))||V_\phi f(Q_\zeta(z_1,z_2))|dz_1dz_2 \\
&\le \sup_{(\zeta_1,\zeta_2)\in\mathbb{R}^{2d}} \|(V_\phi f)\circ Q_\zeta\|_{L^{p,q}_z}\|(V_\phi g)\circ P_\zeta\|_{L^{p',q'}_z} \\
&= \frac{\|V_\phi f\|_{L^{p,q}}}{|\det A_{21}|^{1/p}|\det(A_{22})^{\#}|^{1/q}} \frac{\|V_\phi g\|_{L^{p',q'}}}{|\det A_{11}|^{1/p'}|\det(A_{12})^{\#}|^{1/q'}} \\
&\le C\frac{\|f\|_{M^{p,q}}}{|\det A_{21}|^{1/p}|\det(A_{22})^{\#}|^{1/q}} \frac{\|g\|_{M^{p',q'}}}{|\det A_{11}|^{1/p'}|\det(A_{12})^{\#}|^{1/q'}},
\end{aligned}$$

where the constant C does not depend on f, g or A. On the other hand,

$$\|V_{\Phi_A}\sigma\|_{L^{\infty,1}} \lesssim_A \|\sigma\|_{M^{\infty,1}}. \tag{5.31}$$

We conclude by duality and get the claimed result. □

Remark 5.8.4. *This result broadly generalizes [Grö01, Theorem 14.5.2] and confirms that the Sjöstrand class is a well-suited symbol class leading to bounded operators on modulation spaces. It is worthwhile to stress that the left-regularity assumption for A covers any Cohen-type matrix $A = A_T$.*

Unfortunately, it is not easy to sharpen the estimate (5.30) neither in the case of Cohen-type matrices, the main obstruction being the estimate in (5.31). Nevertheless, a sharp result can be proved for τ-operators as detailed below.

Corollary 5.8.5. *Let $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ and $\tau \in [0,1]$. The pseudodifferential operator σ^τ is bounded on all modulation spaces $M^{p,q}(\mathbb{R}^d)$, $1 \le p,q \le \infty$. In particular, there exists $C > 0$ such that*

$$\|\sigma^\tau\|_{M^{p,q}\to M^{p,q}} \le C\|\sigma\|_{M^{\infty,1}}, \quad \forall\tau\in[0,1].$$

Proof. Let $\phi(t) = e^{-\pi t^2}$ be the Gaussian function on $\mathbb{R}^d$ and set $\Phi_\tau = W_\tau\phi$. From the proof of the previous result we have, for any $f,g \in M^1(\mathbb{R}^d)$,

$$\begin{aligned}
|\langle\sigma^\tau f, g\rangle| &\le \|V_{\Phi_\tau}\sigma\|_{L^{\infty,1}}\|V_{\Phi_\tau}W_\tau(g,f)\|_{L^{1,\infty}} \\
&\le C'\|V_{\Phi_\tau}\sigma\|_{L^{\infty,1}}\|f\|_{M^{p,q}}\|g\|_{M^{p',q'}},
\end{aligned}$$

for some universal constant $C' > 0$. Consider now $\Phi(z) = e^{-\pi z^2}$, $z \in \mathbb{R}^{2d}$; the change-of-window formula (3.3) and Young's inequality for mixed-norm spaces in Proposition 2.3.3 yield

$$\begin{aligned}
\|V_{\Phi_\tau}\sigma\|_{L^{\infty,1}} &\le \||V_\Phi\sigma| * |V_\Phi\Phi_\tau|^\vee\|_{L^{\infty,1}} \\
&\le \|V_\Phi\sigma\|_{L^{\infty,1}}\|V_\Phi\Phi_\tau\|_{L^1} \\
&\le C''\|\sigma\|_{M^{\infty,1}},
\end{aligned}$$

where the universal constant $C'' > 0$ in the last step comes from Lemma 5.5.8 with $v = 1$. The claim follows with $C = C'C''$. $\square$

We now prove a similar boundedness result on Wiener amalgam spaces for T-operators with Sjöstrand symbol. We need first to derive a result similar to the symplectic covariance property satisfied by the Weyl quantization - cf. [CNT19b, Lemma 5.1] for the τ-Wigner case. This identity will provide a natural way to transfer boundedness results from modulation to amalgam spaces.

Lemma 5.8.6. *For any symbol $\sigma \in M^\infty(\mathbb{R}^{2d})$ and any Cohen-type matrix $A = A_T \in \mathrm{GL}(2d, \mathbb{R})$:*

$$\mathcal{F}\sigma^T\mathcal{F}^{-1} = (\sigma \circ J^{-1})^{I-T}$$

Proof. We sketch a formal argument in the spirit of [Grö01, Lemma 14.3.2]. Recall the spreading representation of σ^T from (5.25):

$$\sigma^T f(x) = \int_{\mathbb{R}^{2d}} \widehat{\sigma}(\xi, u) e^{-2\pi i (I-T)u\cdot\xi} T_{-u} M_\xi f(x) du d\xi.$$

Since $\mathcal{F}T_{-u}M_\xi\mathcal{F}^{-1} = e^{2\pi i u\cdot\xi} T_\xi M_u$, we get

$$\mathcal{F}\sigma^T\mathcal{F}^{-1} = \int_{\mathbb{R}^{2d}} \widehat{\sigma}(\xi, u) e^{-\pi i (2T-I)u\cdot\xi} T_\xi M_u du d\xi = (\sigma \circ J^{-1})^{I-T}.$$

$\square$

Remark 5.8.7. *We stress that a comprehensive account of the symplectic covariance for perturbed representations is out of the scope of this dissertation, since it would require to investigate how the metaplectic group should be modified in order to accommodate the perturbations. As an example of the non-triviality of this issue, we highlight the contribution of de Gosson [Gos16, Chapter 13] in the case of τ-operators.*

Theorem 5.8.8. *For any Cohen-type matrix $A = A_T \in \mathrm{GL}(2d, \mathbb{R})$ and any symbol $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$, the operator σ^T is bounded on $W^{p,q}(\mathbb{R}^d)$, with*

$$\|\sigma^T\|_{W^{p,q}\to W^{p,q}} \lesssim_T \|\sigma\|_{M^{\infty,1}}.$$

Proof. The proof follows the pattern of [CNT19b, Theorem 5.6]. Set $\sigma_J = \sigma \circ J$ and consider the following commutative diagram:

$$\begin{array}{ccc} M^{p,q}(\mathbb{R}^d) & \xrightarrow{(\sigma_J)^{I-T}} & M^{p,q}(\mathbb{R}^d) \\ \uparrow \mathcal{F}^{-1} & & \downarrow \mathcal{F} \\ W^{p,q}(\mathbb{R}^d) & \xrightarrow{\sigma^T} & W^{p,q}(\mathbb{R}^d) \end{array}$$

From the identity $V_G(\sigma_J)(z,\zeta) = V_{G\circ J^{-1}}\sigma(Jz, J\zeta)$, where $G \in \mathcal{S}(\mathbb{R}^{2d})$ is an arbitrary non-zero window (cf. Lemma 5.9.6 below), it follows easily that $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ implies $\sigma_J \in M^{\infty,1}(\mathbb{R}^{2d})$; see [CNT19b, Lemma 5.2] for details in the weighted case. The operator $\mathrm{op}_{I-T}(\sigma_J)$ is bounded on $M^{p,q}(\mathbb{R}^d)$ as a consequence of Theorem 5.8.3 and the claim follows at once thanks to the previous lemma. □

Remark 5.8.9. *Note that the result can be made sharp in the case of τ-operators in view of Corollary 5.8.5; in particular, there exists $C > 0$ such that*

$$\|\sigma^\tau\|_{W^{p,q}\to W^{p,q}} \le C\|\sigma\|_{M^{\infty,1}}, \quad \forall \tau \in [0,1].$$

This improves the estimate originally proved in [CNT19b, Theorem 5.6] in the unweighted case.

5.8.1 Almost diagonalization of T-operators

We already mentioned that the backbone of Theorem 5.8.2 is the interplay between the entries of the Gabor matrix of σ^{w} and the short-time Fourier transform of the symbol σ. Theorem 5.8.1 can be extended without difficulty to τ-pseudodifferential operators, see [CNT19b, Theorem 4.1]. We now indicate a further generalization to operators associated with Cohen-type matrices.

Lemma 5.8.10. *Let $A = A_T \in \mathrm{GL}(2d,\mathbb{R})$ be a Cohen-type matrix. Fix a non-zero window $\phi \in M^1(\mathbb{R}^d)$ and set $\Phi_T = W_T\phi$. For any $\sigma \in M^\infty(\mathbb{R}^{2d})$,*

$$|\langle \sigma^T \pi(z)\phi, \pi(w)\phi\rangle| = |V_{\Phi_T}\sigma(\mathcal{T}_T(w,z), J(w-z))| = |V_{\Phi_T}\sigma(x(z,w), y(z,w))|,$$

for any $z, w \in \mathbb{R}^{2d}$, where $\mathcal{T}_T$ is defined in (5.15) *and*

$$x(z,w) = \mathcal{T}_T(w,z) = (I+P_T)w - P_T z, \quad y(z,w) = J(w-z).$$

Moreover,

$$|V_{\Phi_T}\sigma(x,y)| = |\langle \sigma^T \pi(z(x,y))\phi, \pi(w(x,y))\phi\rangle|,$$

for any $x, y \in \mathbb{R}^{2d}$, where

$$z(x,y) = x + (I+P_T)Jy, \qquad w(x,y) = x + P_T J y.$$

Proof. Using the covariance formula (5.13) we have

$$\begin{aligned} |\langle \sigma^T \pi(z)\phi, \pi(w)\phi\rangle| &= |\langle \sigma, W_T(\pi(w)\phi, \pi(z)\phi)\rangle| \\ &= |\langle \sigma, M_{J(w-z)} T_{\mathcal{T}_T(w,z)} W_T\phi\rangle| \\ &= |V_{\Phi_T}\sigma(\mathcal{T}_T(w,z), J(w-z))|. \end{aligned}$$

Now, setting $x = \mathcal{T}_T(w,z)$ and $y = J(w-z)$ we immediately get Eq. (5.8.10). □

Theorem 5.8.11. *Let v be an admissible weight on $\mathbb{R}^{2d}$ and set $v_{-J} = \mathfrak{T}_{J^{-1}}$. Let $g \in M_v^1(\mathbb{R}^d)$ be a non-zero window function and assume that $\Lambda \subset \mathbb{R}^{2d}$ is a lattice such that $\mathcal{G}(g,\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. For any Cohen-type matrix $A = A_T \in \mathrm{GL}(2d,\mathbb{R})$, the following properties are equivalent:*

(i) $\sigma \in M^{\infty,1}_{1\otimes v_{-J}}(\mathbb{R}^{2d})$.

(ii) $\sigma \in M^\infty(\mathbb{R}^{2d})$ *and there exists a function* $H = H_T \in L_v^1(\mathbb{R}^d)$ *such that*

$$|\langle \sigma^T \pi(z)\phi, \pi(w)\phi\rangle| \le H_T(w-z), \qquad w,z\in\mathbb{R}^{2d}.$$

(iii) $\sigma \in M^\infty\big(\mathbb{R}^{2d}\big)$ *and there exists a sequence* $h = h_T \in \ell_v^1(\Lambda)$ *such that*

$$|\langle \sigma^T \pi(\mu)\phi, \pi(\lambda)\phi\rangle| \le h_T(\lambda-\mu), \qquad \lambda,\mu\in\mathbb{R}^{2d}.$$

Proof. The proof faithfully mirrors the one provided for Weyl operators in [Grö06c, Theorem 3.2]. We detail here only the case $(i) \Rightarrow (ii)$, the discrete case for h_T is similar. For $g \in M_v^1(\mathbb{R}^d)$ the T-Wigner distribution $\Phi_T = W_T g$ is in $M^1_{1\otimes v_{-J}}(\mathbb{R}^{2d})$ by Proposition 5.5.6. This implies that the short-time Fourier transform $V_{\Phi_T}\sigma$ is well-defined for $\sigma \in M^{\infty,1}_{1\otimes v_{-J}}\big(\mathbb{R}^{2d}\big)$ (cf. [Grö01, Theorem 11.3.7]). The main insight here is that the controlling function $H_T \in L_v^1(\mathbb{R}^d)$ can be provided by the so-called *grand symbol* associated with σ, which is defined by $Q_T(y) := \sup_{x\in\mathbb{R}^{2d}} |V_{\Phi_T}\sigma(x,y)|$. By definition of $M^{\infty,1}_{1\otimes v_{-J}}\big(\mathbb{R}^{2d}\big)$, we have $Q_T \in L_v^1(\mathbb{R}^{2d})$, so that Lemma 5.8.10 implies

$$\begin{aligned}\big|\big\langle \sigma^A \pi(z)\phi, \pi(w)\phi\big\rangle\big| &= |V_{\Phi_T}\sigma(\mathcal{T}_T(z,w), J(w-z))| \\ &\le \sup_{u\in\mathbb{R}^{2d}} |V_{\Phi_T}\sigma(u, J(w-z))| \\ &= Q_T(J(w-z)).\end{aligned}$$

Setting $H_T := Q_T \circ J$ yields the claim. □

We note that the same claim holds in the standard framework $(\mathcal{S},\mathcal{S}')$ instead of (M^1, M^∞), and similarly all the related results.

The previous characterization is quite strong; we have indeed the following result, which says that T-operators with symbols in Sjöstrand classes are all and only the operators which satisfy such almost-diagonalization estimates of this type.

Corollary 5.8.12. *Under the assumptions of Theorem 5.8.11, fix $T \in \mathbb{R}^{d\times d}$ and assume that the operator $P : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ is continuous and satisfies any of the following conditions:*

1. $|\langle P\pi(z)g, \pi(w)g\rangle| \le H(w-z)$ *for any* $z,w\in\mathbb{R}^{2d}$ *and some* $H \in L_v^1(\mathbb{R}^d)$;

2. $|\langle P\pi(\mu)g, \pi(\lambda)g\rangle| \le h(\lambda-\mu)$ *for any* $\lambda, \mu \in \Lambda$ *and some* $h \in \ell^1_v(\mathbb{Z}^d)$.

Therefore, there exists $\sigma \in M^{\infty,1}_{1\otimes v_{-J}}(\mathbb{R}^{2d})$ *such that* $P = \sigma^T$.

Proof. From Theorem 5.6.3 we have that $P = \sigma^T$ for some symbol $\sigma \in M^\infty(\mathbb{R}^d)$. The claim then follows from Theorem 5.8.11. □

We conclude by highlighting that the properties established in Theorem 5.8.2 extend from Weyl operators to any T-quantization. There are essentially two reasons behind this fact; first, Theorem 5.8.11 can be recast as follows: $\sigma \in M^{\infty,1}_{1\otimes v\circ J^{-1}}$ if and only if the discrete Gabor matrix of σ^T belongs to the class $\mathcal{C}_v(\Lambda)$ of matrices $A = (a_{\lambda,\mu})_{\lambda,\mu\in\Lambda}$ such that there exists a sequence $h \in \ell^1_v$ which almost diagonalizes its entries, i.e.

$$\|a_{\lambda,\mu}\| \le h(\lambda-\mu), \qquad \lambda, \mu \in \Lambda.$$

It is easy to see that $\mathcal{C}_v(\Lambda)$ is indeed a Banach *-algebra under convolution with the norm $\|A\|_{\mathcal{C}_v} = \inf\{\|h\|_{\ell^1_v} : |a_{\lambda,\mu}| \le h(\lambda-\mu),\ \forall \lambda, \mu \in \Lambda\}$, and this insight suggests to considers other matrix algebras and investigates the relation between symbols and the membership of their Gabor matrices in a matrix algebra (cf. for instance [GR08]). The proof of Theorem 5.8.2 in [Grö06c] crucially relies on the properties of $\mathcal{C}_v(\Lambda)$, which is thus identified with the Sjöstrand class.

Moreover, the extension to other quantizations is not surprising, since the operator involved in the change of quantization in Proposition 5.6.6 is a convolution with a chirp-like function (provided that $T_2 - T_1$ is invertible) and all modulation spaces $M^{p,q}$, $1 \le p, q \le \infty$ are invariant under the action of such an operator, cf. [Tof04a, Proposition 1.2(5)] and [Tof04a, Remark 1.5] in the case of τ-quantizations.

We conclude this section by mentioning that the results in [Grö06c] also extend to pseudodifferential operators defined on locally compact Abelian (LCA) groups with symbols in $M^{\infty,1}(G\times\widehat{G})$; the time-frequency analysis approach to this problem shows then its full power, since the original methods of [Sjö94] do not apply here.

5.9 Symbols in Fourier-Sjöstrand classes

Let us further discuss the main trick of the proof of Theorem 5.8.11. The choice of the grand symbol $Q_T(y) = \sup_{x\in\mathbb{R}^{2d}} |V_{\Phi_T}\sigma(x,y)|$ as controlling function is natural in view of the weighted $M^{\infty,1}$-norm; moreover, according to Lemma 5.8.10, the effect of the perturbation matrix T is confined to the window function Φ_T and to the time variable of the short-time Fourier transform of the symbol. A natural question is then the following: what happens if we try to control the time dependence of

$V_{\Phi_\tau}\sigma$? Following the pattern of [CNT19b, Theorem 4.3], this remark is the starting point of a parallel characterization for symbols belonging to the Wiener amalgam space $W^{\infty,1}(\mathbb{R}^{2d}) = \mathcal{F}M^{\infty,1}(\mathbb{R}^{2d})$, which will be referred to as the Fourier-Sjöstrand class. The almost diagonalization of the (continuous) Gabor matrix does not survive the perturbation, but the new result can be interpreted as a measure of the concentration of the time-frequency representation of σ^T along the graph of the map $\mathcal{U}_T$.

Theorem 5.9.1. *Let $A = A_T$ be a right-regular Cohen-type matrix and define $\mathcal{B}_T := (I + P_T)^{-1}$ and $\mathcal{U}_T := (I + P_T^{-1})^{-1}$. Let v be an admissible weight on $\mathbb{R}^{2d}$ and set $v_{\mathcal{B}_T} = \mathfrak{T}_{\mathcal{B}_T} v$, $v_{\mathcal{U}_T} = \mathfrak{T}_{\mathcal{U}_T} v$. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ be a non-zero window function. The following properties are equivalent:*

(i) $\sigma \in W^{\infty,1}_{1\otimes v_{\mathcal{B}_T}}$.

(ii) $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ *and there exists a function* $H = H_T \in L^1_v(\mathbb{R}^d)$ *such that*

$$\left|\left\langle \sigma^T \pi(z)\phi, \pi(w)\phi \right\rangle\right| \le H_T(w - \mathcal{U}_T z), \qquad \forall w, z \in \mathbb{R}^{2d}, \tag{5.32}$$

where

$$\mathcal{U}_T = (I + P_T^{-1})^{-1} = -\begin{bmatrix} (I-T)^{-1}T & 0 \\ 0 & T^{-1}(I-T) \end{bmatrix}.$$

Proof. The proof is a straightforward extension of the one provided for τ-operators in [CNT19b, Theorem 4.3]. Again, we detail here only the case $(i) \Rightarrow (ii)$ for the purpose of tracking the origin of $\mathcal{U}_T$; the proof of $(ii) \Rightarrow (i)$ is an easy consequence of Lemma 5.8.10. If $\phi \in \mathcal{S}(\mathbb{R}^d) \subset M^1_v(\mathbb{R}^d) \cap M^1_{v_{\mathcal{U}_T}}(\mathbb{R}^d)$, $\phi \neq 0$, then $\Phi_T = W_T\phi \in W^{1,1}_{1\otimes v_{\mathcal{B}_T}}(\mathbb{R}^{2d})$ by Proposition 5.5.6. For $\sigma \in W^{\infty,1}_{1\otimes v_{\mathcal{B}_T}}(\mathbb{R}^{2d})$ we have that $V_{\Phi_T}\sigma$ is well defined (cf. Proposition (3.2.10)) and

$$Q_T(x) = \sup_{y\in\mathbb{R}^{2d}} |V_{\Phi_T}\sigma(x,y)| \in L^1_{v_{\mathcal{B}_T}}(\mathbb{R}^{2d}).$$

From Lemma 5.8.10 we infer

$$\begin{aligned} |\langle \sigma^T \pi(z)\phi, \pi(w)\phi\rangle| &= |V_{\Phi_T}\sigma(\mathcal{T}_t(w,z), J(w-z))| \\ &\le \sup_{y\in\mathbb{R}^{2d}} |V_{\Phi_T}\sigma(\mathcal{T}_T(w,z), y)| \\ &= Q_T(\mathcal{T}_T(w,z)). \end{aligned}$$

Recall that if A_T is right-regular, then $I + P_T$ is invertible (and the converse holds too, cf. (5.14)). Recalling the definition of $\mathcal{T}_T$ in (5.15) and setting $\mathcal{B}_T = (I + P_T)^{-1}$ yield

$$\mathcal{B}_T(\mathcal{T}_T(w,z)) = w - (I + P_T)^{-1} P_T z = w - \mathcal{U}_T z,$$

and thus $Q_T(\mathcal{T}_T(w,z)) = Q_T(\mathcal{B}_T^{-1}(w-\mathcal{U}_T z))$. The conclusion follows after setting $H_T = Q_T \circ \mathcal{B}_T^{-1}$. $\square$

Example 5.9.2. *Consider $\sigma = \delta \in W^{\infty,1}(\mathbb{R}^{2d})$. In this case, using formula* (5.13)*,*

$$\begin{aligned} |\langle \delta^T \pi(z)\phi, \pi(w)\phi\rangle| &= |\langle \delta, W_T(\pi(w)\phi, \pi(z)\phi)\rangle| \\ &= |\langle \delta, M_{J(w-z)} T_{\mathcal{T}_T(w,z)} W_T\phi\rangle| \\ &= |T_{\mathcal{T}_T(w,z)} W_T\phi(0)| \\ &= |W_T\phi(-\mathcal{T}_T(w,z)) \\ &= |W_T\phi(-\mathcal{B}_T^{-1}(w-\mathcal{U}_T z))|. \end{aligned}$$

Choosing $H_\tau(z) = |W_T\phi(-\mathcal{B}_T^{-1}(z))|$ we obtain (5.32)*, which reduces to an equality in this case.*

We remark that in this framework the discrete characterization of Theorem 5.8.11 is lost. The main obstruction is the following: for a given lattice Λ, the inclusion $\mathcal{U}_T\Lambda \subseteq \Lambda$ holds if and only if $\mathcal{U}_T = \mathcal{U}_{1/2I} = -I$, the (minus) identity matrix. In particular, $\mathcal{B}_{1/2} = 2I$ and Theorem 5.9.1 can be improved as follows.

Corollary 5.9.3. *Let v be an admissible weight function on $\mathbb{R}^{2d}$ and set $v_2 = \mathfrak{T}_{2I}v$. Consider $\phi \in \mathcal{S}(\mathbb{R}^d)\setminus\{0\}$ such that $\mathcal{G}(\phi,\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. For Weyl operators, the following properties are equivalent:*

1. *$\sigma \in W^{\infty,1}_{1\otimes v_2}(\mathbb{R}^{2d})$.*
2. *$\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ and there exists a function $H \in L^1_v(\mathbb{R}^{2d})$ such that*
$$|\langle \sigma^{\mathrm{w}} \pi(z)\phi, \pi(w)\phi\rangle| \le H(w+z) \qquad \forall w, z \in \mathbb{R}^{2d},$$
3. *$\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ and there exists a sequence $h \in \ell^1_v(\Lambda)$ such that*
$$|\langle \sigma^{\mathrm{w}} \pi(\mu)\phi, \pi(\lambda)\phi\rangle| \le h(\lambda+\mu) \qquad \forall \lambda, \mu \in \Lambda.$$

The statement and the proof of Theorem 5.9.1 require the right-regularity of A_T in order to work. In the degenerate case where A_T is not right-regular, namely if T or $I-T$ is not invertible, one can similarly prove a weaker version of that result.

Theorem 5.9.4. *Let $A = A_T$ be a Cohen-type matrix and v be an admissible weight on $\mathbb{R}^{2d}$. Fix a non-zero window $\phi \in \mathcal{S}(\mathbb{R}^d)$. The following properties are equivalent:*

1. $\sigma \in W^{\infty,1}_{1\otimes v}(\mathbb{R}^{2d})$.

2. $\sigma \in \mathcal{S}(\mathbb{R}^{2d})$ *and there exists a function* $H_T \in L^1_v(\mathbb{R}^{2d})$ *such that*

$$|\langle \sigma^T \pi(z)\phi, \pi(w)\phi\rangle| \le H_T(\mathcal{T}_T(w,z)), \quad \forall w, z \in \mathbb{R}^{2d},$$

where $\mathcal{T}_T$ *is defined in* (5.15).

5.9.1 Boundedness results and other consequences

The almost-diagonalization estimate for the Gabor matrix of a T-operator with symbol in a Fourier-Sjöstrand class can be used to derive boundedness results and algebraic properties in the spirit of Theorem 5.8.2.

Theorem 5.9.5. *Let* $m \in \mathcal{M}_v(\mathbb{R}^{2d})$. *Let* $A = A_T$ *be a right-regular Cohen-type matrix and consider a symbol* $\sigma \in W^{\infty,1}_{1\otimes v_{\mathcal{B}_T}}(\mathbb{R}^{2d})$, *where* $\mathcal{B}_T = (I + P_T)^{-1}$ *and* P_T *defined in* (5.14). *The operator* σ^T *is bounded from* $M^{p,q}_m(\mathbb{R}^d)$ *to* $M^{p,q}_{m\circ\mathcal{U}_T}(\mathbb{R}^d)$, $1 \le p, q \le \infty$.

Proof. The proof uses the techniques developed in [CGNR14, Theorem 3.3]. We fix $g(t) = e^{-\pi t^2}$ and note that $g \in M^1_v(\mathbb{R}^d)$ for every admissible weight v; moreover we have $\|g\|_{L^2} = 1$, so that the inversion formula (3.2) reads $V_g^* V_g = \mathrm{Id}$. Writing T as

$$T = V_g^* V_g T V_g^* V_g,$$

note that $V_g T V_g^*$ is an integral operator with kernel given by the Gabor matrix of T:

$$K_T(w,z) = \langle T\pi(z)g, \pi(w)g\rangle, \quad w, z \in \mathbb{R}^{2d}.$$

Recall that V_g is bounded from $M^p_m(\mathbb{R}^d)$ to $L^p_m(\mathbb{R}^{2d})$ and V_g^* is bounded from $L^p_m(\mathbb{R}^{2d})$ to $M^p_m(\mathbb{R}^d)$. Therefore, if $V_g T V_g^*$ is bounded from $L^p_m(\mathbb{R}^{2d})$ to $L^p_{m_{\mathcal{U}_T}}(\mathbb{R}^{2d})$, then T is bounded from $M^p_m(\mathbb{R}^d)$ to $M^p_{m_{\mathcal{U}_T}}(\mathbb{R}^d)$. Observe that

$$\mathcal{U}_T \mathcal{B}_{I-T} = -\mathcal{B}_T$$

so that $v \circ \mathcal{U}_T \circ \mathcal{B}_{I-T} = v \circ \mathcal{B}_T$, and also note that $\mathcal{U}_{I-T}^{-1} = \mathcal{U}_T$. An application of Theorem 5.9.1 with $I - T$ in place of T and with the admissible weight $v \circ \mathcal{U}_T$ in place of v yields, for $F \in L^p_m(\mathbb{R}^{2d})$,

$$\begin{aligned}
|V_g T V_g^* F(w)| &= \Big| \int_{\mathbb{R}^{2d}} K_T(w,z) F(z)\, dz \Big| \le \int_{\mathbb{R}^{2d}} |F(z)| H_{I-T}(w - \mathcal{U}_{I-T} z) dz \\
&= \int_{\mathbb{R}^{2d}} |F(z)| (H_{I-T} \circ \mathcal{U}_{I-T})(\mathcal{U}_T w - z) dz \\
&= F * (H_{I-T} \circ \mathcal{U}_{I-T})(\mathcal{U}_T w).
\end{aligned}$$

Moreover we have $H_{I-T} \in L^1_{v_{\mathcal{U}_T}}(\mathbb{R}^{2d})$ and hence $H_{I-T} \circ \mathcal{U}_{I-T} L^1_v(\mathbb{R}^{2d})$. Young's inequality then gives $F * (H_{I-T} \circ \mathcal{U}_{I-T}) \in L^p_m(\mathbb{R}^{2d}) * L^1_v(\mathbb{R}^{2d}) \subset L^p_m(\mathbb{R}^{2d})$. This shows that $V_g T V_g^* F \in L^p_{m \circ \mathcal{U}_T}(\mathbb{R}^{2d})$, as desired. $\square$

We now use the partial symplectic covariance of T-calculus to derive boundedness results on amalgam spaces. The following lemma is needed, the proof being a straightforward computation.

Lemma 5.9.6. *For any non-zero window $G \in \mathcal{S}(\mathbb{R}^{2d})$ and $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$,*

$$V_G(\sigma \circ J)(z, \zeta) = (V_{G \circ J^{-1}} \sigma)(Jz, J\zeta).$$

Therefore, for any $1 \leq p, q \leq \infty$, admissible weights u, v on $\mathbb{R}^{2d}$ and right-regular Cohen-type matrices $A = A_T$:

1. *$\sigma_J = \sigma \circ J \in M^{p,q}_{(u \circ J^{-1}) \otimes (v \circ J^{-1})}(\mathbb{R}^{2d})$ if and only if $\sigma \in M^{p,q}_{u \otimes v}(\mathbb{R}^{2d})$. In particular,*
$$\sigma \in M^{\infty,1}_{1 \otimes v}(\mathbb{R}^{2d}) \Leftrightarrow \sigma_J \in M^{\infty,1}_{1 \otimes (v \circ J^{-1})}(\mathbb{R}^{2d}).$$
2. *$\sigma \in W^{p,q}_{(u \circ J^{-1}) \otimes (v \circ J^{-1})}(\mathbb{R}^{2d})$ if and only if $\sigma_J \in W^{p,q}_{u \otimes v}(\mathbb{R}^{2d})$. In particular,*
$$\sigma \in W^{\infty,1}_{1 \otimes (v \circ \mathcal{B}_T \circ J^{-1})}(\mathbb{R}^{2d}) \Leftrightarrow \sigma_J \in W^{\infty,1}_{1 \otimes (v \circ \mathcal{B}_T)}(\mathbb{R}^{2d}).$$

Theorem 5.9.7. *Let $A = A_T$ be a right-regular Cohen-type matrix and $m = m_1 \otimes m_2 \in \mathcal{M}_v(\mathbb{R}^{2d})$. If $\sigma \in W^{\infty,1}_{1 \otimes (v \circ \mathcal{B}_T \circ J^{-1})}(\mathbb{R}^{2d})$ then the operator σ^T is bounded from $W^{p,q}_{m_1 \otimes m_2}(\mathbb{R}^d)$ to $W^{p,q}_{(m_1 \circ \mathcal{U}^1_T) \otimes (m_2 \circ \mathcal{U}^2_T)}(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$, where we set*

$$\mathcal{U}^1_T = -(I-T)^{-1}T, \quad \mathcal{U}^2_T = -T^{-1}(I-T).$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc}
M^{p,q}_m(\mathbb{R}^d) & \xrightarrow{(\sigma_J)^{I-T}} & M^{p,q}_{m \circ \mathcal{U}_T}(\mathbb{R}^d) \\
\uparrow \mathcal{F}^{-1} & & \downarrow \mathcal{F} \\
W^{p,q}_{m_1 \otimes m_2}(\mathbb{R}^d) & \xrightarrow{\sigma^T} & W^{p,q}_{(m_1 \circ \mathcal{U}^1_T) \otimes (m_2 \circ \mathcal{U}^2_T)}(\mathbb{R}^d)
\end{array}$$

Indeed, since $\sigma \in W^{\infty,1}_{1 \otimes (v \circ \mathcal{B}_T \circ J^{-1})}(\mathbb{R}^{2d})$ we have $\sigma_J \in W^{\infty,1}_{1 \otimes (v \circ \mathcal{B}_T)}(\mathbb{R}^{2d})$ by Lemma 5.9.6. The operator $(\sigma_J)^{I-T}$ is bounded by Theorem 5.9.5 with $T' = 1 - T$ and the thesis follows at once thanks to the Lemma 5.8.6. $\square$

Remark 5.9.8. *Note that the same strategy, in particular Lemma 5.9.6, can be used to prove a weighted version of Theorem 5.8.8 above, cf. [CNT19b, Theorem 5.6] for the case of τ-operators.*

The cases $\tau = 0$ and $\tau = 1$. Theorem 5.9.4 can be used to obtain some boundedness results for τ-pseudodifferential operators with $\tau = 0$ or $\tau = 1$, having symbols in Wiener amalgam spaces.

Proposition 5.9.9. *Assume $\sigma \in W^{\infty,1}(\mathbb{R}^d)$. Then the Kohn-Nirenberg operator $\mathrm{op}_0(\sigma)$ is bounded on $M^{1,\infty}(\mathbb{R}^d)$.*

Proof. Consider $H_0(\mathcal{T}_0(w,z))$ with $H_0 \in L^1(\mathbb{R}^{2d})$ as in Theorem 5.9.4. The integral operator T_{H_0} with kernel $H_0(\mathcal{T}_0(w,z)) = H_0(w_1, z_2)$ can be written as

$$T_{H_0}F(w) = \int_{\mathbb{R}^{2d}} H_0 \circ \mathcal{T}_0(w,z)F(z)dz = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H(w_1,z_2)F(z_1,z_2)dz_1dz_2.$$

It is immediate to notice that $T_{H_0} : L^{1,\infty}(\mathbb{R}^{2d}) \to L^{1,\infty}(\mathbb{R}^{2d})$ is a bounded operator. Then, for a fixed non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, we have that

$$T = V_g^* T_{H_0} V_g \,:\, M^{1,\infty}(\mathbb{R}^d) \to M^{1,\infty}(\mathbb{R}^d)$$

is a bounded operator. The claim then follows. □

Proposition 5.9.10. *Assume $\sigma \in W^{\infty,1}(\mathbb{R}^d)$. Then the operator $\mathrm{op}_1(\sigma)$ is bounded on $W1,\infty(\mathbb{R}^d)$.*

Proof. Again, we apply Theorem 5.9.4 and consider $H_1(\mathcal{T}_1(w,z))$ with $H_1 \in L^1(\mathbb{R}^{2d})$. The integral operator T_H with kernel $H_1(\mathcal{T}_1(w,z)) = H_1(z_1,w_2)$ can be written as

$$T_{H_1}F(w) = \int_{\mathbb{R}^{2d}} H_1 \circ \mathcal{T}_1(w,z)F(z)dz = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H_1(z_1,w_2)F(z_1,z_2)dz_1dz_2.$$

It is immediate to notice that $T_{H_1} : L^\infty_{z_1}(L^1_{z_2})(\mathbb{R}^{2d}) \to L^\infty_{w_1}(L^1_{w_2})(\mathbb{R}^{2d})$ is a bounded operator. Then, for a fixed non-zero window $g \in \mathcal{S}\big(\mathbb{R}^d\big)$, we have that

$$T = V_g^* T_{H_1} V_g \,:\, W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \to W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d)$$

is a bounded operator. This concludes the proof. □

Algebra properties. The boundedness and algebraic properties of T-operators with symbols in $W^{\infty,1}$ proved in [CNT19b] rely on the characterization as generalized metaplectic operators according to [CGNR14, Def. 1.1]. In order to benefit from this framework, it is necessary for $\mathcal{U}_T$ to be a symplectic matrix, the latter condition being realized if and only if T is a symmetric matrix (cf. [Gos11, (2.4) and (2.5)] and notice that $T^{-1}(I-T) = (I-T)T^{-1}$).

Therefore, we focus on τ-operators for the sake of concreteness. Note that $v_s \circ \mathcal{B}_\tau \asymp_\tau v_s$; if the symbol σ is in $W^{\infty,1}_{0,s}(\mathbb{R}^{2d})$ then Theorem 5.9.1 says that (5.32) holds for a suitable function $H_\tau \in L^1_{v_s}$, so that the τ-operator σ^τ is a generalized metaplectic operator in the class $FIO(\mathcal{U}_\tau, v_s)$, cf. Section 4.3.5. Moreover, the assumptions of Theorem 4.3.10 are satisfied and we can represent σ^τ as a type-I Fourier integral operator with phase $\Phi(x,\xi) = -(1-\tau)\tau^{-1}x\cdot\xi$ as follows:

$$\sigma^\tau f(x) = \int_{\mathbb{R}^d} e^{-2\pi i \frac{1-\tau}{\tau}\xi\cdot x}\rho(x,\xi)\hat{f}(\xi)\,d\xi,$$

for a suitable symbol $\rho \in M^{\infty,1}_{0,s}(\mathbb{R}^{2d})$. Incidentally, by [CGNR14, Theorem 1.2] we have the following boundedness result.

Corollary 5.9.11. *If $\sigma \in W^{\infty,1}_{0,s}(\mathbb{R}^{2d})$, $s \geq 0$, then the operator σ^τ is bounded on every modulation space $M^p_{v_s}(\mathbb{R}^d)$, for $1 \leq p \leq \infty$ and $\tau \in (0,1)$.*

The connection with the theory of generalized metaplectic operators established above allows us to investigate other properties of τ-operators. First of all, notice that for any $\tau_1, \tau_2 \in (0,1)$,

$$\mathcal{U}_{\tau_1}\mathcal{U}_{\tau_2} = \begin{bmatrix} \frac{\tau_1\tau_2}{(1-\tau_1)(1-\tau_2)}I_d & 0_d \\ 0_d & \frac{(1-\tau_1)(1-\tau_2)}{\tau_1\tau_2}I_d \end{bmatrix}.$$

In particular, $\mathcal{U}_\tau\mathcal{U}_{1-\tau} = \mathcal{U}_{1-\tau}\mathcal{U}_\tau = I_{2d}$. Therefore, composition properties of operators in the class $FIO(\mathcal{A}, v_s)$ (see [CGNR14, Theorems 3.4] and Theorem 5.8.1) yield the following result.

Theorem 5.9.12 (Algebra property)**.** *For any $a, b \in W^{\infty,1}_{0,s}(\mathbb{R}^{2d})$ and $\tau \in (0,1)$, there exists a symbol $c \in M^{\infty,1}_{0,s}$ such that*

$$\mathrm{op}_\tau(a)\mathrm{op}_{1-\tau}(b) = \mathrm{op}_{\mathrm{w}}(c).$$

Remark 5.9.13. *On the other hand, for any choice of $\tau_1, \tau_2 \in (0,1)$, there is no $\tau \in (0,1)$ such that $\mathcal{U}_{\tau_1}\mathcal{U}_{\tau_2} = \mathcal{U}_\tau$. This immediately implies that there is no τ-quantization rule such that composition of τ-operators with symbols in $W^{\infty,1}_{0,s}$ has symbol in the same class. In a similar fashion, given $a \in W^{\infty,1}_{0,s}(\mathbb{R}^{2d})$, $b \in M^{\infty,1}_{0,s}(\mathbb{R}^{2d})$ and $\tau, \tau_0 \in (0,1)$, we have*

$$\mathrm{op}_{\tau_0}(b)\mathrm{op}_\tau(a) = \mathrm{op}_\tau(c_1), \qquad \mathrm{op}_\tau(a)\mathrm{op}_{\tau_0}(b) = \mathrm{op}_\tau(c_2),$$

for some $c_1, c_2 \in W^{\infty,1}_{0,s}(\mathbb{R}^{2d})$. This means that, for fixed quantization rules τ, τ_0, the amalgam space $W^{\infty,1}_{0,s}(\mathbb{R}^{2d})$ is a bimodule over the algebra $M^{\infty,1}_{0,s}(\mathbb{R}^{2d})$ under the laws

$$M^{\infty,1}_{0,s}(\mathbb{R}^{2d}) \times W^{\infty,1}_{0,s}(\mathbb{R}^{2d}) \to W^{\infty,1}_{0,s}(\mathbb{R}^{2d}) \;:\; (b,a) \mapsto c_1,$$

$$W^{\infty,1}_{0,s}(\mathbb{R}^{2d}) \times M^{\infty,1}_{0,s}(\mathbb{R}^{2d}) \to W^{\infty,1}_{0,s}(\mathbb{R}^{2d}) \ : \ (a,b) \mapsto c_2,$$

with c_1 *and* c_2 *as before.*

In conclusion, we provide a result whose proof easily follows by [CGNR14, Theorem 3.7] after noticing that $\mathcal{U}_\tau^{-1} = \mathcal{U}_{1-\tau}$ for any $\tau \in (0,1)$.

Theorem 5.9.14 (Wiener property)**.** *For any* $\tau \in (0,1)$ *and* $a \in W^{\infty,1}_{0,s}(\mathbb{R}^{2d})$ *such that* $\mathrm{op}_\tau(a)$ *is invertible on* $L^2(\mathbb{R}^d)$*, we have*

$$\mathrm{op}_\tau(a)^{-1} = \mathrm{op}_{1-\tau}(b)$$

for some $b \in W^{\infty,1}_{0,s}(\mathbb{R}^{2d})$*.*

Chapter 6

Dispersion, Spreading and Sparsity of Gabor Wave Packets for Metaplectic and Schrödinger Operators

6.1 Preliminary results

A key technical tool for the main results stated in Section 1.3 is the following set of estimates.

Lemma 6.1.1. *Let $s > 1$, $a, b, \sigma \geq 1$ and $v \in \mathbb{R}$. Then*

$$\int_{\mathbb{R}} (a + |\sigma^{-1}u + v|)^{-s}(b + |u|)^{-s} du \lesssim_s (a + |v|)^{-s} b^{-s+1} + a^{-s+1}(b + |v|)^{-s+1}, \tag{6.1}$$

$$\int_{\mathbb{R}} (a + |u - v|)^{-s}(b + \sigma^{-1}|u|)^{-s} du \lesssim_s (a + \sigma^{-1}|v|)^{-s+1} b^{-s+1} + a^{-s+1}(b + \sigma^{-1}|v|)^{-s}. \tag{6.2}$$

Proof. We prove (6.1) under the assumption $|v| \geq 1$, otherwise the estimate is trivial since $\int_{\mathbb{R}} (b + |u|)^{-s} du \lesssim b^{-s+1}$. If $|\sigma^{-1}u + v| \geq |v|/2$ then $(a + |\sigma^{-1}u + v|)^{-s} \lesssim (a + |v|)^{-s}$, hence

$$\int_{\mathbb{R}} (a + |\sigma^{-1}u + v|)^{-s}(b + |u|)^{-s} du \lesssim (a + |v|)^{-s} b^{-s+1}.$$

If $|\sigma^{-1}u+v|\le|v|/2$ then $|u|\ge\sigma|v|/2$, hence

$$\int_{\mathbb{R}}(a+|\sigma^{-1}u+v|)^{-s}(b+|u|)^{-s}du \lesssim a^{-s+1}\sigma(b+\sigma|v|)^{-s} \\ \le a^{-s+1}(b+|v|)^{-s+1}.$$

The proof of (6.2) in the non-trivial case $\sigma^{-1}|v|\ge 1$ follows by similar arguments, by considering separately the cases $|u-v|\ge|v|/2$ and $|u-v|<|v|/2$. □

Remark 6.1.2. *For $s>d$ and $v\in\mathbb{R}^d$, the convolution inequality* (6.2) *with $\sigma=1$ and $a=b$ can be improved in $\mathbb{R}^d$ as follows:*

$$\int_{\mathbb{R}^d}(a+|u-v|)^{-s}(a+|u|)^{-s}du \lesssim_s a^{-s+d}(a+|v|)^{-s}.$$

Notice that for $a=1$ we have $v_{-s}*v_{-s}\lesssim v_{-s}$, cf. [Grö01, Lemma 11.1.1(c)].

Lemma 6.1.3. *Let $\sigma_1,\dots,\sigma_d\ge 1$ and define the matrices*

$$\Sigma=\mathrm{diag}(\sigma_1,\dots,\sigma_d),\quad D'=\Sigma^{-1}\oplus I,\quad D''=I\oplus\Sigma^{-1}.$$

For any $s>2d$

$$\int_{\mathbb{R}^{2d}}(1+|v-D''u|)^{-s}(1+|D'u|)^{-s}du \lesssim_s (1+|D'v|)^{-s+2d},\quad v\in\mathbb{R}^{2d}.$$

Proof. The integral under our attention is

$$\int_{\mathbb{R}^{2d}}\left(1+\sum_{j=1}^{d}|v_j-u_j|+\sum_{j=d+1}^{2d}|v_j-\sigma_{j-d}^{-1}u_j|\right)^{-s}\left(1+\sum_{j=1}^{d}|\sigma_j^{-1}u_j|+\sum_{j=d+1}^{2d}|u_j|\right)^{-s}du.$$

We look at the latter as an iterated integral and we repeatedly apply Lemma 6.1.1; precisely we estimate each of the integrals with respect to $u_1,\dots,u_d$ as in (6.2) and the each one with respect to $u_{d+1},\dots,u_{2d}$ as in (6.1). Careful inspection of the involved quantities reveals that the result after $2d$ steps is dominated by a sum of products of the form $A^{-s+2d}B^{-s+2d}$ with $A,B\ge 1$ such that

$$A+B=2+\sum_{j=1}^{d}\sigma_j^{-1}|v_j|+\sum_{j=d+1}^{2d}|v_j|>1+|D'v|.$$

The claim follows after noticing that $A^{-s+2d}B^{-s+2d}\le(A+B)^{-s+2d}$ since $s>2d$. □

6.2 Proof of the main results

We start this section with the proof of Theorem 1.3.1, namely a pointwise inequality for the Gabor matrix with Gabor atoms in the Schwartz class.

Proof of Theorem 1.3.1. We use the Moyal formula (3.6), the covariance property of Wigner distribution (3.5) and the symplectic covariance of the Weyl calculus (4.9). Hence

$$\begin{aligned}|\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle|^2 &= \int_{\mathbb{R}^{2d}} W(\mu(S)\pi(z)g)(u)W(\pi(w)\gamma)(u)du \\ &= \int_{\mathbb{R}^{2d}} W(\pi(z)g)(S^{-1}u)W\gamma(u-w)du \\ &= \int_{\mathbb{R}^{2d}} Wg(S^{-1}u-z)W\gamma(u-w)du \\ &= \int_{\mathbb{R}^{2d}} Wg(S^{-1}u+S^{-1}w-z)W\gamma(u)du.\end{aligned}$$

Direct application of Proposition 3.1.3 (*ii*) yields, for any $s \geq 0$,

$$|\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle|^2 \lesssim \int_{\mathbb{R}^{2d}} v_{-s}(S^{-1}u+S^{-1}w-z)v_{-s}(u)du.$$

Recall that $S = U^\top DV$, hence $S^{-1} = V^\top D^{-1}U$ and therefore

$$|\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle|^2 \lesssim \int_{\mathbb{R}^{2d}} v_{-s}(D^{-1}u+V(S^{-1}w-z))v_{-s}(u)du.$$

Set $v \coloneqq V(S^{-1}w-z)$. The change of variable $u = D''u'$ leads to

$$|\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle|^2 \lesssim (\det\Sigma)^{-1}\int_{\mathbb{R}^{2d}} (1+|D'u+v|)^{-s}(1+|D''u|)^{-s}du.$$

We fix $s > 2d$ and apply Lemma 6.1.3 with D' and D'' interchanged. The claim then follows after setting $N = (s-2d)/2$, since $s > 2d$ is arbitrarily chosen and $D''v = D'U(w-Sz)$. □

We now prove Theorem 1.3.2, where Gabor atoms in suitable modulation spaces are considered.

Proof of Theorem 1.3.2. Fix $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)\setminus\{0\}$ with $\|\phi\|_{L^2} = \|\psi\|_{L^2} = 1$; the reconstruction formula (3.2) applied to $g \in M^p(\mathbb{R}^d)$, $\gamma \in M^q(\mathbb{R}^d)$ (resp. $g, \gamma \in M^\infty_{v_s}(\mathbb{R}^d)$) yields

$$g = \int_{\mathbb{R}^{2d}} F(u)\pi(u)\phi du, \quad F = V_\phi g \in L^p(\mathbb{R}^{2d}) \text{ (resp. } F = V_\phi g \in L^\infty_{v_s}(\mathbb{R}^{2d}))$$

$$\gamma = \int_{\mathbb{R}^{2d}} G(v)\pi(v)\psi dv, \quad G = V_\psi \gamma \in L^q(\mathbb{R}^{2d}) \text{ (resp. } G = V_\psi\gamma \in L^\infty_{v_s}(\mathbb{R}^{2d})).$$

Then we have

$$\begin{aligned} |\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle| &\le \int_{\mathbb{R}^{4d}} |F(u)||G(v)||\langle \mu(S)\pi(z+u)\phi, \pi(w+v)\psi\rangle| du dv \\ &= \int_{\mathbb{R}^{4d}} |F(u-z)||G(v-w)||\langle \mu(S)\pi(u)\phi, \pi(v)\psi\rangle| du dv. \end{aligned}$$

Direct application of Theorem 1.3.1 with $N > \max\{2d, s\}$ (the reason of this choice will be clear in a moment) yields

$$|\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle| \lesssim_N (\det \Sigma)^{-1/2} \int_{\mathbb{R}^{4d}} |F(u-z)||G(v-w)| v_{-N}(D'Uv - D''Vu) du dv.$$

Set $\widetilde{F} = F \circ (D''V)^{-1}$ and $\widetilde{G} = G \circ (D'U)^{-1}$. Then

$$\begin{aligned} |\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle| &\lesssim_N (\det \Sigma)^{3/2} \int_{\mathbb{R}^{4d}} |\widetilde{F}(u - D''Vz)||\widetilde{G}(v - D'Uw)| v_{-N}(v-u) du dv \\ &= (\det \Sigma)^{3/2} \int_{\mathbb{R}^{4d}} |\widetilde{F}(u)||\widetilde{G}(v + D''Vz - D'Uw)| v_{-N}(v-u) du dv \\ &= (\det \Sigma)^{3/2} (|\widetilde{F}| * v_{-N} * |\widetilde{G}|^\vee)(D'Uw - D''Vz) \\ &= H(D'U(w - Sz)), \end{aligned}$$

where we defined

$$H(u) = (\det \Sigma)^{3/2} (v_{-N} * |\widetilde{F}| * |\widetilde{G}|^\vee)(u), \quad u \in \mathbb{R}^{2d}.$$

For $g \in M^p(\mathbb{R}^d)$ and $\gamma \in M^q(\mathbb{R}^d)$ we apply Young's inequality to prove that $H \in L^r(\mathbb{R}^{2d})$ for $1/p + 1/q = 1 + 1/r$, cf. (4.7). In particular, since $N > 2d$,

$$\begin{aligned} \|H\|_{L^r} &\le (\det \Sigma)^{3/2} \|v_{-N}\|_{L^1} \| |\widetilde{F}| * |\widetilde{G}|^\vee \|_{L^r} \\ &\lesssim (\det \Sigma)^{3/2 - 1/p - 1/q} \|F\|_{L^p} \|G\|_{L^q} \\ &\lesssim (\det \Sigma)^{1/2 - 1/r} \|g\|_{M^p} \|\gamma\|_{M^q}. \end{aligned}$$

For $g, \gamma \in M^\infty_{v_s}(\mathbb{R}^d)$, $s > 2d$, we note that

$$|\widetilde{F}(u)| \le \|g\|_{M^\infty_{v_s}(\mathbb{R}^d)} (1 + |(D'')^{-1}u|)^{-s}, \quad |\widetilde{G}(u)| \le \|\gamma\|_{M^\infty_{v_s}(\mathbb{R}^d)} (1 + |(D')^{-1}u|)^{-s}.$$

Therefore, since $N > s$ and again by Young's inequality,

$$\begin{aligned}\|H\|_{L^\infty_{v_{s-2d}}} &\le (\det\Sigma)^{3/2}\|v_{-N}\|_{L^1_{v_{s-2d}}}\||\widetilde{F}| * |\widetilde{G}|^\vee\|_{L^\infty_{v_{s-2d}}}\\ &\lesssim (\det\Sigma)^{3/2}\|g\|_{M^\infty_{v_s}(\mathbb{R}^d)}\|\gamma\|_{M^\infty_{v_s}(\mathbb{R}^d)}\big\|v_{-s}((D')^{-1}\cdot) * v_{-s}((D'')^{-1}\cdot)\big\|_{L^\infty_{v_{s-2d}}}\\ &\lesssim (\det\Sigma)^{-1/2}\|g\|_{M^\infty_{v_s}(\mathbb{R}^d)}\|\gamma\|_{M^\infty_{v_s}(\mathbb{R}^d)},\end{aligned}$$

where in the last step we used Lemma 6.1.3 with the substitutions $u \mapsto (D')^{-1}(D'')^{-1}u$ and $v \mapsto (D')^{-1}v$. □

Remark 6.2.1. *Notice that after setting $\widetilde{H} = H \circ D'U$ the estimate* (1.25) *reads*

$$|\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle| \le \widetilde{H}(w - Sz),$$

while (1.26) *becomes*

$$\|\widetilde{H}\|_{L^r} \lesssim (\det\Sigma)^{1/2}\|g\|_{M^p}\|\gamma\|_{M^q}.$$

It is then clear that there is a trade-off between the phase-space concentration of $\mu(S)$ along the graph of S and the spreading of wave packets.

We conclude with a result in the same spirit for generalized metaplectic operators.

Proof of Theorem 1.3.3. We assume $\|g\|_{L^2} = \|\gamma\|_{L^2} = 1$ without loss of generality. Denoting by $K_{\mu(S)}(w, z) = \langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle$ the Gabor matrix of $\mu(S)$ and similarly for $K_{a^{\mathrm{w}}}(w, z) = \langle a^{\mathrm{w}}\pi(z)\gamma, \pi(w)\gamma\rangle$, in view of (4.2), by Theorems 4.2.6 and 4.3.10 we have

$$\begin{aligned}|\langle A\pi(z)g, \pi(w)\gamma\rangle| &= |\langle a^{\mathrm{w}}\mu(S)\pi(z)g, \pi(w)\gamma\rangle|\\ &\le \int_{\mathbb{R}^{2d}} |K_{a^{\mathrm{w}}}(w, u)||K_{\mu(S)}(u, z)|du\\ &= \int_{\mathbb{R}^{2d}} |\langle a^{\mathrm{w}}\pi(u)\gamma, \pi(w)\gamma\rangle||\langle \mu(S)\pi(z)g, \pi(u)\gamma\rangle|du\\ &\le \int_{\mathbb{R}^{2d}} H_a(w - u)H_S(D'Uu - D''Vz)du,\end{aligned}$$

where H_S is the controlling function in Theorem 1.3.2 (i) and H_a is the one appearing in Theorem 4.2.6 with $g = \gamma$; in particular $\|H_a\|_{L^1} \asymp \|a\|_{M^{\infty,1}}$. The substitution $y = D'U(w - u)$ yields

$$|\langle A\pi(z)g, \pi(w)\gamma\rangle| \le (\det\Sigma)\big[(H_a \circ (D'U)^{-1}) * H_S\big](D'U(w - Sz)).$$

The claim follows by Young inequality and Theorem 1.3.2 (*i*) after setting $H = (\det\Sigma)(H_a \circ (D'U)^{-1}) * H_S$:

$$\begin{aligned}\|H\|_{L^r} &\leq (\det\Sigma)\big\|H_a \circ (D'U)^{-1}\big\|_{L^1}\|H_S\|_{L^r}\\ &= \|H_a\|_{L^1}\|H_S\|_{L^r}\\ &\lesssim (\det\Sigma)^{1/2-1/r}\|a\|_{M^{\infty,1}}\|g\|_{M^p}\|\gamma\|_{M^q}.\end{aligned}$$

□

We conclude with the proof of Theorem 1.3.4, where we study how the modulation space regularity on a cone in the phase space behaves under the action of a metaplectic operator.

Proof of Theorem 1.3.4. Fix $g, \gamma \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ with $\|g\|_{L^2} = \|\gamma\|_{L^2} = 1$, and Γ and Γ' as in the statement. From (4.2) with $A = \mu(S)$ and Theorem 1.3.1, for any $N > 0$ we have

$$\begin{aligned}|V_\gamma(\mu(S)f)(w)| &\leq \int_{\mathbb{R}^{2d}} |K_{\mu(S)}(w,z)||V_g f(z)|dz\\ &\lesssim_N (\det\Sigma)^{-1/2}\int_{\mathbb{R}^{2d}} v_{-N}(D'U(w-Sz))|V_g f(z)|dz\\ &\lesssim_N (\det\Sigma)^{-1/2}\int_{\mathbb{R}^{2d}} H(w-Sz)|V_g f(z)|dz,\end{aligned}$$

where we set $H = v_{-N} \circ D'U$. After naming $G = H \circ S = v_{-N} \circ D''V$ we apply Hölder's inequality and get

$$\begin{aligned}I &\coloneqq \|\mu(S)f\|_{M^1_{(\gamma)}(S(\Gamma'))}\\ &= \int_{S(\Gamma')} |V_\gamma(\mu(S)f)(w)|dw\\ &= \int_{\Gamma'} |V_\gamma(\mu(S)f)(Sw)|dw\\ &\lesssim (\det\Sigma)^{-1/2}\int_{\Gamma'}\int_{\mathbb{R}^{2d}} G(w-z)|V_g f(z)|dzdw.\end{aligned}$$

We then have $I \lesssim I_1 + I_2$, where

$$I_1 \coloneqq (\det\Sigma)^{-1/2}\int_{\Gamma'}\int_{\Gamma} G(w-z)|V_g f(z)|dzdw,$$

$$I_2 \coloneqq (\det\Sigma)^{-1/2}\int_{\Gamma'}\int_{\Gamma^c} G(w-z)|V_g f(z)|dzdw.$$

Young's inequality yields

$$I_1 \leq \|G\|_{L^1}\|V_g f \cdot 1_\Gamma\|_{L^1} \lesssim (\det\Sigma)^{1/2}\|f\|_{M^1_{(g)}(\Gamma)}.$$

After setting $F(z) = |V_g f(z)| v_{-r}(z)$, the remaining integral is

$$I_2 = (\det\Sigma)^{-1/2}\int_{\Gamma'}\int_{\Gamma^c} G(w-z)v_r(z)F(z)dz.$$

The key point is now that

$$1+|w-z| \asymp \max\{1+|w|, 1+|z|\}, \quad w\in\Gamma',\, z\in\Gamma^c,$$

hence

$$\begin{aligned}
I_2 &\lesssim (\det\Sigma)^{-1/2}\int_{\Gamma'}\int_{\Gamma^c} G(w-z)v_r(w-z)F(z)dz\\
&\leq (\det\Sigma)^{-1/2}\|(G\cdot v_r)*F\|_{L^1}\\
&\lesssim (\det\Sigma)^{-1/2}\|G\cdot v_r\|_{L^1}\|f\|_{M^1_{v_{-r}}}.
\end{aligned}$$

Therefore, the remaining integral to estimate is

$$\|G\cdot v_r\|_{L^1} = \int_{\mathbb{R}^{2d}} (1+|D''z|)^{-N}(1+|z|)^r dz.$$

Recall that $D'' = I \oplus \Sigma^{-1}$, cf. (1.23), and consider the elementary estimates

$$v_{-N}(D''z) \leq v_{-N/2d}(z_1)\cdots v_{-N/2d}(z_d)v_{-N/2d}(\sigma_1^{-1}z_{d+1})\cdots v_{-N/2d}(\sigma_d^{-1}z_{2d}),$$

$$v_r(z) \leq v_r(z_1)\cdots v_r(z_{2d}).$$

As a result, the integral is dominated by $A^d B_1\cdots B_d$, where

$$A \coloneqq \int_{\mathbb{R}} (1+|x|)^{-N/2d+r}dx,$$

$$B_j \coloneqq \int_{\mathbb{R}} (1+\sigma_j^{-1}|x|)^{-N/2d}(1+|x|)^r dx, \quad j=1,\ldots,d.$$

If N is large enough then $A<\infty$ and $B_j \lesssim \sigma_j^{1+r}$, therefore

$$I_2 \lesssim (\det\Sigma)^{1/2+r}\|f\|_{M^1_{v_{-r}}},$$

and the claim follows. □

Remark 6.2.2. *1. Condition* (1.27) *can be generalized to introduce the notion of M^p-regularity, $1 \le p \le \infty$, on the cone Γ with respect to $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. The latter is satisfied for $f \in \mathcal{S}'(\mathbb{R}^d)$ if*

$$\|f\|_{M^p_{(g)}(\Gamma)} := \|V_g f \cdot 1_\Gamma\|_{L^p} < \infty. \tag{6.3}$$

Weighted versions of such conditions can be defined similarly. The proof of Theorem 1.3.4 can be easily modified in order to prove the estimate

$$\|\mu(S)f\|_{M^p_{(\gamma)}(S(\Gamma'))} \lesssim (\det \Sigma)^{1/2} \Big(\|f\|_{M^p_{(g)}(\Gamma)} + (\det \Sigma)^r \|f\|_{M^p_{v_{-r}}} \Big), \tag{6.4}$$

which however is not sharp unless $p = 1$ or $p = \infty$. We postpone further investigations on the issue.

2. The notion of M^p-regularity does not depend on the window g used to compute $V_g f$ in (6.3) *provided that a slightly smaller cone is allowed when changing window. This is indeed a consequence of* (6.4) *in the case where $S = I$. The properties of $M^p_{(g)}(\Gamma)$ as a function space will be object of future studies.*

Corollary 6.2.3. *Consider $1 \le p \le \infty$. There exists $C > 0$ such that, for any $f \in M^p(\mathbb{R}^d)$, $S \in \mathrm{Sp}(d, \mathbb{R})$,*

$$\|\mu(S)f\|_{M^p} \le C(\det \Sigma)^{|1/2-1/p|} \|f\|_{M^p}.$$

Proof. By choosing $\Gamma = \Gamma' = \mathbb{R}^{2d} \setminus \{0\}$ and $r = 0$ in Theorem 1.3.4 we see that the desired estimate holds for $p = 1$. Since $\mu(S)$ is unitary on $L^2(\mathbb{R}^d)$, the operator $\mu(S^{-1})$, and therefore $\mu(S)$, satisfies the same estimate for $p = \infty$. Interpolating with the trivial L^2-estimate, we obtain the desired result, cf. Proposition 3.2.3. □

6.3 Applications to the free particle propagator

Let us consider the free particle propagator $U(t) = e^{i(t/2\pi)\Delta}$ and the corresponding classical flow (4.8); a straightforward computation shows that the largest d singular values of S_t coincide:

$$\sigma_j = \sigma(t) = (1 + 2t^2 + 2(t^2 + t^4)^{1/2})^{1/2} = \sqrt{1 + t^2} + |t|, \quad j = 1, \dots, d.$$

Note in particular that $\sigma(t)$ is comparable to $1 + |t|$, $t \in \mathbb{R}$. An example of Euler decomposition (U_t, V_t, Σ_t) of S_t for $t \ge 0$ is given by

$$U_t = (1 + \sigma(t)^2)^{-1/2} \begin{bmatrix} \sigma(t)I & I \\ -I & \sigma(t)I \end{bmatrix}, \quad V_t = (1 + \sigma(t)^2)^{-1/2} \begin{bmatrix} I & \sigma(t)I \\ -\sigma(t)I & I \end{bmatrix}.$$

Theorem 1.3.1 thus yields

$$\left|\langle e^{i(t/2\pi)\Delta}\pi(z)g, \pi(w)\gamma\rangle\right| \le C(1+|t|)^{-d/2}(1+|D_t' U_t(w-S_t z)|)^{-N}, \quad z,w \in \mathbb{R}^{2d}.$$

The spreading phenomenon manifests itself as a dilation by

$$D_t' U_t = (1+\sigma(t)^2)^{-1/2}\begin{bmatrix} I & \sigma(t)^{-1} I \\ -I & \sigma(t) I \end{bmatrix}.$$

We attempt to shed some light on the apparently unintelligible structure of such matrix by means of a toy example in dimension $d=1$. Let $z=0$ for simplicity and assume that the atom g is concentrated on the box $Q=\{(x,\xi)\in\mathbb{R}^2 : |x| < 1,\ |\xi| < 1\}$ in the time-frequency plane. In view of (1.24) we are lead to consider

$$(D_t' U_t)^{-1}(Q) = \{(x,\xi) : |x+\sigma^{-1}(t)\xi| < \sqrt{1+\sigma(t)^2},\ |x-\sigma(t)\xi| < \sqrt{1+\sigma(t)^2}\}.$$

Therefore, the effect of $D_t' U_t$ on Q ultimately amounts to a horizontal stretch by a factor of approximately $\sigma(t)$. This is consistent with the expected phase-space evolution of a wave packet, as shown in (1.19); see also [CNR09, Figures 1-8] for illuminating graphic representations. We stress that the estimate (1.18) is completely blind to such spreading effect.

Chapter 7

Time-Frequency Analysis of the Dirac Equation

7.1 Proof of the main results

7.1.1 The free case

Consider the Cauchy problem for the free Dirac equation, namely (1.28) with $V = 0$:

$$\begin{cases} i\partial_t \psi(t,x) = \mathcal{D}_m \psi(t,x), \\ \psi(0,x) = \psi_0(x), \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d. \tag{7.1}$$

The solution can be recast in terms of the free Dirac propagator:

$$\psi(t,x) = U_0(t)\psi_0(x), \qquad U_0(t) = e^{-it\mathcal{D}_m}.$$

We can take advantage from vector-valued setting of Gabor analysis by noticing that $U_0(t)$ is an operator-valued Fourier multiplier on the Hilbert space $H = \mathbb{C}^n$, $\mathcal{L}(\mathbb{C}^n) \simeq \mathbb{C}^{n\times n}$, with symbol

$$\mu_t(\xi) = \exp\left[-2\pi i t\left(m\alpha_0 + \sum_{j=1}^{d} \xi_j \alpha_j\right)\right].$$

An explicit expression for this matrix can be derived. After setting $C_j = -2\pi t \xi_j$, $j = 1,\dots,d$, and $C_0 = -2\pi t m$ we have $\mu_t(\xi) = \sum_{n\geq 0} \frac{i^n}{n!}(\sum_{j=0}^d C_j \alpha_j)^n$. The identities (1.30) satisfied by the Dirac matrices imply that

$$\begin{cases} (\sum_{j=0}^d C_j\alpha_j)^n = (-1)^k (\sum_{j=0}^d C_j^2)^k I_n & (n = 2k), \\ (\sum_{j=0}^d C_j\alpha_j)^n = i(-1)^k (\sum_{j=0}^d C_j^2)^k (\sum_{j=0}^d C_j\alpha_j) & (n = 2k+1). \end{cases}$$

A straightforward computation finally yields

$$\mu_t(\xi) = \cos(2\pi t\langle\xi\rangle_m)I_n - 2\pi i\frac{\sin(2\pi t\langle\xi\rangle_m)}{2\pi\langle\xi\rangle_m}\left(m\alpha_0 + \sum_{j=1}^{d}\xi_j\alpha_j\right),$$

from which it is clear that $\mu_t \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^d, \mathbb{C}^{n\times n})$ for any fixed $t \in \mathbb{R}$.

Proof of Theorem 1.4.1. The proof is a direct application of Proposition 4.1.1 ($X = M_{r,s}^{p,q}(\mathbb{C}^n)$) and Proposition 4.1.2 ($X = W_{r,s}^{p,q}(\mathbb{C}^n)$), after noticing that

$$\mu_t \in C_{\mathrm{b}}^{\infty}(\mathbb{C}^{n\times n}) \hookrightarrow M_{0,|r|}^{\infty,1}(\mathbb{C}^{n\times n}) \hookrightarrow W_{|r|,0}^{1,\infty}(\mathbb{C}^{n\times n}), \quad \forall r \in \mathbb{R},$$

the latter embedding being given by the Hausdorff-Young inequality (3.15). □

Proof of estimate (1.32). In order to determine the time dependence of the constant $C_X(t)$, $X = M_{0,s}^{p,q}(\mathbb{C}^n)$, we provide a different proof by making use of the discrete norm (3.14) for modulation spaces. Consider the BUPU in the proof of Theorem 3.2.11. In view of (3.14) we need to provide an estimate for $\left\| \|\Box_k U(t) f\|_{L^p(\mathbb{C}^n)} \right\|_{\ell_s^q}$. We have

$$\|\Box_k U(t) f\|_{L^p(\mathbb{C}^n)} = \sum_{|\ell|_\infty \le 1} \left\|\sigma_{k+\ell}\mu_t\sigma_k\hat{f}\right\|_{\mathcal{F}L^p(\mathbb{C}^n)} \le \sum_{|\ell|_\infty \le 1} \|\sigma_{k+\ell}\mu_t\|_{\mathcal{F}L^1(\mathbb{C}^{n\times n})}\|\Box_k f\|_{L^p(\mathbb{C}^n)},$$

where we used the approximate orthogonality of the frequency-uniform decomposition operators:

$$\Box_k = \sum_{|\ell|_\infty \le 1} \Box_k\Box_{k+\ell}, \quad k \in \mathbb{Z}^d.$$

The multiplier estimate (2.2) implies

$$\|\sigma_{k+\ell}\mu_t\|_{\mathcal{F}L^1(\mathbb{C}^{n\times n})} = \left\|\sigma_0 T_{-(k+\ell)}\mu_t\right\|_{\mathcal{F}L^1(\mathbb{C}^{n\times n})} \lesssim (1+|t|)^{d/2},$$

and complex interpolation with the conservation law $\|\Box_k U(t) f\|_{L^2(\mathbb{C}^n)} = \|\Box_k f\|_{L^2(\mathbb{C}^n)}$ yields

$$\|\Box_k U(t) f\|_{L^p(\mathbb{C}^n)} \lesssim (1+|t|)^{d|1/2-1/p|}\|\Box_k f\|_{L^p(\mathbb{C}^n)}.$$

□

This behaviour is not surprising, given that any component of a solution of the free Dirac equation is also a solution of the free Klein-Gordon equation, for which similar estimates hold [WHHG11, Proposition 6.8]. This connection can be exploited in many ways, as already mentioned in the Introduction; as an example one can easily prove a smoothing estimate for the free Dirac propagator.

Theorem 7.1.1. *Let $\psi(t,x)$ be the solution of* (7.1)*. For any $t>1$, $1\le p,q\le\infty$ and $s\in\mathbb{R}$,*

$$\|\psi(t,\cdot)\|_{M^{p,q}_{0,s}(\mathbb{C}^n)} \lesssim \|\psi_0\|_{M^{p,q}_{0,s}(\mathbb{C}^n)} + |t|^\gamma \|\psi_0\|_{M^{p,q}_{0,s-\gamma}(\mathbb{C}^n)}, \qquad \gamma = d|1/2-1/p|.$$

Proof. Following the same strategy of [KN19, Theorem 1.1], namely projection onto the so-called positive and negative energy subspaces of the Dirac operator (cf. [Tha92]), it turns out that the free Dirac equation (7.1) is unitarily equivalent to a pair of $(n/2)$-dimensional square-root Klein-Gordon equations, namely

$$\begin{cases} i\partial_t \psi_\pm(t,x) = \pm\langle D\rangle_m \psi_\pm(t,x), \\ \psi_\pm(0,x) = (\psi_0)_\pm(x), \end{cases} \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^d.$$

It is then enough to replace the estimate (3.2) in that paper for the Klein-Gordon semigroup $e^{it\langle D\rangle_m}$ with the smoothing one proved in [DDS13, Theorem 1.4]. The proof then proceeds in the same way. □

7.1.2 The case where V is a rough bounded potential

For any $1\le p,q\le\infty$, $\gamma\ge 0$ and $r,s\in\mathbb{R}$ such that $|r|+|s|\le\gamma$, let X denote either $\mathcal{M}^{p,q}_{r,s}(\mathbb{C}^n)$ or $\mathcal{W}^{p,q}_{r,s}(\mathbb{C}^n)$. Let $T>0$ be fixed and consider now the Cauchy problem for the Dirac equation with potential

$$\begin{cases} i\partial_t \psi(t,x) = (\mathcal{D}_m + V(t))\psi(t,x) \\ \psi(0,x) = \psi_0(x) \end{cases} \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^d, \tag{7.2}$$

where $V(t) = \sigma(t,\cdot)^{\mathrm{w}}$, $t\in[0,T]$, and the map $t\mapsto\sigma(t,\cdot)$ is continuous in $M^{\infty,1}_{0,\gamma}(\mathbb{R}^{2d},\mathbb{C}^{n\times n})$ for the narrow convergence. Standard arguments from the theory of operators semigroups (cf. [EN06, Corollary 1.5]) and Theorem 4.2.7 imply that for any fixed $t\in\mathbb{R}$ the propagator $U(t)$ is bounded on X.

Proof of Theorem 1.4.2. The argument is standard, we sketch the strategy for the sake of clarity. Set $\Xi_T = C([0,T];\mathcal{L}_{\mathrm{s}}(X))$; the assumptions on σ and Theorem 4.2.8 imply that $V\in\Xi_T$. A straightforward computation shows that the propagator $U(t)$ corresponding to (7.2) satisfies the following Volterra integral equation:

$$U(t)\psi_0 = U_0(t)\psi_0 - i\int_0^t U_0(t-s)V(s)U(s)\psi_0 ds. \tag{7.3}$$

A solution is given by an iterative scheme: let $\{U_n\}_{n\in\mathbb{N}}$ the sequence of operators

$$U_0(t) = e^{-it\mathcal{D}_m}, \qquad U_n(t)\psi_0 \coloneqq \int_0^t U_0(t-s)V(s)U_{n-1}(s)\psi_0\, ds.$$

We have that $\{U_n\} \subset \Xi_T$, since $U_n = U_0 * VU_{n-1}$ and both convolution and composition are bounded operators on Ξ_T; cf. [EN06, Exercise 1.17.1 and Lemma B.15]. Furthermore, the following estimates hold:

$$\|U_n(t)\|_{\mathcal{L}(X)} \le K(t)^{(n+1)}\frac{t^n}{n!}, \qquad K(t) = \sup_{s\in[0,t]} \|U_0(s)\|\|V(s)\|.$$

It then follows that the Dyson-Phillips series $\sum_n U_n(t)$ converges with respect to the operator norm on $\mathcal{L}(X)$ and also uniformly on $[0,T]$. Therefore $U(t) = \sum_n U_n(t) \in \Xi_T$ and $U(t)$ is a propagator for (7.2). Uniqueness follows by Gronwall's lemma after noticing that a different solution $P(t)$ of (7.3) would satisfy

$$\|(U(t) - P(t))\psi_0\|_X \le K(t)\int_0^t \|(U(\tau) - P(\tau))\psi_0\|_X d\tau.$$

□

7.1.3 The case where V is a rough quadratic potential

Theorem 1.4.3 involves a rough potential V with at most quadratic growth as in (1.33). A key ingredient for the proof of Theorem 1.4.3 is Proposition 3.2.20.

Proof of Theorem 1.4.3. We apply Proposition 3.2.20 twice and get

- $L = L_1 + L_2$, where $L_1 \in C^\infty_{\ge 1}(\mathbb{C}^{n\times n})$ and $L_2 \in M^{\infty,1}(\mathbb{C}^{n\times n})$, and
- $Q = Q_1 + Q_2$, where $Q_1 \in C^\infty_{\ge 2}$ and $Q_2 \in M^{\infty,1}$.

The RHS of (1.28) then becomes

$$\mathcal{H} = (\mathcal{D}_m + L_1 + Q_1) + (L_2 + Q_2 + \sigma^{\mathrm{w}}) =: \mathcal{H}_0 + V'.$$

We see that $e^{-it\mathcal{H}_0}$ is a semigroup of bounded operators on $\mathcal{M}^p(\mathbb{R}^d, \mathbb{C}^n)$ as a consequence of [KN19, Theorem 1.2]. It is understood that we identify the multiplication by a function $f \in M^{\infty,1}$ on $M^{p,q}(\mathbb{C}^n)$ with the operator $fI_n \in \mathbb{C}^{n\times n}$, hence by Remark 3.2.12 we have

$$\|fu\|_{M^{p,q}(\mathbb{C}^n)} \le \|fI_n\|_{M^{\infty,1}(\mathbb{C}^{n\times n})}\|u\|_{M^{p,q}(\mathbb{C}^n)} \asymp \|f\|_{M^{\infty,1}}\|u\|_{M^{p,q}(\mathbb{C}^n)}.$$

The boundedness of $e^{-it\mathcal{H}}$ on $\mathcal{M}^p(\mathbb{C}^n)$ then follows from the fact that V' is a bounded perturbation of $\mathcal{H}_0$ [EN06, Corollary 1.5] by Proposition 4.1.1 and Theorem 4.2.7. The case where $Q = 0$ follows by the same arguments. □

7.2 The nonlinear equation

A standard tool in the study of local well-posedness is the following abstract result.

Theorem 7.2.1 ([Tao06, Proposition 1.38])**.** *Let X and Y be two Banach spaces and $D : X \to Y$ be a bounded linear operator such that*

$$\|Du\|_Y \leq C_0 \|u\|_X, \tag{7.4}$$

for all $u \in X$ and some $C_0 > 0$. Consider then a nonlinear operator $F : Y \to X$, $F(0) = 0$, such that

$$\|F(u) - F(v)\|_X \leq \frac{1}{2C_0} \|u - v\|_Y, \tag{7.5}$$

for all u, v in the ball $B_\epsilon(0) = \{u \in Y : \|u\|_Y \leq \epsilon\}$ for some $\epsilon > 0$. Then for any $u_0 \in B_{\epsilon/2}$ there exists a unique solution $u \in B_\epsilon$ to the equation

$$u = u_0 + DF(u),$$

and the map $u_0 \mapsto u$ is Lipschitz with constant at most 2, that is $\|u\|_Y \leq 2\|u_0\|_Y$.

Lemma 7.2.2. *Let $r, s \geq 0$, $1 \leq p \leq \infty$ and $\epsilon > 0$, and consider a nonlinear function F as in (1.35). Denote by X any of the spaces $M^{p,1}_{0,s}(\mathbb{C}^n)$ or $W^{1,p}_{r,s}(\mathbb{C}^n)$. If $\psi_0 \in X$ then $F(\psi) \in X$ and, for any $\psi, \phi \in B_\epsilon(0) \subset X$ there exists a constant $C_\epsilon > 0$ such that*

$$\|F(\psi) - F(\phi)\|_X \leq C_\epsilon \|\psi - \phi\|_X.$$

Proof. In view of Proposition 3.2.7 (vi) and its counterpart for amalgam spaces the first claim is an easy consequence of the algebra property of X under pointwise multiplication [CN09, Lemmas 2.1-2.2] and the series expansion of each component. The estimate in the second part follows from a straightforward computation (cf. the proof of [CN09, Theorem 4.1]), that is

$$\begin{aligned} F_j(\psi) - F_j(\phi) &= \int_0^1 \frac{d}{dt} F_j(t\psi + (1-t)\phi) dt \\ &= \sum_{k=1}^n \Bigg[(\psi_k - \phi_k) \sum_{\alpha,\beta,\gamma,\delta \in \mathbb{N}^n} c^{j,k}_{\alpha,\beta,\gamma,\delta} \psi^\alpha \bar{\psi}^\beta \phi^\delta \bar{\phi}^\gamma \\ &\qquad + (\bar{\psi}_k - \bar{\phi}_k) \sum_{\alpha,\beta,\gamma,\delta \in \mathbb{N}^n} \tilde{c}^{j,k}_{\alpha,\beta,\gamma,\delta} \psi^\alpha \bar{\psi}^\beta \phi^\delta \bar{\phi}^\gamma \Bigg], \end{aligned}$$

for suitable coefficients $c^{j,k}_{\alpha,\beta,\gamma,\delta}, \tilde{c}^{j,k}_{\alpha,\beta,\gamma,\delta} \in \mathbb{C}$. Again by Proposition 3.2.7 (*iv*) we have

$$\|F(\psi) - F(\phi)\|_X \lesssim \|\psi - \phi\|_X \sum_{j,k=1}^{n} \sum_{\alpha,\beta,\gamma,\delta \in \mathbb{N}^n} C^{j,k}_{\alpha,\beta,\gamma,\delta} \|\psi\|_X^{|\alpha+\beta|} \|\phi\|_X^{|\gamma+\delta|},$$

with $C^{j,k}_{\alpha,\beta,\gamma,\delta} = |c^{j,k}_{\alpha,\beta,\gamma,\delta}| + |\tilde{c}^{j,k}_{\alpha,\beta,\gamma,\delta}|$, and the latter expression is $\leq C_\epsilon \|\psi - \phi\|_X$ whenever $\psi, \phi \in B_\epsilon(0)$. □

Proof of Theorem 1.4.4. The proof is an application of the iteration scheme given in Theorem 7.2.1. In particular we choose either $X = M^{p,1}_{0,s}(\mathbb{C}^n)$ or $X = W^{1,p}_{r,s}(\mathbb{C}^n)$, then $Y = C^0([0,T], X)$, and convert (1.34) in integral form:

$$\psi(t) = U_0(t)\psi_0 - i \int_0^t U_0(t-s) F(\psi(s)) ds,$$

where $U_0 = e^{-it\mathcal{D}_m}$ is the free propagator. It is then enough to prove (7.4) and (7.5) in this setting, where D is the Duhamel operator $D = \int_0^t U_0(t-s) \cdot ds$. First, notice that from Theorem 1.4.1 we have that

$$\|U_0(t)\psi_0\|_X \leq C_T \|\psi_0\|_X, \quad \forall t \in [0,T].$$

Therefore,

$$\left\| \int_0^t U_0(t-s) u(s) ds \right\|_X \leq \int_0^t \|U_0(t-s)u(s)\|_X ds \leq T C_T \sup_{t\in[0,T]} \|u(t)\|_X.$$

Lemma 7.2.2 then provides (7.4) with a constant $C_0 = O(T)$ and also (7.5). The claim follows after choosing $T = T(\|\psi_0\|_X)$ sufficiently small. □

Remark 7.2.3. *A more general version of Theorem 1.4.4, namely a nonlinear variant of Theorem 1.4.2, can be stated. For any* $1 \leq p \leq \infty$ *and* $\gamma \geq 0$ *let* X *denote either* $\mathcal{M}^{p,1}_{0,s}(\mathbb{C}^n)$ *with* $0 \leq s \leq \gamma$ *or* $\mathcal{W}^{1,p}_{r,s}(\mathbb{C}^n)$ *with* $r, s \geq 0$ *such that* $r + s \leq \gamma$*. The differential operator* $L = i\partial_t - \mathcal{D}_m$ *in* (1.34)*, namely* $L\psi = F(\psi)$*, is now extended to* $L = i\partial_t - \mathcal{D}_m - \sigma^{\mathrm{w}}_t$*, where the symbol map* $[0,T] \ni t \mapsto \sigma(t,\cdot) \in M^{\infty,1}_{0,\gamma}(\mathbb{C}^{n\times n})$ *is continuous for the narrow convergence and the nonlinear term is* (1.35)*. We recast the problem in integral form as*

$$\psi(t) = U(t,0)\psi_0 - i \int_0^t U(t,\tau) F(\psi(\tau)) d\tau,$$

where $U(t,\tau)$*,* $0 \leq \tau \leq t \leq T$ *is the linear propagator constructed in the proof of Theorem 1.4.2 corresponding to initial data at time* τ*. In order for the iteration* *scheme in Theorem 7.2.1 to work it is enough to prove that* $U(t,\tau)$ *is strongly continuous on* X *jointly in* (t,τ), $0 \le \tau \le t \le T$*; the latter condition would imply a uniform bound for the operator norm with respect to* t,τ *as a consequence of the uniform boundedness principle. Theorem 1.4.2 yields strong continuity of* $U(t,\tau)$ *in* t *for fixed* τ*. The time-reversibility enjoyed by the equation implies that the same holds after switching* τ *and* t*. Furthermore, for* $\tau' \le \tau \le t$ *we have*

$$\|U(t,\tau)\psi_0 - U(t,\tau')\psi_0\|_X \le C\|\psi_0 - U(\tau,\tau')\psi_0\|_X,$$

hence the map $\tau \mapsto U(t,\tau)\psi_0$ *is continuous in* X*, uniformly with respect to* t *and this gives the desired result.*

Part III

Time-Frequency Analysis of Feynman Path Integrals

Chapter 8

Pointwise Convergence of Integral Kernels in the Feynman-Trotter formula

8.1 Preliminary results

The celebrated Schwartz kernel theorem is usually invoked for proving that any reasonably well-behaved operator is indeed an integral transform (in the distributional sense). In the following we will need this identification but at the topological level [Gas60; Tre67], that is, a linear map $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is continuous if and only if it is generated by a (unique) temperate distribution $K \in \mathcal{S}'(\mathbb{R}^{2d})$, namely:

$$\langle Af, g \rangle = \langle K, g \otimes \overline{f} \rangle, \qquad \forall f, g \in \mathcal{S}(\mathbb{R}^d),$$

and the correspondence $K \mapsto A$ above is a topological isomorphism between $\mathcal{S}'(\mathbb{R}^{2d})$ and the space $\mathcal{L}_{\mathrm{b}}\big(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d)\big)$. Recall from Section 2.1.1 that $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^{2d})$ are endowed with the strong topology by default.

The previous identification provides the following convergence result at the level of integral kernels.

Proposition 8.1.1. *Let $A_n \to A$ in $\mathcal{L}_{\mathrm{s}}\big(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d)\big)$. Then we have convergence in $\mathcal{S}'(\mathbb{R}^{2d})$ of the corresponding distribution kernels.*

Proof. Since $\mathcal{S}(\mathbb{R}^d)$ is a Fréchet space and A_n, being a sequence, defines a filter with countable basis on $\mathcal{L}_{\mathrm{s}}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$, from [Tre67, Corollary at pag. 348] we have that $A_n \to A$ also in $\mathcal{L}_{\mathrm{c}}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$, which is in turn equivalent to convergence in $\mathcal{L}_{\mathrm{b}}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$ since $\mathcal{S}(\mathbb{R}^d)$ is a Montel space - cf. [Tre67, Propositions 34.4 and 34.5]. The desired conclusion then follows from the Schwartz kernel theorem. ☐

The following lemma extends [CGNR14, Lemma 2.2 and Proposition 5.2].

Lemma 8.1.2. *Let X denote either $M^\infty_{0,s}(\mathbb{R}^{2d})$, $s \ge 0$, or $M^{\infty,1}(\mathbb{R}^{2d})$.*

(i) Let $\sigma \in X$ and $t \mapsto S_t \in \mathrm{Sp}(d,\mathbb{R})$ be a continuous mapping defined on the compact interval $[-T,T] \subset \mathbb{R}$, $T > 0$. For any $t \in [-T,T]$, we have $\sigma \circ S_t \in X$, with

$$\|\sigma \circ S_t\|_X \le C(T)\|\sigma\|_X.$$

(ii) Let $\sigma \in X$ and A, B, C be real $d \times d$ matrices with B invertible, and set

$$\Phi(x,y) = \frac{1}{2}Ax \cdot x + Bx \cdot y + \frac{1}{2}Cy \cdot y.$$

There exists a unique symbol $\widetilde{\sigma} \in X$ such that, for any $f \in \mathcal{S}(\mathbb{R}^d)$:

$$\sigma^{\mathrm{w}} \int_{\mathbb{R}^d} e^{2\pi i \Phi(x,y)} f(y)dy = \int_{\mathbb{R}^d} e^{2\pi i \Phi(x,y)} \widetilde{\sigma}(x,y) f(y) dy. \tag{8.1}$$

Furthermore, the map $\sigma \mapsto \widetilde{\sigma}$ is bounded on X.

Proof. The case $X = M^{\infty,1}(\mathbb{R}^{2d})$ is covered by [CGNR14, Lemma 2.2]. We prove here the claim for $X = M^\infty_{0,s}(\mathbb{R}^{2d})$.

(i) For any non-zero window function $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$ and $S \in \mathrm{Sp}(d,\mathbb{R})$ we have

$$\begin{aligned}
\|\sigma \circ S\|_{M^\infty_{0,s}} &= \sup_{z,\zeta \in \mathbb{R}^{2d}} |\langle \sigma \circ S, M_\zeta T_z \Phi\rangle| v_s(\zeta) \\
&= \sup_{z,\zeta \in \mathbb{R}^{2d}} \left|\left\langle \sigma, M_{(S^{-1})^\top \zeta} T_{Sz}\big(\Phi \circ S^{-1}\big)\right\rangle\right| v_s(\zeta) \\
&= \sup_{z,\zeta \in \mathbb{R}^{2d}} \left|\left\langle \sigma, M_\zeta T_z \big(\Phi \circ S^{-1}\big)\right\rangle\right| v_s(S^\top \zeta) \\
&\le \left\|S^\top\right\|^s \|V_{\Phi\circ S^{-1}}\sigma\|_{M^\infty_{0,s}} \\
&\lesssim \|S\|^s \|V_{\Phi \circ S^{-1}}\Phi\|_{L^1_s} \|\sigma\|_{M^\infty_{0,s}},
\end{aligned}$$

where we used the estimate $v_s(S^\top\zeta) \le \|S^\top\|^s v_s(\zeta)$ (here $\|\mathcal{B}\|$ denotes the operator norm of the matrix $\mathcal{B}$) and the change-of-window formula (3.3).

We now prove the uniformity with respect to the parameter t, when $S = S_t$. The subset $\{S_t : t \in [-T,T]\} \subset \mathrm{Sp}(d,\mathbb{R})$ is bounded and thus $\|S_t\| \le C_1(T)$.

Furthermore, $\left\{V_{\Phi\circ S_t^{-1}}\Phi \,:\, t\in[-T,T]\right\}$ is a bounded subset of $\mathcal{S}(\mathbb{R}^{2d})$ (this follows at once by inspecting the Schwartz seminorms of $\Phi\circ S_t^{-1}$), hence $\|V_{\Phi\circ S^{-1}}\Phi\|_{L^1_s}\le C_2(T)$.

(ii) The proof is similar to that of the case $X=M^{\infty,1}(\mathbb{R}^{2d})$ in [CGNR14, Proposition 5.2]. In particular, $\widetilde{\sigma}$ is explicitly derived from σ as follows: $\widetilde{\sigma}=\mathcal{U}_2\mathcal{U}\mathcal{U}_1\sigma$, where $\mathcal{U},\mathcal{U}_1,\mathcal{U}_2$ are the mappings

$$\mathcal{U}_1\sigma(x,y)=\sigma(x,y+Ax),\quad \mathcal{U}_2\sigma(x,y)=\sigma(x,B^\top y),\quad \widehat{\mathcal{U}\sigma}(\xi,\eta)=e^{\pi i\xi\cdot\eta}\widehat{\sigma}(\xi,\eta).$$

$\mathcal{U}_1$ and $\mathcal{U}_2$ are isomorphisms of $M^\infty_{0,s}(\mathbb{R}^{2d})$, as a consequence of the previous item. For what concerns $\mathcal{U}$, an inspection of the proof of [Grö01, Corollary 14.5.5] shows that any modulation space $M^{p,q}_{0,s}(\mathbb{R}^{2d})$ is invariant under the action of $\mathcal{U}$; alternatively, the boundedness of the Fourier multiplier $\mathcal{U}$ can be inferred from Proposition 4.1.1 since $\mu(x,\xi)=e^{-\pi i x\cdot\xi}$ belongs to $W^{1,\infty}(\mathbb{R}^{2d})$ - cf. Lemma 5.4.7. □

We prove now an easy result on exponentials in Banach algebras. Recall the notation introduced in Remark 2.1.1.

Lemma 8.1.3. *Let $(A,\star)$ be a complex Banach algebra with unit $\mathbb{1}$ and consider $a\in A$. For any real t and integer $n\ge 1$ we have*

$$e^{-i\frac{t}{n}a}:=\sum_{k=0}^{\infty}\left(-i\frac{t}{n}\right)^k\frac{a^k}{k!}=\mathbb{1}+i\frac{t}{n}a_0,$$

where $a_0\in A$ and the following estimate holds:

$$\|a_0\|\le\|a\|e^{|t|\|a\|}.$$

Proof. It is enough to set

$$a_0:=-\sum_{k=0}^{\infty}\left(-i\frac{t}{n}\right)^k\frac{a^{k+1}}{(k+1)!}.$$

The desired identity is clearly satisfied and we can estimate the norm of a_0 as follows:

$$\begin{aligned}\|a_0\|&\le\|a\|\left(\sum_{k=0}^{\infty}\frac{|t|^k\|a\|^k}{(k+1)!}\right)\\&=\frac{1}{|t|}\left(e^{|t|\|a\|}-1\right)\le\|a\|e^{|t|\|a\|}.\end{aligned}$$

□

We will repeatedly make use the following result; the proof is an easy consequence of the previous lemma.

Corollary 8.1.4. *Let $X \subset \mathcal{S}'(\mathbb{R}^{2d})$ be a space of symbols such that $(X, \#)$ is a Banach algebra under the Weyl product and* $\mathrm{op_w}(X)$ *be a subalgebra of $\mathcal{L}(L^2(\mathbb{R}^d))$ under composition. The Weyl quantization* $\mathrm{op_w} : X \to \mathcal{L}(L^2(\mathbb{R}^d))$ *is a homomorphism of Banach algebras. In particular, for any $\sigma \in X$, $t \in \mathbb{R}$ and $n \in \mathbb{N}$ we have*

$$e^{-i\frac{t}{n}\sigma^{\mathrm{w}}} = (e^{-i\frac{t}{n}\sigma})^{\mathrm{w}} = I + i\frac{t}{n}\sigma_0^{\mathrm{w}},$$

where $\sigma_0 \in X$ satisfies

$$\|\sigma_0\|_X \leq \|\sigma\|_X e^{|t|\|\sigma\|_X}.$$

8.2 Proof of the main results

Recall that we are dealing with the perturbed problem (1.52), namely

$$\begin{cases} i\partial_t\psi(t,x) = (H_0+V)\psi(t,x) \\ \psi(0,x) = f(x) \end{cases}, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^{2d},$$

where $H_0 = Q^{\mathrm{w}}$ is the Weyl quantization of a real quadratic form on $\mathbb{R}^{2d}$ and $V = \sigma^{\mathrm{w}}$ is a Weyl operator with symbol in a suitable class. The Trotter formula in Theorem 1.5.1 holds: if $U(t) = e^{-it(H_0+V)}$ denotes the evolution operator associated with (1.52), then

$$U(t)\psi = \lim_{n\to\infty} E_n(t)\psi \quad \forall \psi \in L^2(\mathbb{R}^d),$$

where Feynman-Trotter propagators $E_n(t)$ are defined by

$$E_n(t) \coloneqq \left(e^{-i\frac{t}{n}H_0}e^{-i\frac{t}{n}V}\right)^n, \quad n \in \mathbb{N}.$$

Finally, recall that $e_{n,t}(x,y)$ is the distribution kernel of $E_n(t)$, whereas $u_t(x,y)$ is that of $U(t) = e^{-it(H_0+V)}$.

8.2.1 Proof of Theorem 1.6.1

We are assuming $V = \sigma^{\mathrm{w}}$ with $\sigma \in M^{\infty}_{0,s}(\mathbb{R}^{2d})$, with $s > 2d$. The proof will be carried on for $t > 0$, since the case $t < 0$ is similar. Actually, the upper-right block of the matrix $S_{-t} = S_t^{-1}$ is $-B_t^{\top}$ (cf. Proposition 4.3.1), hence $\det B_t \neq 0$ if and only if $\det B_{-t} \neq 0$.

We start from the Trotter formula (1.41). In view of Theorem 4.2.5 and Corollary 8.1.4 we have

$$E_n(t) = \left(e^{-i\frac{t}{n}H_0}e^{-i\frac{t}{n}V}\right)^n = \left(e^{-i\frac{t}{n}H_0}\left(I + i\frac{t}{n}\sigma_0^{\mathrm{w}}\right)\right)^n$$

for a suitable $\sigma_0 = (\sigma_0)_{n,t} \in M^\infty_{0,s}(\mathbb{R}^{2d})$ satisfying

$$\|\sigma_0\|_{M^\infty_{0,s}} \le C(t) \tag{8.2}$$

for some constant $C(t) > 0$ independent of n.

We use the symplectic covariance of Weyl calculus, that is we apply (4.9) repeatedly, and the fact that $e^{isH_0}e^{-isH_0} = I$ for any $s \in \mathbb{R}$ so that the ordered product of operators in $E_n(t)$ can be expanded as

$$\begin{aligned} E_n(t) &= \left[\prod_{k=1}^n \left(I + i\frac{t}{n}\left(\sigma_0 \circ S_{-k\frac{t}{n}}\right)^{\mathrm{w}}\right)\right] e^{-itH_0} \\ &= a^{\mathrm{w}}_{n,t}\, \mu(S_t), \end{aligned}$$

where, for any t and $n \ge 1$,

$$\begin{aligned} \|a_{n,t}\|_{M^\infty_{0,s}} &= \left\| \prod_{k=1}^n \left(1 + i\frac{t}{n}\left(\sigma_0 \circ S_{-k\frac{t}{n}}\right)\right)\right\|_{M^\infty_{0,s}} \\ &\le \prod_{k=1}^n \left(1 + \frac{t}{n}\|\sigma_0 \circ S_{-k\frac{t}{n}}\|_{M^\infty_{0,s}}\right), \end{aligned}$$

where, for any t and $n \ge 1$,

$$\begin{aligned} \|a_{n,t}\|_{M^\infty_{0,s}} &= \left\| \prod_{k=1}^n \left(1 + i\frac{t}{n}\left(\sigma_0 \circ S_{-k\frac{t}{n}}\right)\right)\right\|_{M^\infty_{0,s}} \\ &\le \prod_{k=1}^n \left(1 + \frac{t}{n}\|\sigma_0 \circ S_{-k\frac{t}{n}}\|_{M^\infty_{0,s}}\right), \end{aligned}$$

where in the first product symbol we mean the Weyl product # of symbols - cf. Section 4.2 and Remark 2.1.1. By Lemma 8.1.2 applied with $T = t$ and (8.2), we then have

$$\|a_{n,t}\|_{M^\infty_{0,s}} \le \left(1 + \frac{t}{n}C(t)\right)^n \le e^{C(t)t}, \tag{8.3}$$

for some new locally bounded constant $C(t) > 0$ independent of n.

Since S_t is a free symplectic matrix precisely for $t \in \mathbb{R} \setminus \mathfrak{E}$, by (4.5) and (8.1) we explicitly have

$$\begin{aligned} E_n(t)\psi(x) &= a^{\mathrm{w}}_{n,t}\,\mu(S_t)\psi(x) \\ &= c(t)|\det B_t|^{-1/2} \int_{\mathbb{R}^d} e^{2\pi i \Phi_t(x,y)} \widetilde{a_{n,t}}(x,y)\psi(y)dy, \end{aligned}$$

where Φ_t is given in (1.51) and $c(t) \in \mathbb{C}$ is such that $|c(t)| = 1$.

Therefore, we managed to write $E_n(t)$ as an integral operator (defined up to a sign) with kernel

$$e_{n,t}(x,y) = c(t)|\det B_t|^{-1/2} e^{2\pi i \Phi_t(x,y)} \widetilde{a_{n,t}}(x,y),$$

Now, consider the integral kernel u_t of the propagator $U(t) = e^{-it(H_0+V)}$ and define for consistency $\widetilde{a_t} \in \mathcal{S}'(\mathbb{R}^{2d})$ in such a way that

$$u_t(x,y) = c(t)|\det B_t|^{-1/2} e^{2\pi i \Phi_t(x,y)} \widetilde{a_t}(x,y).$$

Since we know by the usual Trotter formula (1.41) that for any fixed t

$$\|E_n(t)\psi - U(t)\psi\|_{L^2} \to 0, \qquad \forall \psi \in L^2(\mathbb{R}^d),$$

we have $E_n(t) \to U(t)$ in $\mathcal{L}_{\mathrm{s}}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$, because $\mathcal{S}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$. As a consequence of Proposition 8.1.1, we get $e_{n,t} \to u_t$ in $\mathcal{S}'(\mathbb{R}^d)$. This is equivalent to

$$\widetilde{a_{n,t}} \to \widetilde{a_t} \quad \text{in } \mathcal{S}'(\mathbb{R}^{2d}).$$

Therefore, for any non-zero $\Psi \in \mathcal{S}(\mathbb{R}^{2d})$ we have pointwise convergence of the corresponding short-time Fourier transforms: for any fixed $(z,\zeta) \in \mathbb{R}^{4d}$,

$$V_\Psi \widetilde{a_{n,t}}(z,\zeta) = \langle \widetilde{a_{n,t}}, M_\zeta T_z \Psi \rangle \to \langle \widetilde{a_t}, M_\zeta T_z \Psi \rangle = V_\Psi \widetilde{a_t}(z,\zeta). \tag{8.4}$$

By (8.3) and Lemma 8.1.2 we see that the sequence $\widetilde{a_{n,t}}$, for any fixed t, is bounded in $M^\infty_{0,s}(\mathbb{R}^{2d})$. Hence, there exists a constant $C = C(t)$ independent of n such that

$$|V_\Psi \widetilde{a_{n,t}}(z,\zeta)| \le C\langle \zeta \rangle^{-s}, \qquad \forall z,\zeta \in \mathbb{R}^{2d}. \tag{8.5}$$

Combining this estimate with (8.4) immediately yields $\widetilde{a_t} \in M^\infty_{0,s}(\mathbb{R}^{2d})$ as well, hence the first claim of Theorem 1.6.1.

For the remaining part, we argue as follows: choose a non-zero window $\Psi \in C_c^\infty(\mathbb{R}^{2d})$ and set $\Theta \in C_c^\infty(\mathbb{R}^{2d})$ with $\Theta = 1$ on $\operatorname{supp}\Psi$; for any fixed $z \in \mathbb{R}^{2d}$ and $0 < r < s - 2d$, we have

$$\begin{aligned} \big\|\mathcal{F}\big[(e_{n,t} - u_t)\overline{T_z\Psi}\big]\big\|_{L^1_r} &= |\det B_t|^{-1/2} \big\|\mathcal{F}\big[e^{2\pi i\Phi_t}(\widetilde{a_{n,t}} - \widetilde{a_t})\overline{T_z\Psi}\big]\big\|_{L^1_r} \\ &= |\det B_t|^{-1/2} \big\|\mathcal{F}\big[\big(T_z\Theta e^{2\pi i\Phi_t}\big)(\widetilde{a_{n,t}} - \widetilde{a_t})\overline{T_z\Psi}\big]\big\|_{L^1_r} \\ &= |\det B_t|^{-1/2} \big\|\mathcal{F}\big[T_z\Theta e^{2\pi i\Phi_t}\big] * \mathcal{F}\big[(\widetilde{a_{n,t}} - \widetilde{a_t})\overline{T_z\Psi}\big]\big\|_{L^1_r} \\ &\lesssim |\det B_t|^{-1/2} \big\|\mathcal{F}\big[T_z\Theta e^{2\pi i\Phi_t}\big]\big\|_{L^1_r} \big\|\mathcal{F}\big[(\widetilde{a_{n,t}} - \widetilde{a_t})\overline{T_z\Psi}\big]\big\|_{L^1_r}, \end{aligned}$$

where the convolution inequality in the last step is an easy consequence of Peetre's inequality (2.1).

Clearly, $T_z\Theta e^{2\pi i\Psi_t} \in C_c^\infty(\mathbb{R}^{2d})$, while

$$\left\|\mathcal{F}\left[(\widetilde{a_{n,t}}-\widetilde{a}_t)\overline{T_z\Psi}\right]\right\|_{L^1_r} \to 0$$

by dominated convergence, using (8.4) and

$$\begin{aligned}|\mathcal{F}\left[(\widetilde{a_{n,t}}-\widetilde{a}_t)\overline{T_z\Psi}\right]|\langle\zeta\rangle^r &= |V_\Psi(\widetilde{a_{n,t}}-\widetilde{a}_t)(z,\zeta)|\langle\zeta\rangle^r\\ &\le C\langle\zeta\rangle^{r-s} \in L^1(\mathbb{R}^{2d}),\end{aligned}$$

because $s-r>2d$, where in the last inequality we used (8.5) and the fact that $\widetilde{a}_t \in M^\infty_{0,s}(\mathbb{R}^{2d})$. This gives the claimed convergence in $(\mathcal{F}L^1_r)_{\mathrm{loc}}(\mathbb{R}^{2d})$.

To conclude, we have that

$$\left\|(e_{n,t}-u_t)\overline{T_z\Psi}\right\|_{L^\infty} \le \|V_\Psi(e_{n,t}-u_t)(z,\cdot)\|_{L^1} \to 0,$$

and in particular this yields uniform convergence on compact subsets: for any compact $K\subset\mathbb{R}^{2d}$, choose $\Psi\in\mathcal{S}(\mathbb{R}^{2d})$, $\Psi=1$ on K.

8.2.2 Proof of Corollary 1.6.2

The proof of Corollary 1.6.2 is then immediate, since $C_{\mathrm{b}}^\infty(\mathbb{R}^{2d}) = \bigcap_{s\ge 0} M^\infty_{0,s}(\mathbb{R}^{2d})$ and

$$C^\infty(\mathbb{R}^{2d}) = \bigcap_{r>0}\left(\mathcal{F}L^1_r\right)_{\mathrm{loc}}(\mathbb{R}^{2d}). \tag{8.6}$$

The latter characterization is folklore, being a refinement of the standard decay-smoothness trade-off for the Fourier transform; we provide a sketch of the proof for the sake of completeness.

Proof of (8.6). The inclusion $C^\infty \subset \bigcap_{r>0}(\mathcal{F}L^1_r)_{\mathrm{loc}}$ is straightforward. Indeed, let $f\in C^\infty$ and consider arbitrary $r>0$ and $\phi\in C_{\mathrm{c}}^\infty$; then $f\phi\in C_{\mathrm{c}}^\infty$ and

$$\|f\phi\|_{\mathcal{F}L^1_r} = \int_{\mathbb{R}^{2d}} |\mathcal{F}(f\phi)(\zeta)|v_r(\zeta)d\zeta \lesssim_N \int_{\mathbb{R}^{2d}} (1+|\zeta|)^{-N+r}d\zeta,$$

and the latter quantity is finite for $N\in\mathbb{N}$ large enough.

Conversely, assume $f\in(\mathcal{F}L^1_r)_{\mathrm{loc}}$ for any $r>0$ and let $\phi\in C_{\mathrm{c}}^\infty$. For any $\alpha\in\mathbb{N}$, the distribution derivative $\partial^\alpha(f\phi)$ is the (inverse) Fourier transform of a function in $\mathcal{F}L^1$; indeed,

$$\left|\int_{\mathbb{R}^{2d}} e^{2\pi i z\cdot\zeta}(2\pi i\zeta)^\alpha\mathcal{F}(f\phi)(\zeta)d\zeta\right| \lesssim \int_{\mathbb{R}^{2d}} |\mathcal{F}(f\phi)(\zeta)|(1+|\zeta|)^{|\alpha|}d\zeta,$$

and the latter quantity is finite by the assumption with $r=|\alpha|$. ☐

8.2.3 Proof of Theorem 1.6.3

We now assume $V = \sigma^{\mathrm{w}}$ with $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$. Therefore for an arbitrary $\epsilon > 0$, Proposition 3.2.19 allows us to write $\sigma = \sigma_1 + \sigma_2$, with $\sigma_1 \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^{2d})$ and $\sigma_2 \in M^{\infty,1}(\mathbb{R}^{2d})$ with $\|\sigma_2\|_{M^{\infty,1}} \le \epsilon$ and clearly

$$\|\sigma_1\|_{M^{\infty,1}} \le \|\sigma\|_{M^{\infty,1}} + \|\sigma_2\|_{M^{\infty,1}} \le \|\sigma\|_{M^{\infty,1}} + \epsilon \le 1 + \|\sigma\|_{M^{\infty,1}},$$

assuming, from now on, $\epsilon \le 1$. Notice that

$$\begin{aligned} e^{-i\frac{t}{n}\left(\sigma_1^{\mathrm{w}}+\sigma_2^{\mathrm{w}}\right)} &= I + \sum_{k=1}^{\infty} \frac{1}{k!}\left(-i\frac{t}{n}\right)^k (\sigma_1^{\mathrm{w}} + \sigma_2^{\mathrm{w}})^k \\ &= I + i\frac{t}{n}(\sigma_1')^{\mathrm{w}} + i\frac{t}{n}(\sigma_2')^{\mathrm{w}}, \end{aligned}$$

where we set

$$(\sigma_1')^{\mathrm{w}} = -\sum_{k=1}^{\infty} \frac{1}{k!}\left(-i\frac{t}{n}\right)^{k-1} (\sigma_1^{\mathrm{w}})^k,$$

$$(\sigma_2')^{\mathrm{w}} = -\sum_{k=1}^{\infty} \frac{1}{k!}\left(-i\frac{t}{n}\right)^{k-1} ((\sigma_1^{\mathrm{w}} + \sigma_2^{\mathrm{w}})^k - (\sigma_1^{\mathrm{w}})^k).$$

Now, fix once for all $s > 2d$. The norms of the symbols $\sigma_1' = \sigma_{1,n,t}'$ and $\sigma_2' = \sigma_{2,n,t}'$ can be estimated as follows for any $t > 0$ (cf. the proof of Lemma 8.1.3). We have

$$\|\sigma_1'\|_{M^{\infty,1}} \le \|\sigma_1\|_{M^{\infty,1}} e^{t\|\sigma_1\|_{M^{\infty,1}}} \le (1 + \|\sigma\|_{M^{\infty,1}}) e^{t\left(1+\|\sigma\|_{M^{\infty,1}}\right)} =: C_1(t), \quad (8.7)$$

$$\|\sigma_1'\|_{M_{0,s}^{\infty}} \le \|\sigma_1\|_{M_{0,s}^{\infty}} e^{t\|\sigma_1\|_{M_{0,s}^{\infty}}} =: C_2(t,\epsilon). \quad (8.8)$$

Similarly, using the elementary inequality

$$(a+b)^k - a^k \le kb(a+b)^{k-1}, \qquad a, b \ge 0,\ k \ge 1,$$

we obtain

$$\|\sigma_2'\|_{M^{\infty,1}} \le \|\sigma_2\|_{M^{\infty,1}} e^{t\left(\|\sigma_1\|_{M^{\infty,1}} + \|\sigma_2\|_{M^{\infty,1}}\right)} \le \epsilon e^{t\left(2+\|V\|_{M^{\infty,1}}\right)} =: \epsilon\, C_3(t). \quad (8.9)$$

Here $C_1(t)$ and $C_3(t)$ are independent of n and ϵ and $C_2(t,\epsilon)$ is independent of n. The approximate propagator $E_n(t)$ thus becomes

$$\begin{aligned} E_n(t) &= \left(e^{-i\frac{t}{n}H_0} e^{-i\frac{t}{n}\left(\sigma_1^{\mathrm{w}}+\sigma_2^{\mathrm{w}}\right)}\right)^n \\ &= \left(e^{-i\frac{t}{n}H_0}\left(1 + i\frac{t}{n}(\sigma_1')^{\mathrm{w}} + i\frac{t}{n}(\sigma_2')^{\mathrm{w}}\right)\right)^n, \end{aligned}$$

and similar arguments to those of the previous section yield

$$\begin{aligned} E_n(t) &= \left[\prod_{k=1}^{n}\left(I + i\frac{t}{n}\Big(\sigma_1'\circ S_{-k\frac{t}{n}}\Big)^{\mathrm{w}} + i\frac{t}{n}\Big(\sigma_2'\circ S_{-k\frac{t}{n}}\Big)^{\mathrm{w}}\right)\right]e^{-itH_0} \\ &= \big[a^{\mathrm{w}}_{n,t} + b^{\mathrm{w}}_{n,t}\big]\mu(S_t), \end{aligned}$$

where we set

$$a_{n,t} = \prod_{k=1}^{n}\left(1 + i\frac{t}{n}\Big(\sigma_1'\circ S_{-k\frac{t}{n}}\Big)\right),$$

and in the latter product we mean the Weyl product # of symbols.

The term $a^{\mathrm{w}}_{n,t}$ can be estimated as in the proof of Theorem 1.6.1; in particular, using (8.8), we get (cf. (8.3))

$$\|a_{n,t}\|_{M^{\infty}_{0,s}} \le C(t,\epsilon). \tag{8.10}$$

In order to estimate the $M^{\infty,1}$ norm of the remainder $b_{n,t}$, it is useful the following result, which can be easily proved by induction on n.

Lemma 8.2.1. *Let A be a Banach algebra. For any $u_1,\dots,u_n,v_1,\dots,v_n \in A$, with $\|u_i\| \le R$ and $\|v_i\| \le S$ for any $i = 1,\dots,n$ and some $R, S > 0$, and setting $w_k = u_k + v_k$, we have*

$$\prod_{k=1}^{n}(u_k + v_k) = u_1u_2\dots u_n + z_n,$$

where

$$z_n = v_1w_2\dots w_n + u_1v_2w_3\dots w_n + \dots + u_1u_2\dots u_{n-2}v_{n-1}w_n + u_1u_2\dots u_{n-1}v_n,$$

and therefore

$$\|z_n\| \le nS(R+S)^{n-1}.$$

Setting

$$u_k = 1 + i\frac{t}{n}\Big(\sigma_1'\circ S_{-k\frac{t}{n}}\Big), \qquad v_k = i\frac{t}{n}\Big(\sigma_2'\circ S_{-k\frac{t}{n}}\Big), \quad k = 1,\dots,n,$$

and applying Lemma 8.1.2 with $T = t$, and (8.7) and (8.9), we get

$$\|u_k\|_{M^{\infty,1}} = \left\|1 + i\frac{t}{n}\Big(\sigma_1'\circ S_{-k\frac{t}{n}}\Big)\right\|_{M^{\infty,1}} \le 1 + \frac{t}{n}\big\|\sigma_1'\circ S_{-k\frac{t}{n}}\big\|_{M^{\infty,1}} \le 1 + \frac{t}{n}C(t),$$

$$\|v_k\|_{M^{\infty,1}} = \frac{t}{n}\big\|\sigma_2' \circ S_{-k\frac{t}{n}}\big\|_{M^{\infty,1}} \le \frac{t}{n}C(t)\epsilon,$$

for some locally bounded constant $C(t) > 0$ independent of n and ϵ. Therefore, by Lemma 8.2.1,

$$\|b_{n,t}\|_{M^{\infty,1}} \le n\frac{t}{n}C(t)\epsilon\Big(1 + 2\frac{t}{n}C(t)\Big)^{n-1} \le \epsilon t C(t) e^{2tC(t)}. \tag{8.11}$$

Following the pathway of the proof of Theorem 1.6.1, we write $E_n(t)$ as an integral operator with kernel

$$\begin{aligned} e_{n,t}(x,y) &= c(t)|\det B_t|^{-1/2} e^{2\pi i \Phi_t(x,y)}\big(\widetilde{a_{n,t}} + \widetilde{b_{n,t}}\big)(x,y) \\ &= c(t)|\det B_t|^{-1/2} e^{2\pi i \Phi_t(x,y)} k_{n,t}(x,y), \end{aligned}$$

that is $k_{n,t} = \widetilde{a_{n,t}} + \widetilde{b_{n,t}}$, and the Trotter formula (1.41) combined with Proposition 8.1.1 imply that $k_{n,t} \to k_t$ in $\mathcal{S}'(\mathbb{R}^{2d})$, where the distribution k_t is conveniently introduced to rephrase the integral kernel u_t of the propagator $U(t) = e^{-it(H_0+V)}$ as

$$u_t(x,y) = c(t)|\det B_t|^{-1/2} e^{2\pi i \Phi_t(x,y)} k_t(x,y).$$

By repeating this argument with $\sigma_2 = 0$ (hence $\widetilde{b_{n,t}} = 0$ and $k_{n,t} = \widetilde{a_{n,t}}$) we see that $\widetilde{a_{n,t}}$ converges in $\mathcal{S}'(\mathbb{R}^{2d})$ as well, hence $\widetilde{b_{n,t}}$ converges in $\mathcal{S}'(\mathbb{R}^{2d})$ by difference. Therefore, for any non-zero $\Psi \in \mathcal{S}(\mathbb{R}^{2d})$ the functions $\sigma_\Psi \widetilde{a_{n,t}}$ and $\sigma_\Psi \widetilde{b_{n,t}}$ converge pointwise in $\mathbb{R}^{4d}$.

We need a technical lemma at this point.

Lemma 8.2.2. *Let F_n and G_n be two sequences of complex-valued functions on $\mathbb{R}^{2d}$ such that $F_n \to F$, $G_n \to G$ pointwise, and assume $|F_n| \le H \in L^1(\mathbb{R}^{2d})$ and $\|G_n\|_{L^1} \le \epsilon$ for any $n \in \mathbb{N}$. Then,*

$$\limsup_{n\to\infty} \|F_n + G_n - (F+G)\|_{L^1} \le 2\epsilon.$$

Proof. First, notice that $\|G\|_{L^1} \le \epsilon$ by Fatou's lemma. Now,

$$\|F_n + G_n - (F+G)\|_{L^1} \le \|F_n - F\|_{L^1} + \|G_n - G\|_{L^1},$$

where the first term on the right-hand side goes to zero by dominated convergence, while for the other one we have $\|G_n - G\|_{L^1} \le 2\epsilon$. The desired conclusion is then immediate. ☐

For any fixed $z \in \mathbb{R}^{2d}$, set $F_n(\zeta) = \sigma_\Psi \widetilde{a_{n,t}}(z,\zeta)$ and $G_n(\zeta) = \sigma_\Psi \widetilde{b_{n,t}}(z,\zeta)$. By Lemma 8.1.2 and (8.10) we have

$$\sup_{\zeta\in\mathbb{R}^{2d}} \langle\zeta\rangle^s |F_n(\zeta)| \lesssim \|\widetilde{a_{n,t}}\|_{M^\infty_{0,s}} \lesssim \|a_{n,t}\|_{M^\infty_{0,s}} \le C(t,\epsilon).$$

Similarly, by Lemma 8.1.2 and (8.11),

$$\|G_n\|_{L^1} \lesssim \|\widetilde{b_{n,t}}\|_{M^{\infty,1}} \lesssim \|b_{n,t}\|_{M^{\infty,1}} \leq \epsilon\, C(t).$$

These estimates yield two results: on the one hand, the first claim of Theorem 1.6.3 is proved. On the other hand, the assumptions of Lemma 8.2.2 are satisfied: we have $(F_n + G_n)(\zeta) = \sigma_\Psi k_{n,t}(z,\zeta)$ and $(F + G)(\zeta) = \sigma_\Psi k_t(z,\zeta)$, and therefore we obtain

$$\limsup_{n\to\infty} \big\|\mathcal{F}\big[(k_{n,t} - k_t)\overline{T_z\Psi}\big]\big\|_{L^1} \leq 2\epsilon\, C(t).$$

Since ϵ can be made arbitrarily small and the left-hand side is independent of ϵ, we conclude that

$$\lim_{n\to\infty} \big\|\mathcal{F}\big[(k_{n,t} - k_t)\overline{T_z\Psi}\big]\big\|_{L^1} = 0,$$

in particular $k_{n,t} \to k_t$ in $(\mathcal{F}L^1)_{\mathrm{loc}}(\mathbb{R}^{2d})$.

Finally, with the help of a suitable bump function Θ as in the preceding section, for any fixed $z \in \mathbb{R}^{2d}$ we infer

$$\big\|\mathcal{F}\big[(e_{n,t} - u_t)\overline{T_z\Psi}\big]\big\|_{L^1} \leq |\det B_t|^{-1/2} \big\|\mathcal{F}\big[\big(T_z\Theta e^{2\pi i\Phi_t}\big)\big]\big\|_{L^1} \big\|\mathcal{F}\big[(k_{n,t} - k_t)\overline{T_z\Psi}\big]\big\|_{L^1},$$

and thus

$$\big\|\mathcal{F}\big[(e_{n,t} - u_t)\overline{T_z\Psi}\big]\big\|_{L^1} \to 0.$$

This gives $e_{n,t} \to u_t$ in $(\mathcal{F}L^1)_{\mathrm{loc}}(\mathbb{R}^{2d})$ and therefore uniformly on compact subsets of $\mathbb{R}^{2d}$.

8.3 Convergence at exceptional times

Let us commence this section on convergence results for integral kernels at exceptional times with a general result for the kernels of strongly convergent sequences of operators in L^2.

Theorem 8.3.1. *Let $\{A_n\} \subset \mathcal{L}(L^2(\mathbb{R}^d))$, $n \in \mathbb{N}$, be a sequence of bounded linear operators on $L^2(\mathbb{R}^d)$ with associated distribution kernels $\{a_n\} \subset \mathcal{S}'(\mathbb{R}^{2d})$, and $A \in \mathcal{L}(L^2(\mathbb{R}^d))$ with distribution kernel $a \in \mathcal{S}'(\mathbb{R}^{2d})$. Assume that $A_n \to A$ in the strong operator topology. Then:*

1. *a_n, $n \in \mathbb{N}$, and a belong to a bounded subset of $M^\infty(\mathbb{R}^{2d})$;*
2. *$a_n \to a$ in the weak-* topology on $M^\infty(\mathbb{R}^{2d})$.*

In particular we have $a_n \to a$ in $(\mathcal{F}L^\infty)_{\mathrm{loc}}(\mathbb{R}^{2d})$, the latter space endowed with the topology $\sigma((\mathcal{F}L^\infty)_{\mathrm{loc}}(\mathbb{R}^{2d}), (\mathcal{F}L^1)_{\mathrm{comp}}(\mathbb{R}^{2d}))$.

Proof. We have that $\{A_n\}$ is a bounded sequence in $\mathcal{L}(L^2(\mathbb{R}^d))$ as a consequence of the uniform boundedness principle, hence also in $\mathcal{L}(M^1(\mathbb{R}^d), M^\infty(\mathbb{R}^d))$. The Feichtinger kernel theorem (Theorem 3.2.15) yields that the kernels a_n belong to a bounded subset of $M^\infty(\mathbb{R}^{2d})$. Similarly, $A \in \mathcal{L}(L^2(\mathbb{R}^d)) \Rightarrow a \in M^\infty(\mathbb{R}^{2d})$. For the second part of the claim we remark that $A_n \to A$ in the strong operator topology implies that $a_n \to a$ in $\mathcal{S}'(\mathbb{R}^{2d})$. Therefore, for any fixed non-zero $g \in \mathcal{S}(\mathbb{R}^d)$ we have $V_g a_n \to V_g a$ pointwise in $\mathbb{R}^{2d}$. Moreover, we have the estimate $|V_g a_n(x,\xi)| \le C$, for some constant $C > 0$ independent of n by the first part of the claim. Hence, for any $\varphi \in M^1(\mathbb{R}^{2d})$ we have

$$\begin{aligned}\langle a_n, \varphi\rangle &= \int_{\mathbb{R}^{2d}} V_g a_n(x,\xi)\overline{V_g\varphi(x,\xi)}dxd\xi \\ &\to \int_{\mathbb{R}^{2d}} V_g a(x,\xi)\overline{V_g\varphi(x,\xi)}dxd\xi = \langle a, \varphi\rangle,\end{aligned}$$

by the dominated convergence theorem. □

It would be interesting to prove the boundedness of a_n in $M^\infty(\mathbb{R}^{2d})$ in Theorem 8.3.1 without using the uniform boundedness principle, although it could be not immediate.

A straightforward application of this result allows us to prove global-in-time convergence of integral kernels, although in a weaker sense than before.

Corollary 8.3.2. *Assume $V \in L^\infty(\mathbb{R}^d)$ or $V = \sigma^{\mathrm{w}}$ with $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$. Let $e_{n,t} \in \mathcal{S}'(\mathbb{R}^{2d})$ be the distribution kernel of the Feynman-Trotter parametrix $E_n(t)$ in* (1.53) *and $u_t \in \mathcal{S}'(\mathbb{R}^{2d})$ be the kernel of the Schrödinger evolution operator $U(t)$ associated with the Cauchy problem* (1.37)*. For any $n \in \mathbb{N}$ and $t \in \mathbb{R}$ we have $e_{n,t}, u \in M^\infty(\mathbb{R}^{2d})$. Moreover, $e_{n,t} \to u_t$ in the weak-* topology on $M^\infty(\mathbb{R}^{2d})$ for any fixed $t \in \mathbb{R}$.*

For more regular potentials we expect that the conclusion of Corollary 8.3.2 can be improved. We are ready to provide a version of the Trotter formula for potentials in $M^{\infty,1}(\mathbb{R}^d)$, with strong convergence on $M^1(\mathbb{R}^d)$.

Proof of Theorem 1.6.5. We prove that $E_n(t) \to U(t)$ strongly in $\mathcal{L}(M^1(\mathbb{R}^d))$; the claim concerning adjoint operators follows by similar arguments since $U(t)^* = U(-t)$ and $E_n(t)^* = \left(e^{i\frac{t}{n}V}e^{i\frac{t}{n}H_0}\right)^n$.

As already observed, we know that the operator H_0 with domain $D(H_0) = \{f \in L^2(\mathbb{R}^d) : H_0 f \in L^2(\mathbb{R}^d)\}$ is self-adjoint [Hör95]. Let $U_0(t) = e^{-itH_0}$ be the corresponding strongly continuous unitary group on $L^2(\mathbb{R}^d)$. The well-posedness of the Schrödinger equation $i\partial_t\psi = H_0\psi$ in $M^1(\mathbb{R}^d)$ (see e.g. [CNR15b]) implies that the restriction of $U_0(t)$ to $M^1(\mathbb{R}^d)$ defines a strongly continuous group on $M^1(\mathbb{R}^d)$, its generator being the restriction of H_0 to the subspace $\{f \in M^1(\mathbb{R}^d) : H_0 f \in M^1(\mathbb{R}^d)\}$, as a consequence of known results on subspace semigroups, cf. [EN06, Chapter 2, Section 2.3]. Since a Weyl operator with symbol in the Sjöstrand class is a bounded operator on $M^1(\mathbb{R}^d)$ by Theorem 4.2.2, the desired result follows from the classical Trotter theorem on Banach spaces [EN06, Corollary 2.7 and Exercise 2.9]. The second part of the claim is just an equivalent formulation of the previous results for the corresponding integral kernels, whereas the last conclusion follows from the continuous embedding $M^1(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, for every $1 \le p \le \infty$. □

Remark 8.3.3. *We expect other improvements of Theorem 8.3.1 to hold in the case where $A_n = E_n(t)$, $A = U(t)$. In particular, convergence result for the corresponding integral kernels could be investigated in the context of mixed modulation spaces and generalized kernel theorems in the spirit of [CN19].*

8.4 Physics at exceptional times

In spite of the attempts to shed light on the nature of exceptional times and the partial results in the previous section, a physical interpretation of exceptional times is still not clear at the moment. This non-trivial question also appears in the form of an enigmatic exercise in the textbook [FH10, Problem 3-1] by Feynman and Hibbs. While dimensional analysis and heuristic arguments may provide some hints, a precise answer still seems to be missing.

We give our contribution to this discussion with a short argument which elucidates the nature of exceptional times in terms of measurable quantities. Recall that $B(u,r)$ denotes the ball with center $u \in \mathbb{R}^d$ and radius $r > 0$ in $\mathbb{R}^d$. Following the custom in physics we adopt below the bra-ket notation, and we identify states with their wave functions in the position representation.

Fix $x_0, y_0 \in \mathbb{R}^d$ and $a, b > 0$, and consider the normalised wave-packets

$$|A\rangle = \frac{1}{\sqrt{|B(y_0,a)|}} 1_{B(y_0,a)}, \quad |B\rangle = \frac{1}{\sqrt{|B(x_0,b)|}} 1_{B(x_0,b)}.$$

The corresponding transition amplitude from the state $|A\rangle$ to $|B\rangle$ under the Hamiltonian $H = H_0 + V$ as in Theorem 1.6.3, namely

$$I = I(t, x_0, y_0, a, b) = \langle B|U(t)|A\rangle, \quad t \in \mathbb{R},$$

trivially satisfies the estimate

$$|I(t, x_0, y_0, a, b)| \le 1, \quad \forall t \in \mathbb{R},\ x_0, y_0 \in \mathbb{R}^d,\ a, b > 0.$$

This bound cannot be improved at exceptional times: consider for instance the case where $t = 0$, $x_0 = y_0$ and $a = b$, which yields $I = 1$. Nevertheless, we have the following result.

Proposition 8.4.1. *Under the same assumptions of Theorem 1.6.3, for all* $t \in \mathbb{R} \setminus \mathfrak{E}$ *and* $x_0, y_0 \in \mathbb{R}^d$ *we have*

$$\lim_{a,b\to 0} \frac{I(t, x_0, y_0, a, b)}{(ab)^{d/2}} = C\overline{u_t(x_0, y_0)},$$

where $C = C(d) = |B(0, 1)|$.

Proof. An explicit computation yields

$$\frac{I(t, x_0, y_0, a, b)}{C(ab)^{d/2}} = \frac{1}{C^2(ab)^d} \int_{B(x_0,b)} \int_{B(y_0,a)} \overline{u_t(x, y)} dy dx,$$

and the conclusion follows by the continuity of $u_t(x, y)$ in $\mathbb{R}^{2d}$, because $u_t \in \left(\mathcal{F}L^1\right)_{\mathrm{loc}}(\mathbb{R}^{2d})$ for $t \in \mathbb{R} \setminus \mathfrak{E}$ by Theorem 1.6.3. □

This result shows that while $|I| \leq 1$ in general, for a non-exceptional time $t \in \mathbb{R} \setminus \mathfrak{E}$ we have that $|I| \sim (ab)^{d/2}$ as $a, b \to 0$. In particular $|I| \to 0$ as $a, b \to 0$ except (possibly) for exceptional times.

Chapter 9

Approximation of Feynman Path Integrals with non-smooth Potentials

9.1 Short-time action and related estimates

We begin with a brief motivational discussion devoted to explain the structure of formula (1.55) for the approximate action $S^{(N)}(t,s,x,y)$. We refer to [Gos17, Section 4.5] for more details.

It is well known (see [FH10, Section 2.1] and also [FM13; LL76]) that for a classical Hamiltonian

$$H(x,\xi,t)=\frac{1}{2}\xi^2+V(t,x),$$

the action $S(t,s,x,y)$ satisfies the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t}+\frac{1}{2}|\nabla_x S|^2+V(t,x)=0.$$

In order for $\widetilde{E}^{(N)}$ to be a parametrix in a sense to be specified (cf. (9.7) and (9.8) below), we consider the slightly modified equation

$$\frac{\partial S}{\partial t}+\frac{1}{2}|\nabla_x S|^2+V(t,x)+\frac{i\hbar d}{2(t-s)}-\frac{i\hbar}{2}\Delta_x S=0,$$

and look for a solution S in the form $S(t,s,x,y)=\frac{|x-y|^2}{2(t-s)}+R(t,s,x,y)$, $s<t$. This yields an equivalent equation for R, namely

$$\frac{\partial R}{\partial t}+\frac{1}{2}|\nabla_x R|^2+V(t,x)+\frac{1}{t-s}(x-y)\cdot\nabla_x R-\frac{i\hbar}{2}\Delta_x R=0.$$

Assume that

$$R(t,s,x,y) = W_0 + W_1(s,x,y)(t-s) + W_2(s,x,y)(t-s)^2 + \dots,$$

where the functions $W_k(s,x,y)$ will be briefly denoted by $W_k(x,y)$ from now on. We immediately find $W_0 = 0$ and, for $k \geq 1$, by equating to 0 the coefficient of the term $(t-s)^{k-1}$, we obtain the equations

$$kW_k(x,y) + (x-y)\cdot\nabla_x W_k(x,y) = F_k(s,x,y), \tag{9.1}$$

where we set

$$F_k(s,x,y) = -\frac{1}{2}\sum_{\substack{j+\ell=k-1\\ j\geq 1,\ell\geq 1}} \nabla_x W_j \cdot \nabla_x W_\ell - \frac{1}{(k-1)!}\partial_t^{k-1} V(s,x) + \frac{i\hbar}{2}\Delta_x W_{k-1}. \tag{9.2}$$

For brevity we also write $F_k(x,y)$ in place of $F_k(s,x,y)$.

Lemma 9.1.1. *Suppose F_k in* (9.1) *is continuous as a function of $(x,y)\in\mathbb{R}^{2d}$. Then there exists a unique continuous solution of* (9.1)*, namely*

$$W_k(x,y) = \int_0^1 \tau^{k-1} F_k(\tau x + (1-\tau)y, y)\,d\tau. \tag{9.3}$$

Proof. According to the methods of characteristics, along the curves of type $x_u(\lambda) = y + ue^{\lambda}$, where $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}^d$ has unitary norm, the original PDE (9.1) becomes a linear ODE with respect to the variable λ:

$$\frac{d}{d\lambda} W_k(x_u(\lambda), y) + kW_k(x_u(\lambda), y) = F_k(x_u(\lambda), y),$$

whose solutions are given by

$$W_k(x_u(\lambda), y) = e^{-k\lambda}\left(\int_{-\infty}^{\lambda} e^{k\sigma} F_k(x_u(\sigma), y)\,d\sigma + C\right),$$

where $C \in \mathbb{R}$ is an arbitrary constant. Notice that $\lambda = \log|x-y|$ and the change of variable $\sigma = \log(|x-y|\tau)$ thus gives

$$W_k(x,y) = \int_0^1 \tau^{k-1} F_k(\tau x + (1-\tau)y, y)\,d\tau + \frac{C}{|x-y|^k}.$$

It is therefore clear that the unique continuous solution corresponds to $C=0$. □

We now assume that V satisfies Assumption ($\tilde{\mathrm{A}}$) and we prove that we can then solve the equation (9.1) for $k = 1,\dots,N$ by applying repeatedly Lemma 9.1.1 above.

Proposition 9.1.2. *Let V satisfy Assumption* ($\tilde{\mathrm{A}}$). *Then the equation* (9.1) *has, for any $1 \le k \le N$, a unique solution $W_k(s,x,y)$ satisfying*

$$\|\partial_x^\alpha W_k\|_{M^{\infty,1}(\mathbb{R}^{2d})} \le C, \quad \text{for } |\alpha| \le 2(N-k+1),\ s \in \mathbb{R},$$

for some constant $C > 0$.

Proof. First of all we recall that any function in $M^{\infty,1}$ is continuous. Let us first prove the claim for $k=1$. We have $F_1(s,x,y) = -V(s,x)$. Using Lemma 9.1.1 with $k=1$, the STFT of $\partial_x^\alpha W_1(s,\cdot,\cdot)$, $|\alpha| \le 2N$, can be written as

$$|V_g \partial_x^\alpha W_1(z,\zeta)| = \left| \int_0^1 \tau^{|\alpha|} V_g[\partial_x^\alpha V(s, \tau x + (1-\tau)y)](z,\zeta)\, d\tau \right|, \quad z,\zeta \in \mathbb{R}^{2d}.$$

We now think of V as a function on $\mathbb{R}^{2d}$. More precisely, define

$$V'(s,x,y) = V(s,x), \quad s \in \mathbb{R},\ x,y \in \mathbb{R}^d,$$

and notice that V' still satisfies Assumption ($\tilde{\mathrm{A}}$) with $M^{\infty,1}(\mathbb{R}^d)$ replaced by $M^{\infty,1}(\mathbb{R}^{2d})$.

Let us introduce the parametrized matrices $M_\tau = \begin{bmatrix} \tau I & (1-\tau)I \\ 0 & I \end{bmatrix} \in \mathrm{GL}(2d,\mathbb{R})$, with $\tau \in (0,1]$. We can thus write $V(s,\tau x + (1-\tau)y) = V'(s, M_\tau(x,y))$, and by the behaviour of modulation spaces under dilations (cf. Proposition 3.2.4) we have $\partial_x^\alpha V'(s, M_\tau(x,y)) \in M^{\infty,1}(\mathbb{R}^{2d})$. Therefore,

$$\begin{aligned} \|\partial_x^\alpha W_1\|_{M^{\infty,1}} &\lesssim \int_0^1 \tau^{|\alpha|} \|\partial_x^\alpha V'(s, M_\tau \cdot)\|_{M^{\infty,1}}\, d\tau \\ &\lesssim \left(\int_0^1 \tau^{|\alpha|} C_{\infty,1}(M_\tau)\, d\tau \right) \|\partial_x^\alpha V'\|_{M^{\infty,1}} \le C, \end{aligned}$$

where

$$C_{\infty,1}(M_\tau) = \left(\det(I + M_\tau^\top M_\tau)\right)^{1/2}$$

is a continuous (hence bounded) function of the parameter $\tau \in [0,1]$.

Assume now that the claim holds for any W_j up to a certain $k \le N-1$ and consider

$$|V_g \partial_x^\alpha W_{k+1}(z,\zeta)| = \left| \int_0^1 \tau^{k+|\alpha|} V_g[\partial_x^\alpha F_{k+1}(\tau x + (1-\tau)y, y)](z,\zeta)\, d\tau \right|.$$

It is easy to deduce from (9.2) and the hypothesis on $W_1,\dots,W_k$ that $\partial_x^\alpha F_{k+1}(x,y) \in M^{\infty,1}(\mathbb{R}^{2d})$ whenever $|\alpha| \le 2(N-k)$. Again by Proposition

3.2.4 we have $\partial_x^\alpha F_{k+1}(M_\tau(x,y)) \in M^{\infty,1}(\mathbb{R}^{2d})$, and arguing as before we have

$$\begin{aligned}\|\partial_x^\alpha W_{k+1}\|_{M^{\infty,1}} &\lesssim \int_0^1 \tau^{k+|\alpha|}\|\partial_x^\alpha F_{k+1}(M_\tau\cdot)\|_{M^{\infty,1}}d\tau \\ &\lesssim \left(\int_0^1 \tau^{k+|\alpha|}C_{\infty,1}(M_\tau)d\tau\right)\|\partial_x^\alpha W_{k+1}\|_{M^{\infty,1}} \\ &\le C.\end{aligned}$$

The claim is then proved by induction. □

We now define the approximate generating functions as in (1.55), namely

$$S^{(N)}(t,s,x,y) = \frac{|x-y|^2}{2(t-s)} + R^{(N)}(t,s,x,y),$$

where

$$R^{(N)}(t,s,x,y) \coloneqq \sum_{k=1}^{N} W_k(x,y)(t-s)^k \tag{9.4}$$

and $W_k(x,y)$ is defined in (9.3). In particular, the first-order approximation of the action ($N=1$) is

$$S^{(1)}(t,s,x,y) = \frac{|x-y|^2}{2(t-s)} - (t-s)\int_0^1 V(s,\tau x+(1-\tau)y)d\tau.$$

We conclude this section with a uniform estimate for $e^{\frac{i}{\hbar}R^{(N)}}$ in $M^{\infty,1}$ that will be used below.

Proposition 9.1.3. *If the potential function V satisfies Assumption* (A)*, then $e^{\frac{i}{\hbar}R^{(N)}} \in M^{\infty,1}(\mathbb{R}^{2d})$, with $R^{(N)}$ as in* (9.4)*. More precisely,*

$$\|e^{\frac{i}{\hbar}R^{(N)}}\|_{M^{\infty,1}} \le C(T),$$

for $0 \le t-s \le T\hbar$, $0<\hbar\le 1$.

Proof. If V satisfies Assumption ($\tilde{\text{A}}$), Proposition 9.1.2 holds and $\partial^\alpha W_k(x,y) \in M^{\infty,1}(\mathbb{R}^{2d})$ for any $|\alpha| \le 2(N-k+1)$. In particular, $W_k \in M^{\infty,1}(\mathbb{R}^{2d})$ for all $k=1,\dots,N$ and thus $R^{(N)} \in M^{\infty,1}(\mathbb{R}^{2d})$.

Recall from Section 3.2.4 that $M^{\infty,1}(\mathbb{R}^{2d})$ is a Banach algebra for pointwise multiplication (the normalization is such that the unit element has unit norm), hence it is enough to show the desired estimate for $e^{\frac{i}{\hbar}(t-s)^k W_k}$, for any $1 \le k \le N$. We obtain

$$\begin{aligned}\left\| e^{\frac{i}{\hbar}(t-s)^k W_k} \right\|_{M^{\infty,1}} &= \left\| \sum_{n=0}^{\infty} \frac{i^n (t-s)^{kn} (W_k)^n}{\hbar^n n!} \right\|_{M^{\infty,1}} \\ &\le \sum_{n=0}^{\infty} \frac{(t-s)^{kn} \|W_k\|^n_{M^{\infty,1}}}{\hbar^n n!} \\ &= e^{\hbar^{-1}(t-s)^k \|W_k\|_{M^{\infty,1}}} \\ &= e^{T^k \|W_k\|_{M^{\infty,1}}} =: C(T)\end{aligned}$$

where we used $0 \le t - s \le T\hbar$, $0 < \hbar \le 1$. □

9.2 Short-time approximate propagator

Let us first recall that the Cauchy problem for the Schrödinger equation with bounded potentials is globally well-posed in $L^2(\mathbb{R}^d)$. This is an easy and classic result that can be stated as follows.

Proposition 9.2.1. *Assume that V is a real-valued function on $\mathbb{R} \times \mathbb{R}^d$ satisfying $V \in C^\infty(\mathbb{R}, L^\infty(\mathbb{R}^d))$ and let $s \in \mathbb{R}$. Then, the Cauchy problem*

$$\begin{cases} i\hbar\partial_t \psi(t,x) = -\frac{1}{2}\hbar^2 \Delta\psi(t,x) + V(t,x)\psi(t,x) \\ \psi(s,x) = f(x) \end{cases}$$

is (backward and) forward globally well-posed in $L^2(\mathbb{R}^d)$ and the corresponding propagator $U(t,s)$ is a unitary operator on $L^2(\mathbb{R}^d)$.

We also recall from [Bou97, Theorem 2.1] a boundedness result for oscillatory integral operators in terms of the $M^{\infty,1}$ norm of the amplitude.

Lemma 9.2.2. *Consider the oscillatory integral operator*

$$Af(x) = \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{2}} a(x,y) f(y)\, dy, \quad f \in \mathcal{S}(\mathbb{R}^d),$$

with $a \in M^{\infty,1}(\mathbb{R}^{2d})$. Then A extends to a bounded operator in $L^2(\mathbb{R}^d)$ and there exists a constant $C > 0$, depending only on d, such that

$$\|A\|_{L^2 \to L^2} \le C \|a\|_{M^{\infty,1}}.$$

Note that by expanding the phase $|x-y|^2/2$ one could also deduce this result from known boundedness results for Kohn-Nirenberg pseudodifferential operators [Grö01, Corollary 14.5.5] and the Parseval formula for the Fourier transform.

Consider now the parametrix $\widetilde{E}^{(N)}(t,s)$ in (1.54). We have the following result.

Proposition 9.2.3. *For every $T>0$ there exists $C=C(T)>0$ such that, for $0<t-s\le T\hbar$, $0<\hbar\le 1$, we have*

$$\|\widetilde{E}^{(N)}(t,s)\|_{L^2\to L^2}\le C. \tag{9.5}$$

Moreover, for any $f\in L^2(\mathbb{R}^d)$ we have

$$\lim_{t\searrow s}\widetilde{E}^{(N)}(t,s)f=f. \tag{9.6}$$

Proof. First, notice that

$$\widetilde{E}^{(N)}(t,s)f(x)=\frac{1}{(2\pi i(t-s)\hbar)^{d/2}}\int_{\mathbb{R}^d}e^{\frac{i}{\hbar}\frac{|x-y|^2}{2(t-s)}}e^{\frac{i}{\hbar}R^{(N)}(t,s,x,y)}f(y)dy$$

is an OIO with the free-particle-action as phase function and amplitude

$$a^{(N)}(t,s,x,y)\coloneqq e^{\frac{i}{\hbar}R^{(N)}(t,s,x,y)}\in M^{\infty,1}(\mathbb{R}^{2d})$$

by Proposition 9.1.3. We would like to apply Lemma 9.2.2. To this end, we need some preparation. First, using suitable unitary dilation operators (cf. Section 2.2.4 for notation) we rephrase $\widetilde{E}^{(N)}(t,s)$ as follows:

$$\widetilde{E}^{(N)}(t,s)=U_{\frac{1}{\sqrt{\hbar(t-s)}}}B^{(N)}(t,s)U_{\sqrt{\hbar(t-s)}},$$

where

$$B^{(N)}(t,s)f(x)=\frac{1}{(2\pi i)^{d/2}}\int_{\mathbb{R}^d}e^{\frac{i}{2}|x-y|^2}b^{(N)}(t,s,x,y)f(y)dy$$

is an OIO whose phase function is free from time and $\hbar$ dependence and the amplitude is

$$\begin{aligned}b^{(N)}(t,s,x,y)&=e^{\frac{i}{\hbar}\sum_{k=1}^{N}W_k(\sqrt{\hbar(t-s)}x,\sqrt{\hbar(t-s)}y)(t-s)^k}\\&=D_{\sqrt{\hbar(t-s)}}a^{(N)}(x,y).\end{aligned}$$

In particular, by Proposition 3.2.4 we infer $b^{(N)}\in M^{\infty,1}(\mathbb{R}^{2d})$ and

$$\|b^{(N)}\|_{M^{\infty,1}}\le C(T)\|a^{(N)}\|_{M^{\infty,1}}$$

for $0 < \hbar(t-s) \le T$ (in particular for $0 < t-s \le \hbar T$, since $0 < \hbar \le 1$).

Formula (9.5) then will follow from Lemma 9.2.2 and Proposition 9.1.3:

$$\begin{aligned}\|\widetilde{E}^{(N)}(t,s)\|_{L^2\to L^2} &= \|B^{(N)}(t,s)\|_{L^2\to L^2}\\ &\le C\|b^{(N)}\|_{M^{\infty,1}}\\ &\le C(T)\|a^{(N)}\|_{M^{\infty,1}} \le C'(T),\end{aligned}$$

for $0 < t-s \le T\hbar$.

For what concerns strong convergence to the identity as $t \searrow s$, consider the operator

$$H^{(N)}(t,s)f(x) = \frac{1}{(2\pi i(t-s)\hbar)^{d/2}}\int_{\mathbb{R}^d} e^{\frac{i}{\hbar}\frac{|x-y|^2}{2(t-s)}}\left(e^{\frac{i}{\hbar}R^{(N)}(t,s,x,y)}-1\right)f(y)dy$$

and employ again the dilations in order to write

$$H^{(N)}(t,s) = U_{\frac{1}{\sqrt{\hbar(t-s)}}}Q^{(N)}(t,s)U_{\sqrt{\hbar(t-s)}},$$

where

$$Q^{(N)}(t,s)f(x) = \frac{1}{(2\pi i)^{d/2}}\int_{\mathbb{R}^d} e^{\frac{i}{2}|x-y|^2}q^{(N)}(t,s,x,y)f(y)dy$$

is an OIO with amplitude

$$q^{(N)}(t,s,x,y) := b^{(N)}(t,s,x,y) - 1 \in M^{\infty,1}(\mathbb{R}^{2d}).$$

The latter can be expanded as follows:

$$\begin{aligned}q^{(N)}(t,s,x,y) &= e^{\frac{i}{\hbar}R^{(N)}\left(t,s,\sqrt{\hbar(t-s)}x,\sqrt{\hbar(t-s)}y\right)} - 1\\ &= \frac{i}{\hbar}(t-s)\overline{R^{(N)}}\left(t,s,\sqrt{\hbar(t-s)}x,\sqrt{\hbar(t-s)}y\right),\end{aligned}$$

where

$$\begin{aligned}\overline{R^{(N)}}(t,s,x,y) = \sum_{n=1}^{\infty}\frac{i^{n-1}}{n!}\left(\frac{t-s}{\hbar}\right)^{n-1}&\\ \times\left(\sum_{k=1}^{N}W_k\left(\sqrt{\hbar(t-s)}\,x,\sqrt{\hbar(t-s)}\,y\right)(t-s)^{k-1}\right)^n&.\end{aligned}$$

The Banach algebra property of the Sjöstrand class (cf. Proposition 3.2.17) and the properties of modulation spaces under dilation (cf. Proposition 3.2.4) imply that

$\overline{R^{(N)}}$ belongs to a bounded subset of $M^{\infty,1}(\mathbb{R}^{2d})$ for $0 < t-s \le T\hbar$, $0<\hbar\le 1$. It is then clear that $q^{(N)} \to 0$ in $M^{\infty,1}\big(\mathbb{R}^{2d}\big)$ for $t \searrow s$. Therefore, the OIO $Q^{(N)}$ with amplitude $q^{(N)}$ has operator norm converging to 0 as $t \searrow s$ by Lemma 9.2.2. The same holds for the unitarily equivalent operator $H^{(N)}$.

On the other hand, by the very definition of $H^{(N)}$ we have

$$H^{(N)}(t,s) = \widetilde{E}^{(N)}(t,s) - U_0(t,s),$$

where $U_0(t,s)$ is the free propagator, with strong convergence to the identity operator as $t \searrow s$. Hence (9.6) follows. □

A direct check shows that $\widetilde{E}^{(N)}(t,s)$ is a parametrix. Precisely we have

$$\left(i\hbar\partial_t + \frac{1}{2}\hbar^2\Delta - V(t,x)\right)\widetilde{E}^{(N)}(t,s) = G^{(N)}(t,s), \tag{9.7}$$

where

$$G^{(N)}(t,s)f = \frac{1}{(2\pi i(t-s)\hbar)^{d/2}}\int_{\mathbb{R}^d} e^{\frac{i}{\hbar}S^{(N)}(t,s,x,y)} g_N(t,s,x,y) f(y)\, dy. \tag{9.8}$$

From the construction of $S^{(N)}$ (see in particular eqs. (1.55), (9.1) and (9.2)) we see that the amplitude g_N is given by

$$\begin{aligned} g_N(t,s,x,y) &= -\frac{\partial S^{(N)}}{\partial t} - \frac{1}{2}\big|\nabla_x S^{(N)}\big|^2 - V(t,x) - \frac{i\hbar d}{2(t-s)} + \frac{i\hbar}{2}\Delta_x S^{(N)} \\ &= -\frac{1}{2}\sum_{k=N}^{2N}\sum_{\substack{j+\ell=k\\ j,\ell\ge 1}} \nabla_x W_j \cdot \nabla_x W_\ell (t-s)^k \\ &+ \frac{i\hbar}{2}\Delta_x W_N(x,y)(t-s)^N \\ &- \frac{(t-s)^N}{(N-1)!}\int_0^1 (1-\tau)^{N-1}\big(\partial_t^N V\big)((1-\tau)s+\tau t, x)d\tau. \end{aligned}$$

Hence, by Assumption ($\tilde{\mathrm{A}}$) and Proposition 9.1.2 we have

$$\|g_N(t,s,\cdot,\cdot)\|_{M^{\infty,1}(\mathbb{R}^{2d})} \le C(t-s)^N, \tag{9.9}$$

for $0 < t-s \le T$, with a constant $C = C(T) > 0$ independent of $\hbar \in (0,1]$.

The preceding discussion is the bedrock of the following result.

Theorem 9.2.4. *For every $T > 0$, there exists a constant $C = C(T) > 0$ such that*

$$\|\widetilde{E}^{(N)}(t,s) - U(t,s)\|_{L^2\to L^2} \le C\hbar^{-1}(t-s)^{N+1}, \tag{9.10}$$

whenever $0 < t - s \le T\hbar$.

Proof. We can write the operator $G^{(N)}(t,s)$ in (9.8) as

$$G^{(N)}(t,s)f = \frac{1}{(2\pi i(t-s)\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}\frac{|x-y|^2}{2(t-s)}} e^{\frac{i}{\hbar}R^{(N)}(t,s,x,y)} g_N(t,s,x,y) f(y)\, dy.$$

Following the steps of the proof of Proposition 9.2.3, by means of suitable dilations we can see that $G^{(N)}(t,s)$ is unitarily equivalent to an OIO with phase $|x-y|^2/2$ and amplitude

$$\tilde{g}^{(N)}(t,s,x,y) = D_{\sqrt{\hbar(t-s)}}\Big[e^{\frac{i}{\hbar}R^{(N)}(t,s,x,y)} g^{(N)}(t,s,x,y)\Big].$$

Using Proposition 9.1.3, formula (9.9) and the Banach algebra property of $M^{\infty,1}$ we have

$$\Big\| e^{\frac{i}{\hbar}R^{(N)}(t,s,\cdot,\cdot)} g_N(t,s,\cdot,\cdot)\Big\|_{M^{\infty,1}} \le C(T)(t-s)^N$$

for $0 < t - s \le T\hbar$. Again by the dilation properties in Proposition 3.2.4 we obtain

$$\big\|\tilde{g}^{(N)}(t,s,\cdot,\cdot)\big\|_{M^{\infty,1}} \le C(T)(t-s)^N$$

for $0 < t - s \le T\hbar$ and for a new constant $C(T) > 0$. Therefore, by Lemma 9.2.2, $G^{(N)}(t,s)$ extends to a bounded operator on $L^2(\mathbb{R}^d)$ with

$$\big\|G^{(N)} f\big\|_{L^2} \le C\big\|\tilde{g}^{(N)}\big\|_{M^{\infty,1}} \|f\|_{L^2} \le C(T)(t-s)^N \|f\|_{L^2}, \tag{9.11}$$

always for $0 < t - s \le T\hbar$.

Now, the propagator $U(t,s)$ clearly satisfies the equation

$$(i\hbar\partial_t - H)U(t,s)f = 0$$

for all $f \in L^2(\mathbb{R}^d)$, where $H = -(\hbar^2/2)\Delta + V$ is the Hamiltonian operator, with V as in Assumption ($\tilde{\mathrm{A}}$). On the other hand

$$(i\hbar\partial_t - H)\widetilde{E}^{(N)}(t,s)f = G^{(N)}(t,s)f,$$

which can be rephrased in integral form by means of Duhamel's principle as

$$\widetilde{E}^{(N)}(t,s)f = U(t,s)f - i\hbar^{-1}\int_s^t U(t,\tau)G^{(N)}(\tau,s)f d\tau.$$

Therefore, given $f \in L^2(\mathbb{R}^d)$, by (9.11) we have

$$\begin{aligned}
&\left\| U(t,s)f - \widetilde{E}^{(N)}(t,s)f \right\|_{L^2} \\
=&\left\| \hbar^{-1} \int_s^t U(t,\tau)G^{(N)}(\tau,s)f d\tau \right\|_{L^2} \\
\leq&\hbar^{-1} \int_s^t \|U(t,\tau)\|_{L^2\to L^2} \left\| G^{(N)}(\tau,s)f \right\|_{L^2} d\tau \\
\leq&C(T)\hbar^{-1} \int_s^t \|f\|_{L^2}(t-s)^N d\tau \\
\leq&C'(T)\hbar^{-1}(t-s)^{N+1}\|f\|_{L^2},
\end{aligned}$$

for $0 < t - s \leq T\hbar$. □

9.3 An abstract result and proof of the main result

We begin by presenting a convergence result for the approximate propagators in its full generality. In fact, it can be regarded as a generalization of [Fuj80, Lemma 3.2] and in the proof we use some ingenious tricks from that paper.

Theorem 9.3.1. *Assume that for some $\delta > 0$ we have a family of operators $\widetilde{E}^{(N)}(t,s)$ for $0 < t - s \leq \delta$, and $U(t,s)$, $s,t \in \mathbb{R}$, bounded in $L^2(\mathbb{R}^d)$, satisfying the following conditions:*

1. *U enjoys the evolution property $U(t,\tau)U(\tau,s) = U(t,s)$ for every $s < \tau < t$ and for every $T > 0$ there exists a constant $C_0 \geq 1$ such that*

$$\|U(t,s)\|_{L^2\to L^2} \leq C_0 \quad \textit{for} \quad 0 < t-s \leq T. \tag{9.12}$$

2. *There exists $C_1 > 0$ such that*

$$\|\widetilde{E}^{(N)}(t,s) - U(t,s)\|_{L^2\to L^2} \leq C_1(t-s)^{N+1} \quad \textit{for} \quad t-s \leq \delta. \tag{9.13}$$

For any subdivision $\Omega : s = t_0 < t_1 < \ldots < t_L = t$ of the interval $[s,t]$, with $\omega(\Omega) = \sup\{t_j - t_{j-1} : j = 1,\ldots,L\} < \delta$, consider therefore the composition $\widetilde{E}^{(N)}(\Omega,t,s)$ in (1.56).

Then, for every $T > 0$ there exists a constant $C = C(T) > 0$ such that

$$\|\widetilde{E}^{(N)}(\Omega,t,s) - U(t,s)\|_{L^2\to L^2} \leq C\omega(\Omega)^N(t-s) \tag{9.14}$$

for $0 < t - s \le T$. *More precisely,*

$$C = C(T) = C_0^2 C_1 \exp\Big(C_0 C_1 \omega(\Omega)^N T\Big).$$

Proof. Set

$$R^{(N)}(t,s) := \widetilde{E}^{(N)}(t,s) - U(t,s)$$

so that by (9.13) we have

$$\|R^{(N)}(t,s)\| \le C_1 (t-s)^{N+1} \quad \text{for} \quad 0 < t - s \le \delta. \tag{9.15}$$

Hence we can write

$$\begin{aligned}
&\widetilde{E}^{(N)}(\Omega,t,s) - U(t,s)\\
&\qquad = \big(U(t,t_{L-1}) + R^{(N)}(t,t_{L-1})\big)\dots\big(U(t_1,s) + R^{(N)}(t_1,s)\big) - U(t,s).
\end{aligned}$$

One expands the product above and obtains a sum of ordered products of operators, where each product has the following structure: *from right to left* we have, say, q_1 factors of type U, p_1 factors of type $R^{(N)}$, q_2 factors of type U, p_2 factors of type $R^{(N)}$, etc., up to q_k factors of type U, p_k factors of type $R^{(N)}$, to finish with q_{k+1} factors of type U. We can schematically write such a product as

$$\underbrace{U\dots U}_{q_{k+1}}\underbrace{R^{(N)}\dots R^{(N)}}_{p_k}\underbrace{U\dots U}_{q_k}\dots\dots\underbrace{R^{(N)}\dots R^{(N)}}_{p_1}\underbrace{U\dots U}_{q_1}.$$

Here $p_1,\dots,p_k,q_1,\dots q_k,q_{k+1}$ are non negative integers whose sum is L, with $p_j > 0$ and we can of course group together the consecutive factors of type U, using the evolution property assumed for U. Now, for $0 < t-s \le T$ we estimate the $L^2 \to L^2$ norm of the above ordered product using the known estimates for each factor, namely (9.12) and (9.15). In particular, by using the assumption $C_0 \ge 1$, we get

$$\begin{aligned}
&\le C_0^{k+1} \prod_{j=1}^{k}\prod_{i=1}^{p_j} C_1 (t_{J_j+i} - t_{J_j+i-1})^{N+1}\\
&\le C_0 \prod_{j=1}^{k}\prod_{i=1}^{p_j} C_0 C_1 (t_{J_j+i} - t_{J_j+i-1})^{N+1}
\end{aligned}$$

where $J_j = p_1 + \ldots + p_{j-1} + q_1 + \ldots + q_j$ for $j \geq 2$ and $J_1 = q_1$. The sum over $p_1, \ldots, p_k, q_1, \ldots, q_{k+1}$ of these terms is in turn

$$\begin{aligned}
&\leq C_0 \left\{ \prod_{j=1}^{L} (1 + C_0 C_1 (t_j - t_{j-1})^{N+1}) - 1 \right\} \\
&\leq C_0 \left\{ \exp\left(\sum_{j=1}^{L} C_0 C_1 (t_j - t_{j-1})^{N+1} \right) - 1 \right\} \\
&\leq C_0 \big\{ \exp\big(C_0 C_1 \omega(\Omega)^N (t-s) \big) - 1 \big\} \\
&\leq C_0^2 C_1 \omega(\Omega)^N (t-s) \exp\big(C_0 C_1 \omega(\Omega)^N (t-s) \big)
\end{aligned}$$

where in the last inequality we used $e^\tau - 1 \leq \tau e^\tau$, for $\tau \geq 0$.

This gives (9.14) with $C = C(T)$ as in the statement and concludes the proof. □

Proof of Theorem 1.7.1. The claim follows at once from Theorem 9.2.4 and Theorem 9.3.1 applied with T replaced by $T\hbar$, $C_0 = 1$, $C_1 = C\hbar^{-1}$, where C is the constant appearing in (9.10), and using $t - s \leq T\hbar$. □

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