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UNDERSTANDING LINEAR FUNCTIONS IN AN INTERACTIVE DIGITAL LEARNING ENVIRONMENT

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Linear functions are the first important example of mathematical models that students face; crucial to their understanding is the role of the slope, which is a complex concept due to its many different conceptualizations. Problems in understanding the slope are often caused by difficulties in connecting its different meanings. This paper presents an interactive task developed in a Digital Learning Environment aimed at introducing linear functions in grade 8 and approaching an interconnected concept of the slope. The task was proposed to 299 Italian students in a classroom-based context. Through the analysis of a collective class discussion that occurred while solving this task, we show how the emergence of different conceptualizations of the slope can be elicited and supported by interactive technologies in a Digital Learning Environment.

Keywords: Formative assessment, interactive digital learning environment, linear functions, mathematics education, slope.

INTRODUCTION AND THEORETICAL FRAMEWORK

Linear Functions and the Slope

Linear functions are one of the first mathematical models students face in their studies. They emerge within algebra, since they involve simple operations among numbers and variables, and they offer many prompts for reflecting on and understanding mathematical models. The first hurdle that students have to overcome when dealing with linear models is the concept of variable, which is often not well defined in school Mathematics. It can create confusion among the terms variable, unknown, parameter and their relations with numbers and constants (Schoenfeld & Arcavi, 1988). The second hurdle is the dependence between "x" and "y", the variables through which argument and value are usually expressed. The concept of joint variation is one of the most problematic at school teaching. It seems that many difficulties with Mathematics, even at the university level, can be attributed to an underlying misunderstanding of this concept (Carlson et al., 2002). Joint variation is recurrent in secondary school Mathematics since several functions are studied with their properties and representation forms; linear functions are the first example through which this concept is approached. The third point that needs attention is the relationship between "m", the slope, and "c", the intercept, in the standard equation "$y=m\,x+c$" , which determines the trend of the line. A study by Bardini and Stacey (2006), focused on the understanding of m and c in linear functions, shows that, as expected, the slope is a more complex concept than the intercept. However, students tend to omit c as if it is not part of the function, maybe due to the little attention dedicated to the intercept compared to the slope in the classroom activities. Conversions among different semiotic registers (numeric, symbolic, graphic, and real-world context) seem to influence the students' interpretation and understanding of these elements (Bardini & Stacey, 2006).

The concept of slope is crucial for understanding linear functions and for the development of following important mathematical concepts, such as the derivative and differential equations (Rasmussen & King, 2000); its complexity and difficulty is probably due to its many conceptualizations, which could not be properly connected in the students' mind (Stump, 1999).

Based on Stump's work (1999), Moore-Russo and colleagues (2011) distinguished 11 different categories of the conceptualization of slope: geometric ratio (rise above run), algebraic ratio (change in *y* over *x*), physical property (steepness), functional property (constant rate of change), parametric coefficient (*m*), trigonometric conception (tangent of an angle), calculus conception (derivative), real-word situation (static physical situations such as a ramp or dynamic functional situations), determining property (property that determines if the lines are parallel or perpendicular), behavior indicator (real number which indicates the increasing, decreasing, or horizontal trends of a line) and linear constant (the property which shows the lack of curvature on a line). Above all, it seems that developing the concept of slope as rate of change at an early stage is crucial to make sense of the algebraic and graphic-related meanings (Deniz & Kabael, 2017). An interesting vertical study by Gambini et al. (2020) in the Italian context tries to analyze how the understanding (and misunderstanding) of the concept of slope changes from grade 8 to 14 (from lower secondary school to university), using data from national standardized tests and university entry tests. From the results, they observe that students have trouble integrating algebraic thinking and meaning. By grade 8, mainly reasoning numerically, they should have acquired the concept of variation (functional and physical properties) associated with the slope of a linear structure. By grade 10, students should have associated the symbolic and graphic aspects (geometric and algebraic ratio), but it seems that they have abandoned the quantitative reasoning, which helps confer meaning to the involved objects. This split between the different interpretations of the slope continues and is consolidated in grades 13 and 14 when the comprehension of the derivative concept would require integrating the different aspects, joining the mathematical formalism to the numeric, symbolic, and graphic aspects. What too often remains is the algebraic computations, disconnected by their meaning (Gambini et al., 2020).

An Interactive Digital Learning Environment for Mathematics

Interactive technologies are promising to be helpful in understanding dynamical concepts as linear functions and the slope. In this paper, the term "interactive" is intended in the Moreno and Mayer's (2007) meaning, which is a property of the technology through which the student's action is encouraged and where what happens next depends on this action. In previous work, we defined a Digital Learning Environment (DLE) as a learning ecosystem composed of a human part (the learning community), a technological part (constituted by a Learning Management System integrated with tools for doing and assessing Mathematics, populated by activities and resources, and by the devices to access the learning materials) and all the interrelationships among the components (interactions, methodologies, learning and teaching processes) (Barana & Marchisio, 2021). A DLE can enable learning and teaching in classroom-based, online, blended or hybrid modalities. The design of interactive activities in a DLE for this study follows Grabinger and Dunlap's (1995) model, such that they: evolve from and are consistent with constructivist theories; promote study and investigation within authentic (i.e., realistic, meaningful, relevant, complex, and information-rich) contexts; encourage the growth of student responsibility, initiative, decision-making, and intentional learning; cultivate an atmosphere of knowledge-building learning communities that utilize collaborative learning among students and teachers; utilize dynamic, interdisciplinary, generative learning activities that promote high-level thinking processes to help students integrate new knowledge with old knowledge; and, assess student progress in content and learning-to-learn through realistic tasks and performances.

This paper aims at investigating the following research question (RQ): how can interactive activities in a DLE help 8^{th}-grade students approach linear functions and build an integrated concept of the slope? Based on the theoretical framework discussed above, in the following paragraphs, a teaching experiment is presented and discussed, with the purpose of providing an answer to this RQ.

METHODOLOGY

To answer the RQ, we designed an interactive task aimed at approaching linear functions, and in particular the role of the intercept and the slope, using different interacting conceptualizations of the slope. It was implemented in an interactive worksheet using an Advanced Computing Environment (Maple) in an integrated Moodle platform. We proposed the task to 13 8^{th} grade classes (299 students) in Turin (Italy). The experimentation took place in 2018, before the pandemic, in a classroom-based context; the researcher―author of this paper―helped the teachers manage the activities. The task was included in a wider path on formulas and functions (Barana & Marchisio, 2019). The activities were videotaped and successively selected, transcripted and analyzed according to the Moore-Russo and colleagues' (2011) framework in order to identify how the different conceptualizations of the slope emerge in the DLE.

The interactive task on which we focus in this paper is shown in Figure 1. The problem leads to exploring three different linear functions: one passing through the origin, one intersecting the x-axis, and one intersecting the y-axis. Students are asked to explore the numerical representation of the problem filling in the tables in the interactive file, which are initially empty. In the box below, graphs are interactively generated with points and lines using data from the tables. The tables and the interactive graphs help students reason and visualize the trend of the reading of a book by the three friends. The activity engages the learners asking them to insert the graph of their reading, envisaging the speed they would read the book with; thus, it opens up to explorations, comparisons, and discussions. A set of automatically graded questions completes the activity, focused on the graphs' analysis, leading to writing the formulas through which it is possible to express the mathematical models. The questions are: (1) After how many days from the beginning of the Holyday will they end the book? (2) Marco is advantaged because he has already read 30 pages. After how many days will Valentina reach Marco? (3) Who reads faster from the day when they start reading? (4) How many days does it take Luca to read the book? (5) How fast should we read the book to have a vertical line? (6) Write three formulas that express the number of pages read by the three friends as a function of the Holiday days. (7) If the book was 300 pages long and the three friends would keep reading at the same pace, after how many days from the beginning of the holiday would Luca reach Valentina?

This activity was carried out in the classroom: the task was displayed at the Interactive White Board (IWB), and the students worked on one task at the time in small groups, with paper and pen. Each step was discussed with the teacher and the researcher through the IWB, using the interactive worksheet and the automatically assessed questions to drive and support the discussion. Since the experiment was also taken in schools in disadvantaged socio-economic contexts, it was not possible to make students access the activities through digital devices in the classroom; however, the activity, together with other similar ones, was available in the DLE and accessible from home.

RESULTS

In all the classes, the activity started with a verbal description of the real context. Students had to translate it into a numeric register filling in some tables. The students moved from the tables to the graphic register, and drew the points and the lines on a cartesian plane. As the last step, they had to deduce the algebraic formulas for the models. We selected an excerpt, a part of a collective discussion that followed the graphs' drawings, under the input of imagining how students themselves would read the book and add the trend of their reading to the other graphs. We present it in the following lines since it is meaningful to show the students' understanding process of linear functions and slope. The discussion occurred in a school mainly attended by students from lower social classes.

The Holiday Book

For the Summer Holidays the Italian teacher has asked students to read the book. The book is 180 pages long.

- Valentina starts reading it on the first day of Holidays and she reads 15 pages per day.
- Marco has already started to read the book and he has already read 30 pages, so he starts from page 31 on the first day of Holidays. He reads 10 pages per day.
- Luca rests for the first 4 days, then he starts reading the book and he reads 20 pages per day.

Who will be the first to finish the book?

Fill in the following table writing, in the "Pages" column, the book's point that the three friends reach at the end of each day.
Use the "You" table to indicate how you would read this book.
Then observe the graphs below and answer the following questions.

VALENTINA		MARCO		LUCA		YOU	
Days	Pages	Days	Pages	Days	Pages	Days	Pages
0	0	0	30	0	0	0	0
1	15	1	40	1	0	1	30
2	30	2	50	2	0	2	60
3	45	3	60	3	0	3	90
4	60	4	70	4	0	4	120
5	75	5	80	5	30	5	150
Draw the points		Draw the points		Draw the points		Draw the points	
Draw the graph		Draw the graph		Draw the graph		Draw the graph	
Clear		Clear		Clear		Clear	

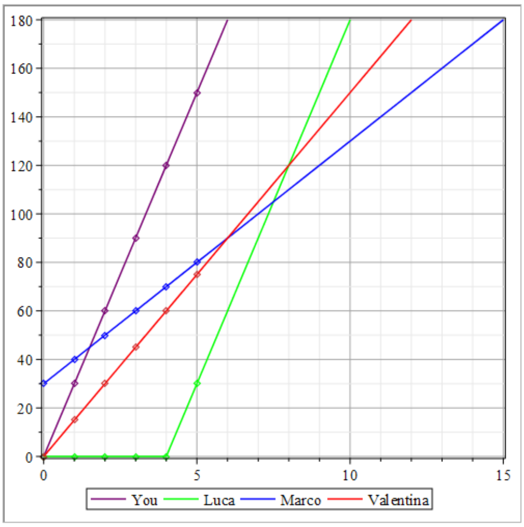

Figure 4. Part of the interactive activity "The Holiday Book" on linear models. The activity, originally in Italian, has been translated into English for the comprehension of the paper.

1	Researcher:	Well, how would you read this book?
2	Luigi:	I would read one page per year.
3	Researcher:	One page per year? Let's say one page per day. How would Luigi's graph be if he reads one page per day?
4	Camilla:	Very little inclined.
5	Researcher:	Yes, how many days does he need to read the whole book?
6	Class:	180 days.

7	Researcher:	180 days. Look how little its values increase from the horizontal axis. [She displays the line through the interactive worksheet at the IWB]. Ok. Is there someone who reads the book a little faster?
8	Simone:	I would read 20 pages per day.
9	Researcher:	So, 20 the first day, 40 the second… [filling in the table at the IWB and displaying the line]. This is the graph. How much time will he take to complete the book?
10	Class:	9 days.
11	Researcher:	Ok. Anyone else?
12	Cecilia:	I would rest for two days, then start the book and read 20 pages per day.
13	Researcher:	[Filling in the table] Two days of rest, so we start from 0 and have 0 for the first two days. Then we reach 20 at the end of the third day, 40, 60, … This is the graph [displaying the graph of the function]. Cecilia, are you faster than Simone?
14	Cecilia:	Yes, I am the fastest one.
15	Researcher:	Are you sure? Indeed you are the first one to end the book.
16	Cecilia:	Yes, I meant that I finished the book before everyone else.
17	Researcher:	Yes, you finish the book one day before Valentina, but what about Simone? Who reads faster?
18	Gianluca:	They are the same.
19	Researcher	How do you understand it from the graph?
20	Gianluca:	Because they are parallel lines.
21	Researcher:	Exactly. The lines are parallel. Even if the book was very much longer, Cecilia would never reach Simone. They increase by the same number of pages each day, but Cecilia started later. Ok, is there anyone else who reads even faster than Cecilia?
22	Mattias:	If I work hard, I think I could read even 35 pages a day.
23	Researcher:	[After filling in the table and displaying the graph]. Ok, you can see that Mattias is faster than the others. How long does he take?
24	Mattias:	[Observing the graph] Less than 5 days.
25	Researcher:	Ok. Would anyone read even faster?
26	Biagio:	One time, I read a whole book in a day.
27	Researcher:	Ok. Let's say that Biagio rests four days and then reads the whole book in one day. How would his graph be?
28	Alessia:	[miming an L] Horizontal until 4, and then vertical.
29	Mattias:	Parallel to the y-axis.
30	Biagio:	Yes, it's like that [miming a vertical line with his hand].
31	Camilla:	No, it's not vertical!
32	Andrea:	She's right. It cannot be vertical. The fourth and fifth points should be connected.
33	Camilla:	Yes, you have to connect the fourth day [pointing at the point (4,0) on the plane] to 180 [pointing at (5,180)].

34 Andrea: Yes, on the fifth day, he reads 180 pages.
35 Alessia: But it’s more or less vertical.
36 Andrea: Almost, but it’s not vertical.
37 Researcher: [displaying the graph at the IWB] It’s very, very steep. At the end of the fourth day, he was at 0 pages, but at the end of the fifth day, he was at 180. If we imagine that he reads the same amount of pages each hour, we have a very steep line.
38 Samuele: So, if he takes one second, would it be vertical?
39 Researcher: How should he read the book to obtain a vertical graph?
40 Luigi: He should have already read the book.
41 Researcher: But if he had already read the book, he would start from 180, not from 0.
42 Gianluca: He should take one second.
43 Simone: Yes, but the graph would be inclined of the space of one second.
44 Researcher: Exactly, there should be a little time which makes the line to be inclined.
45 Samuele: That’s right. If there is a bit of time, there is a bit of inclination.
46 Cecilia: There should not be any time at all.
47 Camilla: Right, the time should be zero.

Through this dialogue, we can notice how different conceptualizations of the slope emerge. We start with a real-word conceptualization (“one page a day”, lines 2 and 3) which translates to a physical property (“very little inclined”, line 4) through a functional property (the constant rate of change of which students had experienced while completing the tables). The students’ intuitions (lines 4 and 6) are confirmed by the interactive graph at the IWB. Thus they can experience the correspondence between a numeric approach to a graphic approach in studying linear functions. The same observations can be repeated in the discussion about Simone’s line. Here the researcher stresses the functional property (line 9) filling the table to build the graph. After that, she elicits a reference to the geometric ratio of the slope, asking students how much time he needs (horizontal shift) to complete the book (vertical shift). Cecilia’s line gives a prompt to speak about parallel lines and see the slope as an invariant for parallel lines (linear constant conceptualization). From here, the discussion focuses on increasing velocity in the real-world situation and seeing what happens to the line, with particular reference to the reduction of the horizontal shift to the limit case. In the end, the impossibility of having a vertical linear function emerges from the impossibility to reduce time to zero. The real-world conceptualization helps attribute a meaning to the functional and geometric properties and to connect different conceptualizations in a unique concept. From the discussion, we can also observe other prompts for analyzing other aspects of linear functions, such as horizontal ones (having zero slope, line 28), intercept (line 41 and previously analyzed drawing Marco’s read), and intersection with the x-axis (line 28 and, previously, Luca’s read). The following analysis, driven by the questions in the worksheet, aimed at also introducing the algebraic relations among variables and coefficients in a linear function, thus leading to the parametric coefficient conceptualization of the slope, and useful to connect also symbolic aspects to the numeric and graphic ones. Similar discussions took place in all the classes when solving this problem. In all the classes, all the students actively participated with interest in the discussion. The problem was comprehensible for everyone. The discussion about how they would have read the book actively engaged even the less interested students, such as Luigi, who usually disturbed his classmates. Thanks to the well-designed contextualization, even Luigi’s provocative answer could become a very interesting prompt for mathematical discussion: lines with a low slope. As a result, Luigi kept concentrated until the end of the discussion, when he proposed a new intuition, this time incorrect.

The interactive worksheet supported the students' discussion, conjectures and argumentations. The possibility to fill the interactive table and immediately generate lines in the graph below helped them visualize the correspondence between different registers and connect different conceptualizations of the slope. The worksheet also supported the teacher and the researcher in orchestrating the discussion and driving the class towards the creation of shared knowledge. Through this discussion, we could observe an interactive DLE composed of the class with the teacher and the researcher; the interactive task displayed at the IWB; interactions among the learning community, mainly consisting in dialogues, and between the community and the technologies. The discussion itself is part of the interactive DLE. The task follows Grabinger and Dunlap's (1995) model. In particular: it promotes active learning; the context is relevant and meaningful; it engages students with their experience; it promotes collaboration and discussion; the activity is dynamic and supports the generation of understanding; the interactivity supports self-assessment. Similar tasks were repeated in the classroom during the following lessons and as online homework to facilitate students to generalize the acquired knowledge and transfer it to new cases. The interactivity and the automatic assessment helped students explore the other problematic situations and check their understanding step-by-step.

CONCLUSION

In conclusion, we can answer the RQ: "how can interactive activities in a DLE help 8^{th}-grade students approach linear functions and build an integrated concept of the slope?". The interactive task presented in this paper allowed students to examine the linear models identified by reading a book at a constant speed. Through the interactive worksheets, they could compare different graphs corresponding to reading with different speeds and observe how the graphs change when the book is started before or after the beginning of the holidays. The classroom discussion selected and shown in this paper allowed us to observe how different conceptualizations of the slope emerge while discussing collectively in a DLE. The interactive activities elicited the emergence of many of the different conceptualizations of the slope identified by Moore-Russo and colleagues (2011), in particular: real-world, functional property, physical property, geometric ratio, and linear constant. Through the following activities, the parametric coefficient and algebraic ratio were also introduced. They are the main conceptualizations accessible at grade 8; developing robust connections among these conceptualizations can open the path to an interconnected understanding of the slope at higher grades. Above all, the stress was posed on the constant rate of change of the function, which, in the literature, is indicated as crucial for the development of a unified concept of the slope. However, in order not to lose the numeric and graphic understanding achieved with this task, similar activities should be repeated in higher grades of instruction and adapted to encompass also the algebraic conceptualization. This is a big challenge, since in Italy, after grade 8, students change school and start their upper secondary path. This discontinuity in the students' school life (change of teacher, class, friends, subjects, and methods) can reflect on the disconnection in building some fundamental concepts as the slope. Thus vertical projects, aimed at sharing vertical learning paths and using similar methodologies at different stages of education, and a focused teacher training, could be helpful to save these achievements and reinforce them to build solid mathematical knowledge. All the materials developed in this experimentation were shared with all the Italian teachers through the national Problem Posing and Solving Project (Brancaccio et al., 2015) with this goal. We aim at developing further this research, on the one hand, also examining the students' results in the final tests, compared with that of a control group; on the other hand, developing similar activities for upper secondary school in order to include other more advanced conceptualizations of the slope.

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