CHAPTER 15

Some metric results in Transcendental Numbers Theory

1. Introduction

In this Chapter we describe some results in the metric theory of transcendental numbers. Let begin with some notation. If $P \in \mathbb{Z}[x_1, \dots, x_m]$ is a non - zero polynomial, we define its *size* $t(P)$ as $h(P) + \deg(P)$. Here, $h(P)$ is the Weil's logarithmic height of P (so, if the gcd of the coefficients of P is 1, then $h(P)$ is the logarithm of the maximum module of the coefficients of P) and $\deg(P)$ is the total degree of P. Let $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{C}^m$ with $\alpha_1, \dots, \alpha_m$ algebraically dependent: we define $t(\alpha)$ as the minimum size of a non - zero polynomial $P \in \mathbb{Z}[x_1, \dots, x_m]$ such that $P(\alpha) = 0$.

We shall prove the following conjecture of Chudnovsky (see [**Chu2**, Problem 1.3, page 178]):

CONJECTURE 1.1. *Let m be an integer, $m \geq 1$. Then for almost all $\omega \in \mathbf{C}^m$ there exists a positive constant C such that*

$$\log |P(\omega)| \geq -Ct(P)^{m+1}$$

for any non - zero $P \in \mathbb{Z}[x_1, \dots, x_m]$.

Let $\alpha \in \mathbf{C}^m$ and let $r > 0$; we denote by $\mathcal{B}_m(\alpha, r)$ the set of points $\omega \in \mathbb{C}^m$ such that $|\omega - \alpha| = \max_{1 \leq j \leq n} |\alpha_j - \omega_j| \leq r$. Given a positive real number τ we also denote by $\mathbf{T}_\tau^m$ the set of $\omega \in \mathcal{B}_m(0,1)$ such that the inequality

$$|P(\omega)| < \exp\{-Ct(P)^\tau\}$$

has nontrivial solutions $P \in \mathbb{Z}[x_1, \dots, x_m]$ for any $C > 0$. Therefore, Chudnovsky's conjecture is equivalent to the statement: $\text{meas}(\mathbf{T}_{m+1}^m) = 0$. We also remark that $\mathbf{T}_{\tau'}^m \subset \mathbf{T}_\tau^m$ if $\tau \leq \tau'$. Moreover, by the box principle (see Lemma 3.2), $\mathbf{T}_\tau^m = \mathcal{B}_m(0,1)$ for any $\tau \in (0, m+1)$.

Define an other subset of the unit ball as follow. Let $\eta > 0$ and let $\mathbf{A}_\eta^m$ be the set of $\omega \in \mathcal{B}_m(0,1)$ such that for any $C' > 0$ the inequality

$$0 < |\omega - \alpha| < \exp\{-C't(\alpha)^\eta\}$$

has a solutions $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{C}^m$ with $\alpha_1, \dots, \alpha_m$ algebraically dependent. As before, we have $\mathbf{A}_{\eta'}^m \subset \mathbf{A}_\eta^m$ if $\eta \leq \eta'$. Moreover, it is easy to see that

$$\mathbf{A}_\eta^m \subset \mathbf{T}_\eta^m$$

(∗) Chapter's author : Francesco AMOROSO.

Y.N. Nesterenko, P. Philippon (Eds.): LNM 1752, pp. 227 - 238, 2001

for $\eta \geq n+1$. Indeed, let ω be in the unit ball of $\mathbb{C}^m$ and assume that there exists $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{C}^m$ with $\alpha_1, \dots, \alpha_m$ algebraically dependent such that

$$0 < |\alpha - \omega| < \exp\{-Ct(\alpha)^\eta\}$$

for some $\eta \geq m+1$. Let $P \in \mathbb{Z}[x_1, \dots, x_m]$ be a non – zero polynomial with integer coefficients, vanishing at α and sucht that $t(P) = t(\alpha)$. Then, using the relation

$$|\alpha_1^{\lambda_1} \cdots \alpha_m^{\lambda_m} - \omega_1^{\lambda_1} \cdots \omega_m^{\lambda_m}| \leq D|\alpha - \omega| \max\{|\alpha|, |\omega|, 1\}^{D-1},$$

which holds for any multiindex lg of weight $\lambda_1 + \dots + \lambda_m \leq D$, we can easily prove that

$$|P(\omega)| < \exp\{-B^{-1}Ct(P)^\eta\}$$

for some positive constant $B = B(m)$, provided that $C \geq B$.

The opposite inclusion $\mathbf{T}_{m+1}^m \subset \mathbf{A}_{m+1}^m$ is also true (Theorem 4.2), but the proof is much more difficult. The proof of the Conjecture 1.1 easily follows from this last inclusion and from the fact that $\mathbf{A}_{m+1}^m$ is a negligeable set (Theorem 4.2).

More generally we can define, for $m \in \mathbb{N}$ and $\tau \geq m+1$ a real number $\eta(\tau, m)$ as

$$\eta(\tau, m) = \sup\{\eta; \quad \mathbf{T}_\tau^m \subset \mathbf{A}_\eta^m\}$$

and ask whenever we have $\eta(\tau, m) = \tau$ ("comparison problem"). The conjectural answer is $\eta(\tau, m) = \tau$, but this is still an open question. If $m = 1$ the conjecture $\eta(\tau, 1) = \tau$ is true. In several dimension, only partial results are known. In [**Amo2**] it is proved that

$$\eta(\tau, m) \geq \max\left\{m + 1 + \frac{\tau - (m+1)}{m}, \tau - 1\right\},$$

which gives $\eta(m+1, m) = m+1$ (and therefore prove Chudnovsky's conjecture). Moreover, if $m \geq 2$

$$\eta(\tau, m) \geq \max\left\{m + \frac{\tau - 2}{m - 1}, \tau - 1\right\}$$

which implies $\eta(\tau, 2) = \tau$.

2. One dimensional results

As mentioned in the introduction, the answer to the comparison problem in dimension 1 is easy:

PROPOSITION 2.1. *Let ω be a complex number. Assume that there exists a polynomial $P \in \mathbb{Z}[x]$ such that*

$$|P(\omega)| < \exp\{-Ct(P)^\tau\}$$

for some $\tau \geq 2$ and some $C > 5 \cdot 2^{\tau+1}$. Then we can find a root α of P such that

$$|\alpha - \omega| < \exp\{-2^{-\tau-1}Ct(\alpha)^\tau\}.$$

PROOF. If $P(\omega) = 0$ the result is obvious, so assume $P(\omega) \neq 0$. Let $P = P_1^{e_1} \cdots P_k^{e_k}$ be the factorisation of P into irreducible factors. We first prove that

$$|P_j(\omega)| < \exp\{-C(t(P_j)/2)^\tau\} \tag{116}$$

for at least one index j. Assume the contrary: then,

$$-\log|P(\omega)| \le C\sum_{j=1}^{k} e_j(t(P_j)/2)^\tau \le C\left(\frac{1}{2}\sum_{j=1}^{k} e_j t(P_j)\right)^\tau.$$

Gelfond's inequality (see [**Gel2**, Ch.3,§4, Lemma 2])

$$e_1h(P_1) + \cdots + e_kh(P_k) \le \deg(P)\log 2 + h(P) \le t(P)$$

gives $e_1t(P_1) + \cdots + e_kt(P_k) \le 2t(P)$. Hence we obtain $-\log|P(\omega)| \le Ct(P)^\tau$ which contradict our assumption.

Let now $P_j(x) = a(x-\alpha_1)\cdots(x-\alpha_d)$ (where $d = \deg(P_j)$) and assume

$$|\omega - \alpha_1| \le \cdots \le |\omega - \alpha_d|.$$

From the inequality $|\omega - \alpha_i| \ge |\alpha_1 - \alpha_i|/2$ $(i = 2, \ldots, d)$ we easily obtain

$$|P_j(\omega)| \ge 2^{-d+1}|\omega - \alpha_1| \cdot |P_j'(\alpha_1)|. \tag{117}$$

To find a lower bound for $|P_j'(\alpha_1)|$, we quote the following well known resultant inequality (see [**Wal1**, p.5.5]): *Let F, G be two co-prime polynomials with integer coefficients and let z be any complex number. Then:*

$$1 < (\deg(F) + \deg(G))\|F\|^{\deg(G)}\|G\|^{\deg(F)}\max\{|F(z)|, |G(z)|\},$$

where $\|\cdot\|$ is the euclidean norm, i.e. the square root of the sum of the square of the coefficients. This inequality, with $F = P_j$, $G = P_j'$ and $z = \alpha_1$, gives (using the upper bounds $\log\|P\| \le \frac{1}{2}\log d + h(P)$ and $h(P') \le \log d + h(P)$ which hold for any polynomial $P \in \mathbf{Z}[x]$ of degree $\le m$):

$$-\log|P_j'(\alpha_1)| \le 3d\log d + 2dh(P_j);$$

Hence, using (116) and (117),

$$\begin{aligned}\log|\omega - \alpha_1| &\le (d-1)\log 2 - \log|P_j'(\alpha_1)| + \log|P_j(\omega)| \\ &\le d + 3d\log d + 2dh(P_j) - C(t(P_j)/2)^\tau \\ &\le 6t(P_j)^2 - C(t(P_j)/2)^\tau \le -2^{-\tau-1}Ct(\alpha_j)^\tau.\end{aligned}$$

□

As a corollary, we find that $\mathbf{T}_\tau^1 = \mathbf{A}_\tau^1$ for any $\tau \ge 2$. We now recall the definition of Hausdorff's dimension.

DEFINITION 2.2. Let Ω be a subset of $\mathbb{R}^n$ and let d a positive integer. We say that Ω has Hausdorff's dimension $< d$ if for any $\varepsilon > 0$ we can find a denumerable set of balls B_i of radii $r_i < \varepsilon$ such that

$$\Omega \subset \bigcup_{i\in\mathrm{N}} \mathcal{B}_i \qquad \sum_i r_i^d < \varepsilon.$$

We also define the Hausdorff's dimension of a set Ω as the infimum of the set of $d > 0$ for which Ω has Hausdorff's dimension $< d$.

If Ω has Hausdorff's dimension $< n$, then Ω is a negligeable set (in the sense of the Lebesgue measure in $\mathbb{R}^n$).

We can now prove the main result of this section (see [**Amo1**, Theorem 2]):

THEOREM 2.3. $\mathbf{T}_2^1$ *has Hausdorff's dimension* 0.

PROOF. Since $\mathbf{T}_2^1 = \mathbf{A}_2^1$, we will show that $\mathbf{A}_2^1$ has Hausdorff's dimension 0. For $t \in \mathbb{N}$ let

$$\Lambda_t = \{P \in \mathbb{Z}[x] \text{ such that } [t(P)] = t\}.$$

Let also, for $k \in \mathbb{N}$,

$$\Omega_k = \bigcup_{t \in \mathbb{N}} \bigcup_{P \in \Lambda_t} \bigcup_{\substack{\alpha \in \mathbb{C} \\ P(\alpha)=0}} B_1(\alpha, \exp\{-kt^2\}).$$

We have

$$\mathbf{A}_2^1 = \bigcap_{k \in \mathbb{N}} \Omega_k.$$

Let d, ε be two positive real numbers. Since

$$\log \operatorname{Card}(\Lambda_t) \le ct^2$$

for some absolute constant c, we also have

$$\sum_{t \in \mathbb{N}} \sum_{P \in \Lambda_t} \sum_{\substack{\alpha \in \mathbb{C} \\ P(\alpha)=0}} \exp\{-dkt^2\} < \varepsilon.$$

This proves that $\mathbf{A}_2^1$ has Hausdorff's dimension $< d$ for any $d > 0$.

□

Using **[Fed]**[1]Corollary 2.10.12, p.176, and **[Rog]**[2], Theorem 4, p.48, and Theorem VII 3, p.104, we deduce:

COROLLARY 2.4. *The set* $\mathbb{C} \backslash \mathbf{T}_2^1$ *is totally disconnected. The set* $\mathbf{T}_2^1$ *is arcwise connected.*

3. Several dimensional results: "comparison Theorem"

The aim of this section is the proof of the following theorem, which is a special case of main result of **[Amo2]** :

THEOREM 3.1. *There exists a constant* $A = A(m) \ge 1$ *having the following property. Let* $k \le m$ *be a positive integer and let* $\omega \in \mathbb{P}^m(\mathbb{C})$ *Let us assume that there exists an homogeneous unmixed ideal* $I \subset \mathbb{Q}[x_0, \dots, x_m]$ *of* rank k *such that*

$$|I(\omega)| < \exp\Big\{ - Ct(I)^{(m+1)/k} \Big\}$$

for some $C \ge (2A)^{(m^2-km+m)/k}$. *Then,* $\exists\, \alpha \in V(I)$ *such that*

$$\|\alpha - \omega\| < \exp\left\{ -\frac{1}{2} A^{-2} C^{k/(m^2-km+m)} t(\alpha)^{m+1} \right\}.$$

We recall that, for $\alpha = [\alpha_0 : \dots : \alpha_m]$, $\omega = [\omega_0 : \dots : \omega_m] \in \mathbb{P}^m(\mathbb{C})$,

$$\|\alpha - \omega\| = \frac{\max\limits_{0 \le i < j \le m} |\omega_i \alpha_j - \omega_j \alpha_i|}{\max\limits_{0 \le i \le m} |\alpha_i| \max\limits_{0 \le i \le m} |\omega_i|}$$

[1]**[Fed]** M. Federer. *Geometric measure theory*, Springer, Berlin, (1969).
[2]**[Rog]** C.A. Rogers, *Hausdorff measures*, Cambridge Univ. Press, Cambridge, (1970).

and $|\alpha| = \max_{0\le j\le m} |\alpha_j|$ (see Chapter 3). We also refer to Chapter 3 for the definitions of $|I(\omega)|$. Finally, the quantity $t(I)$ in the Theorem 3.1 is the *size* of I, i.e. $\deg(I) + h(I)$ (see again Chapter 3 for the definitions of $\deg I$ and $h(I)$).

In the sequel of this section we denote by $c_1, \dots, c_6$ positive constants depending only on m. The proof of Theorem 3.1 splits in several lemmas. We start with an easy consequence of the box – principle. For a non – zero homogeneous polynomial $P \in \mathbb{Q}[x_0, \dots, x_m]$ of degree d and for $\omega \in \mathbb{P}^m(\mathbb{C})$, we denote, as in Chapter 3,

$$\|P\|_\omega = |P(\omega)| \cdot |P|^{-1} \cdot |\omega|^{-d},$$

where $|P|$ is the maximum absolute value of the coefficients of P.

LEMMA 3.2. *Let $m \ge 1$ be an integer and let $\omega \in \mathbb{P}^m(\mathbb{C})$. Then for any real number $T \ge c_1$ there exists a non – zero homogeneous polynomial $P \in \mathbb{Q}[x_0, \dots, x_m]$ with size $\le T$ satisfying*

$$\|P\|_\omega \le \exp\{ - c_1^{-1} T^{m+1} \}.$$

PROOF. Let H and d be two positive integers which will be choosen later and let Λ be the set of homogeneous polynomials $P \in \mathbb{Z}[x_0, \dots, x_m]$ of degree d with non negative coefficients bounded by H. Let $D = \binom{d+m}{m}$ be the number of monomials of total degree d, and remark for further references that

$$(d+1)^m/m! \le D \le (d+m)^m/m!.$$

Let also

$$\delta = |\omega|^{-d} \min_{\substack{P_1, P_2 \in \Lambda, \\ P_1 \ne P_2}} |P_1(\omega) - P_2(\omega)|.$$

Since for any $P \in \Lambda$ we have $|\omega|^{-d} |P(\omega)| \le DH$, the ball of $\mathbb{C}$ with centre at the origin and radius $DH + \delta/2$ contains the disjoint union of the open balls of centre $|\omega|^{-d} P(\omega)$ and radius $\delta/2$, where P runs on Λ. Comparing the areas, we obtain

$$\mathrm{Card}(\Lambda)(\delta/2)^2 \le (DH + \delta/2)^2.$$

The cardinality of Λ is $(H+1)^D$; hence:

$$\delta \le \frac{2DH}{(H+1)^{D/2} - 1} \le 2DH^{1-D/2}$$

and so there exist two polynomials $P_1, P_2 \in \Lambda$, $P_1 \ne P_2$, such that

$$\|P_1 - P_2\|_\omega \le 2DH^{1-D/2}.$$

The polynomial $P = P_1 - P_2$ has degree d, maximum absolute value of the coefficients $\le H$ and satisfies $\|P\|_\omega \le 2DH^{1-D/2}$. The lemma easily follows taking $d = [T/2]$ and $H = [\exp\{T/2\}]$. □

LEMMA 3.3. *Let $C \ge c_2$ be a positive real number and $I \subset \mathbb{Q}[x_0, \dots, x_m]$ be an homogeneous unmixed ideal of rank $k \le m$ such that*

$$|I(\omega)| < \exp\Big\{ - Ct(I)^{(m+1)/k} \Big\}.$$

Then there exists a homogeneous prime ideal $\mathfrak{p} \subset I$ of rank k such that

$$|\mathfrak{p}(\omega)| < \exp\Big\{ - c_2^{-1} Ct(\mathfrak{p})^{(m+1)/k} \Big\}.$$

PROOF. Let $I = I_1 \cap \ldots \cap I_s$ be the reduced primary decomposition of I and let $\mathfrak{p}_j = \sqrt{I_j}$ and e_j be the exponent of I_j. We can assume $\omega \notin V(\mathfrak{p}_j)$ for $j = 1, \ldots, s$. From the Proposition 4.7 of Chapter 3 we obtain:

$$\sum_{j=1}^{s} e_j t(\mathfrak{p}_j) \leq (m^2+1)t(I)$$

and

$$\sum_{j=1}^{s} e_j \log|\mathfrak{p}_j(\omega)| \leq \log|I(\omega)| + m^3 t(I) \leq -Ct(I)^{(m+1)/k} + m^3 t(I).$$

Let now assume $\log|\mathfrak{p}_j(\omega)| > -\tilde{C}t(\mathfrak{p}_j)^{(m+1)/k}$ for $j = 1, \ldots, s$ and for some $\tilde{C} \geq 1$. Then:

$$\begin{aligned} Ct(I)^{(m+1)/k} &< m^3 t(I) + \tilde{C} \sum_{j=1}^{s} e_j t(\mathfrak{p}_j)^{(m+1)/k} \\ &\leq m^3 t(I) + \tilde{C} \left(\sum_{j=1}^{s} e_j t(\mathfrak{p}_j) \right)^{(m+1)/k} \\ &\leq m^3 t(I) + \tilde{C}(m^2+1)^{m+1} t(I)^{(m+1)/k} \\ &\leq 2(m^2+1)^{m+1} \tilde{C} t(I)^{(m+1)/k}. \end{aligned}$$

Hence $\tilde{C} > \frac{1}{2}(m^2+1)^{-m-1}C$. The lemma follows choosing $c_2 = 2(m^2+1)^{m+1}$. □

We also recall two results of Chapter 3. The first one follows immediately from Corollary 4.12 in Chapter 3, while the second one follows from Proposition 4.13 in Chapter 3.

LEMMA 3.4. *Let $\mathfrak{p} \subset \mathbb{Q}[x_0, \ldots, x_m]$ be a homogeneous prime ideal of rank $k \leq n$ and let $Q \in \mathbb{Q}[x_0, \ldots, x_m]$ be a homogeneous polynomials such that $Q \notin \mathfrak{p}$. Let also $\omega \in \mathbb{P}^m(\mathbb{C})$ such that $\omega \notin V(\mathfrak{p})$ and define*

$$\rho = \min_{\alpha \in V(\mathfrak{p})} \|\omega - \alpha\|.$$

Assume that there exist $S > 0$ and $\theta \in \mathbb{N}$ such that

$$|\mathfrak{p}(\omega)| \leq e^{-S}, \qquad \|P\|_\omega \leq e^{-2m \deg P}$$

and

$$-\theta \log \|P\|_\omega \geq 2 \min(S, \log \frac{1}{\rho}).$$

Then, if $k \leq m-1$, there exists a homogeneous unmixed ideal J of rank $k+1$ such that $V(J) = V((\mathfrak{p}, P))$ and

$$\begin{aligned} t(J) &\leq m(m+2)\theta t(Q)t(\mathfrak{p}), \\ \log|J(\omega)| &\leq -S + 13m^2\theta t(Q)t(\mathfrak{p}). \end{aligned}$$

Moreover, if $k = m$ we have

$$S \leq 13m^2\theta t(Q)t(\mathfrak{p}).$$

LEMMA 3.5. *Let $I \subset \mathbb{Q}[x_0, \dots, x_m]$ be a homogeneous unmixed ideal of rank k. Then for any $\omega \in \mathbb{P}^m(\mathbb{C})$ such that $\omega \notin V(I)$ there exists a zero $\alpha \in V(I)$ such that*

$$(\deg I) \log \|\omega - \alpha\| \le \frac{1}{m+1-k} \log |I(\omega)| + 4m^3 t(I).$$

We shall prove Theorem 3.1 by induction on the rank k of I. The following lemma resume the inductive step.

LEMMA 3.6. *There exists a constant $A = A(m) \ge 1$ having the following property. Let C be a positive real number and θ be a positive integer such that $C \ge \theta A$. Let us assume that there exists a homogeneous unmixed ideal $I \subset \mathbb{Q}[x_0, \dots, x_m]$ of rank $k \le m$ such that*

$$|I(\omega)| < \exp\left\{-Ct(I)^{(m+1)/k}\right\}.$$

Then, either $\exists\, \alpha \in V(\mathfrak{p})$, such that

$$\|\alpha - \omega\| < \exp\left\{-A^{-1}\theta t(\alpha)^{m+1}\right\}$$

or there exists an homogeneous unmixed ideal $J \subset \mathbb{Q}[x_0, \dots, x_n]$ of rank $k+1$ such that $V(J) \subset V(I)$ and

$$|J(\omega)| < \exp\left\{-A^{-1}\theta^{-m/(k+1)} C^{k/(k+1)} t(J)^{(m+1)/(k+1)}\right\}.$$

Moreover, if $k = m$ only the first case can occur.

PROOF. We can assume $\omega \notin V(I)$, otherwise we choose $\alpha = \omega$. Lemma 3.3 gives a homogeneous prime ideal $\mathfrak{p} \supset I$ of rank k such that $\log |\mathfrak{p}(\omega)| < -S$ where

$$S = c_2^{-1} C t(\mathfrak{p})^{(m+1)/k}.$$

Let $\alpha \in V(\mathfrak{p})$ such that $\rho = \|\alpha - \omega\|$ is minimal, and define a real number T by the following equality:

$$\theta c_1^{-1} T^{m+1} = 2 \min\left(S, \log \frac{1}{\rho}\right).$$

Assume

$$C \ge 8m^4 c_2;$$

then, by Lemma 3.5,

$$\begin{aligned} \deg \mathfrak{p} \cdot \log \frac{1}{\rho} &\ge -\frac{1}{m} \log |\mathfrak{p}(\omega)| - 4m^3 t(\mathfrak{p}) \\ &\ge \frac{1}{2} m^{-1} c_2^{-1} C t(\mathfrak{p})^{(m+1)/k} \\ &\ge \frac{1}{2} m^{-1} c_2^{-1} C \deg \mathfrak{p}. \end{aligned}$$

Hence,

$$\min\left(S, \log \frac{1}{\rho}\right) \ge \frac{1}{2} m^{-1} c_2^{-1} C$$

and

$$T \ge \left(m^{-1} c_1 c_2^{-1} (C/\theta)\right)^{1/(m+1)} = c_3 (C/\theta)^{1/(m+1)}. \tag{118}$$

We also remark the inequalities:

$$\log \frac{1}{\rho} \geq \frac{1}{2} c_1^{-1} \theta T^{m+1}, \tag{119}$$

$$T \leq \left(2c_1\theta^{-1}S\right)^{1/(m+1)} = c_4 (C/\theta)^{1/(m+1)} t(\mathfrak{p})^{1/k}. \tag{120}$$

If

$$A \geq (c_1/c_3)^{m+1},$$

we have $T \geq c_1$ by (118); hence we can apply Lemma 3.2, which gives a polynomial P of size $\leq T$ such that:

$$\log \|P\|_\omega \leq -c_1^{-1} T^{m+1}. \tag{121}$$

We now distinguish two cases.

• First case: $P \in \mathfrak{p}$.

Then $t(\alpha) \leq t(P) \leq T$, hence, by (119),

$$\log \frac{1}{\rho} \geq \frac{1}{2} c_1^{-1} \theta t(\alpha)^{m+1}.$$

Therefore in this case, the first assertion of Lemma 3.6 is satisfied (if we choose $A \geq \max(1,\ 8m^4c_2,\ (c_1/c_2)^{m+1},\ 1/(2c_1))$.

• Second case: $P \notin \mathfrak{p}$ and $k \leq m-1$.

From the choice of S and from (121) we see that $|\mathfrak{p}(\omega)| \leq e^{-S}$ and $-\theta \log \|P\|_\omega \geq 2\min(S, \log(1/\rho))$. Moreover the other hypothesis of Lemma 3.4,

$$\|P\|_\omega \leq e^{-2m \deg P}$$

is satisfied if $T^m \geq 2c_1 m$ (see (121)) which certainly occur if we choose

$$A \geq c_3^{-m-1} (2c_1 m)^{(m+1)/m}$$

(see (118)). Hence we can apply Lemma 3.4.

Assume first $k \leq m-1$. Then Lemma 3.4 gives a homogeneous unmixed ideal J of rank $k+1$ such that $V(J) = V((\mathfrak{p}, P))$ and

$$\log |J(\omega)| \leq -S + 13m^2\theta T t(\mathfrak{p}), \tag{122}$$

$$t(J) \leq m(m+2)\theta T t(\mathfrak{p}). \tag{123}$$

Assume

$$A \geq (26c_2c_4m^2)^{(m+1)/m}.$$

Then, by (120),

$$\begin{aligned} 13m^2\theta T t(\mathfrak{p}) &\leq 13c_4m^2\theta^{m/(m+1)} C^{1/(m+1)} t(\mathfrak{p})^{(k+1)/k} \\ &\leq \frac{1}{2} c_2^{-1} C t(\mathfrak{p})^{(k+1)/k} \leq \frac{1}{2} S. \end{aligned}$$

Henceforth, from (122),

$$\log |J(\omega)| \leq -\frac{1}{2} c_2^{-1} C t(\mathfrak{p})^{(m+1)/k}. \tag{124}$$

On the other hand, again by (120),

$$m(m+2)\theta T t(\mathfrak{p}) \leq m(m+2) c_4 \theta^{m/(m+1)} C^{1/(m+1)} t(\mathfrak{p})^{(k+1)/k}.$$

Therefore, from (123) we obtain the following lower bound for $t(\mathfrak{p})$:

$$t(\mathfrak{p}) \geq c_5 \theta^{-mk/((m+1)(k+1))} C^{-k/((m+1)(k+1))} t(J)^{k/(k+1)}$$

Inserting this lower bound in (124), we finally found

$$\log |J(\omega)| \leq -c_6\theta^{-m/(k+1)}C^{k/(k+1)}t(J)^{(m+1)/(k+1)}.$$

Therefore, in this case, the second assertion of Lemma 3.6 holds, choosing

$$(125) \quad A \geq \max\left\{1, 8m^4c_2, \left(\frac{c_1}{c_3}\right)^{m+1}, c_3^{-m-1}(2c_1m)^{\frac{m+1}{m}}, (26c_2c_4m^2)^{\frac{m+1}{m}}, \frac{1}{c_6}\right\}.$$

• Third case: $P \notin \mathfrak{p}$ and $k = m$.

As before, we can apply Lemma 3.4 provided that $A \geq c_3^{-m-1}(2c_1m)^{\frac{m+1}{m}}$. Since $k = m$, this lemma gives

$$S \leq 13m^2\theta Tt(\mathfrak{p}).$$

The same computation as before shows that this relation is inconsistent if we assume

$$A > (13c_2c_4m^2)^{(m+1)/m}.$$

So this case can not occurs if A satisfies (125).

Lemma 3.6 follows, choosing

$$A = \max\left\{1, 8m^4c_2, \left(\frac{c_1}{c_3}\right)^{m+1}, (2c_1)^{-1}, c_3^{-m-1}(2c_1m)^{\frac{m+1}{m}}, (26c_2c_4m^2)^{\frac{m+1}{m}}, \frac{1}{c_6}\right\}.$$

□

We can now prove Theorem 3.1 by induction on k. Obviously, we can assume $\omega \notin V(I)$. Let A be the constant which appear in Lemma 3.6.

• **k = m.** Let $\theta = [A^{-1}C]$. By assumption, $C \geq 2A$, hence $\theta \geq \frac{1}{2}A^{-1}C$; moreover $C/\theta \geq A$. Lemma 3.6 gives $\alpha \in V(I)$ such that

$$\log \|\alpha - \omega\| < -A^{-1}\theta t(\alpha)^{m+1} \leq -\frac{1}{2}A^{-2}Ct(\alpha)^{m+1}.$$

• **k < m.** Let

$$\theta = \left[A^{-1}C^{k/(m^2-km+m)}\right].$$

By assumption $C \geq (2A)^{(m^2-km+m)/k}$, hence

$$(126) \quad \frac{1}{2}A^{-1}C^{k/(m^2-km+m)} \leq \theta;$$

moreover we also have

$$(127) \quad \theta \leq A^{-1}C^{k/(m^2-km+m)}.$$

This last inequality gives $C/\theta \geq A$, thus we can apply Lemma 3.6. If there exists $\alpha \in I$ such that

$$\log \|\alpha - \omega\| < -A^{-1}\theta t(\alpha)^{m+1}$$

our assertion follows, since, by (126),

$$A^{-1}\theta \geq \frac{1}{2}A^{-2}C^{k/(m^2-km+m)}.$$

Otherwise, there exists an homogeneous unmixed ideal J of rank $k+1$ such that $V(J) \subset V(I)$ and

$$\log |J(\omega)| < -A^{-1}\theta^{-m/(k+1)}C^{k/(k+1)}t(J)^{(m+1)/(k+1)}.$$

Let $\tilde{C} = A^{-1}\theta^{-m/(k+1)}C^{k/(k+1)}$. By (127) and by the assumption

$$C \geq (2A)^{(m^2-km+m)/k},$$

we have

$$\tilde{C} \geq A^{-1}\left(A^{-1}C^{k/(m^2-km+m)}\right)^{-m/(k+1)} C^{k/(k+1)}$$
$$= A^{(m-k-1)/(k+1)} C^{k(m^2-km)/((k+1)(m^2-km+m))}$$
$$\geq C^{k(m^2-km)/((k+1)(m^2-km+m))} \tag{128}$$
$$\geq (2A)^{(m^2-km)/(k+1)}. \tag{129}$$

Hence, by inductive hypothesis, we can find $\alpha \in V(J)$ such that

$$\log \|\alpha - \omega\| < -\frac{1}{2}A^{-2}\tilde{C}^{(k+1)/(m^2-km)}t(\alpha)^{m+1}$$
$$\leq -\frac{1}{2}A^{-2}C^{k/(m^2-km+1)}t(\alpha)^{m+1},$$

where in the last inequality we have used again (128). □

4. Several dimensional results: proof of Chudnovsky's conjecture

From Theorem 3.1 we easily deduce the following result concerning polynomials:

COROLLARY 4.1. *For any integer $m \geq 1$ and for any real number $\tau \geq m+1$ there exists a positive constant $\tilde{B}$ having the following property. Let ω be in the unit ball of $\mathbb{C}^m$ and assume that there exists a non - zero polynomial $P \in \mathbb{Z}[x_1, \ldots, x_m]$ such that*

$$|P(\omega)| < \exp\left\{-Ct(P)^{m+1}\right\}$$

for some $C \geq \tilde{B}$. Then, there exists $\alpha \in \mathbb{C}^m$ such that $P(\alpha) = 0$ and

$$|\alpha - \omega| < \exp\left\{-\tilde{B}^{-1}C^{1/m^2}t(\alpha)^{m+1}\right\}.$$

PROOF. We can assume $P(\omega) \neq 0$, otherwise we choose $\alpha = \omega$; we also denote by c_7, c_8 two positive constants depending only on m. Let I be the principal ideal generated by the homogenization ${}^hP$ of P and let $\omega' = (1, \omega)$; by Proposition 4.8 of Chapter 3 we have

$$\log |I(\omega')| \leq \log \|P\|_{\omega'} + 2m^2 \deg(P) \leq -Ct(P)^{m+1} + 2m^2t(P)$$

and $t(I) \leq (m^2+1)t(P)$. Hence

$$\log |I(\omega')| \leq \frac{1}{2}(m^2+1)^{-m-1}Ct(I)^{m+1},$$

provided that $C \geq 4m^2$. Theorem 3.1 gives α' such that ${}^hP(\alpha') = 0$ and

$$\log \|\alpha' - \omega'\| < -B^{-1}(C/2)^{1/m^2}t(\alpha')^{m+1}. \tag{130}$$

If $C \geq c_7$, then $\|\alpha' - \omega'\| < 1/2$, hence $\alpha'_0 \neq 0$ and the vector $\alpha \in \mathbb{C}^m$ defined by $\alpha_i = \alpha'_i/\alpha'_0$ $(i = 1, \ldots, m)$ satisfies $P(\alpha) = 0$ and

$$|\alpha - \omega| \leq \max\{1, |\alpha|\}\|\alpha' - \omega'\|.$$

Since $\|\alpha' - \omega'\| < 1/2$, this gives at once $|\alpha| \leq 2$. Thus $|\alpha - \omega| \leq 2\|\alpha' - \omega'\|$, and we deduce from (130) that

$$\log |\alpha - \omega| < -B^{-1}(C/2)^{1/m^2}t(\alpha)^{m+1} + \log 2 \leq -(4B)^{-1}C^{1/m^2}t(\alpha)^{m+1},$$

if $C \geq c_8$. Corollary 4.1 follows, choosing $\tilde{B} = \max\{4m^2, c_7, c_8, 2B\}$. □

As a corollary, we see that $\mathbf{T}^m_{m+1} = \mathbf{A}^m_{m+1}$. We can now prove Chudnovsky's conjecture:

THEOREM 4.2. *The set* $\mathbf{T}^m_{m+1}$ *is negligeable.*

PROOF. By the previous result, it is enough to show that $\mathbf{A}^m_{m+1}$ is negligeable. Given $P \in \mathbb{Z}[x_1, \dots, x_m]$ and $\varepsilon > 0$, denote

$$U_P(\varepsilon) = \{\omega \in \mathcal{B}_m(0,1) \text{ such that } \min_{\substack{\alpha \in \mathbf{C}^{m+1} \\ P(\alpha)=0}} |\omega - \alpha| \leq \varepsilon\}.$$

For $t \in \mathbb{N}$ let

$$\Lambda_t = \{P \in \mathbb{Z}[x_1, \dots, x_m] \text{ such that } [t(P)] = t\}.$$

Let also, for $k \in \mathbb{N}$,

$$\Omega_{m,k} = \bigcup_{t \in \mathbb{N}} \bigcup_{P \in \Lambda_t} U_P(\exp\{-kt^{m+1}\}).$$

We have

$$\mathbf{A}^m_{m+1} = \bigcap_{k \in \mathbb{N}} \Omega_{m,k}.$$

Now, we quote the following lemma:

LEMMA 4.3. *For any* $\varepsilon > 0$ *we have*

$$\operatorname{meas}(U_P(\varepsilon)) \leq \frac{\pi^{2m}}{(m-1)!} \varepsilon^2 \deg(P).$$

PROOF. By a theorem of Lelong

$$\operatorname{meas}(\{P = 0\} \cap \mathcal{B}_m(0,1)) \leq \frac{\pi^{2(m-1)}}{(m-1)!} \deg(P)$$

(see **[Lel1]**[3]Théorème 7). Hence, by a Fubini – Tonelli argument,

$$\operatorname{meas}(U_P(\varepsilon)) \leq \pi^2 \varepsilon^2 \cdot \frac{\pi^{2(m-1)}}{(m-1)!} \deg(P).$$

□

Since

$$\log \operatorname{Card}(\Lambda_t) \leq c(m) t^{m+1}$$

for some constant $c(m)$ depending only on m, we deduce from the lemma above that

$$\operatorname{meas}(\Omega_{m,k}) \leq \sum_{t=1}^{+\infty} \exp\{c(m)t^{m+1}\} \frac{\pi^{2m}}{(m-1)!} \exp\{-kt^{m+1}\} t \to 0$$

as $k \to +\infty$. Theorem 4.2 is proved. □

[3] **[Lel1]** P. Lelong. Propriétés métriques des variétés analytiques complexes définies par une équation, *Ann. École Norm. Sup.* 67, (1950), 393–419.