Pencils of norm form equations and a conjecture of Thomas

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PENCILS OF NORM FORM EQUATIONS AND A CONJECTURE OF THOMAS.

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Abstract. *We introduce a new method to deal with families of norm form equations. These generalize the Thue equations studied first by Thomas using Baker's Method (which however we do not use here). We show that for all large integer values of the parameter t, every solution over $\mathbb{Z}$ arises from specializing a solution over $\mathbb{Z}[T]$ by $T = t$. The results are completely effective.*

1. Introduction

This article is concerned with norm form equations. Classically these are usually considered to have the shape

$$\text{Norm}(x_0\xi^{(0)} + \cdots + x_{d-1}\xi^{(d-1)}) = n \tag{1.1}$$

where the norm is from the number field $\mathbb{Q}\xi^{(0)} + \cdots + \mathbb{Q}\xi^{(d-1)}$ to $\mathbb{Q}$, and the unknowns $x_0, \ldots, x_{d-1}$ may be subject to homogeneous linear conditions over $\mathbb{Q}$ (the case $x_2 = \cdots = x_{d-1} = 0$ is usually called a Thue equation in x_0, x_1). But here we will allow $\xi^{(0)}, \ldots, \xi^{(d-1)}$ to depend on an integer parameter t. For example with $d = 2$ and $\xi^{(0)} = 1, \xi^{(1)} = \sqrt{t}$ we see the general Pell equation

$$x^2 - ty^2 = 1. \tag{1.2}$$

We will then solve the resulting norm form equation uniformly for all sufficiently large positive integers t (in fact this cannot be done for (1.2) above).

Our method is new, and it uses in an essential way the main result of our paper [1], *i. e.* height bounds for the solutions of equations in a multiplicative torus, varying in a pencil.

The main theme goes back to Emery Thomas [17] and diophantine equations with a parameter t, together with his concept of "stably solvable" [18] (p.320). This says roughly that if t is a sufficiently large positive integer, then all integer solutions come from "functional solutions" which are obtained by replacing t in the equation by a variable T and solving the resulting equation in the polynomial ring $\mathbb{Z}[T]$.

A highlight example is given by a family of Thue equations due to Thomas himself [18] (p.322). Let $d \geq 3$ and $A_1(T), \ldots, A_{d-1}(T) \in \mathbb{Z}[T]$ be monic polynomials of degrees satisfying

$$0 < \deg(A_1) < \cdots < \deg(A_{d-1}).$$

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We consider the polynomial

$$\Phi_T(X,Y) = X(X - A_1(T)Y)\cdots(X - A_{d-1}(T)Y) + Y^d.$$

We are interested in the diophantine equation $\Phi_t(x,y) = 1$, to be solved in $t,x,y \in \mathbb{Z}$. Observe that $\Phi_T(X,Y) = 1$ has trivial functional solutions, namely

$$(X,Y) = (1,0), \quad (0,1), \quad (A_1(T),1), \quad \ldots, \quad (A_{d-1}(T),1).$$

Thomas [18, Conjecture 1] (p.322) conjectures that the only solutions of $\Phi_t(x,y) = 1$ are given by specializations of the above functional solutions, provided that t is a sufficiently large natural integer. He proves his conjecture for $d = 3$ under some assumptions on the growth of A_1 and A_2. Later on, Heuberger [9, Theorem 1] (p.377), proves the conclusion of Thomas's general conjecture, under some quite involved degree conditions. Their results are obtained essentially as an application of Baker's estimates for linear forms in logarithms, as for the case of fixed Thue's equations; various devices allow to treat the equations uniformly in the integer parameter t.

Surprisingly enough, Ziegler [21] (p.291) found a *functional* counterexample to Thomas's conjecture. If $A_1(T) = T$ and $A_2(T) = T^4 + 3T$, then

$$X(T) = T^9 + 3T^6 + 4T^3 + 1, \quad Y(T) = T^8 + 3T^5 + 3T^2$$

is a solution of $X(X - A_1Y)(X - A_2Y) + Y^3 = 1$.

In the same paper, Ziegler considers the equation $\Phi_T(X,Y) = a$ for $d = 3$ and proves some counting results (see also [20] for possibilities to improve these). He shows this has no nontrivial functional solutions when $\deg(A_2) > 34\deg(A_1)$. Then he generalizes in [22, Theorem 1] (p.724) this last result to an arbitrary degree d, showing that there are no nontrivial functional solutions if $\deg(A_{d-1}) > c_d \deg(A_{d-2})$ for an explicit $c_d \in \mathbb{N}$.

See also Waldschmidt's recent survey article [19], including an account of work by Levesque and himself.

In this paper we develop a new approach to treat families of norm form equations. We do not use Baker's Method, applying instead a recent specialization theorem of the authors [1]. This allows us to prove that, under suitable assumptions, all solutions of a norm form diophantine equation come from a specialization of T to t of functional solutions of the equation obtained by replacing t with T throughout. For instance for Thomas's cubic equation we get, as a corollary of Theorem 2.2, the following stronger version:

Theorem 1.1. *Let A, B, C be distinct in $\mathbb{Z}[T]$. Then there is an effective finite set (possibly empty) of functional solutions X, Y in $\mathbb{Z}[T]$ of*

$$(X - AY)(X - BY)(X - CY) + Y^3 = 1 \tag{1.3}$$

and an effective t_0 with the following property. If the integer $t \geq t_0$ then every diophantine solution x, y of

$$(x - A(t)y)(x - B(t)y)(x - C(t)y) + y^3 = 1 \tag{1.4}$$

satisfies $x = X(t), y = Y(t)$ *for some functional solution.*

By Ziegler's result [21, Theorem 1] (p.291), Theorem 1.1 confirms Thomas conjecture for $d = 3$, under the assumption $\deg(A_2) > 34\deg(A_1)$, which is weaker than the Heuberger condition on the degrees or Thomas's original growth condition.

Because we do not use Baker's Method, we are able to treat more general norm form equations, which are liable to lead to S-unit equations with many terms (on the functional level see (7.5),(7.17) for example). These are usually handled with the Subspace Theorem (see [13], Section VI, p.153, Theorem 1F), but that is often lacking in effectivity. By contrast, our method always yields effective results. In [1] (p.2604) we treated

$$x^3 - (t^3 - 1)y^3 = 1 \tag{1.5}$$

(see also the explanation at the end of this section) but already things are not quite trivial for the special Pell equation

$$x^2 - (t^2 - 1)y^2 = 1. \tag{1.6}$$

For $m = 0, 1, 2, \ldots$ we have the well-known functional solutions $X = X_m(T)$, $Y = Y_m(T)$ given by

$$\frac{(\sqrt{T^2-1}+T)^m + (-\sqrt{T^2-1}+T)^m}{2} = \sum_{\mu=0}^{[m/2]} \binom{m}{2\mu} (T^2-1)^{\mu} T^{m-2\mu},$$

$$\frac{(\sqrt{T^2-1}+T)^m - (-\sqrt{T^2-1}+T)^m}{2\sqrt{T^2-1}} = \sum_{\mu=0}^{[(m-1)/2]} \binom{m}{2\mu+1} (T^2-1)^{\mu} T^{m-1-2\mu}$$

in $\mathbb{Z}[T]$. And in fact for t large enough it is not hard to prove that the integer solutions of (1.6) are given by

$$(x, y) = (\pm X_m(t), \pm Y_m(t)) \quad (m = 0, 1, 2, \ldots)$$

for independent signs (this is probably classical and holds even for all $t \neq 0$ in $\mathbb{Z}$). Such a result is surely in the spirit of "stably solvable" even though there are infinitely many functional solutions.

Our main result is Theorem 2.2 (with its generalization Theorem 2.5). In fact we allow in (1.1) non-linear algebraic conditions on $x_0, \ldots, x_{d-1}$, even themselves involving the parameter t.

We work out two more examples where we also find all the functional solutions.

First, we can escape from Thue equations like (1.5); for example when t is large enough the only integer solutions of

$$x^3 - (t^3 - 1)y^3 + 3(t^3 - 1)xy + (t^3 - 1)^2 = 1 \tag{1.7}$$

(which cannot be linearly transformed into something homogeneous) are given by

$$(x, y) = (t^2, 2t), \quad (t^2, -t).$$

Second, we can even escape from binary equations; for example when t is large enough the only integer solutions of

$$x^4 - (t^4 - 1)y^4 + (t^4 - 1)^2 z^4 - 2(t^4 - 1)xz(xz - 2y^2) = 1 \tag{1.8}$$

are given by a "Pell family"

$$(x,y,z) = (\pm X_m(t^2), 0, \pm Y_m(t^2)) \quad (m = 0,1,2,\ldots)$$

together with 12 "sporadic"

$$(\pm t, \pm 1, 0), \quad \pm(t^2, \pm 2t, 1), \quad \pm(4t^6 - 5t^2, \pm 4t, -4t^4 - 1)$$

all with independent signs.

Here it is worth noting that the equation

$$x^5 + (t^5-1)y^5 + (t^5-1)^2 z^5 + 5(t^5-1)xyz(xz-y^2) = n \tag{1.9}$$

was considered by Baker [2, Corollary 3] (p.695), who showed using Padé approximations to $(1-t^{-5})^{1/5}$ that if $t > 10^{11}$ then

$$\max\{|x|,|y|,|z|\} < t^{2500}n^2$$

which perhaps suggests functional solutions, at least for $n = 1$. It is not clear if our methods apply to (1.9) for $n = 1$, but if they did, then they would probably allow an extra variable in (1.9).

Let us illustrate our method with (1.7). We already explained (1.5) in [1] with the units $x - \xi_t y$ and $\eta_t = t - \xi_t$ in $\mathbb{Q}(\xi_t)$ for $\xi_t = (t^3-1)^{1/3}$. With (1.7) we find instead that $x - \xi_t y + \xi_t^2$ is a unit and

$$x - \xi_t y + \xi_t^2 = \eta_t^m \tag{1.10}$$

for some integer m. Taking conjugates and eliminating x, y with the usual Siegel Identity we get

$$(t-\xi_t)^m + \omega(t-\omega\xi_t)^m + \omega^2(t-\omega^2\xi_t)^m = 3\xi_t^2 \tag{1.11}$$

with $\omega = e^{2\pi i/3}$ (see also the functional version (7.5) below). This is a four-term S-unit equation. With (1.5) the $3\xi_t^2$ is absent and so only three terms remain. Then the more traditional approach with linear forms in logarithms could have been used. It would already show that $|m|$ is bounded independently of t. In general, linear forms in logarithms do not apply to things like (1.11). We note however that (1.11) for each fixed m is likely to determine t. An obstacle is when (1.11) holds identically in t, and one can verify that this leads back to a functional solution.

Now (1.11) may be compared to

$$\xi^m + (1-\xi)^m + (1+\xi)^m = 1 \tag{1.12}$$

for a general algebraic ξ. When the $(1+\xi)^m$ is absent, already Beukers [4] had shown using Padé approximation that the absolute height of ξ is bounded above independently of $m \neq 1$ (the case $m = 1$ gives a kind of functional solution).

In [1] we imitated Thue's Method of constructing "almost Padé approximations" using Siegel's Lemma. If there are no functional obstacles, then these indeed lead to bounds for the height of (for example) $t - \xi_t$ in (1.11), and so also for the height of t which is of course $|t| = t$. We had already observed that the general method is effective (and that Denz [7] found an explicit bound for the height of ξ in (1.12) independently of any m). It even works for arbitrarily many terms, but then a somewhat involved descent is required.

For (1.11) it is tempting to reach for the Subspace Theorem but as mentioned that is not effective. However the functional analogue is effective, and in principle it can be used to find all the functional solutions.

2. Main Results

Let $P \in \mathbb{Z}[T, X]$ be an irreducible polynomial, monic of degree d in X, and let $\xi \in \overline{\mathbb{Q}(T)}$ be one of its roots. We write $K = \mathbb{Q}(T, \xi)$ and $\widehat{K}$ for its normal closure over $\mathbb{Q}(T)$.

We shall consider various intermediate extension $\mathbb{Q}(T) \subseteq L \subseteq \widehat{K}$ corresponding to (smooth models of) projective curves, denoted $\mathcal{C}_L$. For such a field L and for $t \in \mathbb{N}$ (we exclude the finitely many ramified ones from our discussion[1]) we choose once and for all a point $p_{L,t}$ on $\mathcal{C}_L(\overline{\mathbb{Q}})$ above t in such a way that $p_{L,t}$ is above $p_{L',t}$ if L' is another intermediate extension with $L' \subseteq L$. This $p_{L,t}$ defines a place of a field L; more precisely a place v of L corresponds to $\deg(v)$ geometric points of $\mathcal{C}_L(\overline{\mathbb{Q}})$, conjugates by Galois action. These choices define a specialization map which sends $u \in L$ to $u_t := u(p_{L,t}) \in \overline{\mathbb{Q}}$. We denote by $L_t := \mathbb{Q}(p_{L,t})$ the specialized field, *i. e.* the residue field of L by the place defined by $p_{L,t}$. Thus $L_t = \{u_t \mid u \in L\}$.

Our basic parametric equation is

$$\text{Norm}(x_0 + x_1\xi_t + \cdots + x_{d-1}\xi_t^{d-1}) = 1 \tag{2.1}$$

to be solved in integers $x_0, \ldots, x_{d-1}$ as the integer parameter t varies. As mentioned above, when t is not present, one obtains the so-called "norm form" equation by further restricting $(x_0, x_1, \ldots, x_{d-1})$ to lie in a fixed proper vector space of $\mathbb{Q}^d$. But here we shall allow subspaces - and even subvarieties - of $\mathbb{Q}(t)^d$. As in the work of Thomas, our emphasis is on specializations of the parameter to sufficiently large integers t; and we can clearly assume $t > 0$.

For $t \in \mathbb{N}$ large the numbers r_1 and $2r_2$ of real and imaginary immersions of the number field $\mathbb{Q}(\xi_t)$ are both constants. This can be seen looking at sign changes, see Lemma 3.1. As usual, we let $r = r_1 + r_2 - 1$.

We make the following

Assumption 2.1.

(1) There exist multiplicatively independent elements $u_1, \ldots, u_r$ of $\mathbb{Z}[T, \xi]^*$.
(2) Suppose u is in $\mathbb{Z}[T, \xi]^*$ and the algebraic number v is in the group generated by the conjugates of u. Then v is a root of unity.

The most stringent part is (1), which however is basic to the context that we are studying. Part (2) is more of a technical nature. We will see just after the proof of Lemma 3.3 that it suffices to check (2) for u in the group generated by $u_1, \ldots, u_r$.

Note that $(\mathbb{Z}[T, \xi]^*)_t \subseteq \mathbb{Z}[\xi_t]^*$. Under our assumptions, the index $[\mathbb{Z}[\xi_t]^* : (\mathbb{Z}[T, \xi]^*)_t]$ is finite for large t (see Lemma 3.3(1)), and moreover uniformly bounded (see Theorem 2.3). Let us assume in addition $\mathbb{Z}[\xi_t]^* = (\mathbb{Z}[T, \xi]^*)_t$.

[1]These, arising from the vanishing of the discriminant of P with respect to X, must be excluded for instance to ensure the isomorphism 5.2.

As a first consequence of the specialization theorem [1, Theorem 1.3]) (p.2601) we get:

Theorem 2.2. *Let us assume 2.1. Let $W \subseteq \mathbb{A}^d$ be a proper subvariety defined over $\mathbb{Q}(T)$. Then there exists an effective $t_0 > 0$ such that for $t \geq t_0$ with*

$$\mathbb{Z}[\xi_t]^* = (\mathbb{Z}[T,\xi]^*)_t,$$

the solutions $(x_0,\dots,x_{d-1}) \in W_t(\mathbb{Z})$ of

$$\prod_{j=1}^{d}(x_0 + x_1\sigma_j(\xi_t) + \cdots + x_{d-1}\sigma_j(\xi_t)^{d-1}) = 1 \tag{2.2}$$

are specializations of functional solutions $\mathbf{X} = (X_0,\dots,X_{d-1}) \in W(\mathbb{Z}[T])$ of

$$\prod_{j=1}^{d}(X_0 + X_1\sigma_j(\xi) + \cdots + X_{d-1}\sigma_j(\xi)^{d-1}) = 1. \tag{2.3}$$

For a Thomas equation of degree $d \geq 3$, the result [8, Theorem 2.1] (p.6) of Halter-Koch, Lettl, Pethö and Tichy implies (under some minor conditions) that if t is sufficiently large and $\mathbb{Q}(\xi_t)$ is a primitive extension, then ξ_t, $\xi_t - A_1(t)$, …, $\xi_t - A_{d-2}(t)$ generate a subgroup of uniformly bounded rank of $\mathbb{Z}[\xi_t]^*$. And their result [8, Theorem 3.1] (p.7) implies that primitivity holds for almost all choices of the parameter (in the sense of thin sets).

Here we prove a uniform bound for the index valid for *all* sufficiently large t.

Theorem 2.3. *Let us assume 2.1. Then, there exist effective t_1, $I \in \mathbb{N}$ such that $[\mathbb{Z}[\xi_t]^* : (\mathbb{Z}[T,\xi]^*)_t] \leq I$ for $t \geq t_1$.*

The proof is delicate since we have to consider some degenerate cases (corresponding to intermediate fields in the extension $K/\mathbb{Q}(T)$ above). It requires the following lower bound for the standard absolute logarithmic Weil height h for non-torsion elements in $\mathbb{Z}[\xi_t]^*$.

Theorem 2.4. *Let us assume 2.1. Then, there exist effective t_1 and $c > 0$ such that for $t > t_1$ and any non-torsion $\eta \in \mathbb{Z}[\xi_t]^*$ we have*

$$h(\eta) \geq c \log t.$$

Theorem 2.3 allows us to prove a completely (*i. e.* without assumption on the index) effective description of the solutions of the norm form equation (2.1) at the price of a more technical statement. Let $\sigma_1,\dots,\sigma_d$ be the $\mathbb{Q}(T)$-immersions of $K := \mathbb{Q}(T,\xi)$ in an algebraic closure $\overline{K}$ and consider the linear isomorphism $\Psi\colon \overline{K}^d \to \overline{K}^d$ be the linear isomorphism

$$\Psi(\mathbf{X}) = \Big(\sum_{i=0}^{d-1} X_i\sigma_1(\xi)^{i-1},\dots,\sum_{i=0}^{d-1} X_i\sigma_d(\xi)^{i-1}\Big).$$

Theorem 2.5. *Let us assume 2.1. Let $l = \mathrm{lcm}_{t\geq t_1}[\mathbb{Z}[\xi_t]^* : (\mathbb{Z}[T,\xi]^*)_t]$ and $W \subsetneq \mathbb{A}^d$ be a subvariety defined over $\mathbb{Q}(T)$. Then there exists an effective $t_2 \in \mathbb{N}$ such that for $t \in \mathbb{N}$, $t \geq t_2$, the solutions $(x_0,\dots,x_{d-1}) \in W_t(\mathbb{Z})$*

of (2.2) *are specializations of functional solutions* $\mathbf{X} = (X_0, \ldots, X_{d-1}) \in W(\overline{K})$ *of* (2.3) *satisfying moreover*

$$\Psi^{-1}\big(\Psi(\mathbf{X})^l\big) \in \mathbb{Z}[T]^d. \tag{2.4}$$

Note that $\mathrm{lcm}_{t \geq t_1}[\mathbb{Z}[\xi_t]^* : (\mathbb{Z}[T, \xi]^*)_t]$ in the above theorem is well defined and can be effectively bounded by Theorem 2.3.

Let's come back to the index. The following question arises naturally:

Problem 2.6. *Does there exist a* $t_1 \in \mathbb{N}$ *such that* $\mathbb{Z}[\xi_t]^* = (\mathbb{Z}[T, \xi]^*)_t$ *for* $t \geq t_1$*?*

By Theorem 2.2, a positive answer to this question would give a simple description of the solutions of the norm form equation (2.2).

Our results raise three more questions.

First, can one determine the structure of the functional solutions? Most likely this is possible (and effectively) through the machinery of the function field *abc* (or better *abcd* . . .) theorems, as hinted in Section 7 (see also Mason [10] as well as [11] a bit later); in another article we investigate this in a more general context.

Second, can one determine the solutions when t is "small"? For the Thomas equations and Thue examples in general we can use linear forms in logarithms, but it is not clear if this works for (1.7). Actually that is a curve of genus 1, so the results of Baker and Coates [3] (p.595) apply. However this would not work for analogues of (1.7) of degree 4, where one would expect genus 3.

And (1.8) is a surface (and with an extra variable even a three-fold); further the Padé methods are unlikely to work, as Baker's condition $t > 10^{11}$ suggests.

Third, as noted already in [8], the original Thomas Conjecture can be extended to equations coming from so-called Ankeny-Brauer-Chowla "*ABC*" number fields. This is also punningly suggested by our notation in Theorem 1.1. But can one extend this Theorem in a similarly complete way? A good test case would be

$$(X - AY)(X^2 + BXY + CY^2) + Y^3 = 1$$

assuming that $B^2 - 4C < 0$ for all sufficiently large $T = t > 0$. We already treated the case $A = B = T, C = T^2$ in (1.5).

3. Notations and auxiliary results

Let, as in the previous section $P \in \mathbb{Z}[T, X]$ be an irreducible polynomial, monic of degree d in X and $\xi \in \overline{\mathbb{Q}(T)}$ one of its roots. We recall that $K = \mathbb{Q}(T, \xi)$.

Lemma 3.1. *For large* $t \in \mathbb{N}$*, the number* $r_1(t)$ *of real immersions of the number field* $\mathbb{Q}(\xi_t)$ *is constant.*

Proof. This is an easy consequence of Sturm's Theorem on the number of distinct real roots, located in an interval, of a square-free polynomial. Let us consider P as a polynomial with coefficients in $\mathbb{Q}(T)$ and perform the Sturm's sequence P_i with $P_0 = P$, $P_1 = P'$ and, for $i \geq 1$, P_{i+1} be the remainder of the euclidean division between P_i and P_{i+1}. Then, for each i and for large $t > 0$, the specialization at t of P is square-free and the sign of the specialization at t of the leading coefficient of P_i is constant, proving that the number of real roots of $x \mapsto P(t, x)$ does not depend on t.

□

According to this lemma, we let r_1 and $2r_2$ the numbers of real and imaginary immersions of $\mathbb{Q}(\xi_t)$ (for $t \in \mathbb{N}$ sufficiently large), and $r = r_1 + r_2 - 1$.

For the next lemma, we need the main result of [5], which we state in the following somewhat modified version. We take an irreducible algebraic curve $\mathcal{C}$ defined over $\overline{\mathbb{Q}}$, embedded in some affine space with associated (logarithmic) height h on $\mathcal{C}(\overline{\mathbb{Q}})$, defined with say supremum norms.

Theorem 3.2. *Let Γ be a finitely generated subgroup of $\overline{\mathbb{Q}}(\mathcal{C})^*$ such that the only constants in Γ are roots of unity. Then there exists an effective $h_0 = h_0(\Gamma)$ such that, for $\mathbf{t} \in \mathcal{C}(\overline{\mathbb{Q}})$ with $h(\mathbf{t}) > h_0$, the restriction to Γ of the evaluation at $\mathbf{t}$ is an isomorphism onto its image $\Gamma_{\mathbf{t}}$.*

Proof. We first remark that the restriction is defined outside a finite set and thus for $\mathbf{t}$ of sufficiently large height. We fix basis elements $\gamma^{(1)}, \dots, \gamma^{(k)}$ of $\Gamma/\Gamma_{\mathrm{tors}}$. Since the only constants in Γ are roots of unity, $\gamma^{(1)}, \dots, \gamma^{(k)}$ are multiplicatively independent modulo constants. By [5, Theorem 1' (p.1120)], there exists an effective $h_0 > 0$ such that if $h(\mathbf{t}) > h_0$ then the specializations $\gamma_{\mathbf{t}}^{(1)}, \dots, \gamma_{\mathbf{t}}^{(k)}$ are multiplicatively independent. Let now $\gamma \in \Gamma$ such that $\gamma_{\mathbf{t}} = 1$. Write $\gamma = \zeta(\gamma^{(1)})^{m_1} \cdots (\gamma^{(k)})^{m_k}$ where ζ is a root of unity. Then $1 = \gamma_{\mathbf{t}} = \zeta(\gamma_{\mathbf{t}}^{(1)})^{m_1} \cdots (\gamma_{\mathbf{t}}^{(k)})^{m_k}$. If $h(\mathbf{t}) > h_0$ then $\gamma_{\mathbf{t}}^{(1)}, \dots, \gamma_{\mathbf{t}}^{(k)}$ are multiplicatively independent, and thus $\zeta = 1$, $m_1 = \cdots = m_k = 0$, which implies $\gamma = 1$.

□

Note that the units in $\mathbb{Z}[T, \xi]^*$ form a finitely generated group. We shall apply Theorem 3.2 to various group Γ which are image of subgroups of $\mathbb{Z}[T, \xi]^*$ by homomorphisms, and thus also finitely generated.

Lemma 3.3. *Let us assume 2.1. For $t \in \mathbb{N}$ sufficiently large (effectively) the following holds:*

(1) $\mathrm{rank}\, \mathbb{Z}[\xi_t]^* = \mathrm{rank}\, (\mathbb{Z}[T, \xi]^*)_t = \mathrm{rank}\, \mathbb{Z}[T, \xi]^* = r$.
(2) $K_t = \mathbb{Q}(\xi_t)$ *and* $[K_t : \mathbb{Q}] = d$.
(3) Let $E/\mathbb{Q}(T)$ be a subextension of $K/\mathbb{Q}(T)$. Then $[E_t : \mathbb{Q}] = [E : \mathbb{Q}(T)]$.
(4) The group generated by $u_1, \dots, u_r$ in Assertion 2.1(1) is of finite index in $\mathbb{Z}[T, \xi]^*$.

Proof. To prove (1), let $s_1 \le r_1$ be the number of real roots of the irreducible factor (over $\mathbb{Q}$) of $P(t, X)$ vanishing at ξ_t, whereas let $2s_2$ be the number of complex non-real roots, so $s_2 \le r_2$. By the above inequalities and by the easier part of Dirichlet's theorem, $\mathrm{rank}\, \mathbb{Z}[\xi_t]^* \le s_1 + s_2 - 1 \le r$. By Assumption 2.1(1), $\mathrm{rank}\, \mathbb{Z}[T, \xi]^* \ge r$. Since Assumption 2.1(2) implies that the only constants in $\mathbb{Z}[T, \xi]^*$ are roots of unity, Theorem 3.2, shows that $\mathrm{rank}\, (\mathbb{Z}[T, \xi]^*)_t = \mathrm{rank}\, \mathbb{Z}[T, \xi]^*$. Moreover $(\mathbb{Z}[T, \xi]^*)_t \subseteq \mathbb{Z}[\xi_t]^*$. Putting all these inequalities together,

$$r \le \operatorname{rank} \mathbb{Z}[T,\xi]^* = \operatorname{rank}(\mathbb{Z}[T,\xi]^*)_t \le \operatorname{rank} \mathbb{Z}[\xi_t]^* \le s_1 + s_2 - 1 \le r.$$

Thus $\operatorname{rank} \mathbb{Z}[\xi_t]^* = \operatorname{rank}(\mathbb{Z}[T,\xi]^*)_t = \operatorname{rank} \mathbb{Z}[T,\xi]^* = r$ (and $r_1 = s_1$, $r_2 = s_2$).

The argument above shows that $[\mathbb{Q}(\xi_t):\mathbb{Q}] = s_1 + 2s_2 = r_1 + 2r_2 = d$. Since $\mathbb{Q}(\xi_t) \subseteq K_t$ and $[K_t:\mathbb{Q}] \le [K:\mathbb{Q}(T)] = d$ we have $K_t = \mathbb{Q}(\xi_t)$. This proves (2).

The equality of the degree in (3) for an arbitrary subextension $E/\mathbb{Q}(T)$ follows as a consequence of (2), taking into account that if the degree drops on specializing E, the same would happen for K.

To see (4) we know from (1) that the group generated by $u_1, \dots, u_r$ has the same rank as $\mathbb{Z}[T,\xi]^*$. So it suffices to prove that the torsion of the latter is bounded. But ξ has a Puiseux series, say at $T = \infty$, whose coefficients lie in a number field. Thus any root of unity in $\mathbb{Z}[T,\xi]^*$ must lie in this field and therefore in its finite torsion group.

□

Remark 3.4.
(i) Let $\sigma_1, \dots, \sigma_d$ be the $\mathbb{Q}(T)$-immersions of K in an algebraic closure $\overline{K}$. By (2) we can identify $\sigma_1, \dots, \sigma_d$ with the $\mathbb{Q}$-immersions of $\mathbb{Q}(\xi_t)$ in $\overline{\mathbb{Q}}$ by letting $\sigma_i(\sum a_j \xi_t^j) = \sum a_j (\sigma_i \xi)_t^j$ for $a_0, \dots, a_{d-1} \in \mathbb{Q}$.

(ii) Let E be as in (3) and let u be a primitive element for E over $\mathbb{Q}(T)$, i. e. $E = \mathbb{Q}(T,u)$. Then obviously $\mathbb{Q}(u_t) \subseteq E_t$ and we may not have equality. Note however that equality holds[2] for sufficiently large t, but "sufficiently large" may now depend on the chosen primitive element u . We shall prove a uniform version for units in Proposition 5.4.

(iii) Assertion (4) enables us in Assumption 2.1(2) to restrict to u in the group generated by $u_1, \dots, u_r$.

The next result will be useful in checking this Assumption 2.1(2) for the various examples.

Lemma 3.5. *Let us assume that 2.1(1) holds. Assume further that the Puiseux series at $T = \infty$ of the roots $\xi^{(1)}, \dots, \xi^{(d)}$ of $P(T,X)$ are Laurent series with coefficients in $\mathbf{k}$ where $\mathbf{k}$ is the rational field or an imaginary quadratic field. Then assertion 2.1(2) holds.*

Proof. By 2.1(1), there exist multiplicatively independent $u_1, \dots, u_r \in \mathbb{Z}[T,\xi]^*$. Let u be a monomial in $u_1, \dots, u_r$ and v be an algebraic number in the group generated by the conjugates of u. As already noted (after the proof of Lemma 3.3), it is enough to show that v is a root of unity.

[2]We have equality for t which are not a root of the discriminant of 1, u, …, $u^{[E:\mathbb{Q}(T)]-1}$.

Note that $v \in \mathbf{k}$, since the Puiseux series at $T = \infty$ of $\xi^{(1)}, \dots, \xi^{(d)}$ are Laurent series with coefficients in $\mathbf{k}$. On the other hand, the conjugates of $u_i^{\pm 1}$ are in $\mathbb{Z}[T, \xi^{(1)}, \dots, \xi^{(d)}]$ (since u_i are units) and *a fortiori* in $R := \mathcal{O}_{\mathbf{k}}[T, \xi^{(1)}, \dots, \xi^{(d)}]$. The same holds for v. Since $\mathcal{O}_{\mathbf{k}}^*$ is a set of roots of unity, it is enough to show that

$$R \cap \mathbf{k} = \mathcal{O}_{\mathbf{k}}.$$

We choose a basis $v_1 = 1, v_2, \dots, v_\delta$ of the $\mathbf{k}(T)$-vector space

$$\mathbf{k}(T, \xi^{(1)}, \dots, \xi^{(d)}).$$

Then, there exists non-zero $F \in \mathbb{Z}[T]$ such that

$$F \cdot R \subseteq \mathcal{O}_{\mathbf{k}}[T] + \mathcal{O}_{\mathbf{k}}[T]v_2 + \cdots + \mathcal{O}_{\mathbf{k}}[T]v_\delta.$$

Let $z \in R \cap \mathbf{k}$. A "Landau trick" with $z, z^2, z^3, \dots$ shows that z lies in $\mathcal{O}_{\mathbf{k}}$. Namely, for all $j \in \mathbb{N}$ we have $z^j \in R \cap \mathbf{k}$ and, by the linear independence of $v_1, \dots, v_\delta$, we deduce that $z^j F \in \mathcal{O}_{\mathbf{k}}[T]$. Thus $z \in \mathcal{O}_{\mathbf{k}}$.

□

4. Proofs of Theorems 2.2 and 2.5

The main ingredient in the proof of theorems 2.2 and 2.5 is the specialization theorem [1, Theorem 1.3] (p.2601), which we recall for the reader's convenience.

Let $\mathbb{F}$ be the function field of an algebraic curve $\mathcal{C}$ defined over $\overline{\mathbb{Q}}$. Given a subgroup Γ of $\mathbb{G}_{\mathrm{m}}^d$ defined over $\mathbb{F}$ we say that Γ is constant-free if its image Γ' by any surjective homomorphism $\mathbb{G}_{\mathrm{m}}^d \to \mathbb{G}_{\mathrm{m}}$ satisfies $\Gamma' \cap \overline{\mathbb{Q}}^* = \Gamma'_{\mathrm{tors}}$.

Theorem 4.1 ([1], Theorem 1.3 (p.2601))**.** *Let $\Gamma \subset \mathbb{G}_{\mathrm{m}}^d(\mathbb{F})$ be a finitely generated constant-free subgroup and let V be a subvariety of $\mathbb{G}_{\mathrm{m}}^d$ defined over $\mathbb{F}$. Then the points $t \in \mathcal{C}(\overline{\mathbb{Q}})$, such that for some $\boldsymbol{\gamma} \in \Gamma \setminus V$ the value $\boldsymbol{\gamma}_t$ is defined and lies in V_t, have bounded height.*

We now prove Theorem 2.2. Let us first recall the relevant notations. Let, as in Remark 3.4(i), $\sigma_1, \dots, \sigma_d$ be the $\mathbb{Q}(T)$-immersions of $K := \mathbb{Q}(T, \xi)$ in an algebraic closure $\overline{K}$, which we identify with the $\mathbb{Q}$-immersions of $\mathbb{Q}(\xi_t)$ in $\overline{\mathbb{Q}}$. Let $\Psi \colon \overline{K}^d \to \overline{K}^d$ be the linear isomorphism

$$\Psi(\mathbf{X}) = \Big(\sum_{i=0}^{d-1} X_i \sigma_1(\xi)^{i-1}, \dots, \sum_{i=0}^{d-1} X_i \sigma_d(\xi)^{i-1}\Big).$$

We remark that for $t \in \mathbb{N}$ sufficiently large the specialized map $\Psi_t \colon \overline{\mathbb{Q}}^d \to \overline{\mathbb{Q}}^d$,

$$\Psi_t(\mathbf{x}) = \Big(\sum_{i=0}^{d-1} x_i \sigma_1(\xi_t)^{i-1}, \dots, \sum_{i=0}^{d-1} x_i \sigma_d(\xi)_t^{i-1}\Big).$$

is still an isomorphism.

We consider a solution $\mathbf{x} = (x_0, \dots, x_{d-1}) \in W_t(\mathbb{Z})$ of (2.2). Then

$$\eta := x_0 + x_1\xi_t + \cdots + x_{d-1}\xi_t^{d-1}$$

is a unit of $\mathbb{Z}[\xi_t]$. By assumption $\mathbb{Z}[\xi_t]^* = (\mathbb{Z}[T,\xi]^*)_t$, thus $\eta \in (\mathbb{Z}[T,\xi]^*)_t$ and there exists $\gamma \in \mathbb{Z}[T,\xi]^*$ such that $\eta = \gamma_t$. Let $\boldsymbol{\gamma} = (\sigma_j(\gamma))_{j=1,\dots,d}$ which is in the finite rank subgroup

$$\Gamma = \{(\sigma_1(u), \dots, \sigma_d(u)) \mid u \in \mathbb{Z}[T,\xi]^*\}$$

of $(\overline{K}^*)^d$. A typical element v in the image Γ' of Γ by a surjective homomorphism $\mathbb{G}_{\mathrm{m}}^d \to \mathbb{G}_{\mathrm{m}}$ lies in the group generated by some $\sigma_1(u), \dots, \sigma_d(u)$. So by Assumption 2.1(2) v is a root of unity as soon as it is an algebraic number. This proves that Γ is constant-free.

We have $\mathbf{x} = \Psi_t^{-1}(\boldsymbol{\gamma}_t)$ and thus $\mathbf{X} := \Psi^{-1}(\boldsymbol{\gamma})$ specializes to $\mathbf{x}$. Moreover $\mathbf{X} \in \mathbb{Z}[T]^d$ (since $\boldsymbol{\gamma}$ is stable by the Galois action), $\mathrm{Norm}^K_{\mathbb{Q}(T)}(\gamma) = \pm 1$ and the sign $+$ holds since $\mathbf{X}_t = \mathbf{x}$, *i. e.* $\mathrm{Norm}(\eta) = 1$ and $\mathbf{X}$ satisfies (2.3). Thus we cannot have $\mathbf{X} \in W$.

We denote by V the intersection of $\Psi(W)$ with $\mathbb{G}_{\mathrm{m}}^d$. Observe that $V \neq \mathbb{G}_{\mathrm{m}}^d$ since W is a proper subvariety of $\mathbb{A}^d$. By the previous discussion, $\boldsymbol{\gamma} = \psi(\mathbf{X})$ is not in V and[3] $\boldsymbol{\gamma}_t = \psi_t(\mathbf{x}) \in V_t$. Theorem 4.1 then asserts that $h(t)$ is bounded, hence (t being in $\mathbb{N}$) t is bounded.

□

[3]We can assume $x_i \neq 0$; otherwise, replace W by a subvariety.

We now prove Theorem 2.5. The proof is similar to the proof of Theorem 2.2. First recall that, by Theorem 2.3, $[\mathbb{Z}[\xi_t]^* : (\mathbb{Z}[T,\xi]^*)_t]$ is uniformly bounded for $t \geq t_1$. Thus $l = \mathrm{lcm}_{t\geq t_1}[\mathbb{Z}[\xi_t]^* : (\mathbb{Z}[T,\xi]^*)_t]$ is well defined and can be effectively bounded.

Let $t \in \mathbb{N}$ large enough. We consider a solution $\mathbf{x} = (x_0, \ldots, x_{d-1}) \in W_t(\mathbb{Z})$ of (2.2). Then

$$\eta := x_0 + x_1\xi_t + \cdots + x_{d-1}\xi_t^{d-1}$$

is a unit of $\mathbb{Z}[\xi_t]$ of norm 1 and $\eta^{l_t} \in (\mathbb{Z}[T,\xi]^*)_t$. Let us assume that $\mathbf{x}$ is not a specialization of a functional solutions $\mathbf{X} \in W(\overline{\mathbb{Q}(T)})$ of (2.3) satisfying $\Psi^{-1}\big(\Psi(\mathbf{X})^l\big) \in \mathbb{Z}[T]^d$.

We modify the argument of the proof of Theorem 2.2 as follow. Let $l_t := [\mathbb{Z}[\xi_t]^* : (\mathbb{Z}[T,\xi]^*)_t]$. We select an algebraic function γ such that $\gamma^{l_t} \in \mathbb{Z}[T,\xi]^*$ and we extend the specialization $\Xi \mapsto \Xi_t$ in such a way that $\gamma_t = \eta$. For $j = 1, \ldots, d$ we also extend σ_j to $K(\gamma)$ in such a way that $\sigma_j(\gamma)_t = \sigma_j(\eta)$ for $j = 1, \ldots, d$ and we denote $\boldsymbol{\gamma} = (\sigma_j(\gamma))_{j=1,\ldots,d}$. Then $\boldsymbol{\gamma}$ is in the finite rank group

$$\Gamma_l = \{\boldsymbol{\gamma} \mid \boldsymbol{\gamma}^{l_t} \in \Gamma\}$$

which remains constant-free as Γ. Let $\mathbf{X} := \Psi^{-1}(\boldsymbol{\gamma}) \in \overline{K}^d$ and $\omega := \sigma_1(\gamma)\cdots\sigma_d(\gamma)$. Then $\omega^{l_t} = \mathrm{Norm}^K_{\mathbb{Q}(T)}(\gamma^{l_t}) = \pm 1$. Since $\gamma_t = u$ and $\mathrm{Norm}(u) = 1$ we have $\omega = 1$. Thus $\mathbf{X}$ is a solution of (2.3) which specializes to $\mathbf{x} = \Psi_t^{-1}(\boldsymbol{\gamma}_t)$. Since $\boldsymbol{\gamma}^{l_t}$ is stable by the Galois action, we have $\Psi^{-1}\big(\Psi(\mathbf{X})^{l_t}\big) = \Psi^{-1}(\boldsymbol{\gamma}^{l_t}) \in \mathbb{Z}[T]^d$. Since $l_t \mid l$ we deduce that $\Psi^{-1}\big(\Psi(\mathbf{X})^l\big) \in \mathbb{Z}[T]^d$. Thus $\mathbf{X} \notin W$. This proves that $\boldsymbol{\gamma}$ is not in V and $\boldsymbol{\gamma}_t \in V_t$. By Theorem 4.1, t is bounded.

5. Proof of theorems 2.4 and 2.3

To prove Theorem 2.4 we need some more technical results. Let $E/\mathbb{Q}(T)$ be a subextension of $K/\mathbb{Q}(T)$ of degree $d' > 1$. Remark that $E \cap \mathbb{Q}[T,\xi]$ is a finitely generated torsion-free module of rank d' over the principal domain $\mathbb{Q}[T]$, and therefore there exists a $\mathbb{Q}[T]$-basis, $\omega^{(1)}, \ldots, \omega^{(d')}$. To prove the lower bound for the height in Theorem 2.4 we first relate the discriminant of the order $\mathbb{Z}[\xi_t] \cap E_t$ with the discriminant (ideal) of $(E \cap \mathbb{Q}[T,\xi])/\mathbb{Q}[T]$. This will allow to get, in Proposition 5.3, a lower bound for the maximum (in absolute value) conjugate of a generator $\eta \in \mathbb{Z}[\xi_t] \cap E_t$ of $E_t/\mathbb{Q}$.

Lemma 5.1. *Let us assume 2.1 and let $t \in \mathbb{N}$ be sufficiently large. Let $E/\mathbb{Q}(T)$ be a subextension of $K/\mathbb{Q}(T)$. We consider the $\mathbb{Q}(T)$-vector space*

$$V = \{(c_0, \ldots, c_{d-1}) \in \mathbb{Q}(T)^d \mid c_0 + c_1\xi + \cdots + c_{d-1}\xi^{d-1} \in E\}.$$

and we let V_t be the vector space over $\mathbb{Q}$ obtained by specializing at t the elements of V (those defined at t). Then

$$V_t = \{(b_0, \ldots, b_{d-1}) \in \mathbb{Q}^d \mid b_0 + b_1\xi_t + \cdots + b_{d-1}\xi_t^{d-1} \in E_t\}. \tag{5.1}$$

Thus $\dim_{\mathbb{Q}} V_t = [E : \mathbb{Q}(T)]$.

Proof. To prove the non trivial inclusion "$\supseteq$" in (5.1), let $(b_0, \ldots, b_{d-1}) \in \mathbb{Q}^d$ be a vector in the set on the right, *i. e.* such that the specialization of

$$v := b_0 + b_1\xi + \cdots + b_{d-1}\xi^{d-1} \in K_0$$

is in E_t. Then $v_t = u_t$ for some $u \in E$ regular at t. We write

$$u = c_0 + c_1\xi + \cdots + c_{d-1}\xi^{d-1}, \quad c_i \in \mathbb{Q}[T].$$

Thus $(c_0, \ldots, c_{d-1}) \in V$ and

$$u - v = (c_0 - b_0) + (c_1 - b_1)\xi + \cdots + (c_{d-1} - b_{d-1})\xi^{d-1} \in K_0$$

specializes at 0. Let $m \in \mathbb{Z}$ be the least integer such that $(T-t)^m(c_i - b_i)$ are all regular at t. Thus the specializations $(T-t)^m(c_i - b_i)$ at t are defined and not all zero. Let us assume by contradiction that at least one of the $(c_i - b_i)$'s does not vanish at t. Then $m \geq 0$ and $(T-t)^m u$ specializes to 0 at t. This gives a non trivial $\mathbb{Q}$-linear combination of 1, ξ_t, ..., ξ_t^{d-1} vanishing, contrary to assertion (2) of Lemma 3.3. This shows that $b_i = c_i(t)$ for $i = 0, \ldots, d-1$ as desired.

By Lemma 3.3(2) $[E_t : \mathbb{Q}] = [E : \mathbb{Q}]$ for $t \in \mathbb{N}$ sufficiently large. Thus (5.1) implies $\dim_{\mathbb{Q}} V_t = [E_t : \mathbb{Q}] = [E : \mathbb{Q}]$.

□

Lemma 5.2. *Let us assume 2.1 and let $t \in \mathbb{N}$ be sufficiently large. Let $E/\mathbb{Q}(T)$ be a subextension of $K/\mathbb{Q}(T)$ of degree d' and choose a $\mathbb{Q}[T]$-basis $\omega^{(1)}, \ldots, \omega^{(d')}$ of $E \cap \mathbb{Q}[T, \xi]$. Then there exists an integer $\delta \neq 0$ such that*

$$\delta(\mathbb{Z}[\xi_t] \cap E_t) \subseteq \mathbb{Z}\omega_t^{(1)} + \ldots + \mathbb{Z}\omega_t^{(d')}.$$

Proof. Let V and V_t as in the statement of Lemma 5.1 and let $\Lambda := V \cap \mathbb{Q}[T]^d$ which is a free $\mathbb{Q}[T]$-module of rank d'. Hence, defining $\mathbf{c}^{(1)}, \ldots, \mathbf{c}^{(d')} \in \mathbb{Q}[T]^d$ by

$$\omega_i = \mathbf{c}_0^{(i)} + \mathbf{c}_1^{(i)}\xi + \cdots + \mathbf{c}_{d-1}^{(i)}\xi^{d-1}, \quad i = 1, \ldots, d',$$

we have that actually $\mathbf{c}^{(1)}, \ldots, \mathbf{c}^{(d')}$ is a basis of Λ over $\mathbb{Q}[T]$. Note that Λ is saturated[4] in $\mathbb{Q}[T]^d$. Since $\mathbb{Q}[T]$ is principal, the $d \times d'$ matrix having the transposition of $\mathbf{c}^{(1)}, \ldots, \mathbf{c}^{(d')}$ as column vectors may be completed[5] to a $d \times d$ matrix Γ with entries in $\mathbb{Q}[T]$ and determinant 1. We also remark that for t large enough, the specialization $\mathbf{c}_t^{(1)}, \ldots, \mathbf{c}_t^{(d')}$ remain $\mathbb{Q}$-linearly independent; thus by Lemma 5.1 they form a basis for $V_t/\mathbb{Q}$.

Let now $\eta \in \mathbb{Z}[\xi_t] \cap E_t$. Then we can write $\eta = a_0 + a_1\xi_t + \ldots + a_{d-1}\xi_t^{d-1}$ with $\mathbf{a} = (a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d$. Since $\eta \in E_t$, we have that $\mathbf{a} \in V_t$, so $\mathbf{a} = \sum_{i=1}^{d'} b_i \mathbf{c}_t^{(i)}$, with $b_i \in \mathbb{Q}$. This means that, letting $\mathbf{x}$ be the transpose of $\mathbf{a}$ and $\mathbf{y}$ whose of $(b_1, \ldots, b_{d'}, 0, \ldots, 0)$, we have

$$\mathbf{x} = \Gamma_t \mathbf{y}.$$

[4]This is because Λ is the intersection of $\mathbb{Q}[T]^d$ with $\mathbb{Q}(T)^d$ and $\mathbb{Q}(T)$ is the fraction field of $\mathbb{Q}[T]$.

[5]This amounts to completing $\mathbf{c}^{(1)}, \ldots, \mathbf{c}^{(d')}$ to a $\mathbb{Q}[T]$-basis of $\mathbb{Q}[T]^d$, and for this it suffices to lift a basis of $\mathbb{Q}[T]^d/\Lambda$ to $\mathbb{Q}[T]^d$.

Inverting this equation yields $\mathbf{y} = \Gamma_t^{-1}\mathbf{x}$, and now the conclusion follows because $\mathbf{x}$ has integer coordinates and Γ_t^{-1} has rational entries with denominator bounded independently of t, since $\det \Gamma = 1$ and since the entries of Γ are fixed polynomials in $\mathbb{Q}[T]$.

□

Proposition 5.3. *Let us assume 2.1 and let $t \in \mathbb{N}$ be sufficiently large. Let $E/\mathbb{Q}(T)$ be a subextension of $K/\mathbb{Q}(T)$ of degree $d' > 1$. Then for any generator $\eta \in \mathbb{Z}[\xi_t] \cap E_t$ of $E_t/\mathbb{Q}$, the maximum (in absolute value) conjugate of η over $\mathbb{Q}$ is*

$$\geq c_0 |t|^{\frac{1}{d'(d'-1)}}$$

in absolute value, for some $c_0 > 0$ independent of η and t.

Proof. Let $\omega^{(1)}, \ldots, \omega^{(d')}$ be a $\mathbb{Q}[T]$-basis of $E \cap \mathbb{Q}[T, \xi]$. Lemma 5.2, applied to η^j, $j = 0, 1, \ldots, d' - 1$, implies that

$$\delta \eta^j = c_{1j}\omega_t^{(1)} + \ldots + c_{d'j}\omega_t^{(d')},$$

for a certain integer $\delta \neq 0$ independent of t and η, and suitable integers c_{ij}. Conjugating these equations $d' = [E_t : \mathbb{Q}]$ times we obtain a matrix equation $\delta U = \Omega C$, where the rows of U are the conjugates of the row vector $(1, \eta, \ldots, \eta^{d'-1})$, where Ω has row vectors the conjugates of $(\omega_t^{(1)}, \ldots, \omega_t^{(d')})$ and where C is the matrix of the c_{ij}. Taking determinants we obtain $\delta^{d'} \det U = \det \Omega \det C$.

Now, $\det C$ is a nonzero integer (since η generates $E_t/\mathbb{Q}$), whereas $(\det \Omega)^2$ is the value at t of a nonconstant polynomial, a generator of the discriminant (ideal) of $(E \cap \mathbb{Q}[T, \xi])/\mathbb{Q}[T]$ (this is nonconstant because $d' > 1$ so there must be ramification at some finite point). Hence $|\det U| \gg |t|^{1/2}$, and now the conclusion follows at once.

□

To prove Theorem 2.4 we also need a uniform statement of Remark 3.4(ii) for units $u \in \mathbb{Z}[T, \xi]^*$. We first recall some basic fact on decomposition groups in function fields extensions.

Let $L/\mathbb{Q}(T)$ be a normal extension. Corresponding to t and the chosen point $p_{L,t}$ of $\mathcal{C}_L$ we have a decomposition group

$$\Delta_t = \Delta(p_{L,t}) \subset \mathrm{Gal}(L/\mathbb{Q}(T)).$$

Recall that this is the subgroup of $\mathrm{Gal}(L/\mathbb{Q}(T))$ which stabilizes the Galois orbit of p_t over $\mathbb{Q}$ (and thus fixes the place corresponding to this Galois orbit).

For non ramified points the group Δ_t is isomorphic (see [14, Proposition 20] (p.21) and [15, Theorem 3.8.2] (p.131)) to the Galois group of $L_t/\mathbb{Q}$, the isomorphism being given by

$$\begin{aligned} \Delta_t &\to \mathrm{Gal}(L_t/\mathbb{Q}) \\ \sigma &\mapsto \tilde{\sigma} \end{aligned} \tag{5.2}$$

where

$$\tilde{\sigma}(u_t) = \sigma(u)_t$$

for $u \in L$.

We can now prove our uniform version of Remark 3.4(ii).

Proposition 5.4. *Let us assume 2.1(2). For $t \in \mathbb{N}$ sufficiently large (effectively) and for $\mu \in \mathbb{Z}[T,\xi]^*$ we have $\mathbb{Q}(T,\mu)_t = \mathbb{Q}(\mu_t)$.*

Proof. Let $\widehat{K}$ be the normal closure of K over $\mathbb{Q}(T)$ and let $E = \mathbb{Q}(T,\mu)$. It is clear that $\mu_t \in E_t$. We have to prove that $E_t \subseteq \mathbb{Q}(\mu_t)$, or equivalently by Galois Theory, that $\mathrm{Gal}(\widehat{K}_t/\mathbb{Q}(\mu_t))$ fixes E_t. Let $\sigma \in \mathrm{Gal}(\widehat{K}_t/\mathbb{Q}(\mu_t)) \subseteq \mathrm{Gal}(\widehat{K}_t/\mathbb{Q})$. We identify σ to an element of the decomposition group $\Delta_t = \Delta(p_{\widehat{K},t})$.

Thus $\mu_t = \sigma(\mu_t) = \sigma(\mu)_t$. Let $\gamma := \mu\sigma(\mu)^{-1}$ which stays in the finite rank subgroup

$$\Gamma_\sigma = \{u\sigma(u)^{-1} \mid u \in \mathbb{Z}[T,\xi]^*\}.$$

A typical element $v \in \Gamma_\sigma$ lies in the group generated by the conjugates of some $u \in \mathbb{Z}[T,\xi]^*$. So by Assumption 2.1(2) v is a root of unity as soon as it is an algebraic number. This shows that the only constants in Γ_σ are roots of unity. Since γ specializes at 1, by Theorem 3.2 we have $\sigma(\mu) = \mu$ provided that t is sufficiently large (w.r.t. the finite number of subgroups Γ_τ with $\tau \in \mathrm{Gal}(\widehat{K}/\mathbb{Q})$, and thus uniformly in μ). But then σ fixes E and $\sigma \in \Delta_t \cap \mathrm{Gal}(\widehat{K}/E)$. Thus σ fixes E_t, the fixed field in $\widehat{K}_t$ of $\Delta_t \cap \mathrm{Gal}(\widehat{K}/E)$ (viewed as a subgroup of $\mathrm{Gal}(\widehat{K}_t/\mathbb{Q})$).

□

We will use this proposition coupled with the following remark :

Lemma 5.5. *Let F/E be a finite extension of number fields. Let $\alpha \in F$ such that $E = \mathbb{Q}(\alpha^m)$ for some $m \geq 1$. Then there exists an integer m_0, bounded in terms only on the degree $[F : \mathbb{Q}]$, such that $E = \mathbb{Q}(\alpha^{m_0})$.*

Proof. We define m_0 as the least positive integer such that $\alpha^{m_0} \in E$. Then $m_0 \,|\, m$ and $\mathbb{Q}(\alpha^{m_0}) \subseteq E = \mathbb{Q}(\alpha^m) \subseteq \mathbb{Q}(\alpha^{m_0})$. Hence $\mathbb{Q}(\alpha^m) = \mathbb{Q}(\alpha^{m_0})$. Moreover, since α is a root of $X^m - \alpha^m \in E[X]$ we have $\mathrm{Norm}_E^F(\alpha) = \zeta\alpha^\delta$ where $\delta = [F : E]$ and where ζ is a m-th root of unity. Since ζ is in F, its order k is bounded in terms on $[F : \mathbb{Q}]$. This proves that $\alpha^{\delta k} \in E$. Since $m_0 | \delta k$, the conclusion follows. □

Proof of Theorem 2.4. Let t be large enough and $\eta \in \mathbb{Z}[\xi_t]^*$ be non torsion. To get a lower bound for $h(\eta)$, write $\eta = a_0 + a_1\xi_t + \ldots + a_{d-1}\xi_t^{d-1}$ with $a_0, \ldots, a_{d-1} \in \mathbb{Z}$ and let $u = a_0 + a_1\xi + \ldots + a_{d-1}\xi^{d-1} \in \mathbb{Z}[\xi]$ (note that u may still depend, in a non-algebraic way, on t). For large t we have $\mathrm{rank}\,\mathbb{Z}[\xi_t]^* = \mathrm{rank}\,(\mathbb{Z}[T,\xi]^*)_t = r$ by Lemma 3.3(1). Thus the index $l_t := [\mathbb{Z}[\xi_t]^* : (\mathbb{Z}[T,\xi]^*)_t]$ is finite. By definition of index, we have $\eta^{l_t} = \mu_t \in (\mathbb{Z}[\xi,T]^*)_t$ for some $\mu \in \mathbb{Z}[\xi,T]^*$ (possibly still depending on t). Let $E = \mathbb{Q}(T,\mu)$. Note that $d' := [E : \mathbb{Q}(T)] > 1$ (otherwise $\mu \in \mathbb{Z}[\xi,T]^* \cap \mathbb{Q}(T) = \mathbb{Z}[T]^* = \{\pm 1\}$ and hence η is torsion, contrary to our assumption). By Proposition 5.4 $E_t = \mathbb{Q}(\mu_t) = \mathbb{Q}(\eta^{l_t})$. By Lemma 5.5, $\mathbb{Q}(\eta^{l_t}) = \mathbb{Q}(\eta^{m_0})$

for some m_0 bounded only in term of $[\mathbb{Q}(\eta):\mathbb{Q}]$ and thus independently of t. By Proposition 5.3, we have

$$h(\eta) = h(\eta^{m_0})/m_0 \geq c \log t.$$

$\square$

Remark 5.6. *It would be interesting to extend Theorem 2.4 to arbitrary non constant elements $\eta \in \mathbb{Z}[\xi_t]$, not necessarily units. The obstruction in applying our method depends on the fact that $\mathbb{Q}(\eta)$ is not necessarily the specialization L_t of a subextension $L/\mathbb{Q}(T)$ of $K/\mathbb{Q}(T)$. This holds under stronger assumptions, e.g. if $[\widehat{K}_t : \mathbb{Q}(T)] = [\widehat{K} : \mathbb{Q}]$, which ensures that $\mathrm{Gal}(\widehat{K}_t/\mathbb{Q}(T)) \cong \mathrm{Gal}(\widehat{K}/\mathbb{Q})$.*

We can now prove our uniform bound for the index.

Proof of Theorem 2.3. We denote by c_1, c_2, c_3 positive constants depending only on the algebraic function ξ. For large t we have $\mathrm{rank}\,\mathbb{Z}[\xi_t]^* = \mathrm{rank}\,(\mathbb{Z}[T,\xi]^*)_t = r$ by Lemma 3.3(1). Thus the index $l_t := [\mathbb{Z}[\xi_t]^* : (\mathbb{Z}[T,\xi]^*)_t]$ is finite. The logarithmic embedding $\mathcal{L}\colon K^* \to \mathbb{R}^r$ provides an isomorphism of $\mathbb{Z}[\xi_t]^*/\mathbb{Z}[\xi_t]^*_{\mathrm{tors}}$ with a lattice. It is easy to see that the euclidean norm of a unit η in K is at least $(d/4\sqrt{r})h(\eta)$, and clearly the norm is zero if η is torsion. Thus by Minkowki's Theorem, there exists a non-torsion $\eta \in \mathbb{Z}[\xi_t]^*$ such that

$$h(\eta) \leq c_1 \mathrm{Vol}(\mathbb{Z}[\xi_t]^*)^{1/r} = c_1(\mathrm{Vol}((\mathbb{Z}[T,\xi]^*)_t)/l_t)^{1/r}.$$

Let $\gamma^{(1)}, \ldots, \gamma^{(r)} \in \mathbb{Z}[T,\xi]^*$ be a basis modulo torsion. By Theorem 3.2 $\gamma_t^{(1)}, \ldots, \gamma_t^{(r)}$ is a basis of $(\mathbb{Z}[T,\xi]^*)_t$ modulo torsion. We have $h(\gamma_t^{(j)}) \leq c_2 \log t$, since $\gamma^{(1)}, \ldots, \gamma^{(r)}$ have been fixed independently of t. Thus

$$\mathrm{Vol}((\mathbb{Z}[T,\xi]^*)_t) \leq h(\gamma_t^{(1)}) \cdots h(\gamma_t^{(r)}) \leq c_2^r (\log t)^r$$

and

$$h(\eta) \leq c_3 (\log t)/l_t^{1/r}. \tag{5.3}$$

From (5.3) and from Theorem 2.4 we get $c \log t \leq c_3(\log t)/l_t^{1/r}$ which shows that $l_t \leq (c_3/c)^r$ is bounded independently of t.

$\square$

6. Proof of Theorem 1.1

We shall need the following

Lemma 6.1. *The Puiseux series at $T = \infty$ of the solutions X of*

$$(X-A)(X-B)(X-C)+1 = 0$$

are Laurent series with coefficients in $\mathbb{Q}$.

Proof. We note first that if a, b, c are distinct integers then $f(x) = (x-a)(x-b)(x-c)+1$ has a zero z with

$$|z-a| \leq \frac{9}{|a-b||a-c|}.$$

This is trivial if $p = |a-b||a-c| \leq 9$, as the product of the zeroes of $f(y+a)$ is 1. If $p \geq 10$ we use Rouché with $g(x) = f(x) - 1$, because on $|x-a| = 5/p$ we have $|f-g| = 1$ while

$$|g| \geq \frac{5}{p}\left(|a-b| - \frac{5}{p}\right)\left(|a-c| - \frac{5}{p}\right) \geq \frac{5}{p}\frac{|a-b|}{2}\frac{|a-c|}{2} = \frac{5}{4}.$$

Now we apply this with a, b, c the values of A, B, C at $t > 0$ large enough. We find a zero $\alpha = \alpha(t)$ with $\alpha - A(t) \to 0$ as $t \to \infty$. Similarly β, γ with $\beta - B(t) \to 0$, $\gamma - C(t) \to 0$.

These mean that there must be Puiseux series with principal parts A, B, C; as those are different they are a complete set. If one of them was a series in $T^{-1/e}$ for some (minimal) $e \geq 2$, then applying a non-trivial element of the Galois group of $\mathbb{C}(T^{1/e})$ over $\mathbb{C}(T)$ would induce a non-trivial permutation; also impossible from the principal parts. So they are Laurent series. Finally applying the Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$ to the coefficients shows in a similar way that they are in $\mathbb{Q}$.

□

For Ziegler's example $A = 0, B = T, C = T^4 + 3T$ we find indeed

$$A - \frac{1}{T^5} + \frac{3}{T^8} - \frac{8}{T^{11}} + \frac{22}{T^{14}} - \frac{65}{T^{17}} + \cdots,$$
$$B + \frac{1}{T^5} - \frac{2}{T^8} + \frac{3}{T^{11}} - \frac{3}{T^{14}} + \frac{9}{T^{17}} + \cdots,$$
$$C - \frac{1}{T^8} + \frac{5}{T^{11}} - \frac{19}{T^{14}} + \frac{65}{T^{17}} - \frac{213}{T^{20}} + \cdots$$

even with coefficients in $\mathbb{Z}$ (this does not always happen).

Back to the general proof. For $t > 0$ sufficiently large, the values $A(t), B(t), C(t)$ are different, and it is easily seen that there is a fixed permutation of A, B, C such that $A(t) < B(t) < C(t)$. Thus $r_0 = B(t) - A(t), s_0 = C(t) - A(t)$ satisfy $1 \leq r_0 \leq s_0 - 1$.

If for all large t we have $r_0 = s_0 - 2$ then $C - B = 2$ and (1.4) reduces to $x'(x'-2y)(x'+ry) + y^3 = 1$ with $x' = x - B(t)y$. For large t we have $r_0 > 0$ large and so it comes down to

$$x(x-2y)(x+ty) + y^3 = 1. \tag{6.1}$$

We could not find this in the literature, but it can be handled in the same way as (1.7) and (1.8). Thus we postpone the details until section 7. We did not work out the functional solutions, but Yuri Bilu very kindly found for $t = 116$ the solutions

$$(x, y) = (1, 0), (0, 1), (2, 1), (-116, 1), (3393262, 1700241)$$

of which the last is perhaps unexpected.

Similarly if for all large t we have $r_0 = s_0 - 1$ then it comes down to

$$x(x-y)(x+ty) + y^3 = 1$$

which is treated by Mignotte and Tzanakis [12] (p.49) - there one must replace (y, n) by $(-y, t-1)$. Here for $t > (3.67)10^{32}$ the solutions are

$$(1, 0), \quad (0, 1), \quad (1, 1), \quad (-t, 1), \quad (1, t-1)$$

of which the last is not among the "obvious" ones.

Thus we may assume $1 \le r_0 \le s_0 - 3$. In this case Theorem 3.9 of Thomas [16] (p.39) tells us that the polynomial $z(z-r_0)(z-s_0)+1$ is irreducible over $\mathbb{Q}$ and that for any zero z_0 the pair $z_0, z_0 - r_0$ is a system of fundamental units for the cubic field $\mathbb{Q}(z_0)$. As this field is real, the unit group is in fact generated by $-1, z_0, z_0 - r_0$.

We are going to apply Theorem 2.2 with

$$P(T, X) = (X - A)(X - B)(X - C) + 1$$

the formal norm of $X - \xi$. It is easy to see that this is irreducible over $\mathbb{Q}(T)$ unless A, B, C are congruent modulo $\mathbb{Z}$; but in that case we can take them all in $\mathbb{Z}$ and then Theorem 1.1 is trivial.

We first check Assumption 2.1(1). We can take $u_1 = \xi - A,\ u_2 = \xi - B$ in part 2.1(1). A multiplicative relation would specialize to one between $\xi_t - A(t),\ \xi_t - B(t)$; but these are $z_0, z_0 - r_0$ above, so independent. Therefore 2.1(1) is proved. On the way we see that $(\mathbb{Z}[T, \xi]^*)_t$, already in $\mathbb{Z}[\xi_t]^*$ which in turn is generated by $-1, \xi_t - A(t),\ \xi_t - B(t)$, is in turn inside $(\mathbb{Z}[T, \xi]^*)_t$. Therefore $\mathbb{Z}[\xi_t]^* = (\mathbb{Z}[T, \xi]^*)_t$ as required for Theorem 2.2.

Now Assumption 2.1(2) follows from Lemma 3.5; by Lemma 6.1 we may take $\mathbf{k} = \mathbb{Q}$.

Finally our diophantine equation reduces to $\mathrm{Norm}(x - \xi_t y) = 1$ and therefore we can apply Theorem 2.2 with W defined by the vanishing of the coefficient of ξ_t^2.

7. Examples

We first treat (1.7). We will apply Theorem 2.2 with $P(T, X) = X^3 - (T^3 - 1)$ and

$$\xi = (T^3 - 1)^{1/3} = T - \frac{1/3}{T^2} - \frac{1/9}{T^5} - \frac{5/81}{T^8} - \frac{10/243}{T^{11}} - \cdots . \tag{7.1}$$

We start by finding $\mathbb{Z}[\xi_t]^*$. It contains $\xi_t - t$ and sits in $\mathbb{R}$ so the only roots of unity are ± 1. We proceed to show that $\mathbb{Z}[\xi_t]^*$ is generated by -1 and $\xi_t - t$.

If not, then it would contain $\eta = (\xi_t - t)^{1/l}$ for some integer $l \ge 2$. Writing $\eta = a + b\xi_t + c\xi_t^2$ with coefficients in $\mathbb{Z}$, taking conjugates and solving for c we would get $|c| \ll t^{-3/2}$ with absolute implied constant. So $c = 0$ for large enough t. Similarly $|b| \ll t^{-1/2}$ so $b = 0$. This leaves us with $\eta = a = \pm 1$ an absurdity.

We now proceed to check Assumption 2.1. We can take $u_1 = \xi - T$ in part 2.1(1); and as before we find easily $\mathbb{Z}[\xi_t]^* = (\mathbb{Z}[T, \xi]^*)_t$.

As for part 2.1(2), we can again apply Lemma 3.5, this time with $\mathbf{k} = \mathbb{Q}(\omega)$ for $\omega = \exp(2\pi i/3)$.

Finally (1.7) reduces to $\mathrm{Norm}(x - \xi_t y + \xi_t^2) = 1$ and so we can again apply Theorem 2.2.

It remains to check that the only X, Y in $\mathbb{Z}[T]$ with

$$X^3 - (T^3 - 1)Y^3 + 3(T^3 - 1)XY + (T^3 - 1)^2 = 1 \tag{7.2}$$

are $(X, Y) = (T^2, 2T),\ (T^2, -T)$.

For this we need to know the generic unit group $\mathbb{Z}[T,\xi]^*$. We will show that it is generated by -1 and $\xi - T$ in line with the specialized situation. Here we could apply Theorem 3.2; but the following argument is more elementary and seems more natural.

Let $A + B\xi + C\xi^2$ be such a unit, with coefficients in $\mathbb{Z}[T]$. Specializing, we obtain for each large integer t an integer $m(t)$ and a sign $\varepsilon(t)$ such that

$$A(t) + B(t)\xi_t + C(t)\xi_t^2 = \varepsilon(t)(\xi_t - t)^{m(t)}. \tag{7.3}$$

As $|\xi_t - t| \ll t^{-2}$ (again with absolute implied constant) we see by making $t \to \infty$ that $m(t) \gg -1$. Taking a conjugate gives

$$A(t) + B(t)\omega\xi_t + C(t)\omega^2\xi_t^2 = \varepsilon(t)(\omega\xi_t - t)^{m(t)}.$$

Now making $t \to \infty$ we see that also $m(t) \ll 1$.

Therefore passing to an infinite subsequence we can assume that $\varepsilon = \varepsilon(t)$ and $m = m(t)$ are independent of t. But now both sides of (7.3) are fixed Laurent series that coincide at infinitely many t. Therefore they coincide identically in T. Thus $A + B\xi + C\xi^2 = \varepsilon(\xi - T)^m$ and we have found the desired generators.

Now factorizing the left-hand side of (7.2) we find

$$X - Y\xi + \xi^2 = \varepsilon(\xi - T)^m \tag{7.4}$$

with its conjugates

$$X - Y\omega\xi + \omega^2\xi^2 = \varepsilon(\omega\xi - T)^m,$$

$$X - Y\omega^2\xi + \omega\xi^2 = \varepsilon(\omega^2\xi - T)^m.$$

Eliminating X, Y using the Siegel identities gives z_0, z_1, z_2 in $\mathbb{C}$ with

$$z_0(\xi - T)^m + z_1(\omega\xi - T)^m + z_2(\omega^2\xi - T)^m = \xi^2. \tag{7.5}$$

At this point we could apply Corollary I (p.427) or Corollary II (p.428) of [6], but there are annoying non-degeneracy conditions to be checked as well as a genus to be calculated. Admittedly Theorem B (p.431) simplifies the non-degeneracy condition. But we can also use the basic argument of [6] directly. Thus (7.5) says that $(\xi - T)^m, (\omega\xi - T)^m, (\omega^2\xi - T), \xi^2$ are linearly dependent over $\mathbb{C}$. Therefore their Wronskian W vanishes identically. When the $m-2$ power of $(\xi - T)(\omega\xi - T)(\omega^2\xi - T) = -1$ is ignored, a calculation gives (up to sign)

$$W = m^2(m-1)(m+1)(m-2)\left((m-5)T^3 - \frac{m}{2} + 1\right)\rho$$

for some fixed $\rho \neq 0$ in $\mathbb{C}(T,\xi)$ independent of m. It follows at once that $m = -1, 0, 1, 2$. Now (7.4) is clearly impossible for $m = 0, 1$; and for $m = 2$ we get $\varepsilon(T^2 - 2T\xi + \xi^2)$ leading to $(T^2, 2T)$ and for $m = -1$ we get $-\varepsilon(T^2 + T\xi + \xi^2)$ leading to $(T^2, -T)$. This completes the treatment of (1.7).

We next treat (6.1) which turned up during the proof of Theorem 1.1. We use Theorem 2.2 with $P(T,X) = X(X-2)(X+T) + 1$. We can take

$$\xi = -T - \frac{1}{T^2} + \frac{2}{T^3} - \frac{4}{T^4} + \frac{10}{T^5} - \frac{26}{T^6} + \frac{68}{T^7} - \frac{183}{T^8} + \cdots. \tag{7.6}$$

with conjugates

$$\xi' = \frac{1/2}{T} + \frac{1/8}{T^2} - \frac{3/16}{T^3} - \frac{19/128}{T^4} + \frac{31/256}{T^5} + \frac{201/1024}{T^6} - \frac{139/2048}{T^7} - \cdots,$$

$$\xi'' = 2 - \frac{1/2}{T} + \frac{7/8}{T^2} - \frac{29/16}{T^3} + \frac{531/128}{T^4} - \frac{2591/256}{T^5} + \frac{26423/1024}{T^6} - \cdots.$$

Now $\mathbb{Z}[\xi_t]^*$ contains ξ_t and $\xi_t - 2$, and again the only roots of unity are ± 1. We proceed to show (for the usual large t) that the first two are independent and together with -1 generate $\mathbb{Z}[\xi_t]^*$.

The independence follows from Theorem 3.2, because a relation $z = \xi^m(\xi-2)^n$ in $\mathbb{C}$ implies by (7.6) that

$$z = (-T + \cdots)^m (-2 + \cdots)^n$$

so $m = 0$; however $\xi - 2$ is not in $\mathbb{C}$ so $n = 0$.

Let η_1, η_2 be generators of $\mathbb{Z}[\xi_t]^*$ modulo torsion, and write l for the index of the group generated by ξ_t and $\xi_t - 2$ in $\mathbb{Z}[\xi_t]^*$. Then

$$\eta_1^l = \pm\xi_t^{m_1}(\xi_t - 2)^{n_1}, \quad \eta_2^l = \pm\xi_t^{m_2}(\xi_t - 2)^{n_2}.$$

Further the lattice $\Lambda = \mathbb{Z}(m_1, n_1) + \mathbb{Z}(m_2, n_2)$ has determinant $l^2/l = l$. By Minkowski (as in the proof of Theorem 2.3) there is non-zero (m, n) in Λ with

$$|m| + |n| \le \sqrt{2l}. \tag{7.7}$$

Thus there is $\eta \ne \pm 1$ in $\mathbb{Z}[\xi_t]^*$ with

$$\eta = \pm\xi_t^{m/l}(\xi_t - 2)^{n/l}. \tag{7.8}$$

We find

$$|\eta| \ll t^{|m|/l}, \quad |\eta'| \ll t^{|m|/l}, \quad |\eta''| \ll t^{|n|/l} \tag{7.9}$$

all $\ll t^{\sqrt{2/l}}$. Writing $\eta = a + b\xi_t + c\xi_t^2$ in the usual way, taking conjugates and using (7.7) shows now

$$|c| \ll t^{-1+\sqrt{2/l}}. \tag{7.10}$$

Thus $c = 0$ as soon as $l \ge 3$. Now using just $\eta = a + b\xi_t$, $\eta' = a + b\xi_t'$ we get also $|b| \ll t^{-1+\sqrt{2/l}}$ so also $b = 0$ as soon as $l \ge 3$; and then a contradiction.

If $l = 2$ there are exactly three possibilities for Λ, namely

$$\mathbb{Z}(1,0) + \mathbb{Z}(0,2), \quad \mathbb{Z}(0,1) + \mathbb{Z}(2,0), \quad \mathbb{Z}(1,1) + \mathbb{Z}(0,2). \tag{7.11}$$

In the first two cases we can get $|m| + |n| = 1 < \sqrt{2l}$ and (7.9) now gives $c = 0$ and then $b = 0$.

In the last case we can take $(m, n) = (1, 1)$ and now (7.9) gives again $c = 0$ and then $b = 0$.

Thus we have determined $\mathbb{Z}[\xi_t]^*$; and since $u_1 = \xi$, $u_2 = \xi - 2$ are in $\mathbb{Z}[T, \xi]^*$ this shows $\mathbb{Z}[\xi_t]^* = (\mathbb{Z}[T, \xi]^*)_t$ as usual.

We next check Assumption 2.1; part (1) is already done.

For part (2) we could again apply Lemma 3.5. This finally disposes of (6.1) and so completes the proof of Theorem 1.1.

We turn now to (1.8), again using Theorem 2.2 this time with $P(T,X) = X^4 - (T^4-1)$ and

$$\xi = (T^4-1)^{1/4} = T - \frac{1/4}{T^3} - \frac{3/32}{T^7} - \frac{7/128}{T^{11}} - \frac{77/2048}{T^{15}} - \cdots . \tag{7.12}$$

Now $\mathbb{Z}[\xi_t]^*$ contains $\xi_t - t$ and $-\xi_t - t$, and again the only roots of unity are ± 1. We proceed to show (for the usual large t) that the first two are independent and together with -1 generate $\mathbb{Z}[\xi_t]^*$.

The independence follows again from Theorem 3.2, because a relation $z = (\xi - T)^m(-\xi - T)^n$ in $\mathbb{C}$ implies by (7.12) that

$$z = \left(-\frac{1/4}{T^3} + \cdots\right)^m (-2T + \cdots)^n$$

so $n = 3m$; however $(\xi - T)(-\xi - T)^3$ is not in $\mathbb{C}$ so $m = 0$.

Let η_1, η_2 be generators of $\mathbb{Z}[\xi_t]^*$ modulo torsion, and write l for the index of the group generated by $\xi_t - t$ and $-\xi_t - t$ in $\mathbb{Z}[\xi_t]^*$. Then as above we find non-zero (m,n) in Λ with (7.7), and then $\eta \neq \pm 1$ in $\mathbb{Z}[\xi_t]^*$ with

$$\eta = \pm(\xi_t - t)^{m/l}(-\xi_t - t)^{n/l}. \tag{7.13}$$

Writing $\eta = a + b\xi_t + c\xi_t^2 + d\xi_t^3$ in the usual way, taking conjugates and using (7.7) shows now

$$|d| \ll t^{-3+\sqrt{2/l}}, \quad |c| \ll t^{-2+\sqrt{2/l}}, \quad |b| \ll t^{-1+(|m|+|n|)/l} \leq t^{-1+\sqrt{2/l}}. \tag{7.14}$$

Thus $d = c = b = 0$ as soon as $l \geq 3$; and then a contradiction.

If $l = 2$ we still get $d = c = 0$. In the first two cases of (7.11) we can get $|m| + |n| = 1 < \sqrt{2l}$ and (7.10) now gives also $b = 0$.

In the last case we can take $(m,n) = (1,1)$ and now (7.13) gives

$$a + b\xi_t = \pm(\xi_t - t)^{1/2}(-\xi_t - t)^{1/2}.$$

But $|\xi_t - t| \ll t^{-3}$ so $|a + b\xi_t| \ll t^{-1}$. Taking a single conjugate with $-\xi_t$ gives similarly $|a - b\xi_t| \ll t^{-1}$ and so again $b = 0$ (and even $a = 0$ into the bargain).

Thus we have determined $\mathbb{Z}[\xi_t]^*$; and since $u_1 = \xi - T$, $u_2 = -\xi - T$ are in $\mathbb{Z}[T,\xi]^*$ this shows $\mathbb{Z}[\xi_t]^* = (\mathbb{Z}[T,\xi]^*)_t$ as usual.

We next check Assumption 2.1; part (1) is already done, and part (2) follows as before from Lemma 3.5 now with $\mathbf{k} = \mathbb{Q}(i)$,

Finally (1.8) reduces to $\mathrm{Norm}(x + \xi_t y + \xi_t^2 z) = 1$ and so we can again apply Theorem 2.2.

Thus we must now find all X, Y, Z in $\mathbb{Z}[T]$ with

$$X^4 - (T^4-1)Y^4 + (T^4-1)^2 Z^4 - 2(T^4-1)XZ(XZ - 2Y^2) = 1. \tag{7.15}$$

We first prove that $\mathbb{Z}[T,\xi]^*$ is generated by $-1, \xi - T, -\xi - T$. Again we could appeal to Theorem 3.2 but we use something more elementary. Specializing such a unit $A + B\xi + C\xi^2 + D\xi^3$ gives

$$A(t) + B(t)\xi_t + C(t)\xi_t^2 + D(t)\xi_t^3 = \varepsilon(t)(\xi_t - t)^{m(t)}(-\xi_t - t)^{n(t)}$$

much as before. Making $t \to \infty$ gives $-3m(t) + n(t) \ll 1$. With conjugate $-\xi_t$ we get $m(t) - 3n(t) \ll 1$. And with conjugate $i\xi_t$ we get $m(t) + n(t) \ll 1$. Drawing a picture we see that $|m(t)| + |n(t)| \ll 1$; or, for a bad artist,

$$-8m(t) = 3(-3m(t) + n(t)) + (m(t) - 3n(t)) \ll 1$$

$$4m(t) = (m(t) - 3n(t)) + 3(m(t) + n(t)) \ll 1$$

and similarly for $n(t)$.

So now the argument above with subsequences and Laurent series does the trick.

Next (7.15) implies

$$X + Y\xi + Z\xi^2 = \pm(\xi - T)^m(-\xi - T)^n. \tag{7.16}$$

Taking conjugates and eliminating X, Y, Z gives now the linear dependence of

(7.17)
$$(\xi-T)^m(-\xi-T)^n,\ (-\xi-T)^m(\xi-T)^n,\ (i\xi-T)^m(-i\xi-T)^n,\ (-i\xi-T)^m(i\xi-T)^n$$

over $\mathbb{C}$. This time the Wronskian comes out as (up to sign)

$$(m-n)^2(c_0T^8 + c_1T^4 + c_2)\rho$$

with $\rho \neq 0$ in $\mathbb{C}(T, \xi)$ independent of m, n, where

$$c_0 = 64mn(m-1)(n-1), \quad c_1 = -c_0$$

and

$$c_2 = (m^2 - 2mn + n^2 - m - n)(m^2 - 2mn + n^2 - 3m - 3m + 2).$$

First $m = n$ leads to

$$X + Y\xi + Z\xi^2 = \pm(\xi^2 - T)^m$$

which with the conjugate $-\xi$ shows that $Y = 0$. And then with the conjugate $i\xi$ we find

$$\pm X = \frac{(\xi^2 - T^2)^m + (-\xi^2 - T^2)^m}{2} = X_m(T^2),$$

$$\pm Y = \frac{(\xi^2 - T^2)^m - (-\xi^2 - T^2)^m}{2\xi^2} = Y_m(T^2),$$

in $\mathbb{Z}[T]$ (and we may restrict to $m \geq 0$ by making the two signs independent).

It remains to explore $c_0 = c_1 = c_2 = 0$; and these are easily seen to imply for (m, n) the nine possibilities

$$(0,0),\ (1,0),\ (0,1),\ (2,0),\ (0,2);\ (1,5),\ (5,1);\ (1,3),\ (3,1).$$

For (X, Y, Z) the first five lead to

$$(\pm 1, 0, 0),\ (\pm T, \pm 1, 0),\ \pm(T^2, \pm 2T, 1)$$

with independent signs (of which the first is the above for $m = 0$). The next two lead to

$$\pm(4T^6 - 5T^2, \pm 4T, -4T^4 - 1);$$

and the last two give no solutions. This completes the treatment of (1.8).

□

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