Infinitely many solutions for a class of semilinear elliptic equations in RN

Infinitely many solutions for a class of semilinear elliptic equations in $\mathbb{R}^N$

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In this article we discuss some results obtained by the authors about the semilinear elliptic problem

$$\begin{cases} -\Delta u + u = a(x)f(u) & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{P_a}$$

where $a \in L^\infty(\mathbb{R}^N)$ is such that $\liminf_{|x|\to\infty} a(x) > 0$, and $f \in C^1(\mathbb{R})$ satisfies:

$(f1)$ there exists $C > 0$ such that $|f(s)| \le C(1 + |s|^p)$ for any $s \in \mathbb{R}$, where $p \in (1, \frac{N+2}{N-2})$ if $N \ge 3$, or $p > 1$ if $N = 2$,

$(f2)$ there exists $\theta > 2$ such that $0 < \theta F(s) \le f(s)s$ for any $s \ne 0$, where $F(s) = \int_0^s f(r)\, dr$,

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$(f3)$ $\frac{f(s)}{s} < f'(s)$ for any $s \neq 0$.

Note that $f(s) = |s|^{p-1}s$ verifies $(f1)$–$(f3)$ for a subcritical exponent p, namely, for $p \in (1, \frac{N+2}{N-2})$ if $N \geq 3$, or $p > 1$ if $N = 1, 2$.

Such kind of problem has been widely studied by variational methods. In particular, solutions to (P_a) can be obtained as critical points of the functional $\varphi_a \colon H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\varphi_a(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} a(x)F(u(x))\ dx\ .$$

Here $\|u\|$ denotes the standard norm on the Sobolev space $H^1(\mathbb{R}^N)$. Thanks to the assumptions made on a and f, the functional φ_a turns out to be of class C^1 on $H^1(\mathbb{R}^N)$, and it satisfies the geometric properties of the mountain pass lemma. However, in general, the Palais-Smale condition fails, because of the noncompactness of the Sobolev embedding of $H^1(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$.

We point out that this lack of compactness is not just a technical difficulty for the existence problem, due to the variational approach, but it is an intrinsic feature of the problem itself, that may be responsible of nonexistence results, as it happens in some cases. Nevertheless, as we will discuss in the next section, one can take advantage of the lack of compactness in order to recover even existence of infinitely many solutions to (P_a) in a "generic" situation.

Before explaining this viewpoint in more detail, it is worth giving an overview of some results already known in the literature, which show quite clearly that existence/nonexistence or multiplicity of solutions to (P_a) is strongly affected by the behaviour of the coefficient a.

In the simplest case $a(x) \equiv 1$, discussed in [33], one can recover compactness, and then existence of a positive solution, by restricting to the subspace of radial functions $H^1_{rad}(\mathbb{R}^N)$, which is compactly embedded into $L^q(\mathbb{R}^N)$ for $q > 2$, q subcritical. We also remark that in this case some renowned results by Gidas, Ni, and Nirenberg [21], based on the maximum principle, also guarantee the uniqueness, up to translations, of the positive solution to (P_a).

An important progress in the study of the problem (P_a) has been made with the celebrated papers by P.L. Lions on the concentration-compactness principle [26]. This has supplied a better understanding of the possible ways of losing compactness,

and has permitted to consider the case of a nonnegative coefficient a such that there exists $\lim_{|x|\to\infty} a(x) \in (0, +\infty)$. Under this additional assumption one can prove that the Palais-Smale condition holds true at some levels. This fact has been used to obtain many different existence results. Among many others, we mention, as examples, [19], [34], [9], [10], [13], [23] and [5].

On the other side, small L^∞ perturbations of the coefficient may change from a problem in which compactness holds to a problem with no nontrivial solution. For instance, in [20] it is showed that if a is nonconstant and monotone in one direction then the related problem (P_a) has only the solution $u \equiv 0$.

Another different kind of phenomena appear when the coefficient a has an oscillatory behaviour. For example, when a is periodic, the invariance under translations permits to prove existence, [30], and also multiplicity results, as in [8], [17], [1], [27], where infinitely many solutions (distinct up to translations) are found. In fact, in this case, the noncompactness of the problem can be exploited to set up a new minimax argument, in the spirit of the works [16] and [32], and then to exhibit a rich structure of the set of solutions.

Multiplicity results have been obtained also without periodicity or asymptotic assumptions on a, in some "perturbative" settings, where concentration phenomena occur and a localization procedure can be used to get some compactness in the problem. A first result in this direction is the paper [24] concerning the prescribed scalar curvature problem on $\mathbb{S}^3$ and $\mathbb{S}^4$. We also mention [31], [6], [7], [18], [22], [25] and the references therein, for the case of a nonlinear stationary Schrödinger equation $-\epsilon^2\Delta u + V(x)u = f(u)$ with $\epsilon > 0$ small and $V \in C^1(\mathbb{R}^N)$, $V(x) \geq V_0 > 0$ in $\mathbb{R}^N$, having local maxima or minima or other topologically stable critical points. Similar concentration phenomena occur also considering the equation $-\Delta u + \lambda u = a(x)f(u)$ for $\lambda > 0$ large enough (see [15]) or $-\Delta u + u = a(x)|u|^{p-1}u$ with $p = \frac{N+2}{N-2} - \epsilon$, $\epsilon > 0$ small, and $N \geq 3$, where a blow-up analysis can be done (see [28]).

In the paper [4], we adopt a quite different viewpoint from the ones followed in the works quoted in the Introduction, and we show that the existence of infinitely many solutions for the problem (P_a) is a generic property with respect to $a \in L^\infty(\mathbb{R}^N)$, with $\liminf_{|x|\to\infty} a(x) \geq 0$. Precisely we prove

Theorem 1. *Let $f \in C^1(\mathbb{R})$ satisfy $(f1)$–$(f3)$. Then there exists a set $\mathcal{A}$ open and dense in $\{a \in L^\infty(\mathbb{R}^N) : \liminf_{|x|\to\infty} a(x) \geq 0\}$ such that for every $a \in \mathcal{A}$ the*

problem (P_a) *admits infinitely many solutions.*

In fact, given any $a \in L^\infty(\mathbb{R}^N)$ with $\liminf_{|x|\to\infty} a(x) > 0$, for all $\bar\alpha > 0$ we are able to construct a function $\alpha \in C(\mathbb{R}^N)$, $0 \le \alpha(x) \le \bar\alpha$ in $\mathbb{R}^N$, such that the problem $(P_{a+\alpha})$ admits infinitely many solutions.

The function α is obtained in a constructive way that can be roughly described as follows. First, we introduce the variational setting and we make a careful analysis of the hull of the functionals "at infinity" $\{\varphi_b : b \in H_\infty(a)\}$, where $H_\infty(a)$ is the set of the w^*-L^∞ limits of the sequences $a(\cdot + x_j)$ with $(x_j) \subset \mathbb{R}^N$, $|x_j| \to \infty$. All the functionals at infinity have a mountain pass geometry and, called $c(b)$ the mountain pass level of φ_b, we can show that there exists $a_\infty \in H_\infty(a)$ such that $c(a_\infty) \le c(b)$ for any $b \in H_\infty(a)$ and the corresponding problem (P_{a_∞}) admits a solution characterized as mountain pass critical point. Then, following a suitable sequence $(x_j) \subset \mathbb{R}^N$ such that $a(\cdot + x_j) \to a_\infty$ w^*-L^∞, we construct a family α_ω ($\omega > 0$ small) of perturbations of a by setting:

$$\alpha_\omega(x) = \begin{cases} \bar\alpha(1 - \frac{\omega^2}{4}|x - x_j|^2) & \text{for } |x - x_j| \le \frac{2}{\omega}\ ,\ j \ge j(\omega) \\ 0 & \text{otherwise,} \end{cases}$$

where $\bar\alpha > 0$ is fixed small enough, and $j(\omega)$ is a suitable positive integer (in fact, $j(\omega) \to +\infty$ as $\omega \to 0$). Note that for every $\omega > 0$ small, $\|\alpha_\omega\|_\infty = \bar\alpha$ and the support of α_ω is the union of infinitely many disjoint balls of radius $\frac{2}{\omega}$ and center x_j. Then, we focus on the family of functionals $\varphi_\omega = \varphi_{a+\alpha_\omega}$ and we prove that they satisfy local compactness properties in some sets of the type

$$\begin{aligned} A_j(\omega) = \{u \in H^1(\mathbb{R}^N) \quad : \quad & \varphi_\omega(u) \le c(a_\infty + \bar\alpha) + \epsilon, \\ & \|\varphi'_\omega(u)\| \le \epsilon, \\ & \exists y \in B_{\frac{1}{\omega}}(x_j)\ s.t.\ \|u\|_{H^1(B_1(y))} \ge \bar\rho\ \} \end{aligned}$$

where $\epsilon, \bar\rho > 0$ are some constant independent of ω and j. Finally, the mountain pass structure of the limiting functional $\varphi_\infty = \varphi_{a_\infty+\bar\alpha}$ permits to define local minimax classes for the perturbed functionals φ_ω related to the sets $A_j(\omega)$. This fact, toghether with the compactness properties that hold in $A_j(\omega)$, yields the existence of a critical point in $A_j(\omega)$, for any $\omega > 0$ small enough, and $j \in \mathbb{N}$ sufficiently large. Such as critical point turns out to be "stable" under perturbations of the cofficient $a + \alpha_\omega$ which are small in L^∞.

Hence, the following result holds true:

Theorem 1. *There exists $\hat{\omega} > 0$ such that for every $\omega \in (0, \hat{\omega})$ the problem $(P_{a+\alpha_\omega})$ admits infinitely many solutions. In addition, there exists $\beta_0 > 0$ such that for all $\omega \in (0, \hat{\omega})$ and $\beta \in L^\infty(\mathbb{R}^N)$ with $\|\beta\|_{L^\infty(\mathbb{R}^N)} \leq \beta_0$, also the problem $(P_{a+\alpha_\omega+\beta})$ admits infinitely many solutions.*

We point out that the perturbation α_ω is explicitely known, up to the sequence (x_j) that depends on the behaviour of a at infinity. However, in some simple cases, also the sequence (x_j) can be prescribed *a priori*. For instance, if $a(x) \to a_\infty \in (0, +\infty)$ as $|x| \to \infty$, then any sequence $(x_j) \subset \mathbb{R}^N$ such that $|x_{j+1}| - |x_j| \to +\infty$ works well. Similarly, if a is periodic in each variable, then one can take (x_j) on the period lattice, with the same divergence property as before.

We note that, by a standard argument (taking $\bar{f}$ instead of f, defined by $\bar{f}(t) = 0$ for $t \leq 0$ and $\bar{f}(t) = f(t)$ for $t > 0$), it is possible to show the existence of infinitely many positive classical solutions of the problem (P_a) for any $a \in \mathcal{A}$, a smooth.

Finally we want to point out some possible easy extensions of our result. We observe firstly that with minor change, our argument can be used to prove an analogous result for the class of the nonlinear Schrödinger equations $-\Delta u + b(x)u = a(x)f(u)$ with $b \in L^\infty(\mathbb{R}^N)$, $b(x) \geq b_0 > 0$ for a. e. $x \in \mathbb{R}^N$, and a and f as above. Moreover, we point out that in proving Theorems 1 and 2 we never use comparison theorems based on the maximum principle. Then our argument can be repeated exactly in the same way to study systems of the form $-\Delta u + u = a(x)\nabla F(u)$ where $F \in C^2(\mathbb{R}^N, \mathbb{R}^M)$ satisfies properties analogous to $(f1)$–$(f3)$. In particular the result can be established in the framework of the homoclinic problem for second order Hamiltonian systems in $\mathbb{R}^M$ (see [3] and the references therein).

Secondly we remark that the solutions we find satisfy suitable stability properties. These can be used to prove that in fact the perturbed problem $(P_{a+\alpha})$ admits multibump type solutions (see [32]) with bumps located around the points x_j. We refer in particular to [2] for a proof that can be adapted in this setting.

Finally we mention also the fact that if a is assumed to be positive and almost periodic (see [12]) then it is not known whether or not the problem (P_a) admits solutions. Following [3] it is possible to show that in this case one can construct a perturbation α almost periodic and with L^∞ norm small as we want, in such

a way that the problem $(P_{a+\alpha})$ admits infinitely many (actually multibump type) solutions. Then we get a genericity result (with respect to the property of existence of infinitely many solutions) for the class of problems (P_a) with $a \in C(\mathbb{R}^N)$ positive and almost periodic.

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