**Linear Operator Inequality and Null Controllability with Vanishing Energy for boundary control systems**

# LINEAR OPERATOR INEQUALITY AND NULL CONTROLLABILITY WITH VANISHING ENERGY FOR BOUNDARY CONTROL SYSTEMS

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**Abstract:** We consider a linear boundary control system on a Hilbert space $H$ which is null controllable at some time $T_0 > 0$. Parabolic and hyperbolic PDEs provide several examples of such systems. To every initial state $y_0 \in H$ we associate the minimal "energy" needed to transfer $y_0$ to 0 in a time $T \geq T_0$ ("energy" of a control being the square of its $L^2$ norm). Clearly, it decreases with the control time $T$. We shall prove that, under suitable spectral properties of the linear system operator, the minimal energy converges to 0 for $T \to +\infty$. This extends to boundary control systems a property known for distributed systems (see [30] where the notion of "null controllability with vanishing energy" is introduced).

The proofs for distributed systems depend on properties of the Riccati equation which are not available in the general setting we study in this paper. For this reason we shall base our proofs on the Linear Operator Inequality.

## 1. Introduction and preliminaries

The paper [30] introduced and studied the property of "null controllability with vanishing energy", shortly NCVE, for systems with distributed control action, which is as follows: consider a semigroup control system (cf. [3, 5, 16, 17, 34, 35])

$$\dot{y} = Ay + Bu, \qquad y(0) = y_0 \in H,$$

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which is null controllable in time $T_0 > 0$ (hence also for every larger time $T > T_0$). This null controllable system is NCVE when for every $y_0$ and $\epsilon > 0$ there exist a time $T$ and a control $u$ which steers the initial state $y_0$ to zero in time $T$ and, furthermore, its $L^2(0,T;U)$-norm is less then $\epsilon$. This concept has been already applied in some specific situations (see [12, 13]) and partially extended to the Banach space setting in [24]. Moreover, applications of NCVE property to Ornstein-Uhlenbeck processes are given in [31].

The key result in [30], i.e., Theorem 1.1, shows that, under suitable properties on the operator $A$ recalled below, NCVE holds if and only if the system is null controllable and furthermore the spectrum of $A$ is contained in the *closed* half plane $\{\Re e\, \lambda \leq 0\}$.

The goal of this paper is to extend this result to a large class of boundary control systems (see Hypothesis 1.1), which essentially includes all the classes of systems whose null controllability has been studied up to now. Our main results are Theorems 1.5 and 1.7. The proofs that we give are based on ideas different from those in [30]. Moreover conditions imposed for the necessity part are weaker from those used in [30, Theorem 1.1], in the case of distributed control systems.

Now we describe the notations and the class of systems we are studying.

The spaces in this paper are Hilbert, and are identified with their dual unless explicitly stated. The notations are standard. For example, $\mathcal{L}(H,K)$ denotes the Banach space of all bounded linear operators from $H$ into $K$ endowed with the operator norm.

Let $H$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$ and let $A$ be a generator of a $C_0$-semigroup on $H$. Due to the fact that the spectrum of $A$ has a role in our arguments, we assume from the outset that $H$ is a *complex* Hilbert space.

Let $A^*$ be the Hilbert space adjoint of $A$. Its domain with the graph norm

$$|y|^2 = \langle y, y\rangle + \langle A^*y, A^*y\rangle$$

is a Hilbert space *which is not identified with its dual.* It is well known that $(\mathrm{dom}A^*)'$ (the dual of the Hilbert space $\mathrm{dom}A^*$) is a Hilbert space and

$$(\mathrm{dom}A^*) \subset H = H' \subset (\mathrm{dom}A^*)'$$

(with continuous and dense injections). Moreover, $A$ admits an extension $\mathcal{A}$ to $(\mathrm{dom}A^*)'$, which generates a $C_0$-semigroup $e^{t\mathcal{A}}$ on $(\mathrm{dom}A^*)'$ (cf. [16, Section 0.3], [3, Chapter 3 ] and [34] and Appendix). The domain of such extension is equal to $H$.

The norm in $(\mathrm{dom}A^*)'$ is denoted by $|\cdot|_{-1}$, and it is useful to recall that $|y|_{-1}$ and $|(\omega I - \mathcal{A})^{-1}y|$ are equivalent norms on $(\mathrm{dom}A^*)'$, for every $\omega \in \rho(A) = \rho(\mathcal{A}) = \overline{\rho(A^*)}$ (here $\rho$ indicates the resolvent set). In other words, $(\mathrm{dom}A^*)'$ is the completion of $H$ with respect to the norm $|(\omega I - A)^{-1}\cdot|$, for any $\omega \in \rho(A)$.

Let $B \in \mathcal{L}(U, (\mathrm{dom}\, A^*)')$ and let us consider the control process on $(\mathrm{dom}A^*)'$ described by

$$\dot{y} = \mathcal{A}y + Bu, \qquad y(0) = y_0 \in H\,. \tag{1}$$

In fact, this equation makes sense in $(\mathrm{dom}A^*)'$, for every $y_0 \in (\mathrm{dom}A^*)'$, but we only consider initial conditions $y_0 \in H$. It is known that the transformation

$$u(\cdot) \;\longrightarrow\; (Lu)(t) \qquad \text{where} \quad (Lu)(t) := \int_0^t e^{\mathcal{A}(t-s)}Bu(s)\,\mathrm{d}s \tag{2}$$

is continuous from $L^2(0,T;U)$ into $C([0,T];(\mathrm{dom}A^*)')$, for every $T>0$. The class of systems we study is identified by pairs $(A,B)$ with the following property:

**Hypothesis 1.1.** *We have $B \in \mathcal{L}(U,(\mathrm{dom}\,A^*)')$ and, for every $T>0$, the transformation (2) is linear and continuous from $L^2(0,T;U)$ into $L^2(0,T;H)$.*

Clearly, the case of distributed controls, i.e., $B \in \mathcal{L}(U,H)$, fits Hypothesis 1.1 (in such case, the transformation (2) is linear and continuous from $L^2(0,T;U)$ into $C([0,T];H)$). Examples of boundary control systems which satisfy our condition are in Section 1.1.

From now on we consider $\omega \in \rho(A)$, *which is fixed once and for all,* and introduce the operator

$$D = (\omega I - \mathcal{A})^{-1} B \in \mathcal{L}(U,H)\,. \tag{3}$$

By definition, the solution of system (1) is

$$y^{y_0,u}(t) = e^{At}y_0 + \int_0^t e^{\mathcal{A}(t-s)}Bu(s)\,\mathrm{d}s = e^{At}y_0 + \int_0^t e^{\mathcal{A}(t-s)}(\omega I - \mathcal{A})Du(s)\,\mathrm{d}s\,. \tag{4}$$

This is a continuous $(\mathrm{dom}A^*)'$-valued function and belongs to $L^2_{\mathrm{loc}}(0,+\infty;H)$ thanks to Hypothesis 1.1. Integration by parts shows that:

**Lemma 1.1.** *If $u \in C^1([0,+\infty);U)$ then $y^{y_0,u}$ belongs to $C([0,+\infty);H)$.*

Now we give the definitions of null controllability and NCVE, adapted to our system, by taking into account the fact that if $u \in L^2_{\mathrm{loc}}(0,+\infty;U)$ then the integrals in (4) belong to $L^2_{\mathrm{loc}}(0,+\infty;H)$, and point-wise evaluation in general is meaningless.

**Definition 1.2.** *We say that $y_0 \in H$ can be steered to the rest in time (at most) $T$ if there exists a control $u \in L^2_{\mathrm{loc}}(0,+\infty;U)$ whose support is contained in $[0,T]$ and such that the support of the corresponding solution (4) is contained in $[0,T]$ too.*

*System (1) is null controllable if every $y_0 \in H$ can be steered to the rest in a suitable time $T_{y_0}$ at most.*

*System (1) is null controllable in time (at most) $T$ if every $y_0 \in H$ can be steered to the rest in time at most $T$.*

In connection with this definition see also Lemma 1.9. In particular, if $u$ steers $y_0$ to the rest in time at most $T$, we have

$$\int_0^T e^{\mathcal{A}(t-s)}Bu(s)\,\mathrm{d}s = -e^{At}y_0, \qquad \text{a.e.} \quad t>T,$$

and so the integral is represented by a continuous function for $t>T$.

Controllability in time $T$ implies controllability at every larger time. The control $u$ need not be unique. Then, we define:

**Definition 1.3.** *Let $y_0 \in H$ be an element which can be steered to the rest. We say that this element is NCVE if for every $\epsilon>0$ there exists a control $u_\epsilon$ such that*

- *it steers $y_0$ to the rest in time at most $T_\epsilon$ (here $T=T_\epsilon$ and the support of $u$ is in $[0,T_\epsilon]$);*
- *the $L^2(0,+\infty;U)$ norm of $u$ is less then $\epsilon$:*

$$\int_0^{+\infty} |u(s)|^2\,\mathrm{d}s = \int_0^{T_\epsilon} |u(s)|^2\,\mathrm{d}s \le \epsilon^2\,.$$

*If every element of $H$ is NCVE, then we say that system (1) is NCVE.*

As a variant to Definitions 1.2 and 1.3, we introduce also:

**Definition 1.4.** *Let $\mathcal{D}$ be a subspace of $H$. If every initial condition $y_0 \in \mathcal{D}$ can be steered to the rest in time $T$ then we say that the system is null controllable on $\mathcal{D}$ in time $T$ (note that we don't require that the trajectory which joins $y_0$ to zero remains in the set $\mathcal{D}$).*

*We say that the system is NCVE on $\mathcal{D}$ if for every $y_0 \in \mathcal{D}$ and every $\epsilon > 0$ there exists a control $u_\epsilon$ such that*

- *it steers $y_0$ to the rest in time $T_\epsilon$ (here $T$ depends on $\epsilon$, i.e., $T = T_\epsilon$);*
- *the $L^2(0, +\infty; U)$ norm of $u$ is less then $\epsilon$:*

$$\int_0^{+\infty} |u(s)|^2 \, \mathrm{d}s = \int_0^{T_\epsilon} |u(s)|^2 \, \mathrm{d}s \leq \epsilon^2 \, .$$

1.1. **Classes of systems which fit our framework.** Essentially, controllability has been studied for "parabolic" and "hyperbolic" type systems.

(i) *Parabolic systems* can be described, in a unified way, as follows.

The operator $A$ generates a holomorphic semigroup and, following [16, Section 0.4 and Chapter 1], there exists $\omega \in \rho(A) = \rho(\mathcal{A})$ and $\gamma \in [0, 1)$ such that

$$B \in \mathcal{L}(U, \big(\mathrm{dom}((\omega - A^*)^\gamma)\big)'). \tag{5}$$

Note that (5) implies the estimate

$$\|e^{t\mathcal{A}} B\|_{\mathcal{L}(U,H)} \leq \frac{M e^{\omega_1 t}}{t^\gamma}, \quad t > 0. \tag{6}$$

for some $M > 0$, $\omega_1 \in \mathbb{R}$ (see also Appendix; recall that $\big(\mathrm{dom}((\omega - A^*)^\gamma)\big)' \subset (\mathrm{dom} A^*)'$ with continuous and dense injection).

Using (6), one can show that *Hypothesis 1.1 holds in this case.* Indeed, the integral in (4) does not converge in the space $H$ for every $t$ but, using the Young inequality for convolutions, it defines an $H$-valued locally square integrable function for every locally square integrable input $u$. Formula (4) defines the unique solution of eq. (1) with values in $H$, which however does not have a pointwise sense in general.

The singular inequality (6) holds for certain important class of interconnected systems, as studied for example in [4, 18], even if they do not generate holomorphic semigroups.

(ii) *Hyperbolic systems* are further important examples of systems which fit our framework, see [17, 21, 34]. In spite of the fact that this class lacks of a plain unification, it turns out that in this case the following important property, first proved for the wave equation with Dirichlet boundary control in [14, 15], holds: the function $y(t)$ is even continuous in time. See [34, p. 122] for an abstract setting.

We listed earlier systems which fit our Hypothesis 1.1. However, null controllability cannot be studied "in abstract": it has to be studied separately in concrete cases and these are too many to be cited here. So, we confine ourselves to note that controllability for several hyperbolic type problems is studied in [2, 19, 21, 22]; controllability for parabolic equations is studied in [23, 33, 36] and references therein. Note that controllability for heat-type equations is often achieved using smooth controls, so that the resulting trajectory $y(t)$ is even continuous.

An overview on controllability both of hyperbolic and parabolic type equations is [37].

1.2. **Key results and discussion.** As we have already said, our point of departure is paper [30] which proves the following result, in the case of distributed controls, i.e., the case that $\operatorname{im} D \subseteq \operatorname{dom} A$ (recall that $D$ is defined in (3)) so that $B \in \mathcal{L}(U, H)$: *under suitable assumptions on the spectral properties of the operator $A$, NCVE is equivalent to null controllability at some time $T > 0$. An interesting interpretation of this result is that NCVE does not depend on the control operator provided that this operator is so chosen to guarantee null controllability at a certain time $T$.*

Now we state our main results, which we split in Theorems 1.5 and 1.7. We don't try to unify them, since they are proved using different ideas, see also Remark 1.8.

We recall that an *invariant subspace* $E$ for a $C_0$-semigroup $e^{At}$ on $H$ is a closed subspace of $H$ such that

$$e^{At}x \in E, \qquad \forall x \in E\,,\ \forall t \geq 0\,.$$

It is possible to prove that the restriction of $e^{At}$ is a $C_0$-semigroup on $E$ and that $A\,(E \cap (\operatorname{dom} A)) \subseteq E$ (the restriction of $A$ to $E$ is the infinitesimal generator of $e^{At}$ on $E$).

**Theorem 1.5.** *Assume Hypothesis 1.1 and suppose the existence of an invariant subspace $E$ for $e^{At}$, such that $e^{-At}$ generates a $C_0$-group on $E$ which is* exponentially stable *(for $t \to +\infty$). Then, the system* (1) *is not NCVE.*

A consequence is:

**Corollary 1.6.** *Assume Hypothesis 1.1, If $\sigma(A)$ has an isolated point with* positive *real part, then the system* (1) *is* not *NCVE.*

In fact, [30] proves the existence of the subspace $E$ in Theorem 1.5, under the assumption of the corollary.

Now we come to the second theorem. We recall that $x \in H$ is a generalized eigenvector of $A$ associated to the eigenvalue $\lambda \in \mathbb{C}$ if $x \in \bigcup_{k\geq 1} \operatorname{Ker}[(\lambda - A)^k]$ and we recall the standard notation

$$s(A) = \sup\{\Re e\,\lambda\,, \quad \lambda \in \sigma(A)\}.$$

Now we introduce the following assumption, which slightly generalizes the one in [30, (ii) Hypothesis 1.1].

**Hypothesis 1.2.** *There exist closed linear subspaces $H_s$, $H_1$ of $H$ such that:*

- *their direct sum is $H$, i.e., $H = H_s \oplus H_1$;*
- *for every $x \in H_s$ we have*
$$\lim_{t\to+\infty} e^{At}x = 0$$
- *the subspace $H_1$ is invariant for the semigroup, hence also for $(\omega I - A)^{-1}$ (for every large enough $\omega \in \mathbb{R}$) and furthermore we assume that the set of all generalized eigenvectors of $A$ contained in $H_1$ is linearly dense in $H_1$.*

We note that the assumption in [30] is slightly stronger in that [30] assumes that $H_s$ is an invariant subspace for the semigroup, and that the semigroup restricted to $H_s$ is exponentially stable.

We have:

**Theorem 1.7.** *Assume Hypotheses 1.1 and 1.2 and furthermore suppose that $s(A) \leq 0$. If system (1) is null controllable at some time $T > 0$, then it is NCVE.*

The ideas used in the proof of both Theorems 1.5 and 1.7 are different from those used in the proofs of the corresponding results in [30]. In particular, the proof of Theorem 1.7 relies on the *Yakubovich theory of the regulator problem with stability*, and the corresponding *Linear Operator Inequality,* that can be found in [20, 26, 27, 28].

**Remark 1.8.** *Clearly, the spectral condition in Hypothesis 1.2 is satisfied by most of the systems encountered in practice, when the "dominant part" of the spectrum is a sequence of eigenvalues (in particular, if $A$ has compact resolvent). Hence, for all these systems, Theorems 1.5 and 1.7 can be combined to get a necessary and sufficient condition for NCVE which depends only on the spectrum of $A$, provided that null controllability holds. For example we can state:*

Assume Hypothesis 1.1 and suppose that the operator $A$ has compact resolvent. Then null controllability and NCVE are equivalent properties if $s(A) \leq 0$. When $s(A) > 0$ the system is not NCVE.

*We prefer to keep the theorems distinct, since the two proofs use different ideas.*

We conclude this introduction with the following observation which extends a property of null controllable systems proved by many people for distributed controls (see [7, 32, 24]) and likely known also in the boundary case, in spite of the fact that we cannot give a precise reference:

**Lemma 1.9.** *Assume Hypothesis 1.1 and suppose that every $y \in H$ can be steered to rest in a time $T_y$. Then:*

- *there exists a time $T_0$ such that system (1) can be steered to the rest in time $T_0$;*
- *there is a ball $B(0, r)$ (centered at 0, radius $r > 0$) and a number $N$ such that every element of $B(0, r)$ can be steered to the rest using a control whose $L^2$-norm is less then $N$.*

*Proof.* The proof is the same as for distributed systems: we introduce the sets $E_{T,N}$ of those elements $y \in H$ which can be steered to the rest in time (at most) $T$ and using controls of norm at most $N$. These sets are closed, convex and balanced. Furthermore, they grow both with $T$ and with $N$.

Every $y$ belongs to a suitable $E_{T,N}$ so that

$$H = \cup E_{N,N} \, .$$

Baire Theorem implies the existence of $N_0$ such that $E_{N_0,N_0}$ has interior points.

The set $E_{N_0,N_0}$ being convex and balanced, 0 is an interior point, i.e., any point of a ball centered at zero can be steered to the rest in time $T = N_0$ and the $L^2$-norm of the corresponding control is less then $N_0$. So, every $y \in H$ can be steered to the rest in time $T = N_0$ □

In conclusion, we see that null controllability and null controllability at a fixed time $T > 0$ are equivalent concepts.

## 2. Proof of the main results

First we state three lemmas which have an independent interest. In the proof of Theorem 1.5 we shall use the first two of them.

Let $y(t)$ solve equation (1). Then, $x(t) = (\omega I - \mathcal{A})^{-1} y(t)$ solves the equation

$$\dot{x} = Ax + Du\,, \qquad x(0) = x_0 = (\omega I - A)^{-1} y_0 \in \operatorname{dom} A\,. \tag{7}$$

Consequently, every control which steers $y_0$ to zero, steers also $x_0$ to zero, and conversely. Therefore, we have:

**Lemma 2.1.** *Assume Hypothesis 1.1. There exists $T > 0$ such that system (1) is null controllable in time $T$ if and only if system (7) is null controllable on $\mathcal{D} = \operatorname{dom} A$ in the same time $T$; system (1) is NCVE if and only if system (7) is NCVE on $\mathcal{D} = \operatorname{dom} A$.*

From now on, $\mathcal{D}$ will always denote $\operatorname{dom} A$, i.e.,

$$\mathcal{D} = \operatorname{dom} A\,.$$

The second preliminary result is the following lemma:

**Lemma 2.2.** *Assume Hypothesis 1.1 and suppose that system (1) is null controllable in time $T$. Then there exists a number $M$ such that for every $y_0 \in H$ there exists a control $u^{y_0,T}(t)$ which steers $y_0$ to 0 in time $T$ and such that:*

$$\int_0^T |u^{y_0,T}(t)|^2 \,\mathrm{d}t \le M|y_0|^2\,.$$

*Proof.* We already noted that we can equivalently control system (7) to zero on $\mathcal{D}$, i.e., we can solve

$$-e^{AT}(\omega I - A)^{-1} y_0 = \int_0^T e^{A(T-s)} Du(s)\,\mathrm{d}s \tag{8}$$

and by assumption this equation is solvable for every $y_0 \in H$. We introduce the operator $\Lambda_T : L^2(0,T;U) \to H$ as

$$\Lambda_T u = \int_0^T e^{\mathcal{A}(T-s)} Du(s)\,\mathrm{d}s\,. \tag{9}$$

So, null controllability at time $T$ is equivalent to

$$\operatorname{im} e^{AT}(\omega I - A)^{-1} \subseteq \operatorname{im}\Lambda_T.$$

The operator $\Lambda_T$ is continuous,

$$\Lambda_T \in \mathcal{L}\left(L^2(0,T;U), H\right).$$

Let us introduce the continuous operator $Q_T = \Lambda_T \Lambda_T^*$. Its kernel is closed and its restriction to the orthogonal of the kernel is invertible with closed inverse. Let us denote it as $Q_T^\dagger$ so that the control which steers $(\omega I - A)^{-1} y_0$ to zero in time $T$ and which has minimal $L^2(0,T)$ norm is

$$u^{T,y_0}(t) = -\Lambda_T^* Q_T^\dagger e^{AT}(\omega I - A)^{-1} y_0 = -D^* e^{A^*(T-t)} Q_T^\dagger e^{AT}(\omega I - A)^{-1} y_0\,. \tag{10}$$

The closed operator $Q_T^\dagger e^{AT}(\omega I - A)^{-1}$ being everywhere defined, it is continuous, so that

$$\|u^{T,y_0}\|_{L^2(0,T;U)} \le M|y_0|\,, \qquad M = M_T\,,$$

as wanted. □

**Remark 2.3.** *We note:*

- *The function $\tilde{u}^{T,y_0}(t)$, extended with 0 for $t > T$, produces a solution $y(t)$ to Eq. (1), which has support in $[0,T]$.*

- *We can work with any initial time $\tau$ instead of the initial time 0. If the system is null controllable in time at most $T$, then any "initial condition" assigned at time $\tau$ can be steered to rest on a time interval still of duration $T$, i.e., at the time $T+\tau$ and the previous Lemma 2.2 still holds, with the constant $M$ depending solely on the length of the controllability time, i.e., the same constant $M_T$ can be used for every initial time $\tau$.*

In the proof of Theorem 1.7 we will use use the following result which makes sense since when the control is of class $C^1$ then the function $y^{y_0,u}(t)$ in (4) is continuous.

**Lemma 2.4.** *Assume Hypothesis 1.1 and suppose that system (1) is null controllable in time $T$. Then, for any $y_0 \in H$ and for any $\epsilon > 0$ there exists a control $u_\epsilon \in C^1$ and a time $t_\epsilon$ (depending also on $y_0$) such that the support of $u_\epsilon$ is in $[0, t_\epsilon]$ and*

$$|y^{y_0,u_\epsilon}(t_\epsilon)| < \epsilon. \tag{11}$$

*Proof.* By null controllability, we know that for every, large enough, $T > 0$, there exists a control $u \in L^2$ with support in $[0, T]$ such that $y^{y_0,u}(t)$ has support in $[0, T]$ too. Hence, the class $(Lu)(t) \in L^2(0, +\infty)$, defined in (2), contains a function $z(t)$ which for $t > T$ satisfies

$$z(t) = -e^{At}y_0\,. \tag{12}$$

In particular, $z(t)$ is continuous for $t > T$.

This suggests that we search for $t_\epsilon > T$. Let us consider (12), for example, on $(T, 2T)$. The mapping $u(\cdot) \mapsto (Lu)(\cdot)$ belongs to $\mathcal{L}(L^2(0, \tau; U), L^2(0, \tau; H))$, for any $\tau > 0$ (see Hypothesis 1.1). Therefore there exists a sequence of $C^1$ controls $(u_n)$, with support in $[0, 2T]$, such that

$$\int_0^{2T} |z(t) - z_n(t)|^2 dt \to 0, \quad \text{as} \quad n \to \infty\,.$$

Here, $z_n(t)$ is the continuous function given by $y^{y_0,u_n}(t) - e^{At}y_0$ (cf. Lemma 1.1). So (possibly passing to a subsequence, still denoted $(u_n)$), we have $|z(t) - z_n(t)| \to 0$, as $n \to \infty$, a.e. on $(T, 2T)$. Using (12), it follows that there exists $t_\epsilon \in (T, 2T)$ and a control $u_{n_0}$, with support in $[0, 2T]$ such that (see (12))

$$\left|e^{At_\epsilon}y_0 + z_n(t_\epsilon)\right| = \left|\left(e^{At_\epsilon}y_0 + z(t_\epsilon)\right) - \left(e^{At_\epsilon}y_0 + z_n(t_\epsilon)\right)\right| < \epsilon.$$

This is the assertion with $u_\epsilon = u_{n_0}$. □

We can now prove the first theorem.

2.1. **Proof of Theorem 1.5,** *i.e. if NCVE holds then the invariant subspace $E$ does not exist.* The proof in [30] relays on a precise study of the quadratic regulator problem and the associated Riccati equation. Here we follow a different route: we prove that the existence of the subspace $E$ implies that system (1) is not NCVE.

Let $y_0 \neq 0$ be any point of $E$. If it cannot be steered to 0 then system (1) is not null controllable, hence even not NCVE. So, suppose that there exists a control $u$ which steers $y_0$ to zero in time $T$. Then we have, for every $t > T$,

$$e^{At}y_0 = -\int_0^t e^{\mathcal{A}(t-s)}Bu(s)\,\mathrm{d}s\,.$$

Then we have also

$$e^{AT}(\omega I - A)^{-1}y_0 = -\int_0^T e^{A(T-s)}Du(s)\,\mathrm{d}s\,.$$

Let now $P_E$ be the orthogonal projection of $H$ onto $E$. Then we have

$$e^{AT}(\omega I - A)^{-1}y_0 = -\int_0^T e^{A(T-s)}P_E Du(s)\,\mathrm{d}s - \int_0^T e^{A(T-s)}(I-P_E)Du(s)\,\mathrm{d}s\,.$$

The left hand side belongs to $E$ so that the last integral is zero and we have

$$e^{AT}(\omega I - A)^{-1}y_0 = -\int_0^T e^{A(T-s)}P_E Du(s)\,\mathrm{d}s$$

hence also

$$(\omega I - A)^{-1}y_0 = -\int_0^T e^{-As}P_E Du(s)\,\mathrm{d}s$$

*since $A$ generates a group on $E$.* Note that this equality in particular implies that $P_E D \neq 0$ since the left hand side is not zero.

We assumed that $e^{-At}$ is exponentially stable on $E$, i.e., we assumed the existence of $M > 1$ and $\gamma > 0$ such that

$$\left|e^{-As}y_0\right| \leq Me^{-\gamma s}|y_0| \qquad \forall s > 0 \ \text{ and } y_0 \in E\,.$$

So, using Schwarz inequality we see that:

$$\left|(\omega I - A)^{-1}y_0\right| \leq \frac{M\|P_E D\|_{\mathcal{L}(U,H)}}{\sqrt{2\gamma}}\|u\|_{L^2(0,T;U)}\,.$$

This is an estimate *from below* for the $L^2(0,T)$-norm of any control which steers $y_0$ to the rest, and this estimate does not depend on $T$. Hence, the system is not NCVE, as we wished to prove. □

2.2. **Proof of Theorem 1.7,** *i.e., null controllability and $s(A) \leq 0$ implies NCVE.* We introduce a new notation. Since we need to consider solutions of equation (1) with initial time $\tau$, possibly different from 0, we introduce

$$y(t;\tau,y_0,u)$$

to denote the solution of the problem

$$y' = \mathcal{A}y + Bu \qquad t > \tau\,, \qquad y(\tau) = y_0\,.$$

Furthermore, when $\tau = 0$, we shall write $y(t;y_0,u)$ instead of $y(t;0,y_0,u)$. Comparing with (4), we have

$$y(t;y_0,u) = y^{y_0,u}(t)\,.$$

We first give a different formulation of the problem under study. To this purpose we introduce the following functionals $I(y_0)$ and $Z(y_0)$:

$$I(y_0) = \inf_{u\in\mathcal{U}(y_0)} J(y_0;u)\,, \qquad J(y_0;u) = \int_0^{+\infty}|u(s)|^2\,\mathrm{d}s \tag{13}$$

$$\mathcal{U}(y_0) = \left\{u \in L^2(0,+\infty;U)\ :\ y(\cdot;y_0,u) \in L^2(0,+\infty;H)\right\}$$

and

$$Z(y_0) = \inf_{t>T}\left(\inf\left\{\int_0^t |u(s)|^2 ds\right\}\right), \tag{14}$$

where, for each $t > T$, the infimum in parenthesis is computed on those controls $u$ which steers $y_0$ to the rest in time at most $t$ (i.e., the supports of $u$ and $y^{y_0,u}$ have to be contained in $[0,t]$).

Using Lemma 2.2, we can prove:

**Theorem 2.5.** *Assume Hypothesis 1.1. Let system (1) be null controllable in time $T$. Then we have*

$$I(y_0) = Z(y_0)\,.$$

*Proof.* It is clear that $I(y_0) \le Z(y_0)$ for every $y_0 \in H$. We prove the converse inequality.

Let us fix any $y_0 \in H$. Null controllability implies that $I(y_0) < +\infty$ for every $y_0$ so that for every $\epsilon > 0$ there exist a control $u_\epsilon \in \mathcal{U}(y_0)$ and a time $S_0$ such that for every $S > S_0$ we have

$$\int_0^S |u_\epsilon(t)|^2 \,\mathrm{d}t \le I(y_0) + \epsilon\,.$$

Note that we are computing an infimum, not a minimum; so we can assume $u$ of class $C^1$, so that the function $y^{y_0,u_\epsilon}$ is continuous. Since, in addition, it is square integrable, we have

$$\liminf_{t\to+\infty} y^{y_0,u_\epsilon}(t) = 0.$$

Fix an arbitrary $\sigma > 0$ and a time $S_\sigma > S_0$ such that

$$|y^{y_0,u_\epsilon}(S_\sigma)| = |y(S_\sigma, y_0, u_\epsilon)| < \sigma\,.$$

Null controllability holds also on $[S_\sigma, S_\sigma + T]$ and Lemma 2.2 can be applied on this interval (see also Remark 2.3). Hence, there exists a control $\tilde{u}$ with support in $[S_\sigma, S_\sigma + T]$ which steers to the rest in time $T$ the "initial condition" $y(S_\sigma; y_0, u_\epsilon)$, assigned at the initial time $S_\sigma$. Lemma 2.2 shows that the norm of this control is less then $M\sigma$.

Now we apply in sequence the controls $u_\epsilon$, on $[0, S_\sigma]$ and then the control $\tilde{u}$. This controls steers $y_0$ to zero in time at most $S_\sigma + T$ and its norm is less then $I(y_0) + \epsilon + M^2\sigma^2$. Hence we have

$$Z(y_0) \le I(y_0) + \epsilon + M^2\sigma^2\,.$$

The required inequality follows since $\epsilon > 0$ and $\sigma > 0$ are arbitrary. □

This theorem shows:

**Corollary 2.6.** *Assume Hypothesis 1.1 and suppose that system (1) is null controllable in time $T$. Then System (1) is NCVE if and only if $I(y_0) = 0$, for every $y_0 \in H$.*

So, our goal now is the proof that, under the assumptions of Theorem 1.7, we have $I(y_0) = 0$.

It is not difficult to prove that $I(\lambda y_0) = |\lambda|^2 I(y_0)$, $\lambda \in \mathbb{C}$, $y_0 \in H$. An obvious consequence is

$$\begin{cases} I(x) = I(-x) & \text{and} \quad I(0) = 0\,, \\ |\alpha| = |\beta| & \text{implies} \quad I(\alpha x) = I(\beta x), \quad \alpha, \beta \in \mathbb{C},\ x \in H. \end{cases} \tag{15}$$

First, we give a representation of $I(y_0)$.

**Lemma 2.7.** *Assume Hypothesis 1.1 and suppose that system (1) is null controllable in time $T$. Then there exists an operator $P$ defined on $H$ such that*

$$I(y_0) = \langle y_0, Py_0 \rangle .$$

*The operator $P$ has the following properties:*

- **a)** $\langle Px, \xi \rangle = \langle x, P\xi \rangle \qquad \forall x\,,\ \xi \in H;$
- **b)** $P(x+\xi) = Px + P\xi \qquad \forall x\,,\ \xi \in H;$
- **c)** *the equality $P(qx) = qPx$ holds for all $x \in H$ and every complex number $q$ with* rational *real and imaginary parts.*
- **d)** $\langle y_0, Py_0 \rangle \geq 0$ *for all $y_0 \in H$.*

*Proof.* The proof uses [9, Sect. 9.2] (adapted to complex Hilbert spaces) and it is an adaptation of the proof of [6, Theorem 5].

Recall that $\mathcal{U}(y_0)$ is not empty since system (1) is null controllable. So, null controllability implies that $I(y_0)$ is finite for every $y_0 \in H$.

Let us fix $x_0$ and $\xi_0$ in $\mathcal{D}$ =dom$A$ and controls $u \in \mathcal{U}(x_0)$ and $v \in \mathcal{U}(\xi_0)$ Then we have

$$y(t; x_0 \pm \xi_0, u \pm v) = y(t; x_0, u) \pm y(t; \xi_0, v)$$

and $J$ satisfies the parallelogram identity

$$J(x_0 + \xi_0; u + v) + J(x_0 - \xi_0; u - v) = 2\left[J(x_0; u) + J(\xi_0; v)\right] .$$

We must prove that $I(x)$ satisfies the parallelogram identity too. This part of the proof is the same as that in [6, Theorem 5] and it is reported for completeness.

We fix $x$ and $\xi$ and $\epsilon > 0$ and we choose $u_x$ and $u_\xi$, corresponding to the initial conditions $x$ and $\xi$, such that

$$J(x; u_x) < I(x) + \epsilon/2\,, \qquad J(\xi; u_\xi) < I(\xi) + \epsilon/2.$$

Then

$$\begin{aligned} &J(x+\xi; u_x + u_\xi) + J(x-\xi; u_x - u_\xi) \\ &= 2J(x; u_x) + 2J(\xi; u_\xi) < 2\left[I(x) + I(\xi)\right] + 2\epsilon. \end{aligned}$$

This proves the inequality

$$I(x+\xi) + I(x-\xi) \leq 2\left[I(x) + I(\xi)\right] . \tag{16}$$

We prove that the inequality cannot be strict; i.e., we prove that if $\epsilon$ satisfy

$$I(x+\xi) + I(x-\xi) \leq 2\left[I(x) + I(\xi)\right] - \epsilon \tag{17}$$

then $\epsilon = 0$.

If (17) holds then we can find $\tilde{u}$ and $\tilde{v}$, corresponding to the initial states $x+\xi$ and $x-\xi$, such that

$$J(x+\xi; \tilde{u}) + J(x-\xi; \tilde{v}) \leq 2\left[I(x) + I(\xi)\right] - \epsilon/2\,.$$

For the initial conditions $x, \xi$ we apply, respectively, controls

$$u_0 = \frac{\tilde{u} + \tilde{v}}{2}\,, \qquad v_0 = \frac{\tilde{u} - \tilde{v}}{2}.$$

Then

$$2\left[J(x; u_0) + J(\xi; v_0)\right] = J(x+\xi; u_0+v_0) + J(x-\xi; u_0-v_0) \leq 2\left[I(x) + I(\xi)\right] - \epsilon/2\,.$$

We have also

$$I(x) + I(\xi) \leq J(x; u_0) + J(\xi; v_0) \leq \left[I(x) + I(\xi)\right] - \epsilon/4\,. \tag{18}$$

This shows $\epsilon = 0$ so that parallelogram identity holds.

The operator $P$ is now constructed by polarization (compare with [10]),

$$\langle x, P\xi\rangle = I\left(\frac{1}{2}(x+\xi)\right) - I\left(\frac{1}{2}(x-\xi)\right) + i\left[I\left(\frac{1}{2}(x+i\xi)\right) - I\left(\frac{1}{2}(x-i\xi)\right)\right]. \tag{19}$$

The property $I(x) = \langle x, Px\rangle$ is a routine computation, using (15).

We prove property **a)**. Using (15) we see that the right hand side of (19) is equal to:

$$\begin{aligned} &I\left(\frac{1}{2}(\xi+x)\right) - I\left(\frac{1}{2}(\xi-x)\right) + iI\left(\frac{1}{2}(i\xi+x)\right) - iI\left(\frac{1}{2}(i\xi-x)\right) \\ &\quad = I\left(\frac{1}{2}(\xi+x)\right) - I\left(\frac{1}{2}(\xi-x)\right) + iI\left(\frac{1}{2}(\xi-ix)\right) - iI\left(\frac{1}{2}(\xi+ix)\right) \\ &\qquad = \overline{\langle \xi, Px\rangle} = \langle Px, \xi\rangle\,. \end{aligned}$$

In order to see property **b)** it is sufficient to prove additivity of the real part. In fact, using $4I(y_0) = I(2y_0)$, we check that

$$\Re e\ (4\langle y, P(\xi+x)\rangle) = \Re e\ (\langle 2y, P2(\xi+x)\rangle) = I(x+\xi+y) - I(x+\xi-y) \tag{20}$$

$$= 4\Re e\ (\langle y, P\xi\rangle + \langle y, Px\rangle) = I(\xi+y) - I(\xi-y) + I(x+y) - I(x-y)\,. \tag{21}$$

Using the parallelogram identity for $I(x)$, i.e., (16) with $=$ instead of $\leq$, and associating the terms of equal signs, we see that the right hand side of (21) is equal to

$$\begin{aligned} &\frac{1}{2}\left[I(x+\xi+2y) + I(\xi-x)\right] - \frac{1}{2}\left[I(x+\xi-2y) + I(\xi-x)\right] \\ &\qquad = \frac{1}{2}\left[I(x+\xi+2y) - I(x+\xi-2y)\right] \\ &= \frac{1}{2}\left\{-I(x+\xi) + 2\left[I(x+\xi+y) + I(y)\right] + I(x+\xi) - 2\left[I(x+\xi-y) + I(y)\right]\right\} \\ &\qquad = I(x+\xi+y) - I(x+\xi-y) \end{aligned}$$

as wanted.

Property **c)** for $q$ real rational is consequence of **b)**, as in [9, Sect. 9.2]. When $q = i$ equality follows since **a)** easily shows

$$\langle x, P(i\xi)\rangle = -i\langle x, P\xi\rangle = \langle x, iP\xi\rangle \qquad \text{i.e.,} \qquad P(i\xi) = iP\xi\,.$$

Hence, property **c)** holds also for $iq$ with real rational $q$ and then it holds for every complex number with rational real and imaginary parts.

Property **d)** is obvious. □

Now we prove:

**Lemma 2.8.** *Assume Hypothesis 1.1 and suppose that system (1) is null controllable. Then, there exists a number $M$ such that*

$$I(y) \leq M|y|^2, \quad y \in H.$$

*Proof.* It is sufficient to prove that $I(y)$ is bounded in a ball since $I(\lambda y_0) = |\lambda|^2 I(y_0)$, $\lambda \in \mathbb{C}$, $y_0 \in H$. This is known, see the second statement in Lemma 1.9. □

For the moment, we can't say that the operator $P$ is linear, i.e., that $P(qx) = qPx$ for every *real* $q$. This will be proved below, as a consequence of this version

of Schwarz inequality, which can be proved using solely the properties stated in Lemmas 2.7 and 2.8:

**Lemma 2.9.** *Assume Hypothesis 1.1 and suppose that system (1) is null controllable. Then, we have*

$$|\langle Py, x\rangle| = |\langle y, Px\rangle| \leq M|y||x|, \quad x, y \in H. \tag{22}$$

*Proof.* The inequality is obvious if $\langle Py, x\rangle = 0$. Otherwise, we note the following equality, which holds for every complex number $\lambda$ which has rational real and imaginary parts:

$$0 \leq \langle Px, x\rangle + 2\Re e\ (\lambda\langle Py, x\rangle) + |\lambda|^2\langle Py, y\rangle\,.$$

This inequality is extended to every *real* $\lambda$ by continuity. The usual choice $\lambda = -\left(\langle Px, x\rangle\right)/(\langle Py, x\rangle)$ gives

$$|\langle Py, x\rangle| \leq \sqrt{\langle Px, x\rangle\langle Py, y\rangle} = \sqrt{I(x)I(y)} \leq M|x||y|\,.$$

□

Finally we can prove:

**Theorem 2.10.** *Assume Hypothesis 1.1 and suppose that system (1) is null controllable. Then the operator $P$ defined in Lemma 2.7 is linear and continuous on $H$. Hence, it is selfadjoint and non-negative.*

*Proof.* We first prove that for every *real* $q_0$ and every $\xi \in H$ we have

$$P(q_0\xi) = q_0 P\xi\,.$$

Let $q_n \to q_0$ be a sequence of *rational* numbers. Then we have (Lemma 2.9 is used in the second line)

$$\begin{aligned}
&\lim P(q_n\xi) = \lim q_n P\xi = q_0 P\xi\,,\\
&|P(q_n\xi) - P(q_0\xi)| = |P(q_n - q_0)\xi| = \sup_{|y|=1}\langle y, P(q_n - q_0)\xi\rangle \leq M|q_n - q_0|\,|\xi|.
\end{aligned}$$

So, $q_0 P\xi = \lim q_n P\xi = \lim P(q_n\xi) = P(q_0\xi)$. This gives linearity of the operator $P$ which, from Lemma 2.7 is everywhere defined and symmetric. Continuity follows immediately from (22). □

Combining Corollary 2.6 and Lemma 2.7 we get:

**Corollary 2.11.** *Assume Hypothesis 1.1 and suppose that system (1) is null controllable. Then System (1) is NCVE if and only if $P = 0$.*

So, our goal now is to show that if the system (1) is null controllable and $s(A) \leq 0$ , then $P = 0$.

An obvious but important observation is the following one: the time 0 as initial time has no special role and we can repeat the previous arguments, for every initial time $\tau \geq 0$ and $y_0 \in H$. Hence we can define $P_\tau : H \to H$ such that

$$\langle y_0, P_\tau y_0\rangle = \inf \int_\tau^{+\infty} |u(s)|^2\, \mathrm{d}s, \tag{23}$$

where the infimum is computed on the set

$$\mathcal{U}(y_0, \tau) = \{u \in L^2(\tau, +\infty)\ :\ y(t; \tau, y_0, u) \in L^2(\tau, +\infty)\}\,.$$

We have a family $P_\tau$ of linear operators, and $P_0 = P$ is the operator defined above. The observation is:

**Lemma 2.12.** *Assume Hypothesis 1.1 and suppose that system (1) is null controllable. Then the operator $P_\tau$ does not depend on $\tau$:*

$$P_\tau = P_0 = P\,.$$

*Proof.* Computing infima, we can assume that the inputs $u$ are smooth.

We observe the following equality, which holds for $t > \tau$:

$$y(t;\tau,y_0,u) = y(t-\tau;0,y_0,v)\,, \qquad v(t) = u(t+\tau),$$

and both $v(t)$ and $y(t;0,y_0,v)$, $t \geq 0$, are square integrable on $(0,+\infty)$ if $u(t)$ and $y(t;\tau,y_0,u)$ are square integrable on $(\tau,+\infty)$. Hence, the infimum of the functional in (23) is $\langle y_0, P_\tau y_0\rangle = \langle y_0, Py_0\rangle$, i.e., $P_\tau = P$. □

Now we write

$$\langle y_0, Py_0\rangle \leq \int_0^{+\infty} |u(s)|^2\,\mathrm{d}s = \int_0^{\tau} |u(s)|^2\,\mathrm{d}s + \int_\tau^{+\infty} |u(s)|^2\,\mathrm{d}s\,.$$

We choose a control $u$ which is smooth on $[0,\tau]$ so that $y(\cdot;y_0,u)$ is continuous and we keep the restriction of $u$ to $[0,\tau]$ fixed. Then we compute the infimum of the integral on $[\tau,+\infty)$. We get

$$\langle y_0, Py_0\rangle \leq \int_0^{\tau} |u(s)|^2\,\mathrm{d}s + \langle y(\tau;y_0,u), P_\tau y(\tau;y_0,u)\rangle\,.$$

Using the fact that $P_\tau = P$ is independent of $\tau$, we see that the following inequality holds for every control $u \in C^1$ (cf. Lemma 1.1), every $y_0 \in H$ and $t \geq 0$:

$$\text{(LOI)} \qquad \langle Py(t;y_0,u), y(t;y_0,u)\rangle - \langle Py_0, y_0\rangle + \int_0^t |u(s)|^2\,\mathrm{d}s \geq 0\,. \tag{24}$$

This inequality is known as *Linear Operator Inequality in integral form (LOI)* or *dissipation inequality in integral form.* For special classes of boundary control systems, it has been studied in [26, 27, 28], see also [6].

Now we have all the preliminaries we need to prove Theorem 1.7. Recall that we are assuming that $s(A) \leq 0$ and that system (1) is null controllable in time $T$. Thanks to Corollary 2.11 in order to prove that system (1) is NCVE we need to show that

$$Py_0 = 0, \quad y_0 \in H.$$

We decompose $H = H_s \oplus H_1$ according to Hypothesis 1.2 and show that $P$ restricted respectively to $H_s$ and $H_1$ is identically zero.

If $y_0 \in H_s$, then by (LOI) with $u = 0$ we get

$$\langle Pe^{At}y_0, e^{At}y_0\rangle \geq \langle Py_0, y_0\rangle.$$

By assumption, when $y_0 \in H_s$ we have

$$\lim_{t\to+\infty} e^{At}y_0 = 0\,.$$

Hence, letting $t \to +\infty$, we find $\langle Py_0, y_0\rangle = 0$ and so $Py_0 = 0$.

In order to prove that $P$ is zero on $H_1$, *it is enough to verify that $Pz = 0$ on every generalized eigenvector $z$ of $A$ which belongs to $H_1$.*

Indeed the subspace generated by all generalized eigenvectors of $A$ which belong to $H_1$ is dense in this space and, moreover, $P$ is continuous on $H_1$.

Let us first prove that $Pz_0 = 0$ for any eigenvector $z_0$ associated to an eigenvalue $\lambda$.

We choose $y_0 = z_0$. Let $u$ be smooth. LOI asserts that

$$\begin{aligned}&\left(e^{2(\Re e\,\lambda)t} - 1\right)\langle Pz_0, z_0\rangle + 2\Re e\, e^{\lambda t}\langle P(Lu)(t), z_0\rangle \\ &+ \left\{\int_0^t |u(s)|^2\,\mathrm{d}s + \langle P(Lu)(t), (Lu)(t)\rangle\right\} \geq 0 \qquad \forall t \geq 0 \end{aligned} \tag{25}$$

($Lu$ is defined in (2); note that since $u$ is smooth we know that $Lu$ is continuous). Now if $\Re e\,\lambda < 0$ we insert $u = 0$ in (25) and get

$$\left(e^{2(\Re e\,\lambda)t} - 1\right)\langle Pz_0, z_0\rangle \geq 0.$$

Letting $t \to +\infty$, we infer $Pz_0 = 0$.

Let now $\Re e\,\lambda = 0$. In this case (25) becomes, for any $t \geq 0$, for any $u \in C^1$ and for any $y_0 = z_0$ where $z_0$ is an eigenvector associated to $\lambda$,

$$2\Re e\, e^{\lambda t}\langle P(Lu)(t), z_0\rangle + \left\{\int_0^t |u(s)|^2\,\mathrm{d}s + \langle P(Lu)(t), (Lu)(t)\rangle\right\} \geq 0\,.$$

Let us fix any $t > 0$ and $u \in C^1$. Since we have a linear affine expression of $z_0$, replacing $z_0$ by $\mu z_0$, $\mu \in \mathbb{R}$, we see that the previous inequality implies that

$$\langle P(Lu)(t), z_0\rangle = 0, \tag{26}$$

for every $t \geq 0$ and $u \in C^1$.

We fix now $\epsilon > 0$ and choose a smooth control $u_\epsilon$ from Lemma 2.4 (depending on $y_0 = z_0$ and $\epsilon$). Writing (26) with $u = u_\epsilon$, we obtain with $t = t_\epsilon$ (see (11))

$$0 = \langle z_0, P(Lu_\epsilon)(t_\epsilon)\rangle.$$

Recall that $y(t, y_0, u) = e^{At}y_0 + (Lu)(t) = e^{\lambda t}y_0 + (Lu)(t)$. It follows that

$$|\langle Pz_0, y(t_\epsilon, z_0, u_\epsilon)\rangle| = |\langle Pz_0, e^{\lambda t_\epsilon}z_0\rangle| = \langle Pz_0, z_0\rangle,$$

since $\Re e\,\lambda = 0$. On the other hand, (11) implies that

$$|\langle Pz_0, y(t_\epsilon, z_0, u_\epsilon)\rangle| \leq \epsilon|Pz_0|.$$

We have found that, for any $\epsilon > 0$, $\langle Pz_0, z_0\rangle \leq \epsilon|Pz_0|$. We deduce that $Pz_0 = 0$.

Let us show now that $Pz = 0$, when $z$ is any generalized eigenvector in $H_1$ associated to a fixed eigenvalue $\lambda$.

The proof is done by recurrence, taking into account that, starting from an eigenvector $z_0$ associated to $\lambda$, the generalized eigenvectors are recursively defined by

$$Az_i = \lambda z_i + z_{i-1}\,, \qquad i \geq 1\,. \tag{27}$$

It follows that

$$e^{At}z_i = e^{\lambda t}z_i + q_i(t)\,, \qquad q_i(t) = e^{\lambda t}\sum_{k=1}^{i}\alpha_k z_{i-k}t^k \tag{28}$$

($\alpha_k$ are rational numbers). We already know that the assertion holds for all eigenvectors, i.e., for all $z_k$, when $k = 0$; so we assume that $Pz_k = 0$ when $0 \leq k < i$ and prove that

$$Pz_i = 0. \tag{29}$$

Note that the induction hypothesis implies $Pq_i(t) = 0$ so that (LOI) with initial condition $y_0 = z_i$ takes the form

$$\begin{aligned} 0 &\le \langle P\left[e^{\lambda t} z_i + q_i(t) + (Lu)(t)\right], \left[e^{\lambda t} z_i + q_i(t) + (Lu)(t)\right]\rangle - \langle P z_i, z_i\rangle \\ &+ \int_0^t |u(s)|^2 \, \mathrm{d}s \\ &= \langle P\left[e^{\lambda t} z_i + (Lu)(t)\right], \left[e^{\lambda t} z_i + (Lu)(t)\right]\rangle - \langle P z_i, z_i\rangle + \int_0^t |u(s)|^2 \, \mathrm{d}s. \end{aligned}$$

Now, we argue as before. If $\Re e\, \lambda < 0$, then inserting the control $u = 0$ and letting $t \to +\infty$, we get $Pz_i = 0$.

If $\Re e\, \lambda = 0$, the terms $\langle P z_i, z_i\rangle$ and $\langle P e^{\lambda t} z_i, e^{\lambda t} z_i\rangle$ cancel out and we find

$$2\Re e\, e^{\lambda t} \langle P(Lu)(t), z_i\rangle + \left\{ \int_0^t |u(s)|^2 \, \mathrm{d}s + \langle P(Lu)(t), (Lu)(t)\rangle \right\} \ge 0\,.$$

Repeating the argument before (26) we get first $\langle P(Lu)(t), z_i\rangle = 0$, for any $t \ge 0$, and any smooth control $u$. Using again Lemma 2.4 we deduce that $Pz_i = 0$.

So, we proved $P = 0$ on $H$, as we wanted. □

**Remark 2.13.** *We stress that in the proof of Theorem 1.7 we really need to assume null controllability of the system (and not simply to assume the property stated in Lemma 2.4 (see* (11)*). Indeed null controllability is crucial to prove Theorem 2.5: it is used both to see that $I(y_0)$ is finite and in the definition of $Z(y_0)$.*

## Appendix A. The basic setting for boundary control

Here we present known facts about boundary control which are explained in [34, Sections 2.9 and 2.10]. Useful references are [8, Chapter 2], [3, Chapter 3] and [16, Chapter 1]. However, it seems to us that detailed arguments are not completely presented in standard control references and so we write this appendix for the reader's convenience.

A warning is needed: terms and some settings change in different books. For example [11] uses the same term, adjoint, for Banach space and Hilbert space adjoints, while it is convenient for us to use different terms. More important, the dual spaces and the Banach space adjoints are defined in terms either of linear forms or sesquilinear forms. The use of sesquilinear forms as in [11, 34] is the most convenient for us.

We must introduce few notations. As before, $\langle \cdot, \cdot \rangle$ will be used to denote the inner product in Hilbert spaces (if needed, the spaces are specified with an index; no index is present for the inner product in $H$).

If $V$ is a complex Banach space (possibly Hilbert), $V'$ denotes its topological dual (the Banach space of the continuous *linear* functionals defined on $V$). Thus, if $\omega \in V'$ we can compute $\omega(v)$ for every $v \in V$. We shall use the notation

$${}_{V'}(\omega, v)_V$$

in order to denote the *sesquilinear* pairing of $V$ and $V'$, i.e., ${}_{V'}(\omega, v)_V$ is antilinear in $\omega$ and linear in $v$.

To give an example, we note that the concrete spaces encountered in control theory are complexification of spaces of real functions; i.e., if $V_R$ is a linear space over $\mathbb{R}$, the elements of the corresponding complexified space $V$ have the form

$$v = f + ig\,, \qquad f\,,\ g \in V_R\,.$$

The space $V$ is a linear space on $\mathbb{C}$ and it is simple to construct sesquilinear forms on $V$, using elements of $V'$: let $\omega$ a complex valued linear functional on $V$, i.e. $\omega \in V'$. The associated sesquilinear form is

$$_{V'}(\omega, v)_V =_{V'} (\omega, f + ig)_V = \overline{\omega(f - ig)}\,.$$

We shall use both the Hilbert space adjoint and the dual of an operator in the sense of Banach spaces. The Hilbert space adjoint is defined for densely defined operators $A$ by

$$\langle Ax, y\rangle = \langle x, A^* y\rangle \qquad \forall x \in \mathrm{dom}A\,, \quad \forall y \in \mathrm{dom}A^*$$

(this equality implicitly defines $\mathrm{dom}A^*$ as the set of those $y \in H$ such that $x \to \langle Ax, y\rangle$ is continuous).

This implies in particular that

$$\rho(A^*) = \overline{\rho(A)}\,.$$

The operator $A^*$ is closed and if $A$ is (densely defined) closed then $A^*$ has dense domain too.

The Banach space dual of an operator $A : domA \subset V \to W$ (here $V$ and $W$ are Banach spaces) will be denoted $A'$. It is a linear operator from $W'$ to $V'$. It is (uniquely) defined for densely defined operators $A$ and

$$\mathrm{dom}A' = \{\omega \in W' \ : \ \ v \mapsto \ _{W'}(\omega, Av)_W \text{ is continuous. }\}.$$

By definition,

$$_{V'}(A'\omega, v)_V = \ _{W'}(\omega, Av)_W\,.$$

Sesquilinearity of the pairing implies that

$$\rho(A') = \overline{\rho(A)}$$

(see [11, pg. 184]). Hence, the conjugate of multiplication by $\lambda$ is multiplication by $\bar{\lambda}$ (if instead the conjugate is defined in terms of bilinear forms then the resolvent is not changed). Moreover, $A'$ has dense domain if $A$ has dense domain and it is closed, provided that $V$ is reflexive, in particular if it is a Hilbert space.

If $W = V$ and if $A$ is the infinitesimal generator of a $C_0$ semigroup on $V$ then it might be that $A'$ is not a generator on $V'$. It happens that $A'$ is the infinitesimal generator of a $C_0$-semigroup on $V'$ if $V$ is reflexive, in particular if it is a Hilbert space. In this case $e^{A't} = \left(e^{At}\right)'$. As for the Hilbert space adjoint $A^*$, it generates $e^{A^*t}$ (see [29, Section 1.10]).

With these notations and preliminary information, we can now give the details of the setting used in the analysis of boundary control systems.

A.1. **The operators $A$ and $\mathcal{A} = (A^*)'$.** Let $A$ be the generator of a strongly continuous semigroup $e^{At}$ on a complex Hilbert space $H$ with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$.

We shall identify its topological dual $H'$ with $H$ using the Riesz isomorphism, which we denote $R$: $H \to H'$, defined as:

$$(Rv)(h) = \ _{H'}(Rv, h)_H = \langle h, v\rangle.$$

In practice, $R$ is not explicitly written, hidden behind the equality $H = H'$ but in this appendix the distinction is needed for clarity.

Using the Riesz map $R$: $H \mapsto H'$ and the definition of $A'$, we see that $\mathrm{dom}A' = R\,(\mathrm{dom}A^*)$. In fact

$$_{H'}(Rh, Ak)_H = \langle Ak, h\rangle$$

and the right hand side is a continuous function of $k$ if and only if the same holds for the left side.

For every $h \in \mathrm{dom}A^*$ and every $k \in \mathrm{dom}A$ we have:

$$_{H'}\left(A'Rh, k\right)_H = \ _{H'}\left(R\left(R^{-1}A'Rh\right), k\right)_H = \langle k, R^{-1}A'Rh\rangle\,.$$

The definition of $A'$ is

$$_{H'}\left(A'Rh, k\right)_H = \ _{H'}\left(Rh, Ak\right)_H = \langle k, A^*h\rangle\,.$$

Hence (see [8, Sect. II.7]) we have

$$A^* = R^{-1}A'R\,. \tag{30}$$

The same relation holds for the semigroups

$$e^{A^*t} = R^{-1}e^{A't}R\,.$$

In the sequel we denote by $V$ the Hilbert space $\mathrm{dom}A^*$, with inner product $\langle h, v\rangle_* = \langle h, v\rangle + \langle A^*h, A^*v\rangle$, $h, v \in \mathrm{dom}A^*$. We have

$$\mathrm{dom}A^* = V \subset H \overset{R}{\simeq} H' \subset V' \tag{31}$$

with dense and continuous injections.

Let $j$ be the injection of $V$ into $H$, $j\,v = v \in H$, for $v \in V$. Then, the definition of $j' : H' \to V'$ is

$$\langle j\,v, h\rangle = \ _{H'}(Rh, j\,v)_H = \ _{V'}(j'Rh, v)_V$$

and this shows that $j'Rh$ is the restriction of $Rh$ (acting on $H$) to the subspace $V \subset H$.

As $A^* \in \mathcal{L}(V, H)$ we have $(A^*)' : H' \to V'$ belongs to $\mathcal{L}(H', V')$.

We denote $(A^*)'$ by $\mathcal{A}$, so that $\mathrm{dom}\mathcal{A} = H'$ (or, as usually written when $H$ and $H'$ are identified, $\mathrm{dom}\mathcal{A} = H$). The crucial property used in control theory is expressed by stating that $\mathcal{A}$ *extends* $A$. The precise statement is:

**Lemma A.1.** *If $x \in \mathrm{dom}A$ then we have:*

$$Ax = R^{-1}(j')^{-1}\mathcal{A}\,Rx\,.$$

*Proof.* Indeed, if $x \in \mathrm{dom}A$, $v \in V = \mathrm{dom}A^*$, then

$$\begin{aligned} _{V'}(\mathcal{A}\,Rx, v)_V = \ _{V'}((A^*)'\,Rx, v)_V = \ _{H'}(Rx, A^*v)_H = \langle A^*v, x\rangle \\ = \langle v, Ax\rangle = \ _{H'}(RAx, j\,v)_H = \ _{V'}(j'RAx, v)_V \end{aligned}$$

and so $\mathcal{A}R = j'RA$. When $j'$ and $R$ are not explicitly written, as usual, we get $\mathcal{A}x = Ax$. □

The second property that we want to prove is that $V'$ is an extrapolation space generated by $A$. This means that we can see $V'$ as the completion of $H$ when endowed with the norm $|(\lambda I - A)^{-1}\cdot|$, for any $\lambda \in \rho(A)$. In order to see this, we fix any $\lambda \in \rho(A)$ and we prove that $|\cdot|_{V'}$ restricted to $H$ is equivalent to $|(\lambda I - A)^{-1}\cdot|$, i.e., we prove:

**Lemma A.2.** *On $H$, the norms of $V'$ (more precisely, $h \mapsto |j'Rh|_{V'}$) and the norm $|(\lambda I - A)^{-1}\cdot|$ are equivalent.*

*Proof.* Let $I$ denote the identity in $H$ and also on $V$.

Let $\lambda \in \rho(A)$ and $h \in H$. $|j'Rh|_{V'}$ is computed as follows (recall that $j'Rh$ is the restriction of $Rh$ to $V$).

$$\begin{aligned}|j'Rh|_{V'} &= \sup_{|v|_V \le 1} |_{V'}(j'Rh, v)_V| = \sup_{|v|_V \le 1} |\langle v, h\rangle| \\ &= \sup_{|v|_V \le 1} |\langle \left(\bar{\lambda} I - A^*\right)^{-1} \left(\bar{\lambda} I - A^*\right) v, h\rangle| \\ &= \sup_{|v|_V \le 1} |\langle \left(\bar{\lambda} I - A^*\right) v, (\lambda I - A)^{-1} h\rangle| \le C \left|(\lambda I - A)^{-1} h\right|,\end{aligned}$$

with $C = \sup_{|v|_V \le 1} |(\bar{\lambda} I - A^*)v|$. On the other hand,

$$\left|(\lambda I - A)^{-1} h\right| \le C_0 |Rh|_{H'} = C_0 |(j')^{-1} j'Rh|_{H'} \le C_1 |j'Rh|_{V'}.$$

The proof is complete. □

A.2. **The parabolic case and fractional powers of** $(\omega I - A^*)$**.** Our goal here is to justify inequality (6).

We assume that $A$ (and so $A^*$) *generates a holomorphic semigroup* $e^{At}$ (respectively $e^{A^*t}$). One can also prove that the previous dual semigroup $e^{\mathcal{A}t}$ acting on $V'$ is holomorphic.

There exists $\omega \in \rho(A)$ such that, for any $\gamma \in (0,1)$, $(\omega I - A^*)^\gamma$ is a well defined closed operator with domain $V_\gamma \subset H$ (cf. [29, Section 2.6]).

Arguing as in (31), we have

$$V_\gamma \subset H \simeq H' \subset V'_\gamma$$

and we denote by $\mathcal{A}_\gamma$ the operator $[(\omega - A^*)^\gamma]' : H \to V'_\gamma$. *Let $U$ be another Hilbert space and let $B \in \mathcal{L}(U, V'_\gamma)$.*

Note that we may consider $B : U \to V'$, since $V'_\gamma \subset V'$ with dense and continuous injections. Thus we also have $B \in \mathcal{L}(U, V')$.

Moreover, since $B \in \mathcal{L}(U, V'_\gamma)$, we have $B' \in \mathcal{L}(V_\gamma, U')$ and so, for $t > 0$:

$$\begin{aligned}\|B' e^{A^*t}\|_{\mathcal{L}(H,U')} &= \|B'(\omega I - A^*)^{-\gamma} (\omega I - A^*)^\gamma e^{A^*t}\|_{\mathcal{L}(H,U')} \\ &\le \|B'(\omega I - A^*)^{-\gamma}\|_{\mathcal{L}(H,U')} \|(\omega I - A^*)^\gamma e^{A^*t}\|_{\mathcal{L}(H,H)} \le \frac{M e^{\omega_1 t}}{t^\gamma}, \quad t > 0 \qquad (32)\end{aligned}$$

(in the last line we have used a well known estimate for holomorphic semigroups). Next *we compute the dual operator of $B' e^{A^*t}$ and show that it is $R^{-1} e^{\mathcal{A}t} B$,* usually written as $e^{\mathcal{A}t} B$ when $H$ and $H'$ are identified.

We have, for any $x \in H$, $u \in U$, $t > 0$,

$${}_{U'}(B' e^{A^*t} x, u)_U = {}_{V_\gamma}(e^{A^*t} x, Bu)_{V'_\gamma} = {}_V(e^{A^*t} x, Bu)_{V'},$$

since $Bu \in V'_\gamma \subset V'$ (recall that $V = \mathrm{dom} A^*$) and $e^{A^*t} x \in V$, $t > 0$. It follows that

$${}_{U'}(B' e^{A^*t} x, u)_U = {}_{H'}(e^{\mathcal{A}t} Bu, x)_H = \langle x, R^{-1} e^{\mathcal{A}t} Bu\rangle.$$

and so the claim follows.

The previous assertion implies the identity

$$\|B' e^{A^*t}\|_{\mathcal{L}(H,U')} = \|R^{-1} e^{\mathcal{A}t} B\|_{\mathcal{L}(U,H)}, \quad t > 0, \qquad (33)$$

which together with (32) implies the estimate (6), i.e.,

$$\|e^{\mathcal{A}t} B\|_{\mathcal{L}(U,H)} \le \frac{M e^{\omega_1 t}}{t^\gamma}, \quad t > 0,$$

which has been used in Subsection 1.1.

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