Mean-Extended Gini portfolios personalized to investor's profile

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Abstract

Since Shalit and Yitzhaki (1984) the Mean-Extended Gini (MEG) has been proposed as a workable alternative to the classical Markowitz mean-variance Capital Asset Pricing Model (CAPM). Although the MEG controls the risk belonging to the left-tail of the return distribution, little attention is given to potential gains belonging to the right tail of the return distribution. A generalization of the MEG able to select personalized optimal mean-risk and/or mean-gain portfolios is proposed. We give evidence that if the portfolio distributions are symmetrical and/or the investor has a moderate risk-gain profile, then the efficient mean-risk portfolio always coincides with a not efficient mean-gain portfolio. In more realistic scenarios admitting the existence of asymmetrically distributed assets and/or investors with very defensive or very aggressive investment profiles, portfolios which are optimal under both criteria may exist.

Jel Subject Classification: G10, G11, G12,G29

Key words: Extended Gini Index; MEG; risk-aversion and gain-propensity.

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Purpose - Since Shalit and Yitzhaki (1984), the Mean-Extended Gini (MEG) has been proposed as a workable alternative to the Markowitz mean-variance approach. The challenge is to extend the MEG approach to making customized optimal asset allocation to control down-performance and/or up-performance.

Design/methodology/approach -The MEG approach is used to make strategical allocation tailored to the investor risk aversion and gain propension measured by characteristic parameters of the Extended Gini measures.

Findings - We set up two optimization problems: the former focused on controlling the risk, the latter emphasizing the potential gains. Sufficient conditions such that the efficient MEG-risk frontier coincides with the inefficient MEG-gain frontier are stated. In the realistic scenarios that portfolios have asymmetrical distributions and/or the investor profile is very conservative or very aggressive, the desirable occurrence that a portfolio is optimal under both optimizations may occur.

Originality/value – The main contribution of this research is to have pointed out that optimal allocation must be tailored to both the investor's risk and gain profile. And, the optimality may be not preserved if the investor's risk-gain profile changes. So, the statement "optimal allocation" should be reworded as "optimal allocation personalized to the investor's risk-aversion and gain-propensity".

1 Introduction

Since the seminal contributions of Markowitz (1952), the mean-variance (MV) approach has been extensively used to select efficient portfolios focused on controlling risk. Nevertheless, it is well known in the finance literature that the MV approach is inappropriate when asset returns are not normally distributed or investors' preferences are not characterized by quadratic functions (see Meyer, 1987, 1989 for sufficient conditions for the MV use). In a pioneering paper Shalit and Yitzhaki (1984, 2005) present the Mean-Extended Gini (MEG), as a sound and distribution-free alternative to MV, that is consistent with the rules of stochastic dominance. Its key role in personalized asset allocation has been acknowledged in recent years by both the academia (see Shalit and Yitzhaki , 2010) and investment practitioners (see Sherman Cheung et al., 2005, 2008). But, strategical allocation made through the MEG is more focused on personalized risk control than on potential upside profits. As it has been pointed out in the recent literature, the latter aspect cannot be neglected by ample groups of investors (see Biglova et al., 2004, and recently Neave et al., 2008, and Farinelli et al., 2009). The challenge is not only to match the expected return with the minimum risk, but also to make strategical allocation customized to the investor risk aversion and gain propension.

The aim may be achieved through two different optimizations. The former is based on selecting the classical MEG-risk efficient frontiers, the latter on selecting the so called MEG-gain efficient frontiers where the favorable data dispersion on the right tails of the distributions are maximized.

We show that the optimal investment crucially depends on the investor risk-gain profile, modelled by the orders of the Extended Gini (EG) index used, and the asymmetry of the portfolio distributions. Specifically, if the investor has a "moderate" risk-gain profile using the standard Gini index or the distributions are symmetric and investors have the same level of risk-aversion and gain-propension, then the efficient MEG-risk frontier will coincide with an inefficient MEG-gain one. However, as we skip from this framework and deal with more realistic scenarios, the desirable circumstance that the optimum MEG-risk and MEG-gain portfolios coincide, may occur. A sufficient condition based on a restricted class of feasible portfolios is stated.

The remainder of this paper is organized as follows. Section 2 introduces the definitions of EG-risk and EG-gain measures. In Section 3 we discuss MEG allocation oriented to risk-control and/or profit-gain. Section 4 concludes the paper. An Appendix collects a proof.

2 EG-Risk and EG-Gain measures

Mean-Gini theory was originally developed by Yitzhaki (1982) and afterwards applied to finance by Shalit and Yitzhaki (1984) as an alternative model to MV for evaluating systematic risk and constructing optimal portfolios consistent with expected utility maximization and stochastic dominance. Mean-Gini presents robust results when MV is bound to fail. In particular, this occurs when assets do not have normal distributions or when the regression used to estimate betas by ordinary least-squares provides biased estimators (Shalit and Yitzhaki, 2002).

In the context of measuring social inequality the extension of the Gini coefficient was proposed by Donaldson and Weymark (1980) and Yitzhaki (1983), but only in the pioneering paper of Shalit and Yitzhaki (1984) was the Mean-Extended Gini (MEG) applied to finance. In Shalit and Yitzhaki (2005) the authors go on in this research direction.

Although the different definitions of EG given in the literature coincide in the continuous case, they may differ in the discontinuous one (see Yitzhaki and Schechtman 2005 for details for adjusting discrete variables vs continuous ones). In the following we will use one that fits well with the context and permits the use of results of the theory of extreme value distributions.

Definition 2.1 *Let X be a random variable with cumulative distribution function F. The Extended Gini of X of order k is given by*

$$EG_k(X) = E(X) - E\{\min(X_1, ..., X_k)\} \qquad \text{with } k \text{ a positive integer} \tag{2.1}$$

where $X, X_1, ..., X_k$ are i.i.d. random variables.

In the literature $EG_k(X)$ is used as a personalized risk measure according to the intensity k. The higher[1] k, the more $EG_k(X)$ weights the left tail of X, and the more the investor is risk-averse. Starting from the evidence that the right-tail of X coincides with the left-tail of $-X$, we define the Extended Gini of $-X$ as

$$EG_k(-X) = -E(X) - E\{\min(-X_1, ..., -X_k)\} = E\{\max(X_1, ..., X_k)\} - E(X) \tag{2.2}$$

with k a positive integer (see Cardin et al. 2011) so the higher k, the more $EG_k(-X)$ weights the right tail of X and the more the investor is gain prone. From (2.1) and (2.2), it is easy to check that $EG_k(X)$ and $EG_k(-X)$ assume non-negative values and EG_k can be used to measure the risk and the gain of X.

Definition 2.2 *We call $EG_k(X)$ the EG risk-measure and $EG_k(-X)$ the EG gain-measure of X of order k.*

[1] The fact that EG weights more data on the left-tail of X is clear as soon as we think of the original use of EG as a measure of income inequality. Discrepancies in incomes among the poorest part of the population is weighted more than the discrepancies among the richest part.

Distribution	$EG_k(X)$	$EG_k(-X)$	$E(X)$
Uniform(θ) $\theta>0$	$\frac{\theta(k-1)}{2(k+1)}$	$\frac{\theta(k-1)}{2(k+1)}$	$\frac{\theta}{2}$
Normal(μ,σ^2) $k=2$	$\frac{\sigma}{\sqrt{\pi}}$	$\frac{\sigma}{\sqrt{\pi}}$	μ
Skew-Normal $\left(\xi,\omega^2,\alpha\right)$ $k=2$	$\frac{\omega}{\sqrt{\pi(1+\alpha^2)}}$	$\frac{\omega}{\sqrt{\pi(1+\alpha^2)}}$	$\xi+\sqrt{\frac{2}{\pi(1+\alpha^2)}}\alpha$
Pareto(α,$c=1$)	$\frac{\alpha}{\alpha-1}-\frac{\alpha k}{\alpha k-1}$	$\frac{\alpha^k k!}{(\alpha k-1)\cdots(\alpha-1)}-\frac{\alpha}{\alpha-1}$	$\frac{\alpha}{\alpha-1}$
Weibull ($m=2$, $\lambda=2/\sqrt{\pi}$)	$1-\frac{1}{\sqrt{k}}$	$\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}\frac{1}{\sqrt{j}}-1$	1
Exponential (λ)	$\frac{1}{\lambda}\left(\frac{k-1}{k}\right)$	$\frac{1}{\lambda}\sum_{j=1}^{k}\frac{1}{j}-\frac{1}{\lambda}$	$\frac{1}{\lambda}$

Table 1. EG-risk and EG-gain for common distributions

In practice, the order k of the investor's risk-aversion may differ from that of her gain-propension. In this case the notation k_{risk} and k_{gain} is to be preferred. In Actuarial Science the index $EG_{k_{risk}}$ is called the risk-premium and in analogy with this terminology we call $EG_{k_{gain}}$ the gain-premium.

For the most common distributions used in finance such as the uniform, normal, skew-normal, Pareto, Weibull, exponential these premia can be easily computed (see Cardin et al., 2011). The closed-end formulae are summarized in Table 1.

2.1 A way to measure the customized risk-aversion and gain-propension orders

To associate to an investor her personalized (k_{risk}, k_{gain}), it is sufficient to implement a simple test. Let X be a risky asset with a given distribution. The mean $E(X)$ corresponds to the certainty equivalent of X for a risk and gain neutral investor (i.e. $k_{risk}=k_{gain}=1$), but it can be also interpreted as the value of a safe asset X assuming the single value $E(X)$ with probability 1. A natural way to price the risk-premium $EG_{k_{risk}}(X)$ and gain-premium $EG_{k_{gain}}(-X)$ is to declare these amounts as proportions or percentages of $E(X)$. Denote these proportions by δ_{risk} and δ_{gain}. Note that because X is non-negative, $\delta_{risk}\leq 1$, but if X is unbounded, then δ_{gain} is also unbounded[2].

[2]For pricing a positive asset X it is reasonable that $0\leq\delta_{risk}\leq 1$ and $\delta_{gain}\geq 0$. The former condition guarantees the positiveness of the bid-price of X given by $P_{bid}=E(X)-EG_{risk}(X)=(1-\delta_{risk})E(X)$. Whereas

$$EG_{k_{risk}}(X) = \delta_{risk} E(X) \text{ and } EG_{k_{gain}}(-X) = \delta_{gain} E(X)$$

If δ_{risk} and δ_{gain} are declared, the corresponding k_{risk} and k_{gain} orders come out.

Just for explanatory purposes, let us compute these values for two special distributions.

Let X be uniform on $[0,\theta]$ with $E(X) = \frac{\theta}{2}$. Since the distribution is symmetric, the analytical relation between the percentage δ and the order k is the same for $EG_k(X)$ and $EG_k(-X)$.

$$\delta = \frac{k-1}{k+1} \text{ or equivalently } k = \frac{1+\delta}{1-\delta}$$

Not every value of δ.corresponds to an integer. In such a case, we would describe the risk or gain parameter by the closest such integer k. However, if we start with any value of k, we can find the corresponding δ. We see immediately that if $k = 1$, the risk neutral agent evaluates $\delta = 0$ and $EG_k(X) = 0$; if $k_{risk} = 2$ then $\delta_{risk} = .3$ and the risk-premium is the 33% of the mean; if $k_{risk} = 6$ then $\delta_{risk} = .71$ and so on (see Table 2.)

Uniform Distribution	δ_{risk}	δ_{gain}
$k = 1$	0	0
$k = 2$	.33	.33
$k = 3$	.50	.50
$k = 4$	.60	.60
$k = 6$	.71	.71
$k = 9$	.80	.80
$k = +\infty$	1	1

Table 2. The values of k_{risk} and k_{gain} for the Uniform distribution

Let X be exponential with parameter $\lambda > 0$ and $E(X) = \frac{1}{\lambda}$. It follows that $\delta_{risk} = \frac{k_{risk}-1}{k_{risk}}$ and $\delta_{gain} = \sum_{j=1}^{k_{gain}} \frac{1}{j} - 1$. Note[3] that if $\delta_{gain} < 1$, then $k_{gain} \leq 3$.

to guarantees the positiveness the ask-price $P_{ask} = E(X) + EG_{gain}(-X) = (1 + \delta_{gain}) E(X)$ is sufficient the latter condition (for the definitions of bid and ask price according to the EG certainty equivalent criterium, see Cardin et al., 2011).

[3]If $\delta_{gain} > 1$ it means that the aggressive investor (with $k_{gain} \geq 4$) declares as ask price of X the amount $P_{ask} = (1 + \delta_{gain}) E(X)$, i.e. she is willing to pay more than the double of $E(X)$.

Exponential Distribution	δ_{risk}	δ_{gain}
$k = 1$	0	0
$k = 2$	.50	.50
$k = 3$	.67	.83
$k = 4$	.75	1.08
$k = 6$	.83	1.45
$k = 9$	.89	1.83
$k = +\infty$	1	∞

Table 3. The values of k_{risk} and k_{gain} for the exponential distribution

We can also compare the risk aversion and gain propension as the asset changes. Consider two assets with the same mean but distributed as a uniform and an exponential variable, respectively; at any level of k_{risk} the exponential asset induces a greater risk premium $EG_{k_{risk}}$ than that of the uniform asset. That means that the short fat left-tail of the exponential is perceived riskier than the flat left-tail of the uniform variable. At the same time, the exponential long right-tail promises greater gains than the flat right-tail of the uniform variable. In fact its gain premium $EG_{k_{gain}}$ is greater than the corresponding gain premium achieved with the uniform distribution.

In conclusion, the above examples bring to light the critical "gears" that move the k_{risk} and k_{gain} values: (1) the personal investor's risk/gain attitudes (2) and the distribution of the risky asset under exam.

This latter aspect must be taken under consideration when we aim at categorizing investors in different risk/gain profiles. Firms performing investment services subjected to the Markets in Financial Instruments Directive (MiFID) adopted by the European Commission usually implement a mechanism of categorization of its clients. This transparency directive imposes that any client be classified with a suitable risk/gain profile and consequently any investment advice or suggested financial transaction be preventively checked to be appropriate to the client profile. The standard profiles used are five: (1) very conservative (2) conservative (3) moderate (4) aggressive (5) very aggressive. Using the common loose terminology, a conservative (or a very conservative) profile is characterized by "high" k_{risk} and "low" k_{gain}. Vice versa an aggressive (or a very aggressive) profile by "low" (even close to 1) k_{risk} and "high" k_{gain}. A moderate profile should be characterized by "moderate" k_{risk} and k_{gain} and reasonably around 2. Clearly, the bounds for k_{risk} and k_{gain} in each profile cannot be given in a strict way, because the bounds may change as the distribution used in the test changes. As a first step, we suggest using a test involving a uniform asset representing the case where little information is available.

3 MEG-Risk and MEG-Gain efficient frontiers

Let R_i be the rate of return on asset i ($i = 1, ..., N$) and x_i the share of investor wealth invested in asset R_i. Let $X = \sum_{i=1}^{N} x_i R_i + \left(1 - \sum_{i=1}^{N} x_i\right) r_f$, the return of the portfolio with distribution function F, where r_f is the rate of return on the risk-free asset.

Following the MEG optimal asset allocation introduced by Shalit and Yitzhaki (1984, 2005), we set up the two following optimization problems:

- As the mean return of the portfolio is given, optimize the MEG-risk portfolio, i.e. minimize $EG_{k_{risk}}(X)$ or equivalently:

$$Max\ E\left\{\min\left(X_1, ..., X_{k_{risk}}\right)\right\} \qquad where\ \ X, X_1, ..., X_{k_{risk}}\ are\ i.i.d.\ random\ portfolios. \tag{3.3}$$

- As the mean return of the portfolio is given, optimize the MEG-gain portfolio, i.e. maximize $EG_{k_{gain}}(-X)$:

$$Max\ E\left\{\max\left(X_1, ..., X_{k_{gain}}\right)\right\} \qquad where\ \ X, X_1, ..., X_{k_{gain}}\ are\ i.i.d.\ random\ portfolios. \tag{3.4}$$

$$\text{such that} \qquad E(X) = \sum_{i=1}^{N} x_i E(R_i) + \left(1 - \sum_{i=1}^{N} x_i\right) r_f \text{ is fixed and}$$

$$\sum_{i=1}^{N} x_i = 1 \text{ and } x_i \geq 0.$$

First optimization was originally proposed by Shalit and Yitzhaki (2005). We suggest to add a further optimization aimed at selecting the optimal mean-gain portfolios.

Definition 3.1 *As the mean return of the portfolio varies, the optimal solutions of MEG-risk portfolio and MEG-gain portfolio vary, as well. We call MEG-risk efficient frontiers and MEG-gain efficient frontiers such optimal solutions.*

Clearly, there exist some special cases where MEG-risk efficient frontiers and MEG-gain efficient frontiers coincide. For example, that happens when gain-indifference (e.g. $k_{gain} = 1$) or risk-indifference (e.g. $k_{risk} = 1$) hold, more precisely:

(i) If $k_{gain} = 1$, then the MEG-gain efficient frontier collapses into the MEG-risk efficient frontier.

(ii) If $k_{risk} = 1$, then the MEG-risk efficient frontier collapses into the MEG-gain efficient frontier

The result is very intuitive. In the former case the agent is "very conservative" and indifferent to potential gains, so her efficient frontier coincides with that which minimizes the risk. In the latter case the agent is "very aggressive" and does not care the potential losses, so her efficient frontier coincides with that which maximizes the potential gains.

But in the most common situations the efficient frontiers of MEG-risk and MEG-gain do not coincide. A spontaneous question that arises is whether at least for special values of the portfolio mean return it may happen.

The problem is faced in two steps. First, in Sec. 3.1 we state a number of sufficient conditions under which the efficient MEG-risk frontier coincides with the least efficient MEG-gain frontier. Clearly in such a case, the question is negatively answered. Second, in Sec. 3.2 we state a sufficient condition on investor's attitudes k_{risk} and k_{gain} which guarantee the existence of an portfolio which is optimum according to the both criteria.

3.1 The MEG-risk efficient frontier coincides with the most inefficient MEG-gain frontier and vice versa

We concern the case $k_{risk} = k_{gain} = k$.

Proposition 3.2 *If* $k_{risk} = k_{gain} = k$ *and*

(i) all portfolios are symmetrically distributed; or

(ii) $k = 2$,

then the MEG-risk efficient frontier coincides with an inefficient MEG-gain frontier and vice versa.

Proof. Assume $k_{risk} = k_{gain} = k$. If the portfolio distributions are symmetrical or $k = 2$, then for all X, $EG_k(X) = EG_k(-X)$ (see for example Yitzhaki and Schechtman, 2005). This equality implies that as the mean is fixed, the portfolio with the minimum risk is that with the minimum gain, and vice versa.■

Note that the symmetry of the rate of return R_i of each asset in the portfolio is not a sufficient condition for the symmetry of the portfolio $X = \sum_{i=1}^{N} x_i R_i + \left(1 - \sum_{i=1}^{N} x_i\right) r_f$. For example, assume R_1 takes the values 0 or 2 with probability 1/2. If $R_1 = 0$ assume that R_2 takes the values 0 or 2 with probability 1/2 and if $R_1 = 2$, then $R_2 = 1$ with probability 1. Then R_1 and R_2 are symmetric, but $X = .5R_1 + .5R_2$ is not symmetric. That is, $P(X = 0) = P(X = 1) = 1/4$ and $P(X = 1.5) = .5$.

On the other hand, a number of special distributions of R_i guarantee the symmetry of the portfolios. In general, the symmetry of the portfolio X is guaranteed by a "balance" of conditions on the distribution and on the dependence structure among the assets, the stronger are the former, the looser are the latter, and vice versa. Under the strong assumption of normally distributed asset R_i the portfolios X are guaranteed to be normal and thus symmetric. In this case, if $k_{risk} = k_{gain} = k$, the efficient MEG-risk frontier coincides with the most inefficient MEG-gain frontier. On the other hand, if no conditions are imposed to the distributions of R_i but we assume the severe assumption that $Z_1, \ldots, Z_m$ are independent, symmetric random variables, not necessarily identically distributed, then $\sum_{i=1}^{m} c_i Z_i$ is also symmetric for any $c_i, i = 1$ to m (see the Appendix for a proof). Now suppose that the returns R_i can be modelled as linear combinations of such independent symmetric random variables. This is the case, for example, if the returns are jointly normally distributed, but we could also imagine the case where the returns are linear combinations of independent uniformly distributed assets. We then have that

$$X = \sum_{i=1}^{N} x_i R_i = \sum_{i=1}^{m} c_i Z_i,$$

for some $c_i, i = 1$ to m determined from the $x_i, i = 1$ to N. Hence all such X will be symmetric. In such a case we conclude that the efficient MEG-risk frontier coincides with the most inefficient MEG-gain frontier.

3.2 A sufficient condition under which a portfolio is optimal according to both the MEG-risk and MEG-gain criteria.

In this Section we focus on a sufficient condition such that for a given portfolio mean return, the optimal mean-risk portfolio coincides with the optimal mean-gain portfolio.

Proposition 3.3 *Suppose that X and Y have the same mean μ while X is distributed in [a,c] and Y is distributed on [b,d], where $a < b < c < d$. In the continuous case this means that X and Y have positive densities on (a,c) and (b,d), respectively, and in the discrete case that $P(X = a), P(X = c), P(Y = b)$ and $P(Y = d)$ are all positive. Then*

(1) For large enough k_{risk}, Y is the better MEG-risk portfolio.

(2) For large enough k_{gain}, Y is the better MEG-gain portfolio.

Proof: First note that under the conditions of the proposition, we must have $b < \mu < c$. It also follows that $\min(X_1, \ldots, X_k) \to a$ and $\min(Y_1, \ldots, Y_k) \to b$ as k approaches infinity. Thus as k approaches infinity, $EG_k(X) \to \mu - a$ and $EG_k(Y) \to \mu - b$. Since $\mu - b < \mu - a$, we have that for large enough k, Y is the better MEG-risk portfolio. This proves (1).

Also $EG_k(-X) \to c - \mu$ and $EG_k(-Y) \to d - \mu$. Since $c - \mu < d - \mu$, we have that for large enough k, Y is also the better MEG-gain portfolio. This proves (2). ■

Proposition 3.3 concerns assumptions of a double nature: (1) on the investor's attitude to risk and gains, i.e. the investor must be strongly enough risk-averse (i.e. k_{risk} is large enough) and strongly enough gain-prone (i.e. k_{gain} is large enough), (2) on the domains of the feasible portfolios.

The result has an intuitive interpretation. The investor with above mentioned attitudes will always choose the optimal portfolio on the basis of the lowest and the greatest bounds of the supports of the distributions of the feasible portfolios and not the actual distributions themselves. In conclusion, if there exist a portfolio which has the shortest left-tail and the longest right-tail, that is just the optimal portfolio according to the both criteria.

Note that this selection rule based only on the bounds of the domains may be no longer hold if k_{risk} and k_{gain} assume smaller values, as shown in the following.

Example 3.4 *The optimal MEG-risk portfolio coincides with the optimal MEG-gain portfolio: an unstable preference order.*

Suppose you have two alternative portfolios X and Y with

$$P(X = 47) = 0.1 \text{ and } P(X = 117) = 0.9.$$

while

$$P(Y = 100) = 0.9 \text{ and } P(Y = 200) = 0.1,$$

An easy calculation shows that $E(X) = E(Y) = 110$. For simplicity assume $k_{risk} = k_{gain} = k$. Simple calculations give the following tables of values:

k	$EG_k(X)$	$EG_k(Y)$		$EG_k(-X)$	$EG_k(-Y)$
1	0	0		0	0
2	6.3	9		6.3	9
3	11.97	9.9		6.93	17.1

Table 2. EG-risk and EG-gain

Recall that an investor wants to minimize $EG_k(X)$ and maximize $EG_k(-X)$. It follows that if $k_{risk} = 1$ or $k_{gain} = 1$, then X and Y are equivalent investments in the MEG-risk sense. This corresponds to an investor who is indifferent to gains or losses. If $k_{risk} = 2$, then X is better than Y in the sense of MEG-risk. If $k_{risk} = 3$, then this is reversed and Y is better than X. If $k_{gain} = 2$ or 3, then Y is better than X in the MEG-gain sense.

In particular, we see that if $k_{risk} = 3$, then Y would be the better investment for this investor in the MEG-risk sense as well as the MEG-gain sense for $k_{gain} = 1, 2,$ or 3. On the other hand, if $k_{risk} = 2$ and $k_{gain} = 2$ or 3, then X is the better MEG-risk investment, while Y is the better MEG-gain investment. For large enough k_{risk} and k_{gain} we get a confirmation of the correctness of Proposition 3.3. As $[a, c] = [47, 117]$ and $[b, d] = [100, 200]$. We have $47 < 100 < 117 < 200$, so the conditions are met. Also $E(X) = E(Y)$. It follows that for large enough k_{risk} and k_{gain} that Y is the better MEG-risk and MEG-gain investments. We see that this is in fact true when $k_{risk} = k_{gain} = 3$. Note that $EG_k(X) \to 110 - 47 = 63$, while $EG_k(Y) \to 110 - 100 = 10$ as k approaches infinity. Note also that $EG_k(-X) \to 117 - 110 = 7$, while $EG_k(-Y) \to 200 - 100 = 100$ as k approaches infinity.

A spontaneous question that arises is which would be the preference guideline suggested by the MV criterion. Since $\mathrm{Var}(X) = 441 < \mathrm{Var}(Y) = 900$, it follows that according to the MV criterion the portfolio X should be always preferred to Y. It is worthwhile noting that it reaches the same conclusion as the MEG conclusion in the case where $k_{risk} = 2$ and $k_{gain} = 1$. If $k_{risk} = k_{gain} = 2$, then neither X nor Y is dominant in the MEG risk and gain sense. X has the smaller risk premium, while Y has the greater gain premium. The MEG risk-gain approach would correctly switch the preference order so preferring Y to X if $k_{risk} = k_{gain} = 3$.

These conclusions fit perfectly with the picture of a conservative/moderate investor who prefers a smaller risk in return and chooses an investment with smaller variance compared to an aggressive investor, who might accept greater risk for a greater possible return and chooses an investment with a larger variance.

Example 3.5 ***Optimal MEG-risk coincides with Optimal MEG-gain and MV indifference.***

It is possible to construct a whole class of investments X and Y such that $E(X) = E(Y)$ and $Var(X) = Var(Y)$, so that the investments are indifferent under the MV criterion, but where one is dominant over the other with respect to MEG-gain and MEG-risk as k_{risk} and k_{gain} approach infinity.

Assume that $E(X) = \mu$ and X has positive density over $(0, c)$ with $\mu < c < 2\mu$. Let Y be distributed as $2\mu - X$. Then Y has a positive density over $(2\mu - c, 2\mu)$, $E(Y) = \mu$ and $\mathrm{Var}(Y) = \mathrm{Var}(X)$ Then X and Y meet the assumptions of Proposition 3.3. where $a = 0, b = 2\mu - c, c = c$, and $d = 2\mu$. It follows that X and Y are MV indifferent, but for large enough k_{risk} and k_{gain}, Y dominates X in the MEG-risk and MEG-gain senses.

A further symmetry argument shows that in this case $EG_k(Y) = EG_k(-X)$ and $EG_k(-Y) = EG_k(X)$, which is an interesting relation. That is, the MEG-risk of Y is the MEG-gain of X and vice-versa. This follows since $\min(Y_1, \ldots, Y_n)$ is equal in distribution to $2\mu - \max(X_1, \ldots, X_n)$ and vice-versa.

Example 3.6 *Let X have density $f(x) = 2x/9$ for $0 \leq x \leq 3$. Then $E(X) = 2$. Let $Y = 4 - X$. Then Y has density $g(y) = 2(4 - y)/9$ for $1 \leq y \leq 4$. X and Y have equal means and variances, so they are indifferent according to the MV approach. We know that for large enough k_{risk} and k_{gain} that Y dominates X in the MEG approach. In fact, if $k = 3$, simple calculation shows that $EG_3(X) \approx 0.63$ and $EG_3(-X) \approx 0.57$. Thus $EG_3(Y) \approx 0.57$ and $EG_3(-Y) \approx 0.63$. Then, Y dominates X according to both MEG-risk and MEG-gain criteria for $k_{risk} = k_{gain} = 3$. Again the MV-criterion fails to spotlight the superiority of Y to X for a gain-prone investor interested in the upper bound of the domain.*

4 Conclusion

A generalization of the MEG approach for making customized optimal asset allocation to control both down-performance and/or up-performance is proposed. The investor risk-aversion and gain-propension are captured by personalized parameters of the EG indices. Sufficient conditions such that the efficient MEG-risk frontier coincides with the most inefficient MEG-gain frontier, are stated. On the other hand, if portfolios have asymmetrical distributions and/or the investor profile is aggressive or conservative, the desirable occurrence that a portfolio be optimal under the both criteria may occur. A sufficient condition for that is also stated. A warning is pinpointed: optimal allocation may be not preserved if the investor's risk-gain profile changes. So, the statement "optimal allocation" should be reworded as "optimal tailor-made allocation" to the investor risk-aversion and gain-propensity.

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5 Appendix

Proof of remark in Sec. 3.1 Without loss of generality, we may assume that $Z_1, \ldots, Z_m$ all have mean 0. Otherwise, we can just subtract their means and add a constant at the end. Denote the characteristic function of Z_j by $\phi_j(t) = E(\exp(itZ_j))$ for each j. Then since Z_j is equal in distribution to $-Z_j$ it follows that $\phi_j(t) = \phi_j(-t)$ for each j. Now if $X = \sum_{j=1}^m c_j Z_j$, then X has characteristic function $\phi_X(t) = \prod_{j=1}^m \phi_j(c_j t)$. Thus $-X = \sum_{j=1}^m (-c_j Z_j)$. We have $\phi_{-X}(t) = \prod_{j=1}^m \phi_j(-c_j t) = \prod_{j=1}^m \phi_j(c_j t) = \phi_X(t)$. It follows from the uniqueness theorem for characteristic functions that X and $-X$ have the same distribution. That is, X is a symmetric random variable.■

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