ON GENERALIZED KUMMER OF RANK 3 VECTOR BUNDLES OVER A GENUS 2 CURVE

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1. Introduction.

Let $X$ be a smooth projective complex curve and let $U_X(r, d)$ be the moduli space of semi-stable vector bundles of rank $r$ and degree $d$ on $X$ (see [8]). It contains an open Zariski subset $U_X(r, d)^\ast$ which is the coarse moduli space of stable bundles, i.e. vector bundles satisfying inequality

$$\frac{d_F}{r_F} < \frac{d_E}{r_E}.$$ 

The complement $U_X(r, d) \setminus U_X(r, d)^\ast$ parametrizes certain equivalence classes of strictly semi-stable vector bundles which satisfy the equality

$$\frac{d_F}{r_F} = \frac{d_E}{r_E}.$$ 

Each equivalence class contains a unique representative isomorphic to the direct sum of stable bundles. Furthermore one considers subvarieties $SU_X(r, L) \subset U_X(r, d)$ of vector bundle of rank $r$ with determinant isomorphic to a fixed line bundle $L$ of degree $d$. In this work we study the variety of strictly semi-stable bundles in $SU_X(3, \mathcal{O}_X)$, where $X$ is a genus 2 curve. We call this variety the generalized Kummer variety of $X$ and denote it by $\text{Kum}_3(X)$. Recall that
the classical Kummer variety of $X$ is defined as the quotient of the Jacobian variety $\text{Jac}(X) = U_X(1,0)$ by the involution $L \mapsto L^{-1}$. It turns out that our $\text{Kum}_3(X)$ has a similar description as a quotient of $\text{Jac}(X) \times \text{Jac}(X)$ which justifies the name. We will see that the first definition allows one to define a natural embedding of $\text{Kum}_3(X)$ in a projective space (see section 4). The second approach is useful in order to give local description of $\text{Kum}_3(X)$ by following the theory developed in [1] (section 3).

We point out the use of [4] for local computations.

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2. Generalized Kummer variety.

Let $A$ be an $s$-dimensional abelian variety, $A'$ the $r$-Cartesian product of $A$, and $A^{(r)} := A'/\Sigma_r$ be the $r$-symmetric power of $A$. We can define the usual map $a_r : A^{(r)} \to A$ such that $a_r((x_1, \ldots, x_r)) = x_1 + \cdots + x_r$. This surjective map is just a morphism of varieties since there is no group structure on $A^{(r)}$. However, all fibers of $a_r$ are naturally isomorphic.

Definition 2.1. The generalized Kummer variety associated to an abelian variety $A$ is

$$\text{Kum}_r(A) := a_r^{-1}(0).$$

It is easy to see that

$$\dim(\text{Kum}_r(A)) = s(r - 1).$$

When $\dim A \geq 1$, $A^{(r)}$ is singular. If $\dim A = 2$, $A^{(r)}$ admits a natural desingularization isomorphic to the Hilbert scheme $A^{[r]} := \text{Hilb}(A)^{[r]}$ of 0-dimensional subschemes of $A$ of length $r$ (see [5]). Let $pr : A^{[r]} \to A^{(r)}$ be the usual projection. It is known that the restriction of $pr$ over $\text{Kum}_r(A)$ is a resolution of singularities. Also $\text{Kum}_r(A)$ admits a structure a holomorphic symplectic manifold (see [1]).

2.1 The Kummer variety of Jacobians.

Let $X$ be a smooth connected projective curve of genus $g$ and $\text{SU}_X(r, L)$ be the set of semi-stable vector bundles on $X$ of rank $r$ and determinant which is

\[^1\] here \{$x_1, \ldots, x_r$\} mean an unordered set of $r$ elements.
isomorphic to a fixed line bundle $L$. Let $\text{Jac}(X)$ be the Jacobian variety of $X$ which parametrizes isomorphism classes of line bundles on $X$ of degree 0, or, equivalently the divisor classes of degree 0. We have a natural embedding:

$$\text{Kum}_r(\text{Jac}(X)) \hookrightarrow \text{SU}_X(r, \mathcal{O}_X)$$

where $L_{a_i} := \mathcal{O}_X(a_i)$. Obviously, the condition $a_1 + \ldots + a_r = 0$ means that $\det(L_{a_1} \oplus \ldots \oplus L_{a_r}) = 0$ and $\text{deg}(L_{a_i}) = 0$ for all $i = 1, \ldots, r$. Consequently the Kummer variety $\text{Kum}_r(\text{Jac}(X))$ describes exactly the completely decomposable bundles in $\text{SU}_X(r, \mathcal{O}_X)$ (from now on we’ll write only $\text{SU}_X(r)$ instead of $\text{SU}_X(r, \mathcal{O}_X)$).

In this paper we restrict ourselves with the case $g = 2$ and rank $r = 3$. In this case $\text{Kum}_3(\text{Jac}(X))$ is a 4-fold.

3. Singular locus of $\text{Kum}_3(\text{Jac}(X))$.

From now we let $A$ denote $\text{Jac}(X)$. Let us define the following map:

$$\pi : A^{(2)} \rightarrow \text{Kum}_3(A)$$

$$\{a, b\} \mapsto L_a \oplus L_b \oplus L_{-a-b}.$$

This map is well defined and it is a $(3 : 1)$-covering of $\text{Kum}_3(A)$. Let now $\rho : A^2 \rightarrow A^{(2)}$ be the $(2 : 1)$-map which sends $(x, y) \in A^2$ to $\{x, y\} \in A^{(2)}$. If we consider the map:

$$p := (\pi \circ \rho) : A^2 \rightarrow A^{(2)} \rightarrow \text{Kum}_3(A) \subset A^{(3)}$$

we get a $(6 : 1)$-covering of $\text{Kum}_3(A)$.

**Notations:** Let $X$ and $Y$ be two varieties and $f : X \rightarrow Y$ be a finite morphism. We let $\text{Sing}(X)$ denote the singular locus of $X$, $B_f \subseteq Y$ the branch locus of $f$ and $R_f \subseteq X$ the ramification locus of $f$.

**Observation:** $B_\pi = \pi(B_\rho)$.

**Proof.** Since $B_\rho = \{(x, y) \in A^{(2)} | x = y\}$ and $\pi((x, x)) = \{x, x, -2x\} \in B_\pi$ we obviously get that $\pi(B_\rho) \subset B_\pi$.

Conversely, for any point $\{x, y, z\}$ of $B_\pi$, at least two of the three elements $x, y, z$ are equal to some $t$. Therefore $\pi(\{t, t\}) = \{x, y, z\}$, and hence $B_\pi \subset \pi(B_\rho)$. $\square$
Since $A^2$ is smooth, we have $\text{Sing}(A^{(2)}) \subset B_\rho$. Obviously $B_\rho \subset R_\pi$, hence $\text{Sing}(\text{Kum}_3(A)) \subset B_\pi$. As a consequence we obtain that $\text{Sing}(\text{Kum}_3(A)) \subseteq B_\pi$. Therefore we have to study the $(3 : 1)$-covering $\pi : A^{(2)} \to \text{Kum}_3(A)$.

Since $\pi$ is not a Galois covering, in order to give the local description at every point $Q \in \text{Kum}_3(A)$, we have to consider the following three cases separately:

1. $Q \in \text{Kum}_3(A)$ s.t. $\pi^{-1}(Q)$ is just a point;
2. $Q \in \text{Kum}_3(A)$ s.t. $\pi^{-1}(Q)$ is a set of two different points;
3. $Q \in \text{Kum}_3(A)$ s.t. $\pi^{-1}(Q)$ is a set of exactly three points.

Let us begin studying these cases.

**Case 3.** When $Q \in \text{Kum}_3(A)$ s.t. $\pi(\pi^{-1}(Q)) = 3$ we have that $Q \notin B_\pi$. Since $\pi(B_\rho) = B_\pi$ any point of $\pi^{-1}(Q)$ is smooth in $A^{(2)}$. Then $Q$ is a smooth point of the Kummer variety.

**Case 2.** When $Q \in \text{Kum}_3(A)$ s.t. $\pi(\pi^{-1}(Q)) = 2$ we fix the two points $P_1, P_2 \in A^{(2)}$ s.t. $\pi(P_1) = \pi(P_2) = Q$. In this case $Q = \{x, x, -2x\}$ with $x \neq -2x$: let us fix $P_1 = \{x, x\}, P_2 = \{x, -2x\}$. Let $U \subset \text{Kum}_3(A)$ be a sufficiently small analytic neighborhood of $Q$ such that $\pi^{-1}(U) = U_1 \cup U_2$ where $U_1$ and $U_2$ are respectively analytic neighborhoods of $P_1$ and $P_2$ and also $U_1 \cap U_2 = \emptyset$. Let $\tilde{Q}$ a generic point of $U$, so $\tilde{Q} = \{x + \epsilon, x + \delta, -2x - \epsilon - \delta\}$; the preimage of $\tilde{Q}$ by $\pi$ is $\pi^{-1}(\tilde{Q}) = \{x + \epsilon, x + \delta\}, \{x + \epsilon, -2x - \epsilon - \delta\}, \{x + \delta, -2x - \epsilon - \delta\}$, but $\{x + \epsilon, x + \delta\} \in U_1$ and $\{x + \epsilon, -2x - \epsilon - \delta\}, \{x + \delta, -2x - \epsilon - \delta\} \in U_2$, it means that $P_1$ has ramification order equal to 1 and $P_2$ has ramification order equal to 2. Therefore there is an analytic neighborhood of $P_1$ which is isomorphic by $\pi$ to an analytic neighborhood of $Q$. This allows us to study a generic point of $B_\rho$ instead of a generic point of $B_\pi$.

**Case 1.** When $Q \in \text{Kum}_3(A)$ s.t. $\pi(\pi^{-1}(Q)) = 3$ we consider a point $P \in A^{(2)}$ s.t. $\pi^{-1}(Q) = P \Rightarrow Q = \{x, x, x\}$ s.t. $3x = 0 \Rightarrow x$ is a 3-torsion point of $A$. Now our abelian variety is a complex torus of dimension 2, so we have exactly $3^{2s} = 3^4 = 81$ such points.

**Proposition 3.1.** The singular locus of $\text{Kum}_3(A)$ is a surface which coincides with the branch locus $B_\pi$ of the projection $\pi : A^{(2)} \to \text{Kum}_3(A)$ and it is locally isomorphic at a generic point to $(\mathbb{C}^2 \times Q, \mathbb{C} \times o)$ where $Q$ is a cone over a rational normal curve and $o$ is the vertex of such a cone (see [1]).
Moreover there are exactly 81 points of $\text{Sing}(\text{Kum}_3(X))$ whose local tangent cone is isomorphic to the spectrum of:

$$\mathbb{C}[[u_1, \ldots, u_7]] / I$$

where $I$ is the ideal generated by the following polynomials:

- $u_3^5 - u_4u_6$
- $u_4u_7 - u_5u_6$
- $u_5^2 - u_5u_7$
- $u_3u_4 + u_2u_5 + u_1u_6$
- $u_3u_5 + u_2u_6 + u_1u_7$.

Proof. According to what we saw in Case 2, an analytic neighborhood of $Q \in \text{Kum}_3(A)$ such that $\#(\pi^{-1}(Q)) = 2$ is isomorphic to a generic element of $B_\rho$. We have to study the $(2:1)$-covering $A^2 \to A$. Since $A = \text{Jac}(X)$, $A$ is a smooth abelian variety, this means that $A$ is a complex torus $(\mathbb{C}^g / \mathbb{Z}^g)$ where $g$ is the genus of $X$; in our case $X$ is a genus 2 curve, $A \simeq (\mathbb{C}^2 / \mathbb{Z}^2)$. Thus, in local coordinates at $P \in A$, $\hat{\mathcal{O}}_P \simeq \mathbb{C}[[z_1, z_2]]$, so we consider $U_P$ (a neighborhood of $P \in A$) isomorphic to $\mathbb{C}^2$. Therefore we obtain that locally at $Q \in A^2$, $\hat{\mathcal{O}}_Q \simeq \hat{\mathcal{O}}_P \otimes \hat{\mathcal{O}}_P \simeq \mathbb{C}[[z_1, z_2; z_3, z_4]]$.

We fix a coordinate system $(z_1, z_2; z_3, z_4)$ in $A^2$ such that $A^2 \supset U_P \ni P = (0, 0; 0, 0)$. Let $Q$ be a point in $U_P$, in the fixed coordinate system $Q = (z_1, z_2; z_3, z_4)$. Since $P$ is such that $\rho(P) \in B_\rho$, by definition of $\rho$, we have: $A^{(2)} = A^2 / <i>$, where $i$ is the following involution of $U_P$:

$$i : U_P \to U_P$$

$$i : (z_1, z_2; z_3, z_4) \mapsto (z_3, z_4; z_1, z_2).$$

The involution $i$ is obviously linear and its associated matrix is $M = e_{1,3} + e_{3,1} + e_{2,4} + e_{4,2}$ (where $e_{i,j}$ is the matrix with 1 in the $i, j$ position and 0 otherwise).

Its eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$ have both multiplicity 2, so its diagonal form is:

$$\tilde{M} = (1, 1, -1, -1)$$

which in a new coordinate system:

$$\begin{cases}
  x_1 &= \frac{z_1 + z_2}{2} \\
  x_2 &= \frac{z_3 + z_4}{2} \\
  x_3 &= \frac{z_1 - z_3}{2} \\
  x_4 &= \frac{z_2 - z_4}{2}
\end{cases}$$
corresponds to the linear transformation:

\((x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, -x_3, -x_4)\).

The algebra of invariant polynomials with respect to this actions is generated by the homogeneous forms \((x_1, x_2, x_3, x_4)\). Let us now consider these forms as local coordinates \((s_1, s_2, s_3, s_4, s_5)\) around \(\rho(P)\), here we have that the completion of the local ring is isomorphic to the following one:

\(\mathbb{C}[s_1, \ldots, s_5]/(s_1^2 - s_2 s_3)\).

Therefore \(B_\rho\) at a generic point is locally isomorphic to \((\mathbb{C}^2 \times Q, \mathbb{C} \times o)\) where \(Q\) is a cone over a rational normal curve (we can see this rational normal curve as the image of \(\mathbb{P}^1\) in \(\mathbb{P}^3\) by the Veronese map \(v_2 : (\mathbb{P}^1)^r \to (\mathbb{P}^3)^r\), \(v_2(L) = L^2\)) and \(o\) the vertex of this cone. (What we have just proved in our particular case of \(\text{Kum}_3(A)\) can be found in a more general form in [1].) Therefore we have the same local description of singularity of \(\text{Kum}_3(A)\) out of the correspondent points of the 81 three-torsion points of \(A\).

Now we have to study what happens at those 3-torsion. Let \(Q_0\) be one of them, we already know that \(p^{-1}(Q_0) = (x, x) := P_0\) is such that \(3x = 0\). Let us fix \((z_1, z_2; z_3, z_4) \in \mathbb{C}^2 \times \mathbb{C}^2\) a local coordinate system around \(P_0\) in order to describe locally the \((6 : 1)\)-covering \(p : A^2 \to \text{Kum}_3(A)\). We observe that for a generic \(P\) in that neighborhood, the pre-image of \(p(P)\) is the set of the following 6 points:

\[\begin{align*}
P_1 &:= (z_1, z_2; z_3, z_4), \\
P_2 &:= (z_3, z_4; z_1, z_2), \\
P_3 &:= (z_3, z_4; (z_1 - z_3), (z_2 - z_4)), \\
P_4 &:= ((z_1 - z_3), (z_2 - z_4); z_3, z_4), \\
P_5 &:= ((z_1 - z_3), (z_2 - z_4); z_1, z_2), \\
P_6 &:= (z_1, z_2; (z_1 - z_3), (z_2 - z_4)).
\end{align*}\]

Observe that \(i(P_1) = P_2, i(P_3) = P_4, i(P_5) = P_6\) where \(i\) is the involution defined in (2). We now define a trivolution \(\tau\) of \(\mathbb{C}^2 \times \mathbb{C}^2\) as follows:

\(\tau : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}^2\)

\([(z_1, z_2; z_3, z_4)] \mapsto [(z_3, z_4; (z_1 - z_3), (z_2 - z_4))].\)
It is easy to see that:
\[ P_1 \xrightarrow{\tau} P_3 \xrightarrow{\tau} P_5 \xrightarrow{\tau} P_1, \]
\[ P_2 \xrightarrow{\tau} P_6 \xrightarrow{\tau} P_4 \xrightarrow{\tau} P_2. \]

The matrices that represent \( i \) and \( \tau \) are respectively:
\[
i = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}
\]
furthermore \( \langle \tau, i \rangle \simeq \Sigma_3 \), then the local description of \( \text{Kum}_3(X) \) around \( Q_0 \) is isomorphic to \( \mathbb{A}^2 / \Sigma_3 \).

In what follows we have used [4] program in order to do computations.

First we recall Noether’s theorem ([3] pag. 331)

\[ \text{Theorem 3.2.} \quad \text{Let} \ G \subset GL(n, \mathbb{C}) \ \text{be a given finite matrix group, we have:} \]
\[ \mathbb{C}[z_1, \ldots, z_n]^G = \mathbb{C}[R_G(z^\beta) : |\beta| \leq |G|]. \]

\( \text{where } R_G \text{ is the Reynolds operator.} \)

In other words, the algebra of invariant polynomials with respect to the action of \( G \) is generated by the invariant polynomials whose degree is at most the order of the group. In our case the order of \( G \) is 6, so it is not hard to compute \( \mathbb{C}[z_1, z_2, z_3, z_4]^G \). Then, after reducing the generators, we obtain that \( \mathbb{C}[z_1, z_2, z_3, z_4]^G \) is generated by:

\[
f_1 := z_2^2 + z_2z_4 + z_4^2, \quad f_2 := 2z_1z_2 + z_2z_3 + z_1z_4 + 2z_3z_4, \]
\[
f_3 := z_1^2 + z_1z_3 + z_3^2, \quad f_4 := -3z_2^2z_4 - 3z_1z_4^2, \]
\[
f_5 := z_2z_3 + 2z_1z_2z_4 + 2z_2z_3z_4 + z_1z_3^2, \]
\[
f_6 := -2z_1z_2z_3 - z_2z_3^2 - z_1z_4 - 2z_1z_2z_4, \]
\[
f_7 := 3z_2^2z_3 + 3z_1z_4^2.
\]

Let us now write \( \mathbb{C}[z_1, \ldots, z_4]^G = \mathbb{C}[f_1, \ldots, f_7] \) as:

\[ \mathbb{C}[u_1, \ldots, u_7]/I_G, \]

where \( I_G \) is the syzygy ideal. It is easy to obtain that \( I_G \) is generated by the following polynomials:

\[
u_1(u_2^2 - 4u_1u_3) + 3(u_1^2 - u_4u_6),
\]
\[
u_2(u_2^2 - 4u_1u_3) + 3(u_4u_7 - u_5u_6),
\]
\[
u_3(u_2^2 - 4u_1u_3) + 3(u_6^2 - u_5u_7),
\]
\[
u_4u_4 + u_2u_5 + u_1u_6,
\]
\[
u_5u_5 + u_2u_6 + u_1u_7.\]
and so we have the completion of the local ring at $P$:

$$\widehat{\mathcal{O}}_P \simeq \frac{\mathbb{C}[[u_1, \ldots, u_7]]}{I_G}.$$ 

Let now calculate the tangent cone in $Q_0$ in order to understand which kind of singularity occurs in $Q_0$. With [4] aid we find that this local cone is:

$$\text{Spec}\left(\frac{\mathbb{C}[[u_1, \ldots, u_7]]}{I}\right)$$

where $I$ is the ideal generated by the following polynomials:

- $u_5^2 - u_4u_6$
- $u_4u_7 - u_5u_6$
- $u_6^2 - u_5u_7$
- $u_3u_4 + u_2u_5 + u_1u_6$
- $u_3u_5 + u_2u_6 + u_1u_7$.

The degree of the variety $V(I) \subset \mathbb{P}^6$ is 5, this means that $Q_0$ is a singular point of multiplicity 5.

What we want to do now is to describe the singular locus of the local description. Let us start to calculate the Jacobian of $V(I_G)$, what we find is the following $5 \times 7$ matrix:

$$J_G := \begin{pmatrix}
  u_5^2 - 8u_1u_3 & 2u_1u_2 & -4u_1^2 & -3u_6 & 6u_5 & -3u_4 & 0 \\
  -4u_2u_3 & 3u_2^2 - 4u_1u_3 & -4u_1u_2 & 3u_7 & -3u_6 & -3u_5 & 3u_4 \\
  -4u_2^2 & 2u_2u_3 & u_2^2 - 8u_1u_3 & 0 & -3u_7 & 6u_6 & -3u_5 \\
  u_6 & u_5 & u_4 & u_3 & u_2 & u_1 & 0 \\
  u_7 & u_6 & u_5 & 0 & u_3 & u_2 & u_1
\end{pmatrix}$$

Local equations define a fourfold, so we have to find the locus where the dimension of $\text{Ker}(J_G)$ is at least 5. In order to do it we calculate the minimal system of generators of all $3 \times 3$ minors of $J_G$, we intersect the corresponding variety with $V(I_G)$, we find a minimal base of generators of the ideal corresponding to this intersection and we compute its radical; the polynomials we find define, after suitable change of coordinates, the (reduced) variety of singular locus.
We verified that the only singular point of $V_C$ following polynomials:

$$\text{What we want to find now is the tangent cone in } Q_0 \text{ surface (which is nothing but the graph of (4)) whose projective closure is the Veronese } \tilde{V}\text{ locus. Using [4] we find that its corresponding ideal } \mathcal{I}_C \text{ is generated by following polynomials:}$$

$$\begin{align*}
&u_2^3 - u_5u_7, u_5u_6 - u_4u_7, u_3^2 - u_4u_6, u_3u_5 - u_3u_7, u_2u_6 - u_1u_7, u_3u_4 - u_1u_6, u_2u_5 - u_1u_6, u_2u_4 - u_1u_5, u_2^2 - u_1u_5, u_3^2 - u_2^2, u_2^2 - u_6u_7, u_3^2 - u_5u_7, u_4u_3 - u_4u_7, u_2^3u_3 - u_4u_6, u_2^3u_2 - u_4u_5, u_3^2 - u_2^2).
\end{align*}$$

We verified that the only one singular point of $V(I_S)$ is the origin. Now, let us consider the map from $\mathbb{C}^2$ to $\mathbb{C}^7$ such that:

$$\begin{align*}
(t, s) &\mapsto (t^2, ts, s^2, t^3, t^2s, ts^2, s^3).
\end{align*}$$

This is the parametrization of $V(I_S)$; as we have already done we can find relations between these polynomials and verify that the ideal we get is equal to $I_S$. Now we can consider the following smooth parametrization from $\mathbb{C}^2$ to $\mathbb{C}^9$:

$$\begin{align*}
(t, s) &\mapsto (t, s, t^2, ts, s^2, t^3, t^2s, ts^2, s^3)
\end{align*}$$

(which is nothing but the graph of (4)) whose projective closure is the Veronese surface $\nu_3(\mathbb{P}^2) = V_{2,3}$ where $\nu_3 : (\mathbb{P}^2)^* \to (\mathbb{P}^3)^*$, $\nu_3(L) = L^3$.

What we want to find now is the tangent cone in $Q_0$ seen inside the singular locus. Using [4] we find that its corresponding ideal $\mathcal{I}_C$ is generated by following polynomials:

$$\begin{align*}
&u_2^2 - u_5u_7, u_6u_7, u_5u_7, u_4u_7, u_4u_6,
&u_4u_5, u_2^2 - u_6u_7, u_5u_6, u_5^2,
&u_3u_6 - u_2u_7, u_3u_5 - u_1u_7, u_2u_6 - u_1u_7, u_3u_4 - u_1u_6, u_2u_5 - u_1u_6,
&u_2u_4 - u_1u_5, u_2^2 - u_1u_5.
\end{align*}$$

The ideal $\tilde{I}_C$ has multiplicity 4 (the corresponding variety has degree four) and its radical is the following ideal:

$$I_C = (u_2^2 - u_3u_7, u_4, u_5, u_6, u_7).$$

Then $V(I_C)$ is a cone and $V(\tilde{I}_C)$ is a double cone.

This gives the description of the singularity at one of the 81 3-torsion points.  □

4. Degree of $\text{Kum}_3(A)$.

To find the degree of $\text{Kum}_3(A)$, we have to recall some general facts about theta divisors.

4.1 The Riemann theta divisor.
Let $X$ be a curve of genus $g$ and $\Theta_{\text{Jac}}(X)$ is the Riemann theta divisor. It is known that it is an ample divisor and
\[
\dim |r\Theta_{\text{Jac}}(X)| = r^g - 1
\]
(see [6] Theorem p. 317). Recall that for any fixed point $q_0 \in X$ there exists an isomorphism:
\[
\psi_{g-1,0} : \text{Pic}^{g-1}(X) \rightarrow \text{Jac}(X) = \text{Pic}^0(X).
\]
The set $W_{g-1}$ of effective line bundles of degree $g - 1$ is a divisor in $\text{Pic}^{g-1}(X)$ denoted by $\Theta_{\text{Pic}^{g-1}(X)}$. By Riemann’s Theorem there exists a divisor $k$ of degree 0 such that:
\[
\psi_{g-1,0}(\Theta_{\text{Pic}^{g-1}(X)}) = \Theta_{\text{Jac}}(X) - k.
\]
In a similar way we can define the generalized theta divisor as follows:
\[
\Theta_{\text{SU}}^{\text{gen}}(X, L) = \{ E \in \text{Pic}^{g-1}(X) : h^0(E \otimes L) > 0 \}.
\]
It is known that
\[
\text{Pic}(\text{SU}_X(r, L)) = \mathbb{Z}\Theta_{\text{SU}_X(r, L)}^{\text{gen}},
\]
and there exists a canonical isomorphism:
\[
|r\Theta_{\text{Pic}^{g-1}(X)}| \simeq |\Theta_{\text{SU}_X(r)}^{\text{gen}}|^*.
\]
(see [2]).

4.2 Degree of $\text{Kum}_3(A)$

Let us consider the $(2 : 1)$–map
\[
\phi_3 : \text{SU}_3(X) \rightarrow |3\Theta_{\text{Pic}^1(X)}| \simeq |\Theta_{\text{SU}_X(3)}^{\text{gen}}|^*.
\]
\[
E \mapsto D_E = \{ L \in \text{Pic}^1(X) : h^0(E \otimes L) > 0 \}.
\]

**Definition 4.1.** $\Theta_\eta := \{ E \in \text{SU}_X(3) : h^0(E \otimes \eta) > 0 \} \subset \text{SU}_X(3)$ where $\eta$ is a fixed divisor in $\text{Pic}^1(X)$. 

Observation: \( \phi_3(\Theta_\eta) = H_\eta \subset |3\Theta_{\text{Pic}^1(X)}| \) and \( H_\eta \) is a hyperplane. Since \( \phi_3|_{\text{Kum}_3(A)} : \text{Kum}_3(A) \to \phi_3(\text{Kum}_3(A)) \) is a \((1 : 1)\)-map (it is a well known fact but we will see it in the next section), we have that \( \Theta_\eta \cap \text{Kum}_3(X) \cong H_\eta \cap \phi_3(\text{Kum}_3(X)) \). In order to study the degree of \( \text{Kum}_3(A) \) we have to take four generic divisors \( \eta_1, \ldots, \eta_4 \in \text{Pic}^1(X) \) and consider the respective \( \Theta_{\eta_1}, \ldots, \Theta_{\eta_4} \subset SU_X(3) \). The intersection \( \Theta_{\eta_i} \cap \text{Kum}_3(A) \) is equal to \((L_a \oplus L_b \oplus L_{-a-b}) \subset \text{Kum}_3(X) : h^0(L_a \oplus L_b \oplus L_{-a-b}) > 0) \) for all \( i = 1, \ldots, 4 \). If \( L_a \oplus L_b \oplus L_{-a-b} \) is a generic element of \( \text{Kum}_3(A) \) and \( p \) is the \((6 - 1)\)-covering of \( \text{Kum}_3(A) \) defined as in (1), then \( p^{-1}(L_a \oplus L_b \oplus L_{-a-b}) \subset A^2 \) is a set of 6 points. It’s easy to see that \( p((a, b)) \in \Theta_{\eta_i} \cap \text{Kum}_3(X) \) if and only if \( h^0(L_a \oplus \eta_i) > 0 \) or \( h^0(L_b \oplus \eta_i) > 0 \) or \( h^0(L_{-a-b} \oplus \eta_i) > 0 \) for all \( i = 1, \ldots, 4 \).

Let us recall Jacobi’s Theorem ([6] page: 235):

Jacobi’s Theorem: Let \( X \) be a curve of genus \( g \), \( q_0 \in X \) and \( \omega_1, \ldots, \omega_g \) a basis for \( H^0(X, \Omega^1) \). For any \( \lambda \in \text{Jac}(X) \) there exist \( g \) points \( p_1, \ldots, p_g \in X \) such that

\[
\mu(\sum_{i=1}^{g}(p_i - q_0)) = \lambda,
\]

where

\[
\mu : \text{Div}^0(X) \to \text{Jac}(X)
\]

\[
\sum_i(p_i - q_i) \mapsto \left( \sum_i \int_{q_i}^{p_i} \omega_1, \ldots, \sum_i \int_{q_i}^{p_i} \omega_g \right).
\]

Since \( \text{Jac}(X) \) is isomorphic to \( \text{Pic}^0(X) \), this theorem has the following two corollaries:

1. if \( q_0 \) is a fixed point of \( C \), then for all \( L_a \in \text{Pic}^0(X) \), there are two points \( p_1, p_2 \) in \( X \) such that \( L_a \cong \mathcal{O}_X(P_1 + P_2 - 2q_0) \);
2. Consider the isomorphism

\[
\psi_{1,0} : \text{Pic}^1(X) \cong \text{Pic}^0(X)
\]

\[
\eta \mapsto \eta \otimes \mathcal{O}_X(-q_0).
\]

For every \( i = 1, \ldots, 4 \) there are \( q_{i_1}, q_{i_2} \in C \) such that \( \eta_i \cong \mathcal{O}_X(q_{i_1} + q_{i_2} - q_0) \).
Now these two facts imply that \( h^0(L_a \otimes \eta_i) > 0 \) if and only if \( h^0(\mathcal{O}_X(P_1 + P_2 - 2q_0) \otimes \mathcal{O}_X(q_{i,1} + q_{i,2} - q_0)) > 0 \), and this happens if and only if \( h^0(\mathcal{O}_X(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0)) > 0 \).

**Notations:** \( \Theta_{-k} \) is a translate of theta divisor by \( k \in \text{Pic}^0(X) \).

By Riemann’s Singularity Theorem (see [6], p. 348) the dimension \( h^0(\mathcal{O}_X(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0)) \) is equal to the multiplicity of \( \psi_{1,0}(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0) \) in \( \Theta_{-k} \) (by a suitable \( k \in \text{Pic}^0(X) \)), i.e. it is equal to the multiplicity of \( (P_1 + P_2 + q_{i,1} + q_{i,2} - 4q_0) \) in \( \Theta_{-k} \). It follows from this fact that \( h^0(\mathcal{O}_X(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0)) \) is greater than zero if and only if \( (P_1 + P_2 + q_{i,1} + q_{i,2} - 4q_0) \in \Theta_{-k} \).

**Notations:**

\[
\Theta_i := \Theta_{-k-\eta_i+q_0}; \\
R_i := \{(a, b) \in A^2 : (a + b) \notin \Theta_i\}; \\
\Xi_i := (\Theta_i \times A) \cup (A \times \Theta_i) \cup R_i.
\]

Now \( (P_1 + P_2 + q_{i,1} + q_{i,2} - 4q_0) \in \Theta_{-k} \) iff \( P_1 + P_2 - 2q_0 \in \Theta_i \), which is equivalent to say that \( L_a \) belongs to \( \Theta_i \), but this implies that \( p((a, b)) \in \Theta_{\eta_i} \cap \text{Kum}_3(A) \) if and only if \( L_a \in \Theta_i \) or \( L_b \in \Theta_i \) or \( L_{-a-b} \in \Theta_i \) (or equivalently \( L_{a+b} \) belongs to \( \Theta_i \)), i.e. \( (a, b) \in \Xi_i \).

Therefore we can conclude:

\( (a, b) \in A^2 \) is such that \( p((a, b)) \in \text{Kum}_3(A) \cap \Theta_{\eta_i} \), \( i = 1, \ldots, 4 \) if and only if \( (a, b) \in \Xi_i \).

The last conclusion together with the observation that \( \sharp(pr^{-1}(L_a \oplus L_b \oplus L_{-a-b})) = 0 \) gives the following proposition:

**Proposition 4.2.** \( \deg(\text{Kum}_3(A)) = \frac{1}{6}(\sharp(\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4)) \).

**Proof.** \( \sharp(\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4) = 6 \cdot \sharp(\text{Kum}_3(A) \cap \Theta_{\eta_1} \cap \Theta_{\eta_2} \cap \Theta_{\eta_3} \cap \Theta_{\eta_4}) = 6 \cdot \deg(\text{Kum}_3(A)). \)

**Notations:**

\[
R_{i,1} = \{(a, b) \in A^2 : a \in \Theta_i \text{ and } (a + b) \in \Theta_j\}, \\
R_{i,2} = \{(a, b) \in A^2 : b \in \Theta_i \text{ and } (a + b) \in \Theta_j\}, \\
R_{i,1,2} = \{(a, b) \in A^2 : (a + b) \in \Theta_i \cap \Theta_j\}.
\]

Instead of computing directly \( \Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4 \), we will compute \( (\Xi_1 \cap \Xi_2) \cap \Xi_3 \cap \Xi_4 \)
At the end we will obtain that 

\[ \#(\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4) = 216 \] (see also tables 1. and 2.) and so:

**Proposition 4.3.** \( \deg(\text{Kum}_3(A)) = 36. \)

**Proof.** In the following two tables we write at place \((i, j)\) the cardinality of intersection of the subset of \(\Xi_1 \cap \Xi_2\) which we write at the place \((0, j)\), with the subset of \(\Xi_3 \cap \Xi_4\) which we write at the place \((i, 0)\).

<table>
<thead>
<tr>
<th>(\cap)</th>
<th>((\Theta_1 \cap \Theta_2) \times A)</th>
<th>(A \times (\Theta_1 \cap \Theta_2))</th>
<th>(\Theta_1 \times \Theta_2)</th>
<th>(\Theta_2 \times \Theta_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\Theta_3 \cap \Theta_4) \times A)</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(A \times (\Theta_3 \cap \Theta_4))</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Theta_3 \times \Theta_4)</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>(\Theta_4 \times \Theta_3)</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>(R_{a,3}^4)</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>(R_{a,4}^3)</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>(R_{b,3}^4)</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>(R_{b,4}^3)</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>(R_{3,4})</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 1.

In order to be more clear we show some cases:

\[ R_{a,1}^2 \cap R_{a,3}^4 = \{(a, b) \in A^2 : a \in \Theta_1 \text{ and } b \in \Theta_3 \text{ and } (a + b) \in \{-\Theta_2\} \cap \{-\Theta_4\}\}. \] Recall that \(\Theta_1 \cdot \Theta_2 \cdot \Theta_3 \cdot \Theta_4 = 2. \) So \((a + b) \in \{k_1, k_2\}\) where \(\{k_1, k_2\} = \{-\Theta_2\} \cap \{-\Theta_4\}\). Fix for a moment \((a + b) = k_1\). If we translate \(\Theta_1\) and \(\Theta_3\) by \(-k_1\) we get that \(a \in (\Theta_1)_{-k_1}\) and \(b \in (\Theta_3)_{-k_1}\) and \(a + b = 0\), then \(b\) must be equal to \(-a\) and \(a \in ((\Theta_1)_{-k_1}) \cap ((\Theta_3)_{+k_1})\). Then for
fixed $a + b$ the couple $(a, b)$ has to belong to $\{(h_1, -h_1), (h_2, -h_2)\}$ where $((\Theta_1)_{+k_4}) \cap ((-\Theta_3)_{-k_4}) = \{h_1, h_2\}$. Therefore $\sharp(R_{2,4}^{a,1} \cap R_{4}^{b,3}) = 2 \cdot 2 = 4$.

$(\Theta_1 \times \Theta_2) \cap R_{3,4}$: $(\Theta_1 \times \Theta_2) \cap R_{3,4} = \{(a, b) \in A^2 : a \in \Theta_1, b \in \Theta_2$ and $(a + b) \in \{-\Theta_3\} \cap \{-\Theta_4\}\}$. Then, as in the previous case, we have $\sharp((\Theta_1 \times \Theta_2) \cap R_{3,4}) = 4$.

$R_{2,4}^{a,1} \cap ((\Theta_3 \cap \Theta_4) \times A)$: $R_{2,4}^{a,1} \cap ((\Theta_3 \cap \Theta_4) \times A) = \{(a, b) \in A^2 : a \in \Theta_1 \cap \Theta_3 \cap \Theta_4, (a + b) \in \{-\Theta_2\}\}$, but since $\Theta_j$ are generic curves on a surface, their intersection two by two is the empty set, then $\sharp(R_{2,4}^{a,1} \cap ((\Theta_3 \cap \Theta_4) \times A) = 0$. \hfill \Box

### 4.3 The degree of Sing(Kum$_3$(A))

As we have already seen, the singular locus of Kum$_3$(A) is a surface. What we want to do now is to compute its degree. We use the notation from the previous section.

Let us fix two divisors $\Xi_1$ and $\Xi_2$ in $A^2$. We denote by $\Delta$ the diagonal of $A \times A$.

**Proposition 4.4.** $\text{deg}(\text{Sing}(\text{Kum}_3(A))) = \sharp(\Xi_1 \cap \Xi_2 \cap \Delta)$.

**Proof.** It is sufficient to consider the restriction to $\Delta$ of the map $p$ defined as in (1) and get out the $(1 : 1)$-map $p|_\Delta : \Delta \to \text{Sing}(\text{Kum}_3(A))$. \hfill \Box
Proposition 4.5. \( \deg(\text{Sing(Kum}_3(A))) = 42. \)

**Proof.** The following table is used in the same way as we used Table 1 and Table 2 in the previous section:

<table>
<thead>
<tr>
<th>( \cap )</th>
<th>( \Delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\Theta_1 \cap \Theta_2) \times A)</td>
<td>2</td>
</tr>
<tr>
<td>(A \times (\Theta_1 \cap \Theta_2))</td>
<td>/</td>
</tr>
<tr>
<td>(\Theta_1 \times \Theta_2)</td>
<td>/</td>
</tr>
<tr>
<td>(\Theta_2 \times \Theta_1)</td>
<td>/</td>
</tr>
<tr>
<td>(R_{2}^{a,1})</td>
<td>4</td>
</tr>
<tr>
<td>(R_{1}^{a,2})</td>
<td>4</td>
</tr>
<tr>
<td>(R_{2}^{b,1})</td>
<td>/</td>
</tr>
<tr>
<td>(R_{1}^{b,2})</td>
<td>/</td>
</tr>
<tr>
<td>(R_{1,2})</td>
<td>32</td>
</tr>
</tbody>
</table>

Table 3.

The following list describes Table 3:

\(\Delta \cap A \times (\Theta_1 \cap \Theta_2)\): we have not considered the intersection points between \(\Delta\) and \(A \times (\Theta_1 \cap \Theta_2)\), \(\Theta_1 \times \Theta_2\), \(\Theta_2 \times \Theta_1\) because we have already counted them in \(((\Theta_1 \cap \Theta_2) \times A) \cap \Delta\).

\(\Delta \cap R_{2}^{a,1}\): the previous argument can be used for \(\Delta \cap R_{2}^{b,1}\) and \(\Delta \cap R_{1}^{b,2}\): we have already counted these intersection points respectively in \(R_{2}^{a,1}\) and in \(R_{1}^{a,2}\).

\(R_{2}^{a,1} \cap \Delta\): we have now to show that \(\sharp (R_{2}^{a,1} \cap \Delta) = 4\). The set \(R_{2}^{a,1} \cap \Delta\) is \(\{(a, a) \in A \times A \mid a \in \Theta_1, \ 2a \in (-\Theta_2)\}\) which is equal to \(\{(a, a) \in A \times A : 2a \in ((-\Theta_2) \cap (2 \cdot \Theta_1)) \text{ and } a \in \Theta_1\}\). Let now \(L_1\) be the line bundle on \(A\) associated to \(\Theta_1\). The line bundle \(L_1^2\) is associated to \((2 \cdot \Theta_1)\) and its divisor is linearly equivalent to \(2\Theta_1\). As a consequence of this fact we have that \(2a \in (2\Theta_1 \cap (-\Theta_2))\) then \(\sharp (2\Theta_1 \cap (-\Theta_2)) = 4\). Now, since the map from \(\Theta_1\) to \((2 \cdot \Theta_1)\) is \(1 : 1\) we get the conclusion.

\(R_{1,2} \cap \Delta\): finally we have that \((R_{1,2} \cap \Delta)\) is equivalent to the set \(\{a \in A \mid 2a \in ((-\Theta_1) \cap (-\Theta_2))\}\) whose cardinality is 32. \(\Box\)
5. On action of the hyperelliptic involution and \( \text{Kum}_3(A) \).

Let \( X \) be a curve of genus 2. Consider the degree 2 map:

\[
\phi_3 : \text{SU}_X(3) \xrightarrow{2:1} \mathbb{P}^8 = |3\Theta_{\text{Pic}^1(X)}|
\]

\[
E \mapsto D_E = \{ L \in \text{Pic}^1(X) / h^0(E \otimes L) > 0 \}
\]

(see [7]). Let \( \tau' \) be the involution on \( \text{SU}_X(3) \) acting by the duality:

\[
\tau'(E) = E^*\]

and \( \tau \) the hyperelliptic involution on \( \text{Pic}^1(X) \):

\[
\tau(L) = \omega_X \otimes L^{-1}.
\]

We will use the following well known relation:

\[
\tau \circ \phi_3(E) = \phi_3 \circ \tau'(E).
\]

On \( \text{SU}_X(3) \) there is also the hyperelliptic involution \( h^* \):

\[
E \mapsto h^*(E)
\]

induced by the hyperelliptic involution \( h \) of the curve \( X \). We define \( \sigma := \tau' \circ h^* \). It is the involution which gives the double covering of \( \text{SU}_X(3) \) on \( \mathbb{P}^8 \). The fixed locus of \( \sigma \) is obviously contained in \( \text{SU}_X(3) \) and we recall:

\[
\phi_3(\text{Fix}(\sigma)) = \text{Coble sextic hypersurface}
\]

(see [7]). By definition, the strictly semi-stable locus \( \text{SU}_X(3)^{ss} \) of \( \text{SU}_X(3) \) consists of isomorphism classes of split rank 3 semi-stable vector bundles of determinant \( \mathcal{O}_X \). Its points can be represented by the vector bundles of the form \( F \oplus L \) or \( L_a \oplus L_b \oplus L_c \) with trivial determinant where \( L, L_a, L_b, L_c \) are line bundles and \( F \) is a rank 2 vector bundle. We want to consider the elements of the form \( L_a \oplus L_b \oplus L_c \) (those belonging to \( \text{Kum}_3(A) \)) and actions of previous involutions on them:

- \( \tau'(L_a \oplus L_b \oplus L_c) = (L_a \oplus L_b \oplus L_c)^* = L_{-a} \oplus L_{-b} \oplus L_{-c} \);
- \( \tau'(h^*(L_a \oplus L_b \oplus L_c)) = L_a \oplus L_b \oplus L_c \).
This implies that \( \sigma(\text{Kum}_3(A)) = \text{Kum}_3(A) \subset \text{SU}_X(3) \) which means that \( \text{Kum}_3(A) \subset \text{Fix}(\sigma) \) and then \( \phi_3(\text{Kum}_3(X)) \subset \text{Coble sextic} \) (see 5).

Let us now consider rank 2 semistable vector bundles of trivial determinant: \( \text{SU}_X(2) \). If we take its symmetric square, we obtain a semisable rank three vector bundle with trivial determinant:

\[
\text{SU}_X(2) \to \text{SU}_X(3); \quad E \mapsto \text{Sym}^2(E).
\]

We want to study the action of involutions defined on the beginning of this paragraph on \( \text{Sym}^2(E) \) with \( E \in \text{SU}_X(2) \). Since \( \text{Sym}^2(E)^* = \text{Sym}^2(E) = h^*(\text{Sym}^2(E)) \), then \( \sigma(\text{Sym}^2(E)) = \text{Sym}^2(E) \subset \text{SU}_X(3) \), so \( \text{Sym}^2(\text{SU}_X(2)) \subset \text{Fix}(\sigma) \), and, again by (5), \( \phi_3(\text{Sym}^2(\text{SU}_X(2))) \subset \text{Coble sextic} \).

Now we want to see the action of \( \tau \) on \( |\Theta_{\text{Pic}^1(X)}| \). It is known that \( \text{Fix}(\tau) = \mathbb{P}^4 \sqcup \mathbb{P}^3 \).

**Notations:** We denote by \( \mathbb{P}^3 \) and \( \mathbb{P}^4 \), respectively, the \( \mathbb{P}^3 \) and the \( \mathbb{P}^4 \) which are fixed by action of \( \tau \).

Since the image of \( \text{Sym}^2(\text{SU}_X(2)) \) by \( \phi_3 \) in \( \mathbb{P}^8 \) has dimension 3 and also \( \phi_3(\text{Sym}^2(\text{SU}_X(2))) \subset \text{Fix}(\tau) \), we obtain

\[\phi_3(\text{Sym}^2(\text{SU}_X(2))) \subset \mathbb{P}^4_1.\]

Let \( L_a \oplus L_{-a} \) be an element of \( \text{Kum}_2(X) \subset \text{SU}_X(2) \), then \( \text{Sym}^2(L_a \oplus L_{-a}) = L_{2a} \oplus L_{-2a} \oplus \Theta \in \text{Kum}_3(A) \subset \text{SU}_X(3) \). It means that \( \text{Sym}^2(\text{Kum}_2(A)) \subset \text{Kum}_3(A) \).

**Observation:** Since \( \{L_{2a} \oplus L_{-2a} \oplus \Theta \in \text{SU}_X(3)\} \) is isomorphic to \( S^2((L_a \oplus L_{-a})) \), we can view \( \{L_{2a} \oplus L_{-2a} \oplus \Theta \in \text{SU}_X(3)\} \) as the image of \( \text{Kum}_2(A) \) inside \( \text{SU}_X(3) \) under the symmetric square map. Moreover it follows from the surjectivity of the multiplication by 2 map \( [2] : A \to A \) that the image of \( \text{Kum}_2(A) \) in \( \text{SU}_X(3) \) is isomorphic to \( \text{Kum}_2(A) \).

We have already observed that \( \phi_3|_{\text{Kum}_3(A)} \) is a \((1:1)\) map on the image; this fact allows us to view \( \phi_3(\text{Kum}_3(A)) \) as the \( \text{Kum}_3(A) \) in \( |\Theta_{\text{Pic}^1(X)}| \). For the same reason we can view \( \phi_3(\text{Sym}^2(\text{SU}_X(2))) \) as \( \text{Kum}_2(A) \subset |\Theta_{\text{Pic}^1(X)}| \).

Using this language we can say that \( \text{Kum}_2(A) \) is left fixed by the action of \( \tau \) in \( \text{Kum}_1(A) \subset |\Theta_{\text{Pic}^1(X)}| \) because \( |\Theta_{\text{Pic}^1(X)}| \supset \phi_3(\text{Kum}_3(A)) \supset \phi_3(\text{Sym}^2(\text{SU}_X(2)) = \text{Kum}_2(A) \subset \mathbb{P}^4 \subset \text{Fix}(\tau) \subset |\Theta_{\text{Pic}^1(X)}| \).

**Proposition 5.1.** \( \text{Fix}(\tau) \cap \phi_3(\text{Kum}_3(A)) = \phi_3(\text{Sym}^2(\text{Kum}_2(A))) \).
Proof. By definition $\tau(L_a \oplus L_b \oplus L_c) = L_{-a} \oplus L_{-b} \oplus L_{-c}$ then $L_a \oplus L_b \oplus L_c$ belongs to $\text{Fix}(\tau)$ if and only if $\{a, b, c\} = \{-a, -b, -c\}$.

Let $P$ belong to $\{-a, -b, -c\}$ and $a = P$.

- If $P$ is different from $-a$, suppose that $P = -c$, then $\{-a, -b, -c\} = \{-a, -b, a\}$; moreover $a + b + c = 0$ because $L_a \oplus L_b \oplus L_c \in \text{Kum}_3(A)$, then $b = 0$.

- Now, if $P = -a$ or, equivalently $a = -a$, then $a = 0$ and $b = -c$.

In both cases $L_a \oplus L_b \oplus L_c \in \text{Kum}_3(A)$ such that $\tau(L_a \oplus L_b \oplus L_c) = L_{-a} \oplus L_{-b} \oplus L_{-c}$ are of the form $L_a \oplus L_{-a} \oplus L_0$. This means that they belong to $\text{Kum}_2(A) \subset |3\Theta_{\text{Pic}}^1(X)|$. □

The previous proposition tells us also that $\mathbb{P}^3 \cap \text{Kum}_3 A = \emptyset$. So the projection of $\text{Kum}_3(A) \subset |3\Theta_{\text{Pic}}^1(X)|$ from $\mathbb{P}^3$ to $\mathbb{P}^4$ is a morphism. It would be interesting to find its degree.

Our final observation is the following.

**Proposition 5.2.** $\text{Sing}(\text{Kum}_3(A)) \cap \text{Kum}_2(A) = \text{Sing}(\text{Kum}_2(A))$

**Proof.** Points of $\text{Kum}_2(A) \subset \text{Kum}_3(A)$ are of the form $(P, -P, 0)$. Singular points of $\text{Kum}_3(A)$ are those which have at least two equal components, then $\text{Sing}(\text{Kum}_3(A)) \cap \text{Kum}_2(A) = \{(P, -P, 0)\}$ where $2P = 0$ that are exactly the 15 points of $2$–torsion and one more point $(\mathcal{O}_X, \mathcal{O}_X, \mathcal{O}_X)$ which are singularities of the usual $\text{Kum}_2(A)$. This implies that $\sharp(\text{Sing}(\text{Kum}_3(A)) \cap \text{Kum}_2(A)) = 16$ and $\text{Sing}(\text{Kum}_3(A)) \cap \text{Kum}_2(A) = \text{Sing}(\text{Kum}_2(A))$. □

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