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ON GENERALIZED KUMMER OF RANK 3 VECTOR BUNDLES OVER A GENUS 2 CURVE

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1. Introduction.

Let $X$ be a smooth projective complex curve and let $U_X(r, d)$ be the moduli space of semi-stable vector bundles of rank $r$ and degree $d$ on $X$ (see [8]). It contains an open Zariski subset $U_X(r, d)'$ which is the coarse moduli space of stable bundles, i.e. vector bundles satisfying inequality

$$\frac{d_F}{r_F} < \frac{d_E}{r_E}.$$

The complement $U_X(r, d) \setminus U_X(r, d)'$ parametrizes certain equivalence classes of strictly semi-stable vector bundles which satisfy the equality

$$\frac{d_F}{r_F} = \frac{d_E}{r_E}.$$

Each equivalence class contains a unique representative isomorphic to the direct sum of stable bundles. Furthermore one considers subvarieties $SU_X(r, L) \subset U_X(r, d)$ of vector bundle of rank $r$ with determinant isomorphic to a fixed line bundle $L$ of degree $d$. In this work we study the variety of strictly semi-stable bundles in $SU_X(3, \mathcal{O}_X)$, where $X$ is a genus 2 curve. We call this variety the generalized Kummer variety of $X$ and denote it by $\text{Kum}_3(X)$. Recall that
the classical Kummer variety of $X$ is defined as the quotient of the Jacobian variety $\text{Jac}(X) = U_X(1,0)$ by the involution $L \mapsto L^{-1}$. It turns out that our $\text{Kum}_3(X)$ has a similar description as a quotient of $\text{Jac}(X) \times \text{Jac}(X)$ which justifies the name. We will see that the first definition allows one to define a natural embedding of $\text{Kum}_3(X)$ in a projective space (see section 4). The second approach is useful in order to give local description of $\text{Kum}_3(X)$ by following the theory developed in [1] (section 3).

We point out the use of [4] for local computations.

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2. Generalized Kummer variety.

Let $A$ be an $s$-dimensional abelian variety, $A'$ the $r$-Cartesian product of $A$, and $A^{(r)} := A'/\Sigma_r$ be the $r$-symmetric power of $A$. We can define the usual map $a_r : A^{(r)} \to A$ such that $a_r((x_1, \ldots, x_r)) = x_1 + \cdots + x_r$. This surjective map is just a morphism of varieties since there is no group structure on $A^{(r)}$. However, all fibers of $a_r$ are naturally isomorphic.

**Definition 2.1.** The generalized Kummer variety associated to an abelian variety $A$ is

$$\text{Kum}_r(A) := a_r^{-1}(0).$$

It is easy to see that

$$\dim(\text{Kum}_r(A)) = s(r-1).$$

When $\dim A > 1$, $A^{(r)}$ is singular. If $\dim A = 2$, $A^{(r)}$ admits a natural desingularization isomorphic to the Hilbert scheme $A^{[r]} := \text{Hilb}(A)^{[r]}$ of 0-dimensional subschemes of $A$ of length $r$ (see [5]). Let $pr : A^{[r]} \to A^{(r)}$ be the usual projection. It is known that the restriction of $pr$ over $\text{Kum}_r(A)$ is a resolution of singularities. Also $\text{Kum}_r(A)$ admits a structure a holomorphic symplectic manifold (see [1]).

2.1 The Kummer variety of Jacobians.

Let $X$ be a smooth connected projective curve of genus $g$ and $\text{SU}_X(r, L)$ be the set of semi-stable vector bundles on $X$ of rank $r$ and determinant which is

---

1 here $\{x_1, \ldots, x_r\}$ mean an unordered set of $r$ elements.
isomorphic to a fixed line bundle \( L \). Let \( \text{Jac}(X) \) be the Jacobian variety of \( X \) which parametrizes isomorphism classes of line bundles on \( X \) of degree 0, or, equivalently the divisor classes of degree 0. We have a natural embedding:

\[
\text{Kum}_r(\text{Jac}(X)) \hookrightarrow \text{SU}_X(r, \mathcal{O}_X)
\]

\[
\{a_1, \ldots, a_r\} \mapsto (L_{a_1} \oplus \ldots \oplus L_{a_r})
\]

where \( L_{a_i} := \mathcal{O}_X(a_i) \). Obviously, the condition \( a_1 + \ldots + a_r = 0 \) means that \( \det(L_{a_1} \oplus \ldots \oplus L_{a_r}) = 0 \) and \( \deg(L_{a_i}) = 0 \) for all \( i = 1, \ldots, r \). Consequently the Kummer variety \( \text{Kum}_r(\text{Jac}(X)) \) describes exactly the completely decomposable bundles in \( \text{SU}_X(r) \) (from now on we’ll write only \( \text{SU}_X(r) \) instead of \( \text{SU}_X(r, \mathcal{O}_X) \)).

In this paper we restrict ourselves with the case \( g = 2 \) and rank \( r = 3 \). In this case \( \text{Kum}_3(\text{Jac}(X)) \) is a 4-fold.

### 3. Singular locus of \( \text{Kum}_3(\text{Jac}(X)) \).

From now we let \( A \) denote \( \text{Jac}(X) \). Let us define the following map:

\[
\pi : A^{(2)} \rightarrow \text{Kum}_3(A)
\]

\[
\{a, b\} \mapsto L_a \oplus L_b \oplus L_{-a-b}.
\]

This map is well defined and it is a \((3:1)\)–covering of \( \text{Kum}_3(A) \). Let now \( \rho : A^2 \rightarrow A^{(2)} \) be the \((2:1)\)–map which sends \( (x, y) \in A^2 \) to \( \{x, y\} \in A^{(2)} \). If we consider the map:

\[
p := (\pi \circ \rho) : A^2 \rightarrow A^{(2)} \rightarrow \text{Kum}_3(A) \subset A^{(3)}
\]

we get a \((6:1)\)–covering of \( \text{Kum}_3(A) \).

**Notations:** Let \( X \) and \( Y \) be two varieties and \( f : X \rightarrow Y \) be a finite morphism. We let \( \text{Sing}(X) \) denote the singular locus of \( X \), \( B_f \subseteq Y \) the branch locus of \( f \) and \( R_f \subseteq X \) the ramification locus of \( f \).

**Observation:** \( B_\pi = \pi(B_\rho) \).

**Proof.** Since \( B_\rho = \{(x, y) \in A^{(2)}| x = y \} \) and \( \pi((x, x)) = \{x, x, -2x\} \in B_\pi \), we obviously get that \( \pi(B_\rho) \subseteq B_\pi \).

Conversely, for any point \( \{x, y, z\} \) of \( B_\pi \), at least two of the three elements \( x, y, z \) are equal to some \( t \). Therefore \( \pi(\{t, t\}) = \{x, y, z\} \), and hence \( B_\pi \subseteq \pi(B_\rho) \). \( \square \)
Since $A^2$ is smooth, we have $\text{Sing}(A^{(2)}) \subseteq B_{\rho}$. Obviously $B_{\rho} \subseteq R_{\pi}$, hence $\text{Sing}(\text{Kum}_3(A)) \subseteq B_{\pi}$. As a consequence we obtain that $\text{Sing}(\text{Kum}_3(A)) \subseteq B_{\pi}$. Therefore we have to study the $(3:1)$—covering $\pi : A^{(2)} \to \text{Kum}_3(A)$.

Since $\pi$ is not a Galois covering, in order to give the local description at every point $Q \in \text{Kum}_3(A)$, we have to consider the following three cases separately:

1. $Q \in \text{Kum}_3(A)$ s.t. $\pi^{-1}(Q)$ is just a point;
2. $Q \in \text{Kum}_3(A)$ s.t. $\pi^{-1}(Q)$ is a set of two different points;
3. $Q \in \text{Kum}_3(A)$ s.t. $\pi^{-1}(Q)$ is a set of exactly three points.

Let us begin studying these cases.

**Case 3.** When $Q \in \text{Kum}_3(A)$ s.t. $\sharp(\pi^{-1}(Q)) = 3$ we have that $Q \notin B_{\pi}$. Since $\pi(B_{\rho}) = B_{\pi}$ any point of $\pi^{-1}(Q)$ is smooth in $A^{(2)}$. Then $Q$ is a smooth point of the Kummer variety.

**Case 2.** When $Q \in \text{Kum}_3(A)$ s.t. $\sharp(\pi^{-1}(Q)) = 2$ we fix the two points $P_1, P_2 \in A^{(2)}$ s.t. $\pi(P_1) = \pi(P_2) = Q$. In this case $Q = \{x, x, -2x\}$ with $x \neq -2x$; let us fix $P_1 = \{x, x\}$, $P_2 = \{x, -2x\}$. Let $U \subset \text{Kum}_3(A)$ be a sufficiently small analytic neighborhood of $Q$ such that $\pi^{-1}(U) = U_1 \sqcup U_2$ where $U_1$ and $U_2$ are respectively analytic neighborhoods of $P_1$ and $P_2$ and also $U_1 \cap U_2 = \emptyset$. Let $\tilde{Q}$ a generic point of $U$, so $\tilde{Q} = \{x + \epsilon, x + \delta, -2x - \epsilon - \delta\}$; the preimage of $\tilde{Q}$ by $\pi$ is $\pi^{-1}(\tilde{Q}) = \{x + \epsilon, x + \delta\}$, $\{x + \epsilon, -2x - \epsilon - \delta\}$, but $\{x + \epsilon, x + \delta\} \in U_1$ and $\{x + \epsilon, -2x - \epsilon - \delta\}$, $\{x + \delta, -2x - \epsilon - \delta\} \in U_2$, it means that $P_i$ has ramification order equal to 1 and $P_2$ has ramification order equal to 2. Therefore there is an analytic neighborhood of $P_1$ which is isomorphic by $\pi$ to an analytic neighborhood of $Q$. This allows us to study a generic point of $B_{\rho}$ instead of a generic point of $B_{\pi}$.

**Case 1.** When $Q \in \text{Kum}_3(A)$ s.t. $\sharp(\pi^{-1}(Q)) = 3$ we consider a point $P \in A^{(2)}$ s.t. $\pi^{-1}(Q) = P \Rightarrow Q = \{x, x, x\}$ s.t. $3x = 0 \Rightarrow x$ is a 3—torsion point of $A$. Now our abelian variety is a complex torus of dimension 2, so we have exactly $3^2 \times 3^4 = 81$ such points.

**Proposition 3.1.** The singular locus of $\text{Kum}_3(A)$ is a surface which coincides with the branch locus $B_{\pi}$ of the projection $\pi : A^{(2)} \to \text{Kum}_3(A)$ and it is locally isomorphic at a generic point to $(\mathbb{C}^2 \times Q, \mathbb{C} \times o)$ where $Q$ is a cone over a rational normal curve and $o$ is the vertex of such a cone (see [1]).
Moreover there are exactly 81 points of $\text{Sing}(\text{Kum}_3(X))$ whose local tangent cone is isomorphic to the spectrum of:

$$\mathbb{C}[[u_1, \ldots, u_7]] / I$$

where $I$ is the ideal generated by the following polynomials:

$$u_2^5 - u_4 u_6$$

$$u_4 u_7 - u_5 u_6$$

$$u_6^5 - u_5 u_7$$

$$u_3 u_4 + u_2 u_5 + u_1 u_6$$

$$u_3 u_5 + u_2 u_6 + u_1 u_7.$$ 

**Proof.** According to what we saw in Case 2, an analytic neighborhood of $Q \in \text{Kum}_3(A)$ such that $\#(\pi^{-1}(Q)) = 2$ is isomorphic to a generic element of $B_\rho$. We have to study the $(2:1)$-covering $A^2 \to A^{(2)}$.

Since $A = \text{Jac}(X)$, $A$ is a smooth abelian variety, this means that $A$ is a complex torus $(\mathbb{C}^g / \mathbb{Z}^g)$ where $g$ is the genus of $X$; in our case $X$ is a genus 2 curve, $A \simeq (\mathbb{C}^2 / \mathbb{Z}^4)$. Thus, in local coordinates at $P \in A$, $\hat{\mathcal{O}}_P \simeq \mathbb{C}[[z_1, z_2]]$, so we consider $U_P$ (a neighborhood of $P \in A$) isomorphic to $\mathbb{C}^2$. Therefore we obtain that locally at $Q \in A^2$, $\hat{\mathcal{O}}_Q \simeq \hat{\mathcal{O}}_P \otimes \hat{\mathcal{O}}_P \simeq \mathbb{C}[[z_1, z_2; z_3, z_4]]$.

We fix a coordinate system $(z_1, z_2; z_3, z_4)$ in $A^2$ such that $A^2 \supset U_P \ni P = (0, 0; 0, 0)$. Let $Q$ be a point in $U_P$, in the fixed coordinate system $Q = (z_1, z_2; z_3, z_4)$. Since $P$ is such that $\rho(P) \in B_\rho$, by definition of $\rho$, we have: $A^{(2)} = A^2 / < i >$, where $i$ is the following involution of $U_P$:

$$i : U_P \to U_P$$

$$i : (z_1, z_2; z_3, z_4) \mapsto (z_3, z_4; z_1, z_2).$$

The involution $i$ is obviously linear and its associated matrix is $M = e_{1,3} + e_{3,1} + e_{2,4} + e_{4,2}$ (where $e_{i,j}$ is the matrix with 1 in the $i, j$ position and 0 otherwhere).

Its eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$ have both multiplicity 2, so its diagonal form is:

$$\tilde{M} = (1, 1, -1, -1)$$

which in a new coordinate system:

$$\begin{cases}
  x_1 = \frac{z_1 + z_3}{2} \\
  x_2 = \frac{z_1 - z_3}{2} \\
  x_3 = \frac{z_2 + z_4}{2} \\
  x_4 = \frac{z_2 - z_4}{2}
\end{cases}$$
corresponds to the linear transformation:

\[(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, -x_3, -x_4).\]

The algebra of invariant polynomials with respect to this actions is generated by the homogeneous forms \((x_1, x_2, x_3^2, x_4, x_3x_4)\). Let us now consider these forms as local coordinates \((s_1, s_2, s_3, s_4, s_5)\) around \(\rho(P)\), here we have that the completion of the local ring is isomorphic to the following one:

\[\left( \mathbb{C}[[s_1, \ldots, s_5]] \right) \left[ (s_1^2 - s_2s_3) \right].\]

Therefore \(B_\rho\) at a generic point is locally isomorphic to \((\mathbb{C}^2 \times Q, \mathbb{C} \times o)\) where \(Q\) is a cone over a rational normal curve (we can see this rational normal curve as the image of \(\mathbb{P}^1\) in \(\mathbb{P}^3\) by the Veronese map \(v_2 : (\mathbb{P}^1)^* \to (\mathbb{P}^3)^*, v_2(L) = L^2\) and \(o\) the vertex of this cone. (What we have just proved in our particular case of \(\text{Kum}_3(A)\) can be found in a more general form in [1].) Therefore we have the same local description of singularity of \(\text{Kum}_3(A)\) out of the correspondent points of the 81 three-torsion points of \(A\).

Now we have to study what happens at those 3-torsion. Let \(Q_0\) be one of them, we already know that \(p^{-1}(Q_0) = (x, x) := P_0\) is such that \(3x = 0\). Let us fix \((z_1, z_2; z_3, z_4) \in \mathbb{C}^2 \times \mathbb{C}^2\) a local coordinate system around \(P_0\) in order to describe locally the \((6 : 1)\)-covering \(p : A^2 \to \text{Kum}_3(A)\). We observe that for a generic \(P\) in that neighborhood, the pre-image of \(p(P)\) is the set of the following 6 points:

\begin{align*}
P_1 & := (z_1, z_2; z_3, z_4), \\
P_2 & := (z_3, z_4; z_1, z_2), \\
P_3 & := (z_3, z_4; -z_1 - z_3, -z_2 - z_4), \\
P_4 & := ((-z_1 - z_3), (-z_2 - z_4); z_3, z_4), \\
P_5 & := ((-z_1 - z_3), (-z_2 - z_4); z_1, z_2), \\
P_6 & := (z_1, z_2; (-z_1 - z_3), (-z_2 - z_4)).
\end{align*}

Observe that \(i(P_1) = P_2, i(P_3) = P_4, i(P_5) = P_6\) where \(i\) is the involution defined in (2). We now define a trivolution \(\tau\) of \(\mathbb{C}^2 \times \mathbb{C}^2\) as follows:

\[\tau : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}^2\]

\[(z_1, z_2; z_3, z_4) \mapsto (z_3, z_4; (-z_1 - z_3), (-z_2 - z_4)).\]
It is easy to see that:

\[ P_1 \xrightarrow{\tau} P_3 \xrightarrow{\tau} P_5 \xrightarrow{\tau} P_1, \]
\[ P_2 \xrightarrow{\tau} P_6 \xrightarrow{\tau} P_4 \xrightarrow{\tau} P_2. \]

The matrices that represent \( i \) and \( \tau \) are respectively:

\[
i = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]
\[
\tau = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & -1 & 0 \\
0 & -1 & 0 & -1
\end{pmatrix}
\]

Furthermore \( \langle \tau, i \rangle \simeq \Sigma_3 \), then the local description of \( \text{Kum}_3(X) \) around \( Q_0 \) is isomorphic to \( A^2/\Sigma_3 \).

In what follows we have used [4] program in order to do computations. First we recall Noether’s theorem ([3] pag. 331)

**Theorem 3.2.** Let \( G \subset GL(n, \mathbb{C}) \) be a given finite matrix group, we have:

\[ \mathbb{C}[z_1, \ldots, z_4]^G = \mathbb{C}[R_G(\beta) : |\beta| \leq |G|]. \]

where \( R_G \) is the Reynolds operator.

In other words, the algebra of invariant polynomials with respect to the action of \( G \) is generated by the invariant polynomials whose degree is at most the order of the group. In our case the order of \( G \) is 6, so it is not hard to compute \( \mathbb{C}[z_1, z_2, z_3, z_4]^G \). Then, after reducing the generators, we obtain that \( \mathbb{C}[z_1, z_2, z_3, z_4]^G \) is generated by:

\[
f_1 := z_2^2 + z_2 z_4 + z_4^2,
f_2 := 2z_1 z_2 + z_2 z_3 + z_1 z_4 + 2z_3 z_4,
f_3 := z_1 z_2 + z_1 z_3 + z_2^2,
f_4 := -3z_2 z_4 - 3z_2 z_4,
f_5 := z_2 z_3 + 2z_1 z_2 z_4 + 2z_2 z_3 z_4 + z_1 z_3^2,
f_6 := -2z_1 z_2 z_3 - z_2^2 z_4 - z_1^2 z_4 - 2z_1 z_3 z_4,
f_7 := 3z_1^2 z_3 + 3z_1 z_3^2.
\]

Let us now write \( \mathbb{C}[z_1, \ldots, z_4]^G = \mathbb{C}[f_1, \ldots, f_7] \) as:

\[ \mathbb{C}[u_1, \ldots, u_7]/I_G, \]

where \( I_G \) is the syzygy ideal. It is easy to obtain that \( I_G \) is generated by the following polynomials:

\[
u_1(u_2^2 - 4u_1 u_3) + 3(u_2^2 - u_4 u_6) \\
u_2(u_2^2 - 4u_1 u_3) + 3(u_4 u_7 - u_5 u_6) \\
u_3(u_2^2 - 4u_1 u_3) + 3(u_5^2 - u_4 u_7) \\
u_3 u_4 + u_2 u_5 + u_1 u_6 \\
u_5 u_5 + u_2 u_6 + u_1 u_7 cr
\]
and so we have the completion of the local ring at $P$:

$$\widehat{\mathfrak{O}}_P \simeq \frac{\mathbb{C}[u_1, \ldots, u_7]}{I_G}.$$ 

Let now calculate the tangent cone in $Q_0$ in order to understand which kind of singularity occurs in $Q_0$. With [4] aid we find that this local cone is:

$$\text{Spec} \left( \frac{\mathbb{C}[u_1, \ldots, u_7]}{I} \right)$$

where $I$ is the ideal generated by the following polynomials:

- $u_5^2 - u_4u_6$
- $u_4u_7 - u_5u_6$
- $u_5^2 - u_5u_7$
- $u_3u_4 + u_2u_5 + u_1u_6$
- $u_3u_5 + u_2u_6 + u_1u_7$.

The degree of the variety $V(I) \subset \mathbb{P}^6$ is 5, this means that $Q_0$ is a singular point of multiplicity 5.

What we want to do now is to describe the singular locus of the local description. Let us start to calculate the Jacobian of $V(I_G)$, what we find is the following $5 \times 7$ matrix:

$$J_G := \begin{pmatrix}
    u_5^2 - 8u_1u_3 & 2u_1u_2 & -4u_1^2 & -3u_6 & 6u_5 & -3u_4 & 0 \\
    -4u_2u_3 & 3u_1^2 - 4u_1u_3 & -4u_1u_2 & 3u_7 & -3u_6 & -3u_5 & 3u_4 \\
    -4u_2^2 & 2u_2u_3 & u_7^2 - 8u_1u_3 & 0 & -3u_7 & 6u_6 & -3u_5 \\
    u_6 & u_5 & u_4 & u_3 & u_2 & u_1 & 0 \\
    u_7 & u_6 & u_5 & 0 & u_3 & u_2 & u_1
\end{pmatrix}$$

Local equations define a fourfold, so we have to find the locus where the dimension of $\text{Ker}(J_G)$ is at least 5. In order to do it we calculate the minimal system of generators of all $3 \times 3$ minors of $J_G$, we intersect the corresponding variety with $V(I_G)$, we find a minimal base of generators of the ideal corresponding to this intersection and we compute its radical; the polynomials we find define, after suitable change of coordinates, the (reduced) variety of singular locus
\( V(I_S) \), where \( I_S = (u_2^3 - u_5u_7, u_5u_6 - u_4u_7, u_2^2 - u_4u_6, u_5u_6 - u_2u_7, u_3u_5 - u_1u_7, u_2u_6 - u_1u_7, u_3u_5 - u_1u_6, u_2u_5 - u_1u_5, u_2^2 - u_1u_5, u_3^2 - u_2u_5, u_2u_3^2 - u_6u_7, u_1u_3^2 - u_4u_7, u_3^2u_3 - u_4u_6, u_2^3 - u_4u_5, u_1^3 - u_2^2) \).

We verified that the only one singular point of \( V(\hat{I}_S) \) is the origin. Now, let us consider the map from \( \mathbb{C}^2 \) to \( \mathbb{C}^7 \) such that:

\[(t, s) \mapsto (t^2, ts, s^2, t^3, t^2s, ts^2, s^3).\]

This is the parametrization of \( V(I_S) \); as we have already done we can find relations between these polynomials and verify that the ideal we get is equal to \( I_S \). Now we can consider the following smooth parametrization from \( \mathbb{C}^2 \) to \( \mathbb{C}^9 \):

\[(t, s) \mapsto (t, s, t^2, ts, s^2, t^3, t^2s, ts^2, s^3)\]

(which is nothing but the graph of (4)) whose projective closure is the Veronese surface \( V_{2,3} = V_{2,3}^3 \) where \( v_3 : (\mathbb{P}^2)\times \rightarrow (\mathbb{P}^3)\times \), \( v_3(L) = L^3 \).

What we want to find now is the tangent cone in \( Q_0 \) seen inside the singular locus. Using [4] we find that its corresponding ideal \( \hat{I}_C \) is generated by following polynomials:

\[
\begin{align*}
&u_2^3 - u_5u_7, \\
u_5u_6 - u_4u_7, &u_2^2 - u_4u_6, \quad &u_5u_6 - u_2u_7, \quad &u_3u_5 - u_1u_7, \quad &u_2u_6 - u_1u_7, \quad &u_3u_5 - u_1u_6, \quad &u_2u_5 - u_1u_5, \quad &u_2u_4 - u_1u_5, \quad &u_2^2 - u_1u_5.
\end{align*}
\]

The ideal \( \hat{I}_C \) has multiplicity 4 (the corresponding variety has degree four) and its radical is the following ideal:

\[I_C = (u_2^2 - u_5u_7, u_5u_6 - u_4u_7, u_2^2 - u_4u_6, u_5u_6 - u_2u_7, u_3u_5 - u_1u_7, u_2u_5 - u_1u_5, u_2u_4 - u_1u_5, u_2^2 - u_1u_5).\]

Then \( V(I_C) \) is a cone and \( V(\hat{I}_C) \) is a double cone.

This gives the description of the singularity at one of the 81 3-torsion points. □

4. Degree of Kum\(_3\)(A).

To find the degree of Kum\(_3\)(A), we have to recall some general facts about theta divisors.

4.1 The Riemann theta divisor.
Let $X$ be a curve of genus $g$ and $\Theta_{\text{Jac}}(X)$ is the Riemann theta divisor. It is known that it is an ample divisor and
\[
\dim |r\Theta_{\text{Jac}}(X)| = r^g - 1
\]
(see [6] Theorem p. 317). Recall that for any fixed point $q_0 \in X$ there exists an isomorphism:
\[
\psi_{g-1,0} : \text{Pic}^{g-1}(X) \to \text{Jac}(X) = \text{Pic}^0(X).
\]
The set $W_{g-1}$ of effective line bundles of degree $g-1$ is a divisor in $\text{Pic}^{g-1}(X)$ denoted by $\Theta_{\text{Pic}^{g-1}(X)}$. By Riemann’s Theorem there exists a divisor $k$ of degree 0 such that:
\[
\psi_{g-1,0}(\Theta_{\text{Pic}^{g-1}(X)}) = \Theta_{\text{Jac}}(X) - k.
\]
In a similar way we can define the generalized theta divisor as follows:
\[
\Theta_{\text{SU}^X(r, L)}^{\text{gen}} = \{ E \in \text{Pic}^{g-1}(X) : h^0(E \otimes L) > 0 \}.
\]
It is known that
\[
\text{Pic}(\text{SU}_X(r, L)) = \mathbb{Z}\Theta_{\text{SU}_X(r, L)}^{\text{gen}},
\]
and there exists a canonical isomorphism:
\[
|r\Theta_{\text{Pic}^{g-1}(X)}| \simeq |\Theta_{\text{SU}_X(r)}^{\text{gen}}|^*
\]
(see [2]).

### 4.2 Degree of $\text{Kum}_3(A)$

Let us consider the $(2:1)$—map
\[
\phi_3 : \text{SU}_3(X) \longrightarrow |3\Theta_{\text{Pic}^1(X)}| \simeq |\Theta_{\text{SU}_X(3)}^{\text{gen}}|^*
\]
\[
E \mapsto D_E = \{ L \in \text{Pic}^1(X) : h^0(E \otimes L) > 0 \}.
\]

**Definition 4.1.** $\Theta_\eta := \{ E \in \text{SU}_X(3) : h^0(E \otimes \eta) > 0 \} \subset \text{SU}_X(3)$ where $\eta$ is a fixed divisor in $\text{Pic}^1(X)$. 
Observation: \( \phi_3(\Theta_\eta) = H_\eta \subset |3\Theta_{\text{Pic}^1(X)}| \) and \( H_\eta \) is a hyperplane. Since \( \phi_3|_{\text{Kum}^3(A)} : \text{Kum}^3(A) \rightarrow \phi_3(\text{Kum}^3(A)) \) is a \((1 : 1)\)-map (it is a well known fact but we will see it in the next section), we have that \( \Theta_\eta \cap \text{Kum}^3(X) \simeq H_\eta \cap \phi_3(\text{Kum}^3(X)) \). In order to study the degree of \( \text{Kum}^3(A) \) we have to take four generic divisors \( \eta_1, \ldots, \eta_4 \in \text{Pic}^1(X) \) and consider the respective \( \Theta_{\eta_1}, \ldots, \Theta_{\eta_4} \subset \text{SU}_X(3) \). The intersection \( \Theta_{\eta_i} \cap \text{Kum}^3(A) \) is equal to \( \{L_a \oplus L_b \oplus L_{-a-b} \in \text{Kum}^3(X) : h^0(L_a \oplus L_b \oplus L_{-a-b} \oplus \eta_i) > 0 \} \). For every \( \lambda \in \text{Jac}(X) \) there exist \( g \) points \( p_1, \ldots, p_g \in X \) such that \( \mu(\sum_{i=1}^{g}(p_i - q_0)) = \lambda \),

where

\[ \mu : \text{Div}^0(X) \rightarrow \text{Jac}(X) \]

\[ \sum_i (p_i - q_i) \mapsto \left( \sum_i \int_{q_i}^{p_i} \omega_1, \ldots, \sum_i \int_{q_i}^{p_i} \omega_g \right). \]

Since \( \text{Jac}(X) \) is isomorphic to \( \text{Pic}^0(X) \), this theorem has the following two corollaries:

1. if \( q_0 \) is a fixed point of \( C \), then for all \( L_a \in \text{Pic}^0(X) \), there are two points \( P_1, P_2 \) in \( X \) such that \( L_a \simeq \mathcal{O}_X(P_1 + P_2 - 2q_0) \);
2. Consider the isomorphism

\[ \psi_{1,0} : \text{Pic}^1(X) \xrightarrow{\sim} \text{Pic}^0(X) \]

\[ \eta \mapsto \eta \otimes \mathcal{O}_X(-q_0). \]

For every \( i = 1, \ldots, 4 \) there are \( q_{i1}, q_{i2} \in C \) such that \( \eta_i \simeq \mathcal{O}_X(q_{i1} + q_{i2} - q_0) \).
Now these two facts imply that $h^0(L_a \otimes \eta_i) > 0$ if and only if $h^0(\Omega_X(P_1 + P_2 - 2q_0) \otimes \Omega_X(q_{i,1} + q_{i,2} - q_0)) > 0$, and this happens if and only if $h^0(\Omega_X(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0)) > 0$.

**Notations:** $\Theta_{-k}$ is a translate of theta divisor by $k \in \text{Pic}^0(X)$.

By Riemann’s Singularity Theorem (see [6], p. 348) the dimension $h^0(\Omega_X(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0))$ is equal to the multiplicity of $\psi_{1,0}(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0)$ in $\Theta_{-k}$ (by a suitable $k \in \text{Pic}^0(X)$), i.e. it is equal to the multiplicity of $(P_1 + P_2 + q_{i,1} + q_{i,2} - 4q_0)$ in $\Theta_{-k}$. It follows from this fact that $h^0(\Omega_X(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0))$ is greater than zero if and only if $(P_1 + P_2 + q_{i,1} + q_{i,2} - 4q_0) \in \Theta_{-k}$.

**Notations:**

\[
\begin{align*}
\Theta_i &:= \Theta_{-k - \eta_i + \eta_0}; \\
R_i &:= \{(a, b) \in A^2 : (a + b) \in (\Theta_i)\}; \\
\Xi_i &:= (\Theta_i \times A) \cup (A \times \Theta_i) \cup R_i.
\end{align*}
\]

Now $(P_1 + P_2 + q_{i,1} + q_{i,2} - 4q_0) \in \Theta_{-k}$ iff $P_1 + P_2 - 2q_0 \in \Theta_i$ which is equivalent to say that $L_a$ belongs to $\Theta_i$, but this implies that $p((a, b)) \in \Theta_{\eta_i} \cap \text{Kum}_3(A)$ if and only if $L_a \in \Theta_i$ or $L_b \in \Theta_i$ or $L_{a-b} \in \Theta_i$ (or equivalently $L_{a+b}$ belongs to $\{-\Theta_i\}$, i.e. $(a, b) \in \Xi_i$.

Therefore we can conclude:

\[
(a, b) \in A^2 \text{ is such that } p((a, b)) \in \text{Kum}_3(A) \cap \Theta_{\eta_i}, \ i = 1, \ldots, 4 \text{ if and only if } (a, b) \in \Xi_i.
\]

The last conclusion together with the observation that $\sharp(pr^{-1}(L_a \otimes L_b \oplus L_{a-b})) = 0$ gives the following proposition:

**Proposition 4.2.** $\text{deg} \left( \text{Kum}_3(A) \right) = \frac{1}{6} \left( \sharp(\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4) \right)$.

**Proof.** $\sharp(\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4) = 6 \cdot \sharp(\text{Kum}_3(A) \cap \Theta_{\eta_1} \cap \Theta_{\eta_2} \cap \Theta_{\eta_3} \cap \Theta_{\eta_4}) = 6 \cdot \text{deg} \left( \text{Kum}_3(A) \right).$ \hfill \Box

**Notations:**

\[
\begin{align*}
R_i^{a,i} &:= \{(a, b) \in A^2 : a \in \Theta_i \text{ and } (a + b) \in (\Theta_i)\}, \\
R_i^{b,i} &:= \{(a, b) \in A^2 : b \in \Theta_i \text{ and } (a + b) \in (\Theta_i)\} \text{ and} \\
R_i^{1,2} &:= \{(a, b) \in A^2 : (a + b) \in (\Theta_1) \cap (\Theta_2)\}.
\end{align*}
\]

Instead of computing directly $\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4$, we will compute $(\Xi_1 \cap \Xi_2) \cap \Xi_3 \cap \Xi_4$.
At the end we will obtain that $\#(\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4) = 216$ (see also tables 1. and 2.) and so:

**Proposition 4.3.** $\deg(Kum_3(A)) = 36$.

**Proof.** In the following two tables we write at place $(i, j)$ the cardinality of intersection of the subset of $\Xi_i \cap \Xi_j$ which we write at the place $(0, j)$, with the subset of $\Xi_3 \cap \Xi_4$ which we write at the place $(i, 0)$.

<table>
<thead>
<tr>
<th></th>
<th>$(\Theta_1 \cap \Theta_2) \times A$</th>
<th>$A \times (\Theta_1 \cap \Theta_2)$</th>
<th>$\Theta_1 \times \Theta_2$</th>
<th>$\Theta_2 \times \Theta_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\Theta_3 \cap \Theta_4) \times A$</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$A \times (\Theta_3 \cap \Theta_4)$</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Theta_3 \times \Theta_4$</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$\Theta_4 \times \Theta_3$</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$R^a_{4,3}$</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$R^a_{3,4}$</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$R^b_{4,3}$</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$R^b_{3,4}$</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$R_{3,4}$</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 1.

In order to be more clear we show some cases:

$R^a_{2,1} \cap R^b_{3,3} = \{(a, b) \in A^2 : a \in \Theta_1 \text{ and } b \in \Theta_3 \text{ and } (a + b) \in \Theta_2 \cap \Theta_4\}$. Recall that $\Theta_i \cdot \Theta_j = 2$. So $(a + b) \in \{k_1, k_2\}$ where $\{k_1, k_2\} = \{-\Theta_2\} \cap \{-\Theta_4\}$. Fix for a moment $(a+b) = k_1$. If we translate $\Theta_1$ and $\Theta_3$ by $-k_1$ we get that $a \in (\Theta_1)_{-k_1}$, $b \in (\Theta_3)_{-k_1}$ and $a + b = 0$, then $b$ must be equal to $-a$ and $a \in ((\Theta_1)_{-k_1}) \cap ((-\Theta_3)_{+k_1})$. Then for
fixed $a + b$ the couple $(a, b)$ has to belong to $\{(h_1, -h_1), (h_2, -h_2)\}$ where $((\Theta_1)_{+k_1}) \cap ((-\Theta_3)_{-k_1}) = \{h_1, h_2\}$. Therefore $\sharp(R_2^{a,1} \cap R_4^{b,3}) = 2 \cdot 2 = 4$.

$(\Theta_1 \times \Theta_2) \cap R_{3,4}$: $(\Theta_1 \times \Theta_2) \cap R_{3,4} = \{(a, b) \in A^2 : a \in \Theta_1, b \in \Theta_2$ and $(a + b) \in \{-\Theta_1\} \cap \{-\Theta_4\}\}$. Then, as in the previous case, we have $\sharp((\Theta_1 \times \Theta_2) \cap R_{3,4}) = 4$.

$R_2^{a,1} \cap ((\Theta_3 \cap \Theta_4) \times A)$: $R_2^{a,1} \cap ((\Theta_3 \cap \Theta_4) \times A) = \{(a, b) \in A^2 : a \in \Theta_3 \cap \Theta_4, (a + b) \in \{-\Theta_2\}\}$, but since $\Theta_j$ are generic curves on a surface, their intersection two by two is the empty set, then $\sharp((R_2^{a,1}) \cap ((\Theta_3 \cap \Theta_4) \times A) = 0$. □

### 4.3 The degree of $\text{Sing}(\text{Kum}_3(A))$

As we have already seen, the singular locus of $\text{Kum}_3(A)$ is a surface. What we want to do now is to compute its degree. We use the notation from the previous section.

Let us fix two divisors $\Xi_1$ and $\Xi_2$ in $A^2$. We denote by $\Delta$ the diagonal of $A \times A$.

**Proposition 4.4.** $\deg(\text{Sing}(\text{Kum}_3(A))) = \sharp(\Xi_1 \cap \Xi_2 \cap \Delta)$.

**Proof.** It is sufficient to consider the restriction to $\Delta$ of the map $p$ defined as in (1) and get out the $(1:1)$-map $p|_{\Delta} : \Delta \rightarrow \text{Sing}(\text{Kum}_3(A))$. □
Proposition 4.5. $\deg(\text{Sing}(\text{Kum}_3(A))) = 42$.

Proof. The following table is used in the same way as we used Table 1 and Table 2 in the previous section:

<table>
<thead>
<tr>
<th>$\cap$</th>
<th>$\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\Theta_1 \cap \Theta_2) \times A$</td>
<td>2</td>
</tr>
<tr>
<td>$A \times (\Theta_1 \cap \Theta_2)$</td>
<td>/</td>
</tr>
<tr>
<td>$\Theta_1 \times \Theta_2$</td>
<td>/</td>
</tr>
<tr>
<td>$\Theta_2 \times \Theta_1$</td>
<td>/</td>
</tr>
<tr>
<td>$R^{a,1}_2$</td>
<td>4</td>
</tr>
<tr>
<td>$R^{a,2}_1$</td>
<td>4</td>
</tr>
<tr>
<td>$R^{b,1}_2$</td>
<td>/</td>
</tr>
<tr>
<td>$R^{b,2}_1$</td>
<td>/</td>
</tr>
<tr>
<td>$R_{1,2}$</td>
<td>32</td>
</tr>
</tbody>
</table>

Table 3.

The following list describes Table 3:

- $\Delta \cap A \times (\Theta_1 \cap \Theta_2)$: we have not considered the intersection points between $\Delta$ and $A \times (\Theta_1 \cap \Theta_2)$, $\Theta_1 \times \Theta_2$, $\Theta_2 \times \Theta_1$ because we have already counted them in $((\Theta_1 \cap \Theta_2) \times A) \cap \Delta$.

- $\Delta \cap R^{a,1}_2$: the previous argument can be used for $\Delta \cap R^{b,1}_2$ and $\Delta \cap R^{b,2}_1$: we have already counted these intersection points respectively in $R^{a,1}_2$ and in $R^{b,2}_1$.

- $R^{a,1}_2 \cap \Delta$: we have now to show that $\sharp(R^{a,1}_2 \cap \Delta) = 4$. The set $R^{a,1}_2 \cap \Delta$ is $\{(a, a) \in A \times A \mid a \in \Theta_1, \ 2a \in (-\Theta_2)\}$ which is equal to $\{(a, a) \in A \times A : 2a \in ((-\Theta_2) \cap (2 \cdot \Theta_1)) \}$ and $a \in \Theta_1$. Let now $L_1$ be the line bundle on $A$ associated to $\Theta_1$. The line bundle $L_1^2$ is associated to $(2 \cdot \Theta_1)$ and its divisor is linearly equivalent to $2\Theta_1$. As a consequence of this fact we have that $2a \in (2\Theta_1 \cap (-\Theta_2))$ then $\sharp[2\Theta_1 \cap (-\Theta_2)] = 4$. Now, since the map from $\Theta_1$ to $(2 \cdot \Theta_1)$ is $1 : 1$ we get the conclusion.

- $R^{a,1}_2 \cap \Delta$: finally we have that $(R_{1,2} \cap \Delta)$ is equivalent to the set $\{a \in A | 2a \in ((-\Theta_1) \cap (-\Theta_2))\}$ whose cardinality is 32. $\square$
5. On action of the hyperelliptic involution and Kum$_3(A)$.

Let $X$ be a curve of genus 2. Consider the degree 2 map:

$$\phi_3 : SU_X(3) \xrightarrow{2:1} \mathbb{P}^8 = |3\Theta_{Pic^1(X)}|$$

$$E \mapsto D_E = \{ L \in Pic^1(X) / h^0(E \otimes L) > 0 \}$$

(see [7]). Let $\tau'$ be the involution on $SU_X(3)$ acting by the duality:

$$\tau'(E) = E^*$$

and $\tau$ the hyperelliptic involution on $Pic^1(X)$:

$$\tau(L) = \omega_X \otimes L^{-1}.$$ 

We will use the following well known relation:

$$\tau \circ \phi_3(E) = \phi_3 \circ \tau'(E).$$

On $SU_X(3)$ there is also the hyperelliptic involution $h^*$:

$$E \mapsto h^*(E)$$

induced by the hyperelliptic involution $h$ of the curve $X$. We define $\sigma := \tau' \circ h^*$. It is the involution which gives the double covering of $SU_X(3)$ on $\mathbb{P}^8$. The fixed locus of $\sigma$ is obviously contained in $SU_X(3)$ and we recall:

$$\phi_3(\text{Fix}(\sigma)) = \text{Coble sextic hypersurface}$$

(see [7]). By definition, the strictly semi-stable locus $SU_X(3)_{\text{ss}}$ of $SU_X(3)$ consists of isomorphism classes of split rank 3 semi-stable vector bundles of determinant $\mathcal{O}_X$. Its points can be represented by the vector bundles of the form $F \oplus L$ or $L_a \oplus L_b \oplus L_c$ with trivial determinant where $L, L_a, L_b, L_c$ are line bundles and $F$ is a rank 2 vector bundle. We want to consider the elements of the form $L_a \oplus L_b \oplus L_c$ (those belonging to Kum$_3(A)$) and actions of previous involutions on them:

- $\tau'(L_a \oplus L_b \oplus L_c) = (L_a \oplus L_b \oplus L_c)^* = L_{-a} \oplus L_{-b} \oplus L_{-c}$;
- $\tau'(h^*(L_a \oplus L_b \oplus L_c)) = L_a \oplus L_b \oplus L_c$. 
This implies that $\sigma(\text{Kum}_3(A)) = \text{Kum}_3(A) \subset SU_X(3)$ which means that $\text{Kum}_3(A) \subset \text{Fix}(\sigma)$ and then $\phi_3(\text{Kum}_3(X)) \subset \text{Coble sextic}$ (see 5).

Let us now consider rank 2 semistable vector bundles of trivial determinant: $SU_X(2)$. If we take its symmetric square, we obtain a semisable rank three vector bundle with trivial determinant:

$$SU_X(2) \to SU_X(3); \quad E \mapsto \text{Sym}^2(E).$$

We want to study the action of involutions defined on the beginning of this paragraph on $\text{Sym}^2(E)$ with $E \in SU_X(2)$. Since $\text{Sym}^2(E)^* = \text{Sym}^2(E) = h^*(\text{Sym}^2(E))$, then $\sigma(\text{Sym}^2(E)) = \text{Sym}^2(E) \subset SU_X(3)$, so $\text{Sym}^2(SU_X(2)) \subset \text{Fix}(\sigma)$, and, again by (5), $\phi_3(\text{Sym}^2(SU_X(2))) \subset \text{Coble sextic}.$

Now we want to see the action of $\tau$ on $|3\Theta_{\text{Pic}^1(X)}|$. It is known that $\text{Fix}(\tau) = \mathbb{P}^4 \sqcup \mathbb{P}^3$.

**Notations:** We denote by $\mathbb{P}^3$ and $\mathbb{P}_i^3$, respectively, the $\mathbb{P}^3$ and the $\mathbb{P}^4$ which are fixed by action of $\tau$.

Since the image of $\text{Sym}^2(SU_X(2))$ by $\phi_3$ in $\mathbb{P}^8$ has dimension 3 and also $\phi_3(\text{Sym}^2(SU_X(2))) \subset \text{Fix}(\tau)$, we obtain

$$\phi_3(\text{Sym}^2(SU_X(2))) \subset \mathbb{P}_i^3.$$ 

Let $L_a \oplus L_{-a}$ be an element of $\text{Kum}_2(X) \subset SU_X(2)$, then $\text{Sym}^2(L_a \oplus L_{-a}) = L_{2a} \oplus L_{-2a} \oplus \Theta \in \text{Kum}_3(A) \subset SU_X(3)$. It means that $\text{Sym}^2(\text{Kum}_2(A)) \subset \text{Kum}_3(A)$.

**Observation:** Since $\{L_{2a} \oplus L_{-2a} \oplus \Theta \in SU_X(3)\}$ is isomorphic to $S^2(L_a \oplus L_{-a})$, we can view $\{L_{2a} \oplus L_{-2a} \oplus \Theta \in SU_X(3)\}$ as the image of $\text{Kum}_2(A)$ inside $SU_X(3)$ under the symmetric square map. Moreover it follows from the surjectivity of the multiplication by 2 map $[2] : A \to A$ that the image of $\text{Kum}_2(A)$ in $SU_X(3)$ is isomorphic to $\text{Kum}_3(A)$.

We have already observed that $\phi_3|_{\text{Kum}_3(A)}$ is a $(1:1)$—map on the image; this fact allows us to view $\phi_3(\text{Kum}_3(A))$ as the $\text{Kum}_3(A)$ in $|3\Theta_{\text{Pic}^1(X)}|$. For the same reason we can view $\phi_3(\text{Sym}^2(SU_X(2)))$ as $\text{Kum}_2(A) \subset |3\Theta_{\text{Pic}^1(X)}|$. Using this language we can say that $\text{Kum}_2(A)$ is left fixed by the action of $\tau$ in $\text{Kum}_3(A) \subset |3\Theta_{\text{Pic}^1(X)}|$ because $|3\Theta_{\text{Pic}^1(X)}| \supset \phi_3(\text{Kum}_3(A)) \supset \phi_3(\text{Sym}^2(SU_X(2))) = \text{Kum}_2(A) \subset \mathbb{P}^4 \subset \text{Fix}(\tau) \subset |3\Theta_{\text{Pic}^1(X)}|$. 

**Proposition 5.1.** Fix$(\tau) \cap \phi_3(\text{Kum}_3(A)) = \phi_3(\text{Sym}^2(\text{Kum}_2(A)))$. 


Proof. By definition \( \tau(L_a \oplus L_b \oplus L_c) = L_{-a} \oplus L_{-b} \oplus L_{-c} \) then \( L_a \oplus L_b \oplus L_c \) belongs to \( \text{Fix}(\tau) \) if and only if \( \{a, b, c\} = \{-a, -b, -c\} \). Let \( P \) belong to \( \{-a, -b, -c\} \) and \( a = P \).

- If \( P \) is different from \(-a\), suppose that \( P = -c \), then \( \{a, b, c\} = \{-a, -b, a\} \); moreover \( a + b + c = 0 \) because \( L_a \oplus L_b \oplus L_c \in \text{Kum}_3(A) \), then \( b = 0 \).

- Now, if \( P = -a \) or, equivalently \( a = -a \), then \( a = 0 \) and \( b = -c \).

In both cases \( L_a \oplus L_b \oplus L_c \in \text{Kum}_3(A) \) such that \( \tau(L_a \oplus L_b \oplus L_c) = L_{-a} \oplus L_{-b} \oplus L_{-c} \) are of the form \( L_a \oplus L_{-a} \oplus L_0 \). This means that they belong to \( \text{Kum}_2(A) \subset |3\Theta_{\text{Pic}^1(X)}| \).

The previous proposition tells us also that \( \mathbb{P}^3_\tau \cap \text{Kum}_3(A) = \emptyset \). So the projection of \( \text{Kum}_3(A) \subset |3\Theta_{\text{Pic}^1(X)}| \) from \( \mathbb{P}^3_\tau \) to \( \mathbb{P}^4_\tau \) is a morphism. It would be interesting to find its degree.

Our final observation is the following.

**Proposition 5.2.** \( \text{Sing}(\text{Kum}_3(A)) \cap \text{Kum}_2(A) = \text{Sing}(\text{Kum}_2(A)) \)

Proof. Points of \( \text{Kum}_2(A) \subset \text{Kum}_3(A) \) are of the form \( (P, -P, 0) \). Singular points of \( \text{Kum}_3(A) \) are those which have at least two equal components, then \( \text{Sing}(\text{Kum}_3(A)) \cap \text{Kum}_2(A) = \{(P, -P, 0)\} \) where \( 2P = 0 \) that are exactly the 15 points of 2-torsion and one more point \((O_X, O_X, O_X)\) which are singularities of the usual \( \text{Kum}_2(A) \). This implies that \( \sharp(\text{Sing}(\text{Kum}_3(A)) \cap \text{Kum}_2(A)) = 16 \) and \( \text{Sing}(\text{Kum}_3(A)) \cap \text{Kum}_2(A) = \text{Sing}(\text{Kum}_2(A)) \). □

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