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Fractal Steady States in Stochastic Optimal Control Models*

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The paper is divided into two parts. We first extend the Boldrin and Montrucchio's theorem (JET, 1986) on the inverse control problem, to the Markovian stochastic setting. Given a dynamical system $x_{t+1} = g(x_t, z_t)$, we find a discount factor β^* such that for each $0 < \beta < \beta^*$ a concave problem exists for which the dynamical system is an optimal solution. In the second part we use the previous result for constructing stochastic optimal control systems having fractal attractors. In order to do this, we rely on some results by Hutchinson on fractals and self-similarities. A neo-classical three-sector stochastic optimal growth exhibiting the Sierpinski's carpet as the unique attractor is provided as an example.

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1 Introduction

In the last decade a new emphasis on deterministic optimal control models, especially in the optimal growth literature, has been given from a different perspective: the regularity in dynamic behavior of the economy under standard hypothesis of concavity of the welfare function has been questioned. It has been shown by Boldrin and Montrucchio (see [5] and [22]) that if the infinitely-lived representative agent is impatient enough, an economy characterized by decreasing returns of scale technologies may display optimal dynamics that are very irregular and chaotic (a recent survey on this subject is [23]).

The first part of this paper is concerned with the extension to a stochastic context of Boldrin and Montrucchio's result. In particular, given a policy g, a lower estimate for the individual discount factor β^* will be determined such that,

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if $0 < \beta < \beta^*$, a concave problem characterized by discount factor β exists for which the dynamical system $x_{t+1} = g(x_t, z_t)$ turns out to be the optimal solution. The main argument is similar to that adopted in the deterministic case, following in particular the idea in [22]. A special case arises when g is an affine map: it will be shown that all affine maps are solutions of some quadratic program. Moreover, if the affine map is also a contraction, a quadratic program exists regardless of the magnitude of the discount factor.

The restatement of the theorem for stochastic models keeps its original meaning: anything goes also in the stochastic setting. Clearly this is much less surprising than the analogous result for deterministic models, since one naturally would expect more irregularities in a dynamic system which depends on unpredictable exogenous shocks. However this does not imply necessarily that such models cannot reach any form of stability. Indeed, since the pioneering works of Lucas-Prescott [18] and Brock-Mirman [7], it is well known that a broad class of models converging to a unique "steady state" solution exists (other works about this topic are [19], [20], [6], [9], [13] and, more recently, [15]). Such a stationary solution is expressed in terms of invariant distribution for the stochastic process describing the dynamic of the economy.

By using the inverse control problem it would not be difficult to give examples of models exhibiting optimal policies with multiple invariant measures and thus violating the stability property just mentioned. Similarly, we may provide examples of optimal cyclic sets along the line followed in [4]. We do not pursue these projects here; on the contrary, in the remaining part of the paper we study models in which the optimal process converges to a unique invariant measure. Nevertheless, even if our treatment is in tune with the neo-classical literature on optimal growth cited above, we focus on the fact that the unique attractor of these systems may be very complicated. In particular, we are interested in economies having a singular probability measure as their limiting distribution which is defined on a support that is a fractal set. The results presented here and the examples constructed will be related to those from Hutchinson [16].

A work very close to ours about the stochastic extension of the Boldrin and Montrucchio's indeterminacy theorem has been independently developed at the same time by Mitra [21]. There, an argument similar to that in the first deterministic version of the theorem (see [5]) has been pursued, and a different direction of research has been taken.

The paper is organized as follows. In Section 2 the basic notation is introduced. Section 3 is devoted to the statement of the model and to recall some well known facts about stochastic dynamic programming that will be used in Section 4, where the inverse problem of optimal control is formulated and the main result is proved. Then, Section 5 deals with the problem of constructing fractals by iterating contractive maps; we briefly survey the main results on this field. Finally, in

Section 6 some examples of very simple models converging to fractal attractors are given. In particular, a three-sector model of optimal growth is shown to converge to the Sierpinski's gasket.

2 Notation

For vectors x in $\mathbf{R}^m$, $\|x\|$ denotes the Euclidean norm. The inner product of two vectors is denoted by $\langle x, y\rangle$. Given a metric space (X, d), we recall that the distance between a point x and a set A in X is defined by $d(x, A) = \inf_{y\in A} d(x, y)$. The Hausdorff metric between two sets $A, B \subset X$ is defined as

$$h(A, B) = \max\left[\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right]. \tag{1}$$

The symbol 2_c^X denotes the space of all nonempty closed and bounded subsets of X. $2_k^X \subset 2_c^X$ will be the subspace of all nonempty compact subsets. It is well known that 2_c^X, endowed with the Hausdorff distance h, turns out to be a complete metric space, whenever X is complete (see e.g. [8], page 61, problem 3).

Given the complete metric space X, $\Lambda(X)$ will be the set of all probability measures on X which are Borel regular. The support of $\lambda \in \Lambda(X)$ will be denoted by $spt\lambda$. The space $\Lambda_b(X)$ is the set of regular probability measures having bounded support. We recall that a sequence $\{\lambda_n\}$ of elements in $\Lambda(X)$ converges weakly to λ if $\lim_{n\to\infty} \int f d\lambda_n = \int f d\lambda$ for every bounded continuous function $f : X \to \mathbf{R}$. A useful metric on $\Lambda_b(X)$ related to the weak convergence is the following $\mathcal{L}$ metric:

$$d_H(\mu, \lambda) = \sup_{f\in L_1}\left[\int_X f d\mu - \int_X f d\lambda\right], \ \mu, \lambda \in \Lambda_b(X), \tag{2}$$

where $L_1 = \{f : X \to \mathbf{R}, |f(x) - f(y)| \le d(x, y)\}$ is the set of Lipschitz functions with constant not greater than one. The space $(\Lambda_b(X), d_H)$ turns out to be a complete metric space (see [16]). The metric defined in (2) is a useful criterion to establish weak convergence of probability measures since the $\mathcal{L}$ metric topology and the weak topology coincide on $\Lambda_b(X) \cap \{\lambda; spt\lambda \text{ is compact}\}$. See also [10], [3] and [13]. Therefore, if $X = \mathbf{R}^n$, $\{\lambda_n\} \subset \Lambda_b(X)$ converges weakly to λ if and only if $d_H(\lambda_n, \lambda)$ converges to zero.

3 Markovian Stochastic Dynamic Programming

The uncertainty of the environment is described by an exogenous stochastic process $\{z_t\}_{t=0}^{\infty}$, where each random variable z_t takes values in some measurable space $(Z, \mathcal{Z})$. Such a process is assumed to be Markovian with stationary transition function (stochastic kernel) $Q : Z \times \mathcal{Z} \to [0,1]$. Let $Z^{\infty} = \prod_{t=1}^{\infty} Z_t$, where $Z_t = Z$ for $t = 1, 2, \ldots$, and $\mathcal{Z}_t = \sigma- \{C_1 \times \cdots \times C_t \times Z \times Z \times \cdots\}$, where $C_\tau \in \mathcal{Z}$ for all $1 \leq \tau \leq t$; i.e. $\mathcal{Z}_t$ is the σ-algebra generated by cylinder sets. Given any initial shock $z_0 \in Z$, all finite probabilities on cylinder sets are given by $\mu^t(z_0, C_1 \times \cdots \times C_t) = \int_{C_1} Q(z_0, dz_1) \cdots \int_{C_t} Q(z_{t-1}, dz_t)$.

The state variable takes values within a compact convex set $X \subseteq \mathbf{R}^m$; let $\mathcal{X} \subseteq \mathcal{B}^m$ be the Borel σ-algebra on X. We denote by $(S, \mathcal{S}) = (X \times Z, \mathcal{X} \bigotimes \mathcal{Z})$ the product space representing the state of the system; vector $s_t = (x_t, z_t)$ is an element of the state space at date t. The dynamic constraint is a measurable set $D \subset X \times X \times Z$ such that its sections $D_z \subset X \times X$ are convex for all $z \in Z$. For each $(x, z) \in S$, let $\Gamma : X \times Z \to X$ defined as $\Gamma(x, z) = \{y \in X : (x, y, z) \in D\}$ be the correspondence representing the set of feasible actions when the current state is (x, z). The *one-period return function* $U : D \to \mathbf{R}$ is assumed to be measurable, bounded and with z-sections $U(\cdot, \cdot, z) : D_z \to \mathbf{R}$ concave. The *discount factor* β is a constant parameter belonging to the interval $(0, 1)$.

For each initial condition $s_0 = (x_0, z_0) \in S$, a *feasible plan from* s_0 is a value $\pi_0 \in X$ and a sequence $\{\pi_t\}_{t=1}^{\infty}$ of $\mathcal{Z}_t$-measurable functions $\pi_t : Z^{\infty} \to X$ such that $\pi_0 \in \Gamma(s_0)$ and $\pi_t \in \Gamma(\pi_{t-1}, z_t)$, $\mu^t(z_0, \cdot)-a.e.$, $t = 1, 2, \ldots$. Let $\Pi(s_0)$ denote the set of plans that are feasible from s_0, which we will assume nonempty for all $s_0 \in S$. Then the stochastic optimization problem under investigation is:

$$\text{(P)} \qquad v(s_0) = \sup_{\pi \in \Pi(s_0)} \left\{ U(x_0, \pi_0, z_0) + \mathbf{E}\left[\sum_{t=1}^{\infty} \beta^t U(\pi_{t-1}, \pi_t, z_t) \right] \right\},$$

where expectation is well defined as $\beta < 1$ and U is measurable and bounded.

Markov assumption on the stochastic process of the exogenous shocks establishes an important relationship between the infinite-horizon problem P and the time-independent Bellman equation

$$w(s) = w(x, z) = \sup_{y \in \Gamma(x,z)} \left[U(x, y, z) + \beta \int_Z w(y, z') Q(z, dz') \right], \tag{3}$$

as the next result states.

Define the associated policy correspondence $G : S \to X$ by

$$G(x, z) = \left\{ y \in \Gamma(x, z) : w(x, z) = U(x, y, z) + \beta \int_Z w(y, z') Q(z, dz') \right\}.$$

If there exists a measurable selection $g(x,z) \in G(x,z)$, called *optimal policy*, then we say that a plan $\pi^* = \{\pi_t^*\}_{t=0}^{\infty}$ is generated by g starting at s_0 if $\pi_0^* = g(x_0, z_0)$ and $\pi_t^* = g(\pi_{t-1}^*, z_t)$ $\mu^t(z_0, \cdot) - a.e.$, $t = 1, 2, ...$

Proposition 1
If w is a measurable function satisfying (3) such that

$$\lim_{t\to\infty} \beta^t E\left[w(\pi_{t-1}, z_t)\right] = 0, \text{ for all } \pi \in \Pi(s_0) \text{ and all } s_0 \in S, \tag{4}$$

and G permits a measurable selection g, then w is the value function v of P, and any plan π^* generated by g is optimal.

A good reference for a complete discussion on all the assumptions and the statement of the problem, as well as a proof for Proposition 1, is [26].

4 Indeterminacy of Optimal Policies

We now look at the inverse problem of optimal control[1]: given an arbitrary function $g : S \to X$, we investigate whether there exists a concave problem P whose optimal path is generated by the policy g. Since the discount parameter β will play an important role in constructing the return function U of P, we will henceforth denote it by U_β.

Let us fix the discount factor β, a stochastic kernel $Q(z, dz')$ and a measurable function $w(x,z)$ as the optimal value function of P. Then we obtain U_β by using the Bellman equation, as next proposition states.

Proposition 2
Let $g : S \to X$ and $w : S \to R$ be measurable functions such that $g(s) \in \Gamma(s)$, and w is bounded. Let $V(x,y,z)$ be a scalar function such that $\max_{y\in X} V(x,y,z) = V[x, g(x,z), z] = w(x,z)$. For all $s_0 \in S$, let $\pi^* = \{\pi_t^*\} \in \Pi(s_0)$ be the plan generated by g. Then π^* is an optimal plan for the infinite-horizon problem P characterized by the return function

$$U_\beta(x,y,z) = V(x,y,z) - \beta \int_Z w(y,z')Q(z,dz').$$

Proof
P is well defined for all $\pi \in \Pi(s_0)$ and all $s_0 \in S$, and w satisfies (4). Then, to apply Proposition 1, one has only to show that w is a solution to (3) and that g attains the maximum in (3). We omit here these steps as they are a

[1]This section is taken from Chapter 5 of [25].

straightforward replication of the proof of Lemma 1 in [5] where the deterministic Bellman equation has been replaced with (3). □

It should be noted that some V, satisfying the condition of Proposition 2, does exist. For instance, $V(x,y,z) = -\frac{1}{2}\|y - g(x,z)\|^2 + w(x,z)$ meets this property.

In order to prove the next theorem, let

$$w(x,z) = -(L/2)\|x\|^2 + \langle a, x\rangle + f(z), \tag{5}$$

where $f : Z \to \mathbf{R}$ is measurable and bounded, $L \in \mathbf{R}$ and $a \in \mathbf{R}^m$. Thus, in view of Proposition 2, we define the return function of the infinite-horizon problem as

$$\begin{aligned} U_\beta(x,y,z) = \quad & -\tfrac{1}{2}\|y - g(x,z)\|^2 - \tfrac{L}{2}\|x\|^2 + \langle a, x\rangle + f(z) \\ & +\beta\tfrac{L}{2}\|y\|^2 - \beta\langle a, y\rangle - \beta\textstyle\int_Z f(z')Q(z,dz'). \end{aligned} \tag{6}$$

We need further to restrict the class of policies to be considered and some more notation. Let

$$k_0 = \max_{x,y\in X} \|x - y\|. \tag{7}$$

Assumption 1
For each $z \in Z$, the map $g(\cdot, z)$ is differentiable over an open set containing X and a constant k_1 must exist such that

$$k_1 = \sup_{(x,z)\in S} \|D_1 g(x,z)\|. \tag{8}$$

Assumption 2
For each $z \in Z$, some constant k_2 exists such that

$$\|D_1 g(x,z) - D_1 g(y,z)\| \le k_2 \|x - y\|, \text{ all } x, y \in X.$$

Theorem 3
Let $g : S \to X$ be a function satisfying Assumptions 1 and 2, with $(x, g(x,z), z) \in D$. For any discount factor $0 < \beta < \beta^* = (k_1 + \sqrt{k_0 k_2})^{-2}$, one can find a function $U_\beta(x,y,z)$ strictly concave in x, y so that g turns out to be the optimal policy for a problem P characterized by one-period return U_β and discount factor β. Moreover, U_β can be chosen to be increasing in x and decreasing in y.

Proof
Since $f : Z \to \mathbf{R}$ in (5) is measurable and bounded, Proposition 2 applies. We have thus only to find out for which β a parameter $L \in \mathbf{R}$ exists such that $U_\beta(\cdot,\cdot,z)$ defined in (6) turns out to be strictly concave. Linear terms and terms independent of x, y do not affect concavity of $U_\beta(\cdot,\cdot,z)$, therefore the proof is

identical to that of the deterministic version of this theorem (see Theorem 2.1 in [22]).

We must prove that, given function $g(x,z)$, a real number L and a discount factor $\beta^* > 0$ exist such that for all $0 < \beta < \beta^*$ the function

$$W(x,y,z) = -(1/2)\left\|y - g(x,z)\right\|^2 - (L/2)\left\|x\right\|^2 + \beta(L/2)\left\|y\right\|^2$$

is strictly concave in x and y for each fixed $z \in Z$. In order to do this, we shall show that $W(\cdot,\cdot,z)$ is superdifferentiable over its whole domain, i.e. that

$$\Delta W = W(x+\widehat{x}, y+\widehat{y}, z) - W(x,y,z) - \langle D_1 W(x,y,z), \widehat{x}\rangle - \langle D_2 W(x,y,z), \widehat{y}\rangle \leq 0,$$

for all $x+\widehat{x}$, $y+\widehat{y} \in X$ and for all $z \in Z$. Here $D_1 W$ and $D_2 W$ denote the partial derivatives of W with respect to x and y respectively.

By assuming $\beta L < 1$, the same technique adopted in [22] leads to

$$\begin{aligned}\Delta W \quad &\leq \tfrac{\beta L}{2(1-\beta L)}\left\|g(x+\widehat{x},z) - g(x,z)\right\|^2 \\ &\quad + \langle y - g(x,z), g(x+\widehat{x},z) - g(x,z) - D_1 g(x,z)\widehat{x}\rangle - \tfrac{L}{2}\left\|\widehat{x}\right\|^2.\end{aligned}$$

From Assumption 2 it follows that

$$\left\|g(x,z) - g(y,z) - D_1 g(x,z)(x-y)\right\| \leq (1/2)k_2\left\|x-y\right\|^2, \text{ for all } x,y \in X$$

(see, for example, Theorem 3.2.12 in [24]), and from Assumption 1 we get

$$\left\|g(x+\widehat{x},z) - g(x,z)\right\|^2 \leq k_1^2\left\|\widehat{x}\right\|^2.$$

Since by (7) $\left\|y - g(x,z)\right\| \leq k_0$ holds, it is easily seen that

$$\Delta W \leq (1/2)\left[\beta L(1-\beta L)^{-1}k_1^2 + k_0 k_2 - L\right]\left\|\widehat{x}\right\|^2.$$

Clearly for any pair (β, L) such that $\beta L < 1$ and

$$\beta L(1-\beta L)^{-1}k_1^2 + k_0 k_2 - L \leq 0, \tag{9}$$

$W(\cdot,\cdot,z)$ turns out to be concave. It is readily seen that the set of solutions of (9) is nonempty. More specifically, each pair (β, L^*) such that

$$\begin{gathered}0 \leq \beta \leq \beta^* = (k_1 + \sqrt{k_0 k_2})^{-2}, \text{ and} \\ L^* = (1/2)(\beta^{-1} - k_1^2 + k_0 k_2)\end{gathered}$$

satisfies (9), thus establishing the result on the strict concavity of U_β.

To prove the monotonicity properties of U_β, it will be sufficient to calculate the first order derivatives of U_β:

$$\begin{gathered}D_1 U_\beta(x,y,z) = D_1 g\,(x,z)\,[y - g\,(x,z)] - Lx + a \\ D_2 U_\beta(x,y,z) = g\,(x,z) - (1-\beta L)y - \beta a.\end{gathered}$$

If the components of vector a are positively large enough, then $D_1 U_\beta(x,y,z) > 0$ and $D_2 U_\beta(x,y,z) < 0$ and this completes the proof. □

It is important to remark that the results above still hold under weaker assumptions. Indeed, it would be sufficient to replace "for all $z \in Z$" with "almost for all $z \in Z$" in the statements above. The drawback of such an approach is the more complex notation arising as soon as one tries to define the marginal probability with respect to the zero-probability sets are to be considered[2].

In view of what follows, it is worthwhile to consider the case where the policy function is an affine transformation: $g(x,z) = A(z)\,x + b(z)$, with $A(z)$ an m-order matrix depending on the shock z and $b(z)$ an m-order random vector. In this circumstance, the return function (6) turns out to be quadratic:

$$\begin{aligned} U_\beta(x,y,z) = \; & -\tfrac{1}{2}(1-\beta L)\,\|y\|^2 + \langle y, A(z)\,x\rangle - \tfrac{1}{2}\left\langle x, \left(A'(z)\,A(z) + LI_m\right)x\right\rangle \\ & + \text{linear terms.} \end{aligned}$$

In this case, the strict concavity of $U_\beta(x,y,z)$ can be studied directly without resorting to Theorem 3 and, more importantly, we can relax the boundedness assumption on domain X (not so if we are concerned about the monotonicity of U_β, which requires that X be bounded). It is easily verified that the two conditions: $\beta L < 1$ and $\beta L(1-\beta L)^{-1} A'(z)\,A(z) - LI_m \le -\varepsilon I_m$ for all z and some $\varepsilon > 0$, assure that $U_\beta(.,.,z)$ is strongly concave, uniformly over z. This implies in turn that the fixed point w is the value function. It is readily seen that the condition is $\beta < k_1^{-2}$, where $k_1 = \sup_z \|A(z)\|$, and one may take, for instance, $L = \frac{1}{2}\left(\beta^{-1} - k_1^2\right)$. All that is formalized in the next proposition.

Proposition 4
Given a linear map $g(x,z) = A(z)\,x + b(z)$, for every $0 < \beta < \beta^* = k_1^{-2}$ there exists a strongly concave quadratic programming having g as its optimal policy.

It should be noted that whenever the affine transformations are uniformly contractive, i.e., $\|A(z)\,x_1 - A(z)\,x_2\| \le \alpha\,\|x_1 - x_2\|$ for $\alpha < 1$, then $k_1 = \alpha$ and thus the inverse problem does not require any restriction on the discount factor.

[2]Given any initial probability μ_0 for the random shock z_0 (for example $\mu_0 = \delta_{z_0}$, which denotes the probability that is a unit mass at the point z_0) all marginal probabilities of the random variables z_t are well defined by iterating the adjoint operator associated to Q; that is $\mu_t(\cdot) = \int_Z Q(z,\cdot)\mu_{t-1}(dz)$, $t = 1,2,\ldots$. Since the μ_t's are different probability measures on the same space $(Z,\mathcal{Z})$, also the zero-probability sets are different as t varies. Therefore, in order to restate our results for this broader class of concave models, it is enough to set assumptions that must hold for almost every exogenous shock with respect to all marginal probabilities μ_t, $t = 1,2,\ldots$; or, which is the same, for all exogenous shocks but a set which is the intersection of all zero-probability sets.

5 Constructing Fractals

In this section we study the asymptotic behavior of finite families of contraction maps $\{g_1, ..., g_n\}$ that produce limiting singular measures supported on fractals. Let us suppose that the mappings $g_i : X \to X$, acting over a complete metric space (X, d), have a common contraction factor $\alpha < 1$, i.e.,

$$d\left(g_i(x), g_i(y)\right) \leq \alpha d\left(x, y\right), \text{ for all } x, y, \in X, \text{ and } i = 1, ..., n.$$

The system $\{g_1, ..., g_n\}$ is sometimes called an *iterated function system* and we can associate to it the so-called *Barnsley operator* $g_\#$ defined over the subsets $C \subset X$:

$$g_\#(C) = \bigcup_{i=1}^{n} g_i(C).$$

We can also consider its iterates $g_\#^{t+1}(C) = g_\# \left[g_\#^t(C)\right]$, for all $t \geq 0$.

The asymptotic behavior of $g_\#^t$ is illustrated by the next "collage" theorem.

Theorem 5 (Hutchinson [16])
There exists a unique closed and bounded set A, such that $A = g_\#(A) = \bigcup_{i=1}^{n} g_i(A)$. Furthermore, A is compact and $h\left[g_\#^t(C), A\right] \to 0$ as $t \to \infty$ for all closed bounded sets $C \subset X$, where h is the Hausdorff distance.

Proof
Hutchinson showed that $g_\# : 2_c^X \to 2_c^X$ is a contraction as well as the operator $g_\# : 2_k^X \to 2_k^X$. To be precise,

$$h\left[g_\#(A), g_\#(B)\right] \leq \alpha h(A, B)$$

holds for all $A, B \in 2_c^X$. Therefore, since 2_c^X and 2_k^X are complete metric spaces, the contraction mapping principle applies. □

The invariant set A can be interpreted as the attractor of the system $\{g_1, ..., g_n\}$. Most popular fractals are attractors of a finite number of contractions. As an example, let $X = \mathbf{R}^2$ and consider the linear maps

$$\begin{aligned} g_1(x_1, x_2) &= \left(\tfrac{x_1}{2}, \tfrac{x_2}{2}\right), \\ g_2(x_1, x_2) &= \left(\tfrac{1}{4} + \tfrac{x_1}{2}, \tfrac{1}{2} + \tfrac{x_2}{2}\right), \\ g_3(x_1, x_2) &= \left(\tfrac{1}{2} + \tfrac{x_1}{2}, \tfrac{x_2}{2}\right). \end{aligned}$$

These are similitudes centered in three points of the triangle of vertex $(0, 0)$, $(1/2, 1)$, $(1, 0)$. The invariant set (the attractor) is Sierpinski's gasket which is

shaped by three copies of itself. In the same way, the middle-third Cantor set of the interval $[0,1]$ is generated by the two maps $g_1(x) = \frac{1}{3}x$ and $g_2(x) = \frac{1}{3}x + \frac{2}{3}$. These attractors have Hausdorff dimension $\ln 3/\ln 2$ and $\ln 2/\ln 3$ respectively.

The idea behind the self-similarity for the attractor A is that from $A = \bigcup_{i=1}^{n} g_i(A)$, it turns out that A is shaped by n copies of its miniatures $g_i(A)$, at least if some non-overlapping condition is fulfilled. There are several studies on the topological nature of sets generated by contractive maps and on their Hausdorff dimension. See [16], [1], [27], [12] and [14] for more details. We here mention only the stochastic interpretation of the iterated function systems described above.

Consider the random system $x_{t+1} = g_{\sigma_t}(x_t)$, where the indices σ_t are chosen randomly and independently from the set of indexes $\{1, 2, ..., n\}$ at each date t with probabilities $\Pr(\sigma_t = i) = p_i$, with $p_i > 0$ for all i and $\sum_{i=1}^{n} p_i = 1$. An appealing way to write this, that is consistent with the notation adopted for the policy functions discussed in Section 4, is as follows.

Consider the stochastic process generated by the "policy" $g(x, z^i) = g_i(x)$, where each map g_i is associated with an exogenous shock z^i belonging to a finite set $Z = \{z^1, ..., z^n\}$. Therefore, by construction, the process of exogenous shocks $\{z_t\}$ is i.i.d. with marginal probability distribution given by $\Pr(z^i) = p_i$.

Clearly, $\{x_t\}$ is a Markov process whose law of motion is given by the stationary transition function $P(x, A)$:

$$P(x, A) = \Pr\left\{z^i : g_i(x) \in A\right\} = \sum_{i=1}^{n} p_i \chi_A\left[g_i(x)\right], \text{ for all } x \in X, \text{ all } A \in \mathcal{X}, \quad (10)$$

where $\chi_A(x)$ is the indicator function. The adjoint operator $M : \Lambda(X) \to \Lambda(X)$ associated to transition P is:

$$M(\lambda) = \int_X P(x, \cdot)\lambda(dx). \quad (11)$$

The iterates $M^{t+1}(\lambda_0) = M\left[M^t(\lambda_0)\right]$ define the sequence of marginal probabilities of the process $\{x_t\}$ starting from an initial marginal probability $\lambda_0 \in \Lambda(X)$.

The next theorem can be regarded as the probabilistic counterpart of Theorem 5 and provides the stochastic realization of the attractor A of Theorem 5.

Theorem 6 (Hutchinson [16])
There exists a unique invariant probability measure $\lambda^* \in \Lambda_b(X)$ for the process $\{x_t\}$; i.e., the adjoint operator M, defined in (11), has one and only one fixed point. Moreover, spt $\lambda^* = A$, where A is the unique attractor of the system $\{g_1, ..., g_n\}$. The iterative process $\lambda_{t+1} = M(\lambda_t)$ converges weakly to λ^*, for every starting probability measure $\lambda_0 \in \Lambda_b(X)$.

Proof
Hutchinson proved that the operator M is a contraction; more precisely

$$d_H\left[M(\mu), M(\lambda)\right] \leq \alpha d_H(\mu, \lambda), \text{ for all } \mu, \lambda \in \Lambda_b\left(X\right),$$

where d_H is defined in (2). Since (Λ, d_H) is a complete metric space, the results follow by contraction mapping principle. The fact that *spt* $\lambda^* = A$ is a consequence of Theorem 5. A different proof of this theorem can be found in [17]. □

A consequence of this approach is the following (see [2], [27], [17]).

Proposition 7
With probability one, in the steady state the orbit $\{x_t\}$ *generated by random system* $x_{t+1} = g\left(x_t, z_t\right)$ *is dense on* A.

6 Fluctuations Versus Stability: Fractal Cycles

Here we provide a few examples of the theory developed in the first part of the paper. The first example does not require the inverse theorems of Section 4.

6.1 One-Sector Model with Cantor Attractor

Consider the one-sector growth model with a Cobb-Douglas production function, $f(x) = x^{1/3}$, which already takes into account depreciation of capital. The utility of the representative decision maker is $U(c) = \ln c$. Suppose that an exogenous perturbation may reduce production by some parameter $0 < k < 1$ with probability $p > 0$. This random shock enters multiplicatively the production process; i.e. output is given by $f_z(x) = zx^{1/3}$ where $z \in \{k, 1\}$. Thanks to monotonicity of both production and utility functions, the problem is to maximize $\ln(z_0 x_0^{1/3} - \pi_0) + \mathbf{E}[\sum_{t=1}^{\infty} \beta^t \ln(z_t \pi_{t-1}^{1/3} - \pi_t)]$ over the sequences $\{\pi_t\}$ of random variables such that $0 \leq \pi_t \leq z_t \pi_{t-1}^{1/3}$, $t = 0, 1, ...$, where $0 < \beta < 1$ is the discount factor.

It is well known that the optimal policy for the concave problem just described is $g(x, z) = \frac{1}{3}\beta z x^{1/3}$ (see e.g. [26]); i.e. the plan $\{\pi_t\}$ generated recursively by

$$\pi_t = g(\pi_{t-1}, z_t) = (1/3)\beta z_t \pi_{t-1}^{1/3} \tag{12}$$

is optimal. Consider now the dynamic system obtained by the following logarithmic transformation of $\{\pi_t\}$:

$$y_t = \ln \pi_t - (3/2)\left[\ln(1/3)\beta + \ln k\right]. \tag{13}$$

The new system $\{y_t\}$, conjugated to $\{\pi_t\}$, evolves by the law $\widehat{g}(y, z) = \frac{1}{3}y + \ln(z/k)$:

$$y_t = (1/3)y_{t-1} + \ln(z_t/k),$$

as it is easily seen by substituting (12) into (13). Then the middle third Cantor set on the interval $[0, -\frac{3}{2}\ln k]$ is the invariant set (the attractor) of the system $\{y_t\}$ generated by the linear maps $g_1(y) = \frac{1}{3}y$ and $g_2(y) = \frac{1}{3}y - \ln k$.

Therefore the attractor of the original system $\{\pi_t\}$, i.e. of the optimal dynamic of the one-sector optimal growth model under study, is a Cantor set.

6.2 Stochastic Quadratic Programming and Fractals

As it has been widely discussed, most of fractals are realized by iterating a finite number of contractive mappings. Even better, a good deal of them are obtained by iterating affine mappings. Hence, Theorem 3 and Proposition 4 show that these fractals can be obtained through quadratic programs. To give a flavor of this theory, we treat explicitly the Sierpinski's gasket presented in Section 5. In order to apply this construction to a three-sector model in the next subsection, we modify the original "triangle" attractor by shifting it away from the origin and by shrinking it to let it remain within the square $[0,1]^2$. Thus the policy will be $g(x,z) = \frac{1}{2}x + b(z)$, where $x = (x_1, x_2)$, the shock z can take three values $z \in \{z^1, z^2, z^3\}$ and $b(z^1) = \left(\frac{1}{6}, \frac{1}{6}\right)$, $b(z^2) = \left(\frac{1}{3}, \frac{1}{2}\right)$, $b(z^3) = \left(\frac{1}{2}, \frac{1}{6}\right)$. In this way, the vector function g is constituted of similarities centered in the three points of the triangle of vertex $\left(\frac{1}{3}, \frac{1}{3}\right)$, $\left(\frac{2}{3}, 1\right)$, $\left(1, \frac{1}{3}\right)$.

Let $\beta \in (0,1)$, $a \in \mathbf{R}^2$ and $f : Z \to \mathbf{R}$ be a measurable bounded function. In view of Proposition 4, since $k_1 = \sup_z \|A(z)\| = \frac{1}{2}$, we take $L = \frac{1}{2}\left(\frac{1}{\beta} - \frac{1}{k_1^2}\right) = \frac{1}{2\beta} - \frac{1}{8}$. By replacing these values in (6), we get the following one-period return:

$$\begin{aligned} U_\beta(x,y,z) = \; & -\left(\tfrac{1}{16} + \tfrac{1}{4\beta}\right)\|x\|^2 - \left(\tfrac{1}{4} + \tfrac{\beta}{16}\right)\|y\|^2 + \tfrac{1}{2}\langle y, x\rangle \\ & + \left\langle a + \tfrac{1}{2}b(z), x\right\rangle + \langle b(z) - \beta a, y\rangle - \tfrac{1}{2}\|b(z)\|^2 \\ & + f(z) - \beta \mathbf{E}(f), \end{aligned} \tag{14}$$

where $\mathbf{E}(f)$ is the expectation of f which, as random shocks z_t are i.i.d., does not depend on z. The value function for the model under construction is

$$w(x,z) = -\left[(4\beta)^{-1} - 1/16\right]\|x\|^2 + \langle a, x\rangle + f(z).$$

6.3 Three-Sector Model with Sierpinski Attractor

We turn now to the construction of a stochastic three-sector, no-joint-production, optimal growth model where the one-period welfare function U_β is exactly (14).

Consider an economy with three production sectors: consumption c and two capital goods, k and h. Labor is supplied at two different levels: unskilled work m and skilled work l. Utility is linear in consumption, a fixed amount of labor from both categories is supplied in each period and capital depreciation factor equals

one for each capital good. All factors are employed in the consumption sector while capital k and unskilled work m are used only in the production of capital k. Similarly, capital h is produced by skilled work l and capital h. One can think of sector h as a high-technology sector while sector k as a low-technology sector. The Production Possibility Frontier $T(k,h,k',h',z)$ will be given by

$$\begin{array}{rl} T(k,h,k',h',z) & = \max f_c(k^c,h^c,m^c,l^c,z) \\ s.t.\ \ k' & \leq f_k(k^k,m^k),\ h' \leq f_h(h^h,l^h), \\ k^c & +k^k \leq k,\ h^c+h^h \leq h \\ m^c & +m^k \leq 1,\ l^c+l^h \leq 1, \end{array} \tag{15}$$

where f_c, f_k and f_h are the production functions, k', h' are end-of-period level of the two capital goods, z represents an exogenous shock belonging to the set $\{z^1,z^2,z^3\}$ which is supposed to affect only the consumption good sector. The total amount of work has been normalized at one in each category and all the variables are constrained to non-negative values. Clearly, the dynamic constraint turns out to be $\Gamma(k,h) = \{(k',h') : 0 \leq k' \leq f_k(k,1),\ 0 \leq h' \leq f_h(h,1)\}$, which does not depend on shock z. The consumer's problem then is

$$\begin{array}{c} \max \left\{ T\left(k_0,h_0,\pi_0^k,\pi_0^h,z_0\right) + \mathbf{E}\left[\sum_{t=1}^{\infty} \beta^t T\left(\pi_{t-1}^k,\pi_{t-1}^h,\pi_t^k,\pi_t^h,z_t\right)\right]\right\} \\ \text{s.t.} \quad \left(\pi_t^k,\pi_t^h\right) \in \Gamma\left(\pi_{t-1}^k,\pi_{t-1}^h\right), \end{array} \tag{16}$$

where, as usual, $0 < \beta < 1$.

If we assume that the technologies of capital sectors are Leontief type with coefficients γ and ν respectively, i.e. $f_k(k^k,m^k) = \min\left\{\gamma k^k, m^k\right\}$ and $f_h(h^h,l^h) = \min\left\{\nu h^h, l^h\right\}$, then the solution to (15) is

$$T(k,h,k',h',z) = f_c\left(k-(k'/\gamma), h-(h'/\nu), 1-k', 1-h', z\right).$$

Now, if we use (14) as the Production Possibility Frontier function in (16), i.e., if we let $T(k,h,k',h',z) = U_\beta\left[(k,h),(k',h'),z\right]$, then, by a straightforward substitution of variables, the production function of consumption becomes

$$f_c(k^c,h^c,m^c,l^c,z) = U_\beta\left[(k^c+(1-m^c)/\gamma, h^c+(1-l^c)/\nu), (1-m^c,1-l^c), z\right],$$

which clearly is strictly concave over $[0,1]^2$ for each fixed z. By assuming γ, $\nu \geq 3$, it is easily seen that the dynamic constraint is such that all pairs (k',h') belonging to the square $[0,1]^2$ are feasible whenever k, h belong to the square $\left[\frac{1}{3},1\right]^2$. Furthermore, if γ, ν are such that $\gamma\beta > 1$ and $\nu\beta > 1$, for any vector $a = (a_1,a_2)$ whose components are greater than $\frac{13\gamma}{12(\beta\gamma-1)}$ and $\frac{13\nu}{12(\beta\nu-1)}$ respectively, it is easily seen that f_c turns out to be strictly increasing in k^c, h^c, m^c and l^c. Therefore this neo-classical stochastic optimal growth model converges to the Sierpinski gasket discussed in the previous subsection. By Proposition 7, the trajectories of two capital goods wander densely over this fractal through time.

7 Concluding Remarks

The discussion developed in this paper shows that two different disciplines, stochastic optimal control and chaotic dynamic systems, may interact in the description of the evolution thorough time of an economic system. By joining the two theories we found that stability and complex behavior of economic models turn out to be not mutually incompatible. On one side, standard ergodic theory applied to Markovian systems establishes existence of a unique steady state (a stationary probability defined on an invariant support, the attractor) to which the economy eventually converges; models with optimal policies that are contractive maps are of this type. On the other side, contractive maps generate systems that converge to fractal attractors. Hence, by applying the stochastic version of the Indeterminacy Theorem to affine contractive maps, it is easy to construct economic models converging to invariant (singular) probabilities defined on fractal attractors. Such economies are well shaped as agents have concave, increasing, differentiable utilities, but, in the long run, they evolve through a stationary chaotic cycle.

It remains to find some characterization also for the invariant distribution defined on the fractal support, that is, the stochastic law that moves the system through the points of the attractor after the system entered the steady state. In particular, it would be interesting to study the relationship between the shape of the distribution of the exogenous shocks and the shape of the resulting invariant distribution on the attractor.

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