A weight independence result for quaternionic Hecke Algebras

This is the author's manuscript

Original Citation:

Availability:
This version is available http://hdl.handle.net/2318/140411 since 2016-01-14T16:55:49Z

Published version:
DOI:10.1142/S1793042113500656

Terms of use:
Open Access
Anyone can freely access the full text of works made available as "Open Access". Works made available under a Creative Commons license can be used according to the terms and conditions of said license. Use of all other works requires consent of the right holder (author or publisher) if not exempted from copyright protection by the applicable law.

(Article begins on next page)
This is an author version of the contribution published on:


The definitive version is available at:

A WEIGHT INDEPENDENCE RESULT FOR QUATERNIONIC HECKE ALGEBRAS

LEA TERRACINI

Abstract. Let $p$ be a prime and $B$ be a quaternion algebra indefinite over $\mathbb{Q}$ and ramified at $p$. We consider the space of quaternionic modular forms of weight $k$ and level $p^\infty$, endowed with the action of Hecke operators. By using cohomological methods, we show that the $p$-adic topological Hecke algebra does not depend on the weight $k$. This result provides a quaternionic version of a theorem proved by Hida for classical modular forms; we discuss the relationship of our result to Hida’s theorem in terms of Jacquet-Langlands correspondence.

Keywords: modular forms; quaternion algebras; Hecke algebras; cohomology of Shimura curves.

Mathematics Subject Classification 2010: 11F55, 11F75, 11F25

1. Introduction

Let $p$ be an odd prime. We fix an embedding of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_p$. Let $K_0$ be an algebraic extension of $\mathbb{Q}$, $K$ be its closure in $\overline{\mathbb{Q}}_p$, $\mathcal{O}$ its $p$-adic integer ring. We fix an integer $N$ prime to $p$.

For every pair of positive integers $r, k$, let $S_k(\Gamma_1(Np^r), K_0)$ be the space of cuspidal forms of level $Np^r$, weight $k$ and having $q$-expansion in $K_0$; put $S_k(\Gamma_1(Np^\infty), K) = \lim_{r \to \infty} S_k(\Gamma_1(Np^r), K_0) \otimes_{K_0} K$. Let $H_k(Np^\infty, \mathcal{O})$ be the Hecke algebra generated over $\mathcal{O}$ by Hecke operators $T_n$ for all $n$.

In [5, §1] Hida defines the $K$-vector space

$$S_k(Np^\infty, K) = \lim_{r \to \infty} S_k(\Gamma_1(Np^r), K),$$

obtained by letting the exponent of $p$ vary in the level and keeping the weight fixed. It is equipped with a faithful action of the $\mathcal{O}[\mathbb{Z}_p]$ topological Hecke algebra $H_k(Np^\infty, \mathcal{O}) = \lim_{r \to \infty} H_k(Np^r, \mathcal{O})$; this algebra does not depend on the weight $k$. [5, (1.7)].

This result is proved in [8, Theorem 3.2] and [9, Theorem 2.3] in a more general situation, for the Hecke algebra acting on the space of automorphic forms with respect to groups arising from Eichler orders in a quaternion algebra $B$ on a totally real field, not ramified at finite places.

In this paper, by using the cohomological methods of [8], we shall prove the independence of weight of the Hecke algebra of level $p^\infty$ in the case where $B$ is a quaternion algebra indefinite over $\mathbb{Q}$ and $p$ is a prime dividing the discriminant of $B$.

More precisely, let $\Delta = \Delta'p$ be the discriminant of $B$ and let $R = R(N)$ be an Eichler order of level $N$ in $B$. Let $B^{\times, \infty}_A$ denote the finite part of the
adelization of $B$; we shall consider compact open subgroups of $B^{\times,\infty}_A$, called $(p,N)$-groups, having the form

$$U(N,U_p) = \prod_{l \neq p} R(N)^{\times}_{l} \times U_p$$

where $U_p$ is an open subgroup of $R_p^{\times}$. Let $S^B_k(U(N,U_p))$ be the complex space of automorphic forms of weight $k$ which are right invariant by $U(N,U_p)$. This space is endowed with a standard action of a Hecke ring $\mathcal{H}_k(U(N,U_p))$, generated over $\mathbb{Z}$ by operators $T_q$ for primes $q \neq p$ and $\langle q \rangle$ for primes $q \nmid N\Delta$. It should be noted that the Hecke operator $T_p$ at $p$ is missing.

For a ring $A$ we put $H_k(U(N,U_p),A) = \mathcal{H}_k(U(N,U_p)) \otimes_{\mathbb{Z}} A$. If $U'_p \subseteq U_p$ there are inclusions $S^B_k(U(N,U_p)) \rightarrow S^B_k(U(N,U'_p))$ and restriction maps

$$(1) \quad \mathcal{H}_k(U(N,U_p)) \rightarrow \mathcal{H}_k(U(N,U'_p))$$

give rise to a projective system. We use Matsumura-Shimura isomorphism to switch the action of $H_k(U(N,U_p))$ to the cohomology group $H^1(X(U(N,U_p)), L(n,C))$ where $X(U(N,U_p))$ is the Shimura curve associated to the group $U(N,U_p)$, and $L(n,C)$ is the standard locally constant sheaf associated to weight $k = n+2$. We show that this cohomology group has an integral structure preserved by Hecke action, so that for a suitably $p$-adic integer ring $\mathcal{O}$ the Hecke algebra $\mathcal{H}_k(U(N,U_p),\mathcal{O})$ faithfully acts on $H^1(X(U(N,U_p), L(n,\mathcal{O})).$ The action of diamond operators is compatible with the projective system (1), giving an action of the completed ring $\mathcal{O}[[\mathbb{Z}^\times_p]]$ over $\mathcal{H}_k(p^\infty,N;\mathcal{O}) = \lim_{\leftarrow U_p} H^1(\mathcal{H}_k(U(N,U_p),\mathcal{O}),$)

Our main result is Theorem 8.1, which states that the topological $\mathcal{O}[[\mathbb{Z}^\times_p]]$-algebra $\mathcal{H}_k(p^\infty,N;\mathcal{O})$ does not depend on the weight $k \geq 2$; more precisely we show that there is an homomorphism of $\mathcal{O}[[\mathbb{Z}^\times_p]]$-algebras

$$\iota^* : \mathcal{H}_2(p^\infty,N;\mathcal{O}) \rightarrow \mathcal{H}_k(p^\infty,N;\mathcal{O})$$

sending $T_q$ to $T_q$ for every prime $q \neq p$.

The proof is an adaptation of the Hida’s cohomological proof given in [7]. The main idea, explained in Lemma 7.1, consists roughly in saying that for large enough level any torsion coefficient system becomes trivial. The technical difficulties in applying this method are related to the realization of the Hecke algebra on cohomology groups with divisible coefficients; these problems are addressed in the last sections of the paper.

This work takes place in a general program on quaternionic Hecke algebras, which has been sketched in [18]. The idea is to exploit algebraic and geometric properties of quaternion algebras and Shimura curves in order to recover relevant informations about adelic representations of fixed type at a prime $p$ and Galois representations associated to quaternionic automorphic eigenforms. It is well-known that representations of $B^{\times}_A$ correspond by Jacquet-Langland to representations of $GL_2$ which are special or supercuspidal at primes dividing the discriminant $\Delta$. [11]. A realization of Jacquet-Langlands correspondence as a Hecke-equivariant map from quaternionic automorphic forms and classical forms is performed in [14] in the case of Eichler orders and in [18] in the case of arbitrary level at a prime $p$ dividing...
\(\Delta\). However in this last case the absence of new vectors on the quaternionic side prevents to define the operator \(T_p\) on the space of quaternionic forms. This poses some limitations to the theory of the quaternionic Hecke algebras; for example the space of quaternionic forms and the Hecke algebra are not in general dual each other (not even on \(C\)).

Let us denote by \(U_s\) the \((p, N)\)-group \(U(N, 1 + p^sR_p)\) and by \(U(M, p^s)\) the compact open subgroup of \(GL_2(A)\) defined by \(U(M, p^s) = U_0(M) \cap U_1(p^s)\); as it is shown in [18, Théorème 8.0.11] the Hecke algebra \(\mathcal{H}_k(U_s, O)\) can be canonically regarded as a quotient of the algebra \(\mathcal{H}_k(U(N\Delta', p^{2s+1}), O)\), where the latter is the Hecke algebra generated by operators \(T_q\) for \(q \neq p\) and diamond operators \(\langle q \rangle\) acting on “classical” modular forms: more precisely, \(\mathcal{H}_k(U_s, O)\) can be identified to the \(O\)-algebra obtained by restricting the Hecke operators in \(\mathcal{H}_k(U(N\Delta', p^{2s+1}), O)\) to the space \(W_k^{2s+1}\) spanned by eigenforms \(\varphi\) which generate a representation of \(GL_2(A)\) which is special at \(q|\Delta'\) and special or supercuspidal at \(p\).

We denote by \(\mathcal{H}_k^B(U(M, p^s), O)\) the classical Hecke algebras (including the \(T_p\)-operator), as defined for example in [18, Section 9]. By Hida’s “classical” weight independence, for every integer \(M\) prime to \(p\) there is an isomorphism

\[
\lim_{\leftarrow s} \mathcal{H}_k^B(U(M, p^s), O) \xrightarrow{\sim} \lim_{\leftarrow s} \mathcal{H}_2(U(M, p^s), O)
\]

sending \(T_q\) to \(T_q\) for \(q \in N\) and \(\langle q \rangle\) to \(\langle q \rangle\) for \(q\) prime to \(Mp\). Of course this isomorphism restricts to an isomorphism of subalgebras “deprived of the \(T_p\) operator”

\[
\lim_{\leftarrow s} \mathcal{H}_k(U(M, p^s), O) \xrightarrow{\sim} \lim_{\leftarrow s} \mathcal{H}_2(U(M, p^s), O).
\]

Then Theorem 8.1 implies that there is a commutative diagram

\[
\begin{array}{ccc}
\lim_{\leftarrow s} \mathcal{H}_k(U(N\Delta', p^s), O) & \rightarrow & \lim_{\leftarrow s} \mathcal{H}_2(U(N\Delta', p^s), O) \\
\downarrow \pi_k & & \downarrow \pi_2 \\
\lim_{\leftarrow s} \mathcal{H}_k(U_s, O) & \sim & \lim_{\leftarrow s} \mathcal{H}_2(U_s, O)
\end{array}
\]

where the vertical arrows are surjective and continuous.

Since for every \(k\) the space \(\mathcal{H}_k(U(N\Delta', p^s), O)\) is compact and the space \(\mathcal{H}_k(p^\infty, N; O)\) is Hausdorff, the latter has the quotient topology with respect to the surjection \(\pi_k\).

Notice that by multiplicity one (see [4, Theorem 5.14]) the spaces \(W_k^s\) are also stable by the \(T_p\) operator.

Then we could ask whether the isomorphism of Theorem 8.1 could extend to an isomorphism of complete algebras

\[
\lim_{\leftarrow s} \mathcal{H}_2(U(N\Delta', p^s), O)|_{W_k^{2s+1}} \xrightarrow{\sim} \lim_{\leftarrow s} \mathcal{H}_k(U(N\Delta', p^s), O)|_{W_k^{2s+1}}.
\]

This would mean that the “classical” isomorphism (2) induces an isomorphism on the restrictions of Hecke algebras to modular forms associated to a representation of \(GL_2\) arising by Jacquet-Langlands correspondence from a form over \(D^0_A\) which is invariant by some \((p, N)\)-group (having fixed a maximal order in \(B\)). As in [6, §2] the existence of such an isomorphism
would be equivalent by duality to the weight independence of the Hecke module $\bar{W}_k$ obtained by completing $\lim \to W_k^s$ with respect to the topology of the uniform convergence in $O[[q]]$. However, this would contradict the fact that the eigenvalues for the $T_p$ operator associated to newforms in $W_k$ have $p$-adic valuation $\geq k/2 - 1$, so that the operator $T_p$ is topologically nilpotent over $\bar{W}_k$; and, on the other hand, special forms of weight 2 are ordinary.

The obstruction seems to arise from special forms. A possible way to avoid them could perhaps be to analyze more deeply the possible Hecke operators at $p$ in the quaternionic setup. This consideration is inspired by the fact that, as remarked for example in [2], there is an action of the whole $G(\mathbb{Q}_p)$ on the $p$-adically completed cohomology of a reductive group $G$. Although an operator $T_p$ corresponding to the classical one is not available, there is an action of the unramified $p$-adic integer ring $\mathbb{Z}_p^\times$ on quaternionic automorphic forms, as observed in Remark 5.1. This action commutes with Hecke operators outside $p$, and it is possible that it could help to isolate “supercuspidal components”in the Hecke algebra $\mathcal{H}_k(p^\infty, N, O)$. Since the $T_p$ operator is zero on supercuspidal representation, in this situation the two restricted algebras $\mathcal{H}$ and $\mathcal{H}^\sharp$ would coincide. A step in this direction has been obtained in weight 2 and fixed level in [19, 3], where a Taylor-Wiles system was realized using the cohomology of Shimura curves, in order to show that a suitable local component of a quaternionic Hecke algebra was a universal deformation ring. This result suggest that the action of $\mathbb{Z}_p^\times$ on the direct limit of cohomology groups of Shimura curves deserves a deeper investigation.

A very interesting perspective for future work would be the study of Galois representations over (local components of) the whole quaternionic Hecke algebra $\mathcal{H}_k(p^\infty, N, O)$; this is a challenging problem, because of the high ramification at $p$ of such representations.

Finally, we mention that the methods used in the present paper could probably work over totally real fields.

The structure of the paper is the following: in Section 2 we introduce the spaces of quaternionic automorphic forms, Hecke algebras, Shimura curves and cohomology groups; the action of Hecke algebra on cohomology in the local and global case is described in Section 3. Section 4 indicates some conditions on the level which assure that quaternionic groups are torsion-free; this is important in order to have locally constant sheaves over Shimura varieties. The Hecke algebra $\mathcal{H}_k(p^\infty, N, O)$ is defined in Section 5, and in Section 6 we show that it faithfully acts on the $p$-divisible cohomology of Shimura curves in weight $k$ and level $p^\infty$. Section 7 shows that, up to isogenies, this $p$-divisible Hecke module does not depend on $k$, which implies our main theorem, stated in Section 8. Sections 9 and 10 are devoted to technical proofs.

**Notations and conventions.** We shall denote $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ the natural, integer, rational, real and complex numbers respectively. If $A$ is a ring, $A^\times$ denotes the multiplicative group of invertible elements of $A$; $\mathbb{R}^{\times,+}$ is the subgroup of $\mathbb{R}^\times$ consisting of positive real numbers.
If \( p \) is a prime number, the ring of \( p \)-adic integers and the field of \( p \)-adic numbers are denoted respectively by \( \mathbb{Z}_p \) and \( \mathbb{Q}_p \). The symbol \( \mathbb{A} \) denotes the ring of rational adèles, \( \mathbb{A}^\infty \) the finite adèles, \( \mathbb{Z} = \prod_p \mathbb{Z}_p \). For every prime number \( p \) let \( v_p : \mathbb{Q}_p \to \mathbb{Z} \cup \{ \infty \} \) be the \( p \)-adic valuation. If \( x \in \mathbb{Q}_p \), \( |x|_p = p^{-v_p(x)} \); if \( x \in \mathbb{R} \), \( |x|_\infty = |x| \) is the absolute value. If \( a = (a_v) \in \mathbb{A}^\infty \), we shall denote by \( ||a|| \) the idelic norm of \( a \): \( ||a|| = \prod_v |a_v| \).

Let \( B \) be a quaternion algebra over \( \mathbb{Q} \) and \( R \) be an order in \( B \). For every prime \( q \) (included \( q = \infty \)), we put \( B_q = B \otimes \mathbb{Q}_q \), \( R_q = R \otimes \mathbb{Z}_q \); we shall denote \( B_\mathbb{A} \) the adelization of \( B \), \( B_\mathbb{A}^\times \) the topological group of the invertible elements of \( B_\mathbb{A} \), and \( B_\mathbb{A}^{*,\infty} \) the subgroup of finite idèles. If \( G \) is a multiplicative group in a quaternion algebra, we shall denote by \( G^{(1)} \) the subgroup consisting of element having reduced norm 1.

In the case where \( B = M_2(\mathbb{Q}) \) we put, for every integer \( N \)

\[
U_0(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}_2(\mathbb{Z}) \mid c \equiv 0 \text{ mod } N \right\}
\]

\[
U_1(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in U_0(N) \mid a \equiv 1 \text{ mod } N \right\}
\]

They are compact open subgroups of \( \text{GL}_2(\mathbb{A}) \).

If \( r = p^n \), \( \mathbb{F}_r \) denotes the field with \( r \) elements, \( \mathbb{Q}_r \) is the unramified extension of \( \mathbb{Q}_p \) having degree \( n \), \( \mathbb{Z}_r \) is its ring of \( p \)-adic integers.

If \( V \) is a \( \mathbb{Q} \)-vector space and a lattice \( \Lambda \subset V \) we put \( \Lambda_p = \Lambda \otimes \mathbb{Z}_p \) and \( V_p = V \otimes \mathbb{Q}_p \).

Let \( \text{GL}_2^+(\mathbb{R}) = \{ g \in \text{GL}_2(\mathbb{R}) \mid \det(g) > 0 \} \) and

\[
K_\infty = \mathbb{R}^\times \text{O}_2(\mathbb{R}), \quad K_\infty^+ = \mathbb{R}^\times \text{SO}_2(\mathbb{R}).
\]

The upper complex half-plane \( \mathcal{H} = \{ x + \sqrt{-1}y \in \mathbb{C} \mid y > 0 \} \) can be identified to

\[
\text{GL}_2(\mathbb{R})/K_\infty = \text{GL}_2^+(\mathbb{R})/K_\infty^+
\]

by

\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}_2(\mathbb{R}) \mapsto \left\{ \begin{array}{ll} \frac{a+\sqrt{-1}b}{c+\sqrt{-1}d} & \text{if } ad - bc > 0 \\ \frac{-a+\sqrt{-1}b}{-c+\sqrt{-1}d} & \text{if } ad - bc < 0 \end{array} \right.
\]

By this identification, the group \( \text{GL}_2^+(\mathbb{R}) \) acts on \( \mathcal{H} \) by homographies: if \( g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}_2^+(\mathbb{R}) \) and \( z \in \mathcal{H} \)

\[
g \cdot z = \frac{az + b}{cz + d}.
\]

Let \( j(g, z) = cz + d \).

If \( \Gamma \) is a discrete subgroup of \( \text{SL}_2(\mathbb{R}) \), we shall denote \( \overline{\Gamma} = \Gamma/\Gamma \cap \mathbb{R}^\times \) and \( \mathcal{H}_\Gamma^* = \mathcal{H} \cup \mathcal{P}_I \), where \( \mathcal{P}_I \) is the set of cusps of \( \Gamma \) (see [17, Chap. 1]).

2. Quaternionic Automorphic Forms, Hecke Algebras and Cohomology

We shall fix a quaternion algebra \( B \) over \( \mathbb{Q} \), indefinite, of discriminant \( \Delta \), and a maximal order \( R \) in \( B \). Let \( \nu \) and \( t \) be the reduced norm and trace of \( B \) respectively. Let \( p \) be a prime dividing \( \Delta \), and put \( \Delta = \Delta'p \).
The following lemma will be useful in order to let $B^\times$ act on the standard polynomial representations of $GL_2$:

**Lemma 2.1.** There exists a totally real Galois extension $K_0$ of $\mathbb{Q}$ satisfying the following conditions:

a) $K_0$ contains a quadratic extension $L_0$ in which $p$ is inert;

b) there exists an isomorphism of $K_0$-algebras $\phi_0 : B \otimes \mathbb{Q} K_0 \rightarrow M_2(K_0)$;

c) $\phi_0(R) \subseteq M_2(\mathcal{O}_0)$ where $\mathcal{O}_0$ is the ring of integers of $K_0$.

**Proof.** Let $K'_0$ be a real quadratic field in which $p$ is inert and splitting $B$. There is an isomorphism $\phi' : B \otimes \mathbb{Q} K'_0 \rightarrow M_2(K'_0)$; let $R'_0$ be a maximal order in $M_2(K'_0)$ containing $\phi'(R)$. Then $R'_0$ is conjugated by an element $\beta$ of GL$_2(K'_0)$ to an order of the type $R''_0 = \left( \begin{array}{cc} O'_0 & I' \\ I & O'_0 \end{array} \right)$ where $O'_0$ is the ring of integers of $K'_0$ and $I$ is an ideal of $O'_0$, see [20, Exercice 5.7]. Then we can assume that $\phi'_0(R) \subseteq R''_0$. Let $K_0$ be the Hilbert field of $K'_0$, and $\mathcal{O}_0$ be its ring of integers; then $K_0$ is a totally real extension of $\mathbb{Q}$ and every ideal of $K'_0$ become principal in $K_0$; we put $I\mathcal{O}_0 = (a)$ with $a \in \mathcal{O}_0$. Extending $\phi'$ by linearity we obtain an isomorphism $\phi'' : B \otimes \mathbb{Q} K_0 \rightarrow M_2(K_0)$ such that

$$\phi''(R) \subseteq \left( \begin{array}{cc} O_0 & a^{-1}O_0 \\ a\mathcal{O}_0 & \mathcal{O}_0 \end{array} \right) = \left( \begin{array}{cc} a^{-1} & 0 \\ 0 & 1 \end{array} \right) M_2(\mathcal{O}_0) \left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right).$$

The isomorphism $\phi_0$ we are looking for is the composition $\eta \circ \phi''$, where $\eta$ is the conjugation by the matrix $\left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right)$.

We define the isomorphism $i_\infty : B \otimes \mathbb{Q} \mathbb{R} \rightarrow M_2(\mathbb{R})$ as the isomorphism induced by $\mathbb{R}$-linearity by the isomorphism $\phi_0$ of Lemma 2.1.

For every finite prime $q$ not dividing $\Delta$, we fix an isomorphism $i_q : B_q \rightarrow M_2(\mathbb{Q}_q)$ such that $i_q(R_q) = M_2(\mathbb{Z}_q)$.

If $U$ is a compact open subgroup of $B^{\times, \infty}_A$, we put

$$\Gamma(U) = (GL_2^+(\mathbb{R}) \times U) \cap B^\times.$$  \hfill (3)

By the isomorphism $i_\infty$, the group $\Gamma(U)$ can be identified to a discrete subgroup of SL$_2(\mathbb{R})$ having finite covolume [20, page 104]. Since $\Delta \neq 1$ the quotient space $\Gamma(U) \backslash SL_2(\mathbb{R})$ is compact and $\Gamma(U) \backslash \mathcal{H} = \Gamma(U) \backslash SL_2(\mathbb{R}) / SO_2(\mathbb{R})$ is a compact Riemann surface. By [20, page 107], $\Gamma(U)$ contains a torsion-free subgroup $\Gamma$ of finite index. The group $\Gamma$ acts properly on $\mathcal{H}$, so it is the fundamental group of the compact surface $\Gamma \backslash \mathcal{H}$. This shows that the groups $\Gamma(U)$ are finitely generated.

**Proposition 2.2.** Let $U$ be a compact open subgroup of $B^{\times, \infty}_A$, let $q$ be a finite prime and let $K_q = U \cap B^\times_q$. Then $\Gamma(U)$ is dense in $K^{(1)}_q$ (with the $q$-adic topology).

**Proof.** Let $x \in K^{(1)}_q$, and consider $x$ as an element of $B^{(1)}_A$. Since $B$ is indefinite, by the strong approximation theorem (see [20, page 81]) $x$ can be approximated by elements $x_n = a_n b_n$, where $a_n \in B^{(1)}$, $b_n \in B^{(1)}_\infty$. Therefore
the sequence of $a_n$ tends to $x$ in the $q$-adic topology. Since $U$ is open, $a_n \in U$ for $n \gg 0$, so that $a_n \in \Gamma(U)$. ■

If $U$ is a compact open subgroup of $B^{x,\infty}_A$, let $S_k^B(U)$ denote the complex space of adelic automorphic forms which are right invariant by $U$. We refer to [18, §2] for the definition and the basic properties of this space. If $N$ is an integer prime to $\Delta$, let $R(N)$ be the Eichler order of level $N$ locally defined by:

$$R(N)_l = \begin{cases} R_l & \text{if } l \nmid N \\ \{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}_l \} & \text{if } l \mid N \end{cases}$$

In what follows we shall consider compact open subgroups of $B^{x,\infty}_A$ having the form

$$U = U(N, U_p) = \prod_{l \neq p} R(N)_l^\times \times U_p$$

where $U_p$ is an open subgroup of $R^\times_p$. Such a group will be called a $(p, N)$-group. In [18, §3] a Hecke abstract ring $R(N)$ is defined, acting on $S_k^B(U)$ for every $(p, N)$-group $U$. It is generated by the operators $\tilde{T}(q)$, for prime $q \neq p$ and $T(q, q)$ for prime $q \nmid N \Delta$. For a subring $A$ of $C$ the Hecke algebra $H_k(U, A)$ is defined as the $A$-subalgebra of $\text{End}_C(S_k^B(U))$ generated over by the image of $R(N)$. We denote by $T_q(q)$ the images in $H_k(U, \mathbb{Z})$ of $\tilde{T}(q), \tilde{T}(q, q)$ respectively. For every positive integer $a$ prime to $N \Delta$, write $a = \prod_i q_i^{e_i}$; then we define $(a) = \prod_i q_i^{e_i} \in H_k(U, \mathbb{Z})$.

By strong approximation, if we choose $t_1, \ldots, t_h \in B^{x,\infty}_A$ such that $\nu(t_1), \ldots, \nu(t_h)$ are representatives of $Q^\times \setminus A^\times / R^\times + \nu(U)$ then $B^{x}_A = \prod_{i=1}^h B^+_Q \times \text{GL}_2^+(R) t_i U$. If we fix such a decomposition, we can define for $i = 1, \ldots, h$,

$$\Gamma^i(U) = \Gamma(t_i U t_i^{-1})$$

The complex analytic compact spaces

$$X_i(U) = \Gamma^i(U) \setminus \mathcal{H}$$

are varieties if $\overline{\Gamma(U)}$ has no torsion. We also put

$$X(U) = B^x_Q \setminus B^x_A / \mathcal{K}_\infty^+ U.$$

It is well-known (see for example [8, Proposition 6.1 (i)]) that the map $B^x_A \rightarrow \prod_{i=1}^h \mathcal{H}$ sending $g_Q g_\infty t_j u$ in the element $g_\infty(\sqrt{-1})$ of the $j$-th component induces an isomorphism

$$X(U) \simeq \prod_{i=1}^h X_i(U).$$

If $A$ is a ring, let $L(n, A) = \text{Symm}^n(A^2)$ with an action of $M_2(A)$ gotten from the standard one in $A^2$. It can be represented as the $A$-submodule of $A[X, Y]$ consisting of homogeneous polynomials of degree $n$; if $P(X, Y) \in L(n, A)$ and $\gamma \in M_2(A)$ then

$$P|_\gamma(X, Y) = P((X, Y)^t \gamma).$$
If \( M \) is a \( A \)-module, we define \( L(n, M) = L(n, A) \otimes_A M \), with the action of \( \text{M}_2(A) \) on the first factor. In this way \( L(n, M) \) becomes a right module over the multiplicative semigroup \( \text{M}_2(A) \). We denote \( ^tL(n, M) \) the left \( \text{GL}_2(A) \)-module which is \( L(n, M) \) as \( A \)-module, with the action of \( \text{GL}_2(A) \) given by \( \gamma \cdot P = P|_\gamma \).

We follow [8, Section 6] and [10, Chapter 6] in the description of some sheaves on the variety \( X(U) \) arising from a right module over \( R_p^\times \) or \( \text{GL}_2^+(\mathbb{R}) \). We shall refer to the first case as case \( p \) and to the second as case \( \infty \).

Let \( K \) be a finite extension of \( \mathbb{Q}_p \) in \( \bar{\mathbb{Q}}_p \) containing the field \( K_0 \) of Lemma 2.1 and let \( O \) be the ring of \( p \)-adic integers of \( K \). Since \( K_0 \) contains a quadratic extension in which \( p \) is inert, \( K \) contains \( \mathbb{Q}_p^2 \). The homomorphism \( \phi_0 \) of Lemma 2.1 induces an isomorphism between \( R \otimes_{\mathbb{Z}} \mathbb{Z}_p \) and the \( p \)-adic closure of \( R \) in \( \text{M}_2(O) \); then we can identify \( R_p^\times \) to a subgroup of \( \text{GL}_2(O) \); in this way \( R_p^\times \) acts in a natural way over \( L(n, O) \).

In the case \( \infty \) we assume that \( M \) is a real vector space, in the case \( p \) that it is an \( O \)-module. We define a left action of \( B^\times_\mathbb{Q} \) over \( B^\times_A \times L(n, M) \) by \( \alpha(g, v) = (\alpha g, v) \) and a right action of \( K_\infty^\times U \) by

\[
(g, v)u = \begin{cases} 
(gu, vu_\infty) & \text{in case } \infty \\
(gu, vu_p) & \text{in case } p.
\end{cases}
\]

Giving \( L(n, M) \) the discrete topology we can consider the covering of \( X(U) \)

\[
\mathcal{L}(n, M)/X(U) = B^\times_\mathbb{Q} \backslash B^\times_A \times L(n, M)/K_\infty^\times U.
\]

The groups \( \Gamma^i(U) \) act on \( ^tL(n, M) \) via the natural inclusion \( \Gamma^i(U) \hookrightarrow \text{GL}_2^+(\mathbb{R}) \) in case \( \infty \) and \( \Gamma^i(U) \hookrightarrow R_p^\times \) in case \( p \). Then we can consider the covering of \( X_i(U) \)

\[
\mathcal{L}(n, M)/X_i(U) = \Gamma^i(U) \backslash \mathcal{H} \times ^t L(n, M)
\]

where \( \Gamma^i(U) \) acts diagonally at the left over \( \mathcal{H} \times ^t L(n, M) \). This covering is unramified if \( \Gamma^i(U) \) has no torsion.

The map \( B^\times_A \times L(n, M) \to \prod_{i=1}^h \mathcal{H} \times ^t L(n, M) \) defined by

\[
(gQg_\infty t_iu, m) \mapsto \begin{cases} 
(g_\infty(\sqrt{-1}), g_\infty m)_i & \text{in the case } \infty \\
(g_\infty(\sqrt{-1}), t_iu_p m)_i & \text{in the case } p
\end{cases}
\]

induces an isomorphism of coverings

\[
(6) \quad \mathcal{L}(n, M) \simeq \prod_{i=1}^h \mathcal{L}_i(n, M)
\]

which is compatible with the isomorphism (5), see [8, Proposition 6.1 (ii)]. If \( \Gamma(U) \) has no torsion, then \( \mathcal{L}_i(n, M) \) is locally isomorphic to the variety \( X_i(U) \) and we can consider the sheaf of continuous sections of \( \mathcal{L}_i(n, M) \) over \( X_i(U) \); we shall denote it with the same symbol \( \mathcal{L}_i(n, M)/X_i(U) \). There is also a sheaf \( \mathcal{L}(n, M)/X(U) \).

We need to compare the cohomology of sheaves defined in case \( \infty \) and in case \( p \) by means of a common \( K_0 \)-structure. We define a sheaf \( \mathcal{L}(n, A) \) for a global \( \mathcal{O}_0 \)-algebra \( A \). Let \( A \) be a ring such that \( \mathcal{O}_0 \subseteq A \subseteq C \). For \( i = 1, ..., h \) the group \( \Gamma^i(U) \) is contained in \( R^\times \), so \( \Gamma^i(U) \) acts on the right on \( L(n, A) \). We shall denote \( L_i(n, A) \) the right \( \Gamma^i(U) \)-module \( L(n, A) \) and \( ^tL_i(n, A) \) the
corresponding left $\Gamma_i(U)$-module. If $\Gamma_i(U)$ has no torsion, we can consider the sheaves
\[ L_i(n, A)_{/X_i(U)} = \Gamma_i(U) \backslash H \times^t L_i(n, A) \]
\[ L(n, A)_{/X(U)} = \prod_{i=1}^h L_i(n, A). \]

We shall refer to the latter situation as “global case”, and to cases $\infty$ and $p$ as “local cases”.

Isomorphisms (5) and (6) assure that if $A$ is local or global, there is a fonctorial isomorphism
\[ H^*(X(U), L(n, A)) \simeq \bigoplus_{i=1}^h H^*(X_i(U), L_i(n, A)). \]

Moreover, if $\Gamma_i(U)$ has no torsion
\[ H^*(X_i(U), L_i(n, A)) \simeq H^*(\Gamma_i(U), \check L_i(n, A)), \]
(cf. for example [16, 1.5] and [10, Appendix, Th.1 and Cor. 1]. For every $O_0$-algebra $D$ we define the sheaf $L(n, D)$ over $X(U)$ by
\[ L(n, D)_{/X(U)} = L(n, O_0) \otimes O_0 D. \]

**Remark 2.3.** If $D$ is a $O$-algebra, then the action of $\Gamma(U)$ on $L(n, D)$ via its inclusion in $GL_2(O_0)$ coincides with the action of $\Gamma(U)$ via its inclusion in $R_p^\times \subseteq GL_2(O)$, by Lemma 2.1. Therefore the sheaf $L(n, O_0) \otimes O_0 D_{/X(U)}$ is isomorphic to the sheaf $L(n, D)_{/X(U)}$ obtained in case $p$. In particular there are injective homomorphisms
\[ L(n, O_0)_{/X(U)} \hookrightarrow L(n, C)_{/X(U)} \]
\[ L(n, O_0)_{/X(U)} \hookrightarrow L(n, O)_{/X(U)} \]

The following theorem is well-known [13, §4]

**Theorem 2.4 (Matsushima-Shimura).** There is a canonical isomorphism
\[ S_k^R(U) \oplus \overline{S_k^R(U)} \simeq H^1(X(U), L(n, C)). \]

**3. Hecke operators on cohomology groups**

In [8, Section 7] Hida defines the action of double cosets $U x U$, where $x \in B_{A, \infty}^+$ over the cohomology groups
\[ H^*(X(U), L(n, M)) \bigoplus_{i=0}^h H^*(X_i(U), L_i(n, M)) \bigoplus_{i=0}^h H^*(\Gamma_i(U), \check L_i(n, M)) \]
in local and global cases. We briefly recall the construction.
3.1. Hecke operators over $H^*(X(U), L(n, M))$ in the local case. Let $U, U'$ be compact open subgroups of $B^\times_{A, \infty}$, and $M$ be an $R$-vector space in case $\infty$ or an $O$-module in case $p$. We put $\sigma = \infty$ or $p$. If $x \in B^\times_{A, \infty}$ the map

$$[x] : B^\times_A \times L(n, M) \to B^\times_A \times L(n, M)$$

$(g, m) \mapsto (gx, m \cdot x_\sigma)$ induce an homomorphism

$$B^\times_Q \backslash B^\times_A \times L(n, M) / K^+ V \to B^\times_Q \backslash B^\times_A \times L(n, M) / K^+ V^x$$

where $V = U \cap xU'x^{-1}$ and $V^x = x^{-1}Vx$; therefore there is an homomorphism of sheaves

$$[x] : \mathcal{L}(n, M)_{/X(V)} \to \mathcal{L}(n, M)_{/X(V^x)}$$

which induces a map in cohomology

$$[x] : H^*(X(V), \mathcal{L}(n, M)) \to H^*(X(V^x), \mathcal{L}(n, M)).$$

We define the operator

$$[UxU'] : H^*(X(U), \mathcal{L}(n, M)) \to H^*(X(U'), \mathcal{L}(n, M))$$

by

$$[UxU'] = \text{Tr}_{U'/V^x} \circ [x] \circ \text{res}_{U/V}$$

where $\text{res}_{U/V} : H^*(X(U), \mathcal{L}(n, M)) \to H^*(X(V), \mathcal{L}(n, M))$ is the restriction map, and $\text{Tr}_{U'/V^x} : H^*(X(V^x), \mathcal{L}(n, M)) \to H^*(X(U'), \mathcal{L}(n, M))$ is the trace map [8, Section 7]. It is easy to see that the operator $[UxU']$ depends only on the double coset $UxU'$.

3.2. Hecke operators over $\bigoplus_{j=1}^h H^*(X_j(U), L_i(n, M))$ in local and global cases. For sake of simplicity we shall assume that $U = U'$ is a $(p, N)$-group and that $x \in B^\times_{A, \infty}$ is such that $x_p = 1$. Then $(xUx^{-1})_p = U_p$, so that

$$B^\times_A = \bigcap_{j=1}^h B^\times_Q \backslash \text{GL}_2^+ (R) t_j U = \bigcap_{j=1}^h B^\times_Q \backslash \text{GL}_2^+ (R) t_j V = \bigcap_{j=1}^h B^\times_Q \backslash \text{GL}_2 (R) t_j V^x.$$  

By strong approximation for every $j$ we can write $t_j x = \gamma_j t_i (x^{-1}ux)$ with $u \in V$ and $\gamma_j \in B^\times_Q \ (j \mapsto i$ is a permutation of $\{1, \ldots, h\})$. There is a map

$$L_j(n, M) \to L_i(n, M)$$

$m \mapsto \gamma_j^{-1}m$

and $\Gamma^j(V^x) = \gamma_j^{-1} \Gamma^j(V) \gamma_j$. Therefore for every $j$ there is a sheaves homomorphism

$$[x]_j : \mathcal{L}_j(n, M)_{/X_j(V)} \to \mathcal{L}_i(n, M)_{/X_j(V^x)}$$

which induces maps in cohomology

$$[x]_j : H^*(X_j(V), \mathcal{L}_j(n, M)) \to H^*(X_i(V^x), \mathcal{L}_i(n, M))$$

$$[x]_j : H^*(\overline{\Gamma^j(V)}, L^i n, M)) \to H^*(\overline{\Gamma^j(V^x)}, L^i(n, M)).$$
Then \([x]\) corresponds to \(\bigoplus_{j=1}^{h}[x]_{j}\) via the isomorphism (7). We define operators

\[
[UxU] : \bigoplus_{j=1}^{h} H^{\ast}(X_{j}(U), L_{j}(n, M)) \longrightarrow \bigoplus_{j=1}^{h} H^{\ast}(X_{j}(U), L_{j}(m, M))
\]

\[
[UxU] : \bigoplus_{j=1}^{h} H^{\ast}(\overline{U}(U), \mathcal{L}_{j}(n, M)) \longrightarrow \bigoplus_{j=1}^{h} H^{\ast}(\overline{U}(U), \mathcal{L}_{j}(n, M))
\]

by

\[
[UxU] = \text{Tr}_{U/V^{\ast}} \circ \bigoplus_{j=1}^{h}[x]_{j} \circ \text{res}_{U/V}
\]

where \(\text{res}_{U/V} = \bigoplus_{j=1}^{h}\text{res}_{X_{j}(U)/X_{j}(V)}\) and \(\text{Tr}_{U/V^{\ast}} = \bigoplus_{j=1}^{h}\text{Tr}_{X_{j}(U)/X_{j}(V^{\ast})}\) (and analogously in group cohomology). When \(A\) is local, these operators correspond, by isomorphism (7), to the operator \([UxU]\) defined in the local case.

### 3.3. Action of the Hecke algebra on cohomology.

Let \(U\) be a \((p, N)\)-group. The isomorphism of Theorem 2.4 is compatible by the action of the algebra \(R(N)\). Consequently, if \(A\) is a subring of \(C\), the algebra \(\mathcal{H}_{k}(U, A)\) is isomorphic to the subalgebra of \(\text{End}_{C}(H^{1}(X(U), L(n, C)))\) generated over \(A\) by the images of operators in \(R(N)\).

By the universal coefficient theorem (see for example [1, II, Theorem 15.3] if \(A\) is a principal ring contained in \(C\) and \(B\) is a flat \(A\)-algebra then there is a Hecke-equivariant canonical isomorphism

\[
H^{1}(X(U), L(n, A)) \otimes_{A} B \simeq H^{1}(X(U), L(n, B)).
\]

In particular

\[
H^{1}(X(U), L(n, K_{0})) \otimes_{K_{0}} C \simeq H^{1}(X(U), L(n, C))
\]

(9)

\[
H^{1}(X(U), L(n, K_{0})) \otimes_{K_{0}} K \simeq H^{1}(X(U), L(n, K))
\]

(10)

By (9) the algebra \(\mathcal{H}_{k}(U, \mathcal{O}_{0})\) faithfully acts on \(H^{1}(X(U), L(n, K_{0}))\). Let \(\mathcal{O}'_{0} = \mathcal{O} \cap K_{0}\); then \(\mathcal{O}'_{0}\) is a discrete valuation ring and \(K_{0} \otimes_{\mathcal{O}'_{0}} \mathcal{O} = K\). Then we can consider \(\mathcal{H}_{k}(U, \mathcal{O}) = \mathcal{H}_{k}(U, \mathcal{O}'_{0}) \otimes_{\mathcal{O}'_{0}} \mathcal{O}\) as a subalgebra of \(\text{End}_{K}(H^{1}(X(U), L(n, K)))\), by (10), and obtain the following:

**Proposition 3.1.** Let \(U\) be a \((p, N)\)-group. The Hecke algebra \(\mathcal{H}_{k}(U, \mathcal{O})\) acts faithfully on \(H^{1}(X(U), L(n, K))\).

### 4. Torsion freeness conditions for quaternionic groups

Let \(B, \Delta, R\) be as in Section 2, with \(\Delta = \Delta'p\). Let \(R(N)\) be an Eichler order of level \(N\) in \(R\). For every positive integer \(s\) let \(V_{s}(N)\) be the \((p, N)\)-group whose component at \(p\) is \(U_{p} = 1 + u_{p}^{s}R_{p}\), where \(u_{p}\) is a uniformizer of \(B_{p}^{\times}\).

In order to have locally constant sheaves over Shimura varieties we have to deal with adelic groups \(U\) such that \(\Gamma(U)\) has no torsion. The following proposition gives some conditions over \(N, \Delta, U\) which assure this property. Analogous conditions for modular groups are stated for example in [10, §6.1].
As in the remaining of this paper, \( p \) is an odd prime; we put \( U_{\text{max}} = \prod_i R_i^X \).

**Proposition 4.1.**

a) For every positive integer \( N \), if \( s \geq 1 \) then 
\[ \Gamma(V_s(N)) = \Gamma(V_s(N)) \] does not contain elements of finite order.

b) If there exist prime numbers \( q, r \) dividing \( \Delta \) such that \( q \equiv 1 \mod 3 \) and \( r \equiv 1 \mod 4 \) then \( \Gamma(U_{\text{max}}) \) does not contain elements of finite order.

c) Suppose that either 36 divides \( N \) or there exist two prime numbers \( q, r \) dividing \( N \) such that \( q \equiv 3 \mod 4 \) and \( r \equiv 2 \mod 3 \). Then 
\[ \Gamma(U(N, R_p^X)) \] does not contain elements of finite order.

**Proof.** If \( U \) is a compact open subgroup of \( B_{\mathbb{A}}^{X, \infty} \) then the group \( \Gamma(U) \) is a discrete subgroup of \( \text{SL}_2(\mathbb{R}) \) whose elements have integer trace. By [17, Chap. I], an element \( \alpha \neq \pm 1 \) of \( \Gamma(U) \) has finite order if, and only if, \( |t(\alpha)|_{\infty} \leq 1 \). The possible characteristic polynomials for such an element are \( X^2 + 1, X^2 + X + 1, X^2 - X + 1 \). The roots of \( X^2 + 1 \) have order 2 in \( \Gamma(U) \), those of \( X^2 \pm X + 1 \) have order 3. If \( \alpha \in V_s(N) \) then \( t(\alpha) \equiv 2 \mod p \); this proves a), since we are assuming \( p \neq 2 \). An element of order 2 exists in \( \Gamma(U_{\text{max}}) \) if and only if \( \mathbb{Q}(\sqrt{-1}) \) splits \( B \), and the primes that split in \( \mathbb{Q}(\sqrt{-1}) \) are exactly those congruent to 1 mod 4. Analogously, an element of order 3 exists in \( \Gamma(U_{\text{max}}) \) if and only if \( \mathbb{Q}(\sqrt{-3}) \) splits \( B \), and the primes that split in \( \mathbb{Q}(\sqrt{-3}) \) are exactly those congruent to 1 mod 3. Assertion b) follows. Let \( \alpha \in \Gamma(U(N, R_p^X)) \). For every \( q \) dividing \( N \), \( i_q(\alpha) = \begin{pmatrix} a_q & b_q \\ Nc_q & d_q \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_q) \).

If \( \alpha \) has order 2 in \( \Gamma(U(N, R_p^X)) \) then \( t(\alpha) = 0 \), thus \( a_q^2 + 1 \in \mathbb{N}\mathbb{Z}_q \) for every \( q \) dividing \( N \), so that either \( q = 2 \) and 4 does not divide \( N \), or \( q \equiv 1 \mod 4 \). Analogously, if \( \alpha \) has order 3 in \( \Gamma(U(N, R_p^X)) \) then \( t(\alpha) = \pm 1 \), thus \( a_q^2 \pm a_q + 1 \in \mathbb{N}\mathbb{Z}_q \) for every \( q \) dividing \( N \), so that either \( q = 3 \) and 9 does not divide \( N \), or \( q \equiv 1 \mod 3 \). \( \blacksquare \)

5. The Hecke Algebra of Level \( p^\infty \)

If \( U \) and \( V \) are \((p, N)\)-groups, and \( U \subseteq V \), there is a natural inclusion
\[ H^1(X(V), \mathcal{L}(n, K)) \hookrightarrow H^1(X(U), \mathcal{L}(n, K)) \]
which is invariant for the operators \( T_q, T_{q,q} \). Correspondingly, the restriction of operators in \( \mathcal{H}_k(U, \mathcal{O}) \) to the image of \( H^1(X(V), \mathcal{L}(n, K)) \) gives a surjective homomorphism
\[ \mathcal{H}_k(U, \mathcal{O}) \twoheadrightarrow \mathcal{H}_k(V, \mathcal{O}) \]
mapping \( T_q \) on \( T_q \) and \( T_{q,q} \) on \( T_{q,q} \). Then we can consider the injective limit
\[ \lim_{\substack{U_p \subseteq R_p^\infty \\ U_p \subseteq R_p^\infty}} H^1(U(N, U_p), \mathcal{L}(n, K)) \]
which is endowed with the faithful action of the algebra
\[ \mathcal{H}_k(p^\infty, N; \mathcal{O}) = \lim_{\substack{U_p \subseteq R_p^\infty \\ U_p \subseteq R_p^\infty}} \mathcal{H}_k(U(N, U_p), \mathcal{O}). \]
Since every \((p,N)\)-group contains a subgroup of the type \(V_s(N)\) for some \(s\), the algebra \(\mathcal{H}_k(p^\infty,N;\mathcal{O})\) is also equal to \(\varprojlim_s \mathcal{H}_k(V_s(N),\mathcal{O})\).

If \(U\) is a \((p,N)\)-group, we give to \(\mathcal{H}_k(U,\mathcal{O})\) the \(p\)-adic topology, and to \(\mathcal{H}_k(p^\infty,N;\mathcal{O})\) the topology arising from the projective limit: namely, the weakest topology for which the projections

\[
\mathcal{H}_k(p^\infty,N;\mathcal{O}) \to \mathcal{H}_k(V_s(N),\mathcal{O})
\]

are continuous. Then the algebra \(\mathcal{H}_k(p^\infty,N;\mathcal{O})\) becomes a topological compact algebra.

For every integer \(s\) we put

\[
\begin{align*}
U_s &= V_{2s}(N) = U(N,1+p^s R_p), \\
\Phi_s &= \Gamma(U_s), \\
X_s &= X(U_s), \\
X_s &= X(U_s) = \Phi_s \backslash \mathcal{H}.
\end{align*}
\]

We want to give the Hecke algebra \(\mathcal{H}_k(p^\infty,N;\mathcal{O})\) a structure of continue topological algebra over \(\mathcal{O}[[Z_p^\times]]\).

It is easily verify that if \(a\) is prime to \(N\Delta\) and \(a \equiv 1 \mod p^s\) then \(\langle a \rangle\) trivially acts on \(S_k^B(U_s)\); moreover for every integer \(b\) prime to \(p\) there exists an element \(a\) prime to \(N\Delta\) such that \(a \equiv b \mod p^s\). Therefore the action of diamond operators induce an action of the finite group \((\mathbb{Z}/p^s\mathbb{Z})^\times\) on \(S_k^B(U_s)\) by \(\varphi \mapsto \langle a \rangle \varphi\). There is a commutative diagram

\[
\begin{array}{ccc}
(Z/p^s\mathbb{Z})^\times & \overset{\langle \rangle}{\longrightarrow} & \mathcal{H}_k(U_s,\mathcal{O}) \\
\uparrow & & \uparrow_{\text{res}} \\
(Z/p^{s+1}\mathbb{Z})^\times & \overset{\langle \rangle}{\longrightarrow} & \mathcal{H}_k(U_{s+1},\mathcal{O})
\end{array}
\]

Therefore the map \(a \mapsto \langle \rangle\) gives a continue homomorphism \(Z_p^\times \to \mathcal{H}_k(p^\infty,N,\mathcal{O})\), which can be extended by linearity and continuity to obtain a continue homomorphism of algebras

\[
\langle \rangle : \mathcal{O}[[Z_p^\times]] \longrightarrow \mathcal{H}_k(p^\infty,N,\mathcal{O}),
\]

where we put \(\mathcal{O}[[Z_p^\times]] = \varprojlim_s \mathcal{O}[[Z/p^s\mathbb{Z})^\times]]\). In this way we give \(\mathcal{H}_k(p^\infty,N,\mathcal{O})\) the structure of \(\mathcal{O}[[Z_p^\times]]\)-algebra.

**Remark 5.1.** If we fix an embedding of \(Z_p^\times\) in \(R_p^\times\) we also have an action of \(Z_p^\times\) over \(\varinjlim U_p \to \mathcal{H}_k^\times(U(N,U_p))\). We decide in this work to not include these operators in the Hecke algebra, because we are primarily interested in the the action of the classical Hecke algebra over the forms which are special or supercuspidal at \(p\). However, as discussed in the Introduction, the \(Z_p^\times\)-action seems to be of interest in order to isolate in the cohomology of Shimura curves some Galois representations the having a fixed type at \(p\).

6. **The action of Hecke operators over some \(p\)-divisible modules**

The subgroup \(U_s\) is normal in \(U(N,R_p^\times)\) and \(\nu(U_s) = \prod_{q \neq p} Z_q^\times \times (1+p^s Z_p)\). For every positive integer \(s\) and every \(i \in (\mathbb{Z}/p^s\mathbb{Z})^\times\) let us choose a
representative $t^s_j \in R^\times_p$ such that $\nu(t^s_j) \equiv i \mod p^s$ and such that
\begin{equation}
\text{if } j \in (\mathbb{Z}/p^{s+1}\mathbb{Z})^\times \text{ and } j \equiv i \mod p^s \text{ then } t^{s+1}_j \equiv t^s_i \mod p^s R_p.
\end{equation}
Such a choice is possible: having chosen the representatives at level $s$, for every $j \in (\mathbb{Z}/p^{s+1}\mathbb{Z})^\times$ such that $j \equiv i \mod p^s$ we choose an $u \in R^\times_p$ such that $\nu(u) \equiv j \mod p^{s+1}$. Then $\nu(u(t^s_i)^{-1}) \in 1 + p^s \mathbb{Z}_p$. We choose $\alpha \in 1 + p^s R_p$ such that $\nu(\alpha) = \nu(u(t^s_i)^{-1})$ and put $t^{s+1}_j = t^s_i \alpha$. Then $t^{s+1}_j \equiv t^s_i \mod p^s R_p$ and $\nu(t^{s+1}_j) = \nu(u) \equiv j \mod p^{s+1}$.

**Remark 6.1.** a) Since $U_s$ is normal in $U(N, R^p_p)$, the groups $\Gamma^i(U_s)$ defined in 4 are all equal to $\Phi_s$. Consequently, isomorphisms (7) and (8) establish an isomorphism between $H^1(X_s, \mathcal{L}(n, M))$ and the direct sum of $\varphi(p^i)$ copies of $H^1(\Phi_s, ^1 L(n, M))$ for every $\mathcal{O}$-module $M$.

b) Having chosen the representatives $t^s_i$ as in 11, if $r \geq s$ the restriction map
\[ H^1(X_s, \mathcal{L}(n, M)) \rightarrow H^1(X_r, \mathcal{L}(n, M)) \]
corresponds by (7) and (8) to $\bigoplus_{i \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \text{res}_i$ where
\[ \text{res}_i : H^1(\Gamma^i(U_s), ^1 L(n, M)) \rightarrow \bigoplus_{j \equiv i \mod p^r} H^1(\Gamma^j(U_r), ^1 L(n, M)) \]

sends $x$ in $(\text{res}(x), \ldots, \text{res}(x))$.

For every positive integer $n$, let us consider the exact sequence of $\mathcal{O}[R^p_p]$-modules
\begin{equation}
0 \rightarrow L(n, \mathcal{O}) \rightarrow L(n, K) \rightarrow L(n, K/\mathcal{O}) \rightarrow 0.
\end{equation}

For every positive integer $s$, sequence (12) induces a long exact sequence in sheaves cohomology
\[ 0 \rightarrow H^0(X_s, \mathcal{L}(n, \mathcal{O})) \rightarrow H^0(X_s, \mathcal{L}(n, K)) \rightarrow H^0(X_s, \mathcal{L}(n, K/\mathcal{O})) \rightarrow \]
\[ \rightarrow H^1(X_s, \mathcal{L}(n, \mathcal{O})) \rightarrow H^1(X_s, \mathcal{L}(n, K)) \rightarrow H^1(X_s, \mathcal{L}(n, K/\mathcal{O})) \rightarrow \]
\[ \rightarrow H^2(X_s, \mathcal{L}(n, \mathcal{O})) \rightarrow H^2(X_s, \mathcal{L}(n, K)) \rightarrow H^2(X_s, \mathcal{L}(n, K/\mathcal{O})) \rightarrow 0 \]

because $X_s$ has cohomological dimension 2, being a complete curve. We also have an analogous exact sequence in group cohomology.

Let $V^n_0$ be the image of $H^1(X_s, \mathcal{L}(n, K))$ in $H^1(X_s, \mathcal{L}(n, K/\mathcal{O}))$ and $V^n_0$ be the image of $H^1(\Phi_s, ^1 L(n, K))$ in $H^1(\Phi_s, ^1 L(n, K/\mathcal{O}))$. If $n = 0$ then $\mathcal{L}(0, A)$ is a constant sheaf for every $\mathcal{O}$-module $A$; therefore
\begin{equation}
V^n_0 = H^1(X_s, \mathcal{L}(0, K/\mathcal{O})) \text{ and } V^n_0 = H^1(\Phi_s, ^1 L(0, K/\mathcal{O})).
\end{equation}

Let $\Lambda^n(\mathcal{O})$ be the image of $H^1(X_s, \mathcal{L}(n, \mathcal{O}))$ in $H^1(X_s, \mathcal{L}(n, K))$ and let $\Lambda^n(\mathcal{O})$ be the image of $H^1(\Phi_s, ^1 L(n, \mathcal{O}))$ in $H^1(\Phi_s, ^1 L(n, K))$. By the universal coefficient theorem, since $K$ is $\mathcal{O}$-flat,
\[ H^1(X_s, \mathcal{L}(n, \mathcal{O})) \otimes_{\mathcal{O}} K \simeq H^1(X_s, \mathcal{L}(n, K)). \]

Therefore
\begin{equation}
\Lambda^n(\mathcal{O}) \text{ is a lattice in } H^1(X_s, \mathcal{L}(n, K)), \text{ and } \\
\Lambda^n(\mathcal{O}) \text{ is a lattice in } H^1(\Phi_s, ^1 L(n, K)).
\end{equation}
By definition of Hecke operators $\Lambda^n_s(\mathcal{O})$ is stable under the algebra $\mathcal{H}_k(U_s, \mathcal{O})$; by linearity this algebra faithfully acts on $\mathcal{V}_s^n$.

**Lemma 6.2.** If $n \neq 0$ then the $\Phi_s$-module $^tL(n, K)$ is simple.

**Proof.** It is enough to prove the claim for $\Phi = \Gamma(U_{\max})$ where $U_{\max} = \prod_q R_q^\infty$, because $\Phi_s$ has finite index in $\Phi$. Let $\Phi'$ be the $p$-adic closure of $\Phi$ in $M_2(\mathcal{O})$.

By continuity, it is enough to show the irreducibility of the action of $\Phi'$ over $^tL(n, K)$. Since $p$ divides $\Delta$, $B_p$ is isomorphic to

$$B'_p = \left\{ \left( \begin{array}{cc} a & b \\ pb^\sigma & a^\sigma \end{array} \right) \mid a, b \in \mathbb{Q}_p^2 \right\},$$

where $a$ is the non-trivial element of $\text{Gal}(\mathbb{Q}_p^2/\mathbb{Q}_p)$. The maximal order of $B'_p$ is $S_p = B'_p \cap M_2(\mathbb{Z}_p^2)$. An isomorphism between $B_p$ and $B'_p$ can be extended by $K$-linearity to an automorphism of the $K$-algebra $M_2(K)$, which is the conjugation by an element of $\text{GL}_2(K)$, [20, Théorème 2.1]. Thus it suffices to show that the action of the group $S_p^{(1)}$ over $^tL(n, K)$ is irreducible. But $S_p^{(1)}$ contains a $K$-basis of $M_2(K)$ and it is well-known that symmetric powers of the standard representation of $\text{SL}_2(K)$ are irreducible.

Let $t \geq s$; for each $\Phi_s$-module $M$ there is the inflation-restriction exact sequence

$$0 \longrightarrow H^1(\Phi_s/\Phi_t, M^{\Phi_t}) \xrightarrow{\text{infl}} H^1(\Phi_s, M) \xrightarrow{\text{res}} H^1(\Phi_t, M)^{\Phi_s/\Phi_t} \longrightarrow H^2(\Phi_s, M) \longrightarrow H^2(\Phi_s/\Phi_t, M^{\Phi_t}),$$

where $g \in \Phi_s/\Phi_t$ acts on $H^1(\Phi_t, M)$ by $(g : \xi)(\gamma) = g\xi(g^{-1}g\gamma)$, for every 1-cocycle $\xi : \Phi_t \rightarrow M$. If $M = ^tL(n, \mathcal{O})$ and $n \neq 0$ then by Lemma 6.2 $M^{\Phi_t} = 0$ and thus, if $n \neq 0$

$$\text{Im}(\text{res}) : H^1(\Phi_s, ^tL(n, \mathcal{O})) \rightarrow H^1(\Phi_t, ^tL(n, \mathcal{O})) = H^1(\Phi_t, ^tL(n, \mathcal{O}))^{\Phi_s/\Phi_t}.$$ 

Therefore the restriction map from $\Phi_s$ to $\Phi_t$ sends $\Lambda^n_s(\mathcal{O})$ in $\Lambda^n_t(\mathcal{O})$ and the image is $\Lambda^n_t(\mathcal{O})^{\Phi_s/\Phi_t}$. In particular

$$\Lambda^n_t(\mathcal{O})/\text{res}(\Lambda^n_s(\mathcal{O}))$$

is $\mathcal{O}$-torsion-free if $n \neq 0$.

Then we obtain the following result:

**Proposition 6.3.** If $s \leq t$ and $n \neq 0$ then the restriction map $V^n_s \rightarrow V^n_t$ (and so the restriction map $\mathcal{V}_s^n \rightarrow \mathcal{V}_t^n$) is injective.

**Proof.** By definition $V^n_s \simeq H^1(\Phi_s, ^tL(n, K))/\Lambda^n_s(\mathcal{O})$. Suppose that $x \in H^1(\Phi_s, ^tL(n, K))$ and res($x$) $\in \Lambda^n_s(\mathcal{O})$. By (14) there exists a positive integer $k$ such that $p^k x \in \Lambda^n_s(\mathcal{O})$, so that res($x$) $\in \text{res}(\Lambda^n_s(\mathcal{O}))$ by (15). Notice that res : $H^1(\Phi_s, ^tL(n, K)) \rightarrow H^1(\Phi_t, ^tL(n, K))$ is injective (this is evident, for example, from the fact that $\text{cor} \circ \text{res} = [\Phi_s : \Phi_t]$). Therefore $x \in \Lambda^n_s(\mathcal{O})$.

If $n = 0$ then the $\Phi_s$-action is trivial and the above argument does not work. However, we have the following result, which will be proved in Section 10.
Theorem 6.4. For every \( n \in \mathbb{N} \), the inductive limit of the restriction maps
\[
I_s : H^1(\Phi_s, L(n, K_/O)) \longrightarrow \lim_{t} H^1(\Phi_t, L(n, K_/O))
\]
has finite kernel.

We put \( V^n_s = \lim_{\tau} V^n_s \).

Corollary 6.5. a) For every \( n \), the Hecke algebra \( \mathcal{H}_k(p^{\infty}, N; O) \) acts faithfully on \( V^n_s \). The action is faithful by Theorem 6.4, because \( \mathcal{H}_k(U_s, O) \) has no \( O \)-torsion.

b) If we give \( V^n_s \) the discrete topology, then the topology given by the projective limit and the topology induced by the compact-open topology in \( \text{End}_O(V^n_s) \) coincide over \( \mathcal{H}_k(p^{\infty}, N; O) \).

Proof. a) We saw that \( \mathcal{H}_k(U_s, O) \) faithfully acts on \( V^n_s \), then taking the limit \( \mathcal{H}_k(p^{\infty}, N; O) \) acts on \( V^n_s \). The action is faithful by Theorem 6.4, because \( \mathcal{H}_k(U_s, O) \) has no \( O \)-torsion.

b) Let \( K, A \) be subsets of \( V^n_s \), and suppose \( K \) finite (that is compact in the discrete topology); let \( \Omega_{K,A} \) be the set of endomorphisms \( F \) of \( V^n_s \) such that \( F(K) \subseteq A \). Then the subsets of the form \( \Omega_{K,A} \) provide a basis for the compact-open topology in \( \text{End}_O(V^n_s) \). For each \( K \), we can choose \( s \) and \( r \) such that \( K \subseteq \text{Im}(V^n_s \to V^n_s) \) and \( K \subseteq V^n_{\infty} [p^r] \). Let \( \pi_s \) be the projection \( \mathcal{H}_k(p^{\infty}, N; O) \to \mathcal{H}_k(U_s, O) \). If \( T \in \mathcal{H}_k(p^{\infty}, N; O) \cap \Omega_{K,A} \) then \( T + \pi_s^{-1}(p^r \mathcal{H}_k(U_s, O)) \subseteq \Omega_{K,A} \). This shows that the sets \( \Omega_{K,A} \) are open in \( \mathcal{H}_k(p^{\infty}, N; O) \). Since the latter is compact and the compact-open topology is Hausdorff, claim b) is proved.

The following theorem will be proved in section 10

Theorem 6.6. For every \( n \geq 0 \)
\[
V^n_s = \lim_{\tau} H^1(\Phi_s, \mathcal{L}(n, K_/O)).
\]

7. The evaluation map

Consider the evaluation map
\[
t : L(n, K_/O) \longrightarrow K_/O
\]
\[
P(X, Y) \longrightarrow P(1, 0)
\]
If \( s \geq t \) then the action of \( U_s \) over \( L(n, p^{-t}O_/O) \) is trivial. Thus, the restriction of \( t \) gives rise to a homomorphism of \( U_s \)-modules \( L(n, p^{-t}O_/O) \to p^{-t}O_/O \). This induces an homomorphism of sheaves \( \mathcal{L}(n, p^{-t}O_/O)_{X_s} \to \mathcal{L}(0, p^{-t}O_/O)_{X_s} \), and a map on the cohomology groups
\[
\iota_s : H^1(\Phi_s, \mathcal{L}(n, p^{-t}O_/O)) \longrightarrow H^1(\Phi_s, \mathcal{L}(0, p^{-t}O_/O)).
\]
Since \( X_s \) is compact for every \( s \) there is a canonical identification
\[
H^1(\Phi_s, \mathcal{L}(n, K_/O)) \simeq \lim_{\tau} H^1(\Phi_s, \mathcal{L}(n, p^{-t}O_/O)),
\]
(see [1, II, Corollary 14.5]). We fix \( s \) and consider the map
\[
\tilde{\iota}_s : H^1(\Phi_s, \mathcal{L}(n, K_/O)) \longrightarrow \lim_{\tau} H^1(\Phi_t, \mathcal{L}(0, K_/O))
\]
obtained by composition in the following way:

\[
H^1(X_s, \mathcal{L}(n, K/O)) \xrightarrow{\sim} \lim_{t \to} H^1(X_s, \mathcal{L}(n, p^{-t}O/O))
\]

\[
\xrightarrow{\text{res}} \lim_{t \geq s} H^1(X_t, L(n, p^{-t}O/O))
\]

\[
\xrightarrow{\iota_s} \lim_{t} H^1(X_t, L(0, p^{-t}O/O))
\]

\[
\xrightarrow{\sim} \lim_{t} H^1(X_t, L(0, K/O)).
\]

We shall prove the following result; it is the analogue of [7, Theorem 8.7] in our situation, and our proof will closely follow Hida’s one.

**Theorem 7.1.** The kernel of \( \bar{\iota}_s \) is finite.

**Remark 7.2.**

a) By point b) of Remark 6.1, Theorem 7.1 will be proved if we show the finiteness of the kernel of the map in cohomology of groups

\[
\tilde{j}_s : H^1(\Phi_s, L(n, K/O)) \to \lim_{t} H^1(\Phi_t, K/O)
\]

obtained by composition as follows

\[
H^1(\Phi_s, tL(n, K/O)) \xrightarrow{\sim} \lim_{t} H^1(\Phi_s, tL(n, p^{-t}O/O))
\]

\[
\xrightarrow{\text{res}} \lim_{t} H^1(\Phi_t, tL(n, p^{-t}O/O))
\]

\[
\xrightarrow{\iota_s} \lim_{t} H^1(\Phi_t, p^{-t}O/O)
\]

\[
\xrightarrow{\sim} \lim_{t} H^1(\Phi_t, K/O)
\]

where this time \( \iota_s \) is the map induced by \( \iota \) in cohomology of groups.

b) Since the definition of \( \bar{\iota}_s \) only involves restriction and functoriality of cohomology the map \( \bar{\iota}_s \) is equivariant with respect to the action of Hecke operators \( T_q, T_{q,q} \).

**Lemma 7.3.** There exists a positive integer \( \bar{n} = \bar{n}(s) \) such that \( p^{\bar{n}}V = 0 \) for every \( O \)-submodule \( V \) of \( \ker(\iota) \) (contained in \( L(n, K/O) \)) which is stable for the \( \Phi_s \)-action.

**Proof.** Let \( M \) be the subalgebra of \( \text{End}_K(L(n, K)) \) generated over \( O \) by the action of \( \Phi_s \). We show that for \( j = 1, \ldots, n \) there exists \( \phi_j \in M \otimes K \) such that for each \( P = \sum \lambda_i X^i Y^{n-i} \in L(n, K) \) it holds \( \phi_j(P) = \lambda_j X^n + \ldots \).

If \( \gamma \in \Phi_s \), let \( \phi_\gamma \in \text{End}_K(L(n, K)) \) be the endomorphism corresponding to \( \gamma^{-1} \) and \( A_\gamma \) be the matrix of \( \phi_\gamma \) with respect to the basis \( X^n, X^{n-1}Y, \ldots, Y^n \) of \( L(n, K) \). Since \( L(n, K) \) is simple, the first rows of matrices \( A_\gamma \) generate \( K^{n+1} \); otherwise there would be a non zero element \( P(X, Y) \in L(n, K) \) such that \( \Phi_s P(X, Y) \subseteq KX^{n-1}Y + \ldots + KY^n \). Therefore for \( j = 0, \ldots, n \) the matrices \( A_j \) generate over \( K \) a matrix \( A_j \) having on its first row \( (0, \ldots, 1, \ldots, 0) \).
with 1 in the $j^{th}$ place; $A_j$ is the matrix of the endomorphism $\phi_j$ we were looking for.

Let $\bar{n}$ be such that $\bar{p}\bar{\phi}_j \in M$, for every $j$. Let $P = \sum_i a_i X^i Y^{n-i}$ be an element of $V$; then $p^\bar{n} \bar{\phi}_j(P) = \bar{p}^n(a_j X^n + ...)$ belongs to $V$, for $j = 0, ..., n$, because $V$ is $\Phi_s$-stable. But $V \subseteq \ker(\iota)$, thus $p^n a_j = 0$ for $j = 0, ..., n$; therefore $p^\bar{n} V = 0$. ■

**Proof of Theorem 7.1:** Since the action of $\Phi_t$ over $L(n,p^{-1}O/O)$ is trivial, the image of the restriction map $I_s$ defined in Theorem 6.4 is contained in $\lim_{\xrightarrow{\longleftarrow}} \text{Hom}_{\Phi_s}(\Phi_t, L(n,p^{-1}O/O))$. If $\xi \in \text{Hom}_{\Phi_s}(\Phi_t, L(n,p^{-1}O/O))$, then $\xi(u\bar{w}^{-1}) = u\xi(\delta)$, for every $u \in \Phi_s$, $\delta \in \Phi_t$. Thus if $\iota \circ \xi = 0$ then $\iota(u\xi) = 0$.

Therefore the image of $\xi$ is contained in a $(\Phi_s,O)$-submodule of $\ker(\iota)$ in $L(n,K/O)$. By Lemma 7.3 we deduce that $p^\bar{n} \xi = 0$. In particular, if $\xi$ is a cocycle in $Z^1(\Phi_s, L(n,K/O))$ and the class of $\xi$ in $H^1(\Phi_s, L(n,K/O))$ belongs to $\ker(I_s)$ then $\iota \circ I_s(\xi) = 0$, thus $p^\bar{n} I_s(\xi) = 0$. We deduce that $\ker(I_s) \subseteq p^{-\bar{n}} \ker(I_s)$, which is finite by Theorem 6.4. ■

8. CONSEQUENCES FOR THE HECKE ALGEBRA

**Theorem 8.1.**

a) For every positive integer $s$ the homomorphism $\iota_s$ defined in (16) induces a continuous surjective homomorphism of $O[[Z_p]]$-algebras

$$\iota_s : H_2(p^\infty,N;O) \longrightarrow H_k(U_s,O)$$

such that, for every prime $q \neq p$, $\iota_s(T_q) = T_q$. The homomorphism $\iota_s^\ast$ is compatible with the restriction $H_k(U_1,O) \rightarrow H_k(U_s,O)$, if $t \geq s$.

b) The projective limit of the homomorphisms $\iota_s^\ast$:

$$\iota^\ast : H_2(p^\infty,N;O) \longrightarrow H_k(p^\infty,N;O)$$

is an isomorphism.

**Proof.** a) We put $W_s^n = \iota_s(V_s^n)$. The homomorphism $\iota_s$ is a $R(N)$-equivariant isogeny between $W_s^n$ and $V_s^n$. Since the algebra $H_k(U_s,O)$ has no torsion, it faithfully acts over $W_s^n$, and it has the compact-open topology with respect to the discrete module $W_s^n$. Let $A$ be the dense subalgebra of $H_2(p^\infty,N;O)$ generated over $O$ by $R(N)$; then the restriction of operators in $A$ to the module $W_s^n$ gives a surjective homomorphism $A \rightarrow H_k(U_s,O)$, which is continuous by Corollary 6.5; therefore it extends to a continuous surjective homomorphism of the algebra $H_2(p^\infty,N;O)$ onto $H_k(U_s,O)$. b) The homomorphism $\iota_s^\ast$ is surjective for every $s$ and $H_2(p^\infty,N;O)$ is compact; therefore the image of $\iota^\ast$ is a dense and closed subset of $H_k(p^\infty,N;O)$; thus $\iota^\ast$ is surjective. We prove now that it is injective: define

$$\kappa : p^{-1}O/O \longrightarrow L(n,p^{-1}O/O)$$

$$a \mapsto aX^n;$$

Then $\iota \circ \kappa = id$ and $\kappa$ is a homomorphism of $\Phi_s$-modules if $t \leq s$. Therefore $\iota_s : H^1(X_s,L(n,p^{-1}O/O)) \rightarrow H^1(X_s,L(0,p^{-1}O/O))$ is surjective for every $s$. By Theorem 6.6 and Corollary 6.5 $H_k(p^\infty,N;O)$ faithfully acts
over \( \lim_{\to s} H^1(X_n, L(n, K/O)) \). Then the surjectivity of \( \iota_* \) implies that \( \iota^* \) is injective. ■

9. Proof of Theorem 6.4

If \( n \neq 0 \) then by Proposition 6.3 \( V^n \cap \ker(I_n) = \{0\} \). Therefore \( \ker(I_n) \) is injectively sent in \( H^2(\Phi_s, t L(n, O)) \) in the long exact sequence associated to sequence (12). By [17, Propositions 8.1 and 8.2], since \( \Phi_s \) has no torsion, there is an isomorphism

\[
H^2(\Phi_s, t L(n, O)) \simeq H_0(\Phi_s, t L(n, O)),
\]

and the latter is the group of coinvariants of \( t L(n, O) \), which is finite by Lemma 6.2.

The proof for weight 2 (corresponding to \( n = 0 \)) is the following adaptation of a theorem of Shimura [15, 3.2.1]. We can suppose that \( K \) is the \( p \)-adic closure of \( K_0 \) in \( \overline{Q}_p \) (according to the fixed immersion \( Q \to Q_p \)).

Indeed, let \( K' \) be a finite extension of \( K \) and \( O' \) be its integer ring; then \( H^1(\Phi_s, K'/O') = H^1(\Phi_s, K/O) \otimes O' \) by the universal coefficients theorem. The kernel of the restriction map \( H^1(\Phi_s, K'/O') \to H^1(\Phi_t, K'/O') \) is the image of

\[
\ker(\text{res} : H^1(\Phi_s, K/O) \to H^1(\Phi_t, K/O)) \otimes O',
\]

if the claim is proved for \( K \), then its order is then bounded by a constant which does not depend on \( t \).

Let \( L \) be a finite Galois extension of \( Q \) satisfying

- \( L \) is totally real and linearly disjoint from \( K_0 \) over \( Q \);
- every prime factor of \( N\Delta \text{Disc}(K_0) \) totally split in \( L \);
- \( d = [L:Q] \geq 2 \).

We put \( B' = B \otimes_Q L \); it is a division algebra over \( L \), of discriminant \( \Delta \). Let \( K_0' = L \otimes_Q K_0 \); then \( K_0' \) is a field, because we assumed that \( L \cap K_0 = Q \); let \( O_L \) be the ring of integers of \( L \) and \( O_0' = O_L \otimes Z O_0 \) be the ring of integers of \( K_0' \) (because \( \text{Disc}(K_0) \) is prime to \( \text{Disc}(L) \), see for example [12, III, §3, Prop. 17]). We put \( R' = R \otimes Z O_L \). The \( p \)-adic closure of \( K_0' \) in \( Q_p \) coincide with \( K \). The embedding \( K_0' \hookrightarrow K \) corresponds to a prime ideal \( p \) of \( O_0' \). We put \( O_0' \cap K_0' = O_0'(p) \).

Let \( N = \prod_{j=1}^m q_j^{e_j} \) be the decomposition of \( N \) in a product of powers of primes; we choose for every \( j \) \( 1 \leq j \leq m \) a prime ideal \( q_j \) of \( L \) over \( q_j \). Since the primes dividing \( N \) totally split over \( L \), the isomorphisms \( i_{q_j} \), introduced in Section 2 induce isomorphisms

\[
i_{q_j}^i : R' \otimes_{O_L} O_{L,q_j} \to M_2(Z_{q_j}).
\]

We define

\[
R'(N)_{q_j} = (i_{q_j}^i)^{-1} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(Z_{q_j}) \mid q_j^{e_j} \text{ divides } c \right\}.
\]

Since \( p \) totally splits in \( L \)

\[
B'_p = B' \otimes_{O_L} Q_p = B_p.
\]
We define the compact open subgroup of \( (B'_\mathbf{A}_L)^{\times, \infty} = (B' \otimes_L \mathbf{A}_L^{\infty})^{\times} \)

\[
U'_s = \prod_{q \neq p, q_i} (R'_q)^{\times} \times \prod_{i=1}^m R'(N)_{q_i}^{\times} \times (1 + p^s R_p)
\]

and its subgroup

\[
\Phi'_s = (\text{GL}_2(\mathbf{R})^d \times U'_s) \cap B'^{(1)}.
\]

There is a natural immersion \( B \hookrightarrow B' = B \otimes_{\mathbf{Q}} L \). We can easily see that \( \Phi'_s \cap B = \Phi_s \), for every \( s \). Therefore, if \( t \geq s \), there is a natural injection

\[
\Psi : \Phi_s / \Phi_t \longrightarrow \Phi'_s / \Phi'_t.
\]

**Proposition 9.1.** The map \( \Psi \) is surjective.

**Proof.** By Proposition 2.2 we know that \( \Phi_s \) is dense in \( (1 + p^s R_p)^{(1)} \). Moreover, the groups \( (1 + p^t R_p)^{(1)} \), for \( t \geq s \), are a fundamental system of neighbours of unity in \( (1 + p^s R_p)^{(1)} \). If \( x \in (1 + p^s R_p)^{(1)} \), then there exists an element \( y \in \Phi_s \) such that \( x^{-1} y \in (1 + p^t R_p)^{(1)} \), so that the natural homomorphism \( \Phi_s / \Phi_t \to (1 + p^s R_p)^{(1)} / (1 + p^t R_p)^{(1)} \) is an isomorphism. Proposition 2.2 holds also for a quaternion algebra over the number field \( L \); since \( p \) totally splits in \( L \) we have that \( \Phi'_s \) is dense in \( (1 + p^s R_p)^{(1)} \), so that \( \Phi'_s / \Phi'_t \cong (1 + p^s R_p)^{(1)} / (1 + p^t R_p)^{(1)} \).

There is an isomorphism \( K'_0 / \mathcal{O}'_0 \cong K / \mathcal{O} \); therefore

\[
H^1(\Phi_s / \Phi_t, K / \mathcal{O}) \cong H^1(\Phi'_s / \Phi'_t, K'_0 / \mathcal{O}'_0).
\]

By the inflation-restriction sequence, the kernel of the restriction map \( H^1(\Phi_s, K / \mathcal{O}) \to H^1(\Phi_t, K / \mathcal{O}) \) is isomorphic to \( H^1(\Phi_s / \Phi_t, K / \mathcal{O}) \), because the \( \Phi_t \)-action over \( K / \mathcal{O} \) is trivial. Therefore, in order to prove the finiteness of the kernel of \( I_s \) it suffices to show that

\[
H^1(\Phi_s / \Phi_t, K / \mathcal{O}) \leq M
\]

where \( M \) is a constant not depending on \( t \). By the isomorphism (18) this is equivalent to show that

\[
H^1(\Phi'_s / \Phi'_t, K'_0 / \mathcal{O}'_0) \leq M
\]

with \( M \) not depending on \( t \). Since \( H^1(\Phi'_s / \Phi'_t, K'_0 / \mathcal{O}'_0) \) injects by inflation in \( H^1(\Phi'_s, K'_0 / \mathcal{O}'_0) \), we are reduced to show that the latter is a finite group.

Consider the exact sequence

\[
0 \longrightarrow \mathcal{O}'_0 \longrightarrow K'_0 \longrightarrow K'_0 / \mathcal{O}'_0 \longrightarrow 0.
\]

It induces a long exact sequence in cohomology

\[
H^1(\Phi'_s, K'_0) \longrightarrow H^1(\Phi'_s, K'_0 / \mathcal{O}'_0) \overset{\delta}{\longrightarrow} H^2(\Phi'_s, \mathcal{O}'_0).
\]

The central group is torsion, therefore its image by \( \delta \) is contained in \( H^2(\Phi'_s, \mathcal{O}'_0)^{\text{tors}} \), which finite, because it is a torsion submodule of a finitely generated module over a principal ideal domain. It remains to show that \( H^1(\Phi'_s, K'_0) \) is finite, i.e. that it is zero. Now, \( H^1(\Phi'_s, K'_0) \) maps injectively to \( H^1(\Phi'_s, K'_0 \otimes_{K_0} \mathbf{C}) \). We have \( (B' \otimes_{\mathbf{Q}} \mathbf{R})^{\times} \cong \text{GL}_2(\mathbf{R})^d \) and this isomorphism identifies \( \Phi'_s \) to a discrete irreducible subgroup of \( \text{SL}_2(\mathbf{R})^d \) (see [20, IV, §1]).

By [13, Théorème 7.1] we deduce that \( H^1(\Phi'_s, K'_0 \otimes_{K_0} \mathbf{C}) = 0 \).
Remark 9.2. This proof works for every $n$.

10. Proof of Theorem 6.6

By (13) the theorem is trivially true if $n = 0$.

By applying the inductive limit to the exact sequences

$$0 \rightarrow V^n_s \rightarrow H^1(X_s, \mathcal{L}(n, K/O)) \rightarrow H^2(X_s, \mathcal{L}(n, O)) \rightarrow 0$$

we see that the theorem will be proved if we show that $\lim_{n \rightarrow \infty} H^2(\Phi_s, L(n, O))$ is zero if $n \neq 0$.

10.1. The index of $\Phi_{s+1}$ in $\Phi_s$. By definition, the ring $O$ contains $\mathbb{Z}_{p^2}$.

Let

$$S_p = \left\{ \begin{pmatrix} a & b \\ pb^\sigma & a^\sigma \end{pmatrix} \mid a, b \in \mathbb{Z}_{p^2} \right\} \subseteq M_2(O).$$

As we have seen in the proof of Lemma 6.2, the two rings $R_p$ and $S_p$ are conjugated by a matrix $\delta_0$ of $GL_2(K)$: therefore we have

$$S_p = \delta_0 R_p \delta_0^{-1}, \quad S_p^\times = \delta_0 R_p^\times \delta_0^{-1}, \quad 1 + p^s S_p = \delta_0 (1 + p^s R_p) \delta_0^{-1},$$

for every positive integer $s$. We shall denote by $[a, b]$ the element $\begin{pmatrix} a & b \\ pb^\sigma & a^\sigma \end{pmatrix}$ of $M_2(\mathbb{Z}_{p^2})$. Let $u$ be the matrix $[0, 1] \in S_p$; then $u^2 = p$. The map

$$S_p \rightarrow \mathbb{Z}_{p^2}$$

$$[a, b] \mapsto a$$

induce a isomorphism of rings

$$S_p / uS_p \simeq \mathbb{Z}_{p^2} / p\mathbb{Z}_{p^2} = F_{p^2}.$$

There is also an isomorphism of groups

$$1 + u^s S_p / 1 + u^{s+1} S_p \xrightarrow{\sim} S_p / uS_p$$

$$1 + u^s \alpha \xrightarrow{\sim} \alpha,$$

Therefore $1 + u^s S_p / 1 + u^{s+1} S_p$ is isomorphic to $F_{p^2}$.

If $s = 2r$ is even, then the reduced norm $\nu : 1 + u^s S_p \rightarrow 1 + p^r \mathbb{Z}_p$ is surjective, since $S_p$ contains all matrices of the form $[a, 0]$ with $a \in \mathbb{Z}_{p^2}$. If $s = 2r + 1$ is odd, then $\nu(1 + u^s S_p) = 1 + p^r \mathbb{Z}_p$, because trace($uS_p$) $\subseteq p\mathbb{Z}_p$.

We study now the index $[(1 + u^s S_p)^{(1)} : (1 + u^{s+1} S_p)^{(1)}]$:

- If $s = 2r$ is even, then there is an exact sequence

$$0 \rightarrow C \rightarrow 1 + u^s S_p / 1 + u^{s+1} S_p \rightarrow 1 + p^r \mathbb{Z}_p / 1 + p^{r+1} \mathbb{Z}_p \rightarrow 0$$

where

$$C = \{ \alpha \in 1 + u^s S_p \mid \nu(\alpha) \in 1 + p^{r+1} \mathbb{Z}_p \}/1 + u^{s+1} S_p.$$.

The natural inclusion $(1 + u^s S_p)^{(1)}/(1 + u^{s+1} S_p)^{(1)} \hookrightarrow C$ is surjective; therefore

$$[(1 + u^s S_p)^{(1)} : (1 + u^{s+1} S_p)^{(1)}] = p$$

if $s$ is even.

A system of representatives of the cosets of $(1 + u^{s+1} S_p)^{(1)}$ in $(1 + u^s S_p)^{(1)}$ is given by the elements of the form $\alpha_i \gamma_i$, where $\alpha_i = 1 +
The action of $10.2$. \(G\) and its roots are in $\mathbb{Z}$ and a non-square in $\mathbb{Q}$ then $\Phi \in \mathbb{Z}$.

\[
(1 + u^s S_p) \in (1 + u^{s+1} S_p) \sim 1 + u^s S_p/1 + u^{s+1} S_p
\]

therefore

\[
[(1 + u^s S_p) : (1 + u^{s+1} S_p)] = p^2 \quad \text{if } s \text{ is odd.}
\]

A system of representatives of the cosets of $(1 + u^{s+1} S_p)$ in $(1 + u^s S_p)$ is given by the elements of the form $\alpha \gamma_i$, where $\alpha = 1 + u^{s}[a_i, 0]$, the $a_i$'s vary in $\mathbb{F}_{p^2}$ and the $\gamma_i \in 1 + u^{s+1} S_p$ are such that $\nu(\alpha \gamma_i) = 1$.

If $\Phi'$ is the $p$-adic closure of $\Phi$ in $R_p^\times$, then by Proposition 2.2 we have $\Phi_s / \Phi_{s+1} \simeq \Phi'_s / \Phi'_{s+1} \simeq (1 + p^s R_p) / (1 + p^{s+1} R_p)$, so that

\[
[\Phi_s : \Phi_{s+1}] = p^3.
\]

10.2. **The action of** $S_p^\times$ **over** $L(n, \mathcal{O})$. Let $G$ be a group, $M$ be a $G$-module and let $D(G, M)$ be the subgroup of $M$ generated by the $\gamma m - m, m \in M, \gamma \in G$. Then $H_0(G, M) = M/D(G, M)$ is the biggest quotient of $M$ over which $G$ acts trivially. For every positive integer $s$, let $G_s = (1 + p^s S_p)$. We let $S_p^\times$ act on $L(n, \mathcal{O})$ by the matrices $[a, b]$. We put $D_s' = D(G_s, L(n, \mathcal{O}))$; then $D_s' \subseteq p^s L(n, \mathcal{O})$. If $s \leq t$ we consider the transfer map:

\[
Tr_{G_s, G_t} : L(n, \mathcal{O}) / D_s' \rightarrow L(n, \mathcal{O}) / D_t'
\]

for a decomposition $G_s = \Pi_\gamma G_t$.

**Proposition 10.1.** If $n > 0$ then there exists a positive integer $l_0$, depending only on $n$, such that if $s > l_0$

\[
p^{s+l_0} L(n, \mathcal{O}) \subseteq D_s'.
\]

**Proof.** We choose an element $x \in \mathbb{Z}_{p^2}^\times$ such that $N(1 + p^s x) = 1$, where $N$ denotes the norm of $\mathbb{Q}_p$ over $\mathbb{Q}_p$. Such an element exists: in fact, let $m \in \mathbb{Z}_p$ such that $p \not| m$ and $-m$ is not a square modulo $p$; then the polynomial $X^2 + p^s m X + m$ has discriminant $m(p^{2s}m - 4)$, which is a unity and a non-square in $\mathbb{Q}_p$, because $p$ is odd; therefore it is irreducible over $\mathbb{Q}_p$ and its roots are in $\mathbb{Z}_p$. If $x$ is a root, then $p^s N(x) = p^s m = -tr(x)$; so $1 + p^s x$ has norm $1 + p^s tr(x) + p^{2s} N(x) = 1$. The matrix $\alpha = [1 + p^s x, 0]$ belongs to $G_s$. For $i = 0, \ldots, n$ we have

\[
(21)
\]

\[
\alpha^{-1} X^i Y^{n-i} - X^i Y^{n-i} = (1 + p^s x)^{i(1 + p^s x^o)^{n-i}} X^i Y^{n-i} - X^i Y^{n-i} = (p^s (2i - n)x + p^{2s}(\ldots)) X^i Y^{n-i},
\]

since $x^o \equiv -x \mod p^s$. Let $l$ be such that $p^l > n$; since $|2i - n| \leq n$, if $2i - n \neq 0$, we have $v_p(2i - n) < l$; if $s > l$ then the order at $p$ of the coefficient of $X^i Y^{n-i}$ in (21) is equal to $sv_p(2i - n)$, therefore $p^{s+l} X^i Y^{n-i} \in D_s'$ if $2i - n \neq 0$. Let $L'(n, \mathcal{O})$ be the sub-$\mathcal{O}$-module of $L(n, \mathcal{O})$ generated by the $X^i Y^{n-i}$, for $i \neq n/2$; the argument above shows that $p^{s+l} L'(n, \mathcal{O}) \subseteq D_s'$ if
Since $n > l$; it remains to find $l_0$ such that $p^{s+l_0}X^{n/2}Y^{n/2} \in D'_s$ if $n$ is even and $s > l_0$. Consider the matrix $\beta = [1, p^s] \in 1 + p^sS_p$; then $\nu(\beta) \in 1 + p^{2s+1}\mathbb{Z}_p$; we choose $\delta \in 1 + p^{2s}S_p$ such that $\nu(\beta \delta) = 1$; so $\gamma = \beta \delta$ is in $G_s$. If $P \in L(n, \mathcal{O})$ then $\gamma^{-1}P \equiv \beta^{-1}P \mod p^{2s}L(n, \mathcal{O})$. If $n$ is even, put $i = \frac{n-2}{2}$; then

$$
\gamma^{-1}X^iY^{n-i} - X^iY^{n-i} \equiv \\
\equiv (X + p^sY)(p^{1+1}X+Y)^{n-i} - X^i Y^{n-i} \\
\equiv (X^i + p^n X^{i-1}Y)(Y^{n-i} - p^{2s+1}(n-i)XY^{n-i-1}) - X^i Y^{n-i} \\
\equiv p^{s+1}(n-i)X^{i+1}Y^{n-i-1} + p^iX^{i-1}Y^{n-i+1} \mod p^{2s}L(n, \mathcal{O}).
$$

Since $i + 1 = n/2$ we find that

$$
(n-i)p^{s+1} + p^{2s}z)X^{n/2}Y^{n/2} = \\
= (\gamma^{-1}X^{i}Y^{n-i} - X^i Y^{n-i} + p^{2s}Q - p^iX^{i-1}Y^{n-i+1}
$$

where $z \in \mathcal{O}$, $Q \in L'(n, \mathcal{O})$, therefore

$$
(n-i)p^{s+1} + p^{2s}z)X^{n/2}Y^{n/2} \equiv -p^iX^{i-1}Y^{n-i+1} \mod D'_s \quad \text{if } s > l.
$$

Since $n - 2i = 2$, $n - i$ and $i$ have the same $p$-adic valuation, and $X^{i-1}Y^{n-i+1} \in L'(n, \mathcal{O})$, multiplying by $p^{l_0-2p(i)}$ we find that, if $s > l$ then $p^{s+l_0}X^{n/2}Y^{n/2} \in D'_s$; therefore $l_0 = l + 1$ is the number we were looking for.

We have seen that $[G_s : G_{s+1}] = p^3$; since $G_s$ acts trivially over $L(n, \mathcal{O}/p^s\mathcal{O})$ if $r \leq s$ we have

$$Tr_{G_s/G_{s+1}}L(n, \mathcal{O}) \subseteq p^3L(n, \mathcal{O}) \quad \text{if } s \geq 3
$$

therefore

$$Tr_{G_s/G_{s+k}}L(n, \mathcal{O}) \subseteq p^{3k}L(n, \mathcal{O})
$$

for a suitable choice of representatives of the cosets of $G_{s+k}$ in $G_s$. By Proposition 10.1 we obtain, if $s > l_0$, $s > 3, 2k > s + l_0$

$$Tr_{G_s/G_{s+k}}L(n, \mathcal{O}) \subseteq p^{3k}L(n, \mathcal{O}) \subseteq D'_s.
$$

Therefore we obtain the following result:

**Proposition 10.2.**

$$\lim_{s \to \infty} H_0(G_s, l, L(n, \mathcal{O})) = 0
$$

where the inductive limit is taken with respect to the maps $Tr_{G_s/G_{s+k}}$.

Let now $M$ be a cotorsion $\mathcal{O}$-submodule of $L(n, \mathcal{O})$, stable for the action of $S_p^\times$. For every positive integer $s$, put $D_s^M = D(G_s, M)$. Let $c$ be such that $p^sL(n, \mathcal{O}) \subseteq M$; then $p^sD'_s \subseteq D_s^M$, for every $s$; put $d_0 = l_0 + c$; then

$$p^{s+d_0}M \subseteq p^sD'_s \subseteq D_s^M \quad \text{if } s > l_0.
$$

Since $Tr_{G_s/G_{s+1}}$ multiplies by $p^3$ we have also

$$\lim_{s \to \infty} H_0(G_s, M) = 0.
$$
10.3. **Triviality of** \( \lim_{s \to 0} H^2(\Phi_s, L(n, \mathcal{O})) \) **for** \( n \neq 0 \). **By isomorphism** (17) **the restriction map**

\[
\text{res}_{\Phi_s/\Phi_t} : H^2(\Phi_s, L(n, \mathcal{O})) \longrightarrow H^2(\Phi_t, L(n, \mathcal{O}))
\]

corresponds to the transfer

\[
\text{Tr}_{\Phi_s/\Phi_t} : L(n, \mathcal{O})/D_s \longrightarrow L(n, \mathcal{O})/D_t,
\]

where \( D_s = D(\Phi_s, L(n, \mathcal{O})) \).

**Lemma 10.3.** \( D_s = \tilde{D}_s \).

**Proof.** Of course \( D_s \subseteq \tilde{D}_s \). Since \( L(n, K) \) is a simple \( \Phi_s \)-module, and \( D_s \) is \( \Phi_s \)-invariant, \( H^2(\Phi_s, L(n, \mathcal{O})) \) is finite; therefore \( D_s \) is open in \( L(n, \mathcal{O}) \) and there exists a positive integer \( k \) such that \( p^k L(n, \mathcal{O}) \subseteq D_s \). If \( \gamma \in \Phi_s' \) then by Proposition 2.2 there exists \( \delta \in \Phi_s \) such that \( \delta^{-1} \gamma \in \Phi_s' \). Therefore for every \( P \in L(n, \mathcal{O}) \)

\[
\gamma P - P = \delta(\delta^{-1} \gamma P - P) + \delta P - P \in D_s,
\]

because \( \delta P - P \in D_s \), \( \delta^{-1} \gamma P - P \in p^k L(n, \mathcal{O}) \) and \( D_s \) is \( \Phi_s \)-invariant. ■

**Proposition 10.4.** \( \lim_{t \to s} H^2(\Phi_s, L(n, \mathcal{O})) = 0 \); **therefore**

\[
\lim_{t \to s} H^2(\Phi_s, L(n, \mathcal{O})) = 0.
\]

**Proof.** By structure transfer using diagram (23) it suffices to prove that for every \( s > 0 \) there exists a \( t \geq s \) such that

\[
\text{Tr}_{G_s/G_t} : H_0(G_s, \delta_0^t L(n, \mathcal{O})) \longrightarrow H_0(G_t, \delta_0^s L(n, \mathcal{O}))
\]

is zero. Let \( b \) such that \( p^b \delta_0 L(n, \mathcal{O}) \subseteq L(n, \mathcal{O}) \). Then \( p^b \delta_0 L(n, \mathcal{O}) \) is a sub-module of \( L(n, \mathcal{O}) \) stable for the \( S^p \)-action, and it is isomorphic to the \( R^p \)-module \( L(n, \mathcal{O}) \) by multiplication for \( p^{-b} \delta_0^{-1} \). By 22 there exists \( t \) such that

\[
\text{Tr}_{G_s/G_t} p^b \delta_0 L(n, \mathcal{O}) \subseteq D(G_t, p^b \delta_0 L(n, \mathcal{O})).
\]

Therefore \( \text{Tr}_{\Phi_s/\Phi_t} L(n, \mathcal{O}) \subseteq \tilde{D}_t = D_t \). ■
Acknowledgments

This work was supported by the framework of PRIN 2010/11 “Geometria delle varietà algebriche”, cofinanced by MIUR. The author would like to thank the anonymous referee for his/her thorough review and very useful comments that helped improve the clarity and the relevance of this paper.

References


Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy, e-mail: lea.terracini@unito.it