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# OPTIMAL CONSUMPTION OF A GENERALIZED GEOMETRIC BROWNIAN MOTION WITH FIXED AND VARIABLE INTERVENTION COSTS

A grayscale photograph of a large, multi-story building with a classical facade, featuring arched windows and a prominent entrance. The image is slightly tilted and serves as a background for the author's name.

STEFANO BACCARIN

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# Optimal consumption of a generalized geometric Brownian motion with fixed and variable intervention costs

Stefano Baccarin

Department of Economics and Statistics,  
University of Turin, Corso Unione Sovietica 218/bis, I-10134 Turin, Italy

E-mail: stefano.baccarin@unito.it,  
Tel.: (+39) 011 670.6129 - Fax: (+39) 011 670.6062

## Abstract

We consider the problem of maximizing expected lifetime utility from consumption of a generalized geometric Brownian motion in the presence of controlling costs with a fixed component. Under general assumptions on the utility function and the intervention costs our main result is to show that, if the discount rate is large enough, there always exists an optimal impulse policy for this problem, which is of a Markovian type. We compute explicitly the optimal consumption in the case of constant coefficients of the process, linear utility and a two values discount rate. In this illustrative example the value function is not  $C^1$  and the verification theorems commonly used to characterize the optimal control cannot be applied.

*Keywords:* Stochastic Programming; Markov processes; Impulse control; Quasi-variational inequalities; Consumption-investment problems with fixed intervention costs

## 1 Introduction

In this paper we consider the optimal consumption of a diffusion process  $S_t$ , which is a generalization of the geometric Brownian motion. The evolution of  $S_t$ , in absence of control, is described by the Itô's stochastic differential equation

$$dS_t = S_t\mu(S_t)dt + S_t\sigma(S_t)dB_t \quad (1)$$

where  $B_t$  is a one-dimensional Brownian motion and the functions  $\mu$  and  $\sigma$  are assumed to be bounded and Lipschitz continuous. Starting from an initial value  $S_0 > 0$  the process  $S_t$  remains positive for all  $t$ , and if  $\mu, \sigma$  are constant we have the standard geometric Brownian motion. The agent wants to maximize the

expected utility from consumption of  $S$  over an infinite horizon. Consumption is possible at any time but whenever  $S$  is consumed some quantity of  $S$  is lost as an intervention cost. This cost has a minimum fixed amount  $F$ , and (possibly) a variable part which can depend on the size of the intervention. Because of the fixed cost  $F$ , consumption cannot be done in a continuous way without incurring infinite costs: it is made by finite amounts in separate time instants. We have a problem of impulse control where a control policy is a sequence  $(\tau_i, \xi_i)$  of stopping times and corresponding random variables  $\xi_i$ , where  $\xi_i$  represents the decrease in  $S$  due to the intervention in  $\tau_i$ . We will denote by  $K(\xi)$  the intervention costs related to a displacement of size  $\xi$ . If at time  $\tau_i$  the agent consumes the amount  $c_i \geq 0$  then the level of  $S$  becomes

$$S(\tau_i) = S(\tau_i^-) - \xi_i$$

where  $S(\tau_i^-)$  is the amount of  $S$  just before  $\tau_i$  and  $\xi_i = c_i + K(\xi_i) > 0$  is the sum of consumption and intervention costs ( $\xi_i$  is strictly positive because  $K(\xi) \geq F > 0$ ). The objective is to maximize the discounted expected utility from the (possibly) infinite sequence of interventions

$$E \left[ \sum_{i=1}^{\infty} U(c_i) e^{-\int_0^{\tau_i} \beta(S_t) dt} \right]$$

where  $U$  is the agent's utility function and  $\beta(S_t)$  is the discount rate, which can depend on the current level  $S_t$ . Of course if there was no upper limit in the possible amount of  $c$ , then an agent with an unbounded  $U$  could obtain infinite utility by consuming immediately an unlimited quantity of  $S$ . So we will assume the natural constraint that a policy is admissible only if the level of  $S$ , after consumption, never becomes negative. For a real application, one may think of  $S$  as a financial asset with limited liability, such as a stock or an index fund, whose value evolves according to equation (1) and therefore randomly but with no risk of default. The investor wants to maximize the discounted expected utility from the liquidation of  $S$ , but whenever he sells some quantity of  $S$  he must pay fixed and proportional transaction costs. The nonnegativity constraint on  $S$  means in this case the prohibition to take short positions.

The main focus of this paper will be on proving the existence of an optimal policy and to characterize its form, under general assumptions on the functions  $\mu, \sigma, \beta, K, U$ . Indeed our main result, Theorem 10 and Corollary 11, shows that if the discount rate  $\beta(S)$  is sufficiently large with respect to  $U, \mu$  and  $\sigma$ , then there always exists the optimal consumption. This optimal policy is Markovian and it is characterized by a control region, a complementary continuation region and a set of optimal actions to implement in the control region. The regions and the optimal consumption can be obtained by solving a static optimization problem. To study our problem we will follow the Dynamic Programming methodology. In impulse control theory the value function is shown to be a solution of a quasi-variational inequality (QVI) which plays the same role of the Hamilton-Jacobi-Bellman equation for continuous stochastic control. It is well

known that the value function is not usually a classical  $C^2$  solution of this inequality, and some kind of weak solution must be considered. The most general type of weak solution is certainly the so called viscosity solution, which allows the value function to be even discontinuous (see, for some instances of discontinuous value functions and viscosity solutions, [14] chapter 7, [18], [2]). However the viscosity characterization does not seem of help in showing the existence of an optimal policy. To the best of our knowledge the optimal impulse control is usually obtained by applying some "verification theorems" which presume the existence of a sufficiently regular solution of the QVI (at least a  $C^1$  solution: see, for instance, [6], [21], [22], [11]). We will investigate our model using a variational approach and we will characterize the value function as a weak solution in a weighted Sobolev space. The main advantage of this type of weak solution is that some fundamental results of stochastic calculus, such as the Ito's formula and the Dynkin's formula, can be extended to generalized derivatives, if these generalized derivatives are ordinary functions. Consequently it is possible to prove the existence of an optimal control without assuming a continuously differentiable value function and even without making use of the dynamic programming principle. In section 5 of the paper we will give an example where the value function is not  $C^1$ , but it is possible to characterize the optimal impulse consumption. Our results are based on the functional analysis techniques for stochastic control developed in the monographs of Bensoussan and Lions [4], [5]. Nevertheless the problem we study is different from the impulse control problem studied in [5], chapter 6. The authors considered a process with bounded coefficients and a cost minimization problem where the value function is naturally bounded from below and there is a trade-off between running and controlling costs. This is the most common formulation of the impulse control problem (see, for instance, [1], [3], [12], [15]). In this paper we consider a generalized geometric Brownian motion and our objective has no additive separation between the utility of consumption and the controlling costs. Moreover we face a state constrained maximization problem and the value function is not necessarily finite. The problem is also different from some recent models concerning the optimal distribution of dividends by a firm (see [17], [8], [9]). Since in these models insolvency has always a positive probability to occur, the firm must find the optimal balance between paying more dividends and maintaining enough liquidity to reduce the risk of default. Our setting is more similar to a Merton's model (see [20]), with only one risky asset and in the presence of fixed transaction costs. Indeed the general asset price dynamics considered in [20], section 3 and 4, is the same of equation (1). However the fact that the agent consumes in a discrete way, at an infinite rate by finite amounts in separate time instants, changes the nature of the problem and its solution. The concavity of the utility function is no longer necessary to obtain an optimal policy; on the contrary it is necessary to set a lower bound constraint on the level of  $S$ . If the utility is linear and  $\mu$ ,  $\sigma$ ,  $K$ ,  $\beta$  are constant, it can be shown that the model degenerates to an optimal stopping problem (see Remark 16, in section 5). But when  $\mu$ ,  $\sigma$ ,  $K$ ,  $\beta$  are not constant and we consider various utility functions, a wide range of different solutions is possible. The paper is organized as fol-

lows. Section 2 contains the precise problem formulation and some preliminary results, including a logarithmic transformation that is convenient in order to solve our model. In section 3 of the paper we show the existence of an optimal control for the optimal stopping problem which arises when only one intervention is allowed. Sections 4 contains our main results: the characterization of the value function and the existence and characterization of the optimal impulse control. We obtain these results by reducing our model to an iterative sequence of optimal stopping problems and corresponding variational inequalities. This idea, first introduced in [5], has been used, for instance, in [10], [1], [2], to solve numerically some impulse control problems in mathematical finance. In section 5 we consider the simple case where the utility is linear,  $\mu$ ,  $\sigma$ ,  $K$  are constant but there are two possible values for  $\beta$ , depending on the size of  $S$ . We show that the value function is not continuously differentiable and that the optimal policy is an impulse control with one barrier and one target level, which can be computed explicitly. Section 6 concludes the paper with some final remarks.

## 2 Problem formulation and preliminary results

We consider a standard probability system  $(\Omega, F, P, \mathbb{F}, B_t)$ , where  $B_t$  is a one-dimensional Brownian motion and  $\mathbb{F} = F_t$  is the completed natural filtration of  $B_t$ . We call a generalized geometric Brownian motion  $S(t) : [0, \infty) \times \Omega \rightarrow \mathbb{R}$  on this system, the strong solution of the Itô stochastic differential equation (1) where we assume

$$S_0 > 0, \mu(S), \sigma(S) \in W^{1,\infty}(\mathbb{R}). \quad (\text{A1})$$

Here  $W^{1,\infty}(\mathbb{R})$  is the Sobolev space defined, in the sense of distributions, by

$$W^{1,\infty}(\mathbb{R}) \equiv \{f \in L^\infty \mid f'(x) \in L^\infty\}$$

and it is equivalent in  $\mathbb{R}$  with the functions which are bounded and Lipschitz continuous (in all the sequel the apex denotes a weak derivative). There certainly exists a unique strong solution to (1) because the coefficients  $S\mu(S)$  and  $S\sigma(S)$  are locally Lipschitz and satisfy a linear growth condition (see, for instance, Theorem 4.3.1 in Ikeda and Watanabe [16]). We make the following assumptions on the intervention costs  $K(\xi)$

$$\left\{ \begin{array}{l} K(\xi) : \mathbb{R}_+ \rightarrow (0, +\infty) \text{ is upper-semicontinuous} \\ K(\xi) \geq K(0) = F > 0 \\ \xi - K(\xi) \geq 0 \text{ if and only if } \xi \geq \xi_{\min} > 0. \end{array} \right. \quad (\text{A2})$$

Here  $\xi_{\min}$  is the minimum intervention necessary to obtain the non-negative consumption  $c = \xi - K(\xi)$ . An impulse control  $p = \{(\tau_1, \xi_1); \dots; (\tau_i, \xi_i); \dots\}$  is a sequence of stopping times  $\tau_i$  and corresponding random jumps  $\xi_i$  enforced

into the system. We say that a policy is feasible if it verifies

$$\left\{ \begin{array}{l} \tau_i \text{ is a } \{F_t\} \text{ stopping time} \\ \tau_i \leq \tau_{i+1} \text{ a.s.} \\ \tau_i \rightarrow \infty \text{ a.s. when } i \rightarrow \infty \end{array} \right. \quad (2)$$

$$\left\{ \begin{array}{l} \xi_i \text{ is } F_{\tau_i} \text{ measurable} \\ c_i \equiv \xi_i - K(\xi_i) \geq 0 \end{array} \right. . \quad (3)$$

For a given policy  $p$  and initial state  $S_0$ , the controlled process  $S_p(t)$  can be defined, in a concise way, by the unique solution of the following integral equation (see [5], chapter 6)

$$S_p(t) = S_0 + \int_0^t S_p(r) \mu(S_p(r)) dr + \int_0^t S_p(r) \sigma(S_p(r)) dB_r - \xi_1 - \dots - \xi_{\alpha_t}$$

where  $\alpha_t(\omega) = \max \{n \mid \tau_n(\omega) \leq t\}$ .

We will consider admissible only the feasible policies which verify the non-negativity constraint

$$S_p(t) \geq 0, \quad \forall t \geq 0$$

which implies,  $\forall i \geq 1$ ,

$$S_p(\tau_i^-) - \xi_i - \dots - \xi_{\alpha_i} \geq 0$$

where  $S_p(\tau_i^-) = \lim_{t \uparrow \tau_i} S_p(t)$  and  $\alpha_i(\omega) = \max \{n \geq i \mid \tau_n = \tau_i\}$ .

We denote by  $A_{S_0}$  the set of admissible policies when the process starts in  $S_0$ . To each  $p$  we associate a discounted rewarding functional

$$J(p) = E \left[ \sum_{i=1}^{\infty} U(c_i) e^{-\int_0^{\tau_i} \beta(S_t) dt} \chi_{\tau_i < \infty} \right] \quad (4)$$

where  $\chi_{\tau_i < \infty}(\omega) = \begin{cases} 1 & \text{if } \tau_i(\omega) < \infty \\ 0 & \text{elsewhere} \end{cases}$ .

We make the following assumptions on the utility function

$$\left\{ \begin{array}{l} U(c) : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ is increasing and upper-semicontinuous} \\ U(c) \geq U(0) = 0, \quad U(c) \leq a c^b \quad \text{where } a, b > 0 \end{array} \right. \quad (A3)$$

Thus  $U$  is not necessarily concave but it satisfies a polynomial growth condition. We assume that the discount rate  $\beta$  verifies

$$\beta \in L^\infty, \quad \beta(S) \geq \beta_{\min} > 0 . \quad (A4)$$

Moreover we will assume that  $\beta$  is large enough, for  $S \rightarrow +\infty$ , with respect to  $U, \mu, \sigma$ , in the sense that there exists  $\hat{S} > 0$  such that  $\beta$  verifies

$$\beta \geq \frac{1}{2} \sigma^2 \bar{b}^2 + (\mu - \frac{1}{2} \sigma^2) \bar{b}, \quad \forall S \in [\hat{S}, +\infty) \quad (A5)$$

where  $\bar{b}$  is defined by  $\bar{b} \equiv 1 \vee b \equiv \text{Max}(1, b)$ . To obtain some of our results we will require that this assumption holds in a strict way, that is there exists a constant  $D > 0$  such that

$$\beta - \frac{1}{2}\sigma^2\bar{b}^2 + (\mu - \frac{1}{2}\sigma^2)\bar{b} \geq D > 0, \quad \forall S \in [\hat{S}, +\infty). \quad (\text{A6})$$

Now we define the value function

$$V(S) = \sup_{p \in A_S} J(p)$$

and the problem is to look for the existence of an optimal control  $p^*(S_0)$ , for every initial state  $S_0$ , such that we have

$$V(S_0) = \text{Max}_{p \in A_{S_0}} J(p) = J(p^*(S_0)). \quad (\text{P})$$

Since the variational techniques we will use require to consider a process with bounded coefficients, it is convenient to work with the natural logarithm of the process  $S(t)$ . We define the functions

$$\zeta(x) = \mu(e^x), \quad \delta(x) = \sigma(e^x), \quad \rho(x) = \beta(e^x). \quad (5)$$

From (A1), (A4) and (5) it follows  $\zeta(x), \delta(x) \in W_{loc}^{1,\infty}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ , and

$\rho(x) \in L^\infty(\mathbb{R})$ ,  $\rho(x) \geq \rho_{\min} = \beta_{\min} > 0$ . We consider the auxiliary process  $x_v(t)$ , controlled by policy  $v = (\theta_i, \gamma_i)$ , defined as the solution of

$$x_v(t) = x_0 + \int_0^t (\zeta(x_v(r)) - \frac{1}{2}\delta^2(x_v(r))) dr + \int_0^t \delta(x_v(r)) dB_r - \gamma_1 - \dots - \gamma_{\epsilon_t}$$

where  $\epsilon_t(\omega) = \max\{n(\omega) \mid \theta_n(\omega) \leq t\}$ . The stopping times  $\theta_i$  must meet the same requirements (2) of the  $\tau_i$ . Now we define the functions

$$\begin{cases} d(x, \gamma) = e^x - e^{x-\gamma} \\ g(x, \gamma) = d(x, \gamma) - K(d(x, \gamma)) \end{cases} \quad (6)$$

where  $d, g : \mathbb{R} \times \mathbb{R}_+^* \rightarrow \mathbb{R}$ . Let  $x_{\min} \equiv \ln \xi_{\min}$  and  $\gamma_{\min}^x \equiv x - \ln(e^x - \xi_{\min})$ . Note that we have  $g(x, \gamma) \geq 0$  if and only if  $x \geq x_{\min}$  and  $\gamma \geq \gamma_{\min}^x > 0$ . We will consider admissible the random variables  $\gamma_i > 0$ , (which can take the value  $\gamma_i = +\infty$ , in which case  $x_v(t) = -\infty, \forall t \geq \theta_i$ ), if they verify

$$\begin{cases} \gamma_i \text{ is } F_{\theta_i} \text{ measurable} \\ g(x_v(\theta_i^-), \gamma_i) \geq 0. \end{cases}$$

To policy  $v$  it is associated the rewarding index

$$I(v) = E \left[ \sum_{i=1}^{\infty} U(g(x_v(\theta_i^-), \gamma_i)) e^{-\int_0^{\theta_i} \rho(x_t) dt} \chi_{\theta_i < \infty} \right] \quad (7)$$



and we consider the auxiliary problem

$$\underset{v \in \Gamma_{x_0}}{\text{Max}} I(v) \tag{A}$$

where  $\Gamma_{x_0}$  is the set of admissible policies if  $x_v(0) = x_0$ . Let  $\Phi(x)$  be the value function of problem (A)

$$\Phi(x) = \sup_{v \in \Gamma_x} I(v) .$$

The following lemma shows the equivalence between (A) and our original problem.

**Lemma 1** *If  $S_0 = e^{x_0}$  and if we set  $\tau_i = \theta_i$ ,  $\xi_i = d(x_v(\theta_i^-), \gamma_i)$ ,  $\forall i \geq 1$ , then the processes  $S_p(t)$  and  $e^{x_v(t)}$  are equal almost surely and  $J(p) = I(v)$ . Moreover  $V(S_0) = \Phi(\ln S_0)$  and if there exists an optimal  $v^* = \{(\theta_i^*, \gamma_i^*)\}$  for problem (A) then  $p^* = \{(\theta_i^*, d(x_v(\theta_i^{*-}), \gamma_i^*))\}$  is optimal for problem (P), and vice versa.*

**Proof.** By Itô's formula applied to  $e^{x_v(t)}$  up to  $\tau_1 = \theta_1$  the processes  $e^{x_v(t)}$  and  $S_p(t)$  are equal a.s. in  $[0, \tau_1)$ . In  $\tau_1 = \theta_1$  we have

$$e^{x_v(\tau_1)} = e^{x_v(\tau_1^-) - \gamma_1} = e^{x_v(\tau_1^-)} - d(x_v(\theta_1^-), \gamma_1) = S_p(\tau_1^-) - \xi_1$$

and consequently  $y(\tau_1) = S_p(\tau_1)$ . It is immediate to see recursively that  $e^{x_v(t)} = S_p(t)$ , a.s.,  $\forall t \geq 0$ . Setting  $\theta_i = \tau_i$ ,  $\xi_i = d(x_v(\theta_i^-), \gamma_i)$  a policy  $p \in A_{S_0}$  if and only if  $v \in \Gamma_{\ln S_0}$ , and since, by this correspondence,

$$U(g(x_v(\theta_i^-), \gamma_i)) e^{-\int_0^{\theta_i} \rho(x_t) dt} = U(\xi_i - K(\xi_i)) e^{-\int_0^{\tau_i} \beta(S_t) dt}$$

we also have  $J(p) = I(v)$ . Therefore it follows  $V(S_0) = \Phi(\ln S_0)$  and if  $v^* = \{(\theta_i^*, \gamma_i^*)\}$  verifies  $I(v^*) = \Phi(\ln S_0)$  then  $p^* = \{(\theta_i^*, d(x_v(\theta_i^{*-}), \gamma_i^*))\}$  verifies  $J(p^*) = V(S_0)$  and vice versa. ■

**Remark 2** *The utility of the logarithmic transformation is twofold. The process  $x(t)$  has bounded coefficients, and therefore it is possible to use the variational techniques. Furthermore (A) is a free optimization problem since there is no lower bound constraint on the state variable  $x$ . The only drawback is that the objective functional (7) is slightly more complex with respect to (4) because it also contains the left limits of  $x_v$  at the intervention times  $\theta_i$ . Note that the connection between the two problems lies in the definitions (5), (6) and that problem (A) is not well posed for arbitrary functions  $d$  and  $g$ .*

Thus we can focus on problem (A), under (5), (6) and assumptions (A1)-(A6), since its solution will give us immediately the optimal policy of our original problem. Since the value function may be unbounded in  $\mathbb{R}$  it is convenient to characterize  $\Phi(x)$  as a solution of a variational problem in a weighted Sobolev

space. We consider the regular weight function  $w_\alpha(x) = e^{-\alpha\sqrt{1+x^2}}$  ( $\alpha > 0$ ) and the weighted spaces  $L^{p,\alpha}(\mathbb{R})$ ,  $W^{1,p,\alpha}(\mathbb{R})$ ,  $W^{2,p,\alpha}(\mathbb{R})$ , defined as

$$\left\{ \begin{array}{l} L^{p,\alpha}(\mathbb{R}) = \{f \mid \int w_\alpha^p(x) |f|^p dx < \infty\} \\ W^{1,p,\alpha}(\mathbb{R}) = \{f \in L^{p,\alpha} \mid f'(x) \in L^{p,\alpha}\} \\ W^{2,p,\alpha}(\mathbb{R}) = \{f \in L^{p,\alpha} \mid f'(x), f''(x) \in L^{p,\alpha}\} . \end{array} \right.$$

In particular we have the Hilbert spaces  $H^{0,\alpha} = L^{2,\alpha}$ ,  $H^{1,\alpha} = W^{1,2,\alpha}$ ,  $H^{2,\alpha} = W^{2,2,\alpha}$ . We will denote by

$$\left\{ \begin{array}{l} (u, v)_\alpha = \int_{\mathbb{R}} u v w_\alpha^2 dx \\ (u, v)_{H^{1,\alpha}} = \int_{\mathbb{R}} u v w_\alpha^2 dx + \int_{\mathbb{R}} u' v' w_\alpha^2 dx \end{array} \right.$$

the inner products in  $H^{0,\alpha}$  and  $H^{1,\alpha}$  and by  $|u|_\alpha$ ,  $\|u\|_\alpha$  the corresponding norms. We also define the functions

$$a_2 = \frac{\delta^2}{2}, \quad a_1 = -\left(\zeta - \frac{\delta^2}{2}\right) + a'_2 .$$

For technical reasons we need the following additional assumption on the diffusion coefficient

$$\left\{ \begin{array}{l} \sigma(x) \geq \lambda > 0 \\ \sigma(e^x) = \delta(x) \in W^{1,\infty}(\mathbb{R}) . \end{array} \right. \quad (\text{A7})$$

The former of (A7) is a non-degeneracy assumption typical of the variational approach (see, however, [19] for a treatment of degenerate diffusions). From this assumption it follows  $a_2 \geq \frac{\lambda^2}{2} > 0$  and from (A1), (5) and the latter of (A7) we deduce  $a_2 \in W^{1,\infty}(\mathbb{R})$  and  $a_1 \in L^\infty(\mathbb{R})$ . We define a second order differential operator  $\mathcal{A}$  by

$$\mathcal{A}u = -a_2 u'' - a'_2 u' + a_1 u' + \rho u \quad (8)$$

which corresponds to the infinitesimal generator of the process  $x(t)$ . We will consider frequently the following continuous bilinear form in  $H^{1,\alpha}$

$$a_\alpha(u, v) = \int_{\mathbb{R}} a_2 u' v' w_\alpha^2 dx + \int_{\mathbb{R}} \left(a_1 - \frac{2a_2 x \alpha}{\sqrt{1+x^2}}\right) u' v w_\alpha^2 dx + \int_{\mathbb{R}} \rho u v w_\alpha^2 dx .$$

Since  $\rho$  is given  $a_\alpha(u, v)$  is not necessarily coercive, but choosing  $\vartheta > 0$  large enough, the bilinear form  $a_\alpha(u, v) + \vartheta (u, v)_\alpha$  becomes coercive on  $H^{1,\alpha}$ . It can be easily shown, integrating by parts, that if  $u \in H^{2,\alpha}$  and  $v \in H^{1,\alpha}$ , then we have

$$\int_{\mathbb{R}} (\mathcal{A}u) v w_\alpha^2 dx = a_\alpha(u, v) . \quad (9)$$

Consider now two stopping times  $\tau, \tau'$  (w.r.t.  $F_t$ ), such that  $0 \leq \tau \leq \tau'$  a.s., and  $z(t)$  the strong solution of the following equation

$$z(t) = \eta + \int_0^t \gamma(z(s)) \chi_{s>\tau} ds + \int_0^t \delta(z(s)) \chi_{s>\tau} dB_s$$

where  $\eta$  is  $F_\tau$  measurable,  $z = \eta$  for  $0 \leq t \leq \tau$  and  $\gamma, \delta \in W_{loc}^{1,\infty}(\mathbb{R})$ . We denote by  $I_r$  the open interval  $(-r, r)$  and by  $\tau_r \equiv \inf \{t \geq \tau \mid z(t) \notin I_r\}$  the first exit time after  $\tau$  of the process  $z$  from  $I_r$ . In the sequel we will use frequently the following Lemma. Taking the mathematical expectation in (10) one obtains a generalization of Dynkin's formula in  $H^{2,\alpha}(\mathbb{R})$ .

**Lemma 3** *If  $u \in H^{2,\alpha}(\mathbb{R})$  then we have,  $\forall r > 0$ ,*

$$\begin{aligned} & u(z(\tau))\chi_{\tau < \infty} = \\ & = E \left[ \int_{\tau}^{\tau' \wedge \tau_r} \chi_{\tau < \infty} (\mathcal{A}_z u(z(s))) e^{-\int_{\tau}^s \rho(z(t)) dt} ds + u(z(\tau' \wedge \tau_r)) e^{-\int_{\tau}^{\tau' \wedge \tau_r} \rho(z(t)) dt} \right] | F_\tau \end{aligned} \quad (10)$$

where  $\mathcal{A}_z u = -\frac{\delta^2}{2} u'' - \gamma u' + \rho u$ .

**Proof.** The proof is omitted because this Lemma is an adaptation of some results of chapter 6, in [5]. See, in particular, Lemmas 6.1.1-6.1.2, Corollary 6.1.1 and Theorem 6.1.2. ■

### 3 Study of a variational inequality and of the associated optimal stopping problem

In this section we consider the simplified case when it is possible to stop the process  $x(t)$ , and consequently  $S(t)$ , just one time. It corresponds to set  $\theta_2 = +\infty$  in the previous formulation. A control is now a couple  $v = (\theta, \gamma)$  of a  $\{F_\theta\}$  stopping time  $\theta$  and a random variable  $\gamma$  which is  $F_\theta$  measurable. The control is admissible if it verifies  $g(x(\theta^-), \gamma) \geq 0$ . The value function is

$$\Phi_1(x) \equiv \sup_{v \in \Gamma_{1,x}} I(v) = \sup_{(\theta, \gamma) \in \Gamma_{1,x}} E [U(g(x(\theta^-), \gamma)) e^{-\int_0^\theta \rho(x_t) dt} \chi_{\theta < \infty}]$$

where  $\Gamma_{1,x_0}$  is the set of admissible stopping policies when only one intervention is permitted. From (A5) and (5) it follows that there exists  $\hat{x} = \ln \hat{S}$  such that  $\rho$  verifies  $(\bar{b} = b \vee 1)$

$$\rho \geq \frac{1}{2} \delta^2 \bar{b}^2 + (\zeta - \frac{1}{2} \delta^2) \bar{b}, \quad \forall x \in [\hat{x}, +\infty). \quad (11)$$

Now we define a function  $u^M$  that we will use as an upper bound for  $\Phi_1(x)$  and  $\Phi(x)$ . Let  $N \geq a$  and  $C > 0$  be large enough to verify

$$\left| \begin{array}{l} a(x - F)^b \leq N x^{\bar{b}}, \quad \forall x \geq F \\ N e^{\bar{b}x} (\rho - \frac{1}{2} \delta^2 \bar{b}^2 - (\zeta - \frac{1}{2} \delta^2) \bar{b}) + \rho_{\min} C \geq 0 \quad \forall x \in \mathbb{R}. \end{array} \right. \quad (12)$$

We define

$$u^M(x) = C + N e^{\bar{b}x}. \quad (13)$$

We also consider a function  $\psi(x)$  which represents, for  $x \geq x_{\min}$ , the maximum utility we get if we stop immediately the process in  $x_0 = x$

$$\psi(x) = \begin{cases} \sup_{\gamma \in [\gamma_{\min}^x, +\infty]} U(g(x, \gamma)) & \text{if } x \geq x_{\min} \\ g(x, +\infty) & \text{if } x < x_{\min} . \end{cases} \quad (14)$$

It is always  $\Phi_1(x) \geq \psi(x)$  and we have  $\psi(x) \in L^{2,\alpha}$  if  $\alpha > b$ . The following theorem shows some properties of  $\psi(x)$ .

**Theorem 4** *Given assumptions (A2), (A3), the function  $\psi(x)$  verifies:*

- 1)  $-F < \psi(x) \leq u^M(x)$
- 2)  $\psi(x)$  is continuous
- 3) there exists a Borel measurable function  $\gamma^*(x) : [x_{\min}, \infty) \rightarrow \mathbb{R}$  such that  $(0, \gamma^*(x)) \in \Gamma_{1,x}$  and  $\psi(x) = U(g(x, \gamma^*(x)))$  for  $x \geq x_{\min}$ .

**Proof.** It is as a particular case of the proof of the subsequent theorem 7, setting  $u = 0$  in (29). ■

Let us define the set of functions

$$Z = \{z \in H^{1,\alpha} \mid 0 \leq z \leq u^M\} \quad (15)$$

and the following variational inequality, where  $\varphi \in L^{2,\alpha}$  and  $\alpha > b$

$$\begin{cases} a_\alpha(u, v - u) \geq 0 \\ \forall v \in H^{1,\alpha} \quad \text{such that} \quad v \geq \varphi \\ u \in Z, \quad u \geq \varphi . \end{cases} \quad (16)$$

The next theorem shows that if the discount rate  $\rho$  verifies (11) there always exists a minimum solution of (16).

**Theorem 5** *Given assumptions (A1), (A4), (A5), (A7),  $\varphi \in L^{2,\alpha}$ ,  $\varphi \leq u^M$  the variational inequality (16) has a minimum solution  $u_\varphi(x)$ .*

**Proof.** We consider the following auxiliary variational inequality

$$\begin{cases} a_\alpha(u, v - u) + \vartheta(u, v - u)_\alpha \geq (\vartheta z, v - u)_\alpha \\ \forall v \in H^{1,\alpha} \quad \text{such that} \quad v \geq \varphi \\ u \in H^{1,\alpha}, \quad u \geq \varphi \end{cases} \quad (17)$$

where  $z \in L^{2,\alpha}$  and  $\vartheta > 0$  is large enough to make  $a_\alpha(u, v) + \vartheta(u, v)_\alpha$  coercive in  $H^{1,\alpha}$ . It follows that there exists a unique solution  $u_z$  in  $H^{1,\alpha}$  of (17), (see theorem 1.13, chapter 3, Bensoussan and Lions [4]). We show now that

$$z \in Z \implies u_z \in Z. \quad (18)$$

From (8), the latter of (12) and (13) we deduce  $\mathcal{A}u^M \geq 0$  and from (9) it follows that

$$a_\alpha(u^M, v) = \int_{\mathbb{R}} (\mathcal{A}u^M) v w_\alpha^2 dx \geq 0$$

whenever  $v \in H^{1,\alpha}$  and  $v \geq 0$ . If we choose  $u = u_z$  and  $v = u_z - (u_z - u^M)^+$  in (17), which is an admissible test function because  $\varphi \leq u^M$ , we get

$$-a_\alpha(u_z, (u_z - u^M)^+) - \vartheta(u_z, (u_z - u^M)^+)_\alpha \geq -(\vartheta z, (u_z - u^M)^+)_\alpha .$$

If we add  $a_\alpha(u^M, (u_z - u^M)^+) \geq 0$  to the left side and  $(\vartheta u^M, (u_z - u^M)^+)_\alpha \geq 0$  to both sides of this inequality we obtain

$$-a_\alpha(u_z - u^M, (u_z - u^M)^+) - \vartheta(u_z - u^M, (u_z - u^M)^+)_\alpha \geq \vartheta(u^M - z, (u_z - u^M)^+)_\alpha$$

which is equivalent to

$$\begin{aligned} & a_\alpha((u_z - u^M)^+, (u_z - u^M)^+) + \vartheta((u_z - u^M)^+, (u_z - u^M)^+)_\alpha \leq \\ & \leq -\vartheta(u^M - z, (u_z - u^M)^+)_\alpha . \end{aligned}$$

Since by assumption  $u^M \geq z$  and  $a_\alpha(u, v) + \vartheta(u, v)_\alpha$  is coercive, this inequality implies  $(u_z - u^M)^+ = 0$  that is to say  $u_z \leq u^M$ . Moreover  $u = 0$  is the solution of (17) if  $z = 0$  and  $\varphi \leq 0$  and therefore  $u_z \geq 0$  because  $u_z$  is increasing in  $z$  and  $\varphi$ , and  $z \geq 0$  and  $\varphi \geq -\varphi^-$ . Therefore we have shown the implication (18). Now, we can define an operator  $G : Z \rightarrow Z$  such that  $u_z = Gz$ . From the monotonicity properties of the solution of variational inequalities (see theorem 1.4, chapter 3, in [4]) it follows that  $Gz_1 \leq Gz_2$  if  $z_1 \leq z_2$ . Thus we can consider the increasing sequence of functions

$$\begin{cases} u_n = Gu_{n-1} \\ u_0 = 0 . \end{cases} \quad (19)$$

The sequence  $u_n$  converges in  $L^{2,\alpha}$  to a function  $u_\varphi \in Z$ , with  $u_\varphi \geq u_n \geq \varphi$  by construction, and we aim to show that  $u_\varphi$  is the minimum solution of (16). If we set  $v = u^M$  in (17), which is always admissible, we obtain

$$a_\alpha(u_n, u_n) + \vartheta(u_n, u_n)_\alpha \leq a_\alpha(u_n, u^M) + \vartheta(u_n, u^M)_\alpha - (\vartheta u_{n-1}, u^M - u_n)_\alpha$$

and as  $a_\alpha(u, v) + \vartheta(u, v)_\alpha$  is continuous and coercive we have

$$\|u_n\|_\alpha^2 \leq C \|u_n\|_\alpha \|u^M\|_\alpha .$$

The norms  $\|u_n\|_\alpha \leq C \|u^M\|_\alpha$  stay bounded in  $H^{1,\alpha}$  and there is a subsequence  $u_m$  which converges weakly in  $H^{1,\alpha}$  to a function  $u^*$ . As the injection of  $H^{1,\alpha}$  in  $L^{2,\alpha}$  is compact we also deduce that  $u^* = u_\varphi$ . Since

$$F(v) = a_\alpha(v, v) + \vartheta(v, v)_\alpha$$

is lower semicontinuous in the weak topology of  $H^{1,\alpha}$  (see, for instance, corollary III.8, in Brezis [7])

$$a_\alpha(u_n, v) + \vartheta(u_n, v)_\alpha \geq a_\alpha(u_n, u_n) + \vartheta(u_n, u_n)_\alpha + \vartheta(u_{n-1}, v) - \vartheta(u_{n-1}, u_n)_\alpha$$

implies, as  $n \rightarrow \infty$ , that  $a_\alpha(u_\varphi, v - u_\varphi) \geq 0, \forall v \in H^{1,\alpha}$ . Therefore  $u_\varphi$  is a solution of (16) and it is easy to show that it is the minimum solution. Any solution  $u$  of (16) is a fixed point of  $G$  and from  $u_0 = 0 \leq u$  we deduce recursively  $u_n = Gu_{n-1} \leq u = G^n u$  and consequently  $u_\varphi \leq u$ . ■

The next theorem shows that if we set  $\varphi = \psi$  the value function  $\Phi_1$  is smaller than the minimum solution  $u_\psi$  of (16). Thus if assumption (A5) is verified the discount rate is large enough to assure a finite value function. Furthermore if (A6) holds true, that is, in terms of  $\rho$ ,

$$\rho - \frac{1}{2}\delta^2\bar{b}^2 + (\zeta - \frac{1}{2}\delta^2)\bar{b} \geq D > 0, \quad \forall x \in [\hat{x}, +\infty) \quad (20)$$

then we can show that there always exists an optimal stopping policy, which is Markovian because it depends only on the current state of the process.

**Theorem 6** *If  $\varphi = \psi$  the continuous representative of  $u_\psi(x)$  verifies*

$$u_\psi(x) \geq \Phi_1(x) .$$

*Moreover, if (A6) holds true, then  $u_\psi(x) = \Phi_1(x)$  and for every initial condition  $x_0 \in \mathbb{R}$  there exists an optimal stopping policy  $(\theta_{x_0}^*, \gamma_{x_0}^*)$  given by*

$$\begin{cases} \theta_{x_0}^* = \inf \{s \geq 0 : u_\psi(x(s)) = \psi(x(s))\} \\ \gamma_{x_0}^* = \begin{cases} \gamma^*(x(\theta_{x_0}^{*-})) & \text{if } \theta_{x_0}^* < \infty \\ \text{arbitrary} & \text{if } \theta_{x_0}^* = +\infty . \end{cases} \end{cases}$$

**Proof.** In order to make use of Lemma 2.2, which requires an  $H^2$  regularity, it is useful to consider the penalized equation

$$a_\alpha(u^\varepsilon, v) = \frac{1}{\varepsilon} ((\psi - u^\varepsilon)^+, v)_\alpha, \quad u^\varepsilon \in Z. \quad (21)$$

By an argument similar to that used in theorem 4 to show the existence of  $u_\varphi$  we can prove that there exists a minimum solution  $u_\psi^\varepsilon$  of (21). Indeed, we consider the equation

$$a_\alpha(u, v) + \vartheta(u, v)_\alpha = \frac{1}{\varepsilon} ((\psi - u)^+, v)_\alpha + \vartheta(z, v)_\alpha$$

which has a unique solution  $u_z^\varepsilon \in Z$  if  $z \in Z$ . Starting from  $u_0^\varepsilon = 0$ , the increasing sequence of functions  $u_n^\varepsilon$ , defined by  $u_n^\varepsilon = u_{z_n}^\varepsilon$  where  $z_n = u_{n-1}^\varepsilon$ , converges as  $n \rightarrow \infty$  to the minimum solution  $u_\psi^\varepsilon$  of (21). Moreover, by an adaptation of theorem 4.1.4 in Bensoussan and Lions [5], it can be shown that,

as  $\varepsilon \rightarrow 0$ , we have  $u_n^\varepsilon \uparrow u_n$ , where  $u_n$  is given by (19), and consequently we also have  $u_\psi^\varepsilon \uparrow u_\psi$  when  $\varepsilon \rightarrow 0$ . Let  $(\theta, \gamma) \in \Gamma_{1,x}$  be an admissible policy. Since (21) is an equation, by local regularity it follows that  $u_\psi^\varepsilon \in H^2(I_r)$ ,  $\forall r > 0$ . We can apply formula (10) to the function  $u_\psi^\varepsilon(x)$  and the process  $x(s)$ , considering  $\tau = 0$ ,  $\tau' = \theta$ . Using (9) and (21) and taking the mathematical expectation we obtain

$$u_\psi^\varepsilon(x_0) = E \left[ \frac{1}{\varepsilon} \int_0^{\theta \wedge \tau_r} (\psi - u_\psi^\varepsilon)^+(x(s)) e^{-\int_0^s \rho(z(t)) dt} ds + e^{-\int_0^{\theta \wedge \tau_r} \rho(x(t)) dt} u_\psi^\varepsilon(x(\theta \wedge \tau_r)) \right]. \quad (22)$$

We have  $\tau_r \rightarrow \infty$  a.s. when  $r \rightarrow \infty$ ,  $u_\psi^\varepsilon \uparrow u_\psi$  as  $\varepsilon \rightarrow 0$ ,  $u_\psi \geq u_\psi^\varepsilon$  and  $u_\psi \geq \psi \geq \psi \chi_{\theta < \infty}$ . From (22) we can deduce that for  $\varepsilon \rightarrow 0$  and  $r \rightarrow \infty$

$$u_\psi(x_0) \geq E \left[ e^{-\int_0^\theta \rho(x(t)) dt} \psi(x(\theta)) \chi_{\theta < \infty} \right]. \quad (23)$$

But if  $\theta$  is admissible necessarily  $x(\theta) \geq x_{\min}$  and therefore from (14) and (23) we have

$$u_\psi(x_0) \geq E \left[ U(g(x(\theta^-), \gamma)) e^{-\int_0^\theta \rho(x(t)) dt} \chi_{\theta < \infty} \right].$$

As this inequality is true  $\forall (\theta, \gamma) \in \Gamma_{1,x_0}$  it follows  $u_\psi(x) \geq \Phi_1(x)$ .

We define  $\theta_\varepsilon^* \equiv \inf \left\{ s \geq 0 : u_\psi^\varepsilon(x(s)) \leq \psi(x(s)) \right\}$ . We have  $\theta_\varepsilon^* \leq \theta_{x_0}^*$  because  $u_\psi \geq u_\psi^\varepsilon$  and  $\theta_\varepsilon^* \rightarrow \theta_{x_0}^*$  for  $\varepsilon \rightarrow 0$ . If we set  $\theta = \theta_\varepsilon^*$  in equation (22) and we let  $\varepsilon \rightarrow 0$  we obtain

$$\begin{aligned} u_\psi(x_0) &= E \left[ e^{-\int_0^{\theta_{x_0}^* \wedge \tau_r} \rho(x(t)) dt} u_\psi(x(\theta_{x_0}^* \wedge \tau_r)) \right] = \\ &= E \left[ e^{-\int_0^{\theta_{x_0}^*} \rho(x(t)) dt} u_\psi(x(\theta_{x_0}^*)) \chi_{\theta_{x_0}^* < \tau_r} + e^{-\int_0^{\tau_r} \rho(x(t)) dt} u_\psi(x(\tau_r)) \chi_{\tau_r \leq \theta_{x_0}^*} \right]. \end{aligned} \quad (24)$$

As  $r \rightarrow \infty$  it follows that

$$E \left[ e^{-\int_0^{\theta_{x_0}^*} \rho(x(t)) dt} u_\psi(x(\theta_{x_0}^*)) \chi_{\theta_{x_0}^* < \tau_r} \right] \rightarrow E \left[ e^{-\int_0^{\theta_{x_0}^*} \rho(x(t)) dt} u_\psi(x(\theta_{x_0}^*)) \chi_{\theta_{x_0}^* < \infty} \right]$$

by the monotone convergence theorem. Since  $u_\psi \leq u^M$  we also have

$$\begin{aligned} &E \left[ e^{-\int_0^{\tau_r} \rho(x(t)) dt} u_\psi(x(\tau_r)) \chi_{\tau_r \leq \theta_{x_0}^*} \right] \leq \\ &\leq C E \left[ e^{-\int_0^{\tau_r} \rho(x(t)) dt} \right] + N E \left[ e^{-\int_0^{\tau_r} \rho(x(t)) dt + \bar{b}x(\tau_r)} \right] \leq \\ &\leq C E \left[ e^{-\rho_{\min} \tau_r} \right] + N E \left[ (e^{-\bar{b}r} \chi_{x(\tau_r)=-r} + e^{\bar{b}r} \chi_{x(\tau_r)=r}) e^{-\int_0^{\tau_r} \rho(x(t)) dt} \right]. \end{aligned} \quad (25)$$

Considering the exponential martingale

$$M(t) = \exp[\bar{b}x(t) - ((\zeta(x(t)) - \frac{1}{2}\delta^2(x(t)))\bar{b} + \frac{1}{2}\delta^2(x(t))\bar{b}^2)t]$$

from (20) and  $\tau_r \rightarrow \infty$  a.s., it can be shown that, for  $r \rightarrow \infty$ ,

$$E \left[ e^{\bar{b}r} \chi_{x(\tau_r)=r} e^{-\int_0^{\tau_r} \rho(x(t)) dt} \right] \rightarrow 0 .$$

Therefore from (25), making  $r \rightarrow \infty$ , we obtain

$$E \left[ e^{-\int_0^{\tau_r} \rho(x(t)) dt} u_\psi(x(\tau_r)) \chi_{\tau_r \leq \theta_{x_0}^*} \right] \rightarrow 0 . \quad (26)$$

Finally from (24), (26) and  $u_\psi(x(\theta_{x_0}^*)) = \psi(x(\theta_{x_0}^*))$  it follows

$$\begin{aligned} u_\psi(x_0) &= E \left[ e^{-\int_0^{\theta_{x_0}^*} \rho(x(t)) dt} u_\psi(x(\theta_{x_0}^*)) \chi_{\theta_{x_0}^* < \infty} \right] \\ &= E \left[ e^{-\int_0^{\theta_{x_0}^*} \rho(x(t)) dt} \psi(x(\theta_{x_0}^*)) \chi_{\theta_{x_0}^* < \infty} \right] \\ &= I(\theta_{x_0}^*, \gamma^*(x(\theta_{x_0}^* -))) \end{aligned}$$

and consequently  $u_\psi(x) = \Phi_1(x) = I(\theta_x^*, \gamma^*(x(\theta_x^* -)))$ . ■

The open set  $Q \equiv \{x \in \mathbb{R}: u_\psi(x) > \psi(x)\}$  is the continuation region where the process  $x_v$  evolves freely. The complementary closed region  $Q^C \equiv \{x \in \mathbb{R}: u_\psi(x) = \psi(x)\}$  is the intervention region where it is optimal to consume immediately the quantity  $\psi(x)$ . The set  $Q$  is nonempty because it contains  $(-\infty, x_{\min})$  and  $Q^C$  is closed because  $\psi$  is continuous.

In the next sections we will need the following important additional result on  $u_\psi(x)$ , which is a corollary to theorem 6.

**Corollary 7** *Let  $\theta_i \leq \theta_{i+1}$  be two consecutive stopping times of  $v$  and  $x_v(t)$  the corresponding controlled process. The minimum solution  $u_\psi(x)$  of (16) verifies*

$$E \left[ e^{-\int_{\theta_i}^{\theta_{i+1}} \rho(x_v(t)) dt} u_\psi(x_v(\theta_{i+1}^-)) \chi_{\theta_{i+1} < \infty} \mid F_{\theta_i} \right] \leq u_\psi(x_v(\theta_i)) \chi_{\theta_i < \infty} . \quad (27)$$

Furthermore if

$$\theta_{i+1}^* = \inf \{ t \geq \theta_i \mid u_\psi(x_v(t^-)) = \psi(x_v(t^-)) \}$$

and (20) holds true then we have the equality

$$E \left[ e^{-\int_{\theta_i}^{\theta_{i+1}^*} \rho(x_v(t)) dt} u_\psi(x_v(\theta_{i+1}^* -)) \chi_{\theta_{i+1}^* < \infty} \mid F_{\theta_i} \right] = u_\psi(x_v(\theta_i)) \chi_{\theta_i < \infty} . \quad (28)$$

**Proof.** It is sufficient to apply formula (10) to the minimum solution  $u_\psi^\varepsilon(x)$  of (21) and to the process  $x_v(s)$  in the interval  $[\theta_i, \theta_{i+1} \wedge \tau_r]$ , considering  $\tau = \theta_i$ ,  $\tau' = \theta_{i+1}$ . Using the same argument of the proof of theorem 6, by making  $\varepsilon \rightarrow 0$  and afterwards  $r \rightarrow \infty$  we obtain (27) and if  $\theta' = \theta_{i+1}^*$  we deduce (28). ■



## 4 Existence and characterization of the optimal consumption policy

When an infinity of stopping times are available the maximum reward we can obtain if we stop immediately the process in  $x_0 = x \geq x_{\min}$  is given by

$$\sup_{\gamma \in [\gamma_{\min}^x, +\infty]} U(g(x, \gamma)) + \Phi(x - \gamma) .$$

Then it is natural to define a non local operator  $\mathcal{M}$  on functions  $u : \mathbb{R} \rightarrow \mathbb{R}_+$  such that

$$\mathcal{M}u(x) = \begin{cases} \sup_{\gamma \in [\gamma_{\min}^x, +\infty]} U(g(x, \gamma)) + u(x - \gamma) & \text{if } x \geq x_{\min} \\ g(x, +\infty) & \text{if } x < x_{\min} . \end{cases} \quad (29)$$

$\mathcal{M}$  has the following properties.

**Theorem 8** *Given assumptions (A2), (A3) and  $0 \leq u \leq u^M$ , the function  $\mathcal{M}u$  verifies:*

- 1)  $-F < \mathcal{M}u \leq u^M$
- 2) if  $u$  is continuous then there exists a Borel measurable function  $\gamma_u^* : [x_{\min}, \infty) \rightarrow \mathbb{R}_+$ , such that, for  $x \geq x_{\min}$  we have

$$\mathcal{M}u(x) = U(g(x, \gamma_u^*(x))) + u(x - \gamma_u^*(x)) \quad (30)$$

- 3)  $\mathcal{M}u$  is continuous if  $u$  is continuous.

### Proof.

1) For  $x < x_{\min}$  we have  $-F < \mathcal{M}u(x) < 0$ . For  $x \geq x_{\min}$  we have  $\mathcal{M}u \geq 0$  and from (A3), (12), (13) it follows

$$\begin{aligned} \mathcal{M}u(x) &\leq \sup_{\gamma \in [\gamma_{\min}^x, +\infty]} a(d(x, \gamma) - F)^b + u^M(x - \gamma) \\ &\leq \sup_{\gamma \in [\gamma_{\min}^x, +\infty]} N(d(x, \gamma))^{\bar{b}} + C + Ne^{\bar{b}(x-\gamma)} \\ &\leq \sup_{\gamma \in [\gamma_{\min}^x, +\infty]} C + Ne^{\bar{b}x}((1 - e^{-\gamma})^{\bar{b}} + e^{-\bar{b}\gamma}) < u^M \end{aligned}$$

as  $\bar{b} = \text{Max}(b, 1) \geq 1$ .

2) Since  $U$  and  $g$  are upper-semicontinuous and  $u$  is continuous we have, for  $x \geq x_{\min}$

$$\limsup_{\gamma \rightarrow \infty} U(g(x, \gamma)) + u(x - \gamma) \leq U(e^x - K(e^x))$$

and therefore there certainly exists  $\gamma^*$  such that

$$\mathcal{M}u(x) = U(g(x, \gamma^*)) + u(x - \gamma^*) .$$

By using a selection theorem (see appendix B in Fleming and Rishel [13]) we can define the Borel measurable function  $\gamma_u^*(x)$  which verifies (30),  $\forall x \geq x_{\min}$ .

3) We omit the proof because, when  $U$  and  $g$  are upper-semicontinuous and  $u$  is continuous, this is a well known property of the non local operator in impulse control problems (see, for example, [1]). ■

Let  $\Phi_n(x)$  be the value function of the problem when at the most  $n$  stopping times are available (i.e.  $\theta_{n+1} = +\infty$ ). It is natural to look at  $\Phi_n(x)$  as the solution of (16) with the obstacle  $\psi(x) = \mathcal{M}\Phi_{n-1}(x)$  and to define recursively a sequence starting from  $\Phi_0 = 0$ . We will look for the value function as the limit of  $\Phi_n(x)$  when  $n \rightarrow \infty$ . Now, we consider the following quasi-variational inequality in  $H^{1,\alpha}$

$$\left\{ \begin{array}{l} a_\alpha(u, v - u) \geq 0 \\ \forall v \in H^{1,\alpha} \text{ such that } v \geq \mathcal{M}u \\ u \geq \mathcal{M}u, \quad u \in Z \end{array} \right. \quad (31)$$

Inequality (31) may have many solutions, as it is usual with unbounded domains, but the most relevant solution comes out to be the minimum solution.

**Theorem 9** *Under assumptions (A1)-(A5) and (A7), the quasi variational inequality (31) has a minimum solution  $u_{\min}$ . Moreover, the function  $u_{\min}$  verifies*

$$u_{\min}(x) \geq \Phi(x) = \sup_{v \in \Gamma_x} I(v) \quad \forall x \in \mathbb{R} . \quad (32)$$

**Proof.** We consider an operator  $T : Z \rightarrow Z$ , which relates  $z \in Z$  to the continuous solution  $u_\psi \in Z$  of (16) corresponding to the obstacle  $\psi = \mathcal{M}z$ . This obstacle verifies the assumptions on  $\varphi$  of theorem 5. The operator  $T$  has the fundamental monotonicity property that  $Tz_1 \leq Tz_2$  if  $z_1 \leq z_2$ . This property follows from property 1 of  $\mathcal{M}$  and the fact that the solution of (16) increases when the obstacle  $\psi$  is increasing. We define the increasing sequence of functions

$$\left\{ \begin{array}{l} u_n = Tu_{n-1} \\ u_0 = 0 \end{array} \right. \quad (33)$$

Using the same reasoning of theorem 5 we can show that  $u_n$  converges pointwise to a function  $u \in H^{1,\alpha}$  which verifies  $a_\alpha(u_{\min}, v - u) \geq 0$ ,  $\forall v \in H^{1,\alpha}$  such that  $v \geq \mathcal{M}u$ . Moreover from  $u_n \geq \mathcal{M}u_{n-1}$  it follows  $u_n \geq U(g(x, \gamma)) + u_{n-1}(x - \gamma)$  and since  $\gamma \geq \gamma_{\min}^x$  is arbitrary, for  $n \rightarrow \infty$  we deduce immediately that  $u_{\min} \geq \mathcal{M}u_{\min}$ . Therefore  $u_{\min}$  is a solution of (31) and we can show, in the same way we did in Theorem 5, that it is the minimum solution. If we apply the inequality (27) to  $u_{\min} \geq \mathcal{M}u_{\min}$  and to an arbitrary admissible policy

$v = (\theta_i, \gamma_i)$ , multiplying both sides by  $e^{-\int_0^{\theta_i} \rho(x_v(t)) dt}$  we obtain

$$\begin{aligned} & e^{-\int_0^{\theta_i} \rho(x_v(t)) dt} u_{\min}(x_v(\theta_i)) \chi_{\theta_i < \infty} \\ & \geq E \left[ e^{-\int_0^{\theta_{i+1}} \rho(x_v(t)) dt} \mathcal{M} u_{\min}(x_v(\theta_{i+1}^-)) \chi_{\theta_{i+1} < \infty} \mid F_{\theta_i} \right]. \end{aligned}$$

From the definition of  $\mathcal{M}u$  and  $x_v(\theta_{i+1}) = x_v(\theta_{i+1}^-) - \gamma_{i+1}$  it follows

$$\begin{aligned} & e^{-\int_0^{\theta_i} \rho(x_v(t)) dt} u_{\min}(x_v(\theta_i)) \chi_{\theta_i < \infty} \\ & \geq E \left[ e^{-\int_0^{\theta_{i+1}} \rho(x_v(t)) dt} U(g(x_v(\theta_{i+1}^-), \gamma_{i+1})) \chi_{\theta_{i+1} < \infty} \mid F_{\theta_i} \right] \\ & \quad + E \left[ e^{-\int_0^{\theta_{i+1}} \rho(x_v(t)) dt} u_{\min}(x_v(\theta_{i+1})) \chi_{\theta_{i+1} < \infty} \mid F_{\theta_i} \right]. \end{aligned} \tag{34}$$

If we take the mathematical expectation in (34) and we sum up all the inequalities for  $i$  varying from 0 to  $n-1$ , recalling that  $\theta_0 \equiv 0$ , we get

$$\begin{aligned} u_{\min}(x) & \geq E \left[ \sum_{i=1}^n U(g(x_v(\theta_i^-), \gamma_i)) e^{-\int_0^{\theta_i} \rho(x_t) dt} \chi_{\theta_i < \infty} \right] \\ & \quad + E \left[ e^{-\int_0^{\theta_n} \rho(x_t) dt} u_{\min}(x_v(\theta_n)) \chi_{\theta_n < \infty} \right] \end{aligned}$$

and as  $u_{\min} \geq 0$  and  $x \in \mathbb{R}$ ,  $v \in \Gamma_x$  are arbitrary, we deduce for  $n \rightarrow \infty$   $u_{\min}(x) \geq \Phi(x)$ ,  $\forall x \in \mathbb{R}$ . ■

We describe now the optimal policy  $v^*$ . For this purpose we define in  $\mathbb{R}$  the continuation region  $Q$  where the system evolves freely

$$Q = \{x \in \mathbb{R} : u_{\min}(x) > \mathcal{M}u_{\min}(x)\} \tag{35}$$

and the complementary closed intervention region  $Q^C$ . The first optimal stopping time is defined to be the first exit time of the uncontrolled process from  $Q$

$$\theta_1^*(x_0) = \inf \{t \geq 0 \mid x(t) \notin Q\}. \tag{36}$$

In  $\theta_1^*$  the optimal jump

$$\gamma_1^* = \begin{cases} \gamma^*(x(\theta_1^{*-})) & \text{if } \theta_1^* < \infty \\ \text{arbitrary} & \text{if } \theta_1^* = +\infty \end{cases} \tag{37}$$

is enforced, where  $\gamma^* = \gamma_{u_{\min}}^*(x)$  is the function which verifies (30) with  $u = u_{\min}$  (to simplify the notation we omit the dependence of  $\theta_i^*$  and  $\gamma_i^*$  on the initial condition  $x_0$ ). The subsequent  $(\theta_{i+1}^*, \gamma_{i+1}^*)$  are defined recursively by

$$\left| \begin{aligned} & \theta_{i+1}^* = \inf \{t \geq \theta_i^* \mid x_v(t^-) \notin Q\} \\ & \gamma_{i+1}^* = \begin{cases} \gamma^*(x_{v^*}(\theta_{i+1}^{*-})) & \text{if } \theta_{i+1}^* < \infty \\ \text{arbitrary} & \text{if } \theta_{i+1}^* = +\infty. \end{cases} \end{aligned} \right. \tag{38}$$

The next theorem shows that  $v_x^* = \{(\theta_i^*, \gamma_i^*)\}$  is admissible and optimal and that  $u_{\min}$  is the value function.

**Theorem 10** *Under assumptions (A1)-(A4) and (A6), (A7), the policy  $v_x^* = \{(\theta_i^*, \gamma_i^*)\}$  defined in (36)-(38) is admissible and optimal. Moreover the minimum solution  $u_{\min}$  of (31) is the value function of problem (A), that is to say*

$$u_{\min}(x) = \Phi(x) = I(v_x^*) \quad \forall x \in \mathbb{R} .$$

**Proof.** First we show that  $v_{x_0}^* = \{(\theta_i^*, \gamma_i^*)\}$  is admissible. Suppose there exists  $T > 0$ , such that  $\lim_{i \rightarrow \infty} \theta_i^*(\omega) = T$  on a set  $A$  of positive probability. Since the trajectories of the uncontrolled process  $x(t)$  are a.s. continuous there exists  $\omega_1 \in A$  such that  $x(t, \omega_1) \leq M$  for  $t \in [0, T]$ . Since  $x_v(t) \leq x(t)$  we can deduce that  $\lim_{i \rightarrow \infty} x_v(\theta_i^*(\omega_1)) = -\infty$  because at each  $\theta_i^*$  we have  $\gamma_i^* \gg 0$ . Therefore there exists  $\theta_i^*$  such that  $x_v(\theta_i^*(\omega_1)) \in \text{int}(Q)$ , the interior of  $Q$ , but this contradicts the definition of  $\theta_i^*$  and thus  $\theta_i^* \rightarrow \infty$  a.s. when  $i \rightarrow \infty$ . By using (28) if we consider  $v^*$  in the inequality (34) we obtain an equality and summing up for  $i$  varying from 0 to  $n-1$ , after taking expectations, we get

$$\begin{aligned} u_{\min}(x) &= E \left[ \sum_{i=1}^n U(g(x_v(\theta_i^*^-), \gamma_i)) e^{-\int_0^{\theta_i^*} \rho(x_t) dt} \chi_{\theta_i < \infty} \right] + \\ &+ E \left[ e^{-\int_0^{\theta_n^*} \rho(x_t) dt} u_{\min}(x_v(\theta_n^*)) \chi_{\theta_n < \infty} \right] . \end{aligned} \quad (39)$$

Since  $x_v(t) \leq x(t)$  and  $0 \leq u_{\min} \leq u^M$  we deduce

$$E \left[ e^{-\int_0^{\theta_n^*} \rho(x_t) dt} u_{\min}(x_v(\theta_n^*)) \chi_{\theta_n < \infty} \right] \leq E \left[ e^{-\int_0^{\theta_n^*} \rho(x_t) dt} (C + N e^{\bar{b}x(\theta_n^*)}) \chi_{\theta_n^* < \infty} \right] .$$

For  $n \rightarrow \infty$  we have  $\theta_n^* \rightarrow +\infty$  a.s., and if  $\rho$  verifies (20) we obtain

$$\lim_{n \rightarrow \infty} E \left[ e^{-\int_0^{\theta_n^*} \rho(x_t) dt} u_{\min}(x_v(\theta_n^*)) \chi_{\theta_n < \infty} \right] = 0 . \quad (40)$$

Therefore making  $n \rightarrow \infty$  in (39), from (40) we have  $u_{\min}(x) = I(v_x^*)$ ,  $\forall x \in \mathbb{R}$  and from (32) we conclude that

$$u_{\min}(x) = I(v_x^*) = \Phi(x), \quad \forall x \in \mathbb{R} .$$

■

If we come back to our original problem, it is immediate to obtain the value function and the optimal policy of problem (P).

**Corollary 11** *Under assumptions (A1)-(A4) and (A5)-(A7), the value function of problem (P) is given by*

$$V(S) = \Phi(\ln S) = u_{\min}(\ln S)$$

where  $u_{\min}$  is the minimum solution of (31). Moreover there exists an optimal Markovian control  $p^* = \{(\tau_i^*, \xi_i^*)\}$  which is obtained recursively from ( $i \geq 1, \tau_0 \equiv 0$ )

$$\left| \begin{array}{l} \tau_i^* = \inf \{t \geq \tau_{i-1}^* \mid \ln S_{p^*}(t^-) \notin Q\} \\ \xi_i^* = \begin{cases} S_{p^*}(\tau_i^{*-}) [1 - e^{-\gamma_{u_{\min}}^*(\ln S_{p^*}(\tau_i^{*-}))}] & \text{if } \tau_i^* < \infty \\ \text{arbitrary} & \text{if } \tau_i^* = +\infty \end{cases} \end{array} \right. \quad (41)$$

where  $Q$  and the function  $\gamma_{u_{\min}}^*$  are defined respectively in (35) and (30).

**Proof.** The result follows immediately applying Lemma 1. ■

**Remark 12** As an immediate consequence of Theorem 10 and Corollary 11 we obtain that  $\Phi(x)$  and  $V(S)$  are continuous respectively in  $\mathbb{R}$  and  $\mathbb{R}_+$ . If  $u_{\min} \in H^{2,\alpha}$  than one can easily show the equivalence between (31) and the strong formulation

$$\left\{ \begin{array}{l} \mathcal{A}u \geq 0, \quad u \in H^{2,\alpha} \cap Z \\ u \geq \mathcal{M}u \\ (u - \mathcal{M}u)\mathcal{A}u = 0 \end{array} \right. \quad (42)$$

which is the QVI formulation commonly used to obtain verification theorems for stationary (infinite horizon) impulse control problems. However the existence of a solution to (42) requires more restrictive assumptions than those used in theorem 9. In the next section we give an example where the value function is not smooth enough to be a solution of (42) but it is a solution of (31).

**Corollary 13** If there exists a solution  $u \in H^{2,\alpha}$  of (31), that solution is unique in  $H^{2,\alpha}$  and it is the value function of problem (A), that is  $u = u_{\min} = \Phi$ .

**Proof.** In this case the formula (10) can be applied directly to  $u \in H^{2,\alpha}$  and we obtain (27) and (28). Using the same proof of theorem 10 it follows that  $u$  is the value function. But since the value function is necessarily unique there exists at most one solution  $u \in H^{2,\alpha}$  of (31), and we have  $u = u_{\min}$ . ■

## 5 Linear utility and a two values discount rate

In this section we consider, as an illustrative example, the simple case

$$\left| \begin{array}{l} \mu(S) = \mu > 0, \quad \sigma(S) = \sigma > 0, \quad K(\xi) = F, \quad U(c) = c \\ \beta(S) = \begin{cases} \beta_1 & \text{if } S < 1 \\ \beta_2 & \text{if } S \geq 1 \end{cases} \quad \text{with } \beta_1 < \mu < \beta_2. \end{array} \right. \quad (43)$$

Here the agent is more reluctant to postpone consumption when he is sufficiently rich, specifically when the amount of  $S$  becomes greater than the threshold level

$S = 1$ . The assumptions of Corollary 11 are verified and thus there exists the optimal consumption. We aim to find  $V(S)$  and the optimal policy in an explicit way, assuming  $F$  to be large enough (see condition (45) below). In terms of problem (A) we have

$$\zeta(x) = \mu > 0, \delta(x) = \sigma > 0, \rho(x) = \begin{cases} \beta_1 & \text{if } x < 0 \\ \beta_2 & \text{if } x \geq 0 \end{cases} . \quad (44)$$

We define, for  $\beta > 0$ ,

$$\lambda(\beta) = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\beta\sigma^2}}{\sigma^2} > 0$$

and we denote  $\lambda_1 = \lambda(\beta_1)$ ,  $\lambda_2 = \lambda(\beta_2)$ . From  $\beta_1 < \mu < \beta_2$  it follows

$$0 < \lambda_1 < 1 < \lambda_2$$

and we will assume

$$F > \frac{1}{\lambda_1} - \frac{1}{\lambda_2} . \quad (45)$$

We will need the following result.

**Lemma 14** *Given condition (45) the system of equations*

$$\begin{cases} ke^{\lambda_1 D} - ke^{\lambda_2 d} + e^d - e^D = F \\ k\lambda_2 e^{\lambda_2 d} = e^d \\ k\lambda_1 e^{\lambda_1 D} = e^D \end{cases} \quad (46)$$

has a unique solution  $(k^*, D^*, d^*) \in \mathbb{R}^3$  such that  $k^* > 0$  and  $D^* < 0 < d^*$ .

**Proof.** From (46) we obtain, after some calculation

$$\begin{cases} (1 - \lambda_1)\lambda_1^{\frac{\lambda_1}{1-\lambda_1}} k^{\frac{1}{1-\lambda_1}} + (\lambda_2 - 1)\lambda_2^{-\frac{\lambda_2}{\lambda_2-1}} k^{-\frac{1}{\lambda_2-1}} = F \\ d = \frac{\log k\lambda_2}{1-\lambda_2} \\ D = \frac{\log k\lambda_1}{1-\lambda_1} . \end{cases} \quad (47)$$

We denote by  $G(k) = F$  the first equation in (47) and from  $0 < \lambda_1 < 1 < \lambda_2$  it follows  $\lim_{k \rightarrow 0^+} G(k) = \lim_{k \rightarrow +\infty} G(k) = +\infty$  and  $G''(k) > 0$  for  $k > 0$ . From the last two equations in (47) in order to have  $D < d$  we derive

$$k < \lambda_1^{-\frac{\lambda_2-1}{\lambda_2-\lambda_1}} \lambda_2^{-\frac{\lambda_1-1}{\lambda_2-\lambda_1}} \equiv \bar{k} .$$

We define  $\bar{d} = \frac{\log \bar{k}\lambda_2}{1-\lambda_2}$ ,  $\bar{D} = \frac{\log \bar{k}\lambda_1}{1-\lambda_1}$  and we have  $\bar{d} = \bar{D} < 0$  because  $\bar{k} > 0$  and  $0 < \lambda_1 < 1 < \lambda_2$ . Given the form of  $G(k)$  the equation  $G(k) = F$  has a unique solution  $k^* > 0$  if and only if  $G(\bar{k}) < F$  and  $0 < k^* < \bar{k}$ . But from

$\bar{d} = \bar{D}$  and (46) we obtain  $G(\bar{k}) = \bar{k}e^{\lambda_1 \bar{D}} - \bar{k}e^{\lambda_2 \bar{d}} = e^{\bar{d}}(\frac{1}{\lambda_1} - \frac{1}{\lambda_2})$ . Since  $\bar{d} < 0$  and  $F > \frac{1}{\lambda_1} - \frac{1}{\lambda_2}$  it follows  $G(\bar{k}) < F$ . Therefore the first equation in (47) has a unique solution  $k^* > 0$ , and we set  $d^* = \frac{\log k^* \lambda_2}{1 - \lambda_2}$ ,  $D^* = \frac{\log k \lambda_1}{1 - \lambda_1}$  where  $D^* < d^*$  because  $k^* < \bar{k}$ . As  $(k^*, D^*, d^*)$  is a solution of (46) it must verify  $(\frac{1}{\lambda_1} - 1)e^{D^*} + (1 - \frac{1}{\lambda_2})e^{d^*} = F$ . From (45) and  $D^* < d^*$  it follows  $D^* < 0$  and  $d^* > 0$ . ■

Now we will show that the value function of problem (A), that is  $u_{\min}$ , is given by

$$u(x) = \begin{cases} k^* e^{\lambda_1 x} & \text{if } x \in (-\infty, 0) \\ k^* e^{\lambda_2 x} & \text{if } x \in [0, d^*) \\ e^x - e^{D^*} - F + k^* e^{\lambda_1 D^*} & \text{if } x \in [d^*, +\infty) . \end{cases} \quad (48)$$

This function is continuous in  $\mathbb{R}$  by (46), but it is not continuously differentiable in  $x = 0$  and consequently it is not a solution of (42). Thus, in order to obtain the optimal policy, we cannot apply to  $u$  the usual verification theorems which require a  $C^1$  regularity. Nevertheless we will show that  $u(x) = u_{\min}(x)$  is the minimum solution of (31) and therefore, by using Corollary 11, we are able in this case to solve problem (P) explicitly. We define  $S_{D^*} \equiv e^{D^*} < 1$ ,  $S_{d^*} \equiv e^{d^*} > 1$ .

**Theorem 15** *If (43), (45) hold true then we have*

$$V(S) = \begin{cases} k^* S^{\lambda_1} & \text{if } S \in [0, 1) \\ k^* S^{\lambda_2} & \text{if } S \in [1, S_{d^*}) \\ S - S_{D^*} - F + k^* S_{D^*}^{\lambda_1} & \text{if } S \in [S_{d^*}, +\infty) \end{cases}$$

and the optimal consumption policy  $p^* = \{(\tau_i^*, \xi_i^*)\}$  is given recursively by ( $i \geq 1$ ,  $\tau_0 \equiv 0$ )

$$\left| \begin{array}{l} \tau_i^* = \inf t \geq \tau_{i-1}^* \mid S_{p^*}(t^-) \notin [0, S_{d^*}) \\ \xi_i^* = \left| \begin{array}{l} S_{p^*}(\tau_i^{*-}) - S_{D^*} \text{ if } \tau_i^* < \infty \\ \text{arbitrary if } \tau_i^* = +\infty . \end{array} \right. \end{array} \right.$$

**Proof.** Let  $u$  be defined as in (48). It is sufficient to show that  $u$  is the minimum solution of (31) and then apply Corollary 11. Choosing  $C$  and  $N$  large enough in (13) we have  $0 < u < u^M$  and thus  $u \in Z$ .

Let  $H(x, \gamma) = g(x, \gamma) + u(x - \gamma)$ . From (6), (48) and  $K(\xi) = F > 0$  constant, it follows

$$\frac{\partial H}{\partial \gamma}(x, \gamma) = \begin{cases} e^{x-\gamma} - k^* \lambda_1 e^{\lambda_1(x-\gamma)} & \text{if } x - \gamma \in (-\infty, 0) \\ e^{x-\gamma} - k^* \lambda_2 e^{\lambda_2(x-\gamma)} & \text{if } x - \gamma \in (0, d^*) \\ 0 & \text{if } x - \gamma \in [d^*, +\infty) . \end{cases} \quad (49)$$

By (49) and Lemma 14 we derive

$$\begin{cases} \frac{\partial H}{\partial \gamma}(x, \gamma) > 0 & \text{if } x - \gamma \in (D^*, 0) \cup (0, d^*) \\ \frac{\partial H}{\partial \gamma}(x, \gamma) = 0 & \text{if } x - \gamma \in \{0\} \cup [d^*, +\infty) \\ \frac{\partial H}{\partial \gamma}(x, \gamma) < 0 & \text{if } x - \gamma \in (-\infty, D^*) . \end{cases} \quad (50)$$

As  $K(\xi) = F$  we have  $x_{\min} = \ln F$  and  $x - \gamma_{\min}^x \geq D^*$  when  $x \geq \ln(F + e^{D^*})$ . From the definition (29) of  $\mathcal{M}u$  and (43), (48), (50), we obtain

$$\mathcal{M}u(x) = \begin{cases} e^x - e^{D^*} - F + k^* e^{\lambda_1 D^*} & \text{if } x \in (\ln(F + e^{D^*}), +\infty) \\ k^* e^{\lambda_1(x - \gamma_{\min}^x)} & \text{if } x \in [\ln F, \ln(F + e^{D^*})] \\ e^x - F & \text{if } x \in (-\infty, \ln F) . \end{cases} \quad (51)$$

By the first of (46) it follows  $d^* > \ln(F + e^{D^*})$  and comparing (48) with (51) it is easy to see that  $u = \mathcal{M}u$  if  $x \geq d^*$  and  $u > \mathcal{M}u$  if  $x < d^*$ . For  $x \geq d^*$  we have  $\gamma_u^*(x) = x - D^*$ , where  $\gamma_u^*$  is the function defined by (30). To prove that  $u$  is a solution of (31) it remains to show that  $a_\alpha(u, v - u) \geq 0$ ,  $\forall v \in H^{1,\alpha}$  such that  $v \geq \mathcal{M}u$ . From (8) and (44) we have

$$\mathcal{A}u = \begin{cases} k^* e^{\lambda_1 x} (-\frac{1}{2}\sigma^2 \lambda_1^2 - (\mu - \frac{1}{2}\sigma^2)\lambda_1 + \beta_1) & \text{if } x \in (-\infty, 0) \\ k^* e^{\lambda_2 x} (-\frac{1}{2}\sigma^2 \lambda_2^2 - (\mu - \frac{1}{2}\sigma^2)\lambda_2 + \beta_2) & \text{if } x \in (0, d^*) \\ (e^x - e^{D^*} - F + k^* e^{\lambda_1 D^*})(\beta_2 - \frac{\mu e^x}{e^x - e^{D^*} - F + k^* e^{\lambda_1 D^*}}) & \text{if } x \in (d^*, +\infty). \end{cases}$$

From the definition of  $\lambda_1, \lambda_2$  it follows  $\mathcal{A}u = 0$  in  $(-\infty, 0) \cup (0, d^*)$ , and by  $(\frac{1}{\lambda_1} - 1)e^{D^*} < F$  and  $\beta_2 > \mu$  we obtain  $\mathcal{A}u > 0$  in  $(d^*, +\infty)$ . For any  $v \in H^{1,\alpha}$ ,  $v \geq \mathcal{M}u$  we have  $(Au, v - u)_\alpha = (Au, (v - u)^+)_\alpha - (Au, (v - u)^-)_\alpha \geq 0$  because  $\mathcal{A}u \geq 0$  a.e. and  $\mathcal{A}u = 0$  when  $u > v \geq \mathcal{M}u$ . Integrating by parts in  $(-\infty, 0)$  and in  $(0, d^*)$  it is easy to see that

$$a_\alpha(u, v - u) = (Au, v - u)_\alpha + (v(0) - k^*) \frac{(\lambda_2 - \lambda_1)k^* \sigma^2 w_\alpha^2}{2}$$

and by density it follows that  $a_\alpha(u, v - u) \geq 0$ ,  $\forall v \in H^{1,\alpha}$  such that  $v \geq \mathcal{M}u$ . Finally it is not difficult to show that  $u = u_{\min}$  is the minimum solution of (31). If we consider the sequence of systems ( $n \geq 1$ ) starting with  $k_0 \equiv 0$

$$\begin{cases} k_{n-1} e^{\lambda_1 D_n} - k e^{\lambda_2 d_n} + e^{d_n} - e^{D_n} = F \\ k_n \lambda_2 e^{\lambda_2 d_n} = e^{d_n} \\ k_{n-1} \lambda_1 e^{\lambda_1 D_n} = e^{D_n} \end{cases} \quad (52)$$

we can show, as in Lemma 14, that each system in (52) has a unique solution  $(k_n^*, D_n^*, d_n^*)$  such that  $k^* > k_n^* > 0$ ,  $D_n^* < D^* < 0$ ,  $d_n^* > d^* > 0$ . Furthermore the sequences  $k_n^*, D_n^*$  are increasing while  $d_n^*$  is decreasing and thus we have



$(k_n^*, D_n^*, d_n^*) \rightarrow (k^*, D^*, d^*)$  as  $n \rightarrow \infty$ . Therefore we can show as before that  $(u_0 = 0; D_1^* = -\infty)$

$$u_n(x) \begin{cases} k_n^* e^{\lambda_1 x} & \text{if } x \in (-\infty, 0) \\ k_n^* e^{\lambda_2 x} & \text{if } x \in [0, d_n^*) \\ e^x - e^{D_n^*} - F + k_{n-1}^* e^{\lambda_1 D_n^*} & \text{if } x \in [d_n^*, +\infty) \end{cases}$$

is a solution of (31) when the obstacle  $\mathcal{M}u$  is replaced by  $\mathcal{M}u_{n-1}$ . Since  $u_n$  is increasing and  $u_n \rightarrow u$  we conclude as in Theorem 9 that  $u$  is the minimum solution of (31). ■

**Remark 16** For  $t > 0$  the wealth of the agent will remain in the interval  $(0, S_{d^*}]$ . Apart from the initial consumption in  $t = 0$ , if  $S_0 > S_{d^*}$ , he will always consume the amount  $S_{d^*} - S_{D^*}$  whenever his wealth reaches the barrier  $S = S_{d^*}$ . If  $\beta_1 = \beta_2 > \mu$  it is not difficult to see that our example degenerates to an optimal stopping problem. The solution becomes

$$V(S) = \begin{cases} kS^\lambda & \text{for } S \in (0, \frac{F\lambda}{\lambda-1}) \\ (S - F) & \text{for } S \in [\frac{F\lambda}{\lambda-1}, +\infty) \end{cases}$$

$$\left| \begin{array}{l} \tau_1^* = \inf \left\{ t \geq 0 \mid S(t) \notin (0, \frac{F\lambda}{\lambda-1}) \right\} \\ \xi_1^* = \begin{cases} S(\tau_1^*) \\ \text{arbitrary} \text{ if } \tau_1^* = +\infty \end{cases} \end{array} \right.$$

where  $k = \frac{1}{\lambda^\lambda} (\frac{\lambda-1}{F})^{\lambda-1}$ ;  $\tau_i^* = +\infty$  and  $\xi_i^*$  arbitrary, for  $i > 1$ .

## 6 Conclusions

In this paper we have shown the existence and the structure of the optimal consumption of a generalized geometric Brownian motion under general assumptions on the utility function and the strictly positive intervention costs. The presence of a fixed component in the intervention cost leads to a solution completely different from continuous consumption: the agent consumes only by finite amounts at separated time instants. The generality of our assumptions on the dynamics of  $S$ , on  $K$ ,  $U$  and  $\beta$  allows to consider a great variety of different situations and it is important to deal with realistic applications. In section 5 we have obtained the optimal consumption explicitly in a simple case, giving an example where the value function is not continuously differentiable (it is likely that the value function is not  $C^1$  also if we consider an utility function which is only upper-semicontinuous). In most cases it will not be possible to obtain closed form solutions of our model. However to the extent that we manage to numerically solve the variational inequality (16), the sequence of increasing functions  $u_n$  defined in (33) can be used to compute  $u_{\min}$  and consequently the optimal policy. In this direction the variational techniques can be useful to prove the convergence of a numerical scheme which calculates the QVI solution by a sequence of iterated variational inequalities.

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DEPARTMENT OF ECONOMICS AND STATISTICS  
UNIVERSITY OF TORINO  
Corso Unione Sovietica 218 bis - 10134 Torino (ITALY)  
Web page: <http://esomas.econ.unito.it/>

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