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SECOND-ORDER STRESS RELATIONS FOR HYPERELASTIC CONSTRAINED MATERIALS

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Abstract: In this paper the second-order stress relations for hyperelastic internally constrained materials are derived, both for the Cauchy stress and the two Piola-Kirchhoff stresses. In our approach the constitutive equations are obtained by the corresponding finite constitutive equations by means of suitable expansions. In contrast to the classical approach, our method guarantees the accuracy required by a second-order theory. For incompressible isotropic materials explicit stress relations are derived and compared with those used in classical theory, in order to show that only our constitutive equations are accurate to second order of approximation.

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Key Words: hyperelastic constrained materials, second-order stress relations

1. Introduction

In this paper we derive both for the Cauchy stress and the two Piola-Kirchhoff stresses second-order constitutive equations appropriate for hyperelastic constrained materials; our method is based on suitable expansions which generalize to the second order of approximation the results obtained to first order of approximation by Hoger and Johnson in [3] and by Tonon in [9]. In [3] the so-

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called linearized finite theory of elasticity (for brevity, LFTE in the following) is formulated in order to derive for hyperelastic internally constrained materials linear constitutive equations by linearization of the corresponding finite constitutive equations. In [3] first-order stress relations are obtained for the Cauchy stress and the first Piola-Kirchhoff stress, while in [9] the linear constitutive equation for the second Piola-Kirchhoff stress is derived. In [3], [9] the stress relations provided by LFTE are also compared with those usually adopted for constrained materials in classical linear elasticity (CLTE in the following); for a detailed treatment of CLTE, we refer to [1], Section 58. Comparison shows that only LFTE provides stress relations which are accurate to first order of approximation with respect to the displacement gradient. Many other papers are devoted to LFTE (see [2], [4], [5], [6], [7], [8]). They concern both static problems (see [2], [4], [7], [8]) and dynamical problems (see [5], [6]); in both cases application of LFTE shows that for constrained materials CLTE is inadequate to guarantee the accuracy required by a linear model.

The same occurs for a second-order theory. For this reason, the knowledge of stress relations which are correct to first or second order of approximation is of primary importance; as an example, we recall the fundamental role played by the second Piola-Kirchhoff stress in computational mechanics, since such a tensor is a symmetric tensor referred to the reference configuration.

Finally, it is worth noting that the second-order stress relations obtained in this paper hold for any constraint and for any material symmetry appropriate for the constraint, while this is not true for the second-order constitutive equations which usually occur in the literature concerning constrained materials.

In Section 2 we briefly summarize the method followed in LFTE to derive first-order constitutive equations; in particular, we recall the stress relations obtained in [3], [9] for the Cauchy stress and the two Piola-Kirchhoff stresses. As noted in [9], to first order of approximation the three stress tensors differ by terms which are first order in the strain, while in CLTE they are indistinguishable.

In Section 3 and Section 4 we generalize to the second order of approximation the method exposed in Section 2 and we obtain the second-order constitutive equations for the three stress tensors.

In Section 5 by using the results exposed in Section 3 and Section 4 we provide for incompressible isotropic materials explicit second-order constitutive relations for the three stress tensors. Moreover, for such materials we compare the constitutive equations for the Cauchy stress and the second Piola-Kirchhoff stress obtained by our approach with the corresponding constitutive equations usually adopted in classical theory. We show that both for the Cauchy stress

and the second Piola-Kirchhoff stress only the reaction stress provided by the classical approach is coincidentally correct to the second order of approximation.

2. The first-order stress relations according to the Linearized Finite Theory of Elasticity

In this section we briefly recall the procedure of linearization used in LFTE in order to obtain by the finite theory of elasticity for hyperelastic constrained materials linear constitutive equations for the three stresss tensors.

We refer to [3], [9] for more details.

Let $\mathcal{B}_0$ be a reference configuration of a body and let $\mathcal{B} = \mathbf{f}(\mathcal{B}_0)$ be the deformed configuration, where $\mathbf{f}$ is a deformation function that carries point $\mathbf{X} \in \mathcal{B}_0$ into point $\mathbf{x} = \mathbf{f}(\mathbf{X}) \in \mathcal{B}$.

Denote by

$$\mathbf{F} = \operatorname{Grad} \mathbf{f} \tag{1}$$

the deformation gradient, where Grad is the gradient operator taken with respect to $\mathbf{X}$.

The finite Green strain tensor $\mathbf{E}_G$ is defined through the deformation gradient $\mathbf{F}$ as

$$\mathbf{E}_G = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right), \tag{2}$$

where $\mathbf{I}$ denotes the identity tensor. The displacement $\mathbf{u}$ is related to $\mathbf{f}$ by

$$\mathbf{u}(\mathbf{X}) = \mathbf{f}(\mathbf{X}) - \mathbf{X}, \tag{3}$$

so that the displacement gradient $\mathbf{H} = \operatorname{Grad} \mathbf{u}$ can be expressed in terms of $\mathbf{F}$ as

$$\mathbf{H} = \mathbf{F} - \mathbf{I}. \tag{4}$$

Since this section is devoted to a linear theory of elasticity, the magnitude of the tensor $\mathbf{H}$ is assumed to be small ($\|\mathbf{H}\| \to 0$), and only terms that are most linear in $\mathbf{H}$ are retained in all equations.

Let $\mathcal{B}$ be a constrained finite hyperelastic material; denoting by $W = \hat{W}(\mathbf{E}_G)$ the strain energy function and by

$$\hat{c}(\mathbf{E}_G) = 0 \tag{5}$$

the constraint equation, the finite constitutive equation for the Cauchy stress $\mathbf{T}$ is

$$\mathbf{T} = (\det \mathbf{F})^{-1}\, \mathbf{F}\, \frac{\partial \hat{W}}{\partial \mathbf{E}_G}(\mathbf{E}_G)\mathbf{F}^T + q\mathbf{F}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{E}_G)\mathbf{F}^T, \tag{6}$$

where q is a Lagrange multiplier (see [3], formula (3.3)).

In order to linearize equation (6) about the zero strain state we follow a procedure in which the linearization of the derivative of $\hat{W}$ parallels that of the derivative of $\hat{c}$; moreover according to a linear theory we use the following expansions

$$\det \mathbf{F} \cong 1 + \mathrm{tr}\mathbf{H} \tag{7}$$

$$(\det \mathbf{F})^{-1} \cong 1 - \mathrm{tr}\mathbf{H} \tag{8}$$

$$\mathbf{E}_G \cong \mathbf{O} + \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right) \tag{9}$$

$$\frac{\partial \hat{W}}{\partial \mathbf{E}_G}(\mathbf{E}_G) \cong \frac{1}{2}\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H} + \mathbf{H}^T\right) \tag{10}$$

$$\hat{c}\,(\mathbf{E}_G) \cong \frac{1}{2}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) \cdot \left(\mathbf{H} + \mathbf{H}^T\right) \tag{11}$$

$$\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{E}_G) \cong \frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + \frac{1}{2}\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H} + \mathbf{H}^T\right). \tag{12}$$

In the previous formulas and in the following $\mathbf{O}$ is the zero tensor, while the symbol $\cdot$ denotes scalar product; note that the hypothesis of zero residual stress, that is $\dfrac{\partial \hat{W}}{\partial \mathbf{E}_G}(\mathbf{O}) = \mathbf{O}$, has been used to write expansion (10), while expansion (11) takes into account the condition $\hat{c}\,(\mathbf{O}) = 0$ provided by (5).

The final expression for the Cauchy stress $\mathbf{T}$ appropriate for LFTE is the following

$$\begin{aligned}\mathbf{T} \cong \frac{1}{2}\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\bigg|_{\tilde{c}}\left(\mathbf{H}+\mathbf{H}^T\right) + q\Bigg\{\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + \mathbf{H}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + \\ + \frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O})\mathbf{H}^T + \frac{1}{2}\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H} + \mathbf{H}^T\right)\Bigg\}\end{aligned} \tag{13}$$

(see [3], formula (3.22)); in (13) the subscript $\tilde{c}$ indicates evaluation on the linearized constraint equation

$$\tilde{c}\,(\mathbf{E}) = 0, \tag{14}$$

where

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H}+\mathbf{H}^T\right) \tag{15}$$

is the infinitesimal strain tensor and

$$\tilde{c}\,(\mathbf{E}) = \frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) \cdot \mathbf{E} \tag{16}$$

is the linear constraint function.

If we denote by $l(\mathbf{E}_G)$ the complete list of the polynomial invariants of the strain appropriate for the material symmetry of the body, we have $\hat{W}(\mathbf{E}_G) = \hat{\omega}(l(\mathbf{E}_G))$ and $\hat{c}(\mathbf{E}_G) = \hat{\xi}(l(\mathbf{E}_G))$; then the derivatives of $\hat{W}$ and $\hat{c}$ can be written explicitly in terms of the polynomial invariants as follows

$$\frac{\partial \hat{W}}{\partial \mathbf{E}_G}(\mathbf{E}_G) = \sum_{p=1}^{n} \frac{\partial \hat{\omega}}{\partial I_p}(l(\mathbf{E}_G)) \frac{\partial I_p}{\partial \mathbf{E}_G}(\mathbf{E}_G) \tag{17}$$

$$\begin{aligned} \frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{E}_G) &= \sum_{p=1}^{n} \frac{\partial \hat{\omega}}{\partial I_p}(l(\mathbf{E}_G)) \frac{\partial^2 I_p}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{E}_G) + \\ &+ \sum_{q=1}^{n} \left(\frac{\partial I_q}{\partial \mathbf{E}_G}(\mathbf{E}_G) \otimes \sum_{p=1}^{n} \frac{\partial^2 \hat{\omega}}{\partial I_p \partial I_q}(l(\mathbf{E}_G)) \frac{\partial I_p}{\partial \mathbf{E}_G}(\mathbf{E}_G) \right) \end{aligned} \tag{18}$$

$$\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{E}_G) = \sum_{p=1}^{n} \frac{\partial \hat{\xi}}{\partial I_p}(l(\mathbf{E}_G)) \frac{\partial I_p}{\partial \mathbf{E}_G}(\mathbf{E}_G) \tag{19}$$

$$\begin{aligned} \frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{E}_G) &= \sum_{p=1}^{n} \frac{\partial \hat{\xi}}{\partial I_p}(l(\mathbf{E}_G)) \frac{\partial^2 I_p}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{E}_G) + \\ &+ \sum_{q=1}^{n} \left(\frac{\partial I_q}{\partial \mathbf{E}_G}(\mathbf{E}_G) \otimes \sum_{p=1}^{n} \frac{\partial^2 \hat{\xi}}{\partial I_p \partial I_q}(l(\mathbf{E}_G)) \frac{\partial I_p}{\partial \mathbf{E}_G}(\mathbf{E}_G) \right), \end{aligned} \tag{20}$$

where n is the number of the polynomial invariants and the symbol $\otimes$ denotes tensor product.

It follows that the derivatives of $\hat{W}$ and $\hat{c}$ which appear in (13) take the explicit form

$$\begin{aligned} \frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}) &= \sum_{p=1}^{n} \frac{\partial \hat{\omega}}{\partial I_p}(l(\mathbf{O})) \frac{\partial^2 I_p}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}) + \\ &+ \sum_{q=1}^{n} \left(\frac{\partial I_q}{\partial \mathbf{E}_G}(\mathbf{O}) \otimes \sum_{p=1}^{n} \frac{\partial^2 \hat{\omega}}{\partial I_p \partial I_q}(l(\mathbf{O})) \frac{\partial I_p}{\partial \mathbf{E}_G}(\mathbf{O}) \right) \end{aligned} \tag{21}$$

$$\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) = \sum_{p=1}^{n} \frac{\partial \hat{\xi}}{\partial I_p}(l(\mathbf{O})) \frac{\partial I_p}{\partial \mathbf{E}_G}(\mathbf{O}) \tag{22}$$

$$\begin{aligned}\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}) &= \sum_{p=1}^{n} \frac{\partial \hat{\xi}}{\partial I_p}(l(\mathbf{O})) \frac{\partial^2 I_p}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}) + \\ &+ \sum_{q=1}^{n} \left(\frac{\partial I_q}{\partial \mathbf{E}_G}(\mathbf{O}) \otimes \sum_{p=1}^{n} \frac{\partial^2 \hat{\xi}}{\partial I_p \partial I_q}(l(\mathbf{O})) \frac{\partial I_p}{\partial \mathbf{E}_G}(\mathbf{O}) \right).\end{aligned} \tag{23}$$

In order to derive the constitutive equations for the first Piola-Kirchhoff stress $\mathbf{S}$ and for the second Piola-Kirchhoff stress $\tilde{\mathbf{T}}$ appropriate for LFTE we recall that in finite elasticity the following relations between $\mathbf{S}$ and $\mathbf{T}$ and between $\tilde{\mathbf{T}}$ and $\mathbf{T}$ hold

$$\mathbf{S} = (\det \mathbf{F})\mathbf{T}\mathbf{F}^{-T} \tag{24}$$

$$\tilde{\mathbf{T}} = (\det \mathbf{F})\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}, \tag{25}$$

respectively.

According to a linear theory, from (4) we have

$$\mathbf{F}^{-1} \cong \mathbf{I} - \mathbf{H} \tag{26}$$

$$\mathbf{F}^{-T} \cong \mathbf{I} - \mathbf{H}^T. \tag{27}$$

The relation between $\mathbf{S}$ and $\mathbf{T}$ appropriate for LFTE can be obtained by substituting (7), (27) into (24) and retaining only terms that are most of first order in $\mathbf{H}$; likewise the relation between $\tilde{\mathbf{T}}$ and $\mathbf{T}$ follows by (25) with the use of (7), (26), (27).

Such relations are

$$\mathbf{S} \cong \mathbf{T} + (\mathrm{tr}\mathbf{H})\mathbf{T} - \mathbf{T}\mathbf{H}^T \tag{28}$$

$$\tilde{\mathbf{T}} \cong \mathbf{T} + (\mathrm{tr}\mathbf{H})\mathbf{T} - \mathbf{H}\mathbf{T} - \mathbf{T}\mathbf{H}^T, \tag{29}$$

respectively (see [9], formulas (31), (32)). Since the Cauchy stress $\mathbf{T}$ is given by (13), relations (28), (29) become

$$\begin{aligned}\mathbf{S} \cong \frac{1}{2} \frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}) \bigg|_{\tilde{c}} \left(\mathbf{H} + \mathbf{H}^T\right) + q \Big\{ (1 + \mathrm{tr}\mathbf{H}) \frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + \\ + \mathbf{H} \frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + \frac{1}{2} \frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}) \left(\mathbf{H} + \mathbf{H}^T\right) \Big\}\end{aligned} \tag{30}$$

and

$$\begin{aligned}\tilde{\mathbf{T}} \cong \frac{1}{2} \frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}) \bigg|_{\tilde{c}} \left(\mathbf{H} + \mathbf{H}^T\right) + q \Big\{ (1 + \mathrm{tr}\mathbf{H}) \frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + \\ + \frac{1}{2} \frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}) \left(\mathbf{H} + \mathbf{H}^T\right) \Big\},\end{aligned} \tag{31}$$

respectively (see [3], formula (3.23), and [9], formula (34)).

Equations (13), (30), (31) show that in LFTE the three stress tensors have the same determinate stress, while the reaction stress is different. This is in contrast with CLTE, since in such a theory the three stress tensors are indistinguishable to first order of approximation both for unconstrained and constrained materials.

3. The second-order stress relation for the Cauchy stress tensor

Assuming that to first order of approximation LFTE holds, in this section we write a suitable expansion up to terms of second order in $\mathbf{H}$ for the Cauchy stress $\mathbf{T}$ appropriate for hyperelastic constrained materials.

As usual, the starting-point is the finite constitutive equation for $\mathbf{T}$ given by (6).

The first step is to expand all quantities (except q) appearing in such an equation up to terms of second order in $\mathbf{H}$. By (2), (4), (15) we have for the Green strain tensor the expression

$$\mathbf{E}_G = \mathbf{O} + \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right) + \frac{1}{2}\mathbf{H}^T\mathbf{H}, \tag{32}$$

without approximation.

For the constraint function $\hat{c}\left(\mathbf{E}_G\right)$ we can write the following expansion

$$\begin{aligned} \hat{c}\left(\mathbf{E}_G\right) \;&\cong \frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) \cdot (\mathbf{E}_G - \mathbf{O}) + \\ &+ \frac{1}{2}\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}) \cdot \left((\mathbf{E}_G - \mathbf{O}) \otimes (\mathbf{E}_G - \mathbf{O})\right), \end{aligned} \tag{33}$$

where the condition $\hat{c}(\mathbf{O}) = 0$ has been used.

If we substitute (32) into (33) and we stop our expansion to second-order terms in $\mathbf{H}$, (33) becomes

$$\begin{aligned} \hat{c}\left(\mathbf{E}_G\right) \cong\; &\frac{1}{2}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) \cdot \left(\mathbf{H} + \mathbf{H}^T\right) + \frac{1}{2}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) \cdot \left(\mathbf{H}^T\mathbf{H}\right) + \\ &+ \frac{1}{8}\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}) \cdot \left(\mathbf{H} \otimes \mathbf{H} + \mathbf{H} \otimes \mathbf{H}^T + \right. \\ &\left. + \mathbf{H}^T \otimes \mathbf{H} + \mathbf{H}^T \otimes \mathbf{H}^T\right); \end{aligned} \tag{34}$$

then the second-order constraint equation imposed by constraint equation (5) is

$$\bar{c}(\mathbf{H}) = 0, \tag{35}$$

where we have set

$$\begin{aligned}\bar{c}(\mathbf{H}) &= \frac{1}{2}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) \cdot \left(\mathbf{H}^T\mathbf{H}\right) + \\ &+ \frac{1}{8}\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}) \cdot \left(\mathbf{H}\otimes\mathbf{H} + \mathbf{H}\otimes\mathbf{H}^T + \right. \\ &\left. + \mathbf{H}^T\otimes\mathbf{H} + \mathbf{H}^T\otimes\mathbf{H}^T\right).\end{aligned} \tag{36}$$

Note that if equation (35) holds, also linear constraint equation (14) is satisfied: according to a second-order theory the possible displacement gradients $\mathbf{H}$ must satisfy (35), (36) and also (14), (15), (16).

Now we remark that the second-order expansion of $\det \mathbf{F}$ can be written in the following form

$$\det \mathbf{F} \cong 1 + \mathrm{tr}\mathbf{H} + \frac{1}{2}\left\{(\mathrm{tr}\mathbf{H})^2 - \mathrm{tr}\left(\mathbf{H}^2\right)\right\}, \tag{37}$$

so that

$$(\det \mathbf{F})^{-1} \cong 1 - \mathrm{tr}\mathbf{H} + \frac{1}{2}\left\{(\mathrm{tr}\mathbf{H})^2 + \mathrm{tr}\left(\mathbf{H}^2\right)\right\}. \tag{38}$$

Since the reference configuration is a natural state, we can write for the first derivative of the strain energy function the following expansion

$$\begin{aligned}\frac{\partial \hat{W}}{\partial \mathbf{E}_G}(\mathbf{E}_G) &\cong \frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})(\mathbf{E}_G - \mathbf{O}) + \\ &+ \frac{1}{2}\,\frac{\partial^3 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left((\mathbf{E}_G - \mathbf{O})\otimes(\mathbf{E}_G - \mathbf{O})\right).\end{aligned} \tag{39}$$

Substitution of (32) into (39) provides for $\dfrac{\partial \hat{W}}{\partial \mathbf{E}_G}(\mathbf{E}_G)$ the following expansion to the second order in $\mathbf{H}$

$$\begin{aligned}\frac{\partial \hat{W}}{\partial \mathbf{E}_G}(\mathbf{E}_G) &\cong \frac{1}{2}\,\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H} + \mathbf{H}^T\right) + \\ &+ \frac{1}{2}\,\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H}^T\mathbf{H}\right) + \\ &+ \frac{1}{8}\,\frac{\partial^3 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H}\otimes\mathbf{H} + \mathbf{H}\otimes\mathbf{H}^T + \right. \\ &\left. + \mathbf{H}^T\otimes\mathbf{H} + \mathbf{H}^T\otimes\mathbf{H}^T\right).\end{aligned} \tag{40}$$

Similarly we have

$$\begin{aligned}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{E}_G) \cong\ & \frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + \frac{1}{2}\,\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H}+\mathbf{H}^T\right) + \\ & + \frac{1}{2}\,\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H}^T\mathbf{H}\right) + \\ & + \frac{1}{8}\,\frac{\partial^3 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H}\otimes\mathbf{H}+\mathbf{H}\otimes\mathbf{H}^T + \right. \\ & \left. + \mathbf{H}^T\otimes\mathbf{H}+\mathbf{H}^T\otimes\mathbf{H}^T\right).\end{aligned} \tag{41}$$

In order to obtain the second-order expression for the Cauchy stress $\mathbf{T}$ we substitute into (6) the relations $\mathbf{F} = \mathbf{I}+\mathbf{H}$, $\mathbf{F}^T = \mathbf{I}+\mathbf{H}^T$, given by (4), and the expansions (38), (40), (41); of course after such substitutions we stop our expansions to second-order terms in $\mathbf{H}$.

If we write for $\mathbf{T}$ the decomposition $\mathbf{T} = \mathbf{T}_d + \mathbf{T}_r$, where $\mathbf{T}_d$ is the determinate stress and $\mathbf{T}_r$ is the reaction stress, the final expressions for $\mathbf{T}_d$ and $\mathbf{T}_r$ are

$$\begin{aligned}\mathbf{T}_d \cong\ & \frac{1}{2}\,\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\bigg|_{\bar{c}}\left(\mathbf{H}+\mathbf{H}^T\right) - \\ & -\frac{1}{2}(\mathrm{tr}\mathbf{H})\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\bigg|_{\bar{c}}\left(\mathbf{H}+\mathbf{H}^T\right) + \frac{1}{2}\,\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\bigg|_{\bar{c}}\left(\mathbf{H}^T\mathbf{H}\right) + \\ & + \frac{1}{2}\left(\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\bigg|_{\bar{c}}\mathbf{H}\right)\mathbf{H}^T + \frac{1}{2}\left(\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\bigg|_{\bar{c}}\mathbf{H}^T\right)\mathbf{H}^T + \\ & + \frac{1}{2}\mathbf{H}\,\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\bigg|_{\bar{c}}\mathbf{H} + \frac{1}{2}\mathbf{H}\,\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\bigg|_{\bar{c}}\mathbf{H}^T + \\ & + \frac{1}{8}\,\frac{\partial^3 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\bigg|_{\bar{c}}\left(\mathbf{H}\otimes\mathbf{H}+\mathbf{H}\otimes\mathbf{H}^T + \right. \\ & \left. + \mathbf{H}^T\otimes\mathbf{H}+\mathbf{H}^T\otimes\mathbf{H}^T\right)\end{aligned} \tag{42}$$

and

$$
\begin{aligned}
\mathbf{T}_r \cong q \Bigg\{ & \frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + \mathbf{H}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + \frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O})\mathbf{H}^T + \\
& + \frac{1}{2}\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H} + \mathbf{H}^T\right) + \left(\mathbf{H}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O})\right)\mathbf{H}^T + \\
& + \frac{1}{2}\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H}^T\mathbf{H}\right) + \frac{1}{2}\mathbf{H}\left(\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H} + \mathbf{H}^T\right)\right) + \\
& + \frac{1}{2}\left(\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H} + \mathbf{H}^T\right)\right)\mathbf{H}^T + \\
& + \frac{1}{8}\frac{\partial^3 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H} \otimes \mathbf{H} + \mathbf{H} \otimes \mathbf{H}^T + \right. \\
& \left. + \mathbf{H}^T \otimes \mathbf{H} + \mathbf{H}^T \otimes \mathbf{H}^T\right) \Bigg\},
\end{aligned}
\tag{43}
$$

respectively; in (42) the subscript $\bar{c}$ indicates evaluation on constraint equation (35).

Note that if the terms of second order in $\mathbf{H}$ appearing in formulas (42), (43) are dropped, we obtain for $\mathbf{T}$ expression (13); of course in this case evaluation on (35) reduces to evaluation on (14).

Second-order stress relations (42), (43) hold if the corresponding linear approximations are obtained by the procedure of linearization used in LFTE.

A final remark concerns the possibility to write all derivatives of $\hat{W}$ and $\hat{c}$ appearing in (42) and (43), respectively, in terms of the polynomial invariants of the strain appropriate for the material symmetry. For the explicit expressions of the derivatives $\dfrac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})$, $\dfrac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O})$, $\dfrac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})$, we refer to (21), (22), (23), respectively.

The third-order derivatives of $\hat{W}$ and $\hat{c}$ can be obtained by means of similar calculations by (18), (20), respectively; we can see that in the reference

configuration they take the explicit form

$$
\begin{aligned}
&\frac{\partial^3 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}) = \\
&= 2\sum_{q=1}^{n}\left(\frac{\partial^2 I_q}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}) \otimes \left(\sum_{p=1}^{n} \frac{\partial^2 \hat{\omega}}{\partial I_p \partial I_q}(l(\mathbf{O}))\frac{\partial I_p}{\partial \mathbf{E}_G}(\mathbf{O})\right)\right) + \\
&+ \sum_{q=1}^{n}\left(\frac{\partial I_q}{\partial \mathbf{E}_G}(\mathbf{O}) \otimes \left(\sum_{p=1}^{n} \frac{\partial^2 \hat{\omega}}{\partial I_p \partial I_q}(l(\mathbf{O}))\frac{\partial^2 I_p}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\right)\right) + \\
&+ \sum_{q=1}^{n}\left(\frac{\partial I_q}{\partial \mathbf{E}_G}(\mathbf{O}) \otimes \left(\sum_{r=1}^{n} \frac{\partial I_r}{\partial \mathbf{E}_G}(\mathbf{O}) \otimes \right.\right. \\
&\qquad \left.\left. \otimes \left(\sum_{p=1}^{n} \frac{\partial^3 \hat{\omega}}{\partial I_p \partial I_r \partial I_q}(l(\mathbf{O}))\frac{\partial I_p}{\partial \mathbf{E}_G}(\mathbf{O})\right)\right)\right) + \\
&+ \sum_{p=1}^{n} \frac{\partial \hat{\omega}}{\partial I_p}(l(\mathbf{O}))\frac{\partial^3 I_p}{\partial \mathbf{E}_G \partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})
\end{aligned}
\tag{44}
$$

and

$$
\begin{aligned}
&\frac{\partial^3 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}) = \\
&= 2\sum_{q=1}^{n}\left(\frac{\partial^2 I_q}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}) \otimes \left(\sum_{p=1}^{n} \frac{\partial^2 \hat{\xi}}{\partial I_p \partial I_q}(l(\mathbf{O}))\frac{\partial I_p}{\partial \mathbf{E}_G}(\mathbf{O})\right)\right) + \\
&+ \sum_{q=1}^{n}\left(\frac{\partial I_q}{\partial \mathbf{E}_G}(\mathbf{O}) \otimes \left(\sum_{p=1}^{n} \frac{\partial^2 \hat{\xi}}{\partial I_p \partial I_q}(l(\mathbf{O}))\frac{\partial^2 I_p}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\right)\right) + \\
&+ \sum_{q=1}^{n}\left(\frac{\partial I_q}{\partial \mathbf{E}_G}(\mathbf{O}) \otimes \left(\sum_{r=1}^{n} \frac{\partial I_r}{\partial \mathbf{E}_G}(\mathbf{O}) \otimes \right.\right. \\
&\qquad \left.\left. \otimes \left(\sum_{p=1}^{n} \frac{\partial^3 \hat{\xi}}{\partial I_p \partial I_r \partial I_q}(l(\mathbf{O}))\frac{\partial I_p}{\partial \mathbf{E}_G}(\mathbf{O})\right)\right)\right) + \\
&+ \sum_{p=1}^{n} \frac{\partial \hat{\xi}}{\partial I_p}(l(\mathbf{O}))\frac{\partial^3 I_p}{\partial \mathbf{E}_G \partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}),
\end{aligned}
\tag{45}
$$

respectively.

4. The second-order stress relations for the two Piola-Kirchhoff stress tensors

In this section we obtain the second-order constitutive equations for the first and the second Piola-Kirchhoff stress tensor; such equations hold if the corresponding linear approximations are obtained by the procedure of linearization followed in LFTE. Moreover they apply to any kind of internal constraint and to any kind of material symmetry appropriate for the constraint.

In finite elasticity the relations between $\mathbf{S}$ and $\mathbf{T}$ and between $\tilde{\mathbf{T}}$ and $\mathbf{T}$ are given by (24), (25), respectively. The second-order relations between $\mathbf{S}$ and $\mathbf{T}$ and between $\tilde{\mathbf{T}}$ and $\mathbf{T}$ can be obtained by (24), (25), respectively, with the use of the second-order expansion (37) for $\det \mathbf{F}$ and the following second-order approximations

$$\mathbf{F}^{-1} \cong \mathbf{I} - \mathbf{H} + \mathbf{H}^2 \tag{46}$$

$$\mathbf{F}^{-T} \cong \mathbf{I} - \mathbf{H}^T + \left(\mathbf{H}^T\right)^2 \tag{47}$$

provided by (4). Retaining only terms that are most of second order in $\mathbf{H}$ we have

$$\begin{aligned} \mathbf{S} \cong\ & \mathbf{T} + (\mathrm{tr}\mathbf{H})\,\mathbf{T} - \mathbf{T}\mathbf{H}^T + \mathbf{T}\left(\mathbf{H}^T\right)^2 - \\ & - (\mathrm{tr}\mathbf{H})\,\mathbf{T}\mathbf{H}^T + \frac{1}{2}\left\{(\mathrm{tr}\mathbf{H})^2 - \mathrm{tr}\left(\mathbf{H}^2\right)\right\}\mathbf{T} \end{aligned} \tag{48}$$

and

$$\begin{aligned} \tilde{\mathbf{T}} \cong\ & \mathbf{T} + (\mathrm{tr}\mathbf{H})\,\mathbf{T} - \mathbf{H}\mathbf{T} - \mathbf{T}\mathbf{H}^T + \mathbf{T}\left(\mathbf{H}^T\right)^2 + \\ & + \mathbf{H}\left(\mathbf{T}\mathbf{H}^T\right) + (\mathbf{H})^2\,\mathbf{T} - (\mathrm{tr}\mathbf{H})\,\mathbf{T}\mathbf{H}^T - (\mathrm{tr}\mathbf{H})\,\mathbf{H}\mathbf{T} + \\ & + \frac{1}{2}\left\{(\mathrm{tr}\mathbf{H})^2 - \mathrm{tr}\left(\mathbf{H}^2\right)\right\}\mathbf{T}. \end{aligned} \tag{49}$$

Of course, discarding second-order terms in $\mathbf{H}$ (48), (49) reduce to the corresponding linear relations (28), (29), respectively.

The explicit constitutive equations for $\mathbf{S}$ and $\tilde{\mathbf{T}}$ appropriate for a second-order theory follow from (48), (49), respectively, by using (42), (43) and retaining in the final expressions only terms that are most of second order in $\mathbf{H}$. Also for $\mathbf{S}$ and $\tilde{\mathbf{T}}$ we write the decompositions $\mathbf{S} = \mathbf{S}_d + \mathbf{S}_r$ and $\tilde{\mathbf{T}} = \tilde{\mathbf{T}}_d + \tilde{\mathbf{T}}_r$; for $\mathbf{S}$

to the second order of approximation we have

$$
\begin{aligned}
\mathbf{S}_d \cong & \frac{1}{2}\,\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\bigg|_{\bar{c}} \left(\mathbf{H}+\mathbf{H}^T\right) + \\
& + \frac{1}{2}\,\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\bigg|_{\bar{c}} \left(\mathbf{H}^T\mathbf{H}\right) + \\
& + \frac{1}{2}\mathbf{H}\,\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\bigg|_{\bar{c}} \mathbf{H} + \frac{1}{2}\mathbf{H}\,\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\bigg|_{\bar{c}} \mathbf{H}^T + \\
& + \frac{1}{8}\,\frac{\partial^3 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\bigg|_{\bar{c}} \left(\mathbf{H}\otimes\mathbf{H}+\mathbf{H}\otimes\mathbf{H}^T + \right. \\
& \left. + \mathbf{H}^T\otimes\mathbf{H}+\mathbf{H}^T\otimes\mathbf{H}^T\right)
\end{aligned}
\tag{50}
$$

and

$$
\begin{aligned}
\mathbf{S}_r \cong\, & q\left\{(1+\mathrm{tr}\mathbf{H})\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + \mathbf{H}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + \right. \\
& + \frac{1}{2}\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H}+\mathbf{H}^T\right) + (\mathrm{tr}\mathbf{H})\mathbf{H}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + \\
& + \frac{1}{2}\left\{(\mathrm{tr}\mathbf{H})^2 - \mathrm{tr}(\mathbf{H}^2)\right\}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + \frac{1}{2}\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H}^T\mathbf{H}\right) + \\
& + \frac{1}{2}\mathbf{H}\left(\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H}+\mathbf{H}^T\right)\right) + \\
& + \frac{1}{2}(\mathrm{tr}\mathbf{H})\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H}+\mathbf{H}^T\right) + \\
& + \frac{1}{8}\frac{\partial^3 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H}\otimes\mathbf{H}+\mathbf{H}\otimes\mathbf{H}^T + \right. \\
& \left.\left. + \mathbf{H}^T\otimes\mathbf{H}+\mathbf{H}^T\otimes\mathbf{H}^T\right)\right\};
\end{aligned}
\tag{51}
$$

for $\tilde{\mathbf{T}}$ we find

$$\begin{aligned}\tilde{\mathbf{T}}_d \cong & \frac{1}{2}\left.\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\right|_{\bar{c}} \left(\mathbf{H}+\mathbf{H}^T\right) + \\ & + \frac{1}{2}\left.\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\right|_{\bar{c}} \left(\mathbf{H}^T\mathbf{H}\right) + \\ & + \frac{1}{8}\left.\frac{\partial^3 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\right|_{\bar{c}} \left(\mathbf{H}\otimes\mathbf{H}+\mathbf{H}\otimes\mathbf{H}^T + \right. \\ & \left. + \mathbf{H}^T\otimes\mathbf{H}+\mathbf{H}^T\otimes\mathbf{H}^T\right)\end{aligned} \tag{52}$$

and

$$\begin{aligned}\tilde{\mathbf{T}}_r \cong q \Bigg\{ & (1+\mathrm{tr}\mathbf{H})\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + \frac{1}{2}\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H}+\mathbf{H}^T\right) + \\ & + \frac{1}{2}\left\{(\mathrm{tr}\mathbf{H})^2 - \mathrm{tr}(\mathbf{H}^2)\right\}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + \\ & + \frac{1}{2}\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H}^T\mathbf{H}\right) + \\ & + \frac{1}{2}(\mathrm{tr}\mathbf{H})\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H}+\mathbf{H}^T\right) + \\ & + \frac{1}{8}\frac{\partial^3 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\left(\mathbf{H}\otimes\mathbf{H}+\mathbf{H}\otimes\mathbf{H}^T + \right. \\ & \left. + \mathbf{H}^T\otimes\mathbf{H}+\mathbf{H}^T\otimes\mathbf{H}^T\right)\Bigg\}.\end{aligned} \tag{53}$$

Note that if the second-order terms are dropped, (50), (51) reduce to (30), while (52), (53) reduce to (31).

Finally we obtain the second-order relations involving the three stress tensors. The tensors $\mathbf{S}$ and $\tilde{\mathbf{T}}$ are given in terms of $\mathbf{T}$ by (48), (49), respectively; for the other relations, as usual we start from the corresponding relationships provided by the finite elasticity, that is

$$\tilde{\mathbf{T}} = \mathbf{F}^{-1}\mathbf{S} \tag{54}$$

$$\mathbf{T} = (\det \mathbf{F})^{-1} \mathbf{S}\mathbf{F}^T \tag{55}$$

$$\mathbf{T} = (\det \mathbf{F})^{-1} \mathbf{F}\tilde{\mathbf{T}}\mathbf{F}^T \tag{56}$$

$$\mathbf{S} = \mathbf{F}\tilde{\mathbf{T}}. \tag{57}$$

With the use of (4), (38), (46) they become

$$\tilde{\mathbf{T}} \cong \mathbf{S} - \mathbf{H}\mathbf{S} + \mathbf{H}^2\mathbf{S} \tag{58}$$

$$\begin{aligned} \mathbf{T} \cong\ & \mathbf{S} - (\mathrm{tr}\mathbf{H})\mathbf{S} + \mathbf{S}\mathbf{H}^T - (\mathrm{tr}\mathbf{H})\mathbf{S}\mathbf{H}^T + \\ & + \frac{1}{2}\left\{(\mathrm{tr}\mathbf{H})^2 - \mathrm{tr}(\mathbf{H}^2)\right\}\mathbf{S} \end{aligned} \tag{59}$$

$$\begin{aligned} \mathbf{T} \cong\ & \tilde{\mathbf{T}} - (\mathrm{tr}\mathbf{H})\tilde{\mathbf{T}} + \mathbf{H}\tilde{\mathbf{T}} + \tilde{\mathbf{T}}\mathbf{H}^T + \\ & + \left(\mathbf{H}\tilde{\mathbf{T}}\right)\mathbf{H}^T - (\mathrm{tr}\mathbf{H})\mathbf{H}\tilde{\mathbf{T}} - (\mathrm{tr}\mathbf{H})\tilde{\mathbf{T}}\mathbf{H}^T + \\ & + \frac{1}{2}\left\{(\mathrm{tr}\mathbf{H})^2 - \mathrm{tr}(\mathbf{H}^2)\right\}\tilde{\mathbf{T}} \end{aligned} \tag{60}$$

$$\mathbf{S} = \tilde{\mathbf{T}} + \mathbf{H}\tilde{\mathbf{T}}. \tag{61}$$

If in (58), (59), (60) the second-order terms in $\mathbf{H}$ are discarded, such relations reduce to the corresponding relations obtained in [9] within the framework of LFTE (see [9], formulas (39), (40), (41)), while expression (61) which provides $\mathbf{S}$ in terms of $\tilde{\mathbf{T}}$ coincides with the corresponding expression obtained in LFTE (see [9], formula (42)).

5. Example: incompressible isotropic materials

In this section we apply the results obtained in Section 3 and Section 4 to incompressible isotropic hyperelastic materials. For such materials we obtain the second-order constitutive equations for the three stress tensors, assuming that to first order of approximation LFTE holds. Moreover we compare our constitutive equations for the Cauchy stress and for the second Piola-Kirchhoff stress with those provided by the classical approach and we show that the classical second-order stress relations are not correct to second order of approximation.

For isotropy, the complete list of the polynomial invariants of $\mathbf{E}_G$ is

$$l(\mathbf{E}_G) = \{I_1, I_2, I_3\} = \left\{\mathbf{I}\cdot\mathbf{E}_G, \mathbf{I}\cdot\mathbf{E}_G^2, \mathbf{I}\cdot\mathbf{E}_G^3\right\}. \tag{62}$$

Since

$$\begin{aligned}\frac{\partial I_1}{\partial \mathbf{E}_G} &= \mathbf{I}\\ \frac{\partial I_2}{\partial \mathbf{E}_G} &= 2\mathbf{E}_G\\ \frac{\partial I_3}{\partial \mathbf{E}_G} &= 3\mathbf{E}_G^2\end{aligned} \tag{63}$$

we have

$$\begin{aligned}\frac{\partial I_1}{\partial \mathbf{E}_G}(\mathbf{O}) &= \mathbf{I}\\ \frac{\partial I_2}{\partial \mathbf{E}_G}(\mathbf{O}) &= \mathbf{O}\\ \frac{\partial I_3}{\partial \mathbf{E}_G}(\mathbf{O}) &= \mathbf{O};\end{aligned} \tag{64}$$

note that both $\dfrac{\partial^2 I_1}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})$ and $\dfrac{\partial^2 I_3}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})$ vanish, so that (21) reduces to

$$\begin{aligned}\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}) \;=\; & \frac{\partial^2 \hat{\omega}}{\partial I_1 \partial I_1}(l(\mathbf{O}))\mathbf{I}\otimes\mathbf{I}+\\ & +\frac{\partial \hat{\omega}}{\partial I_2}(l(\mathbf{O}))\frac{\partial^2 I_2}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}),\end{aligned} \tag{65}$$

while (44) becomes

$$\begin{aligned}\frac{\partial^3 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}) = \;& 2\frac{\partial^2 \hat{\omega}}{\partial I_1 \partial I_2}(l(\mathbf{O}))\frac{\partial^2 I_2}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\otimes\mathbf{I}+\\ & +\frac{\partial^2 \hat{\omega}}{\partial I_2 \partial I_1}(l(\mathbf{O}))\,\mathbf{I}\otimes\frac{\partial^2 I_2}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})+\\ & +\frac{\partial^3 \hat{\omega}}{\partial I_1 \partial I_1 \partial I_1}(l(\mathbf{O}))\,\mathbf{I}\otimes\mathbf{I}\otimes\mathbf{I}+\\ & +\frac{\partial \hat{\omega}}{\partial I_3}(l(\mathbf{O}))\frac{\partial^3 I_3}{\partial \mathbf{E}_G \partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O}).\end{aligned} \tag{66}$$

In (65), (66) the components of the two tensors $\dfrac{\partial^2 I_2}{\partial \mathbf{E}_G \partial \mathbf{E}_G}$ and $\dfrac{\partial^3 I_3}{\partial \mathbf{E}_G \partial \mathbf{E}_G \partial \mathbf{E}_G}$ are

$$\frac{\partial^2 I_2}{\partial (\mathbf{E}_G)_{ij} \partial (\mathbf{E}_G)_{kl}} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \tag{67}$$

and

$$\begin{aligned}&\frac{\partial^3 I_3}{\partial (\mathbf{E}_G)_{ij} \partial (\mathbf{E}_G)_{hk} \mathbf{E}_G)_{rs}} =\\ &\quad = \frac{3}{4}\left\{\delta_{hr}\left(\delta_{ki}\delta_{sj} + \delta_{si}\delta_{kj}\right) + \delta_{sk}\left(\delta_{ri}\delta_{hj} + \delta_{hi}\delta_{rj}\right) + \right.\\ &\quad \left. +\delta_{rk}\left(\delta_{hi}\delta_{sj} + \delta_{si}\delta_{hj}\right) + \delta_{sh}\left(\delta_{ri}\delta_{kj} + \delta_{ki}\delta_{rj}\right)\right\},\end{aligned} \tag{68}$$

respectively.

For incompressibility the constraint function is

$$\hat{c}(\mathbf{E}_G) = \det\left(2\mathbf{E}_G + \mathbf{I}\right) - 1, \tag{69}$$

so that in component form we have

$$\frac{\partial \hat{c}}{\partial (\mathbf{E}_G)_{ij}}(\mathbf{O}) = 2\delta_{ij} \tag{70}$$

and

$$\frac{\partial^2 \hat{c}}{\partial (\mathbf{E}_G)_{ij}\partial (\mathbf{E}_G)_{hk}}(\mathbf{O}) = -2\delta_{ih}\delta_{jk} - 2\delta_{ik}\delta_{jh} + 4\delta_{ij}\delta_{hk} \tag{71}$$

(see [9], formula $(49)_2$). Moreover by means of somewhat lengthy calculations we have

$$\begin{aligned}
&\frac{\partial^3 \hat{c}}{\partial (\mathbf{E}_G)_{ij}\partial (\mathbf{E}_G)_{hk}\partial (\mathbf{E}_G)_{rs}}(\mathbf{O}) = \\
&\quad = 8\Big\{-\delta_{ih}\delta_{jk}\delta_{rs} - \delta_{ik}\delta_{jh}\delta_{rs} + \delta_{ij}\delta_{hk}\delta_{rs} - \\
&\quad -\frac{1}{2}\delta_{ij}\delta_{hr}\delta_{ks} - \frac{1}{2}\delta_{ij}\delta_{hs}\delta_{kr} + \frac{1}{4}\delta_{ir}\delta_{hs}\delta_{jk} + \frac{1}{4}\delta_{is}\delta_{hr}\delta_{jk} + \\
&\quad +\frac{1}{4}\delta_{ih}\delta_{kr}\delta_{sj} + \frac{1}{4}\delta_{ih}\delta_{ks}\delta_{rj} + \frac{1}{4}\delta_{ir}\delta_{sk}\delta_{hj} + \frac{1}{4}\delta_{is}\delta_{rk}\delta_{hj} + \\
&\quad + \frac{1}{4}\delta_{ik}\delta_{hr}\delta_{sj} + \frac{1}{4}\delta_{ik}\delta_{rj}\delta_{hs}\Big\}.
\end{aligned} \tag{72}$$

In virtue of (70) linear constraint equation (14) takes the form

$$\mathrm{tr}\mathbf{H} = 0, \tag{73}$$

while second-order condition for isochoric deformations (35) becomes

$$\frac{1}{2}\left\{(\mathrm{tr}\mathbf{H})^2 - \mathrm{tr}\left(\mathbf{H}^2\right)\right\} = 0, \tag{74}$$

according to (37) (see also [10], formulas (66.51), (67.4)).

Then the second derivatives and the third derivatives of $\hat{W}$ given by (65), (66), respectively, must be evaluated on the constraint equations (73), (74). By substituting (65), (66), (67), (68), (73), (74) into (42) we obtain the following form for the determinate stress

$$\begin{aligned}
\mathbf{T}_d \cong\ &\mu\left(\mathbf{H} + \mathbf{H}^T\right) + \mu\,\mathbf{H}\mathbf{H}^T + \\
&+\frac{3}{4}\frac{\partial \hat{\omega}}{\partial I_3}(l(\mathbf{O}))\left(\mathbf{H}^2 + \mathbf{H}\mathbf{H}^T + \mathbf{H}^T\mathbf{H} + \left(\mathbf{H}^T\right)^2\right) + \\
&+\frac{1}{2}\left(\frac{\partial^2 \hat{\omega}}{\partial I_1 \partial I_1}(l(\mathbf{O})) + \frac{\partial^2 \hat{\omega}}{\partial I_2 \partial I_1}(l(\mathbf{O}))\right)\mathrm{tr}\left(\mathbf{H}\mathbf{H}^T\right)\mathbf{I} + \\
&+\mu\left(\mathbf{H}^2 + \mathbf{H}\mathbf{H}^T + \mathbf{H}^T\mathbf{H} + \left(\mathbf{H}^T\right)^2\right),
\end{aligned} \tag{75}$$

where we have set

$$\mu = \frac{\partial \hat{\omega}}{\partial I_2}(l(\mathbf{O})). \tag{76}$$

Equation (43), with the use of (70), (71), (72), (73), (74), provides for the reaction stress the form

$$\mathbf{T}_r \cong 2q\mathbf{I}. \tag{77}$$

We turn now our attention to the first Piola-Kirchhoff stress tensor; with the use of (65), (66), (67), (68), (73), (74), (76) equation (50) becomes

$$\begin{aligned}\mathbf{S}_d \cong\ & \mu\left(\mathbf{H}+\mathbf{H}^T\right)+\\ &+\frac{3}{4}\frac{\partial \hat{\omega}}{\partial I_3}(l(\mathbf{O}))\left(\mathbf{H}^2+\mathbf{H}\mathbf{H}^T+\mathbf{H}^T\mathbf{H}+\left(\mathbf{H}^T\right)^2\right)+\\ &+\frac{1}{2}\left(\frac{\partial^2 \hat{\omega}}{\partial I_1 \partial I_1}(l(\mathbf{O}))+\frac{\partial^2 \hat{\omega}}{\partial I_2 \partial I_1}(l(\mathbf{O}))\right)\operatorname{tr}\left(\mathbf{H}\mathbf{H}^T\right)\mathbf{I}+\\ &+\mu\left(\mathbf{H}^2+\mathbf{H}\mathbf{H}^T+\mathbf{H}^T\mathbf{H}\right).\end{aligned} \tag{78}$$

Equation (51), together with (70), (71), (72), (73), (74), provides

$$\mathbf{S}_r \cong 2q\left(\mathbf{I}-\mathbf{H}^T+\mathbf{H}^T\mathbf{H}\right). \tag{79}$$

Finally, we consider the second Piola-Kirchhoff stress tensor. By substituting (65), (66), (67), (68), (73), (74), (76) into (52) we find

$$\begin{aligned}\tilde{\mathbf{T}}_d \cong\ & \mu\left(\mathbf{H}+\mathbf{H}^T\right)-\mu\left(\mathbf{H}^2+\mathbf{H}\mathbf{H}^T+\left(\mathbf{H}^T\right)^2\right)+\\ &+\frac{3}{4}\frac{\partial \hat{\omega}}{\partial I_3}(l(\mathbf{O}))\left(\mathbf{H}^2+\mathbf{H}\mathbf{H}^T+\mathbf{H}^T\mathbf{H}+\left(\mathbf{H}^T\right)^2\right)+\\ &+\frac{1}{2}\left(\frac{\partial^2 \hat{\omega}}{\partial I_1 \partial I_1}(l(\mathbf{O}))+\frac{\partial^2 \hat{\omega}}{\partial I_2 \partial I_1}(l(\mathbf{O}))\right)\operatorname{tr}\left(\mathbf{H}\mathbf{H}^T\right)\mathbf{I}+\\ &+\mu\left(\mathbf{H}^2+\mathbf{H}\mathbf{H}^T+\mathbf{H}^T\mathbf{H}+\left(\mathbf{H}^T\right)^2\right);\end{aligned} \tag{80}$$

moreover with the use of (70), (71), (72), (73), (74) equation (53) becomes

$$\tilde{\mathbf{T}}_r \cong 2q\left(\mathbf{I}-\left(\mathbf{H}+\mathbf{H}^T\right)+\mathbf{H}^2+\mathbf{H}\mathbf{H}^T+\left(\mathbf{H}^T\right)^2\right). \tag{81}$$

Now we compare our second-order stress relations with those usually adopted for constrained materials. For hyperelastic constrained materials the classical approach followed in CLTE requires that the linear constitutive equation for the Cauchy stress tensor $\mathbf{T}$ can be written in the form

$$\mathbf{T}^{cl} \cong \frac{\partial \tilde{W}_{\tilde{c}}}{\partial \mathbf{E}}(\mathbf{E})+q\frac{\partial \tilde{c}}{\partial \mathbf{E}}(\mathbf{E}), \tag{82}$$

where $\tilde{W}_{\tilde{c}}(\mathbf{E})$ denotes the quadratic strain energy function for the equivalent unconstrained material that has been evaluated on the linear constraint equation $\tilde{c}(\mathbf{E}) = 0$. With the hope of maintaining clarity in this section, in (82) and in the following the symbol "*cl*" indicates classical theory. We refer to [3] for all details concerning the comparison of CLTE with LFTE.

As noted in Section 2, in CLTE the three stress tensors are indistinguishable to first order of approximation, so that also the linear constitutive equations for $\mathbf{S}$ and $\tilde{\mathbf{T}}$ are given by (82). As shown in [3], [9] the first-order stress relations for the three stress tensors provided by CLTE are not correct.

For incompressible isotropic hyperelastic materials equation (82), which holds in CLTE, and equation (13), which holds in LFTE, provide the same linear constitutive equation for $\mathbf{T}$, that is

$$\mathbf{T}^{cl} = \mathbf{T} \cong \mu \left(\mathbf{H} + \mathbf{H}^T\right) + 2q\mathbf{I} \tag{83}$$

(see [3], Section 5); then to first order of approximation the constitutive equation for the Cauchy stress $\mathbf{T}$ obtained according to CLTE by coincidence is correct.

In LFTE the linear constitutive equations for the two Piola-Kirchhoff stress tensors are given by (30), (31), respectively; for incompressible isotropic materials such equations become

$$\mathbf{S} \cong \mu \left(\mathbf{H} + \mathbf{H}^T\right) + 2q \left(\mathbf{I} - \mathbf{H}^T\right) \tag{84}$$

and

$$\tilde{\mathbf{T}} \cong \mu \left(\mathbf{H} + \mathbf{H}^T\right) + 2q \left(\mathbf{I} - \mathbf{H} - \mathbf{H}^T\right), \tag{85}$$

respectively (see [3], formula (4.16) and [9], formula (52)), while according to CLTE we have

$$\mathbf{S}^{cl} = \tilde{\mathbf{T}}^{cl} \cong \mu \left(\mathbf{H} + \mathbf{H}^T\right) + 2q\mathbf{I}. \tag{86}$$

For incompressible isotropic materials second-order stress relations for $\mathbf{T}$ and $\tilde{\mathbf{T}}$ based on a classical approach can be found in [10], Section 67.

According to [10], formula (67.2), the second-order constitutive equations for the determinate stress $\mathbf{T}_d^{cl}$ and the reaction stress $\mathbf{T}_r^{cl}$ are

$$\begin{aligned} \mathbf{T}_d^{cl} \cong\ & \mu \left(\mathbf{H} + \mathbf{H}^T\right) + \mu\, \mathbf{H}\mathbf{H}^T + \\ & + \frac{1}{4}\alpha_6 \mu \left(\mathbf{H}^2 + \mathbf{H}\mathbf{H}^T + \mathbf{H}^T\mathbf{H} + \left(\mathbf{H}^T\right)^2\right) \end{aligned} \tag{87}$$

and

$$\mathbf{T}_r^{cl} \cong 2q\mathbf{I}, \tag{88}$$

respectively.

According to [10], formula (67.10), the constitutive equations for the determinate stress $\tilde{\mathbf{T}}_d^{cl}$ and the reaction stress $\tilde{\mathbf{T}}_r^{cl}$ are

$$\begin{aligned}\tilde{\mathbf{T}}_d^{cl} \cong\ & \mu\left(\mathbf{H}+\mathbf{H}^T\right)-\mu\left(\mathbf{H}^2+\mathbf{H}\mathbf{H}^T+\left(\mathbf{H}^T\right)^2\right)+\\ & +\frac{1}{4}\alpha_6\mu\left(\mathbf{H}^2+\mathbf{H}\mathbf{H}^T+\mathbf{H}^T\mathbf{H}+\left(\mathbf{H}^T\right)^2\right)\end{aligned} \tag{89}$$

and

$$\tilde{\mathbf{T}}_r^{cl} \cong 2q\left(\mathbf{I}-\left(\mathbf{H}+\mathbf{H}^T\right)+\mathbf{H}^2+\mathbf{H}\mathbf{H}^T+\left(\mathbf{H}^T\right)^2\right), \tag{90}$$

respectively.

Classical second-order stress relations (87), (88), (89), (90) are obtained by means of a procedure of approximation which to first order is in agreement to CLTE (see [10], Section 67, for all details), while second-order stress relations (75), (77), (80), (81) are based on a procedure of approximation which starts from LFTE.

Now we turn our attention to the Cauchy stress and we compare (75), (77) with (87), (88), respectively; we see that the classical approach provides a second-order approximation for the determinate stress which is not correct, since there are many terms missing from (75), as compared with (87); by coincidence, for this particular constraint and for this particular material symmetry, the reaction stress is correct.

By comparing (80), (81) with (89), (90), respectively, we can see that the same occurs for the second Piola-Kirchhoff stress.

In conclusion, we can claim that the classical approach applied to incompressible isotropic materials produces second-order stress relations for $\mathbf{T}$ and $\tilde{\mathbf{T}}$ which are not accurate to second order of approximation.

6. Conclusions

In this paper second-order stress relations for hyperelastic constrained materials are derived; by means of suitable expansions we obtain constitutive equations which exhibit the accuracy required by a second-order theory. Even if we stop our analysis to second-order terms with respect to the displacement gradient, our method can be easily extended to higher orders of approximation.

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