On the functoriality of marked families

ON THE FUNCTORIALITY OF MARKED FAMILIES

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Abstract. The application of methods of computational algebra has recently introduced new tools for the study of Hilbert schemes. The key idea is to define flat families of ideals endowed with a scheme structure whose defining equations can be determined by algorithmic procedures. For this reason, several authors developed new methods, based on the combinatorial properties of Borel-fixed ideals, that allow to associate to each ideal J of this type a scheme $\mathbf{Mf}_J$, called J-marked scheme. In this paper we provide a solid functorial foundation to marked schemes and show that the algorithmic procedures introduced in previous papers do give the correct equations defining them. We prove that for all the strongly stable ideals J, the marked schemes $\mathbf{Mf}_J$ can be embedded in a Hilbert scheme as locally closed subschemes, and that they are open under suitable conditions on J. Finally, we generalize a result by Lederer, proving that Gröbner strata are locally closed subschemes of Hilbert schemes for every Hilbert polynomial.

Introduction

This article aims to give a solid functorial foundation to the theory of marked schemes over a strongly stable ideal J introduced in [CR11, BCLR13, BCR12]. We describe them in terms of representable functors and prove that these functors are represented by the schemes constructed in the aforementioned papers. Moreover, under mild additional hypotheses on J, these functors turn out to be subfunctors of a Hilbert functor. Equations defining the marked schemes can be effectively computed, hence these methods allow for effective computations on the Hilbert schemes. In particular, if we only consider algebras and schemes over a field of characteristic zero, marked schemes $\mathbf{Mf}_J$ with J strongly stable provide, up to the action of the linear group, an open cover of the Hilbert scheme.

For a given monomial ideal J in a polynomial ring $A[x_0, \dots, x_n]$, we consider the collection of all the ideals I such that $A[x_0, \dots, x_n] = I \oplus \langle \mathcal{N}(J) \rangle$, where $\mathcal{N}(J)$ denotes the set of monomials not contained in J. In the case where A is a field and J strongly stable, this collection appears for the first time in [CR11], where it is called *J-marked family*, and it is proved that it can be endowed with a structure of scheme (called J-marked scheme) [CR11, BCLR13, BCR12].

All the ideals I of this collection share the same basis $\mathcal{N}(J)$ of the quotient algebra $A[x_0, \dots, x_n]/I$, hence they define subschemes in $\operatorname{Proj} A[x_0, \dots, x_n]$ with the same Hilbert polynomial. These same properties hold for Gröbner strata, which are schemes parametrizing homogeneous ideals having a fixed monomial ideal as their initial ideal with respect to a given term ordering. However, we emphasize that marked schemes and Gröbner strata are not the same objects. Indeed, in general a J-marked scheme strictly contains the Gröbner stratum with initial ideal J w.r.t. a fixed term ordering (or even the union of all Gröbner strata with initial ideal J).

The use of Gröbner strata in the study of Hilbert schemes is very natural and have been discussed since [Bay82, CF88]. Indeed, the ideals of a Gröbner stratum define points on the same Hilbert scheme and Gröbner strata cover set-theoretically the Hilbert scheme. Thus, several authors addressed the question whether a Gröbner stratum can be equipped by a scheme structure and, if so, how this scheme is embedded in the Hilbert scheme.

Notari and Spreafico [NS00] prove that every Gröbner stratum (considering the reverse lexicographic order) is a locally closed subscheme of the support of the Hilbert scheme. Lederer [Led11] obtains a stronger result in the case of Hilbert schemes of points; working in the affine framework, he proves that Gröbner strata are locally closed subschemes of the Hilbert scheme. In [LR11], the authors of the present paper find suitable conditions on the monomial ideal J and on the term ordering that are sufficient to ensure that the Gröbner stratum is an open subscheme of the Hilbert scheme.

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Nevertheless, Gröbner strata are in general not sufficient to obtain an open cover of the Hilbert scheme (see [LR11, CR11]), while we can obtain such an open cover using marked schemes and exploiting the action of the general linear group on the Hilbert scheme. Furthermore, equations defining a J-marked scheme can be computed by some algorithmic procedures developed in [CR11, BCLR13, BLR13]. The key point is a procedure of polynomial reduction, similar to the one for Gröbner bases, but that does not need a term ordering (see Definition 2.8).

In this paper, we prove that the procedure of reduction is also "natural". Indeed, the reduction works independently of the ring A of coefficients of the polynomial ring, so that the schemes introduced in [CR11, BCLR13, BLR13] correctly describe the scheme structure of the Hilbert scheme (Theorem 3.4 and Corollary 4.3).

In the classical construction of the Hilbert scheme, every point is associated to the homogeneous piece of (a sufficiently large) degree r of the ideal defining the corresponding scheme. At first sight, one could be tempted to consider marked scheme over ideals truncated in the same (large) degree. However, explicit computations of these marked schemes turn out to be in general out of reach, due to the huge number of the variables required. As the number of variables depends on the degree of the truncation, we develop the theory of marked functors in a wider generality, considering marked functors over ideals truncated in any degree. In this way we can find marked schemes that correctly describe the local structure of the Hilbert scheme, but that are far easier to compute (Theorem 3.4 and Section 6).

Finally, we discuss the relation between marked schemes and Gröbner strata, also introducing a representable functor whose representing scheme is in fact a Gröbner stratum. For constant Hilbert polynomials, the Gröbner strata we define in the projective case coincide with those introduced by Lederer in the affine case. In this paper, we generalize Lederer result to Hilbert polynomials of any degree, proving that Gröbner strata are closed subschemes of marked schemes, and so locally closed subschemes of the Hilbert scheme (Theorem 5.3).

DA RISCRIVERE DOPO I CAMBIAMENTI: We exhibit several examples in order to show how the term ordering and the degree of the truncation can affect a Gröbner stratum.

1. Marked bases

In this section, we recall the main definitions concerning sets of polynomials marked over a monomial ideal J and we describe some properties of an ideal generated by such a set, assuming that J is strongly stable. First, let us fix some notation. Throughout the paper, we will consider noetherian rings. We will denote by $\mathbb{Z}[\mathbf{x}]$ the polynomial ring $\mathbb{Z}[x_0, \dots, x_n]$ and by $\mathbb{P}^n_{\mathbb{Z}}$ the projective space $\operatorname{Proj} \mathbb{Z}[\mathbf{x}]$. For any ring A, $A[\mathbf{x}]$ will denote the polynomial ring $A \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{x}]$ in $n+1$ variables with coefficients in A and $\mathbb{P}^n_A$ will be the scheme $\operatorname{Proj} A[\mathbf{x}] = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A$. For every integer s, we denote by $A[\mathbf{x}]_s$ the graded component of degree s, and we set $D_s := D \cap A[\mathbf{x}]_s$ for every $D \subseteq A[\mathbf{x}]$.

We denote monomials in multi-index notation. For any element $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$, x^α will be the monomial $x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ and $|\alpha|$ will be its degree. Given a set of homogeneous polynomials H in $A[\mathbf{x}]$, for emphasizing the dependence on the coefficient ring A, we write ${}_A\langle H \rangle$ for the A-module generated by H and ${}_A(H)$ for the ideal in $A[\mathbf{x}]$ generated by H. We will omit this subscript when no ambiguity can arise, for instance when only one ring A is involved.

If J is a monomial ideal in $A[\mathbf{x}]$, then $\mathcal{B}_J$ is its minimal set of generators and $\mathcal{N}(J)$ is the set of monomials not contained in J.

Remark 1.1. A monomial ideal is determined by the set of monomials it contains. In the following, by abuse of notation, we will use the same letter to denote all monomial ideals having the same set of monomials, even in polynomial rings with different rings of coefficients. More formally, if J is a monomial ideal in $\mathbb{Z}[\mathbf{x}]$, we will denote by the same symbol J also all the ideals $J \otimes_{\mathbb{Z}} A$.

Throughout the paper, we assume the variables ordered as $x_0 < \cdots < x_n$. For any monomial x^α, we denote by $\min x^\alpha$ the smallest variable (or equivalently its index) dividing x^α and by $\max x^\alpha$ the greatest variable (or its index) dividing the monomial.

Definition 1.2. *An ideal $J \subseteq A[\mathbf{x}]$ is said* strongly stable *if*

(i) J is a monomial ideal;
(ii) if $x^\alpha \in J$, then $\frac{x_i}{x_j} x^\alpha \in J$, for all $x_j \mid x^\alpha$ and $x_i > x_j$.

These ideals are extensively studied in commutative algebra and widely used in algebraic geometry since they are related to the *Borel-fixed ideals* [Gre98] . Indeed, every strongly stable ideal is Borel-fixed, whereas in general a Borel-fixed ideal does not need to be strongly stable. The two notions coincide in polynomial rings with coefficients in a field of characteristic zero. Borel fixed ideals are involved in some of the most important general results on Hilbert schemes, as for instance the proof of its connectedness given by Hartshorne [Har66].

Combinatorial properties of strongly stable ideals have been successfully used for designing algorithms inspired by the theory of Gröbner bases but not requiring a term ordering. The role of the term ordering, a total ordering on the set of monomials, is played by a partial order called the *Borel ordering*, given as the transitive closure of the relation

$$x^\alpha >_B x^\beta \quad \Longleftrightarrow \quad x_i x^\alpha = x_j x^\beta \text{ and } x_i < x_j.$$

Moving from this order, it is possible to define reduction procedures which turn out to be noetherian. A detailed description of these techniques are contained in the papers [CR11, BCLR13, BLR13, BCR12]. We will now recall some of the main properties needed in the next section.

Definition 1.3. *For a polynomial $f \in A[\mathbf{x}]$, its* support*, denoted by* $\mathrm{Supp}(f)$*, is the set of monomials appearing in f with non-zero coefficient. We refer to the set of non-zero coefficients of f as* $\mathbf{x}$*-coefficients of f. A* monic marked polynomial *is a polynomial $f \in A[\mathbf{x}]$ with a specified monomial* $\mathrm{Ht}(f)$ *from its support, with coefficient 1_A. We call* $\mathrm{Ht}(f)$ *the* head term *of f and we call* $\mathrm{T}(f) := \mathrm{Ht}(f) - f$ *the* tail *of f (so that $f = \mathrm{Ht}(f) - \mathrm{T}(f)$). Throughout the paper we describe marked polynomials adding as subscript the multi-index corresponding to the head term, i.e. we write f_α meaning that* $\mathrm{Ht}(f_\alpha) = x^\alpha$.

Definition 1.4. *Let $J \subseteq A[\mathbf{x}]$ be a strongly stable ideal and let $\mathcal{B}_J$ be the minimal set of generators of J. We call J-*marked set *a set of monic marked polynomials*

$$f_\alpha = x^\alpha - \sum_{x^\beta \in \mathcal{N}(J)_{|\alpha|}} c_{\alpha\beta}\, x^\beta,$$

where $\mathrm{Ht}(f_\alpha) = x^\alpha \in \mathcal{B}_J$ *and $c_{\alpha\beta} \in A$. A J-marked set F_J is called a J-*marked basis *if $A[\mathbf{x}] = {}_A(F_J) \oplus {}_A\langle \mathcal{N}(J)\rangle$, i.e. the monomials of $\mathcal{N}(J)$ freely generate $A[\mathbf{x}]/{}_A(F_J)$.*

We emphasize that the assumption of the head term to be monic is significant only if the coefficients ring A is not a field. Indeed, if A is a field (as done in [CR11, BCLR13]), a set of marked polynomials can always be modified in a set of monic marked polynomials.

If (F_J) is a J-marked basis, then the scheme $\mathrm{Proj}\, A[\mathbf{x}]/(F_J)$ is A-flat as the A-module $A[\mathbf{x}]/{}_A(F_J)$ is free. Therefore, the ideal ${}_A(F_J)$ generated by a J-marked basis F_J has the same Hilbert polynomial as the monomial ideal J, so that J and ${}_A(F_J)$ define schemes corresponding to closed points of the same Hilbert scheme. Therefore, it is interesting to find theoretical conditions and effective procedures in order to state whether a marked set is a marked basis.

Proposition 1.5 ([EK90, Lemma 1.1], [BCLR13, Lemma 1.2])**.** *Let J be a strongly stable ideal.*

(i) Each monomial x^α can be written uniquely as a product $x^\gamma x^\delta$ with $x^\gamma \in \mathcal{B}_J$ and $\min x^\gamma \geqslant \max x^\delta$. Therefore, $x^\delta <_{\mathtt{Lex}} x^\eta$ for every monomial x^η such that $x^\eta \mid x^\alpha$ and $x^{\alpha-\eta} \notin J$. We will write $x^\alpha = x^\gamma *_J x^\delta$ to refer to this unique decomposition.

(ii) Consider $x^\alpha \in J \setminus \mathcal{B}_J$ and let $x_j = \min x^\alpha$. Then, x^α/x_j is contained in J.

(iii) Let x^β be a monomial not contained in J. If $x^\delta x^\beta \in J$, then either $x^\delta x^\beta \in \mathcal{B}_J$ or $x^\delta x^\beta = x^\alpha *_J x^{\delta'}$ with $x^\alpha \in \mathcal{B}_J$ and $x^\delta >_{\mathtt{Lex}} x^{\delta'}$. In particular, if $x_i x^\beta \in J$, then either $x_i x^\beta \in \mathcal{B}_J$ or $x_i > \min x^\beta$.

Definition 1.6. *Let J be a strongly stable ideal and I be the ideal generated by a J-marked set F_J in $A[\mathbf{x}]$. We consider the following sets of polynomials:*

- $F_J^{(s)} := \left\{ x^\delta f_\alpha \;\middle|\; \deg\left(x^\delta f_\alpha\right) = s,\ f_\alpha \in F_J,\ \min x^\alpha \geqslant \max x^\delta \right\}$;
- $\widehat{F}_J^{(s)} := \left\{ x^\delta f_\alpha \;\middle|\; \deg\left(x^\delta f_\alpha\right) = s,\ f_\alpha \in F_J,\ \min x^\alpha < \max x^\delta \right\}$;
- $SF_J^{(s)} := \left\{ x^\delta f_\beta - x^\gamma f_\alpha \;\middle|\; x^\delta f_\beta \in \widehat{F}_J^{(s)},\ x^\gamma f_\alpha \in F_J^{(s)}, x^\delta x^\beta = x^\gamma x^\alpha \right\}$;
- $\mathcal{N}(J, I) := I \cap {}_A\langle \mathcal{N}(J)\rangle$.

Throughout the paper, we use the convention that when multiplying a marked polynomial f by a monomial x^δ, we have $\mathrm{Ht}(x^\delta f) = x^\delta \mathrm{Ht}(f)$. Therefore, for each monomial $x^\gamma \in J_s$, there is a unique polynomial in $F_J^{(s)}$ (resp. in $\widehat{F}_J^{(s)}$) with head term x^γ.

Theorem 1.7. *Let J be a strongly stable ideal and $I \subseteq A[\mathbf{x}]$ be the ideal generated by a J-marked set F_J. For every s,*

(i) $I_s = \langle F_J^{(s)} \rangle + \langle \widehat{F}_J^{(s)} \rangle = \langle F_J^{(s)} \rangle + \langle SF_J^{(s)} \rangle$;

(ii) $A[\mathbf{x}]_s = \langle F_J^{(s)} \rangle \oplus \langle \mathcal{N}(J)_s \rangle$;

(iii) the A-module $\langle F_J^{(s)} \rangle$ is free of rank equal to $\operatorname{rk} J_s$ and is generated by a unique (J_s)-marked set $\widetilde{F}_J^{(s)}$;

(iv) $I_s = \langle F_J^{(s)} \rangle \oplus \mathcal{N}(J,I)_s = \langle \widetilde{F}_J^{(s)} \rangle \oplus \mathcal{N}(J,I)_s$.

Moreover, TFAE:

(v) F_J is a J-marked basis;

(vi) for all s, $I_s = \langle F_J^{(s)} \rangle$;

(vii) for all s, $\langle SF_J^{(s)} \rangle \subseteq \langle F_J^{(s)} \rangle$;

(viii) $\mathcal{N}(J,I) = 0$.

Proof. *(i)* Straightforward from the definition of the homogeneous piece of a given degree of an ideal.

(ii) We start proving that there are no non-zero polynomials in the intersection $\langle F_J^{(s)} \rangle \cap \langle \mathcal{N}(J)_s \rangle$. Let us consider $h := \sum_i b_i x^{\delta_i} f_{\alpha_i}$, where $x^{\delta_i} f_{\alpha_i}$ are distinct elements of $F_J^{(s)}$ and $b_i \in A \setminus \{0\}$. Assume that the polynomials $x^{\delta_i} f_{\alpha_i}$ are indexed so that $x^{\delta_1} \geqslant_{\mathtt{Lex}} x^{\delta_2} \geqslant_{\mathtt{Lex}} \cdots$. Then b_1 turns out to be also the coefficient of the monomial $x^{\delta_1} x^{\alpha_1}$ in h. Indeed, $x^{\alpha_1} x^{\delta_1}$ does not appear either as head term or in the support of the tail of a summand $x^{\delta_i} f_{\alpha_i}$ of h with $i > 1$. The monomial cannot be the head term of $x^{\delta_i} f_{\alpha_i}$, since the head terms in $F_J^{(s)}$ (and so in the summands of h) are all different. Moreover, $x^{\alpha_1} x^{\delta_1}$ cannot appear in $T(x^{\delta_i} f_{\alpha_i})$ with $i > 1$, since it has the unique decomposition $x^{\alpha_1} *_J x^{\delta_1}$, while every monomial $x^{\delta_i} x^\beta \in T(x^{\delta_i} f_{\alpha_i}) \cap J$ has decomposition $x^\alpha *_J x^\eta$ with $x^\eta <_{\mathtt{Lex}} x^{\delta_i} <_{\mathtt{Lex}} x^{\delta_1}$ by Proposition 1.5*(iii)* (note that by definition $x^\beta \in \mathrm{Supp}(T(f_{\alpha_1})) \subseteq \mathcal{N}(J)$). Therefore, no non-zero polynomials in $\langle F_J^{(s)} \rangle$ are contained in $\langle \mathcal{N}(J)_s \rangle$.

To conclude the proof, we show that every monomial x^β of degree s is contained in the direct sum $\langle F_J^{(s)} \rangle \oplus \langle \mathcal{N}(J)_s \rangle$. If $x^\beta \in \mathcal{N}(J)_s$, there is nothing to prove. Now assume that there exists some monomial in J_s not contained in $\langle F_J^{(s)} \rangle \oplus \langle \mathcal{N}(J)_s \rangle$. Among them, choose x^β such that in the unique decomposition $x^\beta = x^\alpha *_J x^\delta$, monomial x^δ is minimum with respect to the $\mathtt{Lex}$ ordering. Since $x^\beta = x^\delta f_\alpha + \mathrm{T}(x^\delta f_\alpha)$, the support of $\mathrm{T}(x^\delta f_\alpha)$ cannot be contained in $\mathcal{N}(J)_s$, i.e. there exists $x^\eta \in \mathrm{Supp}(T(f_\alpha))$ such that $x^\eta x^\delta \in J$. By Proposition 1.5*(iii)*, we have the decomposition $x^\eta x^\delta = x^{\alpha'} *_J x^{\delta'}$ with $x^{\delta'} <_{\mathtt{Lex}} x^\delta$ against the assumption of minimality on x^δ.

(iii) By *(ii)*, we have the short exact sequence

$$0 \to \langle F_J^{(s)} \rangle \hookrightarrow A[\mathbf{x}]_s \xrightarrow{\pi} \langle \mathcal{N}(J)_s \rangle \to 0.$$

For each x^α in J_s, we compute the image $\pi(x^\alpha) = \sum_{x^\beta \in \mathcal{N}(J)_s} a_{\alpha\beta} x^\beta$ and consider the set $\widetilde{F}_J^{(s)} := \{\tilde{f}_\alpha := x^\alpha - \sum_{x^\beta \in \mathcal{N}(J)_s} a_{\alpha\beta} x^\beta \mid x^\alpha \in J_s\} \subseteq \ker \pi = \langle F_J^{(s)} \rangle$. Let $J' := (J_s)$. By construction the set $\widetilde{F}_J^{(s)}$ is a J'-marked set with $\mathrm{Ht}(\tilde{f}_\alpha) = x^\alpha$. Applying *(ii)* to this J'-marked set, we have $\langle \widetilde{F}_J^{(s)} \rangle \oplus \langle \mathcal{N}(J')_s \rangle = A[\mathbf{x}]_s$. Finally, as the A-module generated by $\widetilde{F}_J^{(s)}$ is contained in $\langle F_J^{(s)} \rangle$ and $\mathcal{N}(J)_s = \mathcal{N}(J')_s$, the modules $\langle \widetilde{F}_J^{(s)} \rangle$ and $\langle F_J^{(s)} \rangle$ coincide. Note that $\widetilde{F}_J^{(s)}$ is marked on the monomial ideal J' generated by J_s, but does not need to be a $J_{\geqslant s}$-marked set, since $J_{\geqslant s}$ may have minimal generators of degree $> s$.

(iv) By *(i)* and *(iii)*, we have $I_s = \langle \widetilde{F}_J^{(s)} \rangle + \langle SF_J^{(s)} \rangle$. Since $\langle \widetilde{F}_J^{(s)} \rangle \cap \langle \mathcal{N}(J)_s \rangle = \{0\}$, the module $\mathcal{N}(J, F_J)_s$ can be determined starting from the generators of $\langle SF_J^{(s)} \rangle$ and by replacing each monomial $x^\beta \in J_s$ appearing in some polynomial of $SF_J^{(s)}$ with the tail $\mathrm{T}(\widetilde{f}_\beta)$ of the polynomial $\widetilde{f}_\beta \in \widetilde{F}_J^{(s)}$ with $\mathrm{Ht}(\widetilde{f}_\beta) = x^\beta$. The result of this procedure is a set of polynomials contained both in I_s and $\langle \mathcal{N}(J)_s \rangle$. The sum of $\mathcal{N}(J,I)_s$ and $\langle F_J^{(s)} \rangle$ is direct by *(ii)* and *(iii)*.

The equivalences *(v)*⇔*(vi)*⇔*(vii)*⇔*(viii)* follow directly from the first part of the theorem. In fact, these properties are a rephrasing of the definition of J-marked basis. □

We emphasize that the above result does not hold in general for a monomial ideal J which is not strongly stable, as shown by the following example.

Example 1.8. Let $J = (x_2^2, x_1^2)$ be the monomial ideal in $\mathbb{Z}[x_0, x_1, x_2]$ and I be the ideal generated by the J-marked set $F_J = \{f_{002} = x_2^2 + x_2x_1, f_{020} = x_1^2 + x_2x_1\}$. An easy computation shows that I_3 is freely generated by $F_J^{(3)}$, but $|F_J^{(3)}| = \operatorname{rk} I_3 = 5 < 6 = \operatorname{rk} J_3$ and I_3 does not contain any (J_3)-marked set $\widetilde{F}_J^{(3)}$.

Example 1.9. Consider the strongly stable ideal $J = (x_2^2, x_2x_1, x_1^3) \subseteq \mathbb{Z}[x_0, x_1, x_2]$ and any J-marked set $F_J = \{f_{002}, f_{011}, f_{030}\}$ over a ring A. Let us compute the sets of polynomials $F_J^{(s)}$, $\widehat{F}_J^{(s)}$ and $SF_J^{(s)}$ discussed in Theorem 1.7 for $s = 2, 3, 4$.

$$
\begin{aligned}
(s=2)\qquad & F_J^{(2)} = \{f_{002},\ f_{011}\}, \quad \widehat{F}_J^{(2)} = \emptyset, \quad SF_J^{(2)} = \emptyset,\\
(s=3)\qquad & F_J^{(3)} = \{x_2f_{002},\ x_1f_{002},\ x_0f_{002},\ x_1f_{011},\ x_0f_{011},\ f_{030}\},\\
& \widehat{F}_J^{(3)} = \{x_2f_{011}\}, \quad SF_J^{(3)} = \{x_2f_{011} - x_1f_{002}\},\\
(s=4)\qquad & F_J^{(4)} = \left\{ \begin{array}{l} x_2^2f_{002},\ x_2x_1f_{002},\ x_2x_0f_{002},\ x_1^2f_{002},\ x_1x_0f_{002},\ x_0^2f_{002},\\ x_1^2f_{011},\ x_1x_0f_{011},\ x_0^2f_{011},\ x_1f_{030},\ x_0f_{030} \end{array} \right\},\\
& \widehat{F}_J^{(4)} = \{x_2^2f_{011},\ x_2x_1f_{011},\ x_2x_0f_{011}, x_2f_{030}\},\\
& SF_J^{(4)} = \{x_2^2f_{011} - x_2x_1f_{002}, x_2x_1f_{011} - x_1^2f_{002}, x_2x_0f_{011} - x_1x_0f_{002}, x_2f_{030} - x_1^2f_{011}\}.
\end{aligned}
$$

In order to study the sets of polynomials $\widetilde{F}_J^{(s)}$ and the module $\mathcal{N}(J, I)$, we need to know explicitly the J-marked set, so let us consider for instance:

$$F_J = \left\{f_{002} = x_2^2 + 3x_1^2 - x_2x_0 + x_1x_0,\ f_{011} = x_2x_1 - x_1x_0,\ f_{030} = x_1^3 - 3x_1^2x_0\right\}$$

and let $I := (F_J)$. For $s = 2$, we have $\widetilde{F}_J^{(2)} = F_J^{(2)}$ and $\mathcal{N}(J, I) = \emptyset$.

$(s = 3)$. In order to construct $\widetilde{F}_J^{(3)}$, we have to determine the equivalence classes of monomials in the quotient $A[x_0, x_1, x_2]_3/\langle F_J^{(3)}\rangle \simeq \langle \mathcal{N}(J)_3\rangle$. If $h \in A[x_0, x_1, x_2]_s$, we denote by $\overline{h}$ its class in $A[x_0, x_1, x_2]_s/\langle F_J^{(s)}\rangle$. Following the strategy of the proof of Theorem 1.7, we examine the monomials of J_3 in increasing order with respect to the `Lex` ordering.

$$
\begin{aligned}
& \overline{x_1^3} \overset{-f_{030}}{=} \overline{3x_1^2x_0} \quad \Rightarrow \quad \widetilde{f}_{030} = f_{030},\\
& \overline{x_2x_1x_0} \overset{-x_0f_{011}}{=} \overline{x_1x_0^2} \quad \Rightarrow \quad \widetilde{f}_{111} = x_0f_{011},\\
& \overline{x_2x_1^2} \overset{-x_1f_{011}}{=} \overline{x_1^2x_0} \quad \Rightarrow \quad \widetilde{f}_{021} = x_1f_{011},\\
& \overline{x_2^2x_0} \overset{-x_0f_{002}}{=} \overline{-3x_1^2x_0 + x_2x_0^2 - x_1x_0^2} \quad \Rightarrow \quad \widetilde{f}_{102} = x_0f_{002},\\
& \overline{x_2^2x_1} \overset{-x_1f_{002}}{=} \overline{-3x_1^3 + x_2x_1x_0 - x_1^2x_0} \overset{3\widetilde{f}_{030}}{=} \overline{x_2x_1x_0 - 10x_1^2x_0} =\\
& \qquad \overset{-\widetilde{f}_{111}}{=} \overline{-10x_1^2x_0 + x_1x_0^2} \quad \Rightarrow \quad \widetilde{f}_{012} = x_1f_{002} - 3f_{030} + x_0f_{011},\\
& \overline{x_2^3} \overset{-x_2f_{002}}{=} \overline{-3x_2x_1^2 + x_2^2x_0 - x_2x_1x_0} \overset{3\widetilde{f}_{021}}{=} \overline{x_2^2x_0 - x_2x_1x_0 - 3x_1^2x_0} =\\
& \qquad \overset{-\widetilde{f}_{102}}{=} \overline{-x_2x_1x_0 - 6x_1^2x_0 + x_2x_0^2 - x_1x_0^2} =\\
& \qquad \overset{\widetilde{f}_{111}}{=} \overline{-6x_1^2x_0 + x_2x_0^2 - 2x_1x_0^2} \quad \Rightarrow \quad \widetilde{f}_{003} = x_2f_{002} - 3x_1f_{011} + x_0f_{002} - x_0f_{011}.
\end{aligned}
$$

To determine $\mathcal{N}(J, I)_3$, we can compute the class of the polynomial of $SF_J^{(3)}$ in $A[x_0, x_1, x_2]_3/\langle F_J^{(3)}\rangle = A[x_0, x_1, x_2]_3/\langle \widetilde{F}_J^{(3)}\rangle$:

$$\overline{x_2f_{011} - x_1f_{002}} = \overline{-3x_1^3 - x_1^2x_0} \overset{3\widetilde{f}_{030}}{=} \overline{-10x_1^2x_0}$$

so that $\mathcal{N}(J, I)_3 = \langle 10x_1^2x_0\rangle$.

$(s = 4)$. Repeating the same procedure applied for $s = 3$, we obtain:

$$\begin{aligned}
\widetilde{f}_{130} &= x_0 f_{030} = x_1^3 x_0 - 3x_1^2 x_0^2,\\
\widetilde{f}_{040} &= x_1 f_{030} - b x_0 f_{030} = x_1^4 - 9x_1^2 x_0^2,\\
\widetilde{f}_{211} &= x_0^2 f_{011} = x_2 x_1 x_0^2 - x_1 x_0^3,\\
\widetilde{f}_{121} &= x_1 x_0 f_{011} = x_2 x_1^2 x_0 - x_1^2 x_0^2,\\
\widetilde{f}_{031} &= x_1^2 f_{011} + x_0 f_{030} = x_2 x_1^3 - 3x_1^2 x_0^2,\\
\widetilde{f}_{202} &= x_0^2 f_{002} = x_2^2 x_0^2 + 3x_1^2 x_0^2 - x_2 x_0^3 + x_1 x_0^3,\\
\widetilde{f}_{112} &= x_1 x_0 f_{002} - 3x_0 f_{030} + x_0^2 f_{211} = x_2^2 x_1 x_0 + 10x_1^2 x_0^2 - x_1 x_0^3,\\
\widetilde{f}_{022} &= x_1^2 f_{002} - 3x_1 f_{030} + x_1 x_0 f_{011} - 10x_0 f_{030} = x_2^2 x_1^2 + 29x_1^2 x_0^2,\\
\widetilde{f}_{103} &= x_2 x_0 f_{002} - 3x_1 x_0 f_{011} + x_0^2 f_{002} - x_0^2 f_{011} = x_2^3 x_0 + 6x_1^2 x_0^2 - x_2 x_0^3 + 2x_1 x_0^3,\\
\widetilde{f}_{013} &= x_2 x_1 f_{002} - 3x_1^2 f_{011} + x_1 x_0 f_{002} - x_1 x_0 f_{011} - 6x_0 f_{030} + x_0^2 f_{011} = x_2^3 x_1 + 20x_1^2 x_0^2 - x_1 x_0^3,\\
\widetilde{f}_{004} &= x_2^2 f_{002} - 3x_1^2 f_{002} + 9x_1 f_{030} + x_2 x_0 f_{002} - x_1 x_0 f_{002} - 6x_1 x_0 f_{011} + 33x_0 f_{030} + x_0^2 f_{002} - 2x_0^2 f_{011} =\\
&= x_2^4 - 91x_1^2 x_0^2 - x_2 x_0^3 + 3x_1 x_0^3.
\end{aligned}$$

Moreover, $\mathcal{N}(J, I)_4 = \langle 10x_1^2 x_0^2 \rangle$ as in the quotient $A[x_0, x_1, x_2]_4 / \langle F_J^{(4)} \rangle$ we have

$$\overline{x_2^2 f_{011} - x_2 x_1 f_{002}} = \overline{-10x_1^2 x_0^2}, \qquad \overline{x_2 x_1 f_{011} - x_1^2 f_{002}} = \overline{-30x_1^2 x_0^2},$$
$$\overline{x_2 x_0 f_{011} - x_1 x_0 f_{002}} = \overline{-10x_1^2 x_0^2}, \qquad \overline{x_2 f_{030} - x_1^2 f_{011}} = \overline{0}.$$

Therefore, F_J is not a J-marked basis unless $10 = 0$ in A (cf. Theorem 1.7).

We conclude this section giving a characterization of a J-marked basis F_J that takes into account the homogeneous pieces $(F_J)_s$ of the ideal it generates for a limited number of degrees. The following statement is clearly inspired by Gotzmann's Persistence Theorem, but we emphasize that the proof is independent from that result.

Theorem 1.10. *Let J be a strongly stable ideal, m be the maximum degree of monomials in its minimal monomial basis $\mathcal{B}_J$ and I be the ideal in $A[\mathbf{x}]$ generated by a J-marked set F_J. TFAE:*

(i) F_J is a J-marked basis;
(ii) as an A-module, $I_s = \langle F_J^{(s)} \rangle$ for every $s \leqslant m + 1$;
(iii) as an A-module, $I_s = \langle \widetilde{F}_J^{(s)} \rangle$ for every $s \leqslant m + 1$;
(iv) $\mathcal{N}(J, I)_s = 0$ for every $s \leqslant m + 1$.

Proof. *(i)⇒(ii)* Straightforward by Theorem 1.7*(vi)*.

(ii)⇒(i) We want to prove that for every s, $A[\mathbf{x}]_s = I_s \oplus \langle \mathcal{N}(J)_s \rangle$. This is true for $s \leqslant m + 1$ by hypothesis. By Theorem 1.7*(ii)*-*(iii)*, we know that $A[\mathbf{x}]_s = \langle F_J^{(s)} \rangle \oplus \langle \mathcal{N}(J)_s \rangle$ and $\langle F_J^{(s)} \rangle \subseteq I_s$, so that we need to prove $I_s \subseteq \langle F_J^{(s)} \rangle$. Let us assume that this is not true and let t be the minimal degree for which $I_t \not\subseteq \langle F_J^{(t)} \rangle$. Note that $t \geqslant m + 2 > m$ and $I_t = x_0 I_{t-1} + \cdots + x_n I_{t-1}$.

Since $I_{t-1} = \langle F_J^{(t-1)} \rangle$, there should exist a variable x_i such that $x_i I_{t-1} \not\subseteq \langle F_J^{(t)} \rangle$, or equivalently $x_i F_J^{(t-1)} \not\subseteq \langle F_J^{(t)} \rangle$. Assume that x_i has the minimal index and take a polynomial $x^\delta f_\alpha \in F_J^{(t-1)}$, with $x^\alpha = \mathrm{Ht}(f_\alpha) \in \mathcal{B}_J$, such that $x_i x^\delta f_\alpha \notin \langle F_J^{(t)} \rangle$. The variable x_i has to be greater than $\min x^\alpha$, since otherwise $x_i x^\delta f_\alpha \in F_J^{(t)}$. Moreover, $|\delta| > 0$ since $t - 1 > m$. Let $x_j = \max x^\delta \leqslant \min x^\alpha < x_i$ and $x^{\delta'} = \frac{x^\delta}{x_j}$. The polynomial $x_i x^{\delta'} f_\alpha$ is contained in I_{t-1}, while $x_j(x_i x^{\delta'} f_\alpha) = x_i x^\delta f_\alpha$ is not contained in $\langle F_J^{(t-1)} \rangle$, contradicting the minimality of i.

(ii)⇔(iii)⇔(iv) Straightforward by Theorem 1.7. □

2. Definition and representability of marked functors

We follow the notation for functors used in [HS04]. In particular, for a scheme Z, we denote by $\underline{Z}$ its functor of points.

The main object of interest in the present paper is the set

$$\underline{\mathbf{Mf}}_J(A) := \{\text{ideals } I \subseteq A[\mathbf{x}] \mid A[\mathbf{x}] = I \oplus {}_A\langle \mathcal{N}(J)\rangle\} \tag{2.1}$$

which is defined for every noetherian ring A and every strongly stable ideal $J \subseteq A[\mathbf{x}]$. In this section we will prove that this construction is in fact functorial, i.e. $\underline{\mathbf{Mf}}_J(A)$ is the evaluation in the noetherian ring A of a functor

$$\underline{\mathbf{Mf}}_J : \underline{\text{Noeth-Rings}} \to \underline{\text{Sets}}.$$

Now we will describe the elements of any $\underline{\mathbf{Mf}}_J(A)$ in terms of the notion of J-marked basis, discussed in the previous section. This will be a key point to prove its functoriality.

Proposition 2.1. *Let J be a strongly stable ideal and let I be an element of $\underline{\mathbf{Mf}}_J(A)$.*

(i) The ideal I contains a unique J-marked set F_J.
(ii) $I = (F_J)$ and F_J is the unique J-marked basis contained in I.

Proof. *(i)* Let x^α be a minimal generator of J and consider its image by the projection $A[\mathbf{x}] \xrightarrow{\pi_I} A[\mathbf{x}]/I$. Since $A[\mathbf{x}]_{|\alpha|}/I_{|\alpha|} \simeq \langle \mathcal{N}(J)_{|\alpha|}\rangle$, $\pi_I(x^\alpha)$ is given by a linear combination $\sum c_{\alpha\beta}x^\beta$ of the monomials $x^\beta \in \mathcal{N}(J)_{|\alpha|}$. Therefore, $\ker \pi$ contains a unique homogeneous polynomial $f_\alpha = x^\alpha - \sum c_{\alpha\beta}x^\beta$ with head term x^α. The collection of all f_α, for $x^\alpha \in \mathcal{B}_J$, is the unique J-marked set.

(ii) Starting from F_J we can construct, for every degree s, the sets of polynomials $F_J^{(s)}$ and $\widetilde{F}_J^{(s)}$ as in Theorem 1.10. Recall that they are both contained in the ideal $(F_J) \subseteq I$. In order to show that F_J is a J-marked basis and generates I, we observe that for every s, $\langle F_J^{(s)}\rangle \subseteq I_s$ and $\langle F_J^{(s)}\rangle \oplus \langle \mathcal{N}(J)_s\rangle = A[\mathbf{x}]_s$ by Theorem 1.7*(ii)*. Moreover $I_s \oplus \langle \mathcal{N}(J)_s\rangle = A[\mathbf{x}]_s$, since $I \in \underline{\mathbf{Mf}}_J(A)$. Therefore, $I_s = \langle F_J^{(s)}\rangle = (F_J^{(s)})_s$ in every degree s. Finally, F_J is a J-marked basis by Theorem 1.7*(v)*-*(vi)* and is unique by *(i)*. □

Remark 2.2. We emphasize that uniqueness is not true for a J-marked set generating an ideal $I \notin \underline{\mathbf{Mf}}_J(A)$. For instance, consider the strongly stable ideal $J = (x_2^2, x_2x_1, x_1^3) \subseteq \mathbb{Z}[x_0, x_1, x_2]$. The J-marked set $F_J = \{x_2^2 + x_0^2, x_2x_1, x_1^3\}$ defines an ideal $I = (F_J)$ not contained in $\underline{\mathbf{Mf}}_J(\mathbb{Z})$ as $x_1x_0^2 = x_1(x_2^2+x_0^2) - x_2(x_2x_1) \in I \cap \mathcal{N}(J)$. In fact, the ideal I is generated by infinitely many J-marked sets $\{x_2^2 + x_0^2, x_2x_1, x_1^3 + a\, x_1x_0^2\}$, $a \in \mathbb{Z}$.

As a consequence of the previous result, we are now able to give a new description of $\underline{\mathbf{Mf}}_J(A)$:

$$\underline{\mathbf{Mf}}_J(A) = \{\text{ideal } I \subseteq A[\mathbf{x}] \mid I \text{ is generated by a } J\text{-marked basis}\}.$$

For every strongly stable ideal J, let us consider the map between the category of noetherian rings to the category of sets

$$\underline{\mathbf{Mf}}_J : \underline{\text{Noeth-Rings}} \to \underline{\text{Sets}} \tag{2.2}$$

that associates to a noetherian ring A the set $\underline{\mathbf{Mf}}_J(A)$ and to a morphism $\phi : A \to B$ the map

$$\begin{array}{rccc} \underline{\mathbf{Mf}}_J(\phi): & \underline{\mathbf{Mf}}_J(A) & \longrightarrow & \underline{\mathbf{Mf}}_J(B) \\ & I & \longmapsto & I \otimes_A B. \end{array} \tag{2.3}$$

Proposition 2.3. *For every strongly stable ideal J, $\underline{\mathbf{Mf}}_J$ is a functor.*

Proof. Consider the J-marked basis $F_{J,A}$ generating the ideal $I \in \underline{\mathbf{Mf}}_J(A)$. Any morphism $\phi : A \to B$ gives the structure of A-module to B. Thus, tensoring I by B leads to the following transformation on the J-marked basis $F_{J,A}$:

$$f_{\alpha,A} = x^\alpha - \sum c_{\alpha\beta}x^\beta \in F_{J,A} \quad \longmapsto \quad f_{\alpha,B} = x^\alpha - \sum \phi(c_{\alpha\beta})x^\beta.$$

since $\phi(1_A) = 1_B$, the set $F_{J,B} := \{f_{\alpha,B} \mid f_{\alpha,A} \in F_{J,A}\}$ is still a J-marked set. Finally, $F_{J,B}$ is a J-marked basis since the tensor product by $\otimes_A B$ of a direct sum of free A-modules is a direct sum of free B-modules. □

Now we discuss a necessary condition for this functor to be representable.

Lemma 2.4. *For every strongly stable ideal J, $\underline{\mathbf{Mf}}_J$ is a Zariski sheaf.*

Proof. Let A be a noetherian ring and $U_i = \operatorname{Spec} A_{a_i}$, $i = 1, \dots, s$, an open cover of $\operatorname{Spec} A$, which is equivalent to require that $(a_1, \dots, a_s) = A$. Consider a set of ideals $I_i \in \underline{\mathbf{Mf}}_J(A_{a_i})$ such that for any pair of indices $i \neq j$

$$I_{ij} := I_i \otimes_{A_{a_i}} A_{a_i a_j} = I_j \otimes_{A_{a_j}} A_{a_i a_j} \in \underline{\mathbf{Mf}}_J(A_{a_i a_j}).$$

We need to show that there exists a unique ideal $I \in \underline{\mathbf{Mf}}_J(A)$ such that $I_i = I \otimes_A A_{a_i}$ for every i.

Let us consider the J-marked bases associated to I_i:

$$F_{J,i} = \left\{ f_{\alpha,i} = x^\alpha - \sum_{x^\beta \in \mathcal{N}(J)_{|\alpha|}} c^{(i)}_{\alpha\beta}\, x^\beta \in A_{a_i}[\mathbf{x}] \;\middle|\; x^\alpha \in \mathcal{B}_J \right\}, \quad I_i = (F_{J,i}) \subseteq A_{a_i}[\mathbf{x}], \quad \forall\, i = 1, \dots, s.$$

By assumption, for each $x^\alpha \in \mathcal{B}_J$ and for each pair of indices $i \neq j$, the polynomials $f_{\alpha,i}$ and $f_{\alpha,j}$ coincide on $A_{a_i a_j}[\mathbf{x}]$. By the sheaf axiom for the quasi-coherent sheaf $\widetilde{A[\mathbf{x}]}$ on $\operatorname{Spec} A$, we know that there exists a unique polynomial $f_\alpha \in A[\mathbf{x}]$ whose image in $A_{a_i}[\mathbf{x}]$ is $f_{\alpha,i}$ for every i. The polynomial f_α turns out to be monic. In fact, if c is the coefficient of x^α, then its image in A_{a_i} is $1_{A_{a_i}}$, so that $(c - 1_A)a_i^k = 0$ for some integer k. Thus, $c = 1_A$ since $(a_1^k, \dots, a_s^k) = A$. The collection of polynomials $\{f_\alpha \;:\; x^\alpha \in \mathcal{B}_J\}$ forms a J-marked basis. □

Now we prove that the functor $\underline{\mathbf{Mf}}_J$ is representable finding explicitly the affine scheme $\mathbf{Mf}_J$ representing it. To do that we apply the previous theorems that describe which conditions on the coefficients of polynomials in a J-marked set guarantee that the marked set is a J-marked basis.

We obtain $\mathbf{Mf}_J$ as a closed subscheme of an affine scheme of a suitable dimension depending on J.

Notation 2.5. Let J be any strongly stable monomial ideal in $\mathbb{Z}[\mathbf{x}]$. Then:

- $\mathbf{C}$ is the set of variables of the coordinate ring of the affine scheme $\mathbb{A}^N = \operatorname{Spec} \mathbb{Z}[\mathbf{C}]$, where
$$N = \sum_{x^\alpha \in \mathcal{B}_J} \left| \mathcal{N}(J)_{|\alpha|} \right|.$$
We consider the variables in $\mathbf{C}$ indexed as $C_{\alpha\beta}$ where the multi-index α corresponds to $x^\alpha \in \mathcal{B}_J$ and the multi-index β to $x^\beta \in \mathcal{N}(J)_{|\alpha|}$.
- $\mathcal{I}$ is the ideal in $\mathbb{Z}[\mathbf{C}][\mathbf{x}] = \mathbb{Z}[\mathbf{C}] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{x}]$ generated by the following J-marked set
$$\mathcal{F}_J := \left\{ x^\alpha - \sum_{x^\beta \in \mathcal{N}(J)_{|\alpha|}} C_{\alpha\beta}\, x^\beta \;\middle|\; x^\alpha \in \mathcal{B}_J \right\}. \tag{2.4}$$
- $\mathfrak{I}_J$ the ideal in $\mathbb{Z}[\mathbf{C}]$ generated by the $\mathbf{x}$-coefficients of the polynomials in $\mathcal{N}(J, \mathcal{I})$.
- Every J-marked set $F_J = \{f_\alpha = x^\alpha - \sum_{x^\beta \in \mathcal{N}(J)_{|\alpha|}} c_{\alpha\beta}\, x^\beta \mid x^\alpha \in \mathcal{B}_J\}$ in $A[\mathbf{x}]$ is uniquely identified by the coefficients $c_{\alpha\beta} \in A$, or equivalently, by the ring homomorphism
$$\phi_{F_J} : \mathbb{Z}[\mathbf{C}] \to A : C_{\alpha\beta} \mapsto c_{\alpha\beta}.$$
- Moreover, let $\phi_{F_J}[\mathbf{x}] \colon \mathbb{Z}[\mathbf{C}][\mathbf{x}] \to A[\mathbf{x}]$ be the canonical extension of ϕ_{F_J}.

Theorem 2.6. *In the notation above, the functor $\underline{\mathbf{Mf}}_J$ is represented by $\mathbf{Mf}_J := \operatorname{Spec} \mathbb{Z}[\mathbf{C}]/\mathfrak{I}_J$. Therefore, a J-marked set F_J is a J-marked basis if, and only if, ϕ_{F_J} factors.*

$$\begin{tikzcd} \mathbb{Z}[\mathbf{C}] \arrow[rr] \arrow[dr] & & A \\ & \mathbb{Z}[\mathbf{C}]/\mathfrak{I}_J \arrow[ur] & \end{tikzcd}$$

Proof. Let A be any noetherian ring, F_J be a J-marked set in $A[\mathbf{x}]$ and $I = (F_J) \subseteq A[\mathbf{x}]$. We obtain the statement proving that F_J is a J-marked basis if, and only if, $\ker \phi_{F_J} \supseteq \mathfrak{I}_J$.

By definition, $\phi_{F_J}[\mathbf{x}]$ is the identity on monomials and $\phi_{F_J}[\mathbf{x}](\mathcal{I}) \subseteq I$, so that $\phi_{F_J}[\mathbf{x}]\left(\mathcal{N}(J, \mathcal{I})\right) \subseteq \mathcal{N}(J, I)$. If F_J is a J-marked basis, then $\mathcal{N}(J, I) = 0$, hence $\ker(\phi_{F_J}) \supseteq \mathfrak{I}_J$.

On the other hand, if $\ker(\phi_{F_J}) \supseteq \mathfrak{I}_J$, then for every s we have

$$\widehat{F}_J^{(s)} = \phi_{F_J}[\mathbf{x}](\widehat{\mathcal{F}}_J^{(s)}) \subseteq \phi_{F_J}[\mathbf{x}](\mathcal{I}_s) = \phi_{F_J}[\mathbf{x}]\big(\langle \mathcal{F}_J^{(s)} \rangle \oplus \mathcal{N}(J, \mathcal{I})_s\big) = \phi_{F_J}[\mathbf{x}](\langle \mathcal{F}_J^{(s)} \rangle) \subseteq {}_A\langle F^{(s)} \rangle$$

so that $I_s = {}_A\langle F_J^{(s)} \cup \widehat{F}_J^{(s)} \rangle \subseteq {}_A\langle F_J^{(s)} \rangle \subseteq I_s$ and we conclude applying Theorem 1.7*(v)*-*(vi)*. □

In order to explicitly compute a finite set of generators of the ideal $\mathfrak{I}_J$ we can apply some of the previous results. By Theorem 1.10 we get the following simplification.

Corollary 2.7. *For every strongly stable ideal J, the ideal $\mathfrak{I}_J$ is generated by the $\mathbf{x}$-coefficients of the polynomials in $\mathcal{N}(J,\mathcal{I})_s$ for every $s \leqslant m+1$, where m is the maximum degree of monomials in $\mathcal{B}_J$.*

Proof. Let $\mathfrak{I}'$ the ideal in $\mathbb{Z}[\mathbf{C}]$ generated by $\mathbf{x}$-coefficients of the polynomials in $\mathcal{N}(J,\mathcal{I})_s$ for every $s \leqslant m+1$. Obviously $\mathfrak{I}' \subseteq \mathfrak{I}_J$.

On the other hand, by Theorem 1.10, the image of $\mathcal{F}_J$ in $\mathbb{Z}[\mathbf{C}]/\mathfrak{I}' \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{x}]$ is a J-marked basis. Therefore the map $\mathbb{Z}[\mathbf{C}] \to \mathbb{Z}[\mathbf{C}]/\mathfrak{I}'$ factors through $\mathbb{Z}[\mathbf{C}]/\mathfrak{I}_J$. □

We can obtain a set of generators of $\mathfrak{I}_J$ by computing a set of generators of the $\mathbb{Z}$-module $\mathcal{N}(J,\mathcal{F}_J)_s$ for each $s \leqslant m+1$ through a Gaussian reduction. This is the method applied, for instance, in [CR11].

A more efficient method is the one developed in [BCLR13], which is similar to the Buchberger algorithm for Gröbner bases. We will know describe this method and then prove that it gives in fact a set of generators of $\mathfrak{I}_J$, as claimed in [BCLR13].

Definition 2.8. *Let J be a strongly stable ideal and let F_J be a marked set. We say that a polynomial g is a J-*remainder *if $\mathrm{Supp}(g) \cap J = \emptyset$. Given two polynomials h and g, we say that g can be obtained from h by a step of F_J-reduction if $g = h - c\,f$ where c is the coefficient in h of a monomial $x^\eta \in J_s$ and f is the unique polynomial in $F_J^{(s)}$ with $\mathrm{Ht}(f) = x^\eta$. We write*

$$h \xrightarrow{F_J^{(\cdot)}} g$$

if g arises from h by a finite sequence of reductions as described above. Moreover, we write $h \xrightarrow{F_J^{(\cdot)}}_* g$ if g is a J-remainder.

As proved in [CR11, BCLR13], the procedure $\xrightarrow{F_J^{(\cdot)}}$ is noetherian, i.e. every sequence of F_J-reductions starting on a polynomial h stops after a finite number of steps giving a J-remainder polynomial g. Indeed, each step of reduction $h \mapsto h - cf$ replaces a monomial $x^\eta = x^\alpha *_J x^\gamma$ in the support of h with $x^\gamma T(f_\alpha)$, $f_\alpha \in F_J$. Since $T(f_\alpha) \in \langle \mathcal{N}(J) \rangle$, every monomial appearing in $x^\gamma T(f_\alpha)$ either is in $\mathcal{N}(J)$ or has the decomposition $x^\gamma x^\beta = x^{\alpha'} *_J x^\delta$ with $x^\delta <_{\mathtt{Lex}} x^\gamma$. This allows to conclude since $<_{\mathtt{Lex}}$ is a well-ordering on monomials.

We can find many different sequences of steps of reduction starting from a given polynomial h, but the J-remainder polynomial g is unique. In fact, if h is a homogeneous polynomial of degree s and $h \xrightarrow{F_J^{(\cdot)}}_* g$ and $h \xrightarrow{F_J^{(\cdot)}}_* g'$, then $g - g' = (g-h) - (g'-h) \in \langle F_J^{(s)} \rangle$ since $g-h, g'-h \in \langle F_J^{(s)} \rangle$. By definition of J-remainder, $g - g' \in \langle \mathcal{N}(J)_s \rangle$ and $\langle \mathcal{N}(J)_s \rangle \cap \langle F_J^{(s)} \rangle = \{0\}$ by Theorem 1.7*(ii)*.

Remark 2.9. In general, the marking cannot be performed with respect to a term ordering (see [CR11, Example 3.18]), so that the noetherianity of the procedure is surprising. Indeed, it is well-known that a general reduction process by a set of marked polynomials is noetherian if, and only if, the marking is performed w.r.t. a term ordering (see [RS93]). The ultimate reason for this is our restriction that each monomial $x^\eta \in J$ is reduced by the unique polynomial $x^\gamma f_\alpha \in F_J^{(s)}$ such that $x^\eta = x^\alpha *_J x^\gamma$ — as opposed to *any* polynomial $x^\delta f_\beta \in (F_J)$ such that $x^\delta x^\beta = x^\eta$.

Now we can give a characterization of J-marked basis in terms of this reduction procedure and S-polynomials.

Definition 2.10. *For marked polynomials f_α, f_β in a J-marked set, we call $S(f_\alpha, f_\beta) := x^\eta f_\alpha - x^\nu f_\beta$ the S-*polynomial *of f_α and f_β, where $(x^\eta, -x^\nu)$ is the minimal syzygy between x^α and x^β. We call* EK-polynomial*, and denote by $S^{\mathrm{EK}}(f_\alpha, f_\beta)$, a S-polynomial whose syzygy $(x^\eta, -x^\nu)$ is of Eliahou-Kervaire type, i.e. x^η is a single variable x_j greater that $\min(x^\alpha)$ and $x_j x^\alpha = x^\beta *_J x^\nu$ (see [EK90]).*

Notice that for every EK-polynomial $x_j f_\alpha - x^\nu f_\beta \in A[\mathbf{x}]_s$, we have $x_j f_\alpha \in \widehat{F}_J^{(s)}$ and $x^\nu f_\beta \in F_J^{(s)}$.

Theorem 2.11. *Let J be a strongly stable ideal and let F_J be a J-marked set. TFAE:*

(i) F_J is a J-marked basis;

(ii) $S^{\mathrm{EK}}(f_\alpha, f_\beta) \xrightarrow{F_J^{(\cdot)}}_* 0$, *for all* $S^{\mathrm{EK}}(f_\alpha, f_\beta)$ *with* $f_\alpha, f_\beta \in F_J$;

(iii) $x_i f_\alpha \xrightarrow{F_J^{(\cdot)}}_* 0$, *for all* $f_\alpha \in F_J$ *and* $x_i > \min x^\alpha$.

Proof. *(i)*⇔*(ii)* The EK-syzygies are a basis of the syzygies of J, so that *(ii)* ensures that every other syzygy between two monomials $x^\alpha, x^\beta \in \mathcal{B}_J$ lifted to the corresponding marked polynomials f_α and f_β has J-remainder equal to 0. Since these syzygies are exactly the generators of the module $SF_J^{(s)}$ for all s, *(ii)* is equivalent to $(F_J)_s = \langle F_J^{(s)} \rangle$ for every s, hence to *(i)* by Theorem 1.7.

(iii)⇔*(ii)* The two reductions agree. □

Remark 2.12. Notice that it is possible to prove the equivalence between Theorem 2.11 *(i)* and *(iii)* directly from the properties of the reduction $\xrightarrow{F_J^{(\cdot)}}_*$, as was done in [BCLR13].

We now show how a set of generators of $\mathfrak{I}_J$ can be computed using Theorem 2.11. We consider the J-marked set $\mathcal{F}_J$ given in (2.4) and use the marked sets $\widetilde{\mathcal{F}}_J^{(s)}$ in order to perform the polynomial reduction in each degree s. The elements of $\widetilde{\mathcal{F}}_J^{(s)}$ take the shape

$$\widetilde{f}_\gamma = x^\gamma - \sum_{x^\delta \in \mathcal{N}(J)_s} D_{\gamma\delta}\, x^\delta, \qquad \forall\, x^\gamma \in J_s$$

with coefficients $D_{\gamma\delta} \in \mathbb{Z}[\mathbf{C}]$.

Let $S^{\mathrm{EK}}(f_\alpha, f_{\alpha'})$ be an EK-polynomial with $f_\alpha, f_{\alpha'} \in \mathcal{F}_J$. Assume that $\deg S^{\mathrm{EK}}(f_\alpha, f_{\alpha'}) = s$. We can decompose it as

$$S^{\mathrm{EK}}(f_\alpha, f_{\alpha'}) = x^\eta f_\alpha - x^{\eta'} f_{\alpha'} = \sum_{x^\gamma \in J_s} E_{\alpha\alpha'\gamma}\, x^\gamma + \sum_{x^\delta \in \mathcal{N}(J)_s} E_{\alpha\alpha'\delta}\, x^\delta$$

where the coefficients are equal to

$$E_{\alpha\alpha'\gamma} \text{ (resp. } E_{\alpha\alpha'\delta}) = \begin{cases} 0, & \text{if } x^\gamma \text{ (resp. } x^\delta) \notin \operatorname{Supp}\big(S^{\mathrm{EK}}(f_\alpha, f_{\alpha'})\big), \\ C_{\alpha'\beta'} - C_{\alpha\beta}, & \text{if } x^\gamma \text{ (resp. } x^\delta) \in \operatorname{Supp}(x^\eta f_\alpha) \cap \operatorname{Supp}(x^{\eta'} f_{\alpha'}) \setminus \{x^\eta x^\alpha\}, \\ -C_{\alpha\beta}, & \text{if } x^\gamma \text{ (resp. } x^\delta) \begin{cases} \in \operatorname{Supp}(x^\eta f_\alpha) \setminus \{x^\eta x^\alpha\} \\ \notin \operatorname{Supp}(x^{\eta'} f_{\alpha'}) \setminus \{x^\eta x^\alpha\} \end{cases}, \\ C_{\alpha'\beta'}, & \text{if } x^\gamma \text{ (resp. } x^\delta) \begin{cases} \notin \operatorname{Supp}(x^\eta f_\alpha) \setminus \{x^\eta x^\alpha\} \\ \in \operatorname{Supp}(x^{\eta'} f_{\alpha'}) \setminus \{x^\eta x^\alpha\} \end{cases}. \end{cases}$$

The $\mathcal{F}_J$-reduction of $S^{\mathrm{EK}}(f_\alpha, f_{\alpha'})$ is

$$S^{\mathrm{EK}}(f_\alpha, f_{\alpha'}) \xrightarrow{\mathcal{F}_J^{(\cdot)}}_* \sum_{x^\delta \in \mathcal{N}(J)_s} \left(E_{\alpha\alpha'\delta} + \sum_{x^\gamma \in J_s} E_{\alpha\alpha'\gamma} D_{\gamma\delta} \right) x^\delta.$$

For any α, α' such that $x^\alpha, x^{\alpha'} \in \mathcal{B}_J$ are involved in a syzygy of Eliahou-Kervaire type, we set

$$P_{\alpha\alpha'}^\delta := E_{\alpha\alpha'\delta} + \sum_{x^\gamma \in J_s} E_{\alpha\alpha'\gamma} D_{\gamma\delta}, \qquad s = \deg S^{\mathrm{EK}}(f_\alpha, f_{\alpha'}), \quad \forall\, x^\delta \in \mathcal{N}(J)_s. \tag{2.5}$$

Corollary 2.13. *Let J be a strongly stable ideal and let $\mathfrak{I}_J$ be the ideal in $\mathbb{Z}[\mathbf{C}]$ given in Theorem 2.11. Then $\mathfrak{I}_J$ is the ideal generated by all polynomials $P_{\alpha\alpha'}^\delta$ described in (2.5).*

Proof. Let $\mathfrak{I}'$ be the ideal generated by such polynomials $P_{\alpha\alpha'}^\delta$. The inclusion $\mathfrak{I}' \subseteq \mathfrak{I}_J$ follows directly from the construction, since the above polynomials are $\mathbf{x}$-coefficients of elements in $\mathcal{N}(J, (\mathcal{F}_J))$.

For the opposite inclusion we consider again the J-marked set $\overline{\mathcal{F}}_J$ image of $\mathcal{F}_J$ in $\mathbb{Z}[\mathbf{C}]/\mathfrak{I}'$. By construction $\overline{\mathcal{F}}_J$ satisfy the condition *(iii)* of Theorem 2.11, so that it is a J-marked basis. Therefore $\mathbb{Z}[\mathbf{C}] \to \mathbb{Z}[\mathbf{C}]/\mathfrak{I}'$ factors as $\mathbb{Z}[\mathbf{C}] \to \mathbb{Z}[\mathbf{C}]/\mathfrak{I}_J \to \mathbb{Z}[\mathbf{C}]/\mathfrak{I}'$ and $\mathfrak{I}_J \subseteq \mathfrak{I}'$. □

To determine equations defining $\mathbf{Mf}_J$ we can use Corollary 2.13, namely the criterion for marked bases in terms of syzygies given in Theorem 2.11 that was first introduced in [CR11] and refined in [BCLR13] in terms of EK-syzygies. In particular, Theorem 2.11 and Corollary 2.13 give new proofs in terms of marked functors of [BCLR13, Corollary 4.6] and [CR11, Theorem 4.1].

Example 2.14. Let us compute the equations defining the scheme representing the functor $\underline{\mathbf{Mf}}_J$ with $J=(x_2^2,x_2x_1,x_1^3)\subseteq\mathbb{Z}[x_0,x_1,x_2]$. We start considering the marked set

$$\begin{aligned}
f_{002}&=x_2^2+C_{002,020}x_1^2+C_{002,101}x_2x_0+C_{002,110}x_1x_0+C_{002,200}x_0^2\\
f_{011}&=x_2x_1+C_{011,020}x_1^2+C_{011,101}x_2x_0+C_{011,110}x_1x_0+C_{011,200}x_0^2\\
f_{030}&=x_1^3+C_{030,120}x_1^2x_0+C_{030,201}x_2x_0^2+C_{030,210}x_1x_0^2+C_{030,300}x_0^3.
\end{aligned}$$

There are two EK-polynomials:

$$\begin{aligned}
S^{\mathrm{EK}}(f_{011},f_{002})&=x_2f_{011}-x_1f_{002}=C_{011,020}x_2x_1^2-C_{002,020}x_1^3+C_{011,101}x_2^2x_0+\\
&\qquad(-C_{002,101}+C_{011,110})x_2x_1x_0-C_{002,110}x_1^2x_0+C_{011,200}x_2x_0^2-C_{002,200}x_1x_0^2,\\
S^{\mathrm{EK}}(f_{030},f_{011})&=x_2f_{030}-x_1^2f_{011}=-C_{011,020}x_1^4+(-C_{011,101}+C_{030,120})x_2x_1^2x_0-C_{011,110}x_1^3x_0\\
&\qquad+C_{030,201}x_2^2x_0^2+C_{030,210}x_2x_1x_0^2-C_{011,200}x_1^2x_0^2+C_{030,300}x_2x_0^3.
\end{aligned}$$

Since $\mathrm{Supp}\big(S^{\mathrm{EK}}(f_{011},f_{002})\big)\cap J=\{x_2x_1^2,x_1^3,x_2^2x_0,x_2x_1x_0\}$ and $\mathrm{Supp}\big(S^{\mathrm{EK}}(f_{030},f_{011})\big)\cap J=\{x_1^4,x_2x_1^2x_0,x_1^3x_0,x_2^2x_0^2,x_2x_1x_0^2\}$, to perform the $\xrightarrow{\mathcal{F}_J^{(\cdot)}}_*$ reduction, we need some elements of $\widetilde{\mathcal{F}}_J^{(3)}$ and $\widetilde{\mathcal{F}}_J^{(4)}$. Reducing $S^{\mathrm{EK}}(f_{011},f_{002})$ by

$$\begin{aligned}
&\widetilde{f}_{111}=x_0f_{011},\qquad \widetilde{f}_{102}=x_0f_{002},\qquad \widetilde{f}_{030}=f_{030},\\
&\widetilde{f}_{021}=x_1f_{011}-C_{011,020}\widetilde{f}_{030}-C_{011,101}\widetilde{f}_{111}=x_2x_1^2+(-C_{011,020}C_{011,101}-C_{011,020}C_{030,120}+C_{011,110})x_1^2x_0+\\
&\qquad(-C_{011,101}^2-C_{011,020}C_{030,201})x_2x_0^2+(-C_{011,101}C_{011,110}-C_{011,020}C_{030,210}+C_{011,200})x_1x_0^2+\\
&\qquad(-C_{011,101}C_{011,200}-C_{011,020}C_{030,300})x_0^3
\end{aligned}$$

we obtain

$$\begin{aligned}
&\underbrace{\left(\begin{array}{c}C_{011,020}^2C_{011,101}+C_{011,020}^2C_{030,120}+C_{002,101}C_{011,020}-C_{002,020}C_{011,101}\\-2C_{011,020}C_{011,110}+C_{002,020}C_{030,120}-C_{002,110}\end{array}\right)}_{P_{011,002}^{120}}x_1^2x_0\\
&\qquad+\underbrace{\left(\begin{array}{c}C_{011,020}C_{011,101}^2+C_{011,020}^2C_{030,201}-C_{011,101}C_{011,110}\\+C_{002,020}C_{030,201}+C_{011,200}\end{array}\right)}_{P_{011,002}^{201}}x_2x_0^2\\
&+\underbrace{\left(\begin{array}{c}C_{011,020}C_{011,101}C_{011,110}+C_{011,020}^2C_{030,210}-C_{002,110}C_{011,101}+C_{002,101}C_{011,110}\\-C_{011,110}^2-C_{011,020}C_{011,200}+C_{002,020}C_{030,210}-C_{002,200}\end{array}\right)}_{P_{011,002}^{210}}x_1x_0^2\\
&\qquad+\underbrace{\left(\begin{array}{c}C_{011,020}C_{011,101}C_{011,200}+C_{011,020}^2C_{030,300}-C_{002,200}C_{011,101}\\+C_{002,101}C_{011,200}-C_{011,110}C_{011,200}+C_{002,020}C_{030,300}\end{array}\right)}_{P_{011,002}^{300}}x_0^3.
\end{aligned}$$

To reduce the second EK-polynomial we need

$$\begin{aligned}
&\widetilde{f}_{211}=x_0\widetilde{f}_{111}=x_0^2f_{011},\quad \widetilde{f}_{202}=x_0\widetilde{f}_{102}=x_0^2f_{002},\quad \widetilde{f}_{130}=x_0f_{030},\quad \widetilde{f}_{121}=x_0\widetilde{f}_{021},\\
&\widetilde{f}_{040}=x_1f_{030}-C_{030,120}\widetilde{f}_{130}-C_{030,201}\widetilde{f}_{211}=x_1^4+(-C_{030,120}^2-C_{011,020}C_{030,201}+C_{030,210})x_1^2x_0^2+\\
&\qquad(-C_{011,101}C_{030,201}-C_{030,120}C_{030,201})x_2x_0^3+(-C_{011,110}C_{030,201}-C_{030,120}C_{030,210}+C_{030,300})x_1x_0^3+\\
&\qquad(-C_{011,200}C_{030,201}-C_{030,120}C_{030,300})x_0^4.
\end{aligned}$$

Thus, the reduction of $S^{\mathrm{EK}}(f_{030}, f_{011})$ is

$$\underbrace{(-C_{011,020}C_{011,101}^2 - C_{011,020}^2C_{030,201} + C_{011,101}C_{011,110} - C_{002,020}C_{030,201} - C_{011,200})}_{P^{220}_{030,011}}x_1^2x_0^2$$
$$+\underbrace{\left(\begin{array}{c} -C_{011,101}^3 + C_{011,101}^2C_{030,120} - 2C_{011,020}C_{011,101}C_{030,201} - C_{002,101}C_{030,201} \\ + C_{011,110}C_{030,201} - C_{011,101}C_{030,210} + C_{030,300} \end{array}\right)}_{P^{301}_{030,011}}x_2x_0^3$$
$$+\underbrace{\left(\begin{array}{c} -C_{011,101}^2C_{011,110} + C_{011,101}C_{011,110}C_{030,120} - C_{011,020}C_{011,110}C_{030,201} - C_{011,020}C_{011,101}C_{030,210} \\ + C_{011,101}C_{011,200} - C_{011,200}C_{030,120} - C_{002,110}C_{030,201} + C_{011,020}C_{030,300} \end{array}\right)}_{P^{310}_{030,011}}x_1x_0^3$$
$$+\underbrace{\left(\begin{array}{c} -C_{011,101}^2C_{011,200} + C_{011,101}C_{011,200}C_{030,120} - C_{011,020}C_{011,200}C_{030,201} \\ - C_{011,020}C_{011,101}C_{030,300} - C_{002,200}C_{030,201} - C_{011,200}C_{030,210} + C_{011,110}C_{030,300} \end{array}\right)}_{P^{400}_{030,011}}x_0^4.$$

In order to have a J-marked basis, the J-reduction of the EK-polynomials has to be 0, so that the functor $\underline{\mathbf{Mf}}_J$ is represented by the scheme

$$\mathbf{Mf}_J = \operatorname{Spec}\mathbb{Z}[\mathbf{C}]/\left(P^{120}_{011,002}, P^{201}_{011,002}, P^{210}_{011,002}, P^{300}_{011,002}, P^{220}_{030,011}, P^{301}_{030,011}, P^{310}_{030,011}, P^{400}_{030,011}\right).$$

Now, for any ring A, each element of $\underline{\mathbf{Mf}}_J(A)$ is given by a scheme morphism $\operatorname{Spec} A \to \mathbf{Mf}_J$, or equivalently by a ring morphism $\mathbb{Z}[\mathbf{C}] \to A$ that factors through $\mathbb{Z}[\mathbf{C}] \to \mathbb{Z}[\mathbf{C}]/\mathfrak{I}_J \to A$. For instance, for $A = \mathbb{Z}[t]$, the ring morphism $\mathbb{Z}[\mathbf{C}] \to \mathbb{Z}[t]$ given by

$$\begin{array}{llll} C_{002,020} \mapsto 1-t & C_{002,101} \mapsto 0 & C_{002,110} \mapsto t^3 - t^4 & C_{002,200} \mapsto -t^2 \\ C_{011,020} \mapsto 0 & C_{011,101} \mapsto 0 & C_{011,110} \mapsto t & C_{011,200} \mapsto t^2 - t \\ C_{030,120} \mapsto t^3 & C_{030,201} \mapsto t & C_{030,210} \mapsto 0 & C_{030,300} \mapsto -t^2 \end{array}$$

factors through $\mathbb{Z}[\mathbf{C}] \to \mathbb{Z}[\mathbf{C}]/\mathfrak{I}_J \to \mathbb{Z}[t]$, hence the following is a J-marked basis in $\mathbb{Z}[t][x_0, x_1, x_2]$

$$\begin{array}{l} f_{002} = x_2^2 + (1-t)x_1^2 - (t^4 - t^3)x_1x_0 - t^2\,x_0^2, \\ f_{011} = x_2x_1 + t\,x_1x_0 + (t^2 - t)x_0^2, \\ f_{030} = x_1^3 + t^3\,x_1^2x_0 + t\,x_2x_0^2 - t^2\,x_0^3. \end{array}$$

3. Marked schemes and truncation ideals

An ideal $I \in \underline{\mathbf{Mf}}_J(A)$ defines a quotient algebra $A[\mathbf{x}]/I$ that is a free A-module, so that the family $\operatorname{Proj} A[\mathbf{x}]/I \to \operatorname{Spec} A$ is flat and defines a morphism from $\operatorname{Spec} A$ to a suitable Hilbert scheme, by the universal property of Hilbert schemes. Thus, it is natural to study the relation between marked schemes and Hilbert schemes. Since Hilbert schemes parametrize flat families of subschemes of a projective space, and the same subscheme can be defined by infinitely many different ideals, we first need to investigate the function that associates to every ideal in $\underline{\mathbf{Mf}}_J(A)$ the scheme in $\mathbb{P}^n_A$ it defines. In general, this function can be non-injective, as the following example shows.

Example 3.1 (cf. [BCLR13, Example 3.4])**.** Consider the strongly stable ideal $J = (x_2, x_1^2, x_1x_0)$ in $\mathbb{Z}[x_0, x_1, x_2]$. For any ring A and $a \in A$, consider the J-marked set $F_{J,a} = \{x_2 + a\,x_1, x_1^2, x_1x_0\}$. These marked sets are in fact J-marked bases, since the unique EK-polynomial involving the first generator

$$S^{\mathrm{EK}}(x_2 + a\,x_1, x_1^2) = x_1^2(x_2 + a\,x_1) - x_2(x_1^2) = ax_1^3$$

is clearly contained in $\langle F^{(3)}_{J,a}\rangle$. Moreover, for every a, the ideal $(F_{J,a})_{\geqslant 2}$ coincides with $J_{\geqslant 2} \otimes A$, since $x_2^2 = x_2(x_2 + a\,x_1) - ax_1(x_2 + a\,x_1) + a^2(x_1^2)$, $x_2x_1 = x_1(x_2 + a\,x_1) - a(x_1^2)$ and $x_2x_0 = x_0(x_2 + a\,x_1) - a(x_1x_0)$. Therefore, for all $a \in A$, the ideals $(F_{J,a})$ define the same scheme $\operatorname{Proj} A[\mathbf{x}]/J$.

The following proposition states that non-uniqueness is a consequence of divisibility by x_0.

Proposition 3.2 (cf. [BCLR13, Theorem 3.3])**.** *Let J be a strongly stable ideal and let m be the minimum degree such that $J_m \neq 0$. Assume that no monomial of degree larger than m in the monomial basis $\mathcal{B}_J$ is divisible by x_0 (or equivalently that $x_0^t\mathcal{N}(J)_{\geqslant m} \subseteq \mathcal{N}(J)_{\geqslant m+t}$ for every t). Then for any two different J-marked bases F_J and G_J in $A[\mathbf{x}]$, the schemes $\operatorname{Proj} A[\mathbf{x}]/(F_J)$ and $\operatorname{Proj} A[\mathbf{x}]/(G_J)$ are different.*

Proof. By hypothesis and by Proposition 2.1*(ii)*, there exists a monomial $x^\alpha \in \mathcal{B}_J$ such that the corresponding polynomials $f_\alpha \in F_J$ and $g_\alpha \in G_J$ are different. If $\operatorname{Proj} A[\mathbf{x}]/(F_J) = \operatorname{Proj} A[\mathbf{x}]/(G_J)$, then $(F_J)_{\geqslant s} = (G_J)_{\geqslant s}$ for a sufficiently large s. Therefore, for $s \gg 0$, $x_0^s f_\alpha$ is contained in (G_J) and $x_0^s f_\alpha - x_0^s g_\alpha = x_0^s\big(\mathrm{T}(g_\alpha) - \mathrm{T}(f_\alpha)\big) \in (G_J)$. By definition, the support of $\mathrm{T}(g_\alpha) - \mathrm{T}(f_\alpha)$ is contained in $\mathcal{N}(J)$. Therefore, also the support of $x_0^s\big(\mathrm{T}(g_\alpha) - \mathrm{T}(f_\alpha)\big)$ is in $\mathcal{N}(J)$, due to the hypothesis on J. Finally, by Theorem 1.7*(ii)*-*(vi)*, this implies $x_0^s\big(\mathrm{T}(g_\alpha) - \mathrm{T}(f_\alpha)\big) \in (F_J)_{s+|\alpha|} \cap \langle \mathcal{N}(J)\rangle = \{0\}$, so that $\mathrm{T}(g_\alpha) = \mathrm{T}(f_\alpha)$, against the assumption $f_\alpha \neq g_\alpha$. □

Definition 3.3. *We say that J is an m-*truncation ideal *(m-truncation for short) if $J = J'_{\geqslant m}$ for J' a saturated strongly stable ideal.*

Observe that the monomials divisible by x_0 in the monomial basis of an m-truncation ideal J (if any) are of degree m. Therefore, by Proposition 3.2, different m-truncation ideals define different projective schemes. We emphasize that a priori the truncation degree m can be any positive integer. We will discuss special values of m later in the paper.

We now describe the relations among marked functors (resp. schemes) corresponding to different truncations of the same saturated strongly stable ideal J. We will prove that for integers m larger than a suitable degree depending on J, the $J_{\geqslant m}$-marked schemes are all isomorphic. However, the construction of $\mathbf{Mf}_{J_{\geqslant m}}$ given in Theorem 2.6 depends on m since we obtain it as a closed subscheme of an affine space whose dimension increases with m. From a computational point of view it will be convenient to choose, among isomorphic marked schemes, the one corresponding to the minimal value of m, while for other applications higher values of m can be more convenient.

In order to compare $J_{\geqslant m}$-marked bases in $A[\mathbf{x}]$ for different values of m, we refer to Proposition 3.2. By associating to a marked basis the scheme it defines, we will identify $I = (F_{J_{\geqslant m}}) \in \underline{\mathbf{Mf}}_{J_{\geqslant m}}(A)$ and $I' = (G_{J_{\geqslant m'}}) \in \underline{\mathbf{Mf}}_{J_{\geqslant m'}}(A)$ when $\operatorname{Proj} A[\mathbf{x}]/I = \operatorname{Proj} A[\mathbf{x}]/I'$ in $\mathbb{P}^n_A$, i.e. when $I_{\geqslant s} = I'_{\geqslant s}$, for $s \gg 0$. By Theorem 1.7*(iii)* and Proposition 2.1*(ii)*, this is equivalent to $\widetilde{F}^{(s)}_{J_{\geqslant m}} = \widetilde{G}^{(s)}_{J_{\geqslant m'}}$ for $s \gg 0$.

Theorem 3.4. *Let J be a saturated strongly stable ideal. Then, for every $s > 0$ and for any noetherian ring A, $\underline{\mathbf{Mf}}_{J_{\geqslant s-1}}(A) \subseteq \underline{\mathbf{Mf}}_{J_{\geqslant s}}(A)$. More precisely,*

(i) if J has no minimal generators of degree $s+1$ divisible by the variable x_1 or $J_{\geqslant s-1} = J_{\geqslant s}$, then $\underline{\mathbf{Mf}}_{J_{\geqslant s-1}} = \underline{\mathbf{Mf}}_{J_{\geqslant s}}$;
(ii) otherwise, $\underline{\mathbf{Mf}}_{J_{\geqslant s-1}}$ is a proper closed subfunctor of $\underline{\mathbf{Mf}}_{J_{\geqslant s}}$.

Proof. To prove the inclusion $\underline{\mathbf{Mf}}_{J_{\geqslant s-1}}(A) \subseteq \underline{\mathbf{Mf}}_{J_{\geqslant s}}(A)$, let us consider a $J_{\geqslant s-1}$-marked basis F. The set $G := \widetilde{F}^{(s)} \cup \{f_\alpha \in F \mid x^\alpha \in \mathcal{B}_J \text{ and } |\alpha| > s\}$ is by construction a $J_{\geqslant s}$-marked set. In fact, G is a $J_{\geqslant s}$-marked basis, since $\langle G^{(s)}\rangle = \langle F^{(s)}\rangle$ by Theorem 1.7*(iii)*-*(iv)* and the generators of degree larger than s are the same in the two marked sets.

From now on in this proof we denote by J' the truncation of J in degree $s-1$, by $\mathcal{F}_{J'}$ the marked set analogous to the one given in (2.4) that we use to construct the ideal $\mathfrak{I}_{J'} \subseteq \mathbb{Z}[\mathbf{C}']$ of $\mathbf{Mf}_{J'}$. We also let $A' := \mathbb{Z}[\mathbf{C}']/\mathfrak{I}_{J'}$, $\phi_{F_{J'}} \colon \mathbb{Z}[\mathbf{C}'] \to A'$ the canonical map on the quotient and $\phi_{F_{J'}}[\mathbf{x}]$ the extension to $\mathbb{Z}[\mathbf{C}'][\mathbf{x}] \to A'[\mathbf{x}]$. Moreover, J'' will be the truncation of J in degree s and $\mathcal{F}_{J''}$, $\mathbb{Z}[\mathbf{C}'']$, $\mathfrak{I}_{J''}$, A'', $\phi_{F_{J''}}$ are defined analogously. By the definition of $\mathfrak{I}_{J'}$ and $\mathfrak{I}_{J''}$, we observe that $\phi_{F_{J'}}[\mathbf{x}](\mathcal{F}_{J'})$ is a J'-marked basis in $A'[\mathbf{x}]$ and $\phi_{F_{J''}}[\mathbf{x}](\mathcal{F}_{J''})$ is a J''-marked basis in $A''[\mathbf{x}]$.

We first prove *(ii)*. Let us consider the J''-marked set

$$\mathcal{G} := \widetilde{\mathcal{F}}^{(s)}_{J'} \cup \left\{ f'_\alpha \in \mathcal{F}_{J'} \mid x^\alpha \in \mathcal{B}_J,\ |\alpha| > s \right\}.$$

By Theorem 1.7*(v)*-*(viii)*, $\phi_{F_{J'}}[\mathbf{x}](\mathcal{G})$ is a J''-marked basis of $A'[\mathbf{x}]$, since

$$\mathcal{N}\left(J'', \left(\phi_{F_{J'}}[\mathbf{x}](\mathcal{G})\right)\right) \subseteq \mathcal{N}\left(J', \left(\phi_{F_{J'}}[\mathbf{x}](\mathcal{F}_{J'})\right)\right) = \{0\}.$$

Thus, the ring homomorphism

$$\begin{aligned} \psi \colon \mathbb{Z}[\mathbf{C}''] &\longrightarrow \mathbb{Z}[\mathbf{C}'] \\ C''_{\alpha\beta} &\longmapsto \text{coefficient of } x^\beta \text{ in } g_\alpha \in \mathcal{G}. \end{aligned}$$

induces an homomorphism $\overline{\psi}\colon A'' \to A'$ such that $\phi_{F_{J'}} \circ \psi = \overline{\psi} \circ \phi_{F_{J''}}$. Moreover $\phi_{F_{J'}} \circ \psi$ is surjective, being the composition of two surjective homomorphisms. Indeed,

$$C'_{\alpha\beta} = \begin{cases} \psi(C''_{\alpha\beta}), & \text{if } x^\alpha \in \mathcal{B}_J,\ |\alpha| \geqslant s, \\ \psi(C''_{\eta\gamma}),\ x^\eta = x_0 x^\alpha,\ x^\gamma = x_0 x^\beta, & \text{otherwise} \end{cases}.$$

Under our assumptions, for every $f'_\alpha \in \mathcal{F}_{J'}$ of degree $s-1$, $x_0 T(f'_\alpha)$ is a J''-remainder, so that $x_0 f'_\alpha \in \mathcal{G}$.

Therefore, the epimorphism $\overline{\psi}$ induces an isomorphism between $\mathbf{Mf}_{J'} = \operatorname{Spec} \mathbb{Z}[\mathbf{C}']/\mathfrak{I}_{J'}$ and a closed subscheme of $\mathbf{Mf}_{J''} = \operatorname{Spec} \mathbb{Z}[\mathbf{C}'']/\mathfrak{I}_{J''}$. In order to show that this subscheme is proper, we can look at the Zariski tangent spaces of $\mathbf{Mf}_{J'}$ and $\mathbf{Mf}_{J''}$ at the points corresponding to J' and J'' and see that they have different dimension (see [BCLR13, Theorem 5.7] for the details).

To prove *(i)* we observe that the new condition on J implies that for every $x^\gamma \in \mathcal{N}(J)_s$, either $x_1 x^\gamma \in \mathcal{N}(J)_{s+1}$ or $x_1 x^\gamma = x_0 x^\delta$ with $x^\delta \in J_s$ holds.

Exploiting this property we first prove that $C''_{\eta\gamma} \in \mathfrak{I}_{J''}$ if $x^\eta \in J_s$, $x_0 \mid x^\eta$ and $x_0 \nmid x^\gamma$. Let $x^\epsilon = x_1 x^\eta / x_0$ and consider the EK-syzygy $S^{\mathrm{EK}}(f''_\eta, f''_\epsilon) = x_0 T(f''_\epsilon) - x_1 T(f''_\eta)$ between the elements $f''_\eta, f''_\epsilon \in \mathcal{F}_{J''}$. The J-remainder of this polynomial given by $\xrightarrow{\mathcal{F}^{(\cdot)}_{J''}}$ is of the type $g = x_0 T(f''_\epsilon) - x_1 T(f''_\eta) + x_0 \sum C''_{\eta\delta} f''_\beta$, where $f''_\beta \in \mathcal{F}_{J''}$ and the sum is over the multi-indices β such that $x^\beta := x_1 x^\delta / x_0 \in J_s$ with x^δ divisible by x_0 and contained in the support of $T(f''_\eta)$ and $C''_{\eta\delta}$ the coefficient of x^δ in f''_η. If x^γ is a term in the support of $T(f''_\eta)$ such that $x_0 \nmid x^\gamma$, then $x_1 x^\gamma \in \mathcal{N}(J)_{s+1}$ is contained in the support of g. By definition, $\mathfrak{I}_{J''}$ contains the $\mathbf{x}$-coefficients of g, thus in particular the coefficient $C''_{\eta\gamma}$ of $x_1 x^\gamma$ in g. For every $x^\alpha \in J_{s-1}$ and $x^\eta = x_0 x^\alpha$, let us denote by h_α the polynomial in $\mathbb{Z}[\mathbf{C}''][\mathbf{x}]$ such that $f''_\eta = x_0 h_\alpha + \sum C''_{\eta\gamma} x^\gamma$ with $x_0 \nmid x^\gamma$, so that $\phi_{F_{J''}}[\mathbf{x}](f''_\eta) = \phi_{F_{J''}}[\mathbf{x}](x_0 h_\alpha)$.

Using these polynomials we can define the J'-marked set

$$\mathcal{H} = \{h_\alpha \mid x^\alpha \in J_{s-1}\} \cup \{f''_\eta \in \mathcal{F}_{J''} \mid x^\eta \in \mathcal{B}_J,\ |\eta| \geqslant s\}.$$

By construction, $\phi_{F_{J''}}[\mathbf{x}](x_0 \mathcal{H}) \subseteq \phi_{F_{J''}}[\mathbf{x}](\mathcal{F}_{J''})$, hence $\phi_{F_{J''}}[\mathbf{x}](\mathcal{H})$ is a J'-marked basis by Theorem 1.7 *(v)-(viii)*. In fact, if the support of an element u in the ideal ${}_{A''}\big(\phi_{F_{J''}}[\mathbf{x}](\mathcal{H})\big)$ only contains monomials of $\mathcal{N}(J)$, then $x_0 u$ has the same support and is in ${}_{A''}\big(\phi_{F_{J''}}[\mathbf{x}](\mathcal{F}_{J''})\big)$, so that $u = 0$ since $\mathcal{N}\big(J, (\phi_{F_{J''}}[\mathbf{x}](\mathcal{F}_{J''}))\big) = \{0\}$.

Thus, the ring homomorphism

$$\begin{aligned} \varphi\colon \mathbb{Z}[\mathbf{C}'] &\longrightarrow \mathbb{Z}[\mathbf{C}''] \\ C'_{\alpha\beta} &\longmapsto \text{coefficient of } x^\beta \text{ in } h_\alpha \text{ if } |\alpha| = s-1 \\ C'_{\eta\gamma} &\longmapsto \text{coefficient of } x^\gamma \text{ in } f''_\eta \text{ if } x^\eta \in \mathcal{B}_J,\ |\eta| \geqslant s \end{aligned}$$

induces a homomorphism $\overline{\varphi}\colon A' \to A''$.

Finally, $\overline{\psi}$ and $\overline{\varphi}$ are inverses of each other. Indeed, if we apply to the J'-marked set $\mathcal{H}$ the construction from the first part of the proof, we obtain a J''-marked set $\mathcal{G}'$ such that $\phi_{F_{J''}}[\mathbf{x}](\mathcal{G}')$ is a J''-marked basis and $\phi_{F_{J''}}[\mathbf{x}](\mathcal{G}') \subseteq \phi_{F_{J''}}[\mathbf{x}](\mathcal{F}_{J''})$, hence $\phi_{F_{J''}}[\mathbf{x}](\mathcal{G}') = \phi_{F_{J''}}[\mathbf{x}](\mathcal{F}_{J''})$ by Proposition 2.1. □

4. Marked schemes and Hilbert schemes

We now briefly recall how the Hilbert scheme can be constructed as subscheme of a suitable Grassmannian. For any positive integer n and any numerical polynomial $p(t)$, consider the Hilbert functor

$$\underline{\mathbf{Hilb}}_n^{p(t)} : \underline{\text{Noeth-Rings}} \longrightarrow \underline{\text{Sets}}$$

associating to any noetherian ring A the set

$$\underline{\mathbf{Hilb}}_n^{p(t)}(A) = \{X \subseteq \mathbb{P}^n_A \mid X \to \operatorname{Spec} A \text{ is flat and has fibers with Hilbert polymial } p(t)\}$$

and to any ring homomorphism $f : A \to B$ the map

$$\begin{aligned} \underline{\mathbf{Hilb}}_n^{p(t)}(f) : \underline{\mathbf{Hilb}}_n^{p(t)}(A) &\longrightarrow \underline{\mathbf{Hilb}}_n^{p(t)}(B) \\ X &\longmapsto X \times_{\operatorname{Spec} A} \operatorname{Spec} B. \end{aligned}$$

Grothendieck first defined this functor and showed that it is representable [Gro95]. The Hilbert scheme $\mathbf{Hilb}_n^{p(t)}$ is defined as the scheme representing the Hilbert functor and it is classically constructed as

a subscheme of a suitable Grassmannian. Let us briefly recall how (for a detailed exposition see [IK99, HS04, BLMR14]). By Gotzmann's Regularity theorem ([Got78, Satz (2.9)] and [IK99, Lemma C.23]), there exists a positive integer r only depending on $p(t)$, called *Gotzmann number*, for which the ideal sheaf $\mathcal{I}_X$ of each scheme $X \in \underline{\mathbf{Hilb}}_n^{p(t)}(A)$ is r-regular (in the sense of Castelnuovo-Mumford regularity). This implies that the morphism

$$H^0\big(\mathcal{O}_{\mathbb{P}^n_A}(r)\big) \xrightarrow{\phi_X} H^0\big(\mathcal{O}_X(r)\big)$$

is surjective. By flatness, $H^0(\mathcal{O}_X(r))$ is a locally free module of rank $p(r)$ and, as an A-module, $H^0(\mathcal{O}_{\mathbb{P}^n_A}(r))$ is isomorphic to the homogeneous piece of degree r of the polynomial ring $A[\mathbf{x}]$. Since $A[\mathbf{x}]_r \simeq A^N$, where $N = \binom{n+r}{n}$, the homomorphism ϕ_X may be viewed as an element of the Grassmannian, whose corresponding functor is

$$\underline{\mathbf{Gr}}^N_{p(r)} : \underline{\text{Noeth-Rings}} \longrightarrow \underline{\text{Sets}}$$

associating to any noetherian ring A the set

$$\underline{\mathbf{Gr}}^N_{p(r)}(A) = \left\{ \begin{array}{c} \text{isomorphism classes of epimorphisms } A^N \to Q \\ \text{of locally free modules of rank } p(r) \end{array} \right\}$$

and to any morphism $f : A \to B$ the map

$$\begin{array}{rccc} \underline{\mathbf{Gr}}^N_{p(r)}(f) : & \underline{\mathbf{Gr}}^N_{p(r)}(A) & \longrightarrow & \underline{\mathbf{Gr}}^N_{p(r)}(B) \\ & A^N \to Q & \longmapsto & B^N \to Q \otimes_A B. \end{array}$$

Two epimorphisms $\phi : A^N \to Q$ and $\phi' : A^N \to Q'$ are isomorphic if there exists an isomorphism $\psi : Q \to Q'$ of A-modules such that the diagram

$$\begin{tikzcd} A^N \arrow[r, "\phi"] \arrow[d, "\mathrm{id}"'] & Q \arrow[d, "\psi"] \\ A^N \arrow[r, "\phi'"] & Q' \end{tikzcd}$$

commutes. Equivalently, ϕ and ϕ' are isomorphic if $\ker\phi = \ker\phi'$. Therefore, by identifying isomorphism classes of epimorphisms ϕ with $\ker\phi$, the Grassmann functor sends A to the set

$$\left\{ \begin{array}{l} A\text{-submodules } M \subseteq A^N \text{ such that} \\ A^N/M \text{ is locally free of rank } p(r) \end{array} \right\}.$$

This functor is representable and the representing scheme $\mathbf{Gr}^N_{p(r)}$ is the Grassmannian (see [Vak13, Section 16.7]). Therefore, one of the possible embeddings of the Hilbert scheme into a Grassmannian is given by the natural transformation of functors (introduced by Bayer in [Bay82])

$$\underline{\mathscr{H}} : \underline{\mathbf{Hilb}}_n^{p(t)} \longrightarrow \underline{\mathbf{Gr}}^N_{p(r)} \tag{4.1}$$

given by

$$\begin{array}{ccc} \underline{\mathbf{Hilb}}_n^{p(t)}(A) & \longrightarrow & \underline{\mathbf{Gr}}^N_{p(r)}(A) \\ X & \longmapsto & A[\mathbf{x}]_r \twoheadrightarrow H^0\big(\mathcal{O}_X(r)\big). \end{array}$$

By Yoneda's Lemma, any natural transformation of representable functors is induced by a unique morphism between their representing schemes. The associated morphism $\mathscr{H} : \mathbf{Hilb}_n^{p(t)} \to \mathbf{Gr}^N_{p(r)}$ is a closed embedding and the equations defining the Hilbert scheme $\mathbf{Hilb}_n^{p(t)}$ as a subscheme of $\mathbf{Gr}^N_{p(r)}$ were conjectured by Bayer [Bay82] and proved much later by Haiman and Sturmfels [HS04].

The Grassmannian has the well-known open cover by affine spaces which also defines the Plücker embedding. For any set $\mathcal{N}$ of $p(r)$ distinct monomials of $A[\mathbf{x}]_r$, consider the map

$$\mathrm{i}_{\mathcal{N}} : {}_A\langle \mathcal{N} \rangle \simeq A^{p(r)} \hookrightarrow A[\mathbf{x}]_r \simeq A^N$$

and the subfunctor $\underline{\mathbf{G}}_{\mathcal{N}}$ such that

$$\underline{\mathbf{G}}_{\mathcal{N}}(A) = \Big\{ \text{classes } \phi_Q : A^N \to Q \text{ in } \underline{\mathbf{Gr}}^N_{p(r)}(A) \text{ such that } \phi_Q \circ \mathrm{i}_{\mathcal{N}} \text{ is surjective} \Big\}.$$

Each such subfunctor is open and the family obtained varying the set of monomials $\mathcal{N}$ cover the Grassmann functor [Sta, Lemma 22.22.1]. Since $\phi_Q \circ \mathrm{i}_{\mathcal{N}}$ is an epimorphism between a free module and a locally free module of the same rank, it is in fact an isomorphism. Therefore, each Q in $\underline{\mathbf{G}}_{\mathcal{N}}(A)$ can be identified with the free module ${}_A\langle\mathcal{N}\rangle$ and we can rewrite the functors $\underline{\mathbf{G}}_{\mathcal{N}}$ as

$$\underline{\mathbf{G}}_{\mathcal{N}}(A) = \{\text{epimorphisms } A[\mathbf{x}]_r \to {}_A\langle\mathcal{N}\rangle \text{ of free modules of rank } p(r)\}$$

An epimorphism $\phi : A[\mathbf{x}]_r \to {}_A\langle\mathcal{N}\rangle$ is determined by its values on basis elements, $\phi(x^\alpha) = \sum_{x^\beta\in\mathcal{N}} a_{\alpha,\beta}x^\beta$. Thus its kernel is generated by

$$f_\alpha := x^\alpha - \sum_{x^\beta\in\mathcal{N}} a_{\alpha\beta}\, x^\beta$$

for all x^α of total degree r lying outside $\mathcal{N}$. If J is the ideal generated by the monomials in $A[\mathbf{x}]_r$ not contained in $\mathcal{N}$, then we can describe $\underline{\mathbf{G}}_{\mathcal{N}}$ as

$$\begin{aligned}\underline{\mathbf{G}}_{\mathcal{N}}(A) &= \{\text{free submodules } L \subseteq A[\mathbf{x}]_r \text{ such that } A[\mathbf{x}]_r \simeq L \oplus {}_A\langle\mathcal{N}\rangle\} = \\ &= \{\text{submodules } L \subseteq A[\mathbf{x}]_r \text{ generated by a } J\text{-marked set}\}\end{aligned}$$

We are interested in the open subfunctors of the Hilbert functor $\underline{\mathbf{Hilb}}_n^{p(t)}$ induced by the family of subfunctors $\underline{\mathbf{G}}_{\mathcal{N}}$ by means of $\underline{\mathscr{H}}$. We denote by $\underline{\mathbf{H}}_{\mathcal{N}}$ the subfunctor associating to A the set

$$\underline{\mathbf{H}}_{\mathcal{N}}(A) := \underline{\mathscr{H}}^{-1}\left(\underline{\mathbf{G}}_{\mathcal{N}}(A) \cap \underline{\mathscr{H}}\left(\underline{\mathbf{Hilb}}_n^{p(t)}(A)\right)\right). \tag{4.2}$$

The kernel of the map $A[\mathbf{x}]_r \to H^0(\mathcal{O}_X(r))$ is represented by the global sections of the sheaf $\mathcal{I}_X(r)$, i.e. by the homogeneous piece of degree r of the saturated ideal I_X defining X. Since I_X and $(I_X)_{\geqslant r}$ define the same scheme and $(I_X)_{\geqslant r}$ is generated by the homogeneous piece of degree r, we can rewrite the subfunctor $\underline{\mathbf{H}}_{\mathcal{N}}$ as

$$\begin{aligned}\underline{\mathbf{H}}_{\mathcal{N}}(A) &= \left\{X \in \underline{\mathbf{Hilb}}_n^{p(t)}(A) \;\middle|\; A[\mathbf{x}]_r \simeq H^0\big(\mathcal{I}_X(r)\big) \oplus {}_A\langle\mathcal{N}\rangle\right\} \\ &= \left\{X \in \underline{\mathbf{Hilb}}_n^{p(t)}(A) \;\middle|\; (I_X)_{\geqslant r} \text{ is generated by a } J\text{-marked set}\right\}.\end{aligned} \tag{4.3}$$

It is then natural to relate $\underline{\mathbf{H}}_{\mathcal{N}}(A)$ to $\underline{\mathbf{Mf}}_J(A)$. In general their relations are less obvious than one might expect. However, under suitable conditions on $\mathcal{N}$ and J, we can identify $\underline{\mathbf{H}}_{\mathcal{N}}$ with a marked functor. The following result gives a new proof in terms of functors of [BLR13, Theorem 2.5].

Lemma 4.1. *Let $p(t)$ be a Hilbert polynomial in $\mathbb{P}^n$ with Gotzmann number r and let J be a strongly stable ideal such that $|\mathcal{N}(J)_r| = p(r)$. Then, for every noetherian ring A*

$$\underline{\mathbf{H}}_{\mathcal{N}(J)_r}(A) \neq \emptyset \quad\Longleftrightarrow\quad \textit{the Hilbert polynomial of } A[\mathbf{x}]/J \textit{ is } p(t).$$

Proof. ($\Leftarrow$) If the Hilbert polynomial of $A[\mathbf{x}]/J$ is $p(t)$, then $\operatorname{Proj} A[\mathbf{x}]/J \in \underline{\mathbf{H}}_{\mathcal{N}}(A)$.

($\Rightarrow$) Assume that X is a scheme in $\underline{\mathbf{H}}_{\mathcal{N}}(A)$ and set $I := (I_X)_{\geqslant r}$. By Theorem 1.7*(iii)*, for every $m \geqslant r$, the A-module I_m has a free direct summand with rank equal to that of J_m, hence the value of the Hilbert polynomial of J in every degree $m \geqslant r$ cannot be smaller than $p(m)$. On the other hand, this rank cannot be larger than $p(m)$ by Macaulay's Estimate on the Growth of Ideals [Gre98, Theorem 3.3]. □

Corollary 4.2. *Let J be a saturated strongly stable ideal such that $\mathbb{Z}[\mathbf{x}]/J$ has Hilbert polynomial $p(t)$. Then*

$$\underline{\mathbf{H}}_{\mathcal{N}(J)_r} \simeq \underline{\mathbf{Mf}}_{J_{\geqslant r}}.$$

We can rephrase the statement of the corollary by saying that for every noetherian ring A,

$$\underline{\mathbf{H}}_{\mathcal{N}}(A) = \left\{X \in \underline{\mathbf{Hilb}}_n^{p(t)}(A) \;\middle|\; (I_X)_{\geqslant r} \text{ is generated by a } J\text{-marked basis}\right\}.$$

Therefore, upon identifying ideals and the schemes they define, the isomorphism from the corollary is a canonical identification $\underline{\mathbf{H}}_{\mathcal{N}(J)_r} = \underline{\mathbf{Mf}}_{J_{\geqslant r}}$.

We can then deduce from Corollary 4.2 an isomorphism between the representing schemes. Taking into account Theorem 3.4 we obtain the following result.

Corollary 4.3. *Let J be a saturated strongly stable ideal and r be the Gotzmann number of its Hilbert polynomial $p(t)$. If ρ is the maximal degree of monomials in $\mathcal{B}_J$ divisible by x_1, then*

(i) for $s \geqslant \rho - 1$, $\mathbf{Mf}_{J_{\geqslant s}}$ is an open subscheme of $\mathbf{Hilb}_n^{p(t)}$;

(ii) for $s < \rho - 1$, $\mathbf{Mf}_{J_{\geqslant s}}$ is a locally closed subscheme of $\mathbf{Hilb}_n^{p(t)}$.

Proof. (i) By Theorem 3.4, we have

$$\mathbf{Mf}_{J_{\geqslant \rho-1}} = \mathbf{Mf}_{J_{\geqslant \rho}} = \cdots = \mathbf{Mf}_{J_{\geqslant r}} = \mathbf{H}_{\mathcal{N}(J)_r}.$$

(ii) By Theorem 3.4, for $s < \rho - 1$, we know that in the chain

$$\mathbf{Mf}_{J_{\geqslant s}} \subseteq \mathbf{Mf}_{J_{\geqslant s+1}} \subseteq \cdots \subseteq \mathbf{Mf}_{J_{\geqslant \rho-1}} = \cdots = \mathbf{Mf}_{J_{\geqslant r}} = \mathbf{H}_{\mathcal{N}(J)_r},$$

there is at least one proper closed embedding, so that $\mathbf{Mf}_{J_{\geqslant s}}$ is a locally closed subscheme of the Hilbert scheme. □

Remark 4.4. Though our results only apply to a small amount of the open subsets $\underline{\mathbf{H}}_{\mathcal{N}}$ that are necessary to cover $\mathbf{Hilb}_n^{p(t)}$, in many interesting cases they allow to obtain a different open cover by exploiting the action of the linear group $\mathrm{PGL}(n+1)$. In particular, this is true for the Hilbert scheme $\mathbf{Hilb}_{n,K}^{p(t)} = \mathbf{Hilb}_n^{p(t)} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} K$ representing the Hilbert functor $\underline{\mathbf{Hilb}}_{n,K}^{p(t)} \colon \underline{K\text{-Algebras}} \to \underline{\text{Sets}}$ for every field K of characteristic zero. Indeed, the properties of the generic initial ideal proved by Galligo [Gal74] allow to prove that every point of the Hilbert scheme is contained in an open subset $\mathbf{H}_{\mathcal{N},K}$, where $\mathcal{N} := \mathcal{N}(J)_r$ for a saturated strongly stable ideal J, at least up to the action of a general element in $\mathrm{PGL}(n+1)$. Such open cover of $\mathbf{Hilb}_{n,K}^{p(t)}$ is presented in [BLR13, BLMR14, BBR15].

The set of strongly stable ideals that are necessary to obtain such new open cover of the Hilbert scheme can be effectively determined using the algorithm presented in [CLMR11, Lel12, Lel].

Remark 4.5. The equations of the open subscheme $\mathbf{H}_{\mathcal{N}(J)_r}$ computed as the marked scheme over $J_{\geqslant r}$ are the same equations determined by Iarrobino and Kleiman in [IK99]. Indeed, the Eliahou-Kervaire syzygies among the generators of $J_{\geqslant r}$ are linear, so that imposing $S^{\mathrm{EK}}(f_\alpha, f_\beta) \xrightarrow{F^{(\cdot)}_{J_{\geqslant r}}} 0$ is equivalent to prove that $\langle SF^{(r+1)}_{J_{\geqslant r}} \rangle \subseteq \langle F^{(r+1)}_{J_{\geqslant r}} \rangle$. If we represent the generators $\{x_i f_\alpha \mid \forall\, x^\alpha \in \mathcal{B}_{J_{\geqslant r}},\ i = 0, \ldots, n\}$ of $(F_J)_{r+1}$ by a matrix $\mathcal{M}_J^{(r+1)}$, then the condition $\langle SF^{(r+1)}_{J_{\geqslant r}} \rangle \subseteq \langle F^{(r+1)}_{J_{\geqslant r}} \rangle$ is equivalent to $\operatorname{rk} \mathcal{M}_J^{(r+1)} \leqslant \operatorname{rk} \langle F^{(r+1)}_{J_{\geqslant r}} \rangle = \operatorname{rk} J_{r+1} = \binom{n+r}{n} - p(r)$ and the latter condition is guaranteed by imposing the vanishing of the minors of order $\operatorname{rk} J_{r+1} + 1$. This is how Iarrobino-Kleiman determine local equations of the Hilbert scheme. Notice that using this approach it is possible to deduce that the equations are of degree at most $\binom{n+r}{n} - p(r) + 1$, while constructing the equations applying Theorem 2.11*(ii)* and our reduction procedure, it is possible to deduce that the equations have degree at most $\deg p(t) + 2$ (see [BLR13, Theorem 3.3]).

Remark 4.6. The statements of Corollary 4.3 can be very useful both from a computational and a theoretical point of view. Indeed, for a fixed saturated ideal J, the number of variables involved in the computation of equations defining the marked scheme $\mathbf{Mf}_{J_{\geqslant s}}$ dramatically increases with s. On the other hand, in [BBR15] the equalities of Corollary 4.3*(i)* show that the open subset of $\mathbf{Hilb}_{n,K}^{p(t)}$ of the r'-regular points, for a given $r' < r$, can be embedded as a locally closed subscheme in the Grassmannian $\mathbf{Gr}_{p(r')}^{N(r')}$, smaller than that in which we can embed the entire Hilbert scheme.

Moreover, in several cases marked schemes $\mathbf{Mf}_{J_{\geqslant s}}$ with $s < \rho - 1$ correspond to interesting loci of the Hilbert scheme and our results allow effective computations also on them.

Example 4.7. Consider the strongly stable ideal $J = (x_2^2, x_2x_1, x_1^4) \subseteq \mathbb{Z}[x_0, x_1, x_2]$. The Hilbert polynomial of $\operatorname{Proj} \mathbb{Z}[x_0, x_1, x_2]/J$ is $p(t) = 5$ with Gotzmann number equal to 5. Therefore, the open subscheme $\mathbf{H}_{\mathcal{N}(J)_5} \subseteq \mathbf{Hilb}_2^5$ can be defined as closed subscheme of the affine open subscheme $\mathbf{G}_{\mathcal{N}(J)_5} \subseteq \mathbf{Gr}_5^{21}$ of dimension 80. Applying Corollary 4.3*(i)*, we can define the same open subscheme by means of the isomorphism $\mathbf{Mf}_{J_{\geqslant 3}} \simeq \mathbf{H}_{\mathcal{N}(J)_5}$ with $\mathbf{Mf}_{J_{\geqslant 3}} \subseteq \mathbb{A}^{30}$.

Finally, also the marked scheme associated to the saturated ideal may be very important. For instance, in the special case of zero-dimensional schemes in the projective plane $\mathbb{P}^2$, for each postulation

there is a unique strongly stable ideal J realizing it (see for instance [Eis05, Chapters 1-3]). Therefore, $\mathbf{Mf}_J$ parametrizes the locus of the Hilbert scheme $\mathbf{Hilb}_2^d$ with a fixed Hilbert function (up to the action of the projective linear group). In the example, the scheme $\mathbf{Mf}_J$ parameterize the locus of 5 points in the plane with postulation $(1, 3, 4, 5, \ldots)$.

5. Gröbner strata

Throughout this section, we denote by σ a term ordering on the polynomial ring $A[\mathbf{x}]$ and by $\mathrm{in}_\sigma(I)$ the initial ideal of an ideal $I \subseteq A[\mathbf{x}]$ with respect to such term ordering. We define the *Gröbner functor* $\underline{\mathbf{St}}_J^\sigma : \underline{\text{Noeth-Rings}} \to \underline{\text{Sets}}$ that associates to any ring A the set

$$\underline{\mathbf{St}}_J^\sigma(A) = \{I \subseteq A[\mathbf{x}] \mid \mathrm{in}_\sigma(I) = J\} \tag{5.1}$$

and to any ring homomorphism $\phi : A \to B$ the function

$$\begin{array}{rccc} \underline{\mathbf{St}}_J^\sigma(\phi) : & \underline{\mathbf{St}}_J^\sigma(A) & \longrightarrow & \underline{\mathbf{St}}_J^\sigma(B) \\ & I & \longmapsto & I \otimes_A B. \end{array}$$

Gröbner basis theory over rings is more intricate than Gröbner basis theory over fields (see also [Led11] for a more detailed discussion). A first delicate issue is the definition of initial ideals. Given an ideal $I \subseteq A[\mathbf{x}]$, we can consider the ideal generated by the leading monomials or the ideal generated by leading terms, i.e. monomials with coefficients, of the polynomials in I. In general neither of the two definitions is well-suited for functorial constructions, since taking the initial ideal of a given $I \subseteq A[\mathbf{x}]$ does not commute with base change $\otimes_A B$ unless the initial ideal of I is a monomial ideal. For instance, the initial ideal of $I = (2x_1 + x_0) \subseteq \mathbb{Z}[x_0, x_1]$, $x_1 > x_0$, is $J' = (x_1)$ according to the first definition and $J'' = (2x_1)$ according to the second one; after the extension $\mathbb{Z} \to \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ we obtain $\mathrm{in}_\sigma(I \otimes_{\mathbb{Z}} \mathbb{Z}_2) = (x_0)$, while $J' \otimes_{\mathbb{Z}} \mathbb{Z}_2 = (x_1)$ and $J'' \otimes_{\mathbb{Z}} \mathbb{Z}_2 = (0)$.

Definition 5.1 ([Pau92, Wib07])**.** *Let I be an ideal in a polynomial ring $A[\mathbf{x}]$, with A a noetherian ring, and let σ be a term ordering. The ideal I is called* monic *(with respect to σ) if for all monomials $x^\alpha \in A[\mathbf{x}]$ the set*

$$\mathrm{LC}(I, x^\alpha) = \{a \in A \mid ax^\alpha \text{ is the leading term of } g \in I\} \cup \{0\}$$

is either $\{0\}$ or A.

Therefore, the definition of $\underline{\mathbf{St}}_J^\sigma$ given in (5.1) is correct and non-ambiguous if we assume that J is a monomial ideal and restrict the set of ideals I to those that are monic. To this aim, we follow the line of the definition of marked functor and consider the ideals I that are generated by a suitable set of polynomials, marked on J, that we expect to form a reduced Gröbner basis. Indeed, an ideal $I \subseteq A[\mathbf{x}]$ admits a reduced Gröbner basis if, and only if, I is a monic ideal (see [Asc05, Pau92, Wib07]). We recall that a *reduced* Gröbner basis is a Gröbner basis composed by polynomials with leading coefficient equal to 1_A and such that no term other than the leading one is contained in the initial ideal. More precisely, the reduced Gröbner basis takes the shape

$$G_J = \left\{ x^\alpha - \sum_{\substack{x^\beta \in \mathcal{N}(J)_{|\alpha|} \\ x^\alpha >_\sigma x^\beta}} b_{\alpha\beta}\, x^\beta \;\middle|\; x^\alpha \in \mathcal{B}_J \right\}.$$

This is a J-marked set, considering the marking given by the term ordering, i.e. $\mathrm{Ht}(g) = \mathrm{in}_\sigma(g)$. Furthermore, G_J is a J-marked basis, since for $I = (G_J) \in \underline{\mathbf{St}}_J^\sigma(A)$, the monomials in $\mathcal{N}(\mathrm{in}_\sigma(I)) = \mathcal{N}(J)$ are even a basis of the A-module $A[\mathbf{x}]/I$. Then we can rewrite $\underline{\mathbf{St}}_J^\sigma(A)$ as

$$\begin{aligned} \underline{\mathbf{St}}_J^\sigma(A) &= \{\text{monic ideal } I \subseteq A[\mathbf{x}] \mid \mathrm{in}_\sigma(I) = J\} \\ &= \big\{ I = (G_J) \mid G_J \text{ reduced Gröbner basis and } \mathrm{in}_\sigma\big((G_J)\big) = J \big\}. \end{aligned} \tag{5.2}$$

Thus, $\underline{\mathbf{St}}_J^\sigma(A) \subseteq \underline{\mathbf{Mf}}_J(A)$ for every A, and there is an injection of functors $\underline{\mathbf{St}}_J^\sigma \to \underline{\mathbf{Mf}}_J$.

Lemma 5.2. *Let J be any monomial ideal and σ be a term ordering. Then $\underline{\mathbf{St}}_J^\sigma$ is a functor and a Zariski sheaf.*

Proof. The arguments used in the proofs of Proposition 2.3 and Lemma 2.4 also apply to the case of the Gröbner functor. □

Theorem 5.3. *Let J be an m-truncation strongly stable ideal and σ be a term ordering. Then the Gröbner functor $\underline{\mathbf{St}}^\sigma_J$ is a closed subfunctor of $\underline{\mathbf{Mf}}_J$.*

Using Notation 2.5, $\underline{\mathbf{St}}^\sigma_J$ is represented by the affine scheme $\mathbf{St}^\sigma_J := \operatorname{Spec}\mathbb{Z}[\mathbf{C}]/\mathfrak{I}^\sigma_J$, where $\mathfrak{I}^\sigma_J$ is the sum of the ideal $\mathfrak{I}_J$ described in Theorem 2.6 and the ideal $\mathfrak{G}^\sigma_J := (C_{\alpha\beta} \mid x^\beta >_\sigma x^\alpha)$.

Proof. Straightforward by applying the criterion given in Proposition 2.9 of [HS04] on the inclusion $\iota : \underline{\mathbf{St}}^\sigma_J(A) \hookrightarrow \underline{\mathbf{Mf}}_J(A)$. □

We will call *Gröbner stratum* the scheme representing the Gröbner functor.

Example 5.4. Let us consider the ideal $J = (x_2^2, x_2x_1, x_1^3)$ of Example 2.14 and the term ordering $\mathtt{DegLex}$. There is only one monomial in $\mathcal{N}(J)$ greater than some monomial of the same degree in $\mathcal{B}_J$: $x_2x_0^2 >_{\mathtt{DegLex}} x_1^3$. Therefore, the ideal defining $\mathbf{St}^{\mathtt{DegLex}}_J$ as a subscheme of $\mathbb{A}^{12} = \operatorname{Spec}\mathbb{Z}[\mathbf{C}]$ is the sum of the ideal defining $\mathbf{Mf}_J$ and the principal ideal $(C_{030,201})$ and $\mathbf{St}^{\mathtt{DegLex}}_J$ is an hyperplane section of $\mathbf{Mf}_J$.

An analogue of Theorem 3.4 also holds for Gröbner strata (see [LR11, Theorem 4.7]). In particular, we have an isomorphism $\mathbf{St}^\sigma_{J_{\geqslant s-1}} \simeq \mathbf{St}^\sigma_{J_{\geqslant s}}$ under the assumption of Theorem 3.4*(i)*, leading to the isomorphism $\mathbf{Mf}_{J_{\geqslant s-1}} \simeq \mathbf{Mf}_{J_{\geqslant s}}$. From this property, we can deduce some cases in which marked families and Gröbner strata coincide.

We need the following property.

Proposition 5.5 ([CLMR11, Lemma 3.2])**.** *Let J be a saturated strongly stable ideal. If the truncation $J_{\geqslant m}$ is a gen-segment ideal, then so is $J_{\geqslant m-1}$. In general, the opposite implication is not true.*

Thus, if we consider a strongly stable saturated ideal J without minimal generators divisible by x_1 in degree $s+1$, then $\mathbf{Mf}_{J_{\geqslant s-1}} \simeq \mathbf{Mf}_{J_{\geqslant s}}$ and $\mathbf{St}^\sigma_{J_{\geqslant s-1}} \simeq \mathbf{St}^\sigma_{J_{\geqslant s}}$. If, moreover, we assume that there exists a term ordering σ making $J_{\geqslant s-1}$ a gen-segment ideal, then by Theorem 3.4, we get

$$\mathbf{St}^\sigma_{J_{\geqslant s}} \simeq \mathbf{St}^\sigma_{J_{\geqslant s-1}} = \mathbf{Mf}_{J_{\geqslant s-1}} \simeq \mathbf{Mf}_{J_{\geqslant s}}$$

so that $\mathbf{St}^\sigma_{J_{\geqslant s}}$ and $\mathbf{Mf}_{J_{\geqslant s}}$ coincide, even if $J_{\geqslant s}$ were not a gen-segment ideal. Note that in this last case there exist pairs of monomials $x^\alpha \in J_s$ and $x^\beta \in \mathcal{N}(J)s$ such that $x^\alpha <_\sigma x^\beta$. However, since $\mathfrak{I}_{J_{\geqslant s}}$ and $\mathfrak{I}^\sigma_{J_{\geqslant s}}$ coincide, the variables $C_{\alpha\beta}$ corresponding to those pairs of monomials must be already contained in $\mathfrak{I}_{J_{\geqslant s}}$.

Example 5.6. Let us consider the ideal $J = (x_2^3, x_2^2x_1, x_2x_1^2) \subseteq \mathbb{Z}[x_0, x_1, x_2]$. Its Castelnuovo-Mumford regularity is 3 and its Hilbert polynomial is $p(t) = t + 4$ with Gotzmann number 4.

For $s = 1, 2, 3$, $J_{\geqslant s}$ is a gen-segment ideal with respect to any term ordering σ induced by a refinement of the grading $(4, 3, 1)$, whereas $J_{\geqslant 4}$ cannot be a gen-segment since $x_2x_1^2x_0 \in J_4$, $x_1^4, x_2^2x_0^2 \in \mathcal{N}(J)_4$ and $(x_2x_1^2x_0)^2 = x_1^4 \cdot x_2^2x_0^2$. Since there is no minimal generator in degree 5, the equality $\mathbf{Mf}_{J_{\geqslant 3}} = \mathbf{St}^\sigma_{J_{\geqslant 3}}$ induces the equality $\mathbf{Mf}_{J_{\geqslant 4}} = \mathbf{St}^\sigma_{J_{\geqslant 4}}$ as subschemes of $\mathbf{Hilb}_2^{t+4}$, even if our construction defines them in affine spaces of different dimensions. Indeed, in the construction of $\mathbf{Mf}_{J_{\geqslant 4}}$ we consider the variable $C_{121,040}$ corresponding to the monomial x_1^4 in the tail of the polynomial f_{121} with $\operatorname{Ht}(f_{121}) = x_2x_1^2x_0$, while this variable does not appear in the construction of $\mathbf{St}^\sigma_{J_{\geqslant 4}}$, since $x_1^4 >_\sigma x_2x_1^2x_0$. This means that the variable $C_{121,040}$ must be already contained in the ideal defining $\mathbf{Mf}_{J_{\geqslant 4}}$. We will now check this fact, by a direct computation of $\mathfrak{I}_{J_{\geqslant 4}}$ as done in Corollary 2.13.

Among the EK-polynomials involving f_{121} there is $g := S^{\mathrm{EK}}(f_{121}, f_{130}) = x_1f_{121} - x_0f_{130}$. The only monomials in $\operatorname{Supp}(g) \cap J$ are $x_2^2x_1x_0^2$ and $x_2x_1^2x_0^2$, both divisible by x_0. Then

$$g \xrightarrow{\mathcal{F}^{(\cdot)}_{J_{\geqslant 4}}}{}_* h = g - (C_{121,202} - C_{130,211})\widetilde{f}_{212} - (C_{121,112} - C_{130,121})\widetilde{f}_{212}$$

where $\widetilde{f}_{212} = x_0f_{211}$ and $\widetilde{f}_{122} = x_0f_{121}$. Therefore, we replace the monomials $x_2^2x_1x_0^2$ and $x_2x_1^2x_0^2$ with linear combinations of monomials all divisible by x_0, so that the monomial x_1^5 still appears in the support of h with coefficient $C_{121,040}$. Therefore, $C_{121,040}$ is one of the generators of the ideal $\mathfrak{I}_{J_{\geqslant 4}}$ defining $\mathbf{Mf}_{J_{\geqslant 4}}$.

6. Example: marked schemes and Gröbner strata of $(x_3, x_2^2, x_2x_1^3, x_1^4)$

In the final section, we report some results about marked schemes and Gröbner strata associated to the strongly stable ideal $J = (x_3, x_2^2, x_2x_1^3, x_1^4) \subseteq k[x_0, x_1, x_2, x_3]$ and its truncations. The ideal J defines a point of the Hilbert scheme $\mathbf{Hilb}_3^7$, which is an irreducible scheme of dimension 21 [**?**]. As the Gotzmann number is 7, $\mathbf{Hilb}_3^7$ can be defined as subscheme of the Grassmannian $\mathbf{Gr}_7^{120}$. The Iarrobino-Kleiman equations of the open subscheme $\mathbf{H}_{\mathcal{N}(J)_7} \subseteq \mathbf{G}_{\mathcal{N}(J)_7}$ can be computed considering the marked scheme $\mathbf{Mf}_{J_{\geqslant 7}}$. By direct computation, one can check that $\mathbf{Mf}_{J_{\geqslant 7}} \simeq \mathbf{H}_{\mathcal{N}(J)_7}$ is defined by 2058 quadratic equations in the coordinate ring of the affine space $\mathbb{A}^{791} \simeq \mathbf{G}_{\mathcal{N}(J)_7}$. This embedding is clearly inconvenient because of the huge number of variables and the resulting large codimension of $\mathbf{Mf}_{J_{\geqslant 7}}$.

By Theorem 3.4, the marked scheme $\mathbf{Mf}_{J_{\geqslant 7}}$ is isomorphic to the marked scheme $\mathbf{Mf}_{J_{\geqslant 3}}$. The latter scheme is defined as subscheme of $\mathbb{A}^{105}$, its ideal is generated by 210 quadratic polynomials and it turns out to be isomorphic to a rational hypersurface V in the affine space $\mathbb{A}^{22}$ defined by a degree 6 polynomial. The hardest part of the computation is working out the equations in order to find explicitly the embedding $\mathbf{Mf}_{J_{\geqslant 3}} \hookrightarrow \mathbb{A}^{22}$, since the process of elimination of 83 parameters highly increases the degree of the polynomials. This step can last a few hours (depending on the CPU) and requires large RAM memory. We recall that, in order to overcome this difficulty, an alternative polynomial reduction procedure (the so-called superminimal reduction) has been developed in [BCLR13]. This procedure produce equations defining a marked scheme embedded in an affine space whose dimension is in general far lower than the previous one. . For instance, $\mathbf{Mf}_{J_{\geqslant 3}}$ can be embedded in $\mathbb{A}^{28}$. Considering this embedding, we would need to eliminate only 6 parameters (instead of 83).

The superminimal reduction procedure can be seen as a generalization of the procedure used for computing Gröbner strata of zero-dimensional ideal in the affine framework. However, we emphasize that the open subscheme $\mathbf{H}_{\mathcal{N}(J)_7}$ cannot be studied as a Gröbner stratum. First, the truncation $J_{\geqslant 3}$ is not a gen-segment ideal. Indeed, the polynomials in the marked basis with head terms $x_2^2x_0$ and x_1^4 have respectively $x_2x_1^2$ and $x_2x_1^2x_0$ in the support of their tails, but $x_2x_1^2$ and $x_2x_1^2x_0$ cannot appear at the same time in the generators with initial terms $x_2^2x_0$ and x_1^4 of a reduced Gröbner basis, since we would have $x_2^2x_0 >_\sigma x_2x_1^2$, $x_1^4 >_\sigma x_2x_1^2x_0$ and $x_2^2x_0 \cdot x_1^4 = x_2x_1^2 \cdot x_2x_1^2x_0$. Second, the coefficients of $x_2x_1^2$ and $x_2x_1^2x_0$ in the polynomials of the marked basis with head terms $x_2^2x_0$ and x_1^4 are not contained in the ideal defining $\mathbf{Mf}_{J_{\geqslant 3}}$ so that the Gröbner stratum $\mathbf{St}^\sigma_{J_{\geqslant 3}}$ is a proper subscheme for every σ (Theorem 5.3). In fact, the Gröbner strata correspoding to the term orderings obtained as refinement of the gradings $(13, 6, 4, 1)$ and $(11, 6, 3, 1)$ have codimension 1 in $\mathbf{Mf}_{J_{\geqslant 3}}$ and are both isomorphic to $\mathbb{A}^{20}$. In the generic Gröbner basis of $\mathbf{St}^{(13,6,4,1)}_{J_{\geqslant 3}}$, the monomial $x_2x_1^2$ does not appear in the generator with initial term $x_2^2x_0$, while in the case of $\mathbf{St}^{(11,6,3,1)}_{J_{\geqslant 3}}$, $x_2x_1^2x_0$ does not appear in the generator with initial term x_1^4 (and there are no other differences with the marked basis).

Other proper subschemes of $\mathbf{Mf}_{J_{\geqslant 3}}$ can be obtained considering marked schemes (and Gröbner strata) of truncation of J in degree lower than 3 (the computation in these cases is much simpler and lasts few seconds). In Table 1, we show the comparison between the marked schemes and Gröbner strata w.r.t. the graded lexicographic and reverse lexicographic term orderings of several truncation of J. Notice that we already know by theoretical reasons that $\mathbf{Mf}_{J_{\geqslant 2}} \simeq \mathbf{Mf}_J$ and $\mathbf{St}^\sigma_{J_{\geqslant 2}} \simeq \mathbf{St}^\sigma_J$, for any σ. The case of the reverse lexicographic order is even more special, since the Gröbner strata w.r.t. `RevLex` of all truncations are isomorphic [LR11, Proposition 4.11].

Moreover, by direct computation, we observe that the saturated ideal J is a gen-segment w.r.t. the reverse lexicographic order, so that $\mathbf{Mf}_J = \mathbf{St}^{\mathtt{RevLex}}_J$. The truncation $J_{\geqslant 2}$ is a gen-segment with respect to any term ordering induced by a refinement of the grading $(7, 4, 3, 1)$, so that the marked scheme $\mathbf{Mf}_{J_{\geqslant 2}}$ coincides with the Gröbner stratum $\mathbf{St}^{(7,4,3,1)}_{J_{\geqslant 2}}$. Notice that the term ordering induced by $(7, 4, 3, 1)$ allows two monomials more than the reverse lexicographic order in the tails of the marked set, in fact $\mathbf{St}^{(7,4,3,1)}_{J_{\geqslant 2}} \subseteq \mathbb{A}^{39}$ and $\mathbf{St}^{\mathtt{RevLex}}_{J_{\geqslant 2}} \subseteq \mathbb{A}^{37}$. However, the explicit computation shows that these two monomials cannot appear in the tails of a marked basis, since $\mathbf{St}^{(7,4,3,1)}_{J_{\geqslant 2}} = \mathbf{Mf}_{J_{\geqslant 2}} \simeq \mathbf{Mf}_J = \mathbf{St}^{\mathtt{RevLex}}_J \simeq \mathbf{St}^{\mathtt{RevLex}}_{J_{\geqslant 2}}$.

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	Marked scheme	Gröbner stratum w.r.t. `RevLex`	Gröbner stratum w.r.t. `DegLex`
J	$\mathbf{Mf}_J \subseteq \mathbb{A}^{22}$ 28 equations $\mathbf{Mf}_J \simeq \mathbb{A}^{15}$	$\mathbf{St}_J^{\mathtt{RevLex}} \subseteq \mathbb{A}^{22}$ 28 equations $\mathbf{St}_J^{\mathtt{RevLex}} \simeq \mathbb{A}^{15}$	$\mathbf{St}_J^{\mathtt{DegLex}} \subseteq \mathbb{A}^{19}$ 28 equations $\mathbf{St}_J^{\mathtt{DegLex}} \simeq \mathbb{A}^{12}$
$J_{\geqslant 2}$	$\mathbf{Mf}_{J_{\geqslant 2}} \subseteq \mathbb{A}^{39}$ 77 equations $\mathbf{Mf}_{J_{\geqslant 2}} \simeq \mathbb{A}^{15}$	$\mathbf{St}_{J_{\geqslant 2}}^{\mathtt{RevLex}} \subseteq \mathbb{A}^{37}$ 71 equations $\mathbf{St}_{J_{\geqslant 2}}^{\mathtt{RevLex}} \simeq \mathbb{A}^{15}$	$\mathbf{St}_{J_{\geqslant 2}}^{\mathtt{DegLex}} \subseteq \mathbb{A}^{36}$ 77 equations $\mathbf{St}_{J_{\geqslant 2}}^{\mathtt{DegLex}} \simeq \mathbb{A}^{12}$
$J_{\geqslant 3}$	$\mathbf{Mf}_{J_{\geqslant 3}} \subseteq \mathbb{A}^{105}$ 210 equations $\mathbf{Mf}_{J_{\geqslant 3}} \simeq V \subseteq \mathbb{A}^{22}$ $V \overset{\sim}{\dashrightarrow} \mathbb{A}^{21}$, $\deg V = 6$	$\mathbf{St}_{J_{\geqslant 3}}^{\mathtt{RevLex}} \subseteq \mathbb{A}^{93}$ 204 equations $\mathbf{St}_{J_{\geqslant 3}}^{\mathtt{RevLex}} \simeq \mathbb{A}^{15}$	$\mathbf{St}_{J_{\geqslant 3}}^{\mathtt{DegLex}} \subseteq \mathbb{A}^{102}$ 210 equations $\mathbf{St}_{J_{\geqslant 3}}^{\mathtt{DegLex}} \simeq \mathbb{A}^{18}$

TABLE 1. Marked schemes and Gröbner strata w.r.t. the graded lexicographic and reverse lexicographic term orderings of J, $J_{\geqslant 2}$ and $J_{\geqslant 3}$, where $J = (x_3, x_2^2, x_2x_1^3, x_1^4) \subseteq k[x_0, x_1, x_2, x_3]$.

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