Bornological convergence and shields.

(Article begins on next page)

G. Beer; C. Costantini; S. Levi. Bornological convergence and shields..
MEDITERRANEAN JOURNAL OF MATHEMATICS. 10 (1) pp: 529-560.

Gap topologies in metric spaces *

Gerald Beer
Department of Mathematics, California State University Los Angeles
5151 State University Drive, Los Angeles, California 90032 – USA
gbeer@cslanet.calstatela.edu

Camillo Costantini
Dipartimento di Matematica, Università di Torino
via Carlo Alberto 10, 10123 Torino – Italy
camillo.costantini@unito.it

Sandro Levi
Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca
Via Cozzi 53, 20125 Milano – Italy
sandro.levi@unimib.it

This paper is dedicated to L. Kocinac, on the occasion of his 65th birthday.

Abstract

In this article we study gap topologies on the subsets of a metric space (X, d) induced by a general family $\mathscr{S}$ of nonempty subsets of X. Given two families and two metrics not assumed to be equivalent, we give a necessary and sufficient condition for one induced upper gap topology to be contained in the other. This condition is also necessary and sufficent for containment of the two-sided gap topologies under the mild assumption that the generating families contain the singletons. Coincidence of upper gap topologies in the most important special cases is attractively reflected in the underlying structure of (X, d). First and second countability of upper gap topologies is also characterized. This approach generalizes and unifies results in [12] and [19] and gives rise to a noticeable family of subsets that lie between the totally bounded and the bounded subsets of X.

Introduction

A dominant theme in the study of set convergence and related topologies on subsets of a metric space (X, d) has been the interplay between convergence and geometric extended

*2000 Math. subj. class.: primary 54B20, 54A10, 54E35, 54D70, 54A25; secondary 54A20, 54D65, 54D10, 54D15.

Keywords: metric space, metrizable space, separable metric space, total boundedness, boundedness, hyperspace, gap, gap topology, Wijsman topology, Hausdorff pseudometric topology, Attouch-Wets topology, proximal topology, bounded proximal topology, first countability, second countability, Γ operator

real-valued set functionals. The most important set-functionals are distance functionals, gap functionals, and excess functionals, as perhaps first analyzed systematically as a group in [13]. When the setting is specialized to convex subsets of a normed linear space, other functionals come into play, most notably, support functionals.

Not unexpectedly, distance functionals first received attention. Cornet [16] first suggested a unified program of topologies on nonempty closed subsets by viewing such sets as sitting in $C(X, \mathbb{R})$ equipped with topologies of uniform convergence on various bornologies, identifying each nonempty closed set A with its distance functional $d(\cdot, A)$. For the topology of uniform convergence on X one gets the classical *Hausdorff pseudometric topology* (see, e.g., [4, 26]); for the topology of uniform convergence on finite subsets, one gets nothing more than the topology of pointwise convergence, and the resulting topology is called the *Wijsman topology*, subsequently studied extensively by the authors and their associates [5, 12, 17, 18, 19, 21, 24]; for the topology of uniform convergence on bounded sets, one gets the *Attouch-Wets topology* [1, 4, 26, 27, 28], which plays a fundamental role in convex analysis, as it is stable with respect to duality in arbitrary normed linear spaces, as shown by Beer [2]. Both Hausdorff metric convergence and Attouch-Wets convergence are special cases of so-called bornological convergence as introduced by Lechicki, Levi and Spakowski [25] and studied by the present authors and their associates in subsequent papers [7, 10, 11]. In particular, bornological convergence is reconciled with uniform convergence of distance functions in [6, 14].

As is easily seen, the Wijsman topology is the weakest topology on the nonempty closed subsets such that for each $x \in X$, $A \mapsto d(x, A)$ is continuous. Varying metrics as well as points, one obtains three topologies of both historical and practical importance as weak topologies on the nonempty closed subsets $\mathscr{P}_0(X)$ of X:

- The classical *Vietoris topology* is obtained letting x vary over X and d vary over all metrics compatible with the topology of X [12];

- The *proximal topology*, a uniform variant of the Vietoris topology, is obtained letting x vary over X and d vary over all metrics uniformly equivalent to a initial fixed compatible metric [4, 12];

- The *slice topology* [3, 4, 15] - also known as the *Joly topology* [22] - on the nonempty closed convex subsets of a normed linear space is obtained by letting x vary over X and d vary over all metrics induced by norms equivalent to the initial norm [5].

Now the distance $d(x, A)$ from a point x to a nonempty subset A of X admits two natural extensions replacing x by a nonempty set B, called the *gap* $D_d(B, A)$ between B and A and the *excess* $e_d(B, A)$ of B over A:

$$D_d(B, A) := \inf_{x \in B} d(x, A) \quad \text{and} \quad e_d(B, A) = \sup_{x \in B} d(x, A).$$

Note that while gap is a symmetric functional, excess is not. Some representative results here are the following:

- The weak topology generated by $\{D_d(B,\cdot) : B \text{ nonempty and closed}\}$ agrees with the weak topology generated by $\{\rho(x,\cdot) : x \in X \text{ and } \rho \text{ uniformly equivalent to } d\}$ and is called the *proximal topology* [12];

- The Hausdorff pseudometric metric topology is the weak topology generated by $\{D_d(B,\cdot) : B \text{ nonempty and closed}\} \cup \{e_d(B,\cdot) : B \text{ nonempty and closed}\}$[13];

- The Hausdorff pseudometric topology is also the weak topology generated by $\{e_d(B,\cdot) :$ B nonempty and closed and ρ uniformly equivalent to $d\}$[13];

- The *slice topology* for a normed linear space is also the weak topology generated by $\{D_d(B,\cdot) : B \text{ nonempty, closed, bounded and convex}\}$ where d is the metric induced by the initial norm [3].

The reader can find many other results in this vein in the survey [4] and in the bibliography therein, where the emphasis is on topologies on nonempty subsets. Here we prefer to work more generally, adopting this standard convention: $d(x,\emptyset) = +\infty$, which allows us to define gap topologies on families of subsets of X that include the empty set.

It is the purpose of this paper to study gap topologies, by considering for a given collection $\mathscr{S}$ of nonempty subsets of (X,d) the weakest topology $\mathsf{G}_{\mathscr{S},d}$ on the power set $\mathscr{P}(X)$ of X for which all functionals of the form

$$C \mapsto D_d(S,C) \quad (S \in \mathscr{S})$$

are continuous. In accordance with a customary behaviour of hypertopologists, we may more primitively decompose each such $\mathsf{G}_{\mathscr{S},d}$ into two halves, namely $\mathsf{G}^+_{\mathscr{S},d}$ (its *upper* part) and $\mathsf{G}^-_{\mathscr{S},d}$ (its *lower* part). The upper (resp. lower) part is the weakest topology such that each gap functional is lower (resp. upper) semicontinuous on $\mathscr{P}(X)$. Between the two, the former appears to play a much more influential rôle, inasmuch as whenever the collection $\mathscr{S}$ fulfils the rather reasonable condition of containing all the singleton subsets of X, $\mathsf{G}^-_{\mathscr{S},d}$ turns out to coincide with the classical *lower Vietoris topology* whose structure is completely transparent. Furthermore, for arbitrary metrics d and ρ on X, it turns out that when $\mathscr{S}$ and $\mathscr{T}$ both contain the singletons, the two-sided gap topology $\mathsf{G}_{\mathscr{S},d}$ will be finer than (or equal to) another gap topology $\mathsf{G}_{\mathscr{T},\rho}$ if and only if $\mathsf{G}^+_{\mathscr{S},d}$ is finer than (or equal to) $\mathsf{G}^+_{\mathscr{T},\rho}$. Either condition ensures that d is stronger than ρ, so that if $\mathscr{S}$ and $\mathscr{T}$ both contain the singletons, equivalence of upper gap topologies ensures equivalence of the metrics.

Separately, coincidence of upper gap topologies alone when the two underlying families are chosen from the following three basic bornologies on X

- the family of nonempty finite subsets $\mathscr{F}_0(X)$ of X;

- the family of nonempty bounded subsets $\mathscr{B}_d(X)$ of X;

- the family of nonempty subsets $\mathscr{P}_0(X)$;

is reflected in the structure of the underlying metric space (X, d), as shown in the final section of the paper.

For these reasons, and in the interest of realizing an article of moderate length, we focus our attention almost exclusively on the structure of upper gap topologies.

In §2 we devise and prove a necessary and sufficient condition for an upper gap topology $\mathsf{G}^+_{\mathscr{S},d}$ to be finer than (or equal to) another upper gap topology $\mathsf{G}^+_{\mathscr{T},\rho}$, where d, ρ are metrics on a set X not a priori assumed equivalent and $\mathscr{S}, \mathscr{T}$ are collections of nonempty subsets of X (see Theorem 2.2). It is to be observed that this result generalizes, in a quite faithful way, an analogous result for the upper Wijsman topology [19, Theorem 5′], which could now be deduced as a corollary. Combining this necessary and sufficent condition with its converse, we obtain a characterization of coincidence of $\mathsf{G}^+_{\mathscr{S},d}$ with $\mathsf{G}^+_{\mathscr{T},\rho}$. By our remarks of the preceding paragraph, this yields as corollaries characterizations for containment/coincidence of two-sided gap topologies when both families contain the singletons.

At the heart of our analysis is the introduction of an operator $\mathscr{S} \mapsto \Gamma_{\rho,d}(\mathscr{S})$ in §3 that yields for a family $\mathscr{S}$ of nonempty subsets of X the largest family of subsets $\mathscr{T}$ of X such that $\mathsf{G}^+_{\mathscr{S},d}$ is finer than (or equal to) $\mathsf{G}^+_{\mathscr{T},\rho}$. Of particular interest is the case $d = \rho$, where $\Gamma_{d,d}(\mathscr{S})$ contains $\mathscr{S}$ and is the largest family whose upper gap topology coincides with $\mathsf{G}^+_{\mathscr{S},d}$. In this case, for simplicity, we will simply write Γ for the operator $\Gamma_{d,d}$. Particular attention is given to the operator as applied to $\mathscr{F}_0(X)$, the family of nonempty finite subsets of (X, d).

In §4 we investigate the properties of first and second countability of upper gap topologies, providing characterizations which rely on intrinsic properties of the base space (Theorems 4.4 and 4.7). The condition characterizing second countability, in particular, can be expressed in terms of our operator $\mathscr{S} \mapsto \Gamma(\mathscr{S})$. Then, applying Theorem 4.4 to the upper Wijsman topology, we obtain a result which seems not to be present in the literature: first countability of W^+_d is equivalent to the separability of the base space (X, d) (notice that an analogous result, where we consider instead the two-sided Wijsman topology W_d, is well-known and certainly easier to prove—see [4, Theorem 2.1.5]). We also point out that second countability of the upper Hausdorff pseudometric topology is equivalent to the total boundedness of the base space.

In the final section, §5, we show that each of the following structural properties of a metric space (X, d)

- boundedness of (X, d),
- total boundedness of (X, d),
- total boundedness of each bounded subset of X,

can ultimately be expressed in terms of coincidences of upper gap topologies, noting that the upper Wijsman, the upper Attouch-Wets topology, and the upper Hausdorff pseudometric topology can be so realized. Separately, we can express each of these structural properties in terms of the behavior of $\mathscr{S} \mapsto \Gamma(\mathscr{S})$ applied to $\mathscr{F}_0(X)$ or to $\mathscr{B}_d(X)$, the family of nonempty bounded subsets of X.

1 Preliminaries and basic results

In a metric space (X, d), we denote the open ball with center x and radius $\varepsilon > 0$ by $B_d(x, \varepsilon)$. If $A \subseteq X$, we write $B_d(A, \varepsilon)$ for the ε-*enlargement* of A, that is $\cup_{a \in A} B_d(a, \varepsilon)$. Paralleling our notation for the nonempty bounded subsets of X, we denote the nonempty totally bounded subsets of X by $\mathscr{T}\mathscr{B}_d(X)$. We will denote by $\mathscr{C}(X)$ and $\mathscr{C}_0(X)$ the collections of all closed subsets and all nonempty closed subsets of X, respectively. When we speak of a *hyperspace* in the sequel we mean a family of subsets of X equipped with some topology, and when we speak of *the* hyperspace we mean $\mathscr{P}(X)$ so equipped.

We will have occasion to consider two basic operators on subfamilies of $\mathscr{P}_0(X)$ that we denote by Σ and $\downarrow$. Let $\mathscr{S}$ be a nonempty family of nonempty subsets of X.

$$\Sigma(\mathscr{S}) := \{T \in \mathscr{P}_0(X) : T \text{ is a finite union of elements of } \mathscr{S}\};$$

$$\downarrow \mathscr{S} := \{T \in \mathscr{P}_0(X) : \exists S \in \mathscr{S} \textit{ with } T \subseteq S\}.$$

Notice $\Sigma(\downarrow \mathscr{S}) = \downarrow \Sigma(\mathscr{S})$ is the smallest ideal of nonempty subsets of X that contains $\mathscr{S}$. An ideal of nonempty subsets that is also a cover is called a *bornology* [9], and $\Sigma(\downarrow \mathscr{S})$ is a bornology if and only if $\mathscr{S}$ is a cover.

For $L, M \subseteq X$, we define the *gap* between L and M by:

$$D_d(L, M) = \inf \{d(x, y) \,|\, x \in L,\ y \in M\}.$$

Notice that $D_d(L, M) = +\infty$ if and only if either L or M is empty. For the record, we note the following interesting formula for gap given in [4, p.30] in the hope that some reader might apply it:

$$D_d(L, M) = \inf_{x \in X} d(x, L) + d(x, M)$$

We will use the following notation which is given with respect to the environment $\mathscr{P}(X)$, but is also to be understood with $\mathscr{P}(X)$ replaced by either $\mathscr{C}(X)$ or $\mathscr{P}_0(X)$, or $\mathscr{C}_0(X)$ with the obvious modifications. For $S_1, \ldots, S_n$ nonempty subset of X and $\varepsilon_1, \ldots, \varepsilon_n > 0$, we set

$$\mathscr{A}_d^+(S_1, \ldots, S_n; \varepsilon_1, \ldots, \varepsilon_n) = \{C \in \mathscr{P}(X) \,|\, D_d(C, S_i) > \varepsilon_i \text{ for } i = 1, \ldots, n\}. \qquad (1;1)$$

Let $\mathscr{S}$ be a collection of nonempty subsets of X; then the collection

$$\left\{\mathscr{A}_d^+(S_1, \ldots, S_n; \varepsilon_1, \ldots, \varepsilon_n) \,\middle|\, n \in \mathbb{N},\ S_1, \ldots, S_n \in \mathscr{S},\ \varepsilon_1, \ldots, \varepsilon_n > 0\right\}$$

is closed under finite intersections, and so adjoining $\mathscr{P}(X)$ to the collection, we get a base for a topology on $\mathscr{P}(X)$ which we will call the *upper gap topology* (relative to the metric d and the collection $\mathscr{S}$), and denote by: $\mathsf{G}^+_{\mathscr{S},d}$. Without loss of generality one can assume in the definition that $\mathscr{S} \subseteq \mathscr{C}_0(X)$ because whenever $S \neq \emptyset$, we have $D_d(S,\cdot) = D_d(\overline{S},\cdot)$.

Observe that, for every $A \in \mathscr{P}(X)$, the collection

$$\Big\{\mathscr{A}^+_d(S_1,\ldots,S_n;\varepsilon_1,\ldots,\varepsilon_n) \,\Big|\, n \in \mathbb{N} \ \wedge \ S_1,\ldots,S_n \in \mathscr{S} \ \wedge \ \varepsilon_1,\ldots,\varepsilon_n > 0 \\ \wedge \ \forall i \in \{1,\ldots,n\}\colon D_d(A,S_i) > \varepsilon_i\Big\}$$

plus $\mathscr{P}(X)$ itself is a local base for $\mathsf{G}^+_{\mathscr{S},d}$ at A. It turns out, in particular, that the unique $\mathsf{G}^+_{\mathscr{S},d}$-neighbourhood of X is $\mathscr{P}(X)$, while each open set in the hypertopology is a neighborhood of $\emptyset$.

Let us recall the following standard definition in the theory of hyperspaces.

Definition 1.1 *A topology γ on a nonempty family of subsets $\mathscr{E}$ of a topological space X is an* upper *(resp.* lower*) topology if for every $\mathscr{A} \in \gamma$, for every $C \in \mathscr{A}$, whenever $D \in \mathscr{E}$ and $D \subseteq C$ (resp $C \subseteq D$), then $D \in \mathscr{A}$.*

It is straightforward to verify that upper gap topologies are actually "upper" in the sense of the previous definition. Note that $\mathsf{G}^+_{\mathscr{S},d}$-convergence of a net of sets to a subset B implies convergence of the net to any superset of B and that every net converges to X.

We will freely use in the text the following simple fact: for every $S, A \subseteq X$ and every $\varepsilon > 0$

$$B_d(A,\varepsilon) \cap S = \emptyset \Leftrightarrow D_d(S,A) \geq \varepsilon \tag{1;2}$$

The next result characterizes $\mathsf{G}^+_{\mathscr{S},d}$-convergence of nets of elements of $\mathscr{P}(X)$.

Theorem 1.1 *Let (X,d) be a metric space and $\mathscr{S}$ a collection of nonempty subsets of X. For every net $(A_\sigma)_{\sigma\in\Sigma}$ in $\mathscr{P}(X)$ and every $A \in \mathscr{P}(X)$, the following are equivalent:*

(1) $(A_\sigma)_{\sigma\in\Sigma} \overset{\mathsf{G}^+_{\mathscr{S},d}}{\longrightarrow} A$;

(2) $\forall S \in \mathscr{S}\colon D_d(S,A) \leq \liminf_{\sigma\in\Sigma} D_d(S,A_\sigma)$;

(3) $\forall S \in \mathscr{S}$, $\forall 0 < \mu < \alpha$, $A \cap B_d(S,\alpha) = \emptyset$ implies $A_\sigma \cap B_d(S,\mu) = \emptyset$ eventually.

Proof. We consider two cases: (i) $A \neq \emptyset$, and (ii) $A = \emptyset$.

Case (i):

(1) $\Rightarrow$ (2). Suppose that $(A_\sigma)_{\sigma\in\Sigma} \overset{\mathsf{G}^+_{\mathscr{S},d}}{\longrightarrow} A$, and let $S \in \mathscr{S}$ be given: to prove that $D_d(S,A) \leq \liminf_{\sigma\in\Sigma} D_d(S,A_\sigma)$, we have to show that for every $\varepsilon > 0$ there is $\sigma \in \Sigma$ such that $\inf_{\sigma'\geq\sigma} D_d(S,A_{\sigma'}) \geq D_d(S,A) - \varepsilon$, i.e. such that $D_d(S,A_{\sigma'}) \geq D_d(S,A) - \varepsilon$ for every $\sigma' \geq \sigma$. Actually, for every $\varepsilon > 0$, if $D_d(S,A) - \varepsilon \leq 0$ then the above inequality is obvious for every $\sigma' \in \Sigma$; if, on the contrary, $D_d(S,A) - \varepsilon = \delta > 0$, then the

collection $\mathscr{A}_d^+(S,\delta) = \{M \in \mathscr{P}(X) \,|\, D_d(M,S) > \delta\}$ is a $\mathsf{G}^+_{\mathscr{S},d}$-neighbourhood of A, hence it follows from $(A_\sigma)_{\sigma\in\Sigma} \stackrel{\mathsf{G}^+_{\mathscr{S},d}}{\longrightarrow} A$ that there is $\sigma \in \Sigma$ such that $A_{\sigma'} \in \mathscr{A}(S,\delta)$ for $\sigma' \geq \sigma$, i.e. $D_d(A_{\sigma'}, S) \geq D_d(S,A) - \varepsilon$ for every $\sigma' \geq \sigma$.

(2) $\Rightarrow$ (3). Assume (2) and suppose $\exists S \in \mathscr{S}$, $\exists 0 < \mu < \alpha$ with $A \cap B_d(S,\alpha) = \emptyset$ and $A_\sigma \cap B_d(S,\mu) \neq \emptyset$ frequently. Then $D_d(S,A) \geq \alpha$ and $D_d(S,A_\sigma) \leq \mu$ frequently. Thus $\liminf_{\sigma\in\Sigma} D_d(S,A_\sigma) \leq \mu < \alpha \leq D_d(S,A)$, a contradiction.

(3) $\Rightarrow$ (1). Let $\mathscr{A}_d^+(S_1,\ldots,S_n;\varepsilon_1,\ldots,\varepsilon_n)$ be an arbitrary basic $\mathsf{G}^+_{\mathscr{S},d}$-neighbourhood of A; then $D_d(S_i,A) > \varepsilon_i$ for $i = 1,...,n$. Select, for all $i = 1,...,n$, λ_i and μ_i such that $\varepsilon_i < \lambda_i < \mu_i < D_d(S_i,A)$. Thus $A \cap B_d(S_i,\mu_i) = \emptyset$ and $A_\sigma \cap B_d(S_i,\lambda_i) = \emptyset$ eventually; it follows that $D_d(A_\sigma,S_i) \geq \lambda_i > \varepsilon_i$ eventually, for all $i = 1,...,n$, and A_σ belongs to $\mathscr{A}_d^+(S_1,\ldots,S_n;\varepsilon_1,\ldots,\varepsilon_n)$ eventually.

Case (ii): all three conditions are equivalent to $\lim_\sigma D_d(S,A_\sigma) = +\infty \;\forall S \in \mathscr{S}$. $\square$

Remark 1.1 *(i)* $\mathsf{G}^+_{\mathscr{S},d} = \mathsf{G}^+_{\Sigma(\mathscr{S}),d}$*;*

(ii) $\emptyset$ *is isolated for* $\mathsf{G}^+_{\mathscr{S},d}$ *if and only if* $\exists S,\ \exists r > 0 : B_d(S,r) = X$*;*

(iii) In general $\mathsf{G}^+_{\downarrow\mathscr{S},d}$ *may be strictly stronger than* $\mathsf{G}^+_{\mathscr{S},d}$*;*

(iv) $\mathsf{G}^+_{\mathscr{S},d}$ *is the weakest topology on* $\mathscr{P}(X)$ *such that* $\forall S \in \mathscr{S},\ D_d(S,\cdot)$ *is lower semicontinuous.*

Proof.

(i) follows from the fact that $\mathscr{A}^+(S_1 \cup \cdots \cup S_n;\varepsilon) = \mathscr{A}^+(S_1;\varepsilon) \cap \cdots \cap \mathscr{A}^+(S_n;\varepsilon)$ for every $S_1,\ldots,S_n \in \mathscr{S}$ and every $\varepsilon > 0$;

(ii) suppose $B_d(S,r) = X$ and $(A_\sigma)_{\sigma\in\Sigma} \stackrel{\mathsf{G}^+_{\mathscr{S},d}}{\longrightarrow} \emptyset$; then $A_\sigma \cap B_d(S,r) = \emptyset$ eventually, that is $A_\sigma = \emptyset$ eventually. For the converse, we can suppose, by (i), that $\mathscr{S} = \Sigma(\mathscr{S})$; assume $\forall S,\ \forall r\ B_d(S,r) \neq X$ and pick $x_{S,r} \in X \setminus B_d(S,r)$. Order $\mathscr{S} \times \mathbb{R}$ as follows: $(S,r) \leq (S',r')$ iff $S \subseteq S'$ and $r \leq r'$, so that $(x_{S,r})$ is a net in X based on $\mathscr{S} \times \mathbb{R}$. It is clear that the net of singletons $\{x_{S,r}\}$ $\mathsf{G}^+_{\mathscr{S},d}$-converges to the empty set.

(iii) if $\mathscr{S} = \{X\}$, $\mathsf{G}^+_{\mathscr{S},d}$ is the indiscrete topology on $\mathscr{P}_0(X)$, while $\mathsf{G}^+_{\downarrow\mathscr{S},d}$ is the upper Hausdorff pseudometric topology, as we shall see in Proposition 1.3 below.

(iv) This is an immediate consequence of our last result (apply for instance Theorem 1.2.8 of [4] where each gap functional is viewed as a function with values in $[0,+\infty]$ equipped with the topology consists of all sets of form $\{(\alpha,+\infty] : \alpha \geq 0\}$ in addition to $\emptyset$ and $[0,+\infty]$).

$\square$

We will now show that some well-known upper topologies fit into our framework.

Definition 1.2 *The upper Wijsman topology* W_d^+ *on the hyperspace* $\mathscr{P}(X)$ *of a metric space* (X, d) *is generated by the base*

$$\big\{\mathscr{A}^+(x_1, \ldots, x_n; \varepsilon_1, \ldots, \varepsilon_n) \,\big|\, n \in \mathbb{N},\, x_1, \ldots, x_n \in X,\, \varepsilon_1, \ldots, \varepsilon_n > 0\big\},$$

where $\mathscr{A}^+(x_1, \ldots, x_n; \varepsilon_1, \ldots, \varepsilon_n) = \big\{C \in \mathscr{P}(X) \,\big|\, \forall i \in \{1, \ldots, n\} \colon d(x_i, C) > \varepsilon_i\big\}$ *for every* $x_1, \ldots, x_n \in X$ *and every* $\varepsilon_1, \ldots, \varepsilon_n > 0$ *(of course, this notation is consistent with* $(1; 1)$*, once we identify every element* x *of* X *with* $\{x\}$ *and define* $d(x, \emptyset) = +\infty$*).*

For every net $(A_\sigma)_{\sigma\in\Sigma}$ in $\mathscr{P}(X)$ and every $A \in \mathscr{P}(X)$, we have the well known equivalence:

$$(A_\sigma)_{\sigma\in\Sigma} \xrightarrow{\mathsf{W}_d^+} A \iff \forall x \in X \colon d(x, A) \leq \liminf_{\sigma\in\Sigma} d(x, A_\sigma).$$

If $\mathscr{S}(X)$ is the collection of all singleton subsets of X, we immediately obtain the following:

Proposition 1.2 *Let* (X, d) *be a metric space. Then* $\mathsf{G}^+_{\mathscr{S}(X),d} = \mathsf{W}_d^+$ *on* $\mathscr{P}(X)$*.*

Let (X, d) be a metric space. We now recall the standard presentation of Hausdorff distance H_d as defined on $\mathscr{P}(X)$ (see, e.g., [4]). Let S, T be subsets of X; then

$$H_d(S, T) \;:=\; \inf\{\alpha > 0 : T \subseteq B_d(S, \alpha) \wedge S \subseteq B_d(T, \alpha)\}$$

Hausdorff distance so defined is an infinite-valued pseudometric on $\mathscr{P}(X)$ which, when restricted to the d-closed and d-bounded nonempty subsets of X, becomes a bona fide metric. A local base for the Hausdorff pseudometric topology H_d at $T \in \mathscr{P}(X)$ is given by all sets of the form

$$\{S : T \subseteq B_d(S, \alpha) \wedge S \subseteq B_d(T, \alpha)\} \qquad (\alpha > 0)$$

Analogously, the (pseudometrizable) Attouch-Wets topology AW_d on $\mathscr{P}(X)$ [1, 4, 27] has as a local base at $T \in \mathscr{P}(X)$ all sets of the form

$$\{S : T \cap D \subseteq B_d(S, \alpha) \wedge S \cap D \subseteq B_d(T, \alpha)\} \qquad (\alpha > 0, D \in \mathscr{B}_d(X)).$$

As our next examples, we will consider the upper Hausdorff pseudometric topology H_d^+ and the upper Attouch-Wets topology AW_d^+ on $\mathscr{P}(X)$ which we formally present in terms of convergence (the reader is invited to describe these in terms of a local base at each point of the hyperspace).

Definition 1.3

(a) A net $(A_\sigma)_{\sigma\in\Sigma}$ *in* $\mathscr{P}(X)$ *converges to* $A \in \mathscr{P}(X)$ *for* H_d^+*, provided* $\forall \varepsilon > 0$ $A_\sigma \subseteq B_d(A, \varepsilon)$ *eventually.*

(b) A net $(A_\sigma)_{\sigma\in\Sigma}$ in $\mathscr{P}(X)$ converges to $A \in \mathscr{P}(X)$ for AW_d^+, provided $\forall\varepsilon > 0$ and $\forall D \in \mathscr{B}_d(X)$, $D \cap A_\sigma \subseteq B_d(A,\varepsilon)$ eventually.

Note that $\emptyset$ is isolated for H_d^+ and isolated for AW_d^+ if and only if the metric d is bounded.

Proposition 1.3 *Let (X,d) be a metric space. Then $\mathsf{G}^+_{\mathscr{P}_0(X),d} = \mathsf{H}_d^+$ and $\mathsf{G}^+_{\mathscr{B}_d(X),d} = \mathsf{AW}_d^+$ on $\mathscr{P}(X)$.*

Proof.

(a) $\mathsf{G}^+_{\mathscr{P}_0(X),d} \leq \mathsf{H}_d^+$ on $\mathscr{P}_0(X)$ by Theorem 3 of [11], while $\mathsf{G}^+_{\mathscr{P}_0(X),d} \geq \mathsf{H}_d^+$ on $\mathscr{P}_0(X)$ by Proposition 2, (2), (ii) of [11]; moreover $\emptyset$ is isolated for both topologies;

(b) the proof for AW_d^+ is similar on $\mathscr{P}_0(X)$; moreover, if d is bounded, $\emptyset$ is isolated for both topologies and, if d is unbounded, it is easy to check that a net $(A_\sigma)_{\sigma\in\Sigma}$ converges to $\emptyset$ for $\mathsf{G}^+_{\mathscr{B}_d(X),d}$ if and only if it converges to $\emptyset$ for AW_d^+.

□

The two-sided gap topology $\mathsf{G}_{\mathscr{B}_d(X),d}$ has been called the *bounded proximal topology* in the literature [4, p. 111].

2 Comparison between two upper gap topologies

Definition 2.1 *Let (X,d) be a metric space and $\mathscr{S}$ a collection of nonempty subsets of X. We will say that a subset L of X is strictly $(\mathscr{S},d)$-included in another subset M of X if there exists a finite subset $\{S_1,\dots,S_n\}$ of $\mathscr{S}$, and for every $i \in \{1,\dots,n\}$ there are λ_i, σ_i with $0 < \lambda_i < \sigma_i$, such that:*

$$L \subseteq \bigcup_{i=1}^{n} B_d(S_i,\lambda_i) \subseteq \bigcup_{i=1}^{n} B_d(S_i,\sigma_i) \subseteq M. \tag{2;3}$$

This definition originates in the paper [19], where it was given for the family of singletons.

Lemma 2.1 *Let d,ρ be two metrics on a set X, $S, S_1,\dots,S_n$ subsets of X and $\vartheta, \mu_1,\dots,\mu_n > 0$. Then the following holds:*

(1) if $B_\rho(S,\vartheta) \subseteq \bigcup_{i=1}^n B_d(S_i,\mu_i)$, then $\mathscr{A}_d^+(S_1,\dots,S_n;\mu_1,\dots,\mu_n) \subseteq \mathscr{A}_\rho^+(S,\varepsilon)$ for every $0 < \varepsilon < \vartheta$;

(2) if $\mathscr{A}_d^+(S_1,\dots,S_n;\mu_1,\dots,\mu_n) \subseteq \mathscr{A}_\rho^+(S,\vartheta)$, then $B_\rho(S,\vartheta) \subseteq \bigcup_{i=1}^n B_d(S_i,\lambda_i)$ whenever $\mu_i < \lambda_i$ for every $i \in \{1,\dots,n\}$.

Proof. To prove (1), suppose $C \notin \mathscr{A}_\rho^+(S,\varepsilon)$: then $D_\rho(S,C) \leq \varepsilon < \vartheta$, hence there exists $\bar{x} \in C$ such that $\rho(\bar{x},S) < \vartheta$, i.e. $\bar{x} \in B_\rho(S,\vartheta) \subseteq \bigcup_{i=1}^n B_d(S_i,\mu_i)$. Therefore, $\bar{x} \in B_d(S_i,\mu_i)$ for some $i \in \{1,\ldots,n\}$, whence $D_d(S_i,C) < \mu_i$. Thus, $C \notin \mathscr{A}_d^+(S_1,\ldots,S_n;\mu_1,\ldots,\mu_n)$.

To prove (2), we argue again by considering complementary sets; therefore, suppose $x \notin \bigcup_{i=1}^n B_d(S_i,\lambda_i)$, so that $d(x,S_i) \geq \lambda_i > \mu_i$ for every $i \in \{1,\ldots,n\}$. Then $\{x\} \in \mathscr{A}_d^+(S_1,\ldots,S_n;\mu_1,\ldots,\mu_n) \subseteq \mathscr{A}_\rho^+(S,\vartheta)$, whence $\rho(x,S) > \vartheta$ and hence $x \notin B_\rho(S,\vartheta)$. □

Theorem 2.2 *Let d,ρ be two metrics on a set X, and $\mathscr{S},\mathscr{T}$ two collections of nonempty subsets of X. Then the following are equivalent:*

(a) $\mathsf{G}^+_{\mathscr{S},d} \geq \mathsf{G}^+_{\mathscr{T},\rho}$ *on* $\mathscr{P}(X)$*;*

(b) for every $T \in \mathscr{T}$ and every ε,α with $0 < \varepsilon < \alpha$, the set $B_\rho(T,\varepsilon)$ is strictly $(\mathscr{S},d)$-included in $B_\rho(T,\alpha)$.

Proof. (a)$\Longrightarrow$(b).
Let $T \in \mathscr{T}$ and $0 < \varepsilon < \alpha$ be given and put $C = X \backslash B_\rho(T,\alpha)$: fixing ϑ with $\varepsilon < \vartheta < \alpha$, we see that $C \in \mathscr{A}_\rho^+(T,\vartheta)$. Then there exist $S_1,\ldots,S_n \in \mathscr{S}$ and $\mu_1,\ldots,\mu_n > 0$ such that

$$C \in \mathscr{A}_d^+(S_1,\ldots,S_n;\mu_1,\ldots,\mu_n) \subseteq \mathscr{A}_\rho^+(T,\vartheta). \tag{2;4}$$

For every $i \in \{1,\ldots,n\}$, fix λ_i,σ_i with $\mu_i < \lambda_i < \sigma_i < D_d(S_i,C)$. Then, on the one hand, it follows that

$$C \in \mathscr{A}_d^+(S_1,\ldots,S_n;\sigma_1,\ldots,\sigma_n), \tag{2;5}$$

and this implies the inclusion $\bigcup_{i=1}^n B_d(S_i,\sigma_i) \subseteq B_\rho(T,\alpha)$ (otherwise, there would be $x \in B_d(S_i,\sigma_i) \backslash B_\rho(T,\alpha)$ for some $i \in \{1,\ldots,n\}$, so that $D_d(C,S_i) \leq d(x,S_i) < \sigma_i$, contradicting (2;5)). On the other hand, from the inclusion in (2;4) and the fact that $\mu_i < \lambda_i$ for $i = 1,\ldots,n$, it follows by Lemma 2.1 that $B_\rho(T,\vartheta) \subseteq \bigcup_{i=1}^n B_d(S_i,\lambda_i)$, hence also (as $\varepsilon < \vartheta$) $B_\rho(T,\varepsilon) \subseteq \bigcup_{i=1}^n B_d(S_i,\lambda_i)$. Therefore, we conclude that

$$B_\rho(T,\varepsilon) \subseteq \bigcup_{i=1}^n B_d(S_i,\lambda_i) \subseteq \bigcup_{i=1}^n B_d(S_i,\sigma_i) \subseteq B_\rho(T,\alpha).$$

(b)$\Longrightarrow$(a).
We must show that for every $C \in \mathscr{P}(X)$, every $T \in \mathscr{T}$ and every $0 < \varepsilon < D_\rho(T,C)$, there exists a $\mathsf{G}^+_{\mathscr{S},d}$-neighbourhood of C included in $\mathscr{A}_\rho^+(T,\varepsilon)$.

Put $r = D_\rho(T,C)$, and fix ϑ with $\varepsilon < \vartheta < r$: then it follows from (b) that there exist $S_1,\ldots,S_n \in \mathscr{S}$ and $\lambda_1,\ldots,\lambda_n,\sigma_1,\ldots,\sigma_n$ with $0 < \lambda_i < \sigma_i$ for $1 \leq i \leq n$, such that

$$B_\rho(T,\vartheta) \subseteq \bigcup_{i=1}^n B_d(S_i,\lambda_i) \subseteq \bigcup_{i=1}^n B_d(S_i,\sigma_i) \subseteq B_\rho(T,r). \tag{2;6}$$

We claim that $C \in \mathscr{A}_d^+(S_1,\ldots,S_n;\lambda_1,\ldots,\lambda_n) \subseteq \mathscr{A}_\rho^+(T,\varepsilon)$. Indeed, the inclusion follows from Lemma 2.1 (use the first inclusion in $(2;6)$ and $0<\varepsilon<\vartheta$). On the other hand, $D_\rho(T,C)=r$ implies that $C\cap B_\rho(T,r)=\emptyset$, whence (by the last inclusion in $(2;6)$) $C\cap B_d(S_i,\sigma_i)=\emptyset$, $\forall i=1,...,n$ and hence $D_d(S_i,C)\geq\sigma_i>\lambda_i$, so that $C\in\mathscr{A}_d^+(S_i,\lambda_i)$. □

Denote by $\mathscr{C}_\rho(X)$ the ρ-closed subsets of X; then a further equivalent condition is the following: (a') $\mathsf{G}^+_{\mathscr{S},d}\geq\mathsf{G}^+_{\mathscr{T},\rho}$ on $\mathscr{C}_\rho(X)$. As it is clear that $(a)\Rightarrow(a')$, we only have to show that $(a')\Rightarrow(b)$. Note that the above set C belongs to $\mathscr{C}_\rho(X)$ and that the inclusion in $(2;4)$ still implies (b), even when restricted to the $\mathscr{C}_\rho(X)$-context.

Let us take a look at the case in which $\exists T\in\mathscr{T},\exists r>0: B_\rho(T,r)=X$; this means (Remark 1.1) that $\emptyset$ is isolated for $\mathsf{G}^+_{\mathscr{T},\rho}$; the full strength of condition (b) above then implies that $\emptyset$ is isolated for $\mathsf{G}^+_{\mathscr{S},\rho}$ too, since

$$B_\rho(T,r)=X=\bigcup_{i=1}^n B_d(S_i,\lambda_i)=B_d(S_1\cup....\cup S_n,\max(\lambda_1,...,\lambda_n)),$$

as we can assume that $\mathscr{S}=\Sigma(\mathscr{S})$.

However, consider the following example [19, p. 145]: let d be an unbounded metric on X, put $\rho=\min(1,d)$ and consider W^+_d and W^+_ρ; then $\mathsf{W}^+_d\geq\mathsf{W}^+_\rho$ on $\mathscr{P}_0(X)$, but $\emptyset$ is not isolated for W^+_d, while it is isolated for W^+_ρ. We are thus lead to the next result, whose proof is left to the reader:

Theorem 2.3 *Let d,ρ be two metrics on a set X, and $\mathscr{S},\mathscr{T}$ two collections of nonempty subsets of X. Then the following are equivalent:*

(i) $\mathsf{G}^+_{\mathscr{S},d}\geq\mathsf{G}^+_{\mathscr{T},\rho}$ *on* $\mathscr{P}_0(X)$*;*

(ii) for every $T\in\mathscr{T}$ and every ε,α with $0<\varepsilon<\alpha$, such that $B_\rho(T,\alpha)\neq X$, the set $B_\rho(T,\varepsilon)$ is strictly $(\mathscr{S},d)$-included in $B_\rho(T,\alpha)$.

Taking Proposition 1.2 into account, the two previous results can be considered as generalizations of [19, Theorems 5] and [19, Theorems 5′], respectively. In [19], however, the metrics d and ρ were assumed to be equivalent; here, as a complement, we can state that, if $\mathsf{W}^+_d\geq\mathsf{W}^+_\rho$, then the metric d is stronger than the metric ρ (Corollary 2.6 below) and $\mathscr{B}_\rho(X)\subseteq\mathscr{B}_d(X)$, as each ρ-ball is d-bounded.

The next two corollaries are stated for the base space X, but the proofs go through the hyperspace $\mathscr{P}(X)$ (it is unlikely that the two corollaries would have been otherwise discovered):

Corollary 2.4 *The following are equivalent:*

(1) $\forall T\in\mathscr{P}_0(X),\forall 0<\varepsilon<\alpha,\exists S_1,...,S_n\in\mathscr{P}_0(X),\exists 0<\lambda_i<\sigma_i(i=1,...,n)$ *such that*

$$B_\rho(T,\varepsilon)\subseteq\bigcup_{i=1}^n B_d(S_i,\lambda_i)\subseteq\bigcup_{i=1}^n B_d(S_i,\sigma_i)\subseteq B_\rho(T,\alpha).$$

(2) *the uniformity generated by d is stronger then the uniformity generated by ρ*
(3) $\forall T \in \mathscr{P}_0(X), \forall 0 < \varepsilon < \alpha, \exists S \in \mathscr{P}_0(X), \exists 0 < \lambda < \sigma$ *such that*

$$B_\rho\,(T, \varepsilon) \subseteq B_d\,(S, \lambda) \subseteq B_d\,(S, \sigma) \subseteq B_\rho\,(T, \alpha).$$

Proof. A straightforward modification of Theorem 3.3.2 of [4] shows that condition (2) is equivalent to $\mathsf{H}_d^+ \geq \mathsf{H}_\rho^+$ on $\mathscr{P}_0(X)$ or $\mathscr{P}(X)$; therefore (1) and (2) are equivalent by Theorem 2.2.

Clearly (3) implies (1); now assume (2) and take $\delta > 0$ such that $d(x, w) < \delta$ implies $\rho(x, w) < \alpha - \varepsilon$; put $S = B_\rho\,(T, \varepsilon)$, $\lambda = \frac{1}{2}\delta$, $\sigma = \delta$ and apply the triangle inequality. □

Corollary 2.5 *The following are equivalent:*
(1) $\forall T \in B_\rho(X), \forall 0 < \varepsilon < \alpha$, $\exists S_1, ..., S_n \in \mathscr{B}_d(X), \exists 0 < \lambda_i < \sigma_i (i = 1, ..., n)$ *such that*

$$B_\rho\,(T, \varepsilon) \subseteq \bigcup_{i=1}^{n} B_d\,(S_i, \lambda_i) \subseteq \bigcup_{i=1}^{n} B_d\,(S_i, \sigma_i) \subseteq B_\rho\,(T, \alpha).$$

(2) *Every ρ-bounded subset is d-bounded and the identity mapping $i : (X, d) \to (X, \rho)$ is strongly uniformly continuous on each ρ-bounded subset of X, i.e., whenever $B \in \mathscr{B}_\rho(X)$ and $\varepsilon > 0, \exists \delta > 0$ such that whenever $x \in B$ and $w \in X$ and $d(x, w) < \delta$, then $\rho(x, w) < \varepsilon$.*

Proof. A slight modification of Theorem 3.1 of [8] shows that condition (2) above is equivalent to $\mathsf{AW}_d^+ \geq \mathsf{AW}_\rho^+$ on $\mathscr{P}(X)$; thus, (1) and (2) are equivalent by Theorem 2.2. □

We will now look at the role played by the family of singletons.

Corollary 2.6 *Suppose that $\mathsf{G}^+_{\mathscr{S},d} \geq \mathsf{G}^+_{\mathscr{T},\rho}$ on $\mathscr{P}_0(X)$ and the family $\mathscr{T}$ contains the singletons. Then the metric d is stronger than the metric ρ.*

Proof. Fix $x \in X$ and choose α such that $B_\rho(x, \alpha) \neq X$. By Theorem 2.3 there exist $S_1, ..., S_n \in \mathscr{S}$, $\exists 0 < \lambda_i < \sigma_i (i = 1, ..., n)$ such that

$$B_\rho\left(\{x\},\ \frac{1}{2}\alpha\right) \subseteq \bigcup_{i=1}^{n} B_d\,(S_i, \lambda_i) \subseteq \bigcup_{i=1}^{n} B_d\,(S_i, \sigma_i) \subseteq B_\rho\,(\{x\}, \alpha).$$

In particular, for some $i \leq n$ we have $x \in B_d(S_i, \lambda_i)$ and so for some $w \in S_i$ it follows that $x \in B_d(w, \lambda_i) \subseteq B_\rho(x, \alpha)$ and this shows that d defines a finer topology than ρ. □

By Theorem 1.1, convergence of the net (A_σ) to A for $\mathsf{G}^+_{\mathscr{S},d}$ can be characterized by the condition $\forall S \in \mathscr{S} \colon D_d(S, A) \leq \liminf_{\sigma\in\Sigma} D_d(S, A_\sigma)$. We will say that the net (A_σ) tends to A for the lower gap topology $\mathsf{G}^-_{\mathscr{S},d}$ if $\forall S \in \mathscr{S} \colon D_d(S, A) \geq \limsup_{\sigma\in\Sigma} D_d(S, A_\sigma)$ and that (A_σ) tends to A for the "full" topology $\mathsf{G}_{\mathscr{S},d}$ if $\forall S \in \mathscr{S} \colon D_d(S, A) = \lim_{\sigma\in\Sigma} D_d(S, A_\sigma)$. Of course, the lower gap topology is a lower topology, according to Definition 1.1, and the "full" topology is the weak topology determined by the family of functionals $D_d(S,.)$, where $S \in \mathscr{S}$.

Let us denote by V^- the lower Vietoris topology. Lemma 1 of [11] states that, if the family $\mathscr{S}$ covers X, then $\mathsf{V}^- \geq \mathsf{G}^-_{\mathscr{S},d}$; on the other hand, if $\mathscr{S}$ contains the singletons, it is easy to check that the opposite inequality holds. Therefore, in that case, the lower gap topology is none other than the familiar lower Vietoris topology (which does not depend on the choice of the metric d) and the "full" gap topology is an admissible hyperspace topology (that is, the map $x \to \{x\}$ is an embedding of X into $\mathscr{P}_0(X)$). The next result generalizes Proposition 1 of [19] and has a similar proof:

Proposition 2.7 *Let d and ρ be two metrics on X and suppose the families $\mathscr{S}$ and $\mathscr{T}$ both contain the singletons. Then the inequalities $\mathsf{G}_{\mathscr{S},d} \geq \mathsf{G}_{\mathscr{T},\rho}$ and $\mathsf{G}^+_{\mathscr{S},d} \geq \mathsf{G}^+_{\mathscr{T},\rho}$ are equivalent.*

As a corollary, we obtain that the following condition (a'') is equivalent (on $\mathscr{P}_0(X)$ or $\mathscr{C}_\rho(X)$) to conditions (a) or (a'): (a'') $\mathsf{G}_{\mathscr{S},d} \geq \mathsf{G}_{\mathscr{T},\rho}$.

3 A natural operator on families of subsets of X

We now define an operator which is at the core of our investigation of upper gap topologies.

Definition 3.1 *Let d and ρ be two metrics on X and let $\mathscr{S}$ be a family of nonempty subsets of X. We define the subfamily $\Gamma_{\rho,d}(\mathscr{S})$ of $\mathscr{P}_0(X)$ by*

$$\begin{aligned}\Gamma_{\rho,d}(\mathscr{S}) := \{T \subseteq X : &\forall 0 < \varepsilon < \alpha, \exists n \in \mathbb{N}, \exists S_1, \ldots, S_n \in \mathscr{S}, \\ &\exists \lambda_1, \ldots, \lambda_n, \sigma_1, \ldots, \sigma_n \textit{ such that } \forall i \in \{1, \ldots, n\}, 0 < \lambda_i < \sigma_i \\ &\wedge\ B_\rho(T, \varepsilon) \subseteq \textstyle\bigcup_{i=1}^n B_d(S_i, \lambda_i) \subseteq \bigcup_{i=1}^n B_d(S_i, \sigma_i) \subseteq B_\rho(T, \alpha)\}.\end{aligned}$$

Of course, $\Gamma_{\rho,d}(\mathscr{S})$ is the family of all nonempty subsets T of X such that $B_\rho(T,\varepsilon)$ is strictly $(\mathscr{S},d)$-included in $B_\rho(T,\alpha)$ for every $0 < \varepsilon < \alpha$, so that whenever $\Gamma_{\rho,d}(\mathscr{S})$ is nonempty, it forms by Theorem 2.2 the largest family of nonempty subsets of X whose induced upper gap topology with respect to ρ is contained in $\mathsf{G}^+_{\mathscr{S},d}$. That this operator may produce no subsets whatsoever is illustrated by the next example.

Example. Let $X = \mathbb{N}$ equipped with the Euclidean metric d and let $f : \mathbb{N} \to \mathbb{Q}$ be a bijection. Let ρ be a second metric on $\mathbb{N}$ defined by $\rho(n,k) = |f(n) - f(k)|$. Suppose that $\mathscr{S}$ consists of the singleton subsets of X, so that $\mathsf{G}^+_{\mathscr{S},d}$ is the upper d-Wijsman topology. Notice that each ρ-ball consists of infinitely many points and thus fails to be d-bounded. As a result, whenever $T \neq \emptyset$ and $\varepsilon > 0$, the enlargement $B_\rho(T,\varepsilon)$ is unbounded with respect to the metric d and thus cannot be contained in any d-bounded set of the form $\bigcup_{i=1}^n B_d(S_i, \lambda_i)$ where $S_1, S_2, \ldots, S_n$ are singletons and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are positive scalars. We conclude $\Gamma_{\rho,d}(\mathscr{S})$ is empty.

As a courtesy to the reader, who could easily be intimated by the definition of $\Gamma_{\rho,d}(\mathscr{S})$, we next identify in a very simple setting a nonempty $\Gamma_{\rho,d}(\mathscr{S})$ and its induced upper gap topology which must of course satisfy $\mathsf{G}^+_{\Gamma_{\rho,d}(\mathscr{S}),\rho} \leq \mathsf{G}^+_{\mathscr{S},d}$.

Example. Let $X = \mathbb{R}$, let d be the discrete 0-1 metric, and let ρ be the Euclidean metric. We take for $\mathscr{S}$ the family of nonempty finite subsets of X which we have denoted by $\mathscr{F}_0(X)$. This of course gives rise to the same upper gap topology as determined by the family of singletons. Viewing this as the weakest topology such that each member of $\{D(\cdot, S) : S \in \mathscr{F}_0(X)\}$ is upper semicontinuous, we see that the upper gap topology is generated by all sets of the form

$$\{A \in \mathscr{P}(X) : D_d(A, S) > \varepsilon\} = \begin{cases} \mathscr{P}(X) & \text{if } \varepsilon < 0 \\ \{B : B \subseteq X \backslash S\} & \text{if } 0 \leq \varepsilon < 1 \\ \{\emptyset\} & \text{if } \varepsilon \geq 1 \end{cases}$$

where S runs over $\mathscr{F}_0(X)$.

We intend to show that $\Gamma_{\rho,d}(\mathscr{S})$ coincides with the ρ-dense subsets of $\mathbb{R}$. First, suppose that T is ρ-dense; then whenever $0 < \varepsilon < \alpha$, we have

$$B_\rho(T, \varepsilon) = \mathbb{R} \subseteq B_d(\{0\}, 2) \subseteq B_d(\{0\}, 3) \subseteq B_\rho(T, \alpha)$$

as required. On the other, if $T \neq \emptyset$ is not ρ-dense, choose $\alpha > 0$ with $B_\rho(T, \alpha) \neq \mathbb{R}$. Noting that $B_\rho(T, \alpha/2)$ is not a finite set, we cannot find finite sets $S_1, S_2, \ldots, S_n$ and positive scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$B_\rho(T, \frac{\alpha}{2}) \subseteq \bigcup_{i=1}^{n} B_d(S_i, \lambda_i) \subseteq B_\rho(T, \alpha)$$

as the set in the middle is either finite or is $\mathbb{R}$. We have shown that $\Gamma_{\rho,d}(\mathscr{S}) = \{T : T \text{ is } \rho - \text{dense}\}$; clearly, $\mathsf{G}^+_{\Gamma_{\rho,d}(\mathscr{S}),\rho} = \{\emptyset, \mathscr{P}(X), \{\emptyset\}\}$.

It is evident that the ρ-closure of each set in $\Gamma_{\rho,d}(\mathscr{S})$ again lies in $\Gamma_{\rho,d}(\mathscr{S})$ and that $\Gamma_{\rho,d}(\mathscr{S})$ is stable under taking finite unions. A more interesting stability property is this: $\Gamma_{\rho,d}(\mathscr{S})$ is closed with respect to the topology on $\mathscr{P}_0(X)$ determined by ρ-Hausdorff distance.

Proposition 3.1 *Let d and ρ be metrics on X and let $\mathscr{S}$ be a nonempty family of subsets of X. Then the closure of $\Gamma_{\rho,d}(\mathscr{S})$ in the H_ρ-topology equals $\Gamma_{\rho,d}(\mathscr{S})$.*

Proof. Suppose $A \in \mathscr{P}_0(X)$ is in the H_ρ-closure of $\Gamma_{\rho,d}(\mathscr{S})$. Let $0 < \varepsilon < \alpha$ be given and pick $\delta > 0$ such that $3\delta < \alpha - \varepsilon$. Choose $T \in \Gamma_{\rho,d}(\mathscr{S})$ such that $H_d(T, A) < \delta$. By definition, we can find $S_1, S_2, \ldots, S_n \in \mathscr{S}$, and scalars $0 < \lambda_1 < \sigma_1, 0 < \lambda_2 < \sigma_2, \ldots, 0 < \lambda_n < \sigma_n$ satisfying

$$B_\rho(T,\varepsilon+\delta) \subseteq \bigcup_{i=1}^{n} B_d\left(S_i,\lambda_i\right) \subseteq \bigcup_{i=1}^{n} B_d\left(S_i,\sigma_i\right) \subseteq B_\rho(T,\alpha-\delta).$$

Since $B_\rho(A,\varepsilon) \subseteq B_\rho(T,\varepsilon+\delta)$ and $B_\rho(T,\alpha-\delta) \subseteq B_\rho(A,\alpha)$, we obtain

$$B_\rho(A,\varepsilon) \subseteq \bigcup_{i=1}^{n} B_d\left(S_i,\lambda_i\right) \subseteq \bigcup_{i=1}^{n} B_d\left(S_i,\sigma_i\right) \subseteq B_\rho(A,\alpha)$$

as required. □

Of particular interest is $\Gamma_{d,d}(\mathscr{S})$ which evidently contains $\mathscr{S}$ and so $\Sigma(\mathscr{S})$ as well.

Proposition 3.2 *Let $\mathscr{S}$ be a family of nonempty subsets of X. Then $\Gamma_{d,d}(\mathscr{S})$ is the largest family of subsets $\mathscr{T}$ of X such that $\mathsf{G}^+_{\mathscr{T},d} = \mathsf{G}^+_{\mathscr{S},d}$ on $\mathscr{P}(X)$.*

Proof. Since $\mathscr{S} \subseteq \Gamma_{d,d}(\mathscr{S})$ we have by the monotonicity of $\mathscr{T} \mapsto \mathsf{G}^+_{\mathscr{T},d}$ that $\mathsf{G}^+_{\Gamma_{d,d}(\mathscr{S}),d} \geq \mathsf{G}^+_{\mathscr{S},d}$. By the definition of $\Gamma_{d,d}(\mathscr{S})$ and Theorem 2.2, we get equality.

Now suppose $\mathsf{G}^+_{\mathscr{T},d} = \mathsf{G}^+_{\mathscr{S},d}$. Then clearly $\mathsf{G}^+_{\mathscr{T},d} \leq \mathsf{G}^+_{\mathscr{S},d}$ which means by Theorem 2.2 that $\mathscr{T} \subseteq \Gamma_{d,d}(\mathscr{S})$, showing that $\Gamma_{d,d}(\mathscr{S})$ is the largest such family. □

In view of the last proposition, it is natural to call $\mathscr{S} \mapsto \Gamma_{d,d}(\mathscr{S})$ a *saturation operator* on $\mathscr{P}_0(X)$. We leave the proof of the next elementary but important transitivity property to the reader, in part because it is notationally cumbersome.

Proposition 3.3 *Let d_1, d_2 and d_3 be metrics on X and let $\mathscr{S}$ a family of nonempty subsets of X; then*

$$\Gamma_{d_3,d_2}[\Gamma_{d_2,d_1}(\mathscr{S})] \subseteq \Gamma_{d_3,d_1}(\mathscr{S}). \quad (\textit{transitivity})$$

As an immediate consequence of transitivity and the monotonicty of $\mathscr{S} \mapsto \Gamma_{\rho,d}(\mathscr{S})$, we see that for each pair of metrics d and ρ on X, and each $\mathscr{S} \subseteq \mathscr{P}_0(X)$, we have $\Gamma_{\rho,d}[\Gamma_{d,d}(\mathscr{S})] = \Gamma_{\rho,d}(\mathscr{S})$. This observation plays a role in the next basic result.

Theorem 3.4 *Let $\mathscr{S}$ and $\mathscr{T}$ be families of nonempty subsets of a metric space (X,d). The following conditions are equivalent:*

(a) $\mathsf{G}^+_{\mathscr{S},d} = \mathsf{G}^+_{\mathscr{T},d}$ *on* $\mathscr{P}(X)$;

(b) $\Gamma_{d,d}(\mathscr{S}) = \Gamma_{d,d}(\mathscr{T})$;

(c) For each metric ρ on X, $\Gamma_{\rho,d}(\mathscr{S}) = \Gamma_{\rho,d}(\mathscr{T})$.

Proof. Condition (a) implies condition (b) by Proposition 3.2. If condition (b) holds, then by Proposition 3.2 again, we have

$$\mathsf{G}^+_{\mathscr{S},d} = \mathsf{G}^+_{\Gamma_{d,d}(\mathscr{S}),d} = \mathsf{G}^+_{\Gamma_{d,d}(\mathscr{T}),d} = \mathsf{G}^+_{\mathscr{T},d}.$$

Also by condition (b) and transitivity, for each metric ρ on X,

$$\Gamma_{\rho,d}(\mathscr{S}) = \Gamma_{\rho,d}[\Gamma_{d,d}(\mathscr{S})] = \Gamma_{\rho,d}[\Gamma_{d,d}(\mathscr{T})] = \Gamma_{\rho,d}(\mathscr{T}),$$

i.e., condition (c) holds. On the other if condition (c) holds for all metrics ρ, then in particular it holds for the choice $\rho = d$, so (b) holds. $\square$

We next present a second equally basic result.

Theorem 3.5 *Let $\mathscr{S}$ and $\mathscr{T}$ be families of nonempty subsets of X and let d and ρ be metrics on X. The following conditions are equivalent:*

(a) $\mathsf{G}^+_{\mathscr{S},d} = \mathsf{G}^+_{\mathscr{T},\rho}$ *on* $\mathscr{P}(X)$*;*

(b) $\Gamma_{\rho,\rho}(\mathscr{T}) = \Gamma_{\rho,d}(\mathscr{S})$ *and* $\Gamma_{d,d}(\mathscr{S}) = \Gamma_{d,\rho}(\mathscr{T})$*;*

(c) $\Gamma_{\rho,\rho}(\mathscr{T}) \subseteq \Gamma_{\rho,d}(\mathscr{S})$ *and* $\Gamma_{d,d}(\mathscr{S}) \subseteq \Gamma_{d,\rho}(\mathscr{T})$*.*

Proof. $(a) \Rightarrow (b)$. $\Gamma_{\rho,\rho}(\mathscr{T})$ is the largest family of subsets whose upper gap topology with respect to ρ is contained in $\mathsf{G}^+_{\mathscr{T},\rho}$. On the other hand, $\Gamma_{\rho,d}(\mathscr{S})$ is the largest family of subsets whose upper gap topology with respect to ρ is contained in $\mathsf{G}^+_{\mathscr{S},d}$. Equality of these upper gap topologies ensures equality of the families. Similarly, $\Gamma_{d,d}(\mathscr{S}) = \Gamma_{d,\rho}(\mathscr{T})$.

$(b) \Rightarrow (c)$. This is trivial.

$(c) \Rightarrow (a)$. The first inclusion implies $\mathsf{G}^+_{\mathscr{S},d} \geq \mathsf{G}^+_{\mathscr{T},\rho}$ because always $\mathsf{G}^+_{\Gamma_{\rho,d}(\mathscr{S}),\rho} \leq \mathsf{G}^+_{\mathscr{S},d}$ and $\mathsf{G}^+_{\Gamma_{\rho,\rho}(\mathscr{T}),\rho} = \mathsf{G}^+_{\mathscr{T},\rho}$. Similarly, the second inclusion implies $\mathsf{G}^+_{\mathscr{T},\rho} \geq \mathsf{G}^+_{\mathscr{S},d}$. $\square$

In the rest of this section, we focus our attention exclusively on the case $\rho = d$, and for notational simplicity, we put $\Gamma = \Gamma_{d,d}$ when d is understood. The structural properties of the operator Γ are summarized in the following theorem:

Theorem 3.6 *Let (X, d) be a metric space and let $\mathscr{S} \subseteq \mathscr{P}_0(X)$. Then:*

(a) $\mathscr{S} \subseteq \Gamma(\mathscr{S})$*;*

(b) $\mathscr{S} \subseteq \mathscr{T} \Rightarrow \Gamma(\mathscr{S}) \subseteq \Gamma(\mathscr{T})$*;*

(c) $\Sigma(\Gamma(\mathscr{S})) = \Gamma(\mathscr{S}) = \Gamma(\Sigma(\mathscr{S}))$*;*

(d) $\Gamma(\Gamma(\mathscr{S})) = \Gamma(\mathscr{S})$*;*

(e) $\mathscr{T}\mathscr{B}_d(X) \subseteq \Gamma(\mathscr{F}_0(X)) \subseteq \mathscr{B}_d(X)$.

Proof. (a), (b) and (c) are clear and have been discussed more generally already. Because of (a) and (b), $\Gamma(\mathscr{S}) \subseteq \Gamma(\Gamma(\mathscr{S}))$. On the other hand, by Proposition 3.2, we have $\Gamma(\Gamma(\mathscr{S})) \subseteq \Gamma(\mathscr{S})$ and (d) follows.

To prove (e), let T be a totally bounded subset of X and let $0 < \varepsilon < \alpha$. Select $r < \frac{1}{2}(\alpha - \varepsilon)$; then there exist $x_1, .., x_k \in T$ such that $T \subseteq \bigcup_{i=1}^k B_d(x_i, r)$ and so

$$B_d(T, \varepsilon) \subseteq \bigcup_{i=1}^{k} B_d(x_i, r + \varepsilon) \subseteq \bigcup_{i=1}^{k} B_d(x_i, 2r + \varepsilon) \subseteq B_d(T, \alpha).$$

Note that the first inclusion in (e) also follows from Proposition 3.1. The second inclusion in (e) is clear. $\square$

As we shall see shortly, in any infinite dimensional normed linear space equipped with metric determined by the norm, both inclusions in (e) are proper; further in this context, on the negative side, the family $\Gamma(\mathscr{F}_0(X))$ is not hereditary. These statements fall out of a consideration of $\Gamma(\mathscr{F}_0(X))$ in the context of almost convex metric spaces, a class of spaces which contain the normed linear spaces, and where membership to $\Gamma(\mathscr{F}_0(X))$ is more easily understood.

A subset T of X lies in $\Gamma(\mathscr{F}_0(X))$ provided whenever $0 < \varepsilon < \alpha$, $\exists$ $\{x_1, x_2, \ldots, x_n\} \subseteq X$ and $\lambda_1, \ldots, \lambda_n, \sigma_1, \ldots, \sigma_n$ such that $\forall i \leq n,\ 0 < \lambda_i < \sigma_i$, and

$$(\spadesuit) \qquad B_d(T, \varepsilon) \subseteq \textstyle\bigcup_{i=1}^n B_d(x_i, \lambda_i) \subseteq \bigcup_{i=1}^n B_d(x_i, \sigma_i) \subseteq B_d(T, \alpha).$$

A metric space (X, d) is called an *almost convex metric space* provide whenever $\alpha > 0$ and $\beta > 0$ and $A \in \mathscr{P}_0(X)$, then $B_d(B_d(A, \alpha), \beta) = B_d(A, \alpha + \beta)$. This is known to be equivalent to the following condition [4, p. 108]: whenever $0 < d(x_1, x_2) < \alpha$ and $\beta \in (0, \alpha)$ there exists $x_3 \in X$ with $d(x_1, x_3) < \beta$ and $d(x_3, x_2) < \alpha - \beta$. In an almost convex metric space, it is clear that each open ball lies in $\Gamma(\mathscr{F}_0(X))$, because $B_d(B_d(x_0, \mu), \varepsilon) = B_d(x_0, \mu + \varepsilon)$ and $B_d(B_d(x_0, \mu), \alpha) = B_d(x_0, \mu + \alpha)$. In particular, each open ball in a normed linear belongs to $\Gamma(\mathscr{F}_0(X))$, while such sets fail to be totally bounded whenever the space is infinite dimensional.

Open balls need not belong to $\Gamma(\mathscr{F}_0(X))$ in an arbitrary metric space (X, d).

Example. Our metric space X will consist of a countable set $\{p\} \cup \{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}$ equipped with the metric d defined by cases as follows: (1) $\forall n,\ d(p, x_n) = 2$; (2) $\forall n,\ d(p, y_n) = 2.5$; (3) $d(x_1, y_1) = 1.5$; $d(x_n, y_n) = 1$ for $n \geq 2$; (4) $d(x, y) = 1.5$ for $x \neq y$ otherwise. We intend to show that $T := B_d(p, 2.1) = \{p\} \cup \{x_n : n \in \mathbb{N}\}$ fails to lie in $\Gamma(\mathscr{F}_0(X))$. Note that $B_d(T, 1.1) = B_d(T, 1.3) = T \cup \{y_n : n \geq 2\}$. We claim that this set cannot be realized as a finite union of balls.

First note that any open ball with center p contained in $B_d(T, 1.1)$ can have radius at most 2.5, and thus is a subset of T. On the other hand, each open ball with center x_n or

y_n contained in $B_d(T, 1.1)$ must have radius at most 1.5 and hence contains at most two points. Thus it is impossible to capture $\{y_n : n \geq 2\}$ by a finite union of such balls.

Since $\Sigma(\Gamma(\mathscr{F}_0(X)) = \Gamma(\mathscr{F}_0(X))$, each finite union of balls in an almost convex metric space belongs to $\Gamma(\mathscr{F}_0(X))$. It turns out that in this setting, $\Gamma(\mathscr{F}_0(X))$ consists of all subsets of X that can be approximated in d-Hausdorff distance by a finite union of balls (cf. [4, Lemma 4.1.2 and Corollary 4.18]).

As a prelude to our next result, note that in a general metric space, a subset of X is totally bounded if and only if it can be so approximated by a finite union of balls each with the same arbitrarily small radius.

Theorem 3.7 *Let (X, d) be a metric space.*

(a) If $T \in \Gamma(\mathscr{F}_0(X))$, then $\forall \varepsilon > 0$ there exists a finite collection of open balls $\{B_1, B_2, \ldots, B_n\}$ such that $H_d(T, \cup_{i=1}^n B_i) \leq \varepsilon$;

(b) If (X, d) is an almost convex metric space, then the converse of statement (a) holds.

Proof. For (a), suppose $T \in \Gamma(\mathscr{F}_0(X))$ and let $\varepsilon > 0$ be arbitrary. Using just part of the definition of $\Gamma(\mathscr{F}_0(X))$, we can find $\{x_1, \ldots, x_n\} \subseteq X$ and positive scalars $\{\sigma_1, \ldots, \sigma_n\}$ such that

$$B_d(T, \frac{\varepsilon}{2}) \subseteq \bigcup_{i=1}^{n} B_d(x_i, \sigma_i) \subseteq B_d(T, \varepsilon).$$

It immediately follows that $H_d(\cup_{i=1}^n B_d(x_i, \sigma_i), T) \leq \varepsilon$.

Condition (b) follows from Theorem 3.1, Theorem 3.6(c), and the fact that each open ball in an almost convex metric space belongs to $\Gamma(\mathscr{F}_0(X))$. $\square$

In view of Proposition 1.2, Theorem 2.2 and Proposition 2.7, we may state this corollary.

Corollary 3.8 *Let (X, d) be an almost convex metric space. Let $\mathscr{B}$ be the closure in $\mathscr{P}_0(X)$ with respect to the Hausdorff pseudometric topology of the family of all finite unions of open balls in X. Then $\mathscr{B}$ is the largest subfamily $\mathscr{S}$ of $\mathscr{P}_0(X)$ such that $\mathsf{G}^+_{\mathscr{S},d} = \mathsf{W}^+_d$, equivalently, $\mathsf{G}_{\mathscr{S},d} = \mathsf{W}_d$.*

As we mentioned above, $\Gamma(\mathscr{F}_0(X))$ need not be a hereditary family. In the setting of almost convex metric spaces, we intend to characterize those members of $\Gamma(\mathscr{F}_0(X))$ each of whose nonempty subsets again belongs to $\Gamma(\mathscr{F}_0(X))$. For this purpose, we introduce some notation for iterated enlargements: given $T \in \mathscr{P}_0(X)$ and $\alpha > 0$ put $B_d^1(T, \alpha) = B_d(T, \alpha)$ and $B_d^{n+1}(T, \alpha) = B_d(B_d^n(T, \alpha))$ for $n \in \mathbb{N}$. Almost convexity assures $B_d^n(T, \alpha) = B_d(T, n\alpha)$ for all n.

Proposition 3.9 *Let (X, d) be an almost convex metric space. For a nonempty subset T of X, the following conditions are equivalent:*

(a) each nonempty subset of T belongs to $\Gamma(\mathscr{F}_0(X))$;

(b) T is totally bounded.

Proof. Only $(a) \Rightarrow (b)$ requires proof. Suppose $T \in \Gamma(\mathscr{F}_0(X))$ fails to be totally bounded, yet (a) holds. Then for some $\delta > 0$ we can find a countably infinite subset $E := \{x_m : m \in \mathbb{N}\}$ of T such that $d(x_m, x_j) \geq 3\delta$ for $m \neq j$. Now by the last theorem, we can find a finite family of closed balls $\{B_1, B_2, \ldots, B_n\}$ such that $H_d(E, \cup_{i=1}^n B_i) < \delta$. This means that for each $m \in \mathbb{N}$, some ball hits $B_d(x_m, \delta)$ and that $\cup_{i=1}^n B_i \subseteq B_d(E, \delta)$. By the pigeon-hole principle, we can find distinct indices m_1 and m_2 and a fixed ball - without loss of generality B_1 - such that $B_d(x_{m_1}, \delta)$ contains the center of B_1 and $B_d(x_{m_2}, \delta)$ hits B_1. We now write $B_1 = B_d(y, \alpha)$ and take $w \in B_d(y, \alpha) \cap B_d(x_{m_2}, \delta)$.

Choose $\varepsilon \in (0, \min\{\delta, \alpha - d(y, w)\})$. Let k be the smallest integer such $d(y, w) < k\varepsilon$; this choice ensures that $k\varepsilon < \alpha$. By almost convexity, we can find a finite string $y_0, y_1, y_2, y_3, \ldots, y_k$ with $y_0 = y$ and $y_k = w$ such that $d(y_j, y_{j+1}) < \varepsilon$ for $j = 0, 1, \ldots, k-1$ so that each y_j lies in $B_d(y, \alpha)$. Now $y_0 \in B(x_{m_1}, \delta)$ and by the choice of $\delta, y_k \notin B(x_{m_1}, \delta)$. Thus, there is a first index $j > 0$ such that $y_j \notin B(x_{m_1}, \delta)$. Since $y_{j-1} \in B(x_{m_1}, \delta)$ and $\varepsilon < \delta$, we have $\delta < d(x_{m_1}, y_j) < 2\delta$. From this, $y_j \in \cup_{i=1}^n B_i \backslash B_d(E, \delta)$ which is a contradiction. $\square$

By our last result, each ball in a normed linear space, while lying in $\Gamma(\mathscr{F}_0(X))$, must have a nonempty subset that does not. Equivalently, such a subset cannot be approximated by a finite union of balls in Hausdorff distance.

4 First and second countability

We begin this section with a characterization of first countability of upper gap topologies. We will state this result with respect to the hyperspace $\mathscr{P}_0(X)$ of all nonempty subsets of X, but it will be clear that an analogous characterization holds on every hyperspace of X.

Let (X, d) be a metric space. For a collection $\mathscr{S}$ of nonempty subsets of X, and an $A \in \mathscr{P}_0(X)$, put $\mathscr{S}_A = \{S \in \mathscr{S} \,|\, D_d(S, A) \neq 0\}$. If $\mathscr{S}_A = \emptyset$, it is clear that $\mathscr{P}_0(X)$ is the unique $\mathsf{G}^+_{\mathscr{S},d}$-neighborhood of A. So we will focus on the case $\mathscr{S}_A \neq \emptyset$.

Let us put, for $S_1, \ldots, S_k \in \mathscr{S}$ and $\varepsilon_1, \ldots, \varepsilon_k > 0$,

$$\mathscr{U}(S_1, \ldots, S_k; \varepsilon_1, \ldots, \varepsilon_k) = \{C \in \mathscr{P}_0(X) \,|\, \forall i \in \{1, \ldots, k\} \colon B_d(C, \varepsilon_i) \cap S_i = \emptyset\}.$$

The next result complements Lemma 2.1:

Lemma 4.1

(1) *the family of all $\mathscr{U}(S_1, \dots, S_k; \varepsilon_1, \dots, \varepsilon_k)$, with $S_1, \dots, S_k \in \mathscr{S}_A$ and $D_d(A, S_i) > \varepsilon_i$ for $i = 1, \dots, k$, forms a local base at A for $\mathsf{G}^+_{\mathscr{S},d}$;*

(2) *$\mathscr{U}(S_1, \dots, S_k; \varepsilon_1, \dots, \varepsilon_k) \subseteq \mathscr{U}(T_1, \dots, T_n; r_1, \dots, r_n)$ if and only if*

$$\bigcup_{j=1}^{n} B_d(T_j, r_j) \subseteq \bigcup_{j=1}^{k} B_d(S_j, \varepsilon_j). \tag{4;7}$$

Proof.

(1) To prove (1), note that

$$A \in \mathscr{A}^+(S_1, \dots, S_k; \varepsilon_1, \dots, \varepsilon_k) \subseteq \mathscr{U}(S_1, \dots, S_k; \varepsilon_1, \dots, \varepsilon_k),$$

so that the latter is a neighborhood of A. Now consider, for $i = 1, \dots, k$, $r_i > 0$ such that $\varepsilon_i + r_i < D_d(S_i, A)$: we then have the relations $A \in \mathscr{U}(S_1, \dots, S_k; \varepsilon_1 + r_1, \dots, \varepsilon_k + r_k) \subseteq \mathscr{A}^+(S_1, \dots, S_k; \varepsilon_1, \dots, \varepsilon_k)$.

(2) Suppose the inclusion among neighborhoods holds and there exists $b \in \left(\bigcup_{j=1}^n B_d(T_j, r_j)\right) \setminus \bigcup_{j=1}^k B_d(S_j, \varepsilon_j)$; then $\{b\} \in \mathscr{U}(S_1, \dots, S_k; \varepsilon_1, \dots, \varepsilon_k)$ but $\{b\} \notin \mathscr{U}(T_1, \dots, T_n; r_1, \dots, r_n)$. Suppose, conversely, that (4; 7) holds; then, if $B_d(C, \varepsilon_j) \cap S_j = \emptyset$ $(j = 1, \dots, k)$, we have

$$C \cap \Big(\bigcup_{j=1}^{k} B_d(S_j, \varepsilon_j)\Big) = \emptyset;$$

therefore $C \cap (\bigcup_{j=1}^n B_d(T_j, r_j) = \emptyset$ and $B_d(C, r_j) \cap T_j = \emptyset$ $(j = 1, \dots, n)$.

□

We begin with an important particular case:

Proposition 4.2 *Suppose the family $\mathscr{S}$ is hereditary and has a countable cofinal subfamily; then $\mathsf{G}^+_{\mathscr{S},d}$ is first countable.*

Proof. By Remark 1.1, without loss of generality, we may assume $\sum(\mathscr{S}) = \mathscr{S}$. With this in mind, let (S_n) be an increasing cofinal sequence in $\mathscr{S}$ and A a non-empty subset of X. Suppose $A \in \mathscr{A}^+(S, \varepsilon)$, r is a rational larger than ε and $S \subseteq S_n$; then $A \in \mathscr{A}^+(S_n \cap (A^r)^c, r) \subseteq \mathscr{A}^+(S, \varepsilon)$ and this is enough to prove first countability at A. □

Corollary 4.3 *The topologies AW^+_d and H^+_d are first countable.*

The general case is given by the following theorem. We use in the proof the notation $A^{<\omega}$ for the set of all finite sequences from the nonempty set A.

Theorem 4.4 *Let $\mathscr{S}$ be a collection of nonempty subsets of a metric space (X, d), and let $A \in \mathscr{P}_0(X)$ be such that $\mathscr{S}_A \neq \emptyset$. Then $\mathsf{G}^+_{\mathscr{S},d}$ is first countable at A if and only if there exists a countable subfamily $\mathscr{L}_A = \{S_n \mid n \in \mathbb{N}\} \subseteq \mathscr{S}_A$ with the following property: for every $S \in \mathscr{S}_A$ and for every $0 < \varepsilon < D_d(S, A)$, there exist $k \in \mathbb{N}$, $S_{j_1}, \ldots, S_{j_k} \in \mathscr{L}_A$ and $0 < \varepsilon_r < D_d(S_{j_r}, A)$ for $r = 1, .., k$, such that*

$$B_d(S, \varepsilon) \subseteq \bigcup_{r=1}^{k} B_d(S_{j_r}, \varepsilon_r). \tag{4;8}$$

Proof. i) Suppose the stated condition is verified and let $S \in \mathscr{S}_A$, $0 < \varepsilon < D_d(S, A)$ be given. Then, choosing the S_{j_r} and ε_r as above, inclusion (4; 8) holds and, by Lemma 4.1, $\mathscr{U}(S_{j_1}, \ldots, S_{j_k}; \varepsilon_1, \ldots, \varepsilon_k) \subseteq \mathscr{U}(S, \varepsilon)$.

Let $t_1, .., t_k$ be rationals such that

$$\varepsilon_r < t_r < D_d(S_{j_r}, A) \text{ for } r = 1, \ldots, k;$$

then

$$\mathscr{U}(S_{j_1}, \ldots, S_{j_k}; \varepsilon_1, \ldots, \varepsilon_k) \subseteq \mathscr{U}(S, \varepsilon).$$

As $[L_A]^{<\omega} \times [\mathbb{Q}]^{<\omega}$ is countable, we have found a countable local base at A.

ii) If $\mathsf{G}^+_{\mathscr{S},d}$ is first countable at A, there exists a countable local base $\{\mathscr{U}_n \mid n \in \mathbb{N}\}$ at A. By Lemma 4.1 we can write

$$\mathscr{U}_n = \mathscr{U}(S_1^{(n)}, \ldots, S_{j_n}^{(n)}; {\varepsilon_1}^{(n)}, \ldots, {\varepsilon_{j_n}}^{(n)})$$

with $D_d(A, S_k^{(n)}) > \varepsilon_k$ for $k = 1, \ldots, j_n$. Put $\mathscr{L}_A = \bigcup_{n=1}^{\infty} \{{S_1}^{(n)}, \ldots, S_{j_n}^{(n)}\}$ and let $S \in \mathscr{S}_A$, $0 < \varepsilon < D_d(S, A)$. Then there exists $n \in \mathbb{N}$ such that $\mathscr{U}_n \subseteq \mathscr{U}(S, \varepsilon)$, so that again by Lemma 4.1

$$B_d(S, \varepsilon) \subseteq \bigcup_{k=1}^{j_n} B_d({S_k}^{(n)}, {\varepsilon_k}^{(n)})$$

□

If our family $\mathscr{S}$ contains the singleton subsets of X and A is a nonempty subset, then $\mathscr{S}_A \neq \emptyset$ if and only if $\overline{A} \neq X$. This leads to the following corollary:

Corollary 4.5 *Let $A \in \mathscr{P}_0(X)$ with $\overline{A} \neq X$. The upper Wijsman topology W^+_d is first countable at A if and only if there exists a countable set $L_A = \{x_n \mid n \in \mathbb{N}\} \subseteq X \setminus \overline{A}$ with the following property:*

$$\forall x \in X \setminus \overline{A}\colon \forall 0 < \varepsilon < d(x, A)\colon \exists k \in \mathbb{N}\colon \exists x_1, \ldots, x_k \in L_A\colon \exists \varepsilon_1, \ldots, \varepsilon_k\colon$$
$$\Big(\big(\forall j \in \{1, \ldots, k\}\colon 0 < \varepsilon_j < d(x_j, A)\big) \ \wedge\ B(x, \varepsilon) \subseteq \bigcup_{j=1}^{k} B(x_j, \varepsilon_j) \Big). \tag{4;9}$$

With the above results we are in a position to prove that first countability of the upper Wijsman topology is equivalent to separability of the base space. The analogous result for the lower Wijsman topology or the full Wijsman topology is well-known and much easier to prove. To the authors' knowledge, the next statement does not appear in the literature.

Theorem 4.6 *Let (X, d) be a metric space. The following conditions are equivalent:*

(a) X is separable;

(b) the upper Wijsman topology is first countable on $\mathscr{P}_0(X)$;

(c) the upper Wijsman topology is first countable on $\mathscr{C}_0(X)$.

Proof.

(a)$\Longrightarrow$(b).

Suppose X is separable and let $A \in \mathscr{P}_0(X)$ with $\overline{A} \neq X$; let $\{x_n \mid n \in \mathbb{N}\}$ be a dense subset of $X \setminus \overline{A}$. Fix $x \in X \setminus \overline{A}$ and $0 < \varepsilon < d(x, A)$; choose x_n such that $d(x_n, x) < \frac{1}{2}(d(x, A) - \varepsilon)$ and put $\varepsilon_n = \varepsilon + d(x_n, x)$. We wish to show that $d(x_n, A) > \varepsilon_n$. Suppose, instead, that $d(x_n, A) \leq \varepsilon_n$; then $d(x_n, A) \leq \varepsilon + d(x_n, x) < d(x, A) - d(x_n, x)$ and $d(x_n, x) < d(x, A) - d(x_n, A) \leq d(x_n, x)$, a contradiction. Finally, it is clear that $B(x, \varepsilon) \subseteq B(x_n, \varepsilon_n)$. First countability of W_d^+ at A now follows from Corollary 4.5.

(b)$\Longrightarrow$(c).

This is obvious

(c)$\Longrightarrow$(a).

Suppose (X, d) is a non-separable metric space. Then, as is well-known, it is possible to find a $\vartheta > 0$ and a ϑ-uniformly discrete subset D of X with $|D| = \aleph_1$. (This may be proved, for example, by considering for every $n \in \mathbb{N}^+$ a maximal $\frac{1}{n}$-uniformly discrete subset D_n of X: since the set $\bigcup_{n \in \mathbb{N}^+} D_n$ is easily seen to be dense in X, it must be uncountable, hence at least one of the sets D_n is uncountable, too). Let $D' = \{x \in D \mid x \text{ is isolated in } (X, d)\}$ and $D'' = D \setminus D'$: of course, at least one of the two sets D', D'' will be uncountable.

<u>1st case</u>: $|D''| = \aleph_1$.

For every $x \in D''$ let $a(x)$ be an element of $X \setminus \{x\}$ such that $d\big(x, a(x)\big) < \frac{\vartheta}{3}$, and choose a $\delta(x)$ with

$$0 < \delta(x) < d\big(x, a(x)\big) < \frac{\vartheta}{3}; \tag{4;10}$$

then let $A = X \setminus \bigcup_{x \in D''} B_d(x, \delta(x))$. Notice that A contains $\{a(x) : x \in D''\}$ and clearly A is closed. We claim that

$$\forall x \in D'' \colon a(x) \in A. \tag{4;11}$$

Indeed, on the one hand, $(4;10)$ implies that $a(x) \notin B_d(x, \delta(x))$. On the other hand, if $x' \in D'' \setminus \{x\}$, then we cannot have $a(x) \in B_d(x', \delta(x'))$, because this would imply by $(4;10)$ (applied for $x = x'$) that $d(a(x), x') < \delta(x') < d(x', a(x')) < \frac{\vartheta}{3}$, so that

$d(x, x') \leq d(x, a(x)) + d(a(x), x') < \frac{\vartheta}{3} + \frac{\vartheta}{3} = \frac{2}{3}\vartheta$ (we have applied (4; 10) again, this time for $x = x$), contradicting the fact that D'' is ϑ-uniformly discrete. Therefore, $a(x) \notin \bigcup_{x' \in D''} B_d(x', \delta(x'))$, i.e. $a(x) \in A$.

Now, we claim that A does not satisfy the condition of Corollary 4.5. Indeed, consider an arbitrary countable subset L_A of $X \backslash A$: since $X \backslash A = \bigcup_{x \in D''} B_d(x, \delta(x))$, for every $y \in L_A$ there is $x_y \in D''$ with

$$d(y, x_y) < \delta(x_y). \tag{4;12}$$

This implies that

$$\forall y \in L_A \colon d(y, A) < \frac{2}{3}\vartheta : \tag{4;13}$$

indeed, if $y \in L_A$, then using (4; 11), (4; 12) and (4; 10) we see that $d(y, A) \leq d(y, a(x_y)) \leq d(y, x_y) + d(x_y, a(x_y)) < \delta(x_y) + d(x_y, a(x_y)) < \frac{\vartheta}{3} + \frac{\vartheta}{3} = \frac{2}{3}\vartheta$.

Now, consider the set $D'' \backslash \{x_y \,|\, y \in L_A\}$: since L_A is countable while D'' is not, such a set cannot be empty, hence we may fix a point $\bar{x} \in D'' \backslash \{x_y \,|\, y \in L_A\}$. Clearly, if (4; 10) held for $x = \bar{x}$, then in particular there would be $y_1, \ldots, y_n \in L_A$ and $\varepsilon_1, \ldots, \varepsilon_n$ with $0 < \varepsilon_i < d(y_i, A)$ for $i = 1, \ldots, n$, such that $\bar{x} \in \bigcup_{i=1}^n B_d(y_i, \varepsilon_i)$. However, for every $i \in \{1, \ldots, n\}$, (4; 13) implies that $\varepsilon_i < d(y_i, A) < \frac{2}{3}\vartheta$, while (4; 12) and (4; 10) combine to show that $d(y_i, x_{y_i}) < \delta(x_{y_i}) < \frac{\vartheta}{3}$; therefore $\bar{x} \in B_d(y_i, \varepsilon_i)$ would imply that $d(\bar{x}, x_{y_i}) \leq d(\bar{x}, y_i) + d(y_i, x_{y_i}) < \varepsilon_i + \frac{\vartheta}{3} < \frac{2}{3}\vartheta + \frac{\vartheta}{3} = \vartheta$, which is impossible because $\bar{x}$ and x_{y_i} are distinct elements of D'' and D'' is ϑ-uniformly discrete.

Thus, we conclude that (4; 9) does not hold in this 1st case.

<u>2nd case</u>: $|D'| = \aleph_1$.
Using transfinite induction, we will define for every $\alpha \in \omega_1$ an $x_\alpha \in D'$ and an $M_\alpha \in [D']^{\leq \omega}$ in the following way. Suppose we have defined x'_α and $M_{\alpha'}$ for $\alpha' < \alpha$: then pick any

$$x_\alpha \in D' \backslash \Big(\{x_{\alpha'} \,|\, \alpha' < \alpha\} \cup \bigcup\nolimits_{\alpha' < \alpha} M_{\alpha'} \Big), \tag{4;14}$$

and let M_α be a countable subset of $X \backslash \{x'_\alpha \,|\, \alpha' \leq \alpha\}$ such that

$$d\big(x_\alpha, X \backslash \{x_{\alpha'} \,|\, \alpha' \leq \alpha\}\big) = d(x_\alpha, M_\alpha). \tag{4;15}$$

Consider the set $A = X \backslash \{x_\alpha \,|\, \alpha \in \omega_1\}$ (which is closed, as every point x_α belongs to D' and hence is isolated in (X, d)): we claim that A disproves (4; 9).

First of all, we point out that $M_\alpha \subseteq A$ (i.e., that $M_\alpha \cap \{x_{\alpha'} \,|\, \alpha' \in \omega_1\} = \emptyset$) for every $\alpha \in \omega_1$. Indeed, on the one hand each M_α is by definition a subset of $X \backslash \{x'_\alpha \,|\, \alpha' \leq \alpha\}$, hence it misses the set $\{x'_\alpha \,|\, \alpha' \leq \alpha\}$; on the other hand, for every $\alpha^* > \alpha$ we see by (4; 14) that $x_{\alpha^*} \notin \bigcup_{\alpha' < \alpha^*} M_{\alpha'}$, hence in particular $x_{\alpha^*} \notin M_\alpha$. Therefore M_α misses the whole set $\{x_{\alpha'} \,|\, \alpha' \in \omega_1\}$.

Now, let L_A be any countable subset of $X \backslash \overline{A} = X \backslash A = \{x_\alpha \,|\, \alpha \in \omega_1\}$: to prove that (4; 9) fails it will suffice to show—as in the 1st case—that there is an $\hat{x} \in X \backslash A = \{x_\alpha \,|\, \alpha \in \omega_1\}$ such that for every $y_1, \ldots, y_n \in L_A$ and every $\varepsilon_1, \ldots, \varepsilon_n$ with $0 < \varepsilon_i < d(y_i, A)$ for $i =$

$1, \ldots, n$, we have the relation $\hat{x} \notin \bigcup_{i=1}^{n} B_d(y_i, \varepsilon_i)$. Actually, write $L_A = \{x_\alpha \mid \alpha \in S\}$, with S a countable subset of ω_1, and choose $\hat{x} = x_{\hat{\alpha}}$ with $\hat{\alpha} \in \omega_1$ such that

$$\hat{\alpha} > \alpha \text{ for every } \alpha \in S. \tag{4;16}$$

Now, if $y_1, \ldots, y_n$ and $\varepsilon_1, \ldots, \varepsilon_n$ are as above, then each y_i is $x_{\alpha(i)}$ for some $\alpha(i) \in S$; we see that for every $i \in \{1, \ldots, n\}$ the relation $\hat{x}\,(= x_{\hat{\alpha}}) \in B_d(y_i, \varepsilon_i)\,\big(= B_d\big(x_{\alpha(i)}, \varepsilon_i\big)\big)$ would imply that

$$d(x_{\hat{\alpha}}, x_{\alpha(i)}) < \varepsilon_i < d(y_i, A) = d(x_{\alpha(i)}, A) \leq d(x_{\alpha(i)}, M_{\alpha(i)})$$

(as we have proved before that $M_\alpha \subseteq A$ for every $\alpha \in \omega_1$). Thus, by (4;15), we conclude that $d(x_{\alpha(i)}, x_{\hat{\alpha}}) < d\big(x_{\alpha(i)}, X \backslash \{x_{\alpha'} \mid \alpha' \leq \alpha(i)\}\big)$; but this is a contradiction, as (4;16) and the one-to-one character of $\alpha \mapsto x_\alpha$ (take (4;14) into account) imply that $x_{\hat{\alpha}} \in X \backslash \{x_{\alpha'} \mid \alpha' \leq \alpha(i)\}$.

Therefore, we have shown that $\hat{x} \notin B_d(y_i, \varepsilon_i)$ for $i = 1, \ldots, n$, i.e. that

$$\hat{x} \notin \bigcup_{i=1}^{n} B_d(y_i, \varepsilon_i).$$

$\square$

We will now consider second countability of upper gap topologies.

Theorem 4.7 *Let $\mathscr{S}$ be a collection of subsets of a metric space (X, d). Then the following are equivalent:*

(1) the space $\big(\mathscr{P}_0(X), \mathsf{G}^+_{\mathscr{S},d}\big)$ is second countable;

(2) there exists a countable subcollection $\mathscr{D}$ of $\mathscr{S}$ such that $\mathscr{S} \subseteq \Gamma(\mathscr{D})$.

Proof. (1)$\Longrightarrow$(2).
Since the collection

$$\mathfrak{V} = \big\{\mathscr{A}^+_d(S_1, \ldots, S_n; \varepsilon_1, \ldots, \varepsilon_n) \,\big|\, n \in \mathbb{N},\, S_1, \ldots, S_n \in \mathscr{S},\, \varepsilon_1, \ldots, \varepsilon_n > 0\big\}$$

is a base for $\big(\mathscr{P}_0(X), \mathsf{G}^+_{\mathscr{S},d}\big)$, and this space has weight $\aleph_0$, by a well-known general result (see [20, Theorem 1.1.15]) there exists a countable subcollection $\mathfrak{B}$ of $\mathfrak{V}$ which is still a base for $\big(\mathscr{P}_0(X), \mathsf{G}^+_{\mathscr{S},d}\big)$. Then write $\mathfrak{B}$ as $\big\{\mathscr{P}_0(X)\big\} \cup \{\mathscr{U}^j \mid j \in \mathbb{N}\}$, and for every $j \in \mathbb{N}$ let $S^j_1, \ldots, S^j_{n(j)} \in \mathscr{S}$ and $\varepsilon^j_1, \ldots, \varepsilon^j_{n(j)} > 0$ be such that $\mathscr{U}^j = \mathscr{A}^+_d\big(S^j_1, \ldots, S^j_{n(j)}; \varepsilon^j_1, \ldots, \varepsilon^j_{n(j)}\big)$ (with $n(j) \in \mathbb{N}^+$). Now, if for some $S \in \mathscr{S}$ and $\alpha > 0$ we have the equality $B_d(S, \alpha) = X$, set

$$\mathscr{D} = \{\hat{S}\} \cup \big\{S^j_i \,\big|\, j \in \mathbb{N},\, 1 \leq i \leq n(j)\big\},$$

where $\hat{S}$ is an element of $\mathscr{S}$ for which there is a $\hat{\lambda} > 0$ such that $B_d(\hat{S}, \hat{\lambda}) = X$; otherwise, if $B_d(S, \alpha) \subsetneqq X$ for every $S \in \mathscr{S}$ and $\alpha > 0$, set

$$\mathscr{D} = \left\{ S_i^j \,\middle|\, j \in \mathbb{N},\, 1 \leq i \leq n(j) \right\}.$$

We claim that $\mathscr{S} \subseteq \Gamma(\mathscr{D})$.

Indeed, let $S \in \mathscr{S}$ and $0 < \varepsilon < \alpha$ be arbitrarily given. If $B_d(S, \alpha) = X$, then we are in the case where $\mathscr{D} = \{\hat{S}\} \cup \left\{ S_i^j \,\middle|\, j \in \mathbb{N},\, 1 \leq i \leq n(j) \right\}$ with $B_d\left(\hat{S}, \hat{\lambda}\right) = X$; thus, fixing a $\hat{\sigma} > \hat{\lambda}$, we see that $B_d\left(\hat{S}, \hat{\lambda}\right) = B_d\left(\hat{S}, \hat{\sigma}\right) = B_d(S, \alpha) = X$, and hence, in particular, $B_d(S, \varepsilon) \subseteq B_d\left(\hat{S}, \hat{\lambda}\right) \subseteq B_d\left(\hat{S}, \hat{\sigma}\right) \subseteq B_d(S, \alpha)$.

Thus, we may assume that $B_d(S, \alpha) \neq X$, which means that $C = X \backslash B_d(S, \alpha) \in \mathscr{P}_0(X)$. Since $D_d(S, C) \geq \alpha > \varepsilon$, i.e. $C \in \mathscr{A}_d^+(S, \varepsilon)$, and $\mathfrak{B}$ is a base for $\mathsf{G}^+_{\mathscr{S},d}$, there must exist $\hat{\jmath} \in \mathbb{N}$ such that

$$C \in \mathscr{A}_d^+\big(S_1^{\hat{\jmath}}, \ldots, S_{n(\hat{\jmath})}^{\hat{\jmath}}; \varepsilon_1^{\hat{\jmath}}, \ldots, \varepsilon_{n(\hat{\jmath})}^{\hat{\jmath}}\big) \subseteq \mathscr{A}_d^+(S; \varepsilon). \tag{4;17}$$

Now, on the one hand, the relation $C \in \mathscr{A}_d^+\big(S_1^{\hat{\jmath}}, \ldots, S_{n(\hat{\jmath})}^{\hat{\jmath}}; \varepsilon_1^{\hat{\jmath}}, \ldots, \varepsilon_{n(\hat{\jmath})}^{\hat{\jmath}}\big)$ means that

$$\forall i \in \{1, \ldots, n(\hat{\jmath})\} \colon D_d\big(S_i^{\hat{\jmath}}, C\big) > \varepsilon_i^{\hat{\jmath}},$$

and this allows us to choose for every $i = 1, \ldots, n(\hat{\jmath})$ some $\sigma_i, \lambda_i \in \mathbb{R}$ so that

$$\forall i \in \{1, \ldots, n(\hat{\jmath})\} \colon D_d\big(S_i^{\hat{\jmath}}, C\big) > \sigma_i > \lambda_i > \varepsilon_i^{\hat{\jmath}}; \tag{4;18}$$

consequently, $B_d\big(S_i^{\hat{\jmath}}, \sigma_i\big) \cap C = B_d\big(S_i^{\hat{\jmath}}, \sigma_i\big) \cap \big(X \backslash B_d(S, \alpha)\big) = \emptyset$ for $i = 1, \ldots, n(\hat{\jmath})$, i.e.

$$\bigcup_{i=1}^{n(\hat{\jmath})} B_d\big(S_i^{\hat{\jmath}}, \sigma_i\big) \subseteq B_d(S, \alpha). \tag{4;19}$$

On the other hand, the inclusion $\mathscr{A}_d^+\big(S_1^{\hat{\jmath}}, \ldots, S_{n(\hat{\jmath})}^{\hat{\jmath}}; \varepsilon_1^{\hat{\jmath}}, \ldots, \varepsilon_{n(\hat{\jmath})}^{\hat{\jmath}}\big) \subseteq \mathscr{A}_d^+(S; \varepsilon)$ (from (4;17)) implies, because of lemma 2.1, that

$$B_d(S, \varepsilon) \subseteq \bigcup_{i=1}^{n(\hat{\jmath})} B_d\big(S_i^{\hat{\jmath}}, \lambda_i\big); \tag{4;20}$$

Therefore, (4;19), (4;20) and the inequality $0 < \lambda_i < \sigma_i$ for every $i \in \{1, \ldots, n(\hat{\jmath})\}$ (take again (4;18) into account) combine to show that $S \in \Gamma(\mathscr{D})$.

(2)$\Longrightarrow$(1).
Let $\mathscr{D}$ be a countable subcollection of $\mathscr{S}$ with $\mathscr{S} \subseteq \Gamma(\mathscr{D})$, and let

$$\mathfrak{B} = \big\{\mathscr{P}_0(X)\big\} \cup \big\{ \mathscr{A}_d^+(T_1, \ldots, T_m; \lambda_1, \ldots, \lambda_m) \,\big|\, m \in \mathbb{N}^+,\, T_1, \ldots, T_m \in \mathscr{C}, \\ \lambda_1, \ldots, \lambda_m \in \mathbb{Q} \cap]0, +\infty[\big\}$$

(so that $\mathfrak{B}$ is in its turn countable); we claim that $\mathfrak{B}$ is a base for $\big(\mathscr{P}_0(X), \mathsf{G}^+_{\mathscr{S},d}\big)$.

Indeed, let $C \in \mathscr{P}_0(X)$, and consider an arbitrary $\mathsf{G}^+_{\mathscr{S},d}$-neighbourhood $\mathscr{U}$ of C. If $\mathscr{U} = \mathscr{P}_0(X)$, we are done, as $\mathscr{P}_0(X) \in \mathfrak{B}$. Otherwise, we may assume $\mathscr{U}$ to be of the form $\mathscr{A}^+_d(S_1, \ldots, S_n; \varepsilon_1, \ldots, \varepsilon_n)$ with $n \in \mathbb{N}^+$, $S_1, \ldots, S_n \in \mathscr{S}$ and $0 < \varepsilon_i < D_d(S_i, C)$ for every $i \in \{1, \ldots, n\}$; since $\mathfrak{B}$ is closed under finite intersections, if we can show that for every $i \in \{1, \ldots, n\}$ there is $\mathscr{U}_i \in \mathfrak{B}$ with $C \in \mathscr{U}_i \subseteq \mathscr{A}^+_d(S_i; \varepsilon_i)$, then $\bigcap_{i=1}^n \mathscr{U}_i$ will be an element of $\mathfrak{B}$ such that $C \in \bigcap_{i=1}^n \mathscr{U}_i \subseteq \mathscr{A}^+_d(S_1, \ldots, S_n; \varepsilon_1, \ldots, \varepsilon_n)$.

Actually, given $i \in \{1, \ldots, n\}$, since $0 < \varepsilon_i < D_d(S_i, C)$ we may consider ε^*, α^* such that $0 < \varepsilon_i < \varepsilon^* < \alpha^* < D_d(S_i, C)$. Then the relations $S_i \in \mathscr{S} \subseteq \Gamma(\mathscr{D})$ imply that there exist $T_1, \ldots, T_m \in \mathscr{D}$ and $\lambda_1, \ldots, \lambda_m, \sigma_1, \ldots, \sigma_m$ with $0 < \lambda_\ell < \sigma_\ell$ for $\ell = 1, \ldots, m$, such that

$$B_d(S_i, \varepsilon^*) \subseteq \bigcup_{\ell=1}^m B_d(T_\ell, \lambda_\ell) \subseteq \bigcup_{\ell=1}^m B_d(T_\ell, \sigma_\ell) \subseteq B_d(S_i, \alpha^*); \tag{4;21}$$

moreover, up to replacing each λ_ℓ with a λ'_ℓ such that $\lambda_\ell < \lambda'_\ell < \sigma_\ell$ and $\lambda'_\ell \in \mathbb{Q}$, we may assume that each λ_ℓ in (4; 21) is already in $\mathbb{Q}$.

Now, on the one hand, the inclusion $B_d(S_i, \varepsilon^*) \subseteq \bigcup_{\ell=1}^m B_d(T_\ell, \lambda_\ell)$ entails by Lemma 2.1 (as $0 < \varepsilon_i < \varepsilon^*$) that $\mathscr{A}^+_d(T_1, \ldots, T_m; \lambda_1, \ldots, \lambda_m) \subseteq \mathscr{A}^+_d(S_i; \varepsilon_i)$. On the other hand, since $D_d(S_i, C) > \alpha^*$, and hence in particular $C \cap B_d(S_i, \alpha^*) = \emptyset$, it follows from the inclusion $\bigcup_{\ell=1}^m B_d(T_\ell, \sigma_\ell) \subseteq B_d(S_i, \alpha^*)$ that $C \cap \big(\bigcup_{\ell=1}^m B_d(T_\ell, \sigma_\ell)\big) = \emptyset$, whence $D_d(T_\ell, C) \geq \sigma_\ell > \lambda_\ell$ for $\ell = 1, \ldots, m$; thus, we conclude that $C \in \mathscr{A}^+_d(T_1, \ldots, T_m; \lambda_1, \ldots, \lambda_m)$. Therefore, we may choose $\mathscr{A}^+_d(T_1, \ldots, T_m; \lambda_1, \ldots, \lambda_m)$ as the desired element $\mathscr{U}_i$ of $\mathfrak{B}$ such that $C \in \mathscr{U}_i \subseteq \mathscr{A}^+_d(S_i; \varepsilon_i)$. $\square$

Using the fact that two families $\mathscr{S}$ and $\mathscr{T}$ induce the same upper gap topologies if and only if their saturations $\Gamma(\mathscr{S})$ and $\Gamma(\mathscr{T})$ coincide, in view of Theorem 3.6, we get the following corollary.

Corollary 4.8 *The following conditions are equivalent:*

1) $\big(\mathscr{P}_0(X), \mathsf{G}^+_{\mathscr{S},d}\big)$ is second countable;

2) there exists a countable subfamily $\mathscr{D}$ of $\mathscr{S}$ such that $\Gamma(\mathscr{D}) = \Gamma(\mathscr{S})$;

3) there exists a countable subfamily $\mathscr{D}$ of $\mathscr{P}_0(X)$ such that $\Gamma(\mathscr{D}) = \Gamma(\mathscr{S})$.

Corollary 4.9 *The following conditions are equivalent:*

1) X is separable;

2) W^+_d is first countable;

3) W^+_d is second countable.

Proof. With the definition of the Γ operator in mind, this follows from Theorem 4.6 and Theorem 4.7. $\square$

Proposition 4.10 *The upper Hausdorff pseudometric topology on the hyperspace of a metric space (X,d) is second countable if and only if the space (X,d) is totally bounded.*

Proof. The "if" part is well-known; therefore, we assume that (X,d) is not totally bounded, and prove that $\left(\mathscr{P}_0(X), \mathsf{H}_d^+\right)$ is not second countable.

Actually, since (X,d) is not totally bounded, there exist a $\vartheta > 0$ and a countably infinite subset D of X such that

$$\forall x, y \in D\colon \big(x \neq y \Longrightarrow d(x,y) \geq \vartheta\big). \tag{4;22}$$

Let $\mathscr{E}$ be an uncountable collection of infinite subsets of D such that

$$\forall E_1, E_2 \in \mathscr{E}\colon \big(E_1 \neq E_2 \Longrightarrow (E_1 \nsubseteq E_2 \ \wedge\ E_2 \nsubseteq\ E_1)\big) \tag{4;23}$$

(for instance, we may choose as $\mathscr{E}$ an uncountable almost-disjoint family on D[1]). We know that the upper Hausdorff pseudometric topology on $\mathscr{P}_0(X)$ coincides with the topology $\mathsf{G}^+_{\mathscr{S},d}$, when taking as $\mathscr{S}$ the collection $\mathscr{P}_0(X)$ of all nonempty subsets of X. Therefore, the second countability of $\left(\mathscr{P}_0(X), \mathsf{H}_d^+\right)$ is equivalent to the existence of a countable subcollection $\mathscr{D}$ of $\mathscr{P}_0(X)$ such that $\mathscr{P}_0(X) \subseteq \Gamma(\mathscr{D})$. Now, if we assume towards a contradiction that $\left(\mathscr{P}_0(X), \mathsf{H}_d^+\right)$ is second countable, then in particular it will be possible to associate to every $E \in \mathscr{E}$ some $S_{E,1}, \ldots, S_{E,n(E)} \in \mathscr{D}$ and some $\lambda_{E,1}, \ldots, \lambda_{E,n(E)}, \sigma_{E,1}, \ldots, \sigma_{E,n(E)}$ with $0 < \lambda_{E,i} < \sigma_{E,i}$ for $1 \leq i \leq n(E)$, such that

$$B_d\left(E, \frac{\vartheta}{2}\right) \subseteq \bigcup_{i=1}^{n(E)} B_d\left(S_{E,i}, \lambda_{E,i}\right) \subseteq \bigcup_{i=1}^{n(E)} B_d\left(S_{E,i}, \sigma_{E,i}\right) \subseteq B_d\left(E, \vartheta\right).$$

Picking, for every $E \in \mathscr{E}$ and every $i \in \{1, \ldots, n(E)\}$, a rational number $r_{E,i}$ such that $\lambda_{E,i} < r_{E,i} < \sigma_{E,i}$, we have the inclusions:

$$B_d\left(E, \frac{\vartheta}{2}\right) \subseteq \bigcup_{i=1}^{n(E)} B_d\left(S_{E,i}, r_{E,i}\right) \subseteq B_d\left(E, \vartheta\right). \tag{4;24}$$

We may consider the map:

$$E \mapsto \left(S_{E,1}, \ldots, S_{E,n}, r_{E,1}, \ldots, r_{E,n}\right)$$

[1]Remember that a collection $\mathscr{E}$ of infinite subsets of a countably infinite set D is said to be *almost-disjoint* if the intersection of any two distinct elements of $\mathscr{E}$ is finite. It is well-known that on every countably infinite set there is an almost-disjoint family having the cardinality of the continuum (see, for instance, [23, Theorem 1.3])

as a function from $\mathscr{E}$ to $\mathscr{D}^{<\omega} \times \mathbb{Q}^{<\omega}$. Since the latter set is countable while the former is not, there must exist two distinct $E_1, E_2 \in \mathscr{E}$ with $n(E_1) = n(E_2)$ and $S_{E_1,i} = S_{E_2,i}$, $r_{E_1,i} = r_{E_2,i}$ for every $i \in \{1, \ldots, n(E_1)\}\,(= \{1, \ldots, n(E_2)\}\,)$. Letting $M = \bigcup_{i=1}^{n(E_1)} B_d\,(S_{E_1,i}, r_{E_1,i}) = \bigcup_{i=1}^{n(E_2)} B_d\,(S_{E_2,i}, r_{E_2,i})$, it follows from the first inclusion in (4; 24), for $E = E_1$, that

$$B_d\left(E_1, \frac{\vartheta}{2}\right) \subseteq M,$$

while the last inclusion in (4; 24) gives, for $E = E_2$:

$$M \subseteq B_d\,(E_2, \vartheta);$$

thus we conclude that $E_1 \subseteq B_d\left(E_1, \frac{\vartheta}{2}\right) \subseteq M \subseteq B_d\,(E_2, \vartheta)$. However, this leads to a contradiction, because by (4; 23) the set $E_1 \backslash E_2$ is nonempty, but (according to (4; 22)) no element of $E_1 \backslash E_2$ can be in the ϑ-enlargement of E_2. $\square$

5 Applications

In this final section we use the operator Γ to prove some structural properties of metric spaces. Our first result is in part a remake of Theorem 5.5 of [12] in terms of the operator Γ which shifts the burden of the proof from the base space X to the hyperspace $\mathscr{P}_0(X)$.

Theorem 5.1 *Let (X, d) be a metric space. The following conditions are equivalent:*

(a) (X, d) is totally bounded;

(b) $\Gamma(\mathscr{F}_0(X)) = \mathscr{P}_0(X)$;

(c) $\mathsf{W}_{\mathsf{d}}^+ = \mathsf{H}_d^+$;

(d) the Wijsman topology and the proximal topology coincide.

Proof. By Remark 1.1 and Theorem 3.4, the equality $\Gamma(\mathscr{F}_0(X)) = \mathscr{P}_0(X)$ is equivalent to $\mathsf{W}_d^+ = \mathsf{H}_d^+$, i.e., (b) and (c) are equivalent. Further, if (b) holds, then W_d^+ is first countable because H_d^+ always is, and applying Corollary 4.9, X is separable. Thus W_d^+ is second countable and so is H_d^+. Applying Proposition 4.10, (X, d) is totally bounded. On the other hand if (X, d) is totally bounded, then $\mathsf{W}_d^+ = \mathsf{H}_d^+$ (see, for instance, [21, Proposition 2.5]); to obtain condition (b), we apply Theorem 3.4. Finally, conditions (c) and (d) are equivalent by Proposition 2.7 $\square$

The weaker condition $\downarrow \Gamma(\mathscr{F}_0(X)) = \mathscr{P}_0(X)$ is equivalent to X being d-bounded. This is recorded in the following more comprehensive result.

Theorem 5.2 *Let (X, d) be a metric space. The following conditions are equivalent* *(a)* $\downarrow \Gamma(\mathscr{F}_0(X)) = \mathscr{P}_0(X)$*;*

(b) (X, d) *is bounded;*

(c) $\mathsf{AW}^+_\mathsf{d} = \mathsf{H}^+_d$*;*

(d) the bounded proximal topology and the proximal topology coincide.

Proof. If (X, d) is bounded it is clear that $X \in \Gamma(\mathscr{F}_0(X))$ and therefore $\downarrow \Gamma(\mathscr{F}_0(X)) = \mathscr{P}_0(X)$. If, conversely, $\downarrow \Gamma(\mathscr{F}_0(X)) = \mathscr{P}_0(X)$, then X can be covered by a finite family of open balls and is thus bounded. The equivalence of (b) and (c) is due to Theorem 3.4 and the fact that either condition ensures $\Gamma(\mathscr{B}_d(X)) = \Gamma(\mathscr{P}_0(X)) = \mathscr{P}_0(X)$. Finally, (c) and (d) are equivalent because both $\mathscr{B}_d(X)$ and $\mathscr{P}_0(X)$ contain the singletons. $\square$

The next result generalizes Theorem 3.1.4 in [4] (notice how the introduction of the operator Γ allows such a generalization):

Theorem 5.3 *Let* (X, d) *be a metric space. The following conditions are equivalent:*

(a) $\mathscr{T}\mathscr{B}_d(X) = \mathscr{B}_d(X)$*;*

(b) $\Gamma(\mathscr{F}_0(X)) = \mathscr{B}_d(X)$*;*

(c) $\mathsf{W}^+_d = \mathsf{AW}^+_d$*;*

(d) AW^+_d *is second countable;*

(e) the Wijsman topology and the bounded proximal topology coincide.

Proof. Condition (a) implies (b) by the last property in Theorem 3.6. If condition (b) holds, then by idempotency of the Γ operator, $\Gamma(\mathscr{F}_0(X)) = \Gamma(\mathscr{B}_d(X))$ also holds, so (c) holds by Theorem 3.4. If $\mathsf{W}^+_d = \mathsf{AW}^+_d$, by Corollary 4.3, W^+_d is first countable, and X is therefore separable by Theorem 4.6. Thus W^+_d and AW^+_d are second countable by Corollary 4.9. To see that condition (d) implies (a), assume now second countability of AW^+_d and suppose there exists a subset T of X which is bounded but not totally bounded. The proof presented in Proposition 4.10 applies to T and we reach a contradiction because of the previous theorem. The equivalence of conditions (c) and (e) is argued as before. $\square$

Corollary 5.4 *Suppose* (X, d) *is an almost convex metric space; then the above five conditions are equivalent to the following one:*

(f) $\Gamma(\mathscr{F}_0(X))$ *is hereditary*

Proof. In almost convex spaces balls belong to $\Gamma(\mathscr{F}_0(X))$. $\square$

Note that the previous corollary is valid in any metric space having the property that $\downarrow \Gamma(\mathscr{F}_0(X)) = \mathscr{B}_d(X)$ which is in turn equivalent to each ball of (X, d) being a subset of some set in $\Gamma(\mathscr{F}_0(X))$.

References

[1] H. Attouch, R. Lucchetti, and R. Wets. The topology of the ρ-Hausdorff distance. *Annali di Matematica Pura ed Applicata. Series IV*, 160: 303–320, 1992.

[2] G. Beer. Conjugate convex functions and the epi-distance topology. *Proceedings of the American Mathematical Society*, 108: 117–126, 1990.

[3] G. Beer. The slice topology: a viable alternative to Mosco convergence in nonreflexive spaces. *Nonlinear Analysis*, 19: 271-290, 1992.

[4] G. Beer. *Topologies on closed and closed convex sets.* Kluwer Academic Publishers Group, Dordrecht, 1993.

[5] G. Beer. Wijsman convergence of convex sets under renorming. *Nonlinear Analysis*, 22: 207–216, 1994.

[6] G. Beer. Bornological convergence: a natural generalization of Attouch-Wets convergence. In *Theory and Applications of Proximity, Nearness, and Uniformity* 45–70, G. Di Maio and S. Naimpally, eds., Seconda Università di Napoli, Caserta, 2009.

[7] G. Beer, C. Costantini, and S. Levi. Bornological convergence and shields. *Mediterranean Journal of Mathematics*, published online, November, 2011.

[8] G. Beer and A. DiConcilio. Uniform convergence on bounded subsets and the Attouch-Wets topology. *Proceedings of the American Mathematical Society*, 112: 235-243, 1991.

[9] G. Beer and S. Levi. Total boundedness and bornologies. *Topology and its Applications*, 156(7): 1271–1288, 2009.

[10] G. Beer and S. Levi. Pseudometrizable bornological convergence is Attouch-Wets convergence. *Journal of Convex Analysis*, 15: 439–453, 2008.

[11] G. Beer and S. Levi. Gap, excess and bornological convergence. *Set-Valued Analysis*, 16(4): 489–506, 2008.

[12] G. Beer, A. Lechicki, S. Levi and S. Naimpally. Distance functionals and suprema of hyperspace topologies. *Annali di Matematica Pura ed Applicata. Series IV*, 162: 367–381, 1992.

[13] G. Beer and R. Lucchetti. Weak topologies on the closed subsets of a metrizable space. *Transactions of the American Mathematical Society*, 335: 805–822, 1993.

[14] G. Beer, S. Naimpally, and J. Rodriguez-Lopez. $\mathscr{S}$-topologies and bounded convergences. *Journal of Mathematical Analysis and Applications*, 339(1): 542–552, 2008.

[15] J. Borwein and J. Vanderwerff. Further arguments for slice convergence in nonreflexive spaces. *Set-Valued Analysis*, 2: 529–544, 1994.

[16] B. Cornet. Topologies sur les fermés d'un espace métrique. *Cahiers de mathématiques de la décision* no. 7309, Université de Paris-Dauphine, 1973.

[17] C. Costantini. Every Wijsman topology relative to a Polish space is Polish. *Proceedings of the American Mathematical Society*, 123: 2569–2574, 1995.

[18] C. Costantini, S. Levi and J. Pelant. Infima of hyperspace topologies. *Mathematika*, 42(1): 67–86, 1995.

[19] C. Costantini, S. Levi and J. Ziemińska. Metrics that generate the same hyperspace convergence. *Set-Valued Analysis*, 1(2): 141–157, 1993.

[20] R. Engelking. *General Topology.* Heldermann Verlag, Berlin, revised and completed edition, 1989.

[21] S. Francaviglia, A. Lechicki, and S. Levi. Quasi-uniformization of hyperspaces and convergence of nets of semicontinuous multifunctions. *Journal of Mathematical Analysis and Applications*, 112: 347-370, 1985.

[22] J.-L. Joly. Une famille de topologies sur l'ensemble des fonctions convexes pour lesquelles la polarité est bicontinue. *Journal de Mathématiques Pures et Appliquées. Neuvième Série*, 52: 421–441, 1973.

[23] K. Kunen. *Set Theory. An Introduction to Independence Proofs.* Van Nostrand, Amsterdam, 1980.

[24] A. Lechicki and S. Levi. Wijsman convergence in the hyperspace of a metric space. *Unione Matematica Italiana. Bollettino. B. Serie VII*, 7(I-B): 439–452, 1987.

[25] A. Lechicki, S. Levi and A. Spakowski. Bornological convergences. *Journal of Mathematical Analysis and Applications*, 297(2): 751–770, 2004.

[26] R. Lucchetti. *Convexity and well-posed problems.* Springer Verlag, Berlin, 2006.

[27] J.-P. Penot and C. Zalinescu. Bounded (Hausdorff) convergence: basic facts and applications. In F. Giannessi, A. Maugeri, editors: *Variational Analysis and Applications,* Kluwer Academic Publishers, Dordrecht, 2005.

[28] R. T. Rockafellar and R. Wets. *Variational Analysis.* 2^{nd} Edition. Springer Verlag, New York, 2004.