Wadge degrees

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Wadge Degrees

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Introduction

Descriptive set theory can be roughly described as the study of definable subsets of the Baire space $\omega$ which—as customary in this area—is identified with $\mathbb{R}$. The concept of “definable set” is a bit vague as it depends—essentially—on the amount of strong axioms of set theory that one is willing to accept: if we restrict ourselves to $\text{ZFC}$ alone, then we should diet on Borel sets only, but if we are willing to assume large cardinals, we might consider projective sets, or sets in $L(\mathbb{R})$, or even sets in larger inner models for determinacy. In any case, descriptive set theorists restrict their scope to some nice collection of sets which is immune from all pathologies spawned by the Axiom of Choice, e.g.: sets which are non-Lebesgue measurable, or do not have the property of Baire, etc. An obvious requirement is that such collection of sets (like the Borel sets, the projective sets, etc.) should be closed under continuous pre-images, and this is where the Wadge hierarchy comes into play. For $A, B$ subsets of $\mathbb{R}$, say that $A$ is continuously reducible to $B$ if $A$ is the continuous pre-image of $B$, in symbols $A \leq_W B$. Although the notion of continuous pre-image harks back to the dawn of general topology, and had been used in a variety of situations, it was William W. Wadge, a PhD student of John Addison in Berkeley in the late 60s–early 70s that studied first the structural properties of the relation $\leq_W$ per se. Wadge’s breakthrough relied on a reformulation of $\leq_W$ in terms of games, and it was during that time that games and determinacy hypotheses were becoming a standard tool in descriptive set theory. In Wadge’s own words [Wad83, p. 3]

Yet nowhere (to our knowledge) is the relation $A = f^{-1}(B)$ for some continuous $f$ ever explicitly defined and studied as a partial order, not even in exhaustive work such as Kuratowski (1958) or Sierpiński (1952). In the latter, Sierpiński discusses preimage in general, continuous image and homeomorphic image, but not (explicitly) continuous preimage, which is perhaps the most natural. One possible explanation is that the investigation of $\leq$ naturally involves infinite games, and it is only recently that game methods
have been fully understood and appreciated.¹

Using games, Wadge proved a simple, but fundamental result, now known as Wadge’s Lemma, which has ushered a whole slew of new results in descriptive set theory: Assume AD. Then for all \( A, B \subseteq \mathbb{R} \)

\[
A \leq_W B \quad \lor \quad \mathbb{R} \setminus B \leq_W A.
\]

The equivalence classes of the induced equivalence relation are called Wadge degrees and the pre-order \( \leq_W \) induces a partial order \( \leq \) on them. A degree is self-dual if it is the degree of a set which is Wadge reducible to its complement: for example the degree of a clopen set is self-dual, but the degree of an open, but non-clopen set is non-self-dual. Wadge showed that pairs of non-self-dual degrees and self-dual degrees alternate: immediately above a pair of non-self-dual degrees there is a self-dual degree and immediately above a self-dual degree there is a pair of non-self-dual degrees. Wadge’s Lemma says that there cannot be three mutually incomparable degrees—and hence the antichains have length at most 2. In other words: if we coalesce each non-self-dual degree with its dual, then \( \leq \) becomes a linear order. Wadge then embarked on a thorough analysis of the degrees of Borel sets, showing that the relation \( \leq \) is well-founded. Therefore, to every Borel subset of \( \mathbb{R} \) an ordinal (its Wadge rank) measuring the complexity of the set can be assigned: if \( A \leq B \) then the Wadge rank of \( A \) is no greater than the Wadge rank of \( B \). Wadge also characterized the Borel sets with limit Wadge rank: a set is self-dual if and only if its Wadge rank has countable cofinality. By 1972, Wadge had complete the herculean task of giving a complete analysis of the structure of \( \leq \) restricted to Borel sets, including a computation of the Wadge ranks of the sets involved. All these results were obtained assuming Borel determinacy (before this was proved by Martin in 1974) and are contained in his dissertation [Wad83] which did not appear until 1983.

In the meanwhile other mathematicians started working on Wadge degrees: Tony Martin, building on partial results of Leonard Monk, showed in 1973 that AD implies that \( \leq \) is well-founded, and hence every set of reals can be assigned an ordinal measuring its complexity. (For this reason \( \leq \) is often called the Wadge hierarchy.) The technique of the proof became known as the Martin-Monk method and has been applied time and again in the study of Wadge degrees. For example a couple of years later John Steel and Robert Van Wesep (at the time both students in Berkeley) used the Martin-Monk method to generalize Wadge’s result to all sets (assuming AD) by showing

¹Kuratowski (1958) and Sierpiński (1952) are, respectively, the monographs [Kur58] and [Sie52], and \( \leq \) is our \( \leq_W \).
that a set of limit Wadge rank $\lambda$ is self-dual if and only if $\lambda$ is of countable co-
finality. In the forthcoming years many set theorists developed and extended
Wadge’s results turning the theory of Wadge degrees into a sophisticated area
of set theory. Unfortunately most of this material is not easily accessible (for
example Wadge’s thesis was never published and can only be retrieved at
Wadge’s own web page http://i.csc.uvic.ca/home/hei/WadgePhD.pdf),
and in any case there is no general introduction to the subject.

Which bring us to the reason for this manuscript. I wanted to collect and
organize some of the material that I’ve been studying/working on, so these
notes represent more of a personal journey than a treatise. It is certainly not
meant (nor was ever meant) to be a scholarly work. My apologies to all set
theorists who have worked in this area and whose work hasn’t been properly
presented (or hasn’t been presented at all): this does not mean I consider
certain results/topics unimportant or less important than the ones presented
here. Again, this is just a very personal account of the subject and I make
absolutely no claim of completeness.

Finally, it is not at all clear whether this manuscript will ever materialize
into a real book, so probably you should not refer to it in published journals,
but otherwise you can use and distribute it freely.

Prerequisites and Notation

We assume acquaintance with basic descriptive set theory—all undefined
notions can be found in [Kec95] or [Mos80]. Since we will work under vari-
ous determinacy assumptions, our base theory will be the Zermelo-Fraenkel
set theory $\text{ZF}$. We will state explicitly any (weak) form of the Axiom of
Choice, $\text{AC}$, used in the arguments—among such choice principles, the most
prominent are the Axiom of Countable Choices over the Reals:

$$(\text{AC}_?(\mathbb{R}))$$

If $\emptyset \neq A_n \subseteq \mathbb{R}$, then there is an $f : \omega \to \mathbb{R}$ such that $f(n) \in A_n$, for all $n$,

and the Axiom of Dependent Choices over the Reals:

$$(\text{DC}(\mathbb{R}))$$

If $R \subseteq \mathbb{R} \times \mathbb{R}$ is such that $\forall x \in \mathbb{R} \exists y \in \mathbb{R} (x, y) \in R$, and if $x_0 \in \mathbb{R}$,

then there is a sequence $\langle x_n \mid n \in \omega \rangle$ such that $(x_n, x_{n+1}) \in R$, for all $n$.

These two choice principles can be generalized by replacing $\mathbb{R}$ with a nonempty
set $X$ in the definitions above—these revised statements are denoted by
$\text{AC}_?(X)$ and $\text{DC}(X)$, respectively. $\text{AC}_?$ and $\text{DC}$ are the global versions of
these principles:

\[
\begin{align*}
(\text{AC}_\omega) & \quad \forall X (X \neq \emptyset \Rightarrow \text{AC}_\omega(X)) \\
(\text{DC}) & \quad \forall X (X \neq \emptyset \Rightarrow \text{DC}(X)).
\end{align*}
\]

Since we will refrain from using the full **Axiom of Choice**, \(\text{AC}\), care must be exerted when dealing with the notion of cardinality:

\[
|X| \leq |Y|
\]

means that there is an injection of \(X\) into \(Y\), and

\[
|X| = |Y|
\]

means that \(|X| \leq |Y| \land |Y| \leq |X|\), which by the Shröder–Bernstein theorem is equivalent to the assertion that there is a bijection between \(X\) and \(Y\).

“Finite” means “in bijection with a natural number”, so that “\(X\) is infinite” does not necessarily imply that “\(\omega \leq |X|\)”, unless, of course, the axiom of countable choices \(\text{AC}_\omega\) is assumed.

**Exercise 0.1.** Show that:

1. If \(X\) is the surjective image of \(Y\), then \(\text{AC}_\omega(Y) \Rightarrow \text{AC}_\omega(X)\) and \(\text{DC}(Y) \Rightarrow \text{DC}(X)\).

2. \(\text{AC}_\omega(\mathbb{R})\) implies that the countable union of countable sets of reals is countable, and that \(\omega_1\) is regular.

(These facts will often be used without specific acknowledgment).

Our terminology is quite standard, but for the reader’s convenience we list some of the notation that will be used throughout this book. The **identity function** on \(X\) is the map

\[
id_X : X \to X
\]

such that \(id_X(x) = x\) for every \(x \in X\). The symbols “\(\subseteq\)” and “\(\subset\)” stand for “inclusion” and “proper inclusion”, respectively, while

\[
\mathcal{P}(X) = \{Y \mid Y \subseteq X\}
\]

is the power-set of \(X\). If \(A\) is a subset of the domain of \(f\),

\[
f(A) = \{f(a) \mid a \in A\}
\]

is the pointwise image of \(A\). This should not be source of confusion since we will hardly ever deal with transitive sets, and in those rare cases we shall
use the notation \( f^A \). If \( z \) is a finite or \( \omega \)-sequence then \( \text{lh}(z) \) is the length of \( z \), i.e., its domain \( \text{dom}(z) \). For \( s \) a finite sequence and \( z \) a finite or an \( \omega \)-sequence,

\[
s \cdot z
\]
is the sequence obtained by concatenating \( s \) and \( z \), that is \( \text{lh}(s \cdot z) = \text{lh}(s) + \text{lh}(z) \) and for \( i < \text{lh}(s) + \text{lh}(z) \)

\[
(s \cdot z)(i) = \begin{cases} 
  s(i) & \text{if } i \in \text{dom}(s), \\
  z(i - \text{lh}(s)) & \text{otherwise}.
\end{cases}
\]

When \( z \) is of length 1, i.e. \( z = \langle x \rangle \), we will often write \( s \cdot x \) instead of \( s \cdot \langle x \rangle \), if there is no possible source of confusion; similarly, \( x \cdot s \) denotes \( \langle x \rangle \cdot s \). For any \( x \) the symbol \( \vec{x} \) denotes the \( \omega \)-sequence \( \langle x, x, x, \ldots \rangle \).

\[ N = \{0, 1, 2, \ldots \} \], the set of all natural numbers, is construed as \( \omega \), the first transfinite ordinal. If \( x, y \in \omega X \), then \( x \oplus y \) is the element \( z \in \omega X \) defined by

\[
z(2n) = x(n) \]
\[
z(2n + 1) = y(n).\]

Conversely, given a \( z \in \omega X \), then \( (z)_I \) and \( (z)_I \) are the unique elements of \( \omega X \) such that

\[
(0.2) \quad (z)_I \oplus (z)_I = z.
\]

These notions extend to finite sequences as well, but, obviously, when writing \( s \oplus t \) we implicitly assume that \( \text{lh}(t) \leq \text{lh}(s) \leq \text{lh}(t) + 1 \).

\( \text{Int} \) and \( \text{Cl} \) are, respectively, the interior and closure operators in a topological space. A subset \( X \) of a topological space is

**nowhere dense** if \( \text{Int}(\text{Cl}(X)) = \emptyset \),

**meager** if \( X = \bigcup_n X_n \) with \( X_n \) nowhere dense.

\( \text{MGR} \), the collection of all meager subsets of the given topological space, is closed under subsets and (assuming \( \text{AC} \)) under countable unions.

A topological space is **Polish** if it is separable and completely metrizable, i.e., it admits a complete metric compatible with its topology, and a **Polish metric space** is a complete metric space \( (X, d) \) such that the topology is separable. A topological space is **extremely disconnected** or **0-dimensional** if it has a basis of clopen sets. We will be mostly concerned with the **Baire space** \( \omega^\omega \) and with the **Cantor space** \( \omega^2 \), both of which are Polish and 0-dimensional.
Chapter I

Elementary Results on the Wadge Hierarchy

1 Prologue

If $\mathcal{X}$ and $\mathcal{Y}$ are topological spaces, and $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$, we say that $A$ is continuously reducible to $B$, in symbols

$$A \leq_{\mathcal{X},\mathcal{Y}} B,$$

just in case $A = f^{-1}(B)$ for some continuous $f : \mathcal{X} \to \mathcal{Y}$. The “W” in the sub-script is after William W. Wadge, and the formula above also reads: $A$ is Wadge reducible to $B$ between the spaces $\mathcal{X}$ and $\mathcal{Y}$. Any continuous function witnessing $A \leq_{\mathcal{X},\mathcal{Y}} B$ is called a (continuous) reduction of $A$ to $B$: checking whether “$x \in A$” reduces to the problem of verifying whether “$f(x) \in B$”. When $\mathcal{X} = \mathcal{Y}$, (1) is written as

$$A \leq_{\mathcal{X}} B.$$

The relation $\leq_{\mathcal{X}}$ is a pre-order (i.e., a reflexive, transitive relation) on $\mathcal{P}(\mathcal{X})$ and the intuition behind the definition of (2) is that

the set $A$ is no more complex than the set $B$.

Complexity here refers to some intuitive notion of topological intricacy: we require that no set can be strictly simpler than its complement. Thus if $A \leq_{\mathcal{X}} B$ holds, any upper bound for the complexity of $B$ yields an upper bound for the complexity of $A$ and — conversely — a lower bound for the complexity of $A$ yields a lower bound for the complexity of $B$. For example, if $B$ is Borel (or $F_\sigma$, closed, etc.), then also $A$ must be Borel (resp. $F_\sigma$,}
closed, etc), and if \( A \) is not Borel (or \( F_\sigma \), closed, etc.) then same can be said of \( B \). It is quite natural to ask whether the converse property holds, that is: suppose \( A \) is no more complex than \( B \), must \( A \) be the continuous pre-image of \( B \)? The answer, in general, is negative, as the following example shows.

**Example 1.1.** Let \( X = \mathbb{R} = (-\infty, +\infty) \) be the real line, let \( A = \mathbb{Q} \), let \( C \subseteq [0; 1] \) be the usual 1/3-Cantor set, and let \( B \subseteq C \). We claim that there is no continuous \( f: \mathbb{R} \to \mathbb{R} \) reducing \( A \) to \( B \): since \( B \) can be taken to be not in \( G_\delta \), and since \( A \in F_\sigma \), then \( \mathbb{R} \setminus B \not\leq_W A \), and this will witness the failure of Wadge’s Lemma for \( A \) and \( B \). Towards a contradiction, suppose \( A = f^{-1}(B) \) for some continuous \( f \). Then

\[
\mathbb{R} = \text{Cl}(\mathbb{Q}) = \text{Cl}(f^{-1}(B)) \subseteq f^{-1}\left(\text{Cl}(B)\right) \subseteq f^{-1}(C),
\]

where \( \text{Cl} \) denotes the closure. Thus \( \text{ran}(f) \subseteq C \). But \( \text{ran}(f) \) is connected, and since the connected components of \( C \) are its points, it follows that \( f \) is constant, and hence it cannot be a reduction of \( A \) to \( B \).

Since the same argument applies to any Polish space \( X \) containing a non-empty connected open set, we shall focus on totally disconnected (i.e., 0-dimensional) spaces. Since all 0-dimensional Polish spaces are — up to homeomorphism — closed subsets of \( \omega^\omega \) [Kec95, Theorem 7.8], it is natural to restrict our attention to the Baire space. When \( X = \mathbb{R} = \omega^\omega \), the superscript is dropped and we write

\[ A \leq_W B. \]

In this chapter we will show that the Axiom of Determinacy — introduced in section 2.C below — yields stringent structural properties for the pre-order \( \leq_W \), the most notable ones being that it is well-founded (Theorem 6.1) and that it has antichains of length at most 2 (Lemma 4.5).

## 2 Preliminaries

### 2.A Trees.

A **tree** \( T \) on a set \( X \) is a subset of \( \omega^\omega X \) closed under initial segments, that is \( t \in T \land s \subseteq t \Rightarrow s \in T \). The elements of \( T \) are called **nodes**. Two nodes \( s \) and \( t \) of \( \omega^\omega X \) are said to be **incompatible**, in symbols \( s \perp t \), if

\[ \exists i \in \text{dom}(s) \cap \text{dom}(t) \ (s(i) \neq t(i)). \]

\( \text{tn}(T) \) is the set of all **terminal nodes** of \( T \), that is, all \( t \in T \) such that there is no \( s \in T \) with \( s \supset t \). The **boundary** of \( T \) is the set

\[ \partial T = \{ s \in \omega^\omega X \mid s \notin T \land s \upharpoonright \text{lh}(s) - 1 \in T \}. \]
2. PRELIMINARIES

A branch of $T$ is a function $b: \omega \to X$ such that $b \upharpoonright n \in T$, for all $n \in \omega$. The body of the tree $T$ is the set of all branches of $T$

$$[T] = \{ b \in \omega X \mid \forall n \, (b \upharpoonright n \in T) \}.$$ 

A tree $T$ is

pruned if it has no terminal nodes, i.e., $\text{tn}(T) = \emptyset$,

perfect if every node has two incompatible extensions,

finite branching if every node has a finite number of immediate successors, i.e., for all $t \in T$

$$|\{u \in T \mid t \subset u \land \text{lh}(t) + 1 = \text{lh}(u)\}| < \omega.$$ 

If either $A \subseteq \omega X$ or $A \subseteq \omega^\omega X$, and $s \in \omega^\omega X$ let

$$s \upharpoonright A = \{ s \upharpoonright x \mid x \in A \}$$

and let

$$A_{[s]} = \{ x \mid s \upharpoonright x \in A \}.$$ 

The set $A_{[s]}$ is called the localization of $A$ at $s$. It is immediate to check that if $A \subseteq \omega X$ then $s \upharpoonright A, A_{[s]} \subseteq \omega X$, and that if $T$ is a tree on $X$ then $T_{[s]}$ is also trees on $X$. When $s = \langle x \rangle$ and there is no danger of confusion, we will write $x \upharpoonright A$ and $A_{[x]}$ instead of $\langle x \rangle \upharpoonright A$ and $A_{[\langle x \rangle]}$. We will be mostly concerned with the Baire space $\omega X$ and with the Cantor space $\omega^2$, both of which are Polish and 0-dimensional. In fact the Baire and Cantor spaces are particular cases of the following construction: given $X$ a discrete space (i.e., every subset is open), endow the set $\omega X$ with the product topology, i.e., the topology generated by the sets

$$N_s = N(\omega X; s) = \{ f \in \omega X \mid s \subseteq f \}$$

with $s \in \omega^\omega X$. A complete compatible metric is given by

$$d(f, g) = \begin{cases} 2^{-n} & \text{if } f \upharpoonright n = g \upharpoonright n \text{ and } f(n) \neq g(n), \\ 0 & \text{otherwise.} \end{cases}$$
CHAPTER I. ELEMENTARY RESULTS ON THE WADGE HIERARCHY

The space \( \omega X \) is separable just in case \( |X| \leq \omega \). Since
\[
\omega X \setminus N_s = \bigcup \{N_t \mid t \perp s\},
\]
it follows that each \( N_s \) is clopen and therefore \( \omega X \) is 0-dimensional. If \( T \) is a tree on \( X \) then \([T]\) is a closed subset of \( \omega X \) and, conversely, every closed subset of \( \omega X \) is of the form \([T]\) with \( T \) pruned. The function
\[
(6) \quad h_X : \omega X \times \omega X \to \omega X
\]
defined by
\[
(7) \quad h_X(f, g)(n) = \begin{cases} f(k) & \text{if } n = 2k, \\ g(k) & \text{if } n = 2k + 1, \end{cases}
\]
is a homeomorphism, and we will call it the canonical homeomorphism of \( \omega X \times \omega X \) with \( \omega X \). When \( X \) is clear from the context, we shall write \( h \) instead of \( h_X \). Note that \( X \subseteq Y \) implies that \( h_X = h_Y \mid \omega X \times \omega X \); in particular, the restriction to the Cantor space of the canonical homeomorphism of \( \mathbb{R} \times \mathbb{R} \) and \( \mathbb{R} \) yields the canonical homeomorphism of \( \omega 2 \times \omega 2 \) and \( \omega 2 \).

2. B  Lipschitz and continuous functions.

A function \( \varphi : S \to T \), where \( S \) and \( T \) are non-empty pruned trees, is said to be

- **monotone** if \( s \subseteq s' \Rightarrow \varphi(s) \subseteq \varphi(s') \),
- **Lipschitz** if it is monotone and \( \text{lh}(s) = \text{lh}(\varphi(s)) \),
- **continuous** if it is monotone and \( \lim_{n \to \infty} \text{lh}(\varphi(x \upharpoonright n)) = \infty \), for all \( x \in [S] \),
- **tame** if it is monotone and such that for all \( n \) and \( s \upharpoonright n \in S \),
\[
\varphi(s) \subseteq \varphi(s \upharpoonright n) \Rightarrow \text{lh}(\varphi(s \upharpoonright n)) = \text{lh}(\varphi(s)) + 1.
\]

If \( \varphi \) is monotone, then
\[
D_\varphi = \{ x \in [S] \mid \lim_{n \to \infty} \text{lh}(\varphi(x \upharpoonright n)) = \infty \}
\]
is a \( \Pi^0_2 \) subset of \( \omega X \) and hence of \( [S] \), and every \( \Pi^0_2 \) subset of \( \omega X \) is of this form. The function
\[
f_\varphi : D_\varphi \to [T] , \quad x \mapsto \bigcup_n \varphi(x \upharpoonright n)
\]
is continuous and for every continuous \( f : G \rightarrow [T] \), with \( G \) a \( \mathbf{P}_2 \) subset of \([S]\), there is a monotone, tame \( \varphi : S \rightarrow T \) such that \( D \varphi = G \) and \( f \varphi = f \) (see [Kec95, Proposition 2.6]). Clearly if \( \varphi : S \rightarrow T \) is Lipschitz then it is also continuous, and, for all \( x, y \in [S] \),

\[
x \mid n = y \mid n \Rightarrow f \varphi(x) \mid n = f \varphi(y) \mid n
\]

or equivalently

\[
d^{[T]}(f \varphi(x), f \varphi(y)) \leq d^{[S]}(x, y)
\]

where \( d^{[S]} \) and \( d^{[T]} \) are the two complete metrics induced on \([S]\) and \([T]\) given by (5). Thus \( f \varphi : [S] \rightarrow [T] \) is a Lipschitz function (in the usual sense of Real Analysis) with constant \( \leq 1 \). Conversely, every Lipschitz function \( f : [S] \rightarrow [T] \) with constant \( \leq 1 \) is of the form \( f \varphi \) with \( \varphi : S \rightarrow T \) Lipschitz. A function \( f : [S] \rightarrow [T] \) which is Lipschitz with constant \( \leq 1/2 \), i.e., such that

\[
x \mid n = y \mid n \Rightarrow f(x) \mid n + 1 = f(y) \mid n + 1
\]

it is called a **contraction** and it is induced by a monotone \( \varphi : S \rightarrow T \) such that \( \lh(\varphi(s)) = \lh(s) + 1 \). In this book, we make the following convention: For a function \( f : \mathbb{R} \rightarrow \mathbb{R} \), “Lipschitz” means “Lipschitz with constant \( \leq 1 \),” that is \( f = f \varphi \) for some Lipschitz \( \varphi \) as above. Similarly “\( f \) is a contraction” means “\( f = f \varphi \) for some contraction \( \varphi \).”

Since a continuous (Lipschitz) function \( \mathbb{R} \rightarrow \mathbb{R} \) is completely determined by a continuous (res. Lipschitz) \( \omega \rightarrow \omega \), and since these objects can be coded as subsets of \( \omega \), a map \( x \mapsto f_x \) can be defined so that

\[
\{f_x \mid x \in \mathbb{R}\}
\]

is the collection of all continuous (res. Lipschitz) functions \( f : \mathbb{R} \rightarrow \mathbb{R} \). We call such a map a **parametrization** or coding of the continuous (res. Lipschitz) functions. We now fix once and for all a parametrization of all Lipschitz functions: For any \( x \in \mathbb{R} \) let \( \varphi_x : \omega \rightarrow \omega \) be defined by

\[
\varphi_x(s) = t \iff \lh(s) = \lh(t) \land \forall n \forall i < \lh(s) (s_n = s \mid i + 1 \Rightarrow t(i) = x(n)),
\]

where

\[
(s_n \mid n \in \omega)
\]

is a standard enumeration without repetitions of \( \omega \setminus \{0\} \) such that \( s_n \subseteq s_m \Rightarrow n < m \). This is well-defined since \( s \mid i + 1 = s_n \) for a unique \( n \), and clearly each \( \varphi_x \) is Lipschitz. If \( x, x' \in \omega \) and \( x(n) \neq x'(n) \), then let \( i \) be
such that \( i + 1 = \ln(s_n) \): then \( \varphi_x(s_n) \neq \varphi_{x'}(s_n) \), since these two sequences differ on their last element

\[
(\varphi_x(s_n))(i) = x(n) \neq x'(n) = (\varphi_{x'}(s_n))(i).
\]

In other words: \( x \neq x' \Rightarrow \varphi_x \neq \varphi_{x'} \). Suppose now \( \varphi: \prec \omega \omega \rightarrow \prec \omega \omega \) is Lipschitz: if \( x(n) \) is the last element of \( \varphi(s_n) \), then \( \varphi = \varphi_x \). Therefore

\[
\mathbb{R} \rightarrow \{ \varphi \mid \varphi: \prec \omega \omega \rightarrow \prec \omega \omega \ \text{is Lipschitz} \}, \quad x \mapsto \varphi_x
\]

is a bijection. Letting \( \ell_x: \mathbb{R} \rightarrow \mathbb{R} \) be the Lipschitz functions induced by \( \varphi_x \), it is not hard to see that the evaluation map \( (x, y) \mapsto \ell_x(y) \) is continuous. Therefore we have shown that

**Lemma 2.1.** There is a bijection

\[
\mathbb{R} \rightarrow \{ f \in \mathbb{R} | f \text{ is Lipschitz} \}, \quad x \mapsto \ell_x
\]

such that

\[
\mathbb{R}^2 \rightarrow \mathbb{R}, \quad (x, y) \mapsto \ell_x(y)
\]

is continuous.

**Exercise 2.2.** There is no parametrization \( x \mapsto f_x \) of continuous functions such that the evaluation map \( \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto f_x(y) \), is continuous.

### 2.C Games.

Given a set \( A \subseteq \prec \omega X \), a game \( G(A) \) on \( X \) is defined where two player \( I \) and \( II \) take turns in picking points from \( X \)

\[
\begin{array}{c|ccccc}
I & x_0 & x_1 & \cdots \\
\hline
II & y_0 & y_1 & \cdots \\
\end{array}
\]

Player \( I \) wins just in case the sequence \( \langle x_0, y_0, x_1, y_1, \ldots \rangle \) is in \( A \). In other words, \( I \) and \( II \) cooperatively construct an element \( z \) of \( \prec \omega X \): \( I \)'s goal is to ensure \( z \in A \), while \( II \)'s goal is to ensure \( z \notin A \). We will say that \( x = \langle x_0, x_1, \ldots \rangle \) and \( y = \langle y_0, y_1, \ldots \rangle \) are the plays of \( I \) and \( II \), respectively, while \( x \oplus y \) is the complete play. A run of a game is organized in **rounds** or **innings** with \( I \) and \( II \) playing \( x_n \) and \( y_n \) in the \( n \)-th round. A **position** or **partial play** is simply a finite sequence \( p \in \prec \omega X \), and if it is of even length, we say it is the position after the \( n \)-th round or, equivalently, before
the \( n+1 \)-st round, where \( 2n = \text{lh}(p) \). The sequences \((p)_I\) and \((p)_II\) are called \( I \)'s and \( II \)'s positions associated with \( p \).

A **strategy for** \( I \) **is** any function \( \sigma : \omega X \to X \), and \( \sigma \) is said to be **winning** if
\[
\sigma \ast y = \langle \sigma(\emptyset), y_0, \sigma(\langle y_0 \rangle), y_1, \sigma(\langle y_0, y_1 \rangle), y_2, \sigma(\langle y_0, y_1, y_2 \rangle) \ldots \rangle \in A
\]
for any \( y \in \omega X \). Conversely, a **strategy for** \( II \) **is** any function \( \tau : \omega X \setminus \{\emptyset\} \to X \), and \( \tau \) is said to be **winning** if
\[
x \ast \tau = \langle x_0, \tau(\langle x_0 \rangle), x_1, \tau(\langle x_0, x_1 \rangle), x_2, \tau(\langle x_0, x_1, x_2 \rangle) \ldots \rangle \notin A
\]
for any \( x \in \omega X \). A play \( z \in \omega X \) is said to be **according to** \( \sigma \), strategy for \( I \) (or \( \tau \), strategy for \( II \)) iff \( \exists y \in \omega X (z = \sigma \ast y) \), (res. \( \exists x \in \omega X (z = x \ast \tau) \)).

Suppose \( \tau \) is a strategy for \( II \) for a game on \( X \) and define \( \hat{\tau}(s) \) by induction on \( \text{lh}(s) \)
\[
\hat{\tau}(\emptyset) = \emptyset \\
\hat{\tau}(s \hat{\tau}(x)) = \hat{\tau}(s \rtimes (\tau(s \hat{\tau}(x))))
\]
Then \( \hat{\tau} : \omega X \to \omega X \) is Lipschitz, and, conversely, if \( \varphi : \omega X \to \omega X \) is Lipschitz then
\[
\hat{\varphi}(s) = \varphi(s)(\text{lh}(s) - 1) = \text{the last element of } \varphi(s)
\]
is a strategy for \( II \). Since \( \tau \mapsto \hat{\tau} \) and \( \varphi \mapsto \hat{\varphi} \) are inverse operations, the distinction between Lipschitz functions and strategies for \( II \) will often be blurred. Similarly, strategies for \( I \) can be identified with contractions. Let \( \sigma \) and \( \tau \) be strategies for \( I \) and \( II \), respectively, for games on \( X \):
\[
\sigma \ast \tau
\]
is the element of \( \omega X \) obtained by pitting \( \sigma \) against \( \tau \), i.e., is the unique play according to \( \sigma \) and to \( \tau \). Since the play \( \sigma \ast \tau \) lies either inside or outside a given set \( A \), it follows that in any game at most one player can have a winning strategy. Our definition requires that a strategy acts against the sequence of moves provided by the opponent, but at times it is more convenient to think of a strategy as a recipe suggesting what to play next at any given position where we are to make a move. Therefore we could define strategies for \( I \) and \( II \) as maps
\[
\bar{\sigma} : \bigcup_n^{2n} X \to X \quad \text{and} \quad \bar{\tau} : \bigcup_n^{2n+1} X \to X.
\]
A \( z \in \omega X \) is according to \( \bar{\sigma} \) iff \( \forall n (z_{2n} = \bar{\sigma}(z | 2n)) \), and \( \bar{\sigma} \) is winning for \( I \) in \( G(A) \) just in case every play according to \( \bar{\sigma} \) is in \( A \). (The definition of \( \bar{\tau} \) being winning for \( II \) is analogous.)
Exercise 2.3. Show that the two definitions of strategies are equivalent. In particular, show that \( \sigma \) (or \( \tau \)) is a winning in \( G(A) \) iff the same is true of \( \bar{\sigma} \) (res. \( \bar{\tau} \)).

There is another, apparently more general notion of game on a set \( X \). Let \( T \) be a non-empty pruned tree on \( X \) and let \( A \subseteq [T] \): the game \( G(T; A) \) is just like \( G(A) \), except that both players must move so that the partial play is always in \( T \), the first player to break this rule losing outright. A legal position is just a node of \( T \), and a move that extends a legal position to another legal position is called a legal move. The definition of strategy must be modified accordingly so that they yield only legal moves. Note that all games of the form \( G(A) \) can be considered as games in this new sense, \( G(<\omega X; A) \).

Exercise 2.4. Show that any game \( G(T; A) \) is equivalent to a game \( G(\tilde{A}) \), for some \( \tilde{A} \subseteq <\omega X \), in the sense that either player that has a winning strategy in \( G(T; A) \) has also a winning strategy in \( G(\tilde{A}) \), and conversely.

A game is determined if one (and only one) of the two players admits a winning strategy. The Axiom of Determinacy for games on \( X \) is the statement

\[(\text{AD}_X) \quad \text{For every } A \subseteq \omega X \text{ the game on } X \text{ with pay-off } A \text{ is determined.}\]

Exercise 2.5. Show that \(|X| \leq |Y|\) implies \( \text{AD}_Y \Rightarrow \text{AD}_X \).

If \( X \) is empty or a singleton, then \( \text{AD}_X \) is trivial. When \( X = \omega \) this is the usual Axiom of Determinacy, \( \text{AD} \), and by a result of Mycielski, \( \text{AD}_2 \Rightarrow \text{AD} \). Therefore if \( 2 \leq \alpha < \omega_1 \) then by Exercise 2.5

\[\text{AD}_\alpha \iff \text{AD} .\]

The Axiom of Real Determinacy \( \text{AD}_R \) is (much) stronger than \( \text{AD} \). By deep results of Woodin, assuming large cardinals, both \( \text{AD} \) and \( \text{AD}_R \) are consistent with \( \text{ZF+DC(R)} \) — see [Kan03, pp. 464–468]. On the other hand, both \( \text{AD}_{\omega_1} \) and \( \text{AD}_{\omega(R)} \) are inconsistent with \( \text{ZF} \) — see Exercise 10.2.

\( \text{AD} \) contradicts the Axiom of Choice since, for example, it implies many regularity properties for sets of reals which are known to be inconsistent with \( \text{AC} \) (see section 2.D). Yet \( \text{AD} \) is consistent with weak forms of choice.

Lemma 2.6. \( \text{AD} \Rightarrow \text{AC}_\omega(R) \).

Proof. Given \( \emptyset \neq A_n \subseteq R \) consider the game on \( \omega \) where \( \text{I} \) plays \( n \) (and then his moves are irrelevant) and \( \text{II} \), in order to win, must construct a real in \( A_n \). Then \( \text{I} \) cannot have a winning strategy, and any strategy for \( \text{II} \) yields a choice function for the \( A_n \)s. \( \square \)
2. PRELIMINARIES

On the other hand it is open whether AD implies DC(\( \mathbb{R} \)) — see also the Open Problem 6.4.

**Open problem 2.7.** Does AD \( \Rightarrow \) DC(\( \mathbb{R} \))?  

Kechris has shown [Kec84] that the answer is affirmative under the additional assumption that V = L(\( \mathbb{R} \)), showing thus that the consistency of AD + DC from that of AD, i.e.,

\[ \text{Con}(\text{ZF} + \text{AD}) \Rightarrow \text{Con}(\text{ZF} + \text{AD} + \text{DC}). \]

AD does not imply neither DC nor AC\(_\omega\) by results of [Sol78] and Woodin (unpublished). By unpublished results of Woodin’s the theory \( \text{ZF} + \text{AD} + \neg\text{DC}(\mathbb{R}) \) proves the consistency of \( \text{ZF} + \text{AD}_\mathbb{R} \).

Although AD\(_X\) is inconsistent for any \( X \) containing \( \omega_1 \), the same need not to be true for the weaker statement \( \text{Det}_X(\Gamma) \) requiring only the determinacy of games with payoff sets in some given pointclass \( \Gamma \subseteq \mathcal{P}(\omega \times X) \),

(\( \text{Det}_X(\Gamma) \)) For every \( A \in \Gamma \) the game on \( X \) with pay-off \( A \) is determined.

The pointclass \( \Gamma \) is usually some collection of sets defined in topological terms, e.g., the Borel pointclasses \( \Sigma^0_n, \Pi^0_n, \) and \( \Delta^0_n \), or the projective pointclasses \( \Sigma^1_n, \Pi^1_n, \) and \( \Delta^1_n \). Gale and Stewart [?], showed that AC and \( \forall X \text{ Det}_X(\Pi^0_1) \) are equivalent over the base theory ZF. Thus ZFC proves the determinacy of all games with closed pay-off, on any space \( \omega \times X \). This was extended by Martin [Mar75, ?] to \( \Delta^1_1 \) — for uncountable \( X \) the pointclass \( \Delta^1_1 \) is strictly larger than the collection of Borel sets. Martin’s result is optimal (in ZFC) since the determinacy of games on \( \omega \) with \( \Pi^1_1 \) payoff implies the existence of \( 0^\sharp \) [?].

2.D Regularity properties.

We will now look at three regularity properties for sets of reals: perfect subsets, the property of Baire, and Lebesgue measurability.

The **perfect subset property** is the statement that every set of real numbers is either countable or else it contains a non-empty, perfect closed set. Since any non-empty perfect closed set contains a homeomorphic copy of the Cantor space, the perfect subset property can be stated as

\[ (\text{PSP}) \quad \forall X \subseteq \mathbb{R} (|X| \leq \omega \vee \exists f: ^\omega 2 \to X, f \text{ continuous and injective}). \]

PSP contradicts choice: if \( \mathbb{R} \) is well-orderable then there is a **Bernstein set**, that is a set \( B \subseteq \mathbb{R} \) such that \( B \cap P \neq \emptyset \) and \( P \setminus B \neq \emptyset \) for every non-empty perfect closed set \( P \), [Kec95, Example 8.24]. Clearly the existence of Bernstein sets contradicts PSP.
Lemma 2.8. PSP implies there is no $\omega_1$-sequence of distinct real numbers.

Proof. Towards a contradiction suppose $f: \omega_1 \to \mathbb{R}$ is injective. Then $\text{ran}(f)$, being uncountable, must contain a copy $C$ of the Cantor space. Then $C$, being a subset of $\text{ran}(f)$, is well-orderable, and so is $\mathbb{R}$, since it is in bijection with $C$: a contradiction. \hfill \Box

Since closed sets are determined by pruned trees on $\omega$, and since trees can be identified via their characteristic functions with a real, it follows that PSP implies there is no $\omega_1$-sequence of closed or open sets.

A set $A \subseteq \mathbb{R}$ is said to have the property of Baire if there is an open set $U$ such that $A \Delta U \in \mathbb{MGR}$. We abbreviate the expression “all sets have the property of Baire” with BP. Just as with the perfect subset property, BP implies that $\mathbb{R}$ cannot be well-ordered, and hence it contradicts AC.

The Lebesgue measure on $\mathbb{R}^2$ is the completion $\mu$ of the measure defined on the basic open sets by $\mu(\mathcal{N}(\omega_2; s)) = 2^{-\text{lh}(s)}$, and it is often called the coin-tossing measure. The statement “all subsets of $\mathbb{R}^2$ are Lebesgue measurable” is abbreviated by LM, and by the isomorphism theorem for measures [Kec95, Theorem 17.41] it is equivalent to the statement that every subset of $\mathbb{R}$ is Lebesgue measurable. (Here $\mathbb{R}$ is the usual Euclidean line with the traditional Lebesgue measure studied in analysis.) By a well known theorem of Vitali, if $\mathbb{R}$ is well-orderable, then LM fails. On the other hand, a weak form of the Axiom of Choice, $\text{AC}_\omega(\mathbb{R})$, is needed for the construction of the Lebesgue measure on $\mathbb{R}$ (or for that matter: any non-discrete measure on an uncountable Polish space): the crucial issue is the verification that the outer measure $\mathcal{P}(\mathbb{R}) \to (0; \infty)$

$$A \mapsto \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) \mid \langle I_n \mid n \in \omega \rangle \text{ is a sequence of half-open intervals such that } A \subseteq \bigcup_n I_n \right\}$$

where $\ell(I)$ is the length of the interval $I$, is indeed $\sigma$-subadditive — see [Fre00, Proposition 114D].

If $D$ is non-principal ultrafilter on $\omega$ then, identifying each set of natural numbers with its characteristic function, $D$ can be construed as a subset of $\omega^2$: under this identification, $D$ is an example of a non-measurable set without the property of Baire [Kec95, Exercise 8.50].

The three regularity properties are consistent with and independent of ZF. We summarize here a few known facts

- assuming $\mathbb{R}$ is well orderable we have that $\neg\text{PSP}$, $\neg\text{BP}$, $\neg\text{LM}$;


\textbf{10CHAPTER I. ELEMENTARY RESULTS ON THE WADGE HIERARCHY}
by a celebrated result of Solovay [Sol70], assuming the existence of an inaccessible cardinal it is consistent that
\[ \mathbb{L}(\mathbb{R}) \models DC + PSP + BP + LM \]

both PSP and LM imply the consistency of an inaccessible cardinal;

Shelah in [She84] showed that \( \text{Con}(\text{ZF}) \Rightarrow \text{Con}(\text{ZF} + DC + BP) \), and therefore there is a model satisfying DC + BP + ¬PSP + ¬LM;

Di Prisco and Todorcevic [DPT98] constructed a model of \( \text{ZF} + DC + PSP \) containing a non-principal ultrafilter on \( \omega \), and therefore in this model BP and LM fail.

**Definition 2.9.** A set \( F \subseteq \omega^2 \) is a **flip set** if the truth of the statement “\( z \in F \)” flips between true and false every time the value of \( z \) in one coordinate is changed, that is
\[ \forall z, w \in \omega^2 (\exists k \; z(k) \neq w(k) \Rightarrow (z \in F \iff w \notin F)) . \]

FS is the statement: there is a flip set.

Thus if \( F \) is a flip set and \( |\{ k \mid z(k) \neq w(k)\}| = n \) then
\[
\begin{align*}
n \text{ even} & \Rightarrow (z \in F \iff w \in F) , \\
n \text{ odd} & \Rightarrow (z \in F \iff w \notin F) .
\end{align*}
\]

**Proposition 2.10.** The existence of a non-principal ultrafilter on \( \omega \) implies FS.

**Proof.** For any \( z \in \omega^2 \) let \( \hat{z} \in \omega^2 \) be defined by
\[
\begin{align*}
\hat{z}(0) & = z(0) \\
\hat{z}(n+1) & = \hat{z}(n) + z(n+1)
\end{align*}
\]

where + denotes addition modulo 2. Let \( N = \omega \times \{0, 1\} \) and let
\[ A_z = \{(n, \hat{z}(n)) \in N \mid n \in \omega \} . \]

If \( n_0 \) is the unique natural number such that \( z(n_0) \neq w(n_0) \), then
\[ \forall m \geq n_0 \; ((m, i) \in A_z \iff (m, i + 1) \in A_w) , \]
that is: $A_z$ and $A_w$ partition $N \setminus n_0 \times 2$. Since $N$ is in bijection with $\omega$ we may assume that there is a non-principal ultrafilter $D$ on $N$, and by the equation above

$$A_z \in D \iff A_w \notin D$$

whenever $z$ and $w$ differ in exactly one coordinate. Therefore

$$F = \{z \in \omega^2 \mid A_z \in D\}$$

is a flip-set.

Therefore if $\mathbb{R}$ is well-orderable, then $FS$ holds.

**Proposition 2.11.** No flip set has the property of Baire.

**Proof.** Towards a contradiction, suppose $F$ is a flip set with the property of Baire. Since $\omega^2 \setminus F$ is also a flip set, by the Baire category theorem $F$ and $\omega^2 \setminus F$ cannot be both meager or both co-meager. Therefore — possibly replacing $F$ with its complement — we may assume that $F$ is non-meager. Let $U$ be open in $\omega^2$ and such that $F \triangle U$ is meager. Since $F \notin \text{MGR}$ then $U \neq \emptyset$, so pick $n$ and $s \in \omega^2$ such that $N(\omega^2; s) \subseteq U$. Let $f_n: \omega^2 \to \omega^2$ be the Lipschitz homeomorphism

$$f_n(z)(i) = \begin{cases} 1 - z(i) & \text{if } i = n, \\ z(i) & \text{otherwise.} \end{cases}$$

Then $f_n(N_s) = N_s$ and $F$ is co-meager in $N_s$, hence $\omega^2 \setminus F = f_n(F)$ is co-meager in $N_s$, contradicting the Baire category theorem.

**Proposition 2.12 (ACω(\mathbb{R})).** No flip set is Lebesgue measurable.

**Proof.** Towards a contradiction, suppose $F$ is a Lebesgue measurable flip set. Let $s$ be any finite sequence and let $n = \text{lh}(s)$. Since $\neg F = f_n(F)$, where $f_n$ is as in (10), and since each $f_n$ preserves the measure, then $\mu(F \cap N_s) = \mu(N_s \setminus F) = 1/2 \mu(N_s)$. In particular $F$ is non-null, hence $\mu(N_s \cap F) = \mu(N_s)$ for some $s$: a contradiction.

The Axiom of Determinacy implies all these regularity properties: PSP, BP, and LM (see [Kan03, pp.373–377]). Moreover they are all provable in ZFC if restricted to the realm of analytic sets.
3. Reducibilities

In order to study the theory of Wadge reducibility, it is often necessary to study other forms of reducibilities (like Lipschitz reducibility, or Borel reducibility). Let $X$ be a topological space and let $\mathcal{F} \subseteq \mathcal{X}$ be a family of functions closed under composition and containing the identity and all constant functions $c_x$, i.e.,

$$f, g \in \mathcal{F} \Rightarrow f \circ g \in \mathcal{F},$$

$$\text{id}_\mathbb{R} \in \mathcal{F},$$

$$c_x \in \mathcal{F} \quad (x \in X),$$

where $c_x$ is the constant function $x \in X$. For $A, B \subseteq X$, say that $A$ is $\mathcal{F}$-reducible to $B$, in symbols

$$A \leq^X_{\mathcal{F}} B,$$

just in case

$$\exists f \in \mathcal{F} \forall x \in X \left( x \in A \iff f(x) \in B \right)$$

for some $f \in \mathcal{F}$. By our hypothesis $\leq^X_{\mathcal{F}}$ is reflexive and transitive. Set

$$A \equiv_{\mathcal{F}} B \quad \iff \quad A \leq_{\mathcal{F}} B \land B \leq_{\mathcal{F}} A,$$

and

$$A <_{\mathcal{F}} B \quad \iff \quad A \leq_{\mathcal{F}} B \land B \not\leq_{\mathcal{F}} A.$$ 

An $\mathcal{F}$-degree is an equivalence class of $\equiv_{\mathcal{F}}$, and

$$[A]_\mathcal{F} = \{ B \mid B \equiv_{\mathcal{F}} A \}$$

is the $\mathcal{F}$-degree of $A$. Notice that

$$A \leq_{\mathcal{F}} B \quad \iff \quad \neg A \leq_{\mathcal{F}} \neg B,$$

where for $A \subseteq X$

$$\neg A = X \setminus A$$

is the complement of $A$ in $X$. A set $A$ is $\mathcal{F}$-self-dual iff $A \leq_{\mathcal{F}} \neg A$ iff $A \equiv_{\mathcal{F}} \neg A$, otherwise it is $\mathcal{F}$-non-self-dual. Since self-duality is invariant under $\equiv_{\mathcal{F}}$, it can be applied to $\mathcal{F}$-degrees as well. The dual of $[A]_\mathcal{F}$ is $[\neg A]_\mathcal{F}$, and a pair of distinct degrees of the form $\{[A]_\mathcal{F}, [\neg A]_\mathcal{F}\}$ is a non-self-dual pair. The pre-order $\leq_{\mathcal{F}}$ induces a partial order $\leq$ on the $\mathcal{F}$-degrees:

$$[A]_\mathcal{F} \leq [B]_\mathcal{F} \quad \iff \quad A \leq_{\mathcal{F}} B.$$
Similarly define

\[ [A]_\mathcal{F} < [B]_\mathcal{F} \iff [A]_\mathcal{F} \leq [B]_\mathcal{F} \land [A]_\mathcal{F} \neq [B]_\mathcal{F} \iff A <_\mathcal{F} B. \]

If \( \mathcal{F} \subseteq \mathcal{G} \) are sets of functions like in (11), then the pre-order \( \leq_\mathcal{G} \) is coarser than \( \leq_\mathcal{F} \), hence

\[
\begin{align*}
(12) & \quad A \leq_\mathcal{F} B \Rightarrow A \leq_\mathcal{G} B \\
(13) & \quad A \text{ is } \mathcal{F}\text{-self-dual} \Rightarrow A \text{ is } \mathcal{G}\text{-self-dual} \\
(14) & \quad [A]_\mathcal{F} \subseteq [A]_\mathcal{G}.
\end{align*}
\]

Notice that \([\mathcal{X}]_\mathcal{F} = \{\mathcal{X}\} \text{ and } [\emptyset]_\mathcal{F} = \{\emptyset\} \text{ form a non-self-dual pair, and since } \mathcal{F} \text{ has all constant functions they are } <_{\text{least}} \text{ among } \mathcal{F}\text{-degrees. We say that } [A]_\mathcal{F} \text{ is a successor degree if there is a } [B]_\mathcal{F} < [A]_\mathcal{F} \text{ such that for no } C \subseteq \mathbb{R} \text{ we have } [B]_\mathcal{F} < [C]_\mathcal{F} < [A]_\mathcal{F}. \text{ (In this case the degree } [B]_\mathcal{F} \text{ is an immediate predecessor of } [A]_\mathcal{F}. \) If an \( \mathcal{F}\)-degree is not a successor and it is neither \([\mathcal{X}]_\mathcal{F}\) nor \([\emptyset]_\mathcal{F}\), then we say it is a limit degree. A limit degree is said to be of countable cofinality if it is the least upper bound of an increasing sequence \([A_0]_\mathcal{F} < [A_1]_\mathcal{F} < \ldots\) of \( \mathcal{F}\)-degrees; otherwise it is said to be of uncountable cofinality.

If \( \mathcal{F} \) is the collection of all continuous functions, the “\( \mathcal{F} \)” in all subscripts is replaced by “\( \mathcal{W} \),” while if \( \mathcal{F} \) is the collection of all Lipschitz functions, we use “\( \mathcal{L} \).” If \( \mathcal{F} \) is the set of all functions from \( \mathcal{X} \) to \( \mathcal{X} \), then the structure of the \( \mathcal{F}\)-degrees is trivial, since there are only three degrees: \([\mathcal{X}]_\mathcal{F}, [\emptyset]_\mathcal{F}, \text{ and } \mathcal{P}(\mathcal{X}) \setminus \{\mathcal{X}, \emptyset\}\). On the other hand, AD implies that for any reasonable \( \mathcal{F} \subseteq \mathbb{R} \) there is no largest \( \mathcal{F}\)-degree — Lemma 4.7 below.

## 4 Lipschitz and Wadge games

The Lipschitz game for \( A \) and \( B \)

\[
G_L(A, B)
\]

with \( A, B \subseteq {}^\omega X \), is the game in which I and II play elements of \( X \) \( a_i \) and \( b_i \), respectively,

\[
\begin{array}{c}
\text{I} \\
\hline
a_0 & a_1 & \ldots \\
\text{II} & b_0 & b_1
\end{array}
\]

and player II wins just in case

\[
a = \langle a_i \mid i \in \omega \rangle \in A \iff b = \langle b_i \mid i \in \omega \rangle \in B.
\]
4. LIPSCHITZ AND WADGE GAMES

In the literature the game $G_L$ is often called Wadge game rather than Lipschitz game, but we will use that name for another game to be introduced in section 7.

**Exercise 4.1.** Show that $G_L(A, B)$ is indeed a game on $X$ in the sense of section 22.C.

The notation $G_L(A, B)$ is a tad ambiguous: if $A, B \subseteq \omega^X$ and $X \subseteq Y$, then we may consider the game $G_L(A, B)$ as a game on $X$ or as a game on $Y$, and these can be quite different: for example, if $A, B \subseteq \omega^2 \subseteq \omega^\omega$ with $A \neq \omega^2$ closed and $B = \omega^2$, then $\Pi$ has a winning strategy in $G_L(A, B)$ if this is construed as a game on $\omega$, but not if it is construed as a game on $2$. Therefore whenever there is danger of confusion we will adopt the following convention: if $A, B \subseteq [T]$ where $T$ is a pruned tree, then

$$G^T_L(A, B)$$

is the game described above with the moves restricted to $T$.

Fix a space $\omega^X$. A winning strategy for $\Pi$ in $G_L(A, B)$ is — essentially — a Lipschitz function $f : \omega^X \rightarrow \omega^X$ such that $A = f^{-1}(B)$, hence

$$\Pi \text{ wins } G_L(A, B) \iff A \leq_L B \tag{15}$$

Conversely, a strategy for $I$ yields a contraction thus, in particular a Lipschitz function $f : \omega^X \rightarrow \omega^X$ such that $\neg B = f^{-1}(A)$. Therefore

$$I \text{ wins } G_L(A, B) \Rightarrow \neg B \leq_L A \tag{16}$$

In this case we cannot revert the implication since a Lipschitz function witnessing $\neg B \leq_L A$ isn’t necessarily a contraction.

**Exercise 4.2.** Let $A, B \subseteq \omega^X$.

(a) Let $\tau$ be a winning strategy for $\Pi$ in $G_L(A, B)$ and suppose $p$ and $q$ are positions for $I$ and $\Pi$, respectively, according to $\tau$, that is $q = (p \ast \tau)_I$. Then $A_{[p]} \leq_L B_{[q]}$ and $\bar{\tau}(s) = \tau(p^\ast s)$ is winning for $\Pi$ on $G_L(A_{[p]}, B_{[q]})$.

(b) If $\sigma$ is a winning strategy for $I$ in $G_L(A, B)$, then $\tau = \sigma \setminus \{(\emptyset, \sigma(\emptyset))\}$ is a winning strategy for $\Pi$ in $G_L(B, \neg A_{[\sigma(\emptyset)]})$. Conversely, for any $x \in X$, if $\tau$ is a winning strategy for $\Pi$ in $G_L(B, \neg A_{[x]})$, then $\sigma = \tau \cup \{(\emptyset, x)\}$ is a winning strategy for $I$ in $G_L(A, B)$.

The **Wadge game** for $A$ and $B$,

$$G_W(A, B),$$
is the variant of the Lipschitz game where \( \Pi \) is allowed to pass (i.e., not to play) at any round, but he must play infinitely often otherwise he loses. Formally, this is a game on \( X \cup \{ p \} \), where \( p \) (=the passing move) is an object that does not belong to \( X \). An example of a run of \( GW(A, B) \) is:

\[
\begin{array}{cccccccc}
\Pi & a_0 & a_1 & \ldots & a_n & a_{n+1} & \ldots & a_m & \ldots \\
\text{II} & p & p \ldots p & b_0 & p & p \ldots p & b_1 & \ldots
\end{array}
\]

As in the case of \( GL \), \( \Pi \) wins just in case

\[
a = \langle a_i \mid i \in \omega \rangle \in A \iff b = \langle b_i \mid i \in \omega \rangle \in B.
\]

A strategy for either player for \( GW \) is called a **Wadge strategy**, and by decree, a strategy for \( \Pi \) in must play infinitely often on any given input. Although \( GW(A, B) \) is a game on \( X \cup \{ p \} \), it is more convenient to think of it as a game on \( X \): all moves which are not \( p \) are called active moves, and the position \( q \) of \( \Pi \) at round \( n \) is the sequence of active moves of \( \Pi \) up to round \( n \), disregarding all \( p \)'s (thus \( \text{lh}(q) < n \) is possible). The sequence of length \( n \) recording all of \( \Pi \)'s moves (including passing) is called the **true position** of \( \Pi \) at round \( n \), and has, obviously, length \( n \).

**Exercise 4.3.** Show that if \( \omega \leq |X| \), then the Wadge game for the space \( \omega X \) can be formulated as a game on \( X \).

Just like the case of Lipschitz games, \( GW^T(A, B) \) is the Wadge game for \( A, B \subseteq [T] \), where the moves of the players are restricted to \( T \), a pruned tree on \( X \).

Since a strategy \( \tau \) for \( \Pi \) always produces a sequence with infinitely many entries different from \( p \), the map

\[
\hat{\tau}: \vartriangleleft X \rightarrow \vartriangleleft X, \quad s \mapsto \text{the sequence of active moves of } \tau \text{ on input } s
\]

is continuous. Formally:

\[
\hat{\tau}(s) = \begin{cases} 
\emptyset & \text{if } s = \emptyset, \\
\hat{\tau}(s \upharpoonright \text{lh}(s) - 1) \downharpoonright \tau(s) & \text{if } \tau(s) \neq p, \\
\hat{\tau}(s \upharpoonright \text{lh}(s) - 1) & \text{otherwise},
\end{cases}
\]

Conversely every continuous \( \varphi: \vartriangleleft X \rightarrow \vartriangleleft X \) yields a strategy \( \tau \) for \( \Pi \) in the Wadge game on \( X \) such that \( \hat{\tau} = \varphi \) — just take

\[
\tau(s) = \begin{cases} 
\emptyset & \text{if } s = \emptyset, \\
\varphi(s)(\text{lh}(s) - 1) & \text{if } s \neq \emptyset \text{ and } \varphi(s) \neq \varphi(s \upharpoonright \text{lh}(s) - 1), \\
p & \text{otherwise}.
\end{cases}
\]
For this reason we will often blur the distinction between continuous functions and strategies for $\mathbb{II}$ in Wadge games. On the other hand a strategy for $\mathbb{I}$ in a Wadge game yields a whole family of contractions, since player $\mathbb{II}$ can reveal his moves as slowly as he wants. Summarizing:

\[
\begin{align*}
\mathbb{II} & \text{ wins } G_W(A, B) \iff A \subseteq_W B \tag{18} \\
\mathbb{I} & \text{ wins } G_W(A, B) \Rightarrow \neg B \subseteq_L A. \tag{19}
\end{align*}
\]

The following is the analogue of Exercise 4.2(a) for $G_W$.

**Exercise 4.4.** Let $\tau$ be a winning strategy for $\mathbb{II}$ in $G_W(A, B)$ and suppose $p$ and $q$ are positions for $\mathbb{I}$ and $\mathbb{II}$, respectively, according to $\tau$, that is $q = (p \ast \tau)_{\mathbb{II}}$. Then $A[p] \leq_W B[q]$ and $\tilde{\tau}(s) = \tau(p^s)$ is winning for $\mathbb{II}$ on $G_W(A[p], B[q])$.

The statement that all Lipschitz games on the space $\omega X$ are determined:

\[
(\text{AD}^L_X) \quad \forall A, B \subseteq \omega X \quad G^L_{\omega X}(A, B) \text{ is determined,}
\]

is called **Lipschitz determinacy for the space $\omega X$**, or simply Lipschitz determinacy $(\text{AD}^L)$, when $X = \omega$. Similarly

\[
(\text{AD}^W_X) \quad \forall A, B \subseteq \omega X \quad G_W(A, B) \text{ is determined,}
\]

is called **Wadge determinacy**. Clearly $\text{AD}_X$ implies $\text{AD}^L_X$ and $\text{AD}^W_X$, hence Lipschitz determinacy is a consequence of the Axiom of Determinacy. Just like $\text{AD}_X$, the principles $\text{AD}^L_X$ and $\text{AD}^W_X$ are inconsistent if $\omega_1 \leq |X|$ — see Chapter II, Proposition 11.5. For this reason we will concentrate from now on to the case $X = \omega$. Whether $\text{AD}$ follows from $\text{AD}^L$ is an open problem — see section ??.

Wadge observed the following simple, but crucial fact:

**Lemma 4.5** (Wadge’s Lemma). Assume $\text{AD}$, or even just $\text{AD}^L$. Then for every $A, B \subseteq \mathbb{R}$

\[
A \subseteq_L B \lor \neg B \subseteq_L A. \tag{20}
\]

Clearly (20) implies

\[
A \subseteq_W B \lor \neg B \subseteq_W A, \tag{21}
\]

which is also a consequence of $\text{AD}^W$. Wadge’s Lemma implies that if $A \not\subseteq_L B$ and $B \not\subseteq_L A$, then $\neg B \subseteq_L A$ and $\neg A \subseteq_L B$ and hence $A \equiv_L \neg B$. In
other words: if \(A\) and \(B\) are \(L\)-incomparable, then \(\{[A]_L, [B]_L\}\) is a non-
self-dual pair, and hence if \(C\) is a set not equivalent to either \(A\) or \(B\) (i.e., \(C \notin [A]_L \cup [B]_L\)) then either

\[ C <_L A \land C <_L B \]
or else

\[ A <_L C \land B <_L C. \]

Thus the ordering induced on the Lipschitz degrees is almost a linear-order (if each degree is identified with its dual, then it is indeed linear) and for this reason (20) — the thesis of Wadge’s Lemma — and (21) are called the Semi-Linear Ordering Principle for Lipschitz maps and Semi-Linear Ordering Principle for continuous maps, respectively, and are denoted by

\[ \text{SLO}^L \text{ and } \text{SLO}^W. \]

(We will show in Chapter IV that \(\text{SLO}^W \Leftrightarrow \text{SLO}^L \Leftrightarrow \text{AD}^L\).) More generally, if \(\mathcal{F}\) is a family of functions on a space \(\mathcal{X}\) as in (11), the notion of Semi-Linear Ordering Principle for \(\mathcal{F}\)-maps is

\[(\text{SLO}^\mathcal{F}(\mathcal{X})) \quad \forall A, B \subseteq \mathcal{X} \left( A \leq^\mathcal{F} B \lor \neg B \leq^\mathcal{F} A \right),\]

with the understanding that \(\text{SLO}^\mathcal{F}\) means \(\text{SLO}^\mathcal{F}(\mathbb{R})\). If \(\mathcal{F} \subseteq \mathcal{G}\) then, by (12),

\[(22) \quad \text{SLO}^\mathcal{F}(\mathcal{X}) \Rightarrow \text{SLO}^\mathcal{G}(\mathcal{X}).\]

In particular, if \(\mathcal{F}\) contains all Lipschitz functions and \(\mathcal{X} = \mathbb{R}\), then \(\text{SLO}^\mathcal{F}\) is a consequence of \(\text{AD}^L\), and hence of \(\text{AD}\). Notice that if \(A\) is \(\mathcal{F}\) self-dual, then \(\text{SLO}^\mathcal{F}\) implies that \([A]_\mathcal{F}\) is a node of the ordering of the \(\mathcal{F}\)-degrees, that is it is comparable with all other \(\mathcal{F}\)-degrees

\[(23) \quad [B]_\mathcal{F} \neq [A]_\mathcal{F} \Rightarrow ([B]_\mathcal{F} < [A]_\mathcal{F} \lor [A]_\mathcal{F} < [B]_\mathcal{F}).\]

In order to study the Semi-Linear Ordering principle of different \(\mathcal{X}\)’s, it is convenient to introduce the following terminology: the \(\mathcal{F}\)-structure of \(\mathcal{X}\) is the model-theoretic structure

\[ \langle \mathcal{P}(\mathcal{X}), \neg^\mathcal{X}, \leq^\mathcal{X} \rangle \]

where \(\neg^\mathcal{X}\) is a 1-ary operation of taking complements, \(\neg^\mathcal{X}A = \mathcal{X} \setminus A\). When \(\mathcal{F}\) is the set of all continuous (Lipschitz) functions, we will speak of Wadge structure (Lipschitz structure). Suppose \(\mathcal{X}\) is a retract of \(\mathcal{Y}\), that is, \(\mathcal{X}\) is a closed subset of \(\mathcal{Y}\) and there is a continuous surjection \(\pi: \mathcal{Y} \to \mathcal{X}\) which
is the identity on $\mathcal{X}$. (The map $\pi$ is called a retraction of $\mathcal{Y}$ onto $\mathcal{X}$.) We claim that the map $\Phi: \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{Y})$, $\Phi(A) = \pi^{-1}(A)$ is an embedding of the Wadge structure of $\mathcal{X}$ into the Wadge structure of $\mathcal{Y}$: it is easy to check that it is an injective map that preserves complementation, so it is enough to show that

$$A \leq^W \mathcal{X} B \iff \Phi(A) \leq^W \Phi(B).$$

If $f: \mathcal{X} \to \mathcal{X}$ is continuous and $A = f^{-1}(B)$, then $f \circ \pi: \mathcal{Y} \to \mathcal{X} \subseteq \mathcal{Y}$ witnesses $\Phi(A) \leq^W \Phi(B)$; conversely, if $g: \mathcal{Y} \to \mathcal{Y}$ is continuous and $\Phi(A) = g^{-1}(\Phi(B))$, then $\pi \circ g \mid \mathcal{X}: \mathcal{X} \to \mathcal{X}$ witnesses $A \leq^W \mathcal{X} B$. Therefore we have shown

**Proposition 4.6.** If $\pi: \mathcal{Y} \to \mathcal{X}$ is a retraction, then the map $A \mapsto \pi^{-1}(A)$ is an embedding of the Wadge structure of $\mathcal{X}$ into the Wadge structure of $\mathcal{Y}$. In fact

$$\text{SLO}^W(\mathcal{Y}) \Rightarrow \text{SLO}^W(\mathcal{X}).$$

(The last statement is immediate since $\text{SLO}^W$ is a $\forall$-formula in the language for Wadge structures, and therefore down-ward absolute.) If $S \subseteq T$ are pruned trees, then by [Kec95, Proposition 2.8] (see also (6)) there is a Lipschitz retraction of $[T]$ onto $[S]$, hence $\text{SLO}^L$ implies $\text{SLO}^L(\sim 2)$ and, more generally, $\text{SLO}^L([T])$ for any pruned tree $T$ on a countable set.

If $\mathcal{F} = \mathbb{R} \mathcal{R}$ then $\text{SLO}^\mathcal{F}$ holds for trivial reasons, but if $\mathcal{F}$ is a reasonable collection of functions, then $\text{SLO}^\mathcal{F}$ becomes highly non-trivial: for example, if we restrict ourselves to continuous functions, then the semi-linear ordering principle implies the prefect set property — see Corollary 11.4. The next result shows that if $\mathcal{F}$ is not too large, then there is a uniform way to construct from a set $A \subseteq \mathbb{R}$, a new set which is $<_\mathcal{F}$ larger than $A$ and $\neg A$. Thus, in particular, there is no largest $\mathcal{F}$-degree.

**Lemma 4.7.** Let $\mathcal{F} \subseteq \mathbb{R} \mathcal{R}$ be as in (11) and suppose $\mathcal{F}$ is the surjective image of $\mathbb{R}$ and that $\text{SLO}^\mathcal{F}$ holds. Then there is a map $J = J_\mathcal{F}: \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ such that

$$\forall A \subseteq \mathbb{R} (A <_\mathcal{F} J(A) \land \neg A <_\mathcal{F} J(A)).$$

**Proof.** If $x \mapsto f_x$ is a surjection $\mathbb{R} \to \mathcal{F}$, let

$$J(A) = \{0^\wedge x \mid f_x(0^\wedge x) \notin A\} \cup \{1^\wedge x \mid f_x(1^\wedge x) \in A\},$$

where $\{f_x \mid x \in \mathbb{R}\} = \mathcal{F}$. By diagonalization $J(A) \leq^\mathcal{F} A$ or $J(A) \leq^\mathcal{F} \neg A$ are impossible, hence by $\text{SLO}^\mathcal{F}$, $A, \neg A <_\mathcal{F} J(A)$. \qed
Remarks 4.8. If we replace $\mathbb{R}$ with $\omega X$, then everything so far goes through with minor or no modifications at all, but, of course, we must replace $\mathsf{AD}_X$ with $\mathsf{AD}_X$ in Lemma 4.5, and must assume that $X$ has at least two elements in Lemma 4.7. This last assumption is not an issue since if $X$ is empty or a singleton, then $\omega X$ is utterly uninteresting. Let us take a closer look at the spaces $\omega X$.

(a) If $|X| = n$, then $\omega X$ is homeomorphic to the Cantor space, $\omega 2$, and $\mathsf{AD}_X$ is equivalent to $\mathsf{AD}$.

(b) If $\omega \leq |X| < \omega_1$, then $\omega X$ is homeomorphic to the Baire space, $\omega \omega = \mathbb{R}$, and, again, $\mathsf{AD}_X$ is equivalent to $\mathsf{AD}$.

(c) If $|X| = |\mathbb{R}|$, then $\omega X$ is non-separable, yet $\mathsf{AD}_\mathbb{R}$ yields a theory of its Lipschitz/Wadge degrees completely analogous to the theory of Lipschitz/Wadge degrees for the Baire space.

(d) If $\omega_1 \leq |X|$, then $\mathsf{AD}_X$ is inconsistent. Yet it still might be the case that the analogue of Wadge’s Lemma might hold for spaces like $\omega \omega_1$. In other words, one could ask if (it is consistent that) for any $A, B \subseteq \omega 1$, it is the case that either $A \leq L B$ or $\omega 0 \cap B \leq L A$. We shall show in Chapter II, Proposition 11.5 that this is not the case.

5 Lipschitz degrees

In this section we start our systematic study of Lipschitz degrees.

Lemma 5.1. Suppose $A \leq L B$ and $s, t \in \omega \omega$ are sequences of the same length. Then $s^\omega A \leq L t^\omega B$.

Proof. $\mathsf{II}$ wins $G_L(s^\omega A, t^\omega B)$ as follows. As long as $\mathsf{I}$ enumerates $s$ then $\mathsf{II}$ enumerates $t$, with the understanding that if $\mathsf{I}$ deviates from $s$ at some round $n < \text{lh}(s)$, then $\mathsf{II}$ deviates from $t$ at the same round $n$ and then plays whatever he likes, say, 0. If after the $\text{lh}(s)$-round the two positions are $s$ and $t$, then $\mathsf{II}$ applies his winning strategy in $G_L(A, B)$ from now on. \qed

The preceding proof — like most proofs involving determinacy — is cast in mundane terms and uses freely many intuitions and suggestions borrowed from games that people play in everyday life. Some readers may feel a bit uneasy at first with this game-theoretic jargon, and may want to translate our proofs into a more classical lexicon. As a template, the proof above can be reworded as follows: A formal — but much less intelligible — rewording would be:
Let $\tau$ be a winning strategy for $\mathbb{II}$ in $G_L(A, B)$. Then
\[
\hat{\tau}(u) = \begin{cases} 
 t(lh(u) - 1) & \text{if } u \subset s, \\
 t(lh(u) - 1) + 1 & \text{if } u \not\subset lh(u) - 1 \subset s, \text{ but } u \not\subset s, \\
 0 & \text{if } \exists k < lh(s)(u(k) \neq s(k) \land k + 1 < lh(u)), \\
 \tau(v) & \text{if } u = s^\sim v.
\end{cases}
\]

is a winning strategy for $\mathbb{II}$ in $G_L(s^\sim A, t^\sim B)$.

We feel that this translation is much less intelligible than the original proof since it obscures the simple ideas behind it, and therefore in this book the game-theoretic jargon will always be used.

**Exercise 5.2.** Show that
(i) $A \leq_L s^\sim A$,
(ii) if $lh(s) \leq lh(t)$ and $A \leq_L B$ then $s^\sim A \leq_L t^\sim B$,
(iii) $A[s] \leq_L A$,
(iv) if $A \leq_L \neg A$ then $s^\sim A \leq_L \neg(s^\sim A)$ for all $s$.

Thus $A \leq_L 0^\sim A$, and if $A$ is Lipschitz self-dual, the inequality is strict.

**Lemma 5.3.** Suppose $A \leq_L \neg A$. Then $A <_L 0^\sim A$. Moreover, assuming $AD^L$, $[0^\sim A]_L$ is the immediate successor of $[A]_L$: if $B <_L 0^\sim A$ then $B \leq_L A$.

**Proof.** Let $\tau$ be a winning strategy for $\mathbb{II}$ in $G_L(A, \neg A)$,
\[
\begin{array}{c}
\mathbb{I} & x_0 & \xrightarrow{\tau} & x_1 & \xrightarrow{\tau} & \cdots \\
\mathbb{II} & y_0 & \xrightarrow{\tau} & y_1 & \xrightarrow{\tau} & \cdots
\end{array}
\]

Then $\mathbb{I}$ wins $G_L(0^\sim A, A)$ by playing 0 and then following $\tau$,
\[
\begin{array}{c}
\mathbb{I} & 0 & \xrightarrow{\tau} & y_0 & \xrightarrow{\tau} & y_1 & \xrightarrow{\tau} & \cdots \\
\mathbb{II} & x_0 & \xrightarrow{\tau} & x_1 & \xrightarrow{\tau} & \cdots
\end{array}
\]

Therefore $\mathbb{II}$ does not have a winning strategy in $G_L(0^\sim A, A)$, hence $0^\sim A \not\neq_L A$, and therefore $A <_L 0^\sim A$.

Suppose now $B <_L 0^\sim A$. Since $0^\sim A$ is Lipschitz self-dual, this is equivalent by (23) to $0^\sim A \not\neq_L \neg B$. By $AD^L$ let $\sigma$ be a winning strategy for $\mathbb{I}$ in $G_L(0^\sim A, \neg B)$. Then $\mathbb{II}$ wins $G_L(B, A)$ by pretending that a 0 was played before the play started, and then following $\sigma$. In other words: $\mathbb{II}$’s winning strategy is $\sigma \upharpoonright (\omega \setminus \{0\})$. \qed
This should be contrasted with the continuous case: if \( x \in \neg A \) then the function \( \varphi: \omega \omega \rightarrow \omega \omega \)

\[
\varphi(s) = \begin{cases} 
\emptyset & \text{if } s = \emptyset, \\
 t & \text{if } s = 0 \land t, \\
x \mid \ln(s) & \text{otherwise,}
\end{cases}
\]

is continuous and witnesses \( 0^\omega A \leq_W A \). Therefore

(24) \( \forall s \in \omega (A \neq \mathbb{R} \Rightarrow s^\omega A \leq_W A) \).

**Proposition 5.4.** If \( A \leq_L \neg A \) then \( \forall n (A_{\lfloor n \rfloor} \leq_L A) \). Conversely, assuming \( \text{AC}_\omega(\mathbb{R}) + \text{AD}^L \), if \( \forall n (A_{\lfloor n \rfloor} <_L A) \), then \( A \leq_L \neg A \).

**Proof.** Suppose \( A \leq_L \neg A \) and \( A_{\lfloor n \rfloor} \equiv_L A \), for some \( n \in \omega \). Then \( A_{\lfloor n \rfloor} \) is Lipschitz self-dual hence \( A_{\lfloor n \rfloor} \leq_L n^\omega A_{\lfloor n \rfloor} \leq_L A \): a contradiction.

Vice versa, suppose \( \forall n (A_{\lfloor n \rfloor} <_L A) \). Since \( A \nless_L A_{\lfloor n \rfloor} \), choose \( \sigma_n \) a winning strategy for \( \mathbf{I} \) in \( G_L(A, A_{\lfloor n \rfloor}) \). Then \( \mathbf{II} \) wins \( G_L(A, \neg A) \) as follows: if \( \mathbf{I} \) plays \( n \) at round 0 then \( \mathbf{II} \) follows \( \sigma_n \) from now on

\[
\begin{array}{cccc}
\mathbf{I} & n & a_0 & a_1 \\
\mathbf{II} & b_0 = \sigma_n(0) & b_1 = \sigma_n(a_0) & b_2 = \sigma_n(a_0, a_1) \\
\vdots
\end{array}
\]

so that

\[
n^\omega a \in A \iff a \in A_{\lfloor n \rfloor} \iff b \in \neg A,
\]

hence \( A \) is Lipschitz self-dual. \( \square \)

We now introduce an infinitary operation on sets: for \( A_n \subseteq \mathbb{R} \) let

(25) \( \bigoplus_n A_n = \bigcup_n n^\omega A_n \)

and for \( A, B \subseteq \mathbb{R} \) let

(26) \( A \oplus B = \bigoplus_n A_n \)

where \( A_{2n} = A \) and \( A_{2n+1} = B \).

**Exercise 5.5.** (i) Show that \( A, B \leq_L A \oplus B \) and \( A_i \leq_L \bigoplus_n A_n \), for all \( i \in \omega \).

(ii) Show that \( A \leq_L A' \land B \leq_L B' \Rightarrow A \oplus B \leq_L A' \oplus B' \). Assuming \( \text{AC}_\omega(\mathbb{R}) \), show that \( A_n \leq_L A_n' \Rightarrow \bigoplus_n A_n \leq_L \bigoplus_n A_n' \).

(iii) Show that \( \neg (A \oplus B) \equiv_L \neg A \oplus \neg B \) and \( \neg \bigoplus_n A_n = \bigoplus_n \neg A_n \).

(iv) Show that \( A \oplus B \equiv_L B \oplus A \) and that \( \bigoplus_n A_n \equiv_L \bigoplus_n A_{j(n)} \) for any surjection \( j: \omega \rightarrow \omega \).
Thus the operations $\oplus$ and $\bigoplus$ can be extended to degrees:

**Definition 5.6.** Let $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$ be a family of functions satisfying (11) and containing all Lipschitz functions.\(^1\) Define

$$[A]_L \oplus [B]_L = [A \oplus B]_L$$
$$\bigoplus_n [A_n]_L = [\bigoplus_n A_n]_L.$$

It is easy to check that these operations are commutative.

If $I$ is a countable set, then let

$$\bigoplus_{i \in I} [A_i]_L = \bigoplus_n [A_{j(n)}]_L$$

for some/any surjection $j: \omega \to I$.

Suppose $\varphi_n: <\omega, \omega \to <\omega, \omega$ is continuous and witnesses that $A_n \leq_W A$. Then

$$\varphi(s) = \begin{cases} 0 & \text{if } s = \emptyset, \\ \varphi_n(t) & \text{if } s = n \cap t, \end{cases}$$

is continuous and witnesses $\bigoplus_n A_n \leq_W A$. Therefore $\text{AC}_\omega(\mathbb{R})$ implies that

$$\forall n \ (A_n \leq_W A) \Rightarrow \bigoplus_n A_n \leq_W A.$$

**Lemma 5.7.** $A \oplus \neg A$ is Lipschitz self-dual. Moreover if $\text{AD}^L$ holds, then $[A]_L \oplus [-A]_L$ is the least degree above $[A]_L$ and $[-A]_L$.

**Proof.** Consider the strategy $\tau$ for $\Pi$ that in the first round plays

- 0, if $I$ has played 1,
- 1, if $I$ has played 0,
- $k + 2$, if $I$ has played $k + 2,$

and then copies $I$’s moves from now on. Then $\tau$ is winning in $G_L(A \oplus \neg A, \neg(A \oplus \neg A))$.

Suppose now $A$ is Lipschitz non-self-dual. Then $A, \neg A <_L A \oplus \neg A$ hence it is enough to show that if $B <_L A \oplus \neg A$ then

$$B \leq_L A \vee B \leq_L \neg A.$$ 

\(^1\)The $\mathcal{F}$ that we will consider in the next few sections are the set of all Lipschitz functions and the set of all continuous functions.
By assumption II does not win $G_L (A \oplus \neg A, B)$ hence I has a winning strategy $\sigma$. If $B = \mathbb{R}$ then (28) follows at once, so we may as well assume otherwise. If in the first round $\sigma$ plays an integer $\neq 0, 1$, then I will not produce a real in $A \oplus \neg A$, hence II can win by enumerating an arbitrary real not in $B$. Therefore $\sigma(\emptyset) = 0$ or $\sigma(\emptyset) = 1$. For the sake of definiteness assume the former. Then $\sigma$ yields a winning strategy $\tau$ for II in $G_L (B, \neg A)$ by letting $\tau(s) = \sigma(0^s)$.

Lemma 5.8. Assume SLO$^L$ and AC$^\omega(\mathbb{R})$ and suppose $A_n \subseteq \mathbb{R}$ are such that

$$\forall n \exists m > n \left(A_n <_L A_m\right).$$

Then $\bigoplus_n A_n$ is Lipschitz self-dual and $A_i <_L \bigoplus_n A_n$ for each $i$. If moreover we assume AD$^L$ then $\left[\bigoplus_n A_n\right]$ is the least-upper bound of the $\left[A_i\right]$'s: if $A_n <_L B$ for all $n$, then $\bigoplus_n A_n \leq_L B$.

Proof. Let $f : \omega \to \omega$

$$f(n) = \text{ the least } m > n \text{ such that } A_n <_L A_m.$$  

By SLO$^L$ $A_n <_L \neg A_{f(n)}$ holds, so, by AC$^\omega(\mathbb{R})$, let us fix winning strategies $\tau_n$ for II in $G_L (A_n, \neg A_{f(n)})$. Then II wins $G_L (\bigoplus_n A_n, \neg \bigoplus_n A_n)$ as follows: in the first round, if I plays $n$ then II responds $f(n)$; after this first round II follows $\tau_n$.

Towards a contradiction, suppose $A_n \leq_L B$, for all $n$, and that $\bigoplus_n A_n \not\leq_L B$. Then player I has a winning strategy $\sigma$ in $G_L (\bigoplus_n A_n, B)$. Then $B \leq_L \neg A_n$ for some $n$, and since SLO$^L$ implies that $\neg A_n \not\leq_L A_{f(n)}$, a contradiction is reached.

The assumption (11) can be relaxed a bit, but cannot be completely removed: assuming $A_n = A$ for all $n$, if $A$ is non-self-dual, then $\bigoplus_n A_n$ is non-self-dual and if $A$ is self-dual, then $\bigoplus_n A_n \equiv_L 0^* A$ is self-dual, but it is not Lipschitz equivalent to $A$, the least upper bound of the $A_n$'s.

Exercise 5.9. (i) Show that if $A_n <_L A_{n+1}$, for all $n$, and $f : \omega \to \omega$ is strictly increasing, then $\bigoplus_n A_n \equiv_L \bigoplus_n A_{f(n)}$.

(ii) Suppose $\forall n \exists m \left(A_n \leq_L B_m\right)$ and $\forall n \exists m \left(B_n \leq_L A_m\right)$. Then $\bigoplus_n A_n \equiv_L \bigoplus_n B_n$.

Proposition 5.10. Assume AD$^L$ + AC$^\omega(\mathbb{R})$. A limit Lipschitz degree is self-dual if and only if it is of countable cofinality.
5. LIPSCHITZ DEGREES

Proof. If the \( A_n \)'s witness that \([A]_L\) is of countable cofinality, then Lemma 5.8 implies that \( \bigoplus_n A_n \equiv_L A \) is self-dual.

Suppose \([A]_L\) is self-dual, so that by Lemma 5.4 \( A_{[n]} \leq_L A \) for all \( n \). Let 
\[
B_0 = A_{[0]} \text{ and }
\]
\[
B_{n+1} = \begin{cases} 
A_{[n+1]} & \text{if } B_n <_L A_{[n+1]}, \\
0^\infty (B_n \oplus \neg B_n) & \text{otherwise.}
\end{cases}
\]

By induction \( A_{[n]} \leq_L B_n \leq_L A \) and \( B_n <_L B_{n+1} \), hence \( \bigoplus_n B_n \) is Lipschitz self-dual and \( A = \bigoplus_n A_{[n]} \leq_L \bigoplus_n B_n \).

But \( \bigoplus_n B_n \) is the least upper bound of the \( B_n \leq_L A \) and thus \( \bigoplus_n B_n \leq_L A \). Therefore \( \bigoplus_n B_n \equiv_L A \), and the \([B_n]_L\)'s witness that \([A]_L\) is of countable cofinality.

Lemma 5.3 shows that if \([A]_L\) is self-dual, then \([0^\infty A]_L\) is its immediate successor and it is self-dual. Therefore we can define the successor operator \( \mathcal{L} \) on self-dual Lipschitz degrees
\[
(30) \quad \mathcal{L}([A]_L) = [0^\infty A]_L.
\]

This operation can be iterated through the countable ordinals:
\[
\begin{align*}
\mathcal{L}^0([A]_L) &= [A]_L, \\
\mathcal{L}^{\alpha+1}([A]_L) &= \mathcal{L}(\mathcal{L}^\alpha([A]_L)), \\
\mathcal{L}^\lambda([A]_L) &= \bigoplus_n [A_n]_L, & \text{if } \lambda \text{ is limit},
\end{align*}
\]

where \( A_n \in \mathcal{L}^{\alpha_n}([A]_L) \), and the \( \alpha_n \)'s are increasing and cofinal in \( \lambda \). Thus above a self-dual Lipschitz degree there is an \( \omega_1 \)-sequence of self-dual Lipschitz degrees.

Remark 5.11. (a) If in the definition of \( \mathcal{L}^\lambda([A]_L) \) we choose another increasing cofinal sequence \( \alpha'_n \) and sets \( A'_n \in \mathcal{L}^{\alpha'_n}([A]_L) \), then \( \bigoplus_n A_n \equiv_L \bigoplus_n A'_n \) by Exercise 5.9, so the operation \( \mathcal{L}^\lambda \) is well-defined.

(b) In order to choose the \( A_n \in \mathcal{L}^{\alpha_n}([A]_L) \) we only need to appeal to \( \text{AC}_\omega(\mathbb{R}) \). To see this it is enough to verify by induction on \( \alpha < \omega_1 \) that
\[
(31) \quad B \in \mathcal{L}^\alpha([A]_L) \Rightarrow B \equiv_W A.
\]

In fact if (31) holds at \( \lambda \) and \( \{f_x \mid x \in \mathbb{R} \} \) is an enumeration of all continuous functions, then
\[
X_n = \{x \in \mathbb{R} \mid f_x^{-1}(A) \in \mathcal{L}^{\alpha_n}([A]_L)\}.
\]
is non-empty, so by $\mathsf{AC}_\omega(\mathbb{R})$ we can choose $x_n \in X_n$ and hence $A_n = f_{x_n}^{-1}(A)$. To show (31) notice that the successor case follows from (24), while the limit case follows from (27).

(c) Had we defined $\mathcal{L}$ on sets rather than degrees (by putting, say, $\mathcal{L}(A) = 0^\mathcal{L}A$) we would have run into troubles when trying to iterate the operation in the transfinite, since there is no uniform, definable way to pick an increasing cofinal sequence converging to a limit ordinal. In fact, $\mathsf{PSP}$ (which is a consequence of $\mathsf{AD}$, and in fact of $\mathsf{SLO}^W$) implies that there is no $\omega^1$-sequence of distinct clopen sets — see Lemma 2.8. (The argument above shows that there is an $\omega^1$-sequence of distinct Lipschitz degrees of clopen sets.)

(d) If we look at the Lipschitz degrees in the Cantor space — rather than in the Baire space — then Lemmas 5.3 and 5.7 imply that $[A \oplus \neg A]_L$ is self-dual and the immediate successor of $\{[A]_L, [\neg A]_L\}$ and that above a self-dual degree there are $\omega$ self-dual degrees. Proposition 5.4 and its proof hold also for $\omega^1$ (if in the statement we understand $n$ to vary in $\{0, 1\}$), and this implies that every limit degree is non-self-dual: if $[A]_L$ is self-dual, then $A = A_{[0]} \oplus A_{[1]}$ is the least upper bound of $A_{[0]}, A_{[1]}$, hence $[A]_L$ cannot be limit.

6 The well-foundedness of the Lipschitz hierarchy

Wadge’s Lemma implies that if every Lipschitz degree is combined with its dual then $\leq_L$ is a linear order. In fact under this identification it is a well-order.

Theorem 6.1 (Martin–Monk). Assume $\mathsf{AD}$. There is no sequence $\langle A_n \mid n < \omega \rangle$ of sets of reals such that $\forall n \in \omega (A_{n+1} <_L A_n)$ or $\forall n \in \omega (A_{n+1} <_W A_n)$.

Corollary 6.2. Assume $\mathsf{AD} + \mathsf{DC}(\mathbb{R})$. Then $<_L$ and $<_W$ are well-founded on $\mathcal{P}(\mathbb{R})$.

Proof. Since $\leq_L$ refines $\leq_W$ it is enough to prove the result for the Lipschitz case. Towards a contradiction suppose $\emptyset \neq \mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$ does not have an $<_L$ minimal element, and let $A \in \mathcal{A}$. Fix an enumeration $\{f_x \mid x \in \mathbb{R}\}$ of all Lipschitz functions and let

$$T = \{\langle x_0, x_1, \ldots, x_n \rangle \in <\omega \mathbb{R} \mid \forall i < n \left( f_{x_{i+1}}(A) <_L f_{x_i}(A) \right) \}.$$ 

$T$ is the tree on $\mathbb{R}$ searching for an infinite $<_L$-descending chain $\langle f_{x_i}(A) \mid i \in \omega \rangle$ in $\mathcal{A}$. By the Theorem such a sequence does not exist, hence $T$ cannot
have an infinite branch. But by the assumption on $A$, the tree $T$ is pruned, hence by DC(ℝ) it must have an infinite branch: a contradiction. \[ \square \]

In view of this result we can give the following

**Definition 6.3 (AD + DC(ℝ)).** The **Lipschitz rank** of $A \subseteq \mathbb{R}$ is the rank of $A$ in the well-founded relation $<_L$ and it is denoted by

$$\|A\|_L.$$  

Similarly

$$\|A\|_W$$

denotes the **Wadge rank** of $A$ in the $<_W$ pre-order.

For technical reasons, Lipschitz and Wadge ranks take values in the non-zero ordinals, that is

$$\|\emptyset\|_L = \|\mathbb{R}\|_L = 1 = \|\emptyset\|_W = \|\mathbb{R}\|_W.$$  

Therefore AD + DC(ℝ) yields a fine calibration of sets of reals that enables us to prove results about $\mathcal{P}(\mathbb{R})$ by induction on the Wadge rank. This is most remarkable since AD forbids any kind of well ordering of $\mathbb{R}$.

It is a well-known open problem in descriptive set theory whether AD + DC(ℝ). The following also seems to be open:

**Open problem 6.4.** Is DC(ℝ) needed for Corollary 6.2?

**Proof of Theorem 6.1.** Towards a contradiction suppose $A_{n+1} <_L A_n$ for all $n$. Then $A_n \not<_L A_{n+1}$ and $A_n \not<_L \neg A_{n+1}$ so that by AD and AC\(_L\) and AC\(_\omega(\mathbb{R})\) there are winning strategies $\sigma^0_n$ and $\sigma^1_n$ for $I$ in $G_L(A_n, A_{n+1})$ and in $G_L(A_n, \neg A_{n+1})$, respectively. For $z \in \omega^2$ let

$$G^z_n = \begin{cases} G_L(A_n, A_{n+1}) & \text{if } z(n) = 0, \\ G_L(A_n, \neg A_{n+1}) & \text{if } z(n) = 1. \end{cases}$$

The games $\{G^z_n \mid n \in \omega\}$ will be played simultaneously with $I$ using $\sigma^z_n$ in $G^z_n$ and $II$ copying $I$’s moves from the next game, $G^z_{n+1}$. To see how this is done consider the diagram of Figure 1 where $\sigma_n$ is the strategy $\sigma^z_n$ and $\equiv$ is the copying strategy. $I$ starts by filling-in the first column, then $II$ copies and fills-in the second column. Now $I$ can use his strategies to fill-in the third column. And so on. Let $a^z_n = \langle a^z_n(i) \mid i \in \omega \rangle$ be the real played by $I$ in $G^z_n$. Then $a^z_{n+1}$ is also the real played by $I$ in $G^z_n$. So we have the following

$$z(n) = 0 \Rightarrow (a^z_n \in A_n \Leftrightarrow a^z_{n+1} \notin A_{n+1})$$

$$z(n) = 1 \Rightarrow (a^z_n \in A_n \Leftrightarrow a^z_{n+1} \in A_{n+1}).$$

(32)
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\[ I \quad \sigma_0(0) = a_0^z(0) \quad \sigma_0(1) = a_0^z(1) \quad \sigma_0(2) = a_0^z(2) \quad \sigma_0(3) = a_0^z(3) \]

\[ G_0^z \]

\[ II \]

\[ a_1^z(0) \quad a_1^z(1) \quad a_1^z(2) \]

\[ \sigma_0 \]

\[ \sigma_1 \]

\[ \sigma_1 \]

\[ \sigma_1 \]

\[ I \quad \sigma_1(0) = a_1^z(0) \quad \sigma_1(1) = a_1^z(1) \quad \sigma_1(2) = a_1^z(2) \]

\[ G_1^z \]

\[ II \]

\[ a_2^z(0) \quad a_2^z(1) \]

\[ \sigma_1 \]

\[ G_2^z \]

\[ II \]

\[ a_3^z(0) \]

\[ \sigma_2 \]

\[ G_3^z \]

\[ II \]

\[ I \quad \sigma_2(0) = a_2^z(0) \quad \sigma_2(1) = a_2^z(1) \]

\[ G_5^z \]

\[ II \]

\[ I \quad \sigma_3(0) = a_3^z(0) \]

\[ G_6^z \]

Figure 1: A Martin-Monk diagram

Notice that if \( z \) and \( w \) are eventually equal, that is for some \( m_0 \)

\[ \forall m \geq m_0 \left( z(m) = w(m) \right), \]

then

\[ \forall m \geq m_0 \left( a_m^z = a_m^w \right). \]

Claim 6.2.1. Suppose \( z, w \in \omega^2 \) are such that \( \exists k \in \omega \left( z(k) \neq w(k) \right) \). Then

\[ a_0^z \in A_0 \iff a_0^w \notin A_0. \]

Proof. As \( z \) and \( w \) agree after \( k \), then \( a_{k+1}^z = a_{k+1}^w \). By (32) above, \( a_k^z \in A_k \iff a_k^w \notin A_k \). As \( z \upharpoonright k = w \upharpoonright k \) and again by repeated applications of (32), \( a_0^z \in A_0 \iff a_0^w \notin A_0. \)

Therefore \( F = \{ z \in \omega^2 \mid a_0^z \in A_0 \} \) is a flip set, contradicting BP and LM, and hence AD.

Notice that the existence of a flip set implies the ill-foundedness of \( <_L \) in the Cantor space — see Exercise 10.3.
Remarks 6.5. (a) The strategies $\sigma_n^i$ in the proof of Theorem 6.1 are (or better: induce) contractions $f_n$, and having them play against each other amounts to construct a some sort of projective limit
\[ \ldots \rightarrow \mathbb{R} \rightarrow \mathbb{R} \rightarrow \mathbb{R} \]
and the real in the first row of the diagram of Figure 1 is the unique element of $\bigcap_n B_n$, where
\[ B_n = f_0 \circ \cdots \circ f_n(\mathbb{R}) \, . \]
(b) By looking at Figure 1 it is clear that if $z \restriction n = w \restriction n$ then the diagrams relative to $z$ and to $w$ agree in the upper triangular part, that is
\[ i + j \leq n \Rightarrow a_i^z(j) = a_i^w(j) \, , \]
so that the map $z \mapsto a_i^z$ is Lipschitz and $F \leq_L A_0$. Therefore, in order to reach the desired contradiction, we only need the property of Baire to hold of sets $\leq_L A_0$.
(c) Determinacy was used in a “local” way in the proofs of Wadge’s Lemma and the Martin-Monk Theorem. For example, in order to prove Wadge’s Lemma for sets in some pointclass $\Gamma$ it is enough to assume the determinacy of all sets contained in the smallest Boolean algebra containing $\Gamma$. So, for example, the well-foundedness of the Wadge hierarchy restricted to Borel sets can be proved in $\text{ZFC}$. In fact it can be proved in second order arithmetic [LSR88], while Borel determinacy is not provable in Zermelo’s set theory with choice, $\text{ZC}$, that is $\text{ZFC}$ without replacement [Fri71]. On the other hand by results of Harrington and Martin the determinacy of $\Pi^1_3$ sets implies the determinacy of boolean combination of $\Pi^1_3$ sets, hence it is enough to prove $\text{SLO}^L$ and the well-foundedness of $<_L$ for $\Pi^1_3$ and $\Sigma^1_3$ sets.
(d) The statements and proofs of Theorem 6.1 and Corollary 6.2 were given for $\mathbb{R}$, but work for any space $[T]$, with $T$ a pruned tree on $\omega$.

Corollary 6.6. Assume $\text{AD} + \text{DC}(\mathbb{R})$. The order $\leq$ on the Lipschitz degrees in any space of the form $[T]$ with $T$ a pruned tree on $\omega$ is well-founded and the anti-chains have size at most 2.

(a) If the space is $\mathbb{R}$, there are blocks of length $\omega_1$ of consecutive self-dual degrees and non-self-dual pairs at levels of uncountable cofinality. There-
fore the Lipschitz hierarchy looks like this:

\[ \begin{align*}
\cdots & \cdots \cdots \cdots \\
\omega_1 & \cdots \cdots \cdots \\
\cdots & \cdots \cdots \cdots \\
\cdots & \cdots \cdots \cdots \\
\text{cof} = \omega & \text{cof} > \omega
\end{align*} \]

with non-self-dual pairs occurring exactly at levels \( \omega_1 \cdot \alpha \), with \( \alpha \) successor or limit of cofinality > \( \omega \).

(b) If the space is \( \omega^k \), there are blocks of length \( \omega \) of consecutive self-dual

degrees, and non-self-dual pairs occur at all limit levels.

\[ \begin{align*}
\cdots & \cdots \cdots \\
\omega & \cdots \cdots \cdots \\
\cdots & \cdots \cdots \\
\cdots & \cdots \cdots \\
\cdots & \cdots \cdots
\end{align*} \]

7 Wadge games

The analogue of Lemma 4.5 for \( G_W \) is also called Wadge’s Lemma:

**Lemma 7.1.** Assume AD, or even just \( AD^W \). Then for every \( A, B \subseteq \mathbb{R} \)

\[ (SLO^W) \quad A \leq_W B \quad \lor \quad \neg B \leq_W A. \]

Thus \( AD^W \) and \( AD^L \) imply \( SLO^W \) and \( SLO^L \), respectively, and are both

consequences of AD. Keeping in mind (22) we have that

\[ (34) \quad AD^L \implies SLO^L \quad \quad AD^W \implies SLO^W \]

In Chapter IV we shall prove that, assuming \( \neg FS + DC(\mathbb{R}) \),

\[ SLO^W \iff SLO^L \iff AD^W \iff AD^L. \]

For the time being let us point out that many of the preceding proofs did go

through under weaker assumptions. For example:

**Theorem 7.2.** Assume \( AC_\omega(\mathbb{R}) + \neg FS + AD^L \). Then there is no sequence

\[ \langle A_n \mid n \in \omega \rangle \text{ of sets of reals such that } A_{n+1} <_L A_n. \]
and

Corollary 7.3. Assume $\text{DC}(\mathbb{R}) + \neg \text{FS} + \text{AD}^L$. Then $<_L$ is well-founded on $\mathcal{P}(\mathbb{R})$.

The following seems to be open:

Open problem 7.4. Does $\text{SLO}^W$ imply $\neg \text{FS}$?

Many of the results in the forthcoming pages are stated under rather technical assumptions like the ones in 7.2 and 7.3, since these assumptions will be of importance in the subsequent Chapters, but on first reading, the reader should probably replace them with $\text{AD}$.

8 The Wadge hierarchy

As was observed in part (b) of Remark 5.11, each $\omega_1$-block of consecutive self-dual Lipschitz degrees is included in the same self-dual Wadge degree. The next result — whose proof is deferred to the next section — implies that this Wadge degree contains no other Lipschitz degrees.

Theorem 8.1 (Steel, Van Wesep). Assume $\text{AD}^L + \neg \text{FS}$. For all $A \subseteq \mathbb{R}$

$$A \leq_W \neg A \Rightarrow A \leq_L \neg A.$$  

Corollary 8.2. Assume $\text{AD}^L + \text{AC}_\omega(\mathbb{R}) + \neg \text{FS}$.

(a) $[A]_L$ is self-dual iff $[A]_W$ is self-dual, and in this case $[A]_W = [A]_L$ and

$$\{B \subseteq \mathbb{R} \mid B \leq_L A\} = \{B \subseteq \mathbb{R} \mid B \leq_W A\}.$$  

(b) Every self-dual degree $[A]_W$ is the union of an $\omega_1$-block of consecutive Lipschitz degrees and

$$\{B \subseteq \mathbb{R} \mid B \leq_L A\} \subset \{B \subseteq \mathbb{R} \mid B \leq_W A\}.$$  

(c) Every self-dual Wadge degree has a non-self-dual pair of immediate successors.

(d) Every non-self-dual pair of Wadge degrees has a self-dual degree as immediate successor.

(e) Assume also $\text{DC}(\mathbb{R})$.  

• If we work with the Baire space \( \mathbb{N} \), then at limit levels of countable cofinality there is a self-dual Wadge degree; at levels of uncountable cofinality there is a non-self-dual pair.

• If we work with the Cantor space \( 2^{\mathbb{N}} \), then at all limit levels there is a non-self-dual pair of degrees.

Therefore the Wadge degrees on \( \mathbb{N} \) look like this:

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \bullet & \cdots \\
\uparrow & \text{cof} = \omega & & & & & & \\
\end{array}
\]

and on \( 2^{\mathbb{N}} \) the Wadge degrees look like this:

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \cdots & \bullet & \cdots \\
\end{array}
\]

8.A The length of the Wadge hierarchy

The two diagrams (35) and (36) suggest the following question: what is the length of the Wadge hierarchy? A binary relation \( \preceq \) is a \textbf{pre-well-order} (pwo for short) on \( X \) if it is reflexive, transitive, well-founded and connected, i.e.,

\[
\forall x, y \in X \ (x \preceq y \land y \preceq x).
\]

Any pwo yields a \textbf{norm} on \( X \), i.e., an ordinal valued function \(|\cdot|_{\preceq} : X \to \text{Ord}:

\[
|x|_{\preceq} = \text{the rank of } x \text{ in the well-founded relation } \preceq.
\]

Conversely any norm \( f : X \to \text{Ord} \) gives rise to a pwo on \( X \). A norm is regular if its range is an ordinal — all canonical norms associated to pre-well-orders are regular. The length \( \text{lh}(\preceq) \) of a pwo \( \preceq \) is the range of its canonical norm.

**Definition 8.3.** The ordinal \( \Theta \) is defined by

\[
\Theta = \sup \{ \text{lh}(\preceq) \mid \preceq \text{ is a pwo on } \mathbb{R} \}
\]

\[
= \sup \{ \alpha \mid \exists f : \mathbb{R} \to \alpha \text{ is surjective} \}.
\]

**Exercise 8.4.** Show in ZF that \( \Theta \) is a cardinal and \( \Theta \geq \omega_2 \).
Assuming AD, Θ is a fixed point of the N-functions, i.e., NΘ = Θ, but it is not the least such fixed point; in fact it is larger than the least fixed point of the enumerating function of all fixed points of the N-function, and so on (see [Kan03, pp. 396–399]).

**Theorem 8.5.** Assume AD + DC(\(\mathbb{R}\)). Then Θ is the length of the Wadge hierarchy:

\[
Θ = \sup\{|A|_W \mid A \subseteq \mathbb{R}\}.
\]

**Proof.** Let \(f: \mathbb{R} \rightarrow \alpha\) be a surjection. Define \(\langle A_\nu \mid \nu < \alpha \rangle\) by

\[
A_\nu = J(\{x \oplus y \in \mathbb{R} \mid f(x) < \nu \land y \in A_{f(x)}\})
\]

where \(J = J_\mathcal{F}\) is as in Lemma 4.7 and \(\mathcal{F}\) is the collection of all continuous functions. Then \(\nu < \xi \Rightarrow A_\nu <_W A_\xi\), hence

\[
Θ \leq \sup\{|A|_W \mid A \subseteq \mathbb{R}\}.
\]

For the other direction, fix \(\{g_x \mid x \in \mathbb{R}\}\) a parametrization of all continuous functions: if \(|A|_W = \alpha\) then \(f: \mathbb{R} \rightarrow \alpha + 1\)

\[
f(x) = \|[g_x^{-1}(A)]_W
\]

is a surjection. □

Since Θ is a cardinal bigger than \(\omega_1\) then \(\omega_1 \cdot Θ = Θ\), and Θ is also the length of the Lipschitz hierarchy.

### 8.B Boldface pointclasses

The theory of Wadge degrees can be seen as the general theory of boldface pointclasses. A **pointclass** in \(\mathcal{X}\) is simply a non-empty \(\mathcal{X} \subseteq \mathcal{P}(\mathcal{X})\). If \(\mathcal{X}\) is a topological space a **boldface pointclass** is a pointclass closed under continuous pre-images; boldface pointclasses are denoted by capital boldface Greek letters, like \(\Gamma, \Lambda\), etc. If \(\mathcal{X}\) is not mentioned, it is understood to be \(\mathbb{R}\). Assuming SLO\(^W\), every boldface pointclass \(\Gamma\) (on \(\mathbb{R}\) or \(\omega_2\)) is down-ward closed under \(\leq_W\) and it is of the form

(37) \[\Gamma = \{A \subseteq \mathbb{R} \mid A <_W C\}\]

or

(38) \[\Gamma = \{A \subseteq \mathbb{R} \mid A \leq_W C\}\].
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In the latter case $\Gamma$ is said to be principal and generated by $C$, and the set $C$ (and hence any other $C' \in [C]_W$) is said to be $\Gamma$-complete, or complete for $\Gamma$. Not every pointclass is principal for example $\Delta^1_1$, the collection of Borel sets, is not. The dual of a pointclass $\Gamma$ is the pointclass

$$\bar{\Gamma} = \Gamma^\sim = \{ \neg A \mid A \in \Gamma \}.$$  

A pointclass $\Gamma$ is self-dual if $\bar{\Gamma} = \Gamma$, that is, if it is closed under complements; otherwise it is called non-self-dual. If $\Gamma$ is principal and generated by $A$, then

$$\Gamma \text{ is self-dual } \iff A \text{ is self-dual.}$$

**Exercise 8.6.** Show that $\text{SLO}^W$ implies Wadge’s Lemma for boldface pointclasses, i.e. for any $\Gamma$ and $A$,

$$\Gamma \subseteq A \quad \lor \quad \bar{\Gamma} \subseteq \Gamma.$$  

Show that the non-self-dual pointclasses are exactly those which are principal and generated by a non-self-dual set.

Give an example of self-dual pointclass which is principal, and one which is non-principal.

For $A \subseteq X \times Y$, $\bar{x} \in X$ and $\bar{y} \in Y$,

$$A_{\{\bar{x}\}} = \{ y \mid (\bar{x}, y) \in A \}$$

$$A^{\{\bar{y}\}} = \{ x \mid (x, \bar{y}) \in A \},$$

are, respectively the vertical and horizontal sections of $A$ through $\bar{x}$ and $\bar{y}$. A set $U \subseteq \mathbb{R} \times \mathbb{R}$ is universal for $\Gamma$ if

$$\Gamma = \{ U_{\{x\}} \mid x \in \mathbb{R} \}$$

and $h(U) \in \Gamma$, where $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the canonical homeomorphism of (7).

**Proposition 8.7.** Assume $\text{SLO}^L$. Every non-self-dual $\Gamma$ admits a universal set $U$, and $h(U) \in \Gamma$ is complete.

**Proof.** Let $\Gamma$ be non-self-dual and generated by $A$, and let

$$U = \{ (x, y) \mid \ell_x(y) \in A \}.$$  

Then $h(U) \in \Gamma$, and since every Lipschitz function is continuous

$$\{ B \mid B \leq_L A \} \subseteq \Gamma.$$  

Conversely, if $B \in \Gamma$, and, towards a contradiction, $B \not\leq_L A$, then $A \leq_L \neg B$: since $B \leq_W A$, then by transitivity $B \leq_W \neg B$ and hence $A \equiv_W B$ would be Wadge self-dual, contrarily to our assumption. Therefore $U$ is universal for $\Gamma$.  

□
8. THE WADGE HIERARCHY

8.C An overview of the Wadge hierarchy

A clopen partition of a topological space $\mathcal{X}$ is a family $\{D_i \mid i < N\}$ such that

(i) $1 \leq N \leq \omega$

(ii) each $D_i$ is clopen and non-empty,

(iii) $i < j < N \Rightarrow D_i \cap D_j = \emptyset$,

(iv) $\bigcup_{i<N} D_i = \mathcal{X}$.

The partition is said to be finite iff $N < \omega$. In the Baire space any non-empty clopen set $D$ can be written as $D = \bigcup_{n=\omega} D_n$ with each $D_n$ clopen, and $n \neq m \Rightarrow D_n \cap D_m = \emptyset$, so any clopen partition can be refined to an infinite clopen partition.

Exercise 8.8. Let $\mathcal{X}$ be a topological space. Show that:

(i) If $\{D_i \mid i < N\}$ is a clopen partition of $\mathcal{X}$ and $f_i : D_i \rightarrow \mathcal{X}$ is continuous, then $\bigcup_{i<N} f_i : \mathcal{X} \rightarrow \mathcal{X}$ is continuous.

(ii) If $D \subseteq \mathcal{X}$ is clopen, $f : D \rightarrow \mathcal{X}$ is continuous, $B \subseteq \mathcal{X}$, and $A = f^{-1}(B)$, then $A \leq^\mathcal{X}_W B$.

(iii) If $\{D_i \mid i < N\}$ is a clopen partition of $\mathcal{X}$, then

$$A \leq^\mathcal{X}_W B \iff \forall i < N \left( A \cap D_i \leq^\mathcal{X}_W B \right).$$

Proposition 8.9. Let $A, B_n \subseteq \mathbb{R}$.

(a) $A \leq_W \bigoplus_{n=\omega} B_n$ iff there is a clopen partition $\{D_n \mid n < \omega\}$ of $\mathbb{R}$ such that

$$\forall n \left( A \cap D_n \leq^{D_n, \mathbb{R}}_W B_n \right).$$

(b) $A \leq_W B_0 \oplus B_1$ iff there is a clopen partition $\{D_0, D_1\}$ of $\mathbb{R}$ such that $A \cap D_n \leq_W B_n$, for all $n = 0, 1$.

Remark 8.10. The definition of $\leq^{D_n, \mathcal{X}}_W$ is given in (1), and when $B_n \neq \mathbb{R}$, (39) is equivalent to $A \cap D_n \leq_W B_n$ by Exercise 8.8(ii).
CHAPTER I. ELEMENTARY RESULTS ON THE WADGE HIERARCHY

Proof of 8.9. If \( f \) is continuous, then \( D_n = f^{-1}(\mathcal{N}^{(n)}) \) is clopen and \( \{D_n \mid n \in \omega\} \) is partition. (For \( b \) use \( D_0 = \bigcup_n f^{-1}(\mathcal{N}^{(2n)}) \) and \( D_1 = \bigcup_n f^{-1}(\mathcal{N}^{(2n+1)}) \).) Then

\[
g_n \overset{\text{def}}{=} f \upharpoonright D_n : D_n \to \mathbb{R}
\]
is continuous and such that \( g_n^{-1}(B_n) = A \cap D_n \), so we are done by part (iii) of Exercise 8.8. \( \square \)

The very same proof yields

**Proposition 8.11.** If \( A, B_0, B_1 \subseteq \omega^2 \), then \( A \leq_{\mathcal{W}}^{\omega^2} B_0 \oplus B_1 \) iff there is a clopen partition \( \{D_0, D_1\} \) of \( \omega^2 \) such that \( A \cap D_n \leq^{\mathcal{W}}_{\omega^2} B_n \), for \( n = 0, 1 \).

If \( T \) is a well-founded tree on \( \omega \), then

\[
\forall x \in \mathbb{R} \exists s \in \partial T (s \subset x)
\]
so that \( \mathbb{R} = \bigcup_{s \in \partial T} \mathcal{N}_s \), and since

\[
\forall s, t \in \partial T (s \neq t \Rightarrow \mathcal{N}_s \cap \mathcal{N}_t = \emptyset),
\]
it follows that \( \{\mathcal{N}_s \mid s \in \partial T\} \) is a clopen partition of \( \mathbb{R} \). Conversely, if \( \{D_n \mid n < N\} \) is a clopen partition of \( \mathbb{R} \), then

\[
T = \{t \in <\omega \omega \mid \forall n (\mathcal{N}_t \not\subseteq D_n)\}
\]
is a well-founded tree on \( \omega \). Since every real belongs to a unique \( D_n \), then \( \{\mathcal{N}_s \mid s \in \partial T\} \) is a clopen partition refining \( \{D_n \mid n < N\} \). We shall refer to the \( \{\mathcal{N}_s \mid s \in \partial T\} \) thus obtained as the **canonical clopen partition refining** \( \{D_n \mid n < N\} \).

Suppose \( T \) is a tree on \( \omega \), and suppose \( \bar{A} = \langle A_s \mid s \in \partial T \rangle \) is a sequence of sets of reals. Then let

\[
(40) \quad S(T; \bar{A}) = \bigcup_{s \in \partial T} s^{-}A_s.
\]

There are two cases when this definition is problematic:

- if \( T = \emptyset \), then \( \partial T = \{\emptyset\} \), so \( S(T; \langle A_\emptyset \rangle) = A_\emptyset \),
- and if \( T = <\omega \omega \) then \( \partial T = \emptyset \), so \( \bar{A} = \emptyset \) and \( S(T; \bar{A}) = \emptyset \).

Proposition 8.9 can be strengthened to
Proposition 8.12. Let $B_n \subseteq \mathbb{R}$. The pointclass generated by $\bigoplus_n B_n$ is

$$\{ S(T; \vec{A}) \mid T \text{ is well-founded, and } \forall s \in \partial T \exists n (A_s \leq_W B_n) \}.$$ 

Similarly, then pointclass generated by $B_0 \oplus B_1$ is

$$\{ S(T; \vec{A}) \mid T \text{ is well-founded, and } \forall s \in \partial T \exists i \in \{2\} (A_s \leq_W B_i) \}.$$ 

Proof. For the sake of definiteness, let us focus on the pointclass generated by $\bigoplus_n B_n$, the other case being left as an exercise. By Proposition 8.9, its elements are sets $A$ such that $A \cap D_n \equiv^{D_n, \mathbb{R}} B_n$, for some carefully chosen clopen partition $\{D_n \mid n < N\}$: if $\{N_s \mid s \in \partial T\}$ is the canonical refinement of $\{D_n \mid n < N\}$, and if $N_s \subseteq D_n$, then $A = S(T; \langle A_s \mid s \in \partial T \rangle)$ and $s^\Delta A_s \equiv^{\mathbb{R}, N_s} B_n$. Since $A_s \equiv^{\mathbb{R}, N_s} s^\Delta A_s$, the result follows. \qed

We are now going to look at the first few Wadge degrees or — equivalently — at the first few boldface pointclasses. By the general remark after (14) on page 14, at the bottom of the Wadge hierarchy there is the non-self-dual pair $[\emptyset]_W = \{\emptyset\}$ and $[\mathbb{R}]_W = \{\mathbb{R}\}$. (For the sake of definiteness, these facts are stated for $\mathbb{R}$ only, but they also hold for $\omega_2$.) Notice that these are also boldface pointclasses, albeit trivial ones. The self-dual degree immediately above it is $[\emptyset \oplus \mathbb{R}]_W$, and the pointclass it generates is $\Delta^0_1$, the collection of all clopen sets. Thus

$$[\emptyset \oplus \mathbb{R}]_W = \Delta^0_1 \setminus \{\emptyset, \mathbb{R}\}.$$ 

The two non-self-dual pointclasses immediately above are $\Sigma^0_1$ and $\Pi^0_1$, the collection of open and closed sets. To see this notice that if $B <_W A$ and $A$ is open, then $B$ is open and by $\text{SLO}^W$, $\neg B <_W A$ and hence $B <_W \neg A$, which implies $B$ is closed. Therefore $B$ is clopen. Notice that the appeal to $\text{SLO}^W$ is not necessary since the games involved are Borel and therefore determined in $\text{ZFC}$.

The next $\omega_1$ levels of the Wadge hierarchy are occupied by the Hausdorff difference pointclasses over $\Pi^0_1$ sets (see Exercise ??) so that the sets of countable Wadge rank are the $\Delta^0_2$ sets. In other words, Hausdorff’s analysis of $\Delta^0_2$ sets is exhaustive: all non-self-dual boldface pointclasses between $\Pi^0_1$ and $\Delta^0_2$ are the $\alpha$-$\Pi^0_1$ and their dual. On the other hand between $\Sigma^0_2$ and $\Pi^0_2$ (which live at level $\omega_1$) and $\Sigma^0_3$ and $\Pi^0_3$ there are many more non-self-dual pointclasses than the $\alpha$-$\Pi^0_1$ and their dual. In fact the Wadge rank of a complete $\Sigma^0_3$ (or $\Pi^0_3$) set is not $\omega_1 + \omega_1$, but $\omega_1^{\omega_1}$, see section ?? for more on this.
9 Proof of the Steel-Van Wesep Theorem

The proof of Theorem 8.1 uses a variant of the technique in the proof of Theorem 6.1, usually called the Martin–Monk method. This technique figures prominently in the theory of Wadge degrees, so we start with some remarks on how to formulate it in general terms.


Given a sequence \( \sigma_n \) of strategies for \( \mathbf{I} \) in a Lipschitz game, we can pit them against each other, pretending that the opponent of the player using \( \sigma_n \) is the player using \( \sigma_{n+1} \). Therefore the diagram of Figure 1 can be conveyed in a more compact form

(41)

\[
\begin{array}{cccc}
\sigma_0 & a_0^0 = \sigma_0(\emptyset) & a_0^1 = \sigma_0(a_0^1) & a_0^2 = \sigma_0(a_0^2) & \cdots \\
\sigma_1 & a_1^0 = \sigma_1(\emptyset) & a_1^1 = \sigma_1(a_0^2) & a_1^2 = \sigma_1(a_0^2) & \cdots \\
\sigma_2 & a_2^0 = \sigma_2(\emptyset) & a_2^1 = \sigma_2(a_0^2) & a_2^2 = \sigma_2(a_0^2) & \cdots \\
\end{array}
\]

The player using \( \sigma_n \) operates on row \( n \) and contrasts the moves of his opponent on row \( n + 1 \)

(42)

\[
\begin{array}{cccc}
\mathbf{I} & \sigma_n & a_0^0 & a_0^1 & \cdots & a_0^{n-1} & a_0^n \\
\mathbf{II} & a_0^{n+1} & a_1^{n+1} & \cdots \\
\end{array}
\]

while the player on row \( n + 1 \) is using \( \sigma_{n+1} \) to contrast the moves of the next player on row \( n + 2 \), and so on. Thus Remark 6.5(b) says that knowing \( \sigma_0, \ldots, \sigma_n \) (or even just a finite part of them) allows us to fill-in the upper-left triangular part of the Martin-Monk diagram with entries \( a_j^i \), with \( i + j \leq n \),

(43)

\[
\begin{array}{cccc}
a_0^0 & a_0^1 & \cdots & a_0^{n-1} & a_0^n \\
a_0 & a_1 & \cdots & a_{n-1} & a_n \\
\vdots \\
a_0^{n-1} & a_1^{n-1} \\
a_0^n \\
\end{array}
\]

(Formally, the finite region filled-in by \( \sigma_0, \ldots, \sigma_n \) is an \( f : D \to \omega \) with \( D \subset \omega \times \omega \) finite, and with the obvious requirement that the values \( f(k, 0), f(k, 1), \ldots \) are the output \( a_k^0, a_k^1, \ldots \) on row \( k \) computed by \( \sigma_k \) on the input given by row \( k + 1 \).)

Suppose now we want to repeat the construction when some of the \( \sigma_n \) are strategies for \( \mathbf{II} \). To this end let’s convene that the player using \( \sigma_n \) always
makes his moves are on row $n$ while his opponent moves on row $n + 1$. Thus if $\sigma_n$ is a strategy for $I$ we have (42), if $\sigma_n$ is a strategy for $II$ we have

\[
\begin{array}{c|c|c|c}
\sigma_n & a^n_0 & a^n_1 & \cdots \\
\hline
I & a^1_0 & a^1_1 & \cdots \\
\hline
II & a^{n+1}_0 & a^{n+1}_1 & \cdots \\
\end{array}
\]

Therefore if row $n + 1$ has the first $k$ entries filled-in, then row $n$ has the first $k + 1$ or $k$ entries filled in, depending whether $\sigma_n$ is a strategy for $I$ or $II$. The presence of strategies for $II$ slows-down the filling-in procedure, and the diagram cannot be completely filled-in unless $\sigma_n$ is a strategy for $I$ for infinitely many $n$. A more topological way to say this is that, if $A_n$ are as in equation (33) then $\text{diam}(A_{n+1}) < \text{diam}(A_n)$ if and only if $f_{n+1}$ is a contraction, i.e., $\sigma_{n+1}$ is a strategy for $I$. The region filled in by a finite number of strategies $\sigma_0, \ldots, \sigma_n$ is, in general, smaller than the triangular region of (43). For example, if among $\sigma_0, \ldots, \sigma_5$ only $\sigma_0, \sigma_2$, and $\sigma_3$ are strategies for $II$, then the region of the Martin-Monk diagram filled-in is

\[
\begin{array}{c|c|c|c|c|c}
\sigma_0 & a^1_0 & a^1_1 & a^1_2 & \cdots & \sigma_3 \\
\hline
\sigma_1 & a^1_0 & a^1_1 & a^1_2 \\
\sigma_2 & a^0_0 & a^1_1 \\
\sigma_3 & a^0_0 & a^1_1 \\
\sigma_4 & a^0_0 & a^1_1 \\
\sigma_5 & a^0_0 \\
\end{array}
\]

Finally, and this is the most interesting case, suppose each $\sigma_n$ is either a strategy for $I$ or a Wadge strategy for $II$ (and this second case includes the possibility that $\sigma_n$ is actually a strategy in a Lipschitz game). In this case it is not true anymore in the region filled in by $\sigma_0, \ldots, \sigma_n$ that rows with smaller indexes are at least as long as rows with larger indexes. If, for example, in the example of (45) above $\sigma_3$ passes in the first two innings, the region filled in by $\sigma_0, \ldots, \sigma_3$ is

\[
\begin{array}{c|c|c|c|c|c}
\sigma_0 & a^1_0 & a^1_1 & a^1_2 \\
\hline
\sigma_1 & a^1_0 \\
\sigma_2 & a^0_0 \\
\sigma_3 & a^0_0 & a^1_1 \\
\sigma_4 & a^0_0 & a^1_1 \\
\sigma_5 & a^0_0 \\
\end{array}
\]

Suppose $\langle \sigma_n | n \in \omega \rangle$ is a sequence of Wadge strategies for $II$ and Lipschitz strategies for $I$, and let $f_n : D_n \to \omega$ be the finite region of the Martin-Monk diagram filled-in by $\sigma_0, \ldots, \sigma_n$. The following is a tedious, but straightforward verification.
CHAPTER I. ELEMENTARY RESULTS ON THE WADGE HIERARCHY

Exercise 9.1. Show that $n < m$ implies $D_n \subseteq D_m$ and $f_m \upharpoonright D_n = f_n$.

We will say that $(\sigma_n \mid n \in \omega)$ admits a global play iff $\bigcup_n D_n = \omega \times \omega$, and $\bigcup_n f_n$ is the fill-in of the Martin-Monk diagram. It is not hard to see that the existence of infinitely many $n$’s such that $\sigma_n$ is a strategy for $I$ does not guarantee that there is a global play, i.e., that the diagram can be completely filled-in, since a Wadge strategy for $II$ can take a lot of naps and can drive the filling-in procedure to a grinding halt. The cure for this is to guarantee that the rows on which a Wadge strategy for $II$ is used are sparse enough compared to the rows where a Lipschitz strategy for $I$ is used. We will see an example of such a situation in the next proof.


Suppose, towards a contradiction, that $\sigma_0$ is a winning strategy for $II$ in $G_W(A, \neg A)$ and $\sigma_1$ be $II$’s copying strategy, i.e., $\sigma_1(s \cap n) = n$. Let also $\sigma_2$ be a winning strategy for $I$ in $G_L(A, \neg A)$. We shall define an increasing sequence of natural numbers

$$0 = M_0 < M_1 < M_2 < \ldots$$

and let

$$G_n = \begin{cases} G_W(A, \neg A) & \text{if } n \in \{M_i \mid i \in \omega\}, \\ G_L(A, \neg A) & \text{otherwise.} \end{cases}$$

When $n = M_i$, player $II$ uses either $\sigma_0$ or $\sigma_1$ in the game $G_n = G_W(A, \neg A)$ on row $n$: the choice will depend on the value of $z(i)$, with $z \in \omega^2$. On all remaining rows, player $I$ uses $\sigma_2$. If the $M_i$’s are sparse enough, the diagram can be filled-in, no matter what $z$ is used, and an argument as in the proof of Theorem 6.1 will yield the desired contradiction. Since $M_0 = 0$, player $II$ can use either $\sigma_0$ or $\sigma_1$ on the 0-th row — for the sake of definiteness, let’s assume that the former holds. Choose $n'$ large enough so that if $\sigma_2$ is used on rows 1, 2, \ldots, $n'$ and hence $n'$ values of row 1 are determined, then $\sigma_0$ makes a move in row 0 (remember that $\sigma_0$ may pass.) Let $M_1 = n' + 1$. Since $\sigma_1$ is the copying strategy, we can conclude that no matter what $I$ uses on row $M_0$, then at least the first value of this row is determined. Choose now $n'' > M_1$ large enough such that if on rows $M_1 + 1 < M_1 + 2 < \ldots < n''$ player $I$ uses $\sigma_2$, then at least the first two entries on row $M_0$ will be filled-in, no matter what strategies $II$ uses on rows $M_0$ and $M_1$. Let $M_2 = n'' + 1$, and so on (see picture ??).

Here is the formal construction. Suppose $M_0, \ldots, M_k$ have been defined, and let $s \in M_{k+13}$ be such that

$$\forall n \leq M_k (s(n) \in \{0, 1\} \iff n \in \{M_i \mid i \leq k\}).$$
Claim 9.1.1. There is an \( m \) such that letting \( t = s^{-0}(m) \), the \( \sigma_{t(0)}, \ldots, \sigma_{t(M_k+m)} \) fill-in (at least) the first \( k+1 \) entries in the 0-th row, i.e., if \( f : D \to \omega \) is the region filled-in by the \( \sigma_{t(0)}, \ldots, \sigma_{t(M_k+m)} \), then \( n \leq M_k \Rightarrow (0, n) \in D \).

Proof. Consider the element \( w = s^{-0} \in \omega^3 \). Then \( \langle \sigma_{w(n)} | n \in \omega \rangle \) admits a global play, since all strategies from some point on are for \( I \). Let \( x \) be the real in the \( M_k+1 \)-st row. Since each \( \sigma_{w(0)}, \ldots, \sigma_{w(M_k)} \) are (or better: yield) continuous functions, there is an \( m \) such that \( x \upharpoonright m \) is enough to compute the value of the real in the 0-th row up to \( k+1 \). This \( m \) will do.

Let \( m_s \) be the least \( m \) as in the Claim and let

\[
M_{k+1} = M_k + \sup\{m_s | s \in M_{k+1} \cap s \text{ satisfies (47)}\}.
\]

This concludes the definition of \( \langle M_k | k \in \omega \rangle \). Finally for any \( z \in \omega^2 \) let \( w_z \in \omega^3 \) be obtained by coping the values of \( z \) on the set \( \{M_k | k \in \omega \} \) and set the value 2 in the other spots, that is

\[
w_z(n) = \begin{cases} 
  z(i) + 1 & \text{if } n = M_i, \\
  2 & \text{otherwise.}
\end{cases}
\]

The sequence of strategies \( \langle \sigma_{w_z(n)} | n \in \omega \rangle \) admits a global play \( f^z : \omega \times \omega \to \omega \), for each \( z \in \omega^2 \), since the first \( M_k \) strategies are enough to determine the first \( k+1 \) entries of row \( M_0 \). Let \( h : \omega^2 \to \omega \)

\[
h(z) = \langle f^z(0, m) | m \in \omega \rangle
\]

= the real in the 0-th row of the diagram.

Arguing as is the proof of Theorem 6.1, the set \( \{z \in \omega^2 | h(z) \in A \} \) is a flip set, contradicting \( \neg \text{FS} \). This concludes the proof of Theorem 8.1.

In view of Corollary 8.2 above we will say that a set \( A \) is (non-)self-dual iff the Wadge degree \([A]_W\), or equivalently the Lipschitz degree \([A]_L\), is (non-)self-dual. Notice that Theorem 8.1 is non-trivial even for Borel sets. For example it implies that if \( A \in \Sigma^0_\alpha \setminus \Pi^0_\alpha \) and \( B = f^{-1}(A) \) for some continuous \( f \) then \( B = g^{-1}(A) \) for some Lipschitz \( g \).

Exercise 9.2. Assume \( \neg \text{FS} \). Show that if \( A \leq_W \neg A \) then there is no contraction \( f \) such that \( A = f^{-1}(A) \).
10 Other ambient spaces

In the previous sections we have seen how AD (or some of its consequences) plus DC(\(\mathbb{R}\)) yields a complete description of the Lipschitz and Wadge hierarchies on \(\mathbb{R}\) and on the Cantor spaces \(2^\omega\). Assuming the stronger AD\(_R\), the same analysis can be carried out for \(2^\omega\), where \(\mathbb{R}\) is given the discrete topology: in this case the Lipschitz and Wadge hierarchies look like their analogues on \(\mathbb{R}\). Suppose now we work in the space \([T]\), where \(T\) is a perfect pruned tree on \(\omega\). These spaces are not (in general) homogeneous, in the sense that
\[ [T_{\{s\}}] = [T] \quad \text{and} \quad s^{-1}[T] = \mathcal{N}([T]; s) \]
need not to hold.

Additional exercises

Exercise 10.1. Let \(T\) be a pruned, perfect, finite branching tree. Construct a \(\varphi: T \to ^\omega 2\) such that \(f_\varphi: [T] \to ^\omega 2\) is a homeomorphism.

Show that \(^\omega h\) and \(^\omega k\) are not Lipschitz isometric when \(2 \leq h < k < \omega\).

Exercise 10.2. Consider the game \(G\) on \(\omega_1\) where \(I\) plays an ordinal \(\alpha \geq \omega\) in his first move (and then his other moves are irrelevant) and \(II\) plays natural numbers \(n_i\), with the condition that \(II\) wins iff \(\langle n_i \mid i \in \omega\rangle\) codes a well-order of length \(\alpha\). Show that the determinacy of \(G\) implies the failure of the PSP.

Conclude that AD\(_{\omega_1}\) is inconsistent.

Show that AD\(_{\mathcal{P}(\mathbb{R})}\) is inconsistent.

Exercise 10.3. Show that a flip set \(F\) is Lipschitz self-dual, and that \(F_{\{s\}}\) is a flip set, for all \(s \in ^\omega 2\). Conclude that the existence of a flip set implies that \(<_L\) is ill-founded.

Notes and references

The basic results of this Chapter are from Wadge’s Ph.D. thesis [Wad83]. Our account follows closely that of [VW78], where Theorems 6.1 and 8.1 are presented: Martin never published Theorem 6.1, but other accounts of this result are in [Mos80, Exercise 7D.14s] and [Kec95, Theorem 21.15], while Theorem 8.1 was proved independently by Steel and Van Wesep and recorded in their dissertations [Ste77], [VW77]. Lemma 4.7 and Theorem 8.5 are from [Sol78]. Example 1.1 is taken from [Woo99, Remark 9.26, page 624]. Proposition 2.10 is due to Camillo Costantini, while the notion of Wadge structure and Proposition 4.6 are due to Alberto Marcone.
In this Chapter we start a systematic investigation of the consequences of \( \text{SLO}^W \). It turns out that many results proved in Chapter I under various assumptions like \( \text{AD} \), \( \text{AD}^L \), \( \text{AD}^W \), and \( \text{SLO}^L \) are already consequences of \( \text{SLO}^W \) (see diagram (34)). For example the axiom of countable choices over the reals and the perfect subset property follow from \( \text{SLO}^W \), and hence from \( \text{AD}^L \), \( \text{SLO}^L \) and \( \text{AD}^W \). In fact \( \text{SLO}^W \) is strong enough to recover the basic structure (diagram (35)) of the Wadge hierarchy. The detailed analysis of the Wadge hierarchy will be carried out under the assumption of \( \text{SLO}^W \) or some “local” version of it—the reason for this technical twists will be clear in Chapter IV.

11 Countable choices over \( \mathbb{R} \) and perfect subsets

The axiom of countable choices over the reals, \( \text{AC}_\omega(\mathbb{R}) \), is used all the time in descriptive set theory (and many other parts of mathematics). It is equivalent to a seemingly weaker statement: given any family of non-empty sets of reals \( A_n \), it is possible to choose an element from \( A_n \), for infinitely many \( n \).

**Lemma 11.1.** The following are equivalent:

(a) \( \text{AC}_\omega(\mathbb{R}) \).

(b) For any sequence \( \langle A_n \mid n \in \omega \rangle \) of non-empty sets of reals, there are \( \langle a_n \mid n \in \omega \rangle \) and \( k_0 < k_1 < \ldots \) such that \( \forall n \ (a_n \in A_{k_n}) \).

**Proof.** (a)\( \Rightarrow \) (b) is trivial, so we may assume (b) towards proving (a). Given \( \langle A_n \mid n \in \omega \rangle \) be as in the statement, let \( B_0 = A_0 \) and

\[
B_{n+1} = \{ x \oplus y \mid x \in A_{n+1} \land y \in B_n \}.
\]
By induction on \( n \), \( \emptyset \neq B_n \); hence fix an increasing sequence of integers \( k_0 < k_1 < \ldots \) and elements \( b_n \in B_{k_n} \). By composing the surjections

\[
B_{n+1} \to A_{n+1}, \quad z \mapsto (z)_I
\]

and

\[
B_{n+1} \to B_n, \quad z \mapsto (z)_{II}
\]

where \((z)_I\) and \((z)_{II}\) are as in (0.2), canonical surjections

\[
\pi_{m,n} : B_m \to A_n \quad (n \leq m)
\]

are obtained. Define for \( k_{m-1} < n \leq k_m \) (and \( k_{-1} = -1 \)),

\[
a_n = \pi_{k_m,n}(b_m) \in A_n.
\]

\(\square\)

For \( A \subseteq \mathbb{R} \) let

\[
A^\dagger = \bigcup_n 0^{(n)} \cap 1 \cap A \quad \text{and} \quad A^\circ = \{\emptyset\} \cup A^\dagger.
\]

The operations \( A \mapsto A^\dagger \) and \( A \mapsto A^\circ \) will be studied extensively in section 15. For the time being it is enough to notice that \( A \leq_L A^\dagger \) and \( A \leq_L A^\circ \) via the function \( x \mapsto 1^\cap x \).

**Theorem 11.2.** \( \text{SLO}^W \Rightarrow \text{AC}_\omega(\mathbb{R}) \).

**Proof.** Let \( \langle A_n \mid n \in \omega \rangle \) be a sequence of non-empty sets of reals. We want to replace the \( A_n \)'s with sets \( B_n \) that code the \( A_n \) and are \( <_W \)-increasing: using the function \( J_W \) of Lemma 4.7, define inductively

\[
B_0 = \{x \oplus y \mid x \in A_0 \wedge y \in \mathbb{R}\},
\]

\[
B_{n+1} = \{x \oplus y \mid x \in A_{n+1} \wedge y \in J_W(B_n^\dagger \oplus B_n^\circ)\}.
\]

Since \( B_n \leq_W B_n^\dagger \oplus B_n^\circ \) via \( x \mapsto \langle 0,1 \rangle^\cap x \), since \( B_n^\dagger \oplus B_n^\circ <_W J_W(B_n^\dagger \oplus B_n^\circ) \) by definition of \( J_W \), and since \( J_W(B_n^\dagger \oplus B_n^\circ) \leq_W B_{n+1} \) via \( y \mapsto x \oplus y \), where \( x \) is some fixed element of \( A_{n+1} \), it follows that

\[
\emptyset, \mathbb{R} <_W B_0 <_W B_1 <_W \ldots .
\]

Notice that for any fixed \( \bar{n} \)

\[
B_{\bar{n}}^\dagger, B_{\bar{n}}^\circ, \neg B_{\bar{n}}^\dagger, \neg B_{\bar{n}}^\circ <_W B_{\bar{n}+1}.
\]
Let

\[ C = \bigcup_n n \neg B^c_n \]

and

\[ D = \bigcup_{h,k \in \omega} 0^{(h)} \neg k + 1^c B^\updownarrow_k. \]

**Claim 11.2.1.** \( D \not\preceq_W -C. \)

**Proof of the Claim.** Suppose otherwise, and let \( \sigma \) be a winning strategy for \( \mathbf{II} \) in \( G_D^W(D, -C). \) Let \( \mathbf{I} \) play 0’s as long as \( \mathbf{II} \) passes: since \( \sigma \) is winning, there is a first round \( h \) when \( \mathbf{II} \) plays some natural number \( \bar{n}, \) that is \( \sigma(0^{(h)}) = \bar{n}. \) Let \( \mathbf{I} \) answer \( \bar{n} + 2, \) so that after this round the true positions of the two players in \( G_D^W(D, -C) \) are

\[
\begin{array}{cccc}
\mathbf{I} & \overbrace{0 \cdots 0}^h & \bar{n} + 2 \\
\mathbf{II} & \underbrace{p \cdots p}_{h-i} & \bar{n} & i
\end{array}
\]

where \( i \in \omega \cup \{p\}. \) Since \( \sigma \) is winning, \( \mathbf{II} \) can use \( \sigma \) to win what’s left of the game, namely \( G_D^W(D, 0^{(h)} \neg \bar{n} + 2, \neg C \{|(\bar{n}, i)\}). \) But

\[
(-C)_{\{|(\bar{n}, i)\}} = \begin{cases} 
- B^c_{\bar{n}} & \text{if } i \in \{p, 0\}, \\
- B_{\bar{n}} & \text{if } i = 1, \\
\emptyset & \text{otherwise,}
\end{cases}
\]

so in any case, by (2)

\[ (-C)_{\{|(\bar{n}, i)\}} <_W B^n_{\bar{n}+1} \leq_W B^\updownarrow_{\bar{n}+1} = D_{0^{(h)} \neg \bar{n} + 2}, \]

a contradiction. \( \square \)

By \( \text{SLO}^W \) \( C \leq_W D \) holds so let \( \tau \) be a winning strategy for \( \mathbf{II} \) in \( G^W_D(C, D). \) Since \( n \neg 0 \in C \) but \( 0 \not\in D, \) for every \( n \) there is a least \( m \) such that \( \tau\{n \neg 0^{(m)}\} \) is a non-zero natural number \( k_n + 1 \) and let \( h \) be the number of 0’s played so far by \( \tau; \) that is \( \tau \) responds with \( 0^{(h)} \neg k_n + 1 \) to \( \mathbf{I} \) playing \( n \neg 0^{(m)}). \)

**Claim 11.2.2.** \( n \leq k_n. \)

**Proof of the Claim.** Suppose otherwise. Then \( \tau \) would yield a continuous reduction of \( B^c_n \) to \( D_{0^{(h)} \neg k_n + 1} = B^\updownarrow_{k_n} <_W B_n \leq W B^\updownarrow_n: \) a contradiction. \( \square \)
For every $n$ let $m$ and $h$ be as above: since $0^{(h)}k_n + 1\bar{\bar{0}} \notin D$ and $n\bar{\bar{0}} \in C$, and since $\tau$ is winning, there are $i \in \omega$ and $g(n) \in \mathbb{R}$ be such that

$$0^{(h)}k_n + 1\bar{\bar{0}} 0^{(i)} 1g(n) \in D$$

is the result of applying $\tau$ to $n\bar{\bar{0}} \in C$. Then $\forall n \in \omega (g(n) \in B_{k_n})$ hence $a_n = g(n)1 \in A_{k_n}$. Therefore we have constructed a sequence of integers $k_n$ and reals $a_n \in A_{k_n}$. By Claim 11.2.2 $\{k_n \mid n \in \omega\}$ is infinite, so by passing to a subsequence we may assume that the $k_n$’s are increasing, so we are done by Lemma 11.1.

We now turn to the PSP. Let

$$P = \{x \in \mathbb{R} \mid \exists \infty n \ x(n) = 0\}$$

be the set of all sequences containing infinitely many 0s. It is easy to check that $P \in \Pi^0_2$.

**Lemma 11.3.** For every $A \subset \mathbb{R}$, if $P \leq_w A$ then $A$ contains a perfect set.

**Proof.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and such that $f^{-1}(A) = P$. Let also $\varphi : \omega \omega \to \omega \omega$ be such that $f = f_\varphi$. We will construct a complete binary tree $T$ such that $[T] \subseteq P$ and $f \mid [T]$ is injective, and therefore $f([T])$ will be a perfect subset of $A$. The tree will be

$$T = \{t \in \omega \omega \mid \exists w \in \omega 2 (t \subseteq s_w)\},$$

where the map $\omega 2 \to \omega \omega$, $w \mapsto s_w$ is such that

- $z \subseteq w$ $\Rightarrow$ $s_z \subseteq s_w$
- $z \perp w$ $\Rightarrow$ $s_z \perp s_w$.

Set $s_\emptyset = \emptyset$. Since $\bar{0} \in P$ and $\bar{1} \notin P$, then $f(\bar{0}) \in A$ and $f(\bar{1}) \notin A$, hence there are $n, m$ such that $\varphi(0^{(n)}) \perp \varphi(1^{(m)})$. Let $s_0 = 0^{(n)}$ and $s_1 = 1^{(m)}\bar{0}$: since $\varphi$ is monotone $\varphi(s_0) \perp \varphi(s_1)$. Arguing as before, $f(s_i\bar{0}) \neq f(s_i\bar{1})$, for $i = 0, 1$, so there are $n', m'$ such that $\varphi(s_{i,0}) \perp \varphi(s_{i,0})$, where $s_{i,0} = s_i\bar{0}^{(m')}\bar{0}$ and $s_{i,0} = s_i\bar{1}^{(m')}\bar{0}$. We can now repeat the argument for $s_{0,0}, s_{0,1}, s_{1,0}, s_{1,1}$, constructing all $s_w$ for $w$ of length 3, and so on.

**Corollary 11.4.** $SLO^W \Rightarrow$ PSP. In particular $SLO^W$ implies that $\mathbb{R}$ cannot be well-ordered.
11. COUNTABLE CHOICES OVER $\mathbb{R}$ AND PERFECT SUBSETS

Proof. We must show that if $A \subseteq \mathbb{R}$ is uncountable then it contains a perfect set. By $\text{SLO}^W$

$$P \leq_W A \lor A \leq_W \neg P.$$ If the former holds, then we are done by the Lemma. Otherwise $A \in \Sigma^0_2$, and hence $A = \bigcup_n C_n$, where each $C_n$ is closed. By the Cantor-Bendixson theorem (which is provable in ZF) each $C_n$ is either countable, or it contains a perfect set: if some $C_n$ contains a perfect set, then so does $A$; otherwise each $C_n$ is countable, hence by Theorem 11.2 and the remark following it, $A$ is countable.

Let us consider the notion of continuous reducibility in the generalized Baire space $\mathbb{W} \omega_1$. In this space, the games $G_\omega$ are games on $\mathbb{W} \omega_1$ and we know that $\text{AD}_{\omega_1}$ is false in ZF (Exercise 10.2). Therefore this does not bode too well for the consistency of the statement: “all games $G_\omega(A, B)$, with $A, B \subseteq \omega_1$, are determined.” In order to avoid unpleasant surprises we might just retreat to $\text{SLO}^W(\mathbb{W} \omega_1)$, the analogue of $\text{SLO}^W$ for $\omega_1$. Unfortunately, even this move won’t save us:

**Theorem 11.5.** $\text{SLO}^W(\omega_1)$ is inconsistent with ZF.

Proof. The proof is an elaboration of the ideas behind Theorem 11.2. Suppose $\text{SLO}^W(\omega_1)$ holds, and let $\pi : \omega_1 \to \omega_1$ be the retraction

$$\pi(x)(n) = \begin{cases} x(n) & \text{if } x(n) < \omega, \\ 2 & \text{otherwise.} \end{cases}$$

(The rationale for the 2 in the formula above is explained in (4) below.) Then, by Proposition 4.6, $\text{SLO}^W$ follows and hence Theorem 11.2 holds. We will show that there exist an $\omega_1$ sequence of reals, contradicting PSP and hence Corollary 11.4. In order to simplify the notation, let’s agree that $\mathcal{G}$ is the Wadge game on the space $\mathbb{W} \omega_1$, i.e., $\mathcal{G} = G^T_\omega$ where $T = \prec \omega_1$. Similarly $\leq_W^{\omega_1}$ and $\prec_W^{\omega_1}$ are abbreviated with $\leq$ and $\prec$. The definition of $A^\omega$ and $A^o$ (1) work also in the case of $A \subseteq \omega_1$. If $A \subseteq \omega_2$, then $A^\omega, A^o \subseteq \omega_2$ and

$$(4) \quad \pi^{-1}(A^\omega) = A^\omega \land \pi^{-1}(A^o) = A^o.$$ (This is the point of the specific definition of $\pi$.) For any $x \in \omega_2$ let $E_x$ be the binary relation on $\omega$ given by

$$n E_x m \iff x(\langle n, m \rangle) = 1$$ where $\langle \cdot, \cdot \rangle : \omega \times \omega \to \omega$ is a standard bijection, e.g., $\langle n, m \rangle = 2^n(2m + 1)$. As usual, $\text{WO}_\alpha$ is the set of all reals coding a well-order of type $\alpha \geq \omega$, i.e.,

$$\text{WO}_\alpha = \{ x \in \omega_1 \mid \langle \omega, E_x \rangle \cong \alpha \}.$$
CHAPTER II. SLO$^W$ AND WADGE DEGREES

By a theorem of J. Stern [Ste78]

$$W_\alpha \overset{\text{def}}{=} \omega_\alpha \in \Sigma_{2\alpha+2}^0 \setminus \Sigma_{2\alpha+1}^0.$$ 

Since $W_\alpha$ is a countable union of sets in $\Sigma_{2\alpha+2}^0$, then $\mathbb{R} \setminus W_\alpha \in \Pi_{2\alpha+2}^0$, and therefore

$$\alpha < \beta \Rightarrow (W_\beta \not\in W \setminus W_\alpha^\omega, \mathbb{R} \setminus W_\alpha, W_\alpha^\omega).$$

Let

$$A = \bigcup_{1 \leq \alpha < \omega_1} \alpha \cap W_\alpha^\omega \quad \text{and} \quad B = \bigcup_{1 \leq \alpha < \omega_1} 0^{(h)} \cap \alpha \cap W_\alpha^\omega.$$ 

We claim that

$$A \not\in B \quad \text{and} \quad B \not\in \omega_1 \setminus A,$$

and hence $\text{SLO}(\omega_1)$ fails.

Lemma 11.6. $B \not\in \omega_1 \setminus A$.

Proof. Suppose otherwise, and let $\tau$ be a winning strategy for $\text{II}$ in $G(B, \omega_1 \setminus A)$. Let $\text{I}$ play 0’s as long as $\text{II}$ passes: since $\tau$ is winning, there is a first round $h$ when $\text{II}$ plays some ordinal number $\bar{\alpha}$, that is $\tau(0^{(h)}) = \bar{\alpha}$. Let $\text{I}$ answer $\bar{\alpha} + 1$, so that after this inning is over, the positions of the two players in are

$$\begin{array}{c}
\text{I} & 0 \cdots \cdots 0 \\
\text{II} & p \cdots p \\
\end{array}$$

where $i \in \omega_1 \cup \{p\}$ and $p$ denotes “passing”. Since $\tau$ is winning, $\text{II}$ can use $\tau$ to win what’s left of the game. In other words, recalling the notion of localization of a set at a given sequence (3):

$$W_{\bar{\alpha}+1}^\omega = B_{0^{(h)}^{\bar{\alpha}+1}} \preceq (\omega_1 \setminus A)_{(\bar{\alpha}, \bar{\alpha})},$$

and this last set is

$$(\omega_1 \setminus A)_{(\bar{\alpha}, i)} = \begin{cases} 
\omega_1 \setminus W_\alpha^\omega & \text{if } i \in \{p, 0\}, \\
\omega_1 \setminus W_\alpha & \text{if } i = 1, \\
\omega_1 & \text{if } i < 1.
\end{cases}$$

The third case is impossible, since the only $X$ such that $X \preceq \omega_1$ is $\omega_1$ itself and $W_{\bar{\alpha}+1}^\omega \not\in \omega_1$. Therefore

$$W_{\bar{\alpha}+1}^\omega \preceq \omega_1 \setminus W_\alpha^\omega \quad \text{and} \quad W_{\bar{\alpha}+1}^\omega \preceq \omega_1 \setminus W_\alpha.$$
Proposition 4.6 and (4) imply that

\[ W_{\alpha+1} \leq_{W} R \setminus W_{\alpha} \] 

\[ \lor \quad W_{\alpha+1}^{\sim} \leq_{W} R \setminus W_{\alpha} , \]

and since \( W_{\alpha+1} \leq_{W} W_{\alpha+1}^{\sim} \) either of these contradicts (5). \( \square \)

Suppose now \( A \preceq B \) and let \( r \) be a winning strategy for \( \Pi \) in \( G(A, B) \). If \( I \) plays \( \alpha \sim \bar{0} \in A \), then \( \Pi \) cannot respond with \( \bar{0} \) since \( \bar{0} \notin B \), so for any \( \alpha \geq 1 \) let \( m_{\alpha} \in \omega \) be least such that \( r(\alpha \sim 0(m_{\alpha})) = \gamma_{\alpha} > 0 \), and let \( h_{\alpha} \) be the number of \( 0 \)s played so far; that is, after the \( m_{\alpha} \)-th inning is completed, the sequences of actual moves\(^1\) played by \( I \) and \( \Pi \) are \( \alpha \sim 0(m_{\alpha}) \) and \( 0(h_{\alpha}) \sim \gamma_{\alpha} \), respectively. (Notice that \( h_{\alpha} < m_{\alpha} \) is possible, since \( r \) can pass.) If \( \gamma_{\alpha} < \alpha \) then

\[ W_{\alpha} \preceq W_{\alpha}^{\sim} \]

\[ = A[\alpha \sim 0(m_{\alpha})] \]

\[ \preceq B[0(h_{\alpha}) \sim \gamma_{\alpha}] \]

\[ = W_{\gamma_{\alpha}} \]

and hence, by Proposition 4.6, \( W_{\alpha} \leq_{W} W_{\gamma_{\alpha}} \), contradicting (5).

Therefore \( \alpha \leq \gamma_{\alpha} \), for all \( 1 \leq \alpha < \omega_{1} \). Since \( \alpha \sim \bar{0} \in A \) but \( 0(h_{\alpha}) \sim \gamma_{\alpha} \notin B \), player \( \Pi \) cannot go on playing \( 0 \)s forever, and arguing as before, the first non-zero ordinal played must be a \( 1 \). Let \( g(\alpha) \in W_{\gamma_{\alpha}} \) be such that

\[ 0(h_{\alpha}) \sim \gamma_{\alpha} \sim 0(i) \sim 1 \sim g(\alpha) \]

is \( r \)'s answer to \( I \) playing \( \alpha \sim \bar{0} \). Let \( C \subseteq \omega_{1} \) be unbounded and such that the map \( \alpha \mapsto \gamma_{\alpha} \) is injective. Since the \( W_{\alpha} \)'s are disjoint, then the map \( C \to \mathbb{R}, \alpha \mapsto g(\alpha) \), yields an \( \omega_{1} \) sequence of distinct reals, contradicting \( \text{PSP} \).

This concludes the proof of Theorem 11.5. \( \square \)

Some remarks about the proof. Recall that a set \( A \subseteq \omega X \) is \( \Sigma_{1}^{1} \) if it is the projection of a closed subset of \( \omega X \times \mathbb{R} \), and it is \( \Delta_{1}^{1} \) if it and its complement \( \omega X \setminus A \) are both in \( \Sigma_{1}^{1} \). Every Borel set of \( \omega X \) is \( \Delta_{1}^{1} \) and if \( X \) is countable the converse holds by a theorem of Suslin [Kec95, Theorem 14.11]. For \( X \) uncountable Suslin's theorem fails and \( \Delta_{1}^{1} \) is strictly larger than the collection of Borel sets, and an alternative definition of \( \Delta_{1}^{1} \) is the following: it is the smallest \( \mathcal{D} \subseteq \mathcal{P}(\omega X) \) containing all open sets, closed under complements, countable unions, and open-separated unions, i.e., for any family \( A_{i} \in \mathcal{D} \) \( (i \in I) \) such that there are open disjoint sets \( U_{i} \supseteq A_{i} \), then \( \bigcup_{i \in I} A_{i} \in \mathcal{D} \). Working in \( \text{ZFC} \), Martin showed in [Mar90] that all \( \Delta_{1}^{1} \)

\(^{1}\)That is, disregarding the \( p \)'s.
games with pay-off in \( \omega X \) are determined and therefore under AC, no pair of \( \Delta^1_1 \) sets can witness the failure of \( \text{SLO}^W(\omega_1) \). On the other hand, the determinacy of closed set (and hence of \( \Delta^1_1 \) sets) of \( \omega X \), for any \( X \), implies AC, so Martin’s theorem does not apply in ZF. Our construction gave (in \( \text{ZF} + \text{AC}_\omega(\mathbb{R}) \)) a pair of \( \Delta^1_1 \) sets \( A \) and \( B \) such that \( B \not\subseteq \omega_1 \setminus A \) and such that \( A \preceq B \) implies the existence of an \( \omega_1 \) sequence of distinct reals: this contradicts PSP, but it is a consequence of AC, and in fact \( \text{ZFC} \vdash A \preceq B \).

12 \( \text{SLO}^W \) and the well-foundedness of \( <_W \)

As we already mentioned before on page 19, if \( S \supseteq T \) are pruned trees on some set \( X \), then there is a retraction of \( S \) onto \( T \), i.e., there is a Lipschitz map \( \varrho : S \to T \) which is the identity on \( T \). Obviously there are many retractions of \( S \) onto \( T \), so we will now define a specific one, called the canonical retraction of \( S \) onto \( T \), when \( S \supseteq T \) are non-empty pruned trees on \( \omega \).

\[(6) \quad \varrho_{S,T} : S \to T \]

by setting \( \varrho_{S,T}(s) = t \), if \( s \in T \), and for \( s \prec n \notin T \)

\[
\varrho_{S,T}(s \prec x) = \begin{cases} 
\varrho_{S,T}(s \prec n) & \text{if } \varrho_{S,T}(s \prec n) \in T, \smallskip \\
\varrho_{S,T}(s \prec m) & \text{if } \varrho_{S,T}(s \prec n) \notin T \text{ and } m \text{ is least } \\
& \text{such that } \varrho_{S,T}(s \prec m) \in T. 
\end{cases}
\]

Notice that if

\[(7) \quad t \in T|_{\varrho_{S,T}(s)} \Rightarrow \varrho_{S,T}(s \prec t) = \varrho_{S,T}(s \prec t).\]

The Lipschitz surjection induced by \( \varrho_{S,T} \) is denoted by

\[(8) \quad r_{S,T} : [S] \to [T],\]

and it is also called the canonical retraction of \([S]\) onto \([T]\). When \( S = \prec_\omega \omega \) we shall drop it from the index and write \( \varrho_T : \prec_\omega \omega \to T \) and \( r_T : \mathbb{R} \to [T] \).

In order to state many results in this book at the right level of generality we need a “local form” of the semilinar-ordering principle: for any \( A \subseteq \mathbb{R} \), \( \text{SLO}^W_{\leq A} \) holds iff

\[(\text{SLO}^W_{\leq A}) \quad \forall B, C \preceq_W A (B \leq_W C \lor \neg C \preceq_W B).\]

Exercise 12.1. Show that:
12. SLO$^W$ AND THE WELL-FOUNDEDNESS OF $<_W$

(i) \( \text{SLO}^W_{\leq A} \iff \text{SLO}^W_{\preceq A} \).

(ii) \( \text{SLO}^W_{\leq A} \implies \forall B <_W A (B <_W \neg A) \).

(iii) If \( B_{[n]} <_W B \leq A \) for all \( n \), and \( \text{SLO}^W_{\preceq A} \) holds, then \( B \) is self-dual.

Lemma 12.2. (a) If \( A \leq_W B \) then there is \( A' \equiv_W A \) such that \( A' \leq_L A \) and \( A' \leq_L B \).

(b) Assume \( \text{SLO}^W_{\leq B} \) and \( A <_W B \). Then there is \( A' \equiv_W A \) such that \( A' \leq_L B \) and \( A' \leq_L \neg B \) via a contraction, i.e., \( \mathbf{I} \) has a winning strategy in \( G_L(\neg B, A') \) and \( G_L(B, A') \).

Proof. (a) Let \( \varphi : <^\omega \omega \to <^\omega \omega \) be a tame map whose induced function witnesses \( A \leq_W B \). Then

\[
S = \{ s \in <^\omega \omega \mid \forall t \subset s \ \text{lh}(\varphi(t)) < \text{lh}(\varphi(s)) \}
\]

is the set of all sequences where \( \varphi \) properly extends its previous values. The idea is to replace \( S \) with \( <^\omega \omega \) via some enumeration of \( <^\omega \omega \), so that the resulting map will be Lipschitz. Recall from (9) that \( \langle s_i \mid i \in \omega \rangle \) is some fixed enumeration without repetitions of \( <^\omega \omega \) such that \( s_0 = \emptyset \). Let \( \gamma : <^\omega \omega \to <^\omega \omega \) be the continuous map

\[
\gamma(t) = \begin{cases} 
  s_{k_0+1} \wedge \cdots \wedge s_{k_n+1} & \text{if } t = \langle k_0, \ldots, k_n \rangle, \\
  \emptyset & \text{if } t = \emptyset,
\end{cases}
\]

and let \( g = f_\gamma : \mathbb{R} \to \mathbb{R} \) be its induced function. Let

\[
C = \{ z \in \mathbb{R} \mid \forall n \ \text{lh}(\varphi(z \mid n)) = n \}
= \{ z \in \mathbb{R} \mid \forall n \ \gamma(z \mid n) \in S \}.
\]

Given a real \( x \in \mathbb{R} \) we can construct, continuously in \( x \), a real \( h(x) = z \in C \) such that \( x = s_{z(0)+1} \wedge s_{z(1)+1} \wedge \cdots \) if \( z \mid n \) has been defined, let \( s \) be of least length (which exists since \( \varphi \) is continuous) such that \( s_{z(0)+1} \wedge \cdots \wedge s_{z(n-1)+1} \wedge s \subset x \) and \( s_{z(0)+1} \wedge \cdots \wedge s_{z(n-1)+1} \wedge s \in S \), that is \( \text{lh}(\varphi(s_{z(0)+1} \wedge \cdots \wedge s_{z(n-1)+1} \wedge s)) = n + 1 \), and set \( z(n) = \) the unique \( k \) such that \( s_{k+1} = s \). \( C \) is closed since \( C = [T] \) where

\[
T = \{ s \mid n \mid s \in S \land n \in \omega \}
\]

is a pruned tree. Clearly the function \( g \mid C : C \to \mathbb{R}, z \mapsto s_{z(0)+1} \wedge s_{z(1)+1} \wedge \cdots \) is Lipschitz. Let

\[
A' = (g \circ r_T)^{-1}(A),
\]
where \( r_T : \mathbb{R} \to [T] = C \) is the Lipschitz retraction of (8). Then \( A' \leq_L A \), and 
\( A \leq_W A' \) since \( A = h^{-1}(A') \). Moreover the Lipschitz map \( t \mapsto \varphi(\gamma(r_T(t))) \) 
witnesses \( A' \leq_L B \).

(b) If \( \forall k \, (A \notin_W B_{[k]}) \) then, by \( \text{SLO}^W_{\leq B} \), \( \forall n \, (B_{[k]} \leq_W -A) \) hence \( B \leq_W -A \), a contradiction. Similarly \( \forall m \, (A \notin_W -B_{[m]}) \) does not hold, so fix \( k \) and \( m \) such that \( A \leq_W B_{[k]}, -B_{[m]} \). Choose \( \varphi, \psi : \omega^\omega \to \omega^\omega \) \( \text{continuous, tame and such that } f_{\varphi}^{-1}(B_{[k]}) = A \) and \( f_{\psi}^{-1}(B_{[m]}) = A \). Let \( \gamma \) 
and \( g : \mathbb{R} \to \mathbb{R} \) be as above, and let

\[
C = \{ z \in \mathbb{R} \mid \forall n \, \min\{lh(\varphi(\gamma(z \upharpoonright n))), lh(\psi(\gamma(z \upharpoonright n)))\} = n \}
\]

Arguing as in part (a), \( C \) is closed, so \( C = [T] \) for some pruned tree \( T \), and let \( \pi = r_T : \mathbb{R} \to [T], \quad h : \mathbb{R} \to C \), and let \( A' = (g \circ r_T)^{-1}(A) \) be as above. Then \( A' \leq_L A \) via \( g \circ r_T \) and \( A \leq_W A' \) via \( h \), and \( t \mapsto \varphi(\gamma(r_T(t))) \) 
and \( t \mapsto \psi(\gamma(r_T(t))) \) witness \( A' \leq_L B_{[i]} \) and \( A' \leq_L -B_{[m]} \), respectively. Therefore \( A' \leq_L B, -B \) via contractions. \( \square \)

We can now re-prove Martin’s Theorem 7.2 for \( \prec_W \):

**Theorem 12.3.** Assume \( \neg \text{FS} + \text{DC}(\mathbb{R}) + \text{SLO}^W \). There is no sequence \( \langle A_n \mid n \in \omega \rangle \) such that \( A_{n+1} \prec_W A_n \) for all \( n \in \omega \).

**Proof.** Using Lemma 12.2(b), by induction on \( n \) choose \( A'_n \in [A_n]_W \) such 
that \( I \) has winning strategies \( \sigma^0_n \) in \( G_L(A'_n, A_{n+1}) \) and \( \sigma^1_n \in G_L(-A'_n, A_{n+1}) \).
(\( \sigma^i_n \) are reals and the \( A'_n \) are of the form \( L_{<\omega}^{-1}(A_0) \), so can be coded \nby reals; therefore choosing \( A'_n, \sigma^0_n, \) and \( \sigma^1_n \) requires only \( \text{DC}(\mathbb{R}) \).) We can 
now repeat \textit{verbatim} the argument of the proof in Theorem 6.1 and reach a \ncontradiction. \( \square \)

In fact the proof shows that

**Theorem 12.4.** Assume \( \text{DC}(\mathbb{R}) + \text{SLO}^W_{\leq A} \). Then there is no sequence \( \langle A_n \mid n \in \omega \rangle \) such that \( A_{n+1} \prec_W A_n \prec_W A \) for all \( n \in \omega \).

**Exercise 12.5.** Suppose \( A_0 \prec_W A_1 \prec_W A_2 \prec_W \ldots \) and assume that \( \text{SLO}^W_{\leq A_n} \) holds, for all \( n \). Show that if \( \bigoplus_n A_n \) is self-dual and \( [\bigoplus_n A_n]_W \) is the least upper bound of the \( A_n \)’s.

13 The Wadge tree of a set

Wadge introduced the following definition:

\[
T(A) = \{ s \in \omega^\omega \mid A_{[s]} \equiv_W A \}.
\]
Exercise 5.2((iii)) implies that $T(A)$ is a tree, and since $A_{[0]} = A$, it is a non-empty tree.

**Exercise 13.1.** Show that $T(\neg A) = \neg T(A)$ and that for $t \in T(A)$

$$T(A_{[t]}) = T(A)_{[t]}.$$  

If $s$ is a terminal node of $T(A)$, then $A_{[s-n]} <_W A \equiv_W A_{[s]}$ for all $n$, which by $\text{SLO}^W_{\leq A}$ implies that $\neg A_{[s-n]} <_W A_{[s]}$ for all $n$, and hence $\bigoplus_n \neg A_{[s-n]} \leq_W A_{[s]} = \bigoplus_n A_{[s-n]}$ is self-dual. Therefore we have shown that

**Proposition 13.2.** Assume $\text{SLO}^W_{\leq A} + \text{AC}_\omega(\mathbb{R})$. If $T(A)$ is not pruned then $A$ is self-dual. In particular, if $T(A)$ is well-founded, then $A$ is self-dual.

Conversely:

**Theorem 13.3.** Assume $\text{AC}_\omega(\mathbb{R}) + \neg \text{FS}$. If $A$ is Wadge-self-dual, then $T(A)$ is well-founded.

**Corollary 13.4.** Assume $\text{SLO}^W_{\leq A} + \text{AC}_\omega(\mathbb{R}) + \neg \text{FS}$. Then

- $A$ is self-dual $\iff$ $T(A)$ is well-founded $\iff$ $T(A)$ has a terminal node,
- $A$ is non-self-dual $\iff$ $T(A)$ is ill-founded $\iff$ $T(A)$ is pruned.

**Proof of 13.3.** Towards a contradiction, let $x \in [T(A)]$ and $A \leq_W \neg A$. Clearly for every $n \in \omega$, $\Pi$ wins $G_W(A_{[x\upharpoonright n]}, \neg A_{[x\upharpoonright n]})$ via some strategy $\tau_n$, and $I$ wins $G_L(A_{[x\upharpoonright n]}, \neg A_{[x\upharpoonright n+1]})$ via the strategy $\sigma_n$ that plays $x(n)$ in the first round and then copies $\Pi$’s moves. Let $\tau$ be $\Pi$’s copying strategy, i.e., $\tau(s) = \tau(n)$ is the last element of $s$. We are now going to apply the Martin-Monk method. The basic idea is to pit $\tau$ against the $\tau_n$’s so that a flip set is constructed, contradicting our assumption $\neg \text{FS}$. The problem is that $\tau$ and $\tau_n$ are strategies for $\Pi$, while we need strategies for $I$ in order to start the filling-in procedure of the Martin-Monk diagram. This is where the $\sigma_n$’s come into the picture: an increasing sequence of natural numbers $0 = M_0 < M_1 < M_2 < \cdots$ is defined so that $M_k + 1 < M_{k+1}$ and on the $M_k$-th row $\Pi$ plays using either $\tau$ or $\tau_n$ for some $n$, and on the $m$-th row with $m \notin \{M_k \mid k \in \omega\}$ $I$ plays using $\sigma_n$ for an appropriate $n$. More precisely, letting $\langle N_j \mid j \in \omega \rangle$ be the enumeration of $\omega \setminus \{M_k \mid k \in \omega\}$

- on the $M_k$-th row $\Pi$ is playing $G_W(\neg A_{[x\upharpoonright i]}, A_{[x\upharpoonright i]})$ using $\tau_i$ or $\tau$, where $i$ is such that $N_i = M_k + 1$;
- on the $N_j$-th row $I$ is playing $G_L(A_{[x\upharpoonright j]}, \neg A_{[x\upharpoonright j+1]})$ using $\sigma_j$. 

The choice of using $\tau$ versus $\tau_i$ on the $M_k$-th row, with $M_k + 1 = N_i$, will be given by $z(k)$ where $z \in \omega^2$: $\tau$ will be used when $z(k) = 0$, and $\tau_i$ will be used when $z(k) = 1$. For example, if the first few $M_k$’s are $(0, 3, 5, 8, \ldots)$ and $z = (0, 1, 1, 0, \ldots)$ then the strategies in the Martin-Monk diagram are: 
\[
\langle \tau, \sigma_0, \sigma_1, \tau_2, \sigma_2, \tau_3, \sigma_3, \tau_4, \sigma_4, \tau, \sigma_5, \ldots \rangle.
\]
Since $\tau$ is the copying strategy, for any $s \in \omega^\omega$, $\tau$’s response to $s$ is a sequence of the same length as $s$, that is $\text{lh}((s * \tau)_\Pi) = \text{lh}(s)$. On the hand, the $\tau_n$’s might pass, so it can happen that $\tau_n$’s response to $s$ is a sequence of shorter length, $\text{lh}((s * \tau_n)_\Pi) < \text{lh}(s)$. Therefore the $M_k$’s are to be taken sparse enough so that the filling-in procedure does not come to a grinding halt. Arguing as in the proof of Theorem 8.1, suppose $\langle M_k \mid k \leq n \rangle$ has been defined and for every $s \in ^{n+1}2$ the finite diagram with $n + 1$ rows where $\tau$ is used on the $M_k$-th row iff $s(k) = 0$, has the first $n$ entries of the 0-th row are filled-in. Fix any $s \in ^{n+1}2$. Suppose the $M_n + 1$-st row is filled-in with $\langle x(m + i) \mid m \in \omega \rangle$, where $i$ is such that $N_i = M_n + 1$. Then the 0-th row is completely determined, so by continuity of the strategies there is a $j = j(s) > 0$ such that $\langle x(i), x(i + 1), \ldots, x(i + j - 1) \rangle$ is enough to fill-in the first $n + 1$ entries of the 0-th row. Therefore if the $\sigma$’s are used in the rows $M_n + 1, \ldots, M_n + j$, then the first $n + 1$ entries of the 0-th row are filled-in. Let
\[
M_{n+1} = \sup \{M_n + j(s) + 1 \mid s \in ^{n+1}2 \}.
\]
By construction of the $M_k$’s, for any $z \in \omega^2$ the Martin-Monk diagram relative to $z$ can be filled-in. Then $\{z \in \omega^2 \mid$ the real in the 0-th row of the Martin-Monk diagram relative to $z$ is in $A\}$ is a flip set, contradicting $\neg\text{FS}$. $\square$

**Lemma 13.5.** Assume $\neg\text{FS} + \text{AC}_\omega(\mathbb{R})$ and that $A \equiv_W B$.

(a) If $f : \mathbb{R} \to \mathbb{R}$ witnesses $A \leq_W B$ then $f([T(A)]) \subseteq [T(B)]$.

(b) $A \cap [T(A)] \equiv_W B \cap [T(B)]$

**Proof.** (a) follows from Exercise 4.4.

(b) If $A$ and $B$ are self-dual or if $A = B = \mathbb{R}$, then result follows for trivial reasons, so we may assume that $A$ and $B$ are non-self-dual and different from $\mathbb{R}$. Let $\tau$ be $\Pi$’s winning strategy in $G_W(A, B)$. The $\Pi$ wins $G_W(A \cap [T(A)], B \cap [T(B)])$ as follows:

As long as $\Pi$ plays inside $T(A)$ then $\Pi$ follows $\tau$, so that if $I$ ends up with a real in $[T(A)]$, then $x \in A \Leftrightarrow (x * \tau)_\Pi \in B \cap [T(B)]$. If at some round the positions of $I$ and $\Pi$ are $p \in T(A)$ and $q \in T(B)$, respectively, and if $I$ plays $n$ such that $p^n q \not\in T(A)$,
then \( \mathbf{II} \) starts enumerating a real outside \( B_{[q]} \), which exists since \( B_{[q]} \equiv_W B \neq \mathbb{R} \).

\[ \square \]

**Proposition 13.6.** Assume \( AC_\omega(\mathbb{R}) + \neg FS + SLO^W_{\leq A} \). Then \( A \cap [T(A)] \leq_L A \).

**Proof.** If \( A \) is self-dual then the result holds trivially, so we may assume that \( A \) is non-self-dual and hence \( T(A) \) is pruned. We may also assume that \( A \neq \mathbb{R} \). We claim that if \( t \cap n \in \partial T(A) \), then \( N_t \setminus A \neq \emptyset \). In fact if \( N_t \subseteq A \) then \( \mathbb{R} = A_{[t]} \equiv_W A \) so that \( A = \mathbb{R} \), contrarily to our assumption. Then \( \mathbf{II} \) wins \( G_L(A \cap [T(A)], A) \) as follows:

\( \begin{align*}
& \mathbf{II} \text{ copies } \mathbf{I}'s \text{ moves as long as they yield positions inside } T(A), \\
& \text{so that if } \mathbf{I}'s \text{ play is in } [T(A)] \text{ then } \mathbf{II} \text{ wins this game. Suppose} \\
& \text{at some round } \mathbf{I} \text{ plays } n \text{ so that his resulting position is } t \cap n \in \\
& \partial T(A): \text{ then } \mathbf{II} \text{ follows from now on a real in } N_t \setminus A.
\end{align*} \)

\[ \square \]

In general, \( A \cap [T(A)] \) is much simpler than \( A \) — see Lemma 15.2(c). If \( A \cap [T(A)] \) is of the same degree as \( A \), then the operation \( T \) applied to \( A \) or to \( A \cap [T(A)] \) yields the same tree.

**Proposition 13.7.** Assume \( \neg FS + AC_\omega(\mathbb{R}) \). Then

\[ A \equiv_W A \cap [T(A)] \Rightarrow T(A) = T(A \cap [T(A)]) . \]

**Proof.** Suppose \( A \equiv_W A \cap [T(A)] \). If \( A = \mathbb{R}, \emptyset \) then the result follows at once, so we may assume \( A \neq \mathbb{R}, \emptyset \). If \( t \in T(A) \) then

\[ \begin{align*}
(A \cap [T(A)])_{[t]} &= A_{[t]} \cap [T(A)]_{[t]} \\
&= A_{[t]} \cap [T(A_{[t]})] \quad \text{by Exercise 13.1} \\
&\equiv_W A \cap [T(A)]
\end{align*} \]

the third line following from \( A_{[t]} \equiv_W A \) and Lemma 13.5(b). Therefore \( t \in T(A \cap [T(A)]) \).

Conversely, suppose \( t \notin T(A) \): then \( \emptyset = (A \cap [T(A)])_{[t]} \leq_W A \cap [T(A)] \), and by case assumption the inequality is strict. Hence \( t \notin T(A \cap [T(A)]) \). \( \square \)

A set \( A \subseteq \mathbb{R} \) is said to be **homogeneous** just in case \( T(A) = <^\omega A \). For example \( \mathbb{R} \) and \( \emptyset \) are homogeneous, and so is the set \( P \) of (3). On the other hand no open or closed set other than \( \mathbb{R} \) and \( \emptyset \) is homogeneous, nor is any self-dual set. Since \( T(\neg A) = T(A) \) then homogeneity is preserved by taking
complements; on the other hand it is not preserved by Wadge equivalence: for example, if \( A \) is homogeneous, then \( A \equiv_W 0^\perp A \) but \( 0^\perp A \) is not homogeneous.

Van Wesep characterized in [VW77] all degrees which contain a homogeneous set—see Chapter ?? For the time being let us prove the following result.

**Lemma 13.8.** If \( A \) is non-self-dual and \( A \equiv_W A \cap T(A) \), then there is \( B \equiv_L A \cap [T(A)] \) such that \( B \) is homogeneous.

**Proof.** If \( A \) is closed the result is trivial, so we may assume otherwise. Let

\[ \rho = \varphi_{T(A)} : \omega \to T(A) \]

be the canonical surjection of \( \omega \) onto \( T(A) \) and let \( r : \mathbb{R} \to [T(A)] \) be the induced Lipschitz retraction—see (6),(8). By (7) \( s \in T(A)_{[\rho(t)]} \Rightarrow \rho(t) \circ s = \rho(t) \circ s. \) Let \( B = r^{-1}(A \cap [T(A)]) \).

**Claim 13.8.1.** \( \forall t \in \omega \left( A_{[\rho(t)]} \cap [T(A)]_{[\rho(t)]} \leq_L B_{[t]} \right) \).

**Proof of Claim.** Fix a \( t \in \omega \). Player II wins \( G_W(A_{[\rho(t)]} \cap [T(A)]_{[\rho(t)]}, B_{[t]}) \) as follows:

As long as I plays inside \( T(A)_{[\rho(t)]} \), then II copies I’s moves: if I’s complete play is \( a \in [T(A)_{[\rho(t)]}] \) then II’s complete play will also be \( a \) and

\[ a \in A_{[\rho(t)]} \iff a = r^{-1}(a) \in B, \]

since \( r \restriction A_{[\rho(t)]} \cap [T(A)]_{[\rho(t)]} \) is the identity.

Suppose that at some round I plays an \( n \) such that \( s \circ n \in \partial(T(A)_{[\rho(t)]}) \), that is, \( \rho(t) \circ s \circ n \in \partial T(A) \). Since I’s play will be for sure outside \( A_{[\rho(t)]} \cap [T(A)]_{[\rho(t)]} \) we must guarantee that II’s play will be outside \( B_{[t]} \). Since \( \rho(t) \circ s \in T(A) \), then \( A \equiv_W A_{[\rho(t)] \circ s} \) and hence

\[ A \cap [T(A)] \equiv_W A_{[\rho(t)] \circ s} \cap [T(A)_{[\rho(t)] \circ s}] \]

by Lemma 13.5(b), and since \( T(A)_{[\rho(t)] \circ s} = T(A)_{[\rho(t)]} \) by Exercise 13.1 and \( A \cap [T(A)] \equiv_W A \) is not closed by case assumption, then \( [T(A)]_{[\rho(t)] \circ s} \backslash A_{[\rho(t)] \circ s} \neq \emptyset. \) Then II can enumerate a real \( \bar{a} \in [T(A)]_{[\rho(t)] \circ s} \backslash A_{[\rho(t)] \circ s} \) from now on, so that his final play will be \( s \circ \bar{a} \). Since \( s \circ \bar{a} \in [T(A)]_{[\rho(t)]} \), then \( r(s \circ \bar{a}) = s \circ \bar{a} \) and therefore \( s \circ \bar{a} \in B_{[t]} \) would imply \( s \circ \bar{a} \in A_{[\rho(t)]} \), contrarily to the choice of \( \bar{a} \). Therefore \( s \circ \bar{a} \notin B_{[t]} \).
By taking $t = \emptyset$ in the Claim, we have that $A \cap [T(A)] \leq_L B$ and since $r$ witnesses that $B \leq_W A \cap [T(A)]$,

$$B \equiv_L A \cap [T(A)].$$

By Proposition 13.7 $T(A \cap [T(A)]) = T(A)$, and since $\rho(t) \in T(A)$ for every $t$, then

$$\forall t \in \omega(A \cap [T(A)]) \equiv_W A \cap [T(A)] \equiv_W A.$$ 

Therefore by the Claim $\forall t \in \omega(B \equiv_W A)$, that is $T(B) = \omega$, which is what we had to prove.

14 SLO$^W$ and the Wadge hierarchy

**Proposition 14.1.** (a) Assume SLO$^W_{\leq A} + AC_\omega(\mathbb{R})$. If there is $\{D_n \mid n < N\}$ a clopen partition of $\mathbb{R}$ such that $\forall n < N (A \cap D_n < W A)$, then $A \leq_W \neg A$.

(b) Assume $AC_\omega(\mathbb{R}) + \neg FS$. If $A$ is not clopen and $A \leq_W \neg A$ then there is $\{D_n \mid n < \omega\}$ a clopen partition of $\mathbb{R}$ such that $\forall n < \omega (A \cap D_n < W A)$.

**Proof.** (a) $A \cap D_n \leq_W \neg A$ by SLO$^W_{\leq A}$, so we can choose continuous reductions $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n^{-1}(\neg A) = A \cap D_n$. Then $\bigcup_n (f_n \upharpoonright D_n) : \mathbb{R} \to \mathbb{R}$ is continuous and witnesses $A \leq_W \neg A$.

(b) By Theorem 13.3 $T(A)$ is well-founded so

$$A = \bigcup_{t \in \partial T(A)} t^\ast A_{[t]}.$$ 

Let $\{D_n \mid n < \omega\}$ be an enumeration without repetitions of $\{N_t \mid t \in \partial T(A)\}$ so that it is a clopen partition of $\mathbb{R}$. Then each $A \cap D_n$ is of the form $t^\ast A_{[t]}$ with $t \in \partial T(A)$: if $A_{[t]} \neq \mathbb{R}$ then $t^\ast A_{[t]} \equiv_W A_{[t]} < W A$, and if $A_{[t]} = \mathbb{R}$ then $t^\ast A_{[t]}$ is clopen, hence $<_W A$. Therefore $\forall n < \omega (A \cap D_n < W A)$. 

**Exercise 14.2.** Assume SLO$^W_{\leq A} + AC_\omega(\mathbb{R}) + \neg FS$. Then $[A]_W$ is self-dual iff the clopen partition of $\mathbb{R}$ can be taken to be finite—in fact of size 2.

**Lemma 14.3.** Assume $\neg FS + DC(\mathbb{R})$. If $A < W B$ are self-dual and SLO$^W_{\leq B}$ holds, then there is a non-self-dual $C$ such that $A < W C < W B$.  

Proof. It is enough to show that there is some \( C \) such that \( A <_W C <_W B \), since if every such \( C \) were self-dual, an infinite descending chain would result

\[
A <_W \ldots <_W C_2 <_W C_1 <_W C_0 <_W B,
\]

contrarily to Theorem 12.3. Towards a contradiction, suppose there is no \( C \) such that \( A <_W C <_W B \). If \( t \in \text{tn}(T(B)) \), then for every \( n \in \omega \)

\[
B_{[t\downarrow n]} <_W B_{[t]} \equiv_W B
\]

and by \( \text{SLO}^W_{\leq B} \) and \( A \equiv_W \neg A \),

\begin{equation}
B_{[t\downarrow n]} \leq_W A.
\end{equation}

By Theorem 13.3 \( T(B) \) is well-founded. \( \square \)

The following is the analogue of Lemma 5.8 for the Wadge hierarchy.

**Exercise 14.4.** Assume \( \text{SLO}^W_{\leq A} \) and \( \text{AC}_\omega(\mathbb{R}) \) and suppose \( A_n <_W A \) are such that

\begin{equation}
\forall n \exists m > n (A_n <_W A_m).
\end{equation}

Then \( [\bigoplus_n A_n]_W \) is self-dual and it is the least-upper bound of the \( [A_i]_W \)'s.

We are now able to prove Corollary 8.2 under weaker hypotheses.

**Theorem 14.5.** Assume \( \text{SLO}^W + \neg \text{FS} + \text{DC}(\mathbb{R}) \). Then

(a) \( \prec_W \) is well-founded,

(b) immediately above a self-dual Wadge degree there is a non-self-dual pair of Wadge degrees,

(c) immediately above a non-self-dual pair of Wadge degrees there is a self-dual Wadge degree,

(d) at limit levels of countable cofinality there is a single self-dual Wadge degree, and at all other limit levels there is a non-self-dual pair of Wadge degrees.

**Proof.** (a) and (b) follow from Theorem 12.3 and Lemma 14.3, respectively. If \( A \not\leq_W \neg A \), then, arguing as in Lemma 5.7, \( A \oplus \neg A \) is Wadge self-dual and above \( A \) and \( \neg A \). If \( A \leq_W B \) and \( \neg A \leq_W B \) via reductions \( \varphi \) and \( \psi \), then \( \Pi \) wins \( G_W(A \oplus \neg A, B) \) by passing in the first round, and then following the appropriate reduction depending on whether \( I \) played an even
or an odd number. Therefore $[A \oplus \neg A]_W$ is the self-dual degree immediately above $[A]_W$ and $[-A]_W$. This settles (c).

Now for (d). If $A_0 <_W A_1 <_W \cdots$ witness $[A]_W$ is of countable cofinality, then by Exercise 12.5, $\bigoplus_n A_n$ is Wadge self-dual and $\bigoplus_n A_n \equiv_W A$. Conversely, suppose $[A]_W$ is limit and self-dual, so that by Lemma 13.3 $T(A)$ is well-founded. Then $\{N_s \mid s \in \partial T(A)\}$ is a clopen partition of $\mathbb{R}$. If $B$ is such that

$$\forall s \in \partial T(A) \left(A_{[s]} \leq_W B \right),$$

then $A = \bigcup_{s \in \partial T(A)} s^c A_{[s]} \leq_W B$ by Exercise 8.8, and therefore $\|A\|_W = \sup \{|A_{[s]}|_W \mid s \in \partial T(A)\}$, witnessing that $[A]_W$ is of countable cofinality.

Note that the same proof shows that the Wadge hierarchy has the expected behavior below $A$, if $\mathsf{SLO}^W_{\leq A}$ rather than the full $\mathsf{SLO}^W$ is assumed.

### 15 The next non-self-dual pair

Recall from (1) the definition of $A^0 = \bigcup_n 0^{(n)} A$ and $A^0 = A^0 \cup \{\bar{0}\}$.

**Exercise 15.1.** Show that:

(i) $A \leq_L A^0, A^0$;

(ii) $A \leq_W B \Rightarrow A^0 \leq_W B^0 \wedge A^0 \leq_W B^0$, and similarly for $\leq_L$ instead of $\leq_W$;

(iii) $(A^0)^0 \equiv_L A^0$ and $(A^0)^0 \equiv_L A^0$;

(iv) $\mathbb{R}^0$ is open and $\mathbb{R}^0$ is the closure of $\mathbb{R}^0$;

(v) $\emptyset^0 = \{\bar{0}\}$ and $\emptyset^0 = \emptyset$.

There is an annoying lack of symmetry between ((iv)) and ((v)) due to the specific definition of $A^0$ in (1): had we defined it as

$$\bigcup_{n \geq 0} 0^{(n)} m^c A$$

all of the previous result would still hold (with minor changes in the proofs) and ((iv)) would become

$$(iv)^* \mathbb{R}^0 = \mathbb{R} \setminus \{\bar{0}\} \text{ and } \mathbb{R}^0 = \mathbb{R}.$$
The reason for the specific definition in (1) is that it makes sense even if $A$ is a subset of the Cantor space.

In view of ((ii)) the $\lor$ and $\cap$ operations can be defined on Lipschitz and Wadge degrees:

$$
[A]_L^\lor = [A^\lor]_L \quad \text{and} \quad [A]_L^\cap = [A^\cap]_L.
$$

$$
[A]_W^\lor = [A^\lor]_W \quad \text{and} \quad [A]_W^\cap = [A^\cap]_W.
$$

**Lemma 15.2.** (a) If $A$ is self-dual, then $A^\lor \equiv_W \neg A^\cap$.

(b) If $\neg \text{FS} + \text{AC}_o(\mathbb{R})$ holds then $A^\lor$ and $A^\cap$ are non-self-dual and $\{[A]_W^\lor, [A]_W^\cap\}$ is a non-self-dual pair above $[A]_W$.

(c) If $\neg \text{FS} + \text{AC}_o(\mathbb{R})$ hold and $A \neq \emptyset$, then $[T(A^\lor)] \cap A^\lor <_W A^\lor$.

**Proof.** (a) Let $\tau$ be the winning strategy for $\Pi$ in $G_W(A, \neg A)$. Then $\Pi$ wins $G_W(A^\lor, \neg A^\cap)$ as follows:

As long as $I$ plays 0 then $\Pi$ answers 0. If at some round $I$ plays a non-zero integer $n$, then $\Pi$ plays $n$, and after that follows $\tau$.

The proof that $A^\cap \leq_W A^\lor$ is similar.

(b) Since $T(A^\lor) = T(A^\cap) = \{0^{(n)} \mid n \in \omega\}$ is ill-founded, Lemma 13.3 implies that $A^\lor$ and $A^\cap$ are non-self-dual. Therefore $\{[A]_W^\lor, [A]_W^\cap\}$ is a non-self-dual pair above $[A]_W$.

(c) By part (b) $A^\lor$ is non-self-dual and $A^\lor \neq \emptyset$. By Proposition 13.6, $\emptyset = [T(A^\lor)] \cap A^\lor \leq_L A^\lor$, and the result follows at once. $\square$

**Proposition 15.3.** Assume $\text{SLO}^W_{\leq B}$ with $B$ non-self-dual, and let $A \equiv_W \neg A <_W B$.

(a) If $[T(B)] \setminus B \neq \emptyset$ then $A^\lor \leq_W B$.

(b) If $[T(B)] \cap B \neq \emptyset$ then $A^\cap \leq_W B$.

**Proof.** Suppose $x \in [T(B)] \setminus B$. Then $\Pi$ wins $G_W(B, A^\lor)$ as follows:

$\Pi$ enumerates $x$ as long as $I$ plays 0s. Suppose $n$ is the first round when $I$ plays $k > 0$: up to this round $\Pi$’s position is $x \upharpoonright n \in T(B)$ and

$$
B \equiv_W B_{[x[n]}.
$$

We distinguish two cases:
Exercise 15.6.

This completes the proof. 

II show that $B$ as each enumeration of alialities, suppose $B$ wins.

Proof. Suppose $B$ wins. Then either $A$ wins or $B$ wins. Then either Proposition 15.5.

This proves (a). The proof of (b) is analogous and it is left to the reader. □

Corollary 15.4. Suppose $A$ is self-dual and assume SLO$^W_{\leq A^o}$ or, equivalently, SLO$^W_{\leq A^o}$. Then $\{[A^o]^W, [A^o]^W\}$ is the least non-self-dual pair above $[A]^W$, and for any $B \equiv_W A^o$ and $C \equiv_W A^o$

$$[T(B)] \cap B = \emptyset \land [T(C)] \subseteq C.$$  

Proof. Suppose $B$ is non-self-dual and, say, $A <_W B \leq_W A^o$, so that SLO$^W_B$ holds. (The case of $A <_W B \leq_W A^o$ is analogous.) If there is an $x \in [T(B)] \cap B$ then $A^o \leq_W B$ and hence $A^o \leq_W A^o$, contradicting Lemma 15.2, so $[T(B)] \cap B = \emptyset$. Therefore $A^o \leq_W B$, which implies $A^o \equiv_W B$ and $A^o \equiv_W \neg B$. □

The hypothesis SLO$^W_{\leq A^o}$ in the Corollary can be weakened to SLO$^W_{\leq A}$—see section 17. Conversely

Proposition 15.5. Suppose $A$ is non-self-dual and SLO$^W_{\leq A} + ACW(\mathbb{R})$ holds. Then either $[T(A)] \subseteq A$ or $[T(A)] \cap A = \emptyset$ imply $[A]^W$ is a successor degree or $A \in \{\mathbb{R}, \emptyset\}$.

Proof. For the sake of definiteness, suppose $[T(A)] \subseteq A$ and, to avoid trivialities, suppose $A \notin \{\mathbb{R}, \emptyset\}$, so that $T(A) \neq ^\omega \omega$. Let $\{B_n \mid n \in \omega\}$ be an enumeration of $\{A_{[s]}, \neg A_{[s]} \mid s \in \partial T(A)\}$. The set $B = \bigoplus_n B_n$ is self-dual, as each $B_n$ is the dual of some $B_{n}$. By SLO$^W_{\leq A}$ each $B_n <_W A$ and therefore $B \leq_W A$. As $A$ is non-self-dual the inequality is strict, i.e., $B <_W A$. We will show that $A \leq_W B^o$ and therefore $A \equiv_W B^o$, completing the proof. Player II wins $G_W(A, B^o)$ as follows:

As long as $I$ plays in $T(A)$ then $II$ plays $0$. Suppose at some round $I$ reaches a position $s \in \partial T(A)$: then $II$ plays $n$, where $n$ is such that $B_n = A_{[s]}$ and copies $I$’s moves from now on.

This completes the proof. □

Exercise 15.6. Show that:

(a) If $T$ is a tree on $\omega$, $B_s \leq_W A$ for all $s \in \partial T$, and SLO$^W_{\leq A} + ACW(\mathbb{R})$ holds, then

$$S(T; B) \leq_W A^7 \land S(T; B) \cup |T| \leq_W A^o.$$
(b) If \( A \leq W \neg A \) and \( B \leq W A^\circ \), then \( B = S(T; \bar{B}) \) for some sequence \( \bar{B} = \langle B_s \mid s \in \partial T \rangle \) such that \( B_s \leq W A \).

Similarly, if \( A \leq W \neg A \) and \( B \leq W A^\circ \), then \( B = S(T; \bar{B}) \cup [T] \) for some sequence \( \bar{B} = \langle B_s \mid s \in \partial T \rangle \) such that \( B_s \leq W A \).

Using this we can characterize the sets in non-self-dual successor degrees.

**Proposition 15.7.** Assume \( \neg FS + AC_\omega(\mathbb{R}) \) and \( A \leq W \neg A \). Then \( B \in [A^\circ]_W \) iff \( B = S(T; \bar{B}) \) for some tree \( T \) on \( \omega \) and some \( \bar{B} = \langle B_s \mid s \in \partial T \rangle \) such that

\[
\exists \bar{x} \in [T] \forall A' <_W A \forall n \in \omega \exists s \in \partial T (s \supset \bar{x} \mid n \wedge A' \leq W B_s).
\]

Similarly, \( B \in [A^\circ]_W \) iff \( B = S(T; \bar{B}) \cup [T] \) for some \( T \), \( \bar{B} \) satisfying (13)

**Proof.** We consider only the case of \( A^\circ \), the other one being similar. All sets \( S(T; \bar{B}) \) as above are \( \leq W A^\circ \) by the Exercise 15.6, so we must check that II wins \( G_W(A^\circ, S(T; \bar{B})) \):

As long as I plays 0’s, II enumerates \( \bar{x} \). If at some point

\[ \square \]

Assume \( SLO^W + \neg FS \), and let \([A]_W \) be self-dual. Then \([A]^\circ_W \), \([A]_W^\circ \) are the immediate successors of \([A]_W \), and by Exercise 15.1

\[ [A]^\circ_W \leq [A]_W^\circ, [A]_W^{\circ^\circ} \quad \text{and} \quad [A]_W^\circ \leq [A]_W^{\circ^\circ}, [A]_W^{\circ^\circ} \]

and since \([A]_W^{\circ^\circ}, [A]_W^\circ \) are dual to each other, the inequality is strict. By Lemma 15.2 \([A]_W^\circ \) and \([A]_W^{\circ^\circ} \) are non-self-dual so

\[ [A]_W^\circ \oplus [A]_W^\circ < [A]_W^{\circ^\circ}, [A]_W^{\circ^\circ} \]

By Exercise 15.1

\[ ([A]_W^\circ \oplus [A]_W^\circ)^{\circ^\circ} \leq [A]_W^{\circ^\circ^\circ} = [A]_W^{\circ^\circ} \leq ([A]_W^\circ \oplus [A]_W^\circ)^{\circ^\circ} \]

hence \([A]_W^\circ \oplus [A]_W^\circ)^{\circ^\circ} = [A]_W^{\circ^\circ} \) and, similarly \([A]_W^\circ \oplus [A]_W^\circ)^{\circ^\circ} = [A]_W^{\circ^\circ}.\]

Therefore \( \{[A]_W^\circ, [A]_W^\circ\} \) is the least non-self-dual pair above \( \{[A]_W^\circ, [A]_W^\circ\} \)

The following diagram summarizes the situation:

![Diagram](image-url)

The computations above are done under the assumption of \( SLO^W \) for simplicity reasons, but by the results in the next section, only \( SLO^W_{\leq A} \) is needed.
Proposition 15.8. Suppose $A$ is self-dual and $\text{SLO}_W^W \leq_A$ holds. If $B, C \in [A^\omega]_W$ and $B \cap C = \emptyset$, then there is $D \leq_W A$ such that $B \subseteq D$ and $D \cap C = \emptyset$.

Proof. By Corollary 15.4 $[T(B)] \subseteq B$ and $[T(C)] \subseteq C$, and $[T(B)] \cap [T(C)] = \emptyset$ so that $T = T(B) \cap T(C)$ is well-founded. We will define

$$D = \bigcup_{t \in \partial T} t \cdot D_{[t]}$$

as follows. Let $t \in \partial T$: we consider two cases.

Case 1 $t \in \partial T(B)$. Then $B_{[t]} \leq_W A^\omega$ so $B_{[t]} \leq_W A$. Let $D_{[t]} = B_{[t]}$ so that $D_{[t]} \cap C_{[t]} = \emptyset$.

Case 2 otherwise, that is, $t \in T(C) \setminus T(B)$. Then $C_{[t]} \leq_W A^\omega$ so $C_{[t]} \leq_W A$. Let $D_{[t]} = \neg C_{[t]}$ so that $D_{[t]} \supseteq B_{[t]}$.

It is easy to check that the set $D$ is as required. \qed

16 Addition of degrees

For any $x \in ^\omega \omega \cup \mathbb{R}$, let $x + 1 = \langle x(n) + 1 \mid n \in \text{lh}(x) \rangle$, and for $A \subseteq ^\omega \omega \cup \mathbb{R}$, let

$$A^+ = \{x + 1 \mid x \in A\}.$$

The addition of two sets $A$ and $B$ is defined as

$$A + B = \{s + 1 \cdot (0)^{\omega} x \mid x \in A\} \cup B^+.$$

Notice that the definition of $A^+$, and hence of $A + B$ does not make sense in the Cantor space.

Exercise 16.1. Show that:

(i) $B^+ = \emptyset + B$;

(ii) $(A + B)_{[s]} = A + B_{[s+1]}$;

(iii) $\neg (A + B) = \neg A + \neg B$;

(iv) $\bigoplus_n (A + B_n) \equiv_L A + \bigoplus_n B_n$;

(v) $A \leq_L A + B$;
(vi) \( A \leq_L A' \land B \leq_L B' \Rightarrow A + B \leq_L A' + B' \);

(vii) \((A + B) + C \equiv_L A + (B + C)\);

and similarly for the \( \leq_W \) pre-order.

**Lemma 16.2.** If \( A \subseteq \mathbb{R} \) and \( \text{Int}(A) = \emptyset \), then \( A \equiv_L A^+ \).

**Proof.** The map \( x \mapsto x^+ \) witnesses that \( A \leq_L A^+ \) so it is enough to check that \( \text{II} \) wins \( G_L(A^+, A) \). Here is a strategy:

As long as \( \text{I} \) plays integers of the form \( n + 1 \) then \( \text{II} \) answers \( n \).

Suppose at some round \( \text{I} \) plays 0, and let \( p \) be \( \text{II} \)'s position before this round. Since \( \text{Int}(A) = \emptyset \), there is an \( x \not\in A_{[p]} \), and \( \text{II} \) can follow \( x \) from now on.

\( \square \)

The addition operation can be defined on Wadge degrees by

\[ [A]_W + [B]_W = [A + B]_W. \]

This definition makes sense also for Lipschitz degrees or, more generally, for \( \mathcal{F} \)-degrees, where \( \mathcal{F} \subseteq \mathbb{R} \) is a family of functions containing all Lipschitz functions and satisfying (11), but we have no use for such generalizations. Although addition on degrees is associative, it is far from commutative; in fact, as we shall see, addition of sets pretty much resembles ordinal addition.

**Exercise 16.3.** Show that

(i) \( \mathbb{R} + \mathbb{R} = \mathbb{R} \) and \( \emptyset + \emptyset = \emptyset \).

(ii) Assume \( \mathsf{AC}_\omega(\mathbb{R}) \) and \( A \not\subseteq \mathbb{R} \). Then \( A + \emptyset \equiv_L A^\omega \) and \( A + \mathbb{R} \equiv_L A^\omega \).

(iii) Suppose \( A \leq_W \neg A \). Assume \( \mathsf{AC}_\omega(\mathbb{R}) \) and \( \text{SLO}^W_{\leq_A + \mathbb{R}} \) or—equivalently—\( \text{SLO}^W_{\leq_A + \mathbb{R}} \). Then \( \forall B (A <_W A + B) \).

The assumption in (iii) that \( A \) be self-dual is crucial, since there are non-self-dual \( B \) such that \( B + B \equiv_W B \)—see (??)—while by the results of section 17 the hypothesis \( \text{SLO}^W_{\leq_A + \mathbb{R}} \) could be weakened to \( \text{SLO}^W_{\leq_A} \).

Suppose that \( A \) is self-dual and that \( \tau \) is a winning strategy for \( \text{II} \) in \( G_W(A + B, A + B') \). We claim that if \( \text{II} \) follows \( \tau \) then he is not the first player to play a 0. Suppose otherwise: then at some round \( \text{I} \) reaches a position \( s + 1 \) and \( \text{II} \) plays 0 reaching a position \( (t + 1)^\omega(0) \). Then \( \text{II} \) must be able to win what’s left of the game and hence

\[ A + B_{[s]} = (A + B)_{[s+1]} \leq_W (A + B')_{[(t+1)^\omega(0)]} = A, \]
contradicting Exercise 16.3(iii). Therefore $\tau$ yields a winning strategy $s \mapsto \tau(s+1) - 1$ for $\Pi$ in $G_W(B, B')$. Since the implication $B \leq_W B' \Rightarrow A + B \leq_W A + B'$ is immediate, we have shown that

**Proposition 16.4.** Assume $\mathcal{AC}_\omega(\mathbb{R}) + \text{SLO}^W_{\leq_A}$, and that $A \leq_W \neg A$. Then

$$B \leq_W B' \iff A + B \leq_W A + B'.$$

Thus set-addition is strictly increasing in the second variable; on the other hand, it need not be strictly increasing in the first variable, as the next result shows.

**Lemma 16.5.** Suppose $B$ is non-self-dual and $B + B \equiv_W B$. Then

$$\neg B + B \equiv_W (B \oplus \neg B) + B.$$

**Proof.** For notational ease, let $A = B \oplus \neg B$. By Exercise 16.1 $\neg B \leq_W A$ implies $\neg B + B \leq_W A + B$, so it is enough to show that $A + B \leq_W \neg B + B$.

Again by Exercise 16.1 it is enough to prove that $\Pi$ has a winning strategy in $G_W(A + B, \neg B + (B + B))$. Here is such a strategy:

As long as $I$ plays non-0 natural numbers, $\Pi$ copies and adds 1. If at some round $I$ plays 0 for the first time, then $\Pi$ passes. In the next round, if $I$ plays an odd integer (choosing the $\neg B$-side of $A$) then $\Pi$ plays 0 and then copies; if otherwise $I$ plays an even integer (choosing the $B$-side of $A$) then $\Pi$ plays 1 and then copies and adds 1.

**Lemma 16.6** (Wadge). Suppose $A <_W X$ with $A$ self-dual, and assume $\text{SLO}^W_{\leq_X}$. Then there is a $Y \leq_W X$ such that $X \equiv_W A + Y$.

**Proof.** The tree

$$T = \{ t \in {}^{<\omega} \mathbb{N} \mid X[t] \not\leq_W A \}$$

is pruned, since $X|_{\{\langle n \rangle \}} \leq_W A$ for all $n$ implies $X[t] \leq_W A$. Let $\varphi_T : {}^{<\omega} \mathbb{N} \rightarrow T$ and $r_T : \mathbb{N} \rightarrow [T]$ be as in (6) and (8). Then

$$Y = r_T^{-1}(X \cap [T]) \leq_W X.$$

It is enough to show that $A + Y \equiv_W X$.

To show that $A \equiv_W A + Y$ consider the following strategy for $\Pi$ in $G_W(X, A + Y)$:
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As long as $I$ plays in $T$, then $\Pi$ copies and adds 1. If $I$ ever reaches a $t \in \partial T$, then $X_{[t]} \leq_W A$. Then $\Pi$ plays 0 and from now on follows a strategy witnessing $X_{[t]} \leq_W A$.

To show that $A + Y \leq_W X$ consider the following strategy for $\Pi$ in $G_W(A + Y, X)$

As long as $I$ plays non-zero integers, then $\Pi$ subtracts 1 and uses $\phi_T$. Suppose at some round $I$ plays a 0, and let $s$ and $t$ be $I$'s and $\Pi$'s positions reached before this round: then $t \in T$, hence $X_{[t]} \not\leq_W A = (A + Y)_{[s-0]}$. Since $X_{[t]} \leq_W X$ and $A \leq_W X$, then SLO$_{\leq_X}^W$ implies that

\[(A + Y)_{[s-0]} <_W B_{[t]}\]

and therefore $\Pi$ can easily win the game from this point on.

Therefore, by (??), if $A$ is self-dual and SLO$_{\leq_A+}^W$ holds, then

\[\|A + B\|_W = \|A\|_W + \|B\|_W.\]

Suppose SLO$_{\leq_A}^W$ holds and that $\|A\|_W \geq \omega$. Let $D$ be clopen. By Lemma 16.6 there is a $C \leq_W A$ such that $A \equiv_W D + C$ and hence $\|A\|_W = 2 + \|C\|_W$. Thus $\|A\|_W = \|C\|_W$, which implies $A \equiv_W C$ and hence $D + A \equiv_W A$. Since $A \leq_W \emptyset + A \leq_W D + A$, then $A \equiv_W \emptyset + A = A^+$. Since Int($A^+$) = $\emptyset$, we have that

\[\text{SLO}_{\leq_A}^W \land \|A\|_W \geq \omega \Rightarrow \exists B \equiv_W A \text{ (Int}(B) = \emptyset).\]

Using this and (??) we get

\[\text{SLO}_{\leq_A}^W \land \|A\|_W \geq \omega \Rightarrow A \equiv_W A^+.\]

For the next result we need to recall the definition of \textit{additively indecomposable ordinal}: a non-zero ordinal $\lambda$ is said to be additively indecomposable iff $\forall \alpha, \beta < \lambda (\alpha + \beta \neq \lambda)$ iff $\forall \alpha, \beta < \lambda (\alpha + \beta < \lambda)$. The additively indecomposable ordinals are exactly those of the form $\omega^\gamma$, for some $\gamma$.

\textbf{Lemma 16.7.} Assume SLO$_{\leq_A}^W + \text{DC}(\mathbb{R})$ so that $\|A\|_W$ is defined. Let $\delta \leq \|A\|_W$ be an additively indecomposable ordinal. If $C \leq_W A$ and $\|C\|_W < \delta$, then $C + A \equiv_W A$. 

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Proof. Assume first that $C$ is self-dual. Let $D \leq_W A$ be such that $\|D\|_W = \delta$; by Lemma 16.6, there is a $D' \leq_W D$ such that $C + D' \equiv_W D$. By (16) $\|C\|_W + \|D'\|_W = \|C + D'\|_W = \|D\|_W$ by indecomposability. Thus $D' \equiv_W C + D$. If $\delta = \|A\|_W$, then $D \equiv_W A$ and we are done, therefore we may assume $\delta < \|A\|_W$. If $D$ is self-dual then, by Lemma 16.6 again, $D + A' \equiv_W A$ for some $A'$, hence

$$C + A \equiv_W C + (D + A') \equiv_W (C + D) + A' \equiv_W D + A' \equiv_W A.$$  

If $D$ is non-self-dual, it is enough to show that $C + (D + \neg D) \equiv_W D + \neg D$, and since $D + \neg D$ is self-dual we can proceed as before: by (16) $\|C + (D + \neg D)\|_W = \|C\|_W + \delta + 1 = \delta + 1 = \|D + \neg D\|_W$, hence we are done.

Assume now $C$ is non-self-dual. Then $C + A \leq_W (C + \neg C) + A \equiv_W A$ by the previous paragraph, and since the reverse inequality holds trivially, we are done. \hfill \Box

**Lemma 16.8.** Let $T$ be a (non-necessarily pruned) tree on $\omega$ such that $[T] \neq \emptyset$, and let $X, Y, Z \subseteq \mathbb{R}$.

(a) If $X \cap [T] \leq_W Z$ and $\forall s \in \partial T \left( X_{[s]} \leq_W Y \right)$, then $X \leq_W Y + Z$.

(b) Suppose $Z \leq_W X \cap [T]$, $\text{Int}([T]) = \emptyset$, and $\forall s \in \partial T \left( Y \leq_W X_{[s]} \right)$, then $Y + Z \leq_W X$.

(c) Suppose $X = f^{-1}\circ(Y + Z)$ with $f : \mathbb{R} \to \mathbb{R}$ continuous, and $[T] \subseteq f^{-1}\circ\mathbb{R}^+$ and $\text{Int}(Z) = \emptyset$. Then $X \cap [T] \leq_W Z$.

(d) Suppose $X = f^{-1}\circ(Y + Z)$ with $f : \mathbb{R} \to \mathbb{R}$ continuous, and suppose $T$ is the pruned tree such that $[T] = f^{-1}\circ\mathbb{R}^+$. Then $\forall s \in \partial T \left( X_{[s]} \leq_W Y \right)$.

**Proof.** (a) Let $\tau$ be a winning strategy for $\mathbf{II}$ in $G_W(X \cap [T], Z)$. Then $\mathbf{II}$ wins $G_W(X, Y + Z)$ by following $\tau + 1$ as long as $\mathbf{I}$'s position is in $T$, and if $\mathbf{I}$ ever reaches a position $s \in \partial T$, then $\mathbf{II}$ plays 0 and then uses the reduction $X_{[s]} \leq_W Y$.

(b) Let $\tau$ be a winning strategy for $\mathbf{II}$ in $G_W(Z, X \cap [T])$. Here is a winning strategy for $\mathbf{II}$ in $G_W(Y + Z, X)$: as long as $\mathbf{I}$ plays non-0 integers then $\mathbf{II}$ subtracts 1 and follows $\tau$; if at some round $\mathbf{I}$ plays 0, then $\mathbf{II}$ reaches a position $s \in \partial T$ (and this is possible, since $\text{Int}([T]) = \emptyset$) and then follows the reduction $Y \leq_W X_{[s]}$.  

(c) Let $\tau$ be $\Pi$’s winning strategy in $G_W(X, Y + Z)$ defining $f$. Then $\Pi$ wins $G_W(X \cap [T], Z)$ as follows: As long as $I$ plays in $T$, then $\Pi$ follows $\tau - 1$. If $I$ reaches $\partial T$, then $\Pi$ starts enumerating a real not in $Z$—here is where we need that $\text{Int}(Z) = \emptyset$.

(d) Let $\tau$ be a winning strategy for $\Pi$ in $G_W(X, Y + Z)$ and suppose that at round $n$ $I$ reaches a position $s \in \partial T$. Since $\tau$ can pass, it may happen that at round $n$ $\Pi$’s position be still in $(<\omega)\omega$, although any play extending $s$ will eventually bring $\Pi$’s play outside $\mathbb{R}^+$. Therefore

$$U = \{p \mid ((s \setminus p) * \tau)_H \in (<\omega)\omega\}$$

is a well-founded (possibly empty) tree. Then $\Pi$ wins $G_W(X_{[s]}, Y)$ by passing, until $I$ reaches a position $s' \supseteq s$, $s' \notin U$, where $\tau$ plays 0 for the first time. Then $\Pi$ follows $\tau$. \hfill $\square$

**Theorem 16.9.** Assume $\text{DC}(\mathbb{R})$ and suppose $\text{SLO}_W^{\leq A}$ holds, with $A$ non-self-dual and $\|A\|_W$ limit. Then

$$\|A\|_W \text{ is additively indecomposable} \iff A \equiv_W [T(A)] \cap A.$$ 

**Proof.** Suppose $A$ is non-self-dual so that $T(A)$ is pruned. If $\|A\|_W$ is limit and additively decomposable, and $B, C <_W A$ are such that $A \equiv_W B + C$, then $T(B + C) = T(C)^+$ and by (??)

$$A \cap [T(A)] \equiv_W (B + C) \cap [T(B + C)] = C^+ \cap [T(C)^+] \leq_W C^+.$$ 

Since $\|C\|_W$ is limit, then $C^+ \equiv_W C$ by (18), hence $A \cap [T(A)] <_W A$ by (??). Conversely, assume $\|A\|_W$ is additively indecomposable. For any $s \in \partial T(A)$ (if any such $s$ exist) $A_{[s]} \leq_W A$, and since $\text{cof}(\|A\|_W) > \omega$, there is a self-dual $B <_W A$ such that $\forall s \in \partial T(A) (A_{[s]} \equiv_W B)$. Lemma 16.8(a) implies $A \leq_W B + ([T(A)] \cap A)$, so that $\|A\|_W \leq \|B\|_W + \|T(A)\|_W$ and therefore $A \equiv_W [T(A)] \cap A$ by (??) and case assumption. \hfill $\square$

### 17 Propagation of SLO$^W$

In this section we will prove that if $A$ is self-dual then $\text{SLO}_W^{< A} \Rightarrow \text{SLO}_W^{< A + A}$.

Since $A + A$ is also self-dual, this argument can be iterated to prove $\text{SLO}_W^{< A}$ for all $n$, where

$$A \cdot n = A + (A + (\cdots (A + A) \cdots)) = A + A + \cdots + A.$$
In fact this construction can be pushed through the countable ordinals, but in order to state it in a convenient form it is better to introduce the following notation: since $\mathcal{SLO}^W$ holds of all the sets of rank $< ||A||_W \cdot \alpha$, which is what the theorem aims to show. The correct reformulation is

**Theorem 17.1.** Assume $\text{DC}(\mathbb{R}) + \neg \text{FS} + \mathcal{SLO}^W_{\leq |A|_W}$ where $[A]_W$ is self-dual. Then $\mathcal{SLO}^W_{\leq |A|_W \cdot \alpha}$ holds, for $1 \leq \alpha < \omega_1$.

Strictly speaking, the statement of Theorem 17.1 is not quite correct, since the definition of $[A]_W \cdot \alpha$ (with $\alpha$ limit) assumes that $\mathcal{SLO}^W_{\leq |A|_W}$. Theorem 17.2. Assume $\text{DC}(\mathbb{R}) + \neg \text{FS} + \mathcal{SLO}^W_{\leq |A|_W}$ where $[A]_W$ is self-dual. Then for each $1 \leq \gamma < \omega_1$ there is a set $C$ such that $\mathcal{SLO}^W_{\leq C}$ holds and $||C||_W = ||A||_W \cdot \gamma$.

First of all, let us notice that it is enough to prove this when $||A||_W$ is an additively indecomposable ordinal, or the successor of an additively indecomposable ordinal. (Recall that a limit ordinal $\lambda$ is additively indecomposable iff $\forall \xi, \eta < \lambda (\xi + \eta < \lambda)$ and that all such ordinals are of the form $\omega^\alpha$.)

**Theorem 17.3.** Assume $\text{DC}(\mathbb{R}) + \neg \text{FS} + \mathcal{SLO}^W_{\leq |A|_W}$ where $[A]_W$ is self-dual and either $||A||_W$ is additively indecomposable, or else is the successor of an additively indecomposable. Then for each $1 \leq \gamma < \omega_1$ there is a self-dual set $C$ such that $\mathcal{SLO}^W_{\leq |C|_W}$ holds and $||C||_W = ||A||_W \cdot \gamma$. 
In fact, suppose \([A]_W\) is self-dual, but \(\|A\|_W\) is neither additively indecomposable, nor the successor of an additively indecomposable ordinal. Let \(\omega^\alpha\) be the largest additively indecomposable ordinal below \(\|A\|_W\). If \(\text{cof}(\alpha) = \omega\), let \(A'\) be a set of rank \(\omega^\alpha\), and if \(\text{cof}(\alpha) > \omega\), let \(A'\) be a set of rank \(\omega^\alpha + 1\). In either case, \([A']_W\) is self-dual and \(\text{SLO}^W_{\leq A} = \text{SLO}^W_{\leq A'}\) holds. Since
\[
\sup_{\gamma < \omega_1} \|A'\|_W : \gamma = \sup_{\gamma < \omega_1} \|A\|_W : \gamma
\]
then Theorem 17.2 follows at once from Theorem 17.3.

Towards proving Theorem 17.3, let us start with a few preliminary results.

**Lemma 17.4.** \(\text{SLO}^W_{\leq A} \Rightarrow \text{SLO}^W_{\leq \forall A} = A\).

**Proof.** Suppose \(X = f^{-1}(A \oplus \neg A)\) with \(f : \mathbb{R} \rightarrow \mathbb{R}\) continuous. For ease of notation let \(A_0 = A\) and \(A_1 = \neg A\), let
\[
D_i = f^{-1}\left(\bigcup_n N_{(2n+i)}\right)
\]
and let \(X_i = X \cap D_i\), for \((i = 0, 1)\). If \(A_i = X_i\) for both \(i = 0\) and \(i = 1\), then \(I\) wins \(G_W(A \oplus \neg A, X)\) by passing in the first round and then choosing the right reduction. If \(A_i \not\subseteq W X_i\) for some \(i = 0, 1\), then \(\text{SLO}^W_{\leq B}\) implies that \(X_i \subseteq B_{1-i}\) and therefore \(\forall I\) wins \(G_W(X, A_{1-i})\) by passing until \(I\) has gotten into \(D_0\) or \(D_1\) and then chooses the right reduction. Therefore if \(X, Y \subseteq W A \oplus \neg A\) then \(X \subseteq W Y\) or \(Y \subseteq W \neg X\).

**Lemma 17.5.** Suppose \(A_0 \subseteq W A_1 \subseteq W \ldots\) and let \(A = \bigoplus_n A_n\). If \(\text{SLO}^W_{\leq A_n}\) holds for every \(n\), then \(\text{SLO}^W_{\leq A}\) holds.

**Proof.** The assumption \(\forall n < \omega \ \text{SLO}^W_{\leq A_n}\) is enough to prove that each \(A_n \subseteq W \neg A_{n+1}\) hence \(A = \bigoplus_n A_n\) is self-dual. Let \(X = f^{-1}(A)\) with \(f : \mathbb{R} \rightarrow \mathbb{R}\) continuous, and let \(D_n = f^{-1}(N_{(n)})\). If
\[
\forall n \exists m \ A_n \subseteq W X \cap D_m,
\]
then let \(m(n)\) be the least \(m\) such that there is an \(f_n : \mathbb{R} \rightarrow \mathbb{R}\) witnessing \(A_n \subseteq W X \cap D_m\): since \(f_n(\text{Cl}(A_n)) \subseteq D_{m(n)}\) and since \(D\) is clopen, we may modify \(f_n\) (if needed) so that \(\text{ran}(f_n) \subseteq D_{m(n)}\) while preserving the fact that \(A_n\) is reducible to \(X \cap D_{m(n)}\); therefore \(I\) can win \(G_W(A, X)\) by using the reduction \(f_n\) if \(I\) plays \(n\) in the first round. If instead
\[
\exists n \ \forall m \ A_n \not\subseteq W X \cap D_m,
\]
then since \(X \cap D_m \subseteq W A_m\), the hypothesis \(\text{SLO}^W_{\leq A_n}\) (when \(n < m\)) implies that \(X \cap D_m \subseteq W \neg A_n \subseteq W A_{n+1}\) and hence \(\forall I\) wins \(G_W(X, A_{n+1})\) by passing
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until $I$ enters in some $D_m$ and then uses the reduction. Therefore we have shown that

$$\forall X \leq W A \left( X \equiv W A \vee \exists n < \omega (X \leq W A_n) \right).$$

Thus if $X, Y \leq W A$ then either $X \equiv W A \equiv W Y$, or $X < W A \equiv W Y$, or $Y < W A \equiv W X$, or else $X, Y < W A_n$ for some $n$. This shows that $\text{SLO}^W_{\leq A}$ holds.

**Lemma 17.6.** Assume $A$ is self-dual and that $\text{SLO}^W_{\leq A \cdot n}$ holds. Suppose that

$$(\star_{A \cdot n}) \quad \forall X \leq W A \cdot (n+1) \left( X \leq W A \vee \exists Y \leq W A \cdot n (X \equiv W A + Y) \right).$$

Then $\text{SLO}^W_{\leq A \cdot (n+1)}$ holds.

**Proof.** We must show that for any $X_1, X_2 \leq W A \cdot (n+1)$

$$X_1 \leq W X_2 \vee X_2 \leq W \lnot X_1.$$

If both $X_1 \leq W A$ and $X_2 \leq W A$ then (20) follows from $\text{SLO}^W_{\leq A}$. If $X_1 \leq W A$ and $X_2 \equiv W A + Y_2$, then $X_1 \leq W X_2$ by (16.3). If $X_1 \equiv W A + Y_1$ (with $Y_i \leq W A \cdot n$ and $i = 1, 2$) then $Y_1 \leq W Y_2$ or $Y_2 \leq W \lnot Y_1$ by $\text{SLO}^W_{\leq A \cdot n}$ hence (20) holds.

**Lemma 17.7.** Assume $\text{SLO}^W_{\leq A \cdot n}$ with $A$ self-dual and suppose either $\|A\|_W$ is an additively indecomposable limit ordinal or else $A \equiv W B \oplus \lnot B$ with $B$ non-self-dual, $B + B \equiv W B$, and $\|B\|_W$ additively indecomposable limit ordinal. Then $(\star_{A \cdot n})$ holds.

The proof is word-by-word that of Lemma 16.6: there the assumption $\text{SLO}^W_{\leq B}$ was only used to prove (15), while here we replace it with the following

**Lemma 17.8.** Assume $\text{SLO}^W_{\leq A \cdot n}$ with $A$ self-dual, and suppose either $\|A\|_W$ is an additively indecomposable limit ordinal or else $A \equiv W B \oplus \lnot B$ with $B$ non-self-dual, $B + B \equiv W B$, and $\|B\|_W$ additively indecomposable limit ordinal. Suppose also $X \leq W A \cdot (n+1)$ and $X \not\leq W A$. Then either $A + 0 \leq W X$, or else $A + \mathbb{R} \leq W X$.

**Proof.** Since $\|A \cdot n\|_W \geq \omega$, then by (17) we may assume that $\text{Int}(A \cdot n) = \emptyset$. Let $X = f^{-1}(A \cdot (n+1))$, with $f : \mathbb{R} \to \mathbb{R}$ continuous, and suppose $X \not\leq W A$. Suppose $X \leq W A \cdot n$: then $n = 1$ implies that $X \leq W A$ (contradicting our assumption), and $n > 1$ implies $A < W X$ by $\text{SLO}^W_{\leq A \cdot n}$, and therefore $A + 0 \leq W X$ or $A + \mathbb{R} \leq W X$. Therefore we may assume that $X \not\leq W A \cdot n$. 

CHAPTER II. SLO\textsuperscript{w} AND WADGE DEGREES

The set $f^{-1}(\mathbb{R}^+)$ is closed and let $\hat{T} \supseteq T$ be the pruned trees such that $[\hat{T}] = f^{-1}(\mathbb{R}^+)$ and $[T] = [\hat{T}] \setminus \text{Int}([\hat{T}])$. By Lemma 16.8(c)

\[(21) \quad X \cap [\hat{T}] \leq_{\text{w}} A \cdot n \quad \text{and} \quad X \cap [T] \leq_{\text{w}} A \cdot n.
\]

Suppose $s \in \partial T$. If $s \in \partial \hat{T}$ then $X_{[s]} \leq_{\text{w}} A$ by Lemma 16.8(d), and if $s \in \hat{T}$ then $X_{[s]} \subseteq [\hat{T}_{[s]}] = \mathbb{R}$ so

$X_{[s]} = (X \cap [T])_{[s]} \leq_{\text{w}} X \cap [T] \leq_{\text{w}} A \cdot n$

Therefore

\[(22) \quad \forall s \in \partial T \left( X_{[s]} \leq_{\text{w}} A \cdot n \right)
\]

and that For each $t \in T$ let

$\varphi(t) = \sup\{\|X_{[s]}\|_W \mid s \in \partial T \land s \supseteq t\}$

and let

$U = \{t \in T \mid \varphi(t) \geq \|A\|_W\}$.

Since $s \subseteq t \Rightarrow \varphi(s) \geq \varphi(t)$, then $U$ is a (possibly empty) subtree of $T$. (Notice that the Wadge ranks mentioned above are defined, since the sets involved are reducible to $A \cdot n$ and SLO\textsuperscript{w}_{\leq A \cdot n} holds.)

**Claim 17.8.1.** Suppose $x_0 \in [U]$. Then $x_0 \in X \Rightarrow A + \mathbb{R} \leq_{\text{w}} X$ and $x_0 \notin X \Rightarrow A + \emptyset \leq_{\text{w}} X$.

**Proof of the Claim.** Suppose $x_0 \in X$. Then $\text{II}$ wins $G_W(A + \mathbb{R}, X)$ as follows

As long as $\text{I}$ does not play 0, then $\text{II}$ follows $x_0$. If, at round $n$ Player $\text{I}$ first plays 0, then $\text{II}$ passes until $\text{I}$ reaches a position $q = (t + 1)^0 \cdot p$ such that $(A + \mathbb{R})_{[q]} = A_{[p]} \leq_{\text{w}} A$ (and this is bound to happen, since $A$ is self-dual). Since $[T]$ has empty interior, $\text{II}$ can reach a position $s \supseteq x_0 \mid n$, with $s \in \partial T$, such that $(A + \mathbb{R})_{[q]} \leq_{\text{w}} X_{[s]}$ and then follows this reduction.

The proof when $x_0 \notin X$ is completely analogous. □

Therefore we may assume that

\[(23) \quad U \text{ is well-founded}.
\]

**Case 1:** $\|A\|_W$ is an additively indecomposable ordinal.

**Claim 17.8.2.** $\forall u \in \partial U \cap T \left( X_{[u]} \leq_{\text{w}} A \cdot n \right)$.
Proof of the Claim. Let \( u \in \partial U \cap T \) and let \( \tau \) be II’s winning strategy for \( G_W(X_{[u]} \cap [T_{[u]}], A \cdot n) \), which exists by (21) and by \( X_{[u]} \cap [T_{[u]}] = (X \cap [T])_{[u]} \leq_W X \cap [T] \). Let \( C \) be such that \( \varphi(u) < \|C\|_W < \|A\|_W \) —such a set exists since \( \|A\|_W \) is limit. By Lemma 16.7, \( C + A \cdot n \equiv_W A \cdot n \), so it is enough to show that II wins \( G_W(X_{[u]}, C + A \cdot n) \):

II follows the reduction \( \tau \) and adds 1, as long as I is in \( T \); if I ever reaches a position \( s \in \partial T \), then II plays 0 and follows a reduction witnessing \( X_{[s]} \leq_W C \), which exists, since \( \|X_{[s]}\|_W \leq \varphi(u) < \|C\|_W \).

\[ \square \]

The Claim yields that \( X \leq_W A \cdot n \), contradicting our assumption. In fact II wins \( G_W(X, A \cdot n) \) as follows:

II passes as long as I plays in \( U \). Suppose at round \( n \) I reaches a position \( u \in \partial U \). If \( u \in T \) then \( X_{[u]} \leq_W A \cdot n \) by the Claim. If \( u \in \partial T \), then \( X_{[u]} \leq_W A \cdot n \) by (21). In either case II has a winning strategy.

Case 2: \( A \equiv_W B \oplus \neg B \) with \( B \) non-self-dual, \( B \equiv_W B + B \) and \( \|B\|_W \) is additively indecomposable.

Let \( T_i = \{ t \in T \mid \exists s \ni t (s \in \partial T \land B_i \leq_W X_{[s]} ) \} \),

where, for notational simplicity, \( B_0 = B \) and \( B_1 = \neg B \). Then \( T_0 \) and \( T_1 \) are (possibly empty) subtrees of \( T \). Notice that our assumption implies that \( B_i + B_i \equiv_W B_i \), for \( i = 0, 1 \), hence by Lemma 16.5

\[ (24) \quad B_i + B_{1-i} \equiv_W A + B_{1-i} . \]

Suppose \( s \in T \setminus (T_0 \cup T_1 \cup U) \). Then \( \varphi(s) < \|B\|_W \). Since \( [B]_W \) is limit and non-self-dual, then \( SLO^W_A \) implies that \( \text{cof}(\|B\|_W) > \omega \), hence there is a set \( C_s \) such that

\[ (25) \quad t \ni s \land t \in \partial T \Rightarrow X_{[t]} \leq_W C_s . \]

Arguing as in Claim 17.8.1 we get

Claim 17.8.3. Suppose \( x_0 \in [T_0] \cap [T_1] \). Then \( x_0 \in X \Rightarrow A + \mathbb{R} \leq_W X \) and \( x_0 \in X \Rightarrow A + 0 \leq_W X \).
Therefore we may assume that

\[(26) \quad T_0 \cap T_1 \text{ is pruned.}\]

Suppose \(\tau\) is a winning strategy for \(\Pi\) in \(G_W(B_{1-i}, X \cap [T_i])\). Then \(\Pi\) wins \(G_W(B_i + B_{1-i}, X)\) by (subtracting 1 and) following \(\tau\), as long as \(I\) does not play 0; if at some round \(I\) plays 0, then, by the definition of \(T_i\), \(\Pi\) can reach a position \(s \in \partial T\) such that \(B_i \leq_W X_{[s]}\) and then follow this reduction. Using (24) we get that

\[
B_{1-i} \leq_W X \cap [T_i] \Rightarrow A + B_{1-i} \leq_W X.
\]

Since \(A + \mathbb{R}, A + \emptyset \leq_W A + B_i\) by monotonicity, then the result follows from \(B_{1-i} \leq_W X \cap [T_i]\). Therefore we may assume that \(B_{1-i} \not\leq_W X \cap [T_i]\). Keeping in mind that \(B_{1-i}, X \cap [T_i] \leq_W A \cdot n\), \(\text{SLO}^W_{A \cdot n}\) implies \(X \cap [T_i] \leq_W B_i\). Thus we may assume that

\[(27) \quad X \cap [T_i] \leq_W B_i.\]

We are now going to show that \(X \leq_W A \cdot n\), contradicting our initial assumption. Here is a winning strategy for \(\Pi\) in \(G_W(X, A \cdot n)\):

By (26) and (23) \(\Pi\) passes until \(I\) reaches a position \(s \notin (T_0 \cap T_1) \cup U\), and consider three cases:

**Case A** \(s \notin T\). Then \(s \in \partial T\) and therefore \(X_{[s]} \leq_W A \cdot n\) by (22).

**Case B** \(s \in T \setminus (T_0 \cup T_1)\). Let \(C_s\) be as in (25). Then \(\Pi\) reduces \(X_{[s]}\) to \(C_s + A \cdot n\) by following the reduction \(X_{[s]} \cap [T] \leq_W A \cdot n\) by (21) as long as \(I\) plays in \(T\), and then using (25) as soon as \(I\) reaches \(\partial T\). Since \(\|C_s\|_W < \|B\|_W\) and \(\|B\|_W\) is additively indecomposable, then \(C_s + A \cdot n \equiv_W A \cdot n\), and therefore we are done.

**Case C** \(s \in T_i \setminus T_{i-1}\), for some \(i \in \{0, 1\}\). If \(T_i\) is wellfounded, then \(\Pi\) passes until \(I\) reaches a position \(t \in \partial T_i\) and we follow Case A or B depending wether \(t \notin T\) or \(t \in T\). So we may assume \([T_i] \neq \emptyset\). We will show that \(X \leq_W B_i + B_i \equiv_W B_i\), and since \(B_i \leq_W A \leq_W A \cdot n\), we are done. The winning strategy for \(\Pi\) in \(G_W(X, B_i + B_i)\) is as follows: as long as \(I\)'s position is in \(T_i\), use that \(X \cap [T_i] \leq B_i\) by (27), if \(I\) reaches \(t \in \partial T_i\), then \(\Pi\) plays 0 and follows the reduction \(X_{[t]} <_W B_i\).

This concludes the proof of Lemma 17.8. \(\square\)

By Lemmata 17.6 and 17.7 we get
Corollary 17.9. Suppose $A$ is self-dual and SLO$^{W}_{A}$ holds. Suppose also that either $\|A\|^{W}_A$ is an additively indecomposable limit ordinal or else $A \equiv W B \oplus \neg B$ with $B$ non-self-dual, $B + B \equiv W B$, and $\|B\|^{W}_W$ additively indecomposable limit ordinal. Then, for every $n \geq 1$, SLO$^{W}_{A,n}$ holds.

Finally, we prove the “propagation of SLO$^{W}$” result mentioned before.

Theorem 17.10. Suppose $A$ is self-dual and SLO$^{W}_{A}$ holds. Then, for every $1 \leq \gamma < \omega_1$ there is a self-dual $C$ such that $A \leq W C$, SLO$^{W}_{C}$ holds, and $\|C\|^{W}_C = \|A\|^{W}_A \cdot \gamma$.

Proof. Let us first prove this under the additional assumption $(\ast)$ that:

either $\|A\|^{W}_A$ is an additively indecomposable ordinal or else $A \equiv W B \oplus \neg B$, with $B$ non-self-dual, $B + B \equiv W B$, and $\|B\|^{W}_W$ additively indecomposable of uncountable cofinality.

The result clearly holds for $\gamma = 1$ by taking $C = A$, so assume $\gamma > 1$ and that the result holds for ordinals $< \gamma$.

If $\gamma$ is limit, then choose an increasing sequence $\gamma_n \to \gamma$ and, by inductive hypothesis, choose $C_n$ such that SLO$^{W}_{\leq C_n}$ holds and $\|C_n\|^{W}_C = \|A\|^{W}_A \cdot \gamma_n$. By Lemma ??, $\bigoplus_n C_n$ witnesses the theorem for $\gamma$.

Suppose $\gamma = \gamma' + 1$. If $\gamma' \in \omega$, then we are done by Corollary 17.9, so we may assume that $\gamma' \geq \omega$. Let $\delta \leq \gamma'$ be the largest additively indecomposable ordinal. By inductive hypothesis, SLO$^{W}_{\leq D}$ holds and $\|D\|^{W}_D = \|A\|^{W}_A \cdot \delta$, for some self-dual set $D$, since cof$(\delta) = \omega$. Let $n$ be least such that $\delta \cdot n \geq \gamma$ and let $\bar{D} = D \cdot n$: then by Corollary 17.9 SLO$^{W}_{\leq \bar{D}}$ holds and therefore there is a self-dual set $C \leq W \bar{D}$ such that SLO$^{W}_{\leq C}$ holds and $\|C\|^{W}_C = \|A\|^{W}_A \cdot \gamma$.

Now for the general case: Let $\gamma \leq \|A\|^{W}_A$ be the largest additively indecomposable ordinal. Suppose that the set $A'$ of Wadge rank $\gamma$ is self-dual; then by $(\ast)$ SLO$^{W}_{\leq C}$ holds for every $C$ such that $\|C\|^{W}_C < \gamma \cdot \omega_1$, and since $\gamma \cdot \omega_1 = \|A\|^{W}_A \cdot \omega_1$, we are done. Suppose instead that the set $B$ of Wadge rank $\gamma$ is non-self-dual, and let $A' = B \oplus \neg B$. By $(\ast)$ SLO$^{W}_{\leq C}$ holds for every $C$ such that $\|C\|^{W}_C < (\gamma + 1) \cdot \omega_1 = \|A\|^{W}_A \cdot \omega_1$, hence we are done. \qed

18 Jumping $\omega_1$-many Steps

We will now introduce two new operations on sets, $A^\uparrow$ and $A^\downarrow$ such that $([A^\uparrow]^{W}_A, [A^\downarrow]^{W}_A)$ is the least non-self-dual pair above the $[A]^{W}_A \cdot \alpha$’s, $(\alpha < \omega_1)$, when $A$ is self-dual.
CHAPTER II. SLO\textsuperscript{W} AND WADGE DEGREES

Definition 18.1. For $A \subseteq \mathbb{R}$

$$A^\ddagger = \{s \cdot 0^\omega(x + 1) \mid s \in \langle \omega \rangle \land x \in A\} \cup A^+$$

and

$$A^\bowtie = A^\ddagger \cup \{x \in \mathbb{R} \mid \exists n x(n) = 0\}.$$  

$A^\ddagger$ can be seen as some sort of infinite sum of copies of $A$, and since $A^\ddagger = A^\ddagger + A$, then it is natural to think of it as $A^\ddagger = \cdots + A + A$.

Exercise 18.2. Show that for $A, B \subseteq \mathbb{R}$, $s \in \langle \omega \rangle$, $x \in \mathbb{R}$$$

(i) $A \leq_W A^\ddagger, A^\bowtie$;

(ii) $A \leq_W B \Rightarrow A^\ddagger \leq_W B^\ddagger$ and $A^\bowtie \leq_W B^\bowtie$. Thus we can define the $\ddagger$ and $\bowtie$ operations on Wadge degrees by setting

$$[A]^\ddagger_W = [A]_W \quad \text{and} \quad [A]^\bowtie_W = [A]_W.$$

(iii) $x \in A^\ddagger \iff s \cdot 0^\omega x \in A^\ddagger$ and $x \in A^\bowtie \iff s \cdot 0^\omega x \in A^\bowtie$;

(iv) $A^\ddagger_{[s \cdot 0]} = A^\ddagger$ and $A^\bowtie_{[s \cdot 0]} = A^\bowtie$;

(v) $A^\ddagger \equiv_W A^\ddagger$ and $A^\bowtie \equiv_W A^\bowtie$.

Lemma 18.3. For any $A \subseteq \mathbb{R}$ and $1 \leq \alpha < \omega_1$,

$$[A]_W \cdot \alpha \leq_W [A]^\ddagger_W, [A]^\bowtie_W$$

Proof. By induction on $\alpha$. \hfill \square

Theorem 18.4. Assume $\neg \text{FS} + \text{AD}^L$. For a self-dual $([A]_W, [A]_W)$ is the least non-self-dual pair above the $[A]_W \cdot \alpha$'s, ($1 \leq \alpha < \omega_1$), and therefore, assuming DC(\mathbb{R}),

$$\|A^\ddagger\|_W = \|A^\bowtie\|_W = \|A\|_W \cdot \omega_1.$$  

Proof. We first show that $B \in \bigcup_{1 \leq \alpha < \omega_1} [A]_W \cdot \alpha \Leftrightarrow B \leq_W A^\ddagger, A^\bowtie$. One direction follows from 22.15(iii) so it is enough to prove that $B \leq_W A^\ddagger, A^\bowtie$ implies that $B \in \bigcup_{1 \leq \alpha < \omega_1} [A]_W \cdot \alpha$. We need the following

Lemma 18.5. Assume $\neg \text{FS} + \text{AD}^L$. For any $A, B \subseteq \mathbb{R}$, if $B \leq_W A^\ddagger, A^\bowtie$ then there is a winning strategy $\tau$ for $\Pi$ in $G_W(B, A^\ddagger)$ such that any play for $\Pi$ according to $\tau$ belongs to $\mathbb{R}^\ddagger = \{x \in \mathbb{R} \mid \forall n x(n) \neq 0\}$. 
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Proof. Let $\tau_0, \tau_1$ be $\Pi$’s winning strategies for $G_W(B, A^\omega)$ and $G_W(B, A^\omega)$, respectively. The plan is to use alternatively $\tau_0$ or $\tau_1$, switching each time the strategy being used requires to play a 0. We dove-tail $\tau_0$ and $\tau_1$ as follows:

for any $s \in \omega$ let $u_0, \ldots, u_n, v_0, \ldots, v_m \in \omega$ be such that

$$(s \ast \tau_0)_\Pi = (u_0 + 1)^\ast (0)^\ast \cdots (0)^\ast (u_n + 1)$$

and

$$(s \ast \tau_1)_\Pi = (v_0 + 1)^\ast (0)^\ast \cdots (0)^\ast (v_m + 1)$$

then let

$$(s \ast \tau)_\Pi = ((u_0 + 1)^\ast (0)^\ast (v_0 + 1)^\ast (0)^\ast \cdots (0)^\ast (u_k + 1)^\ast (0)^\ast (v_k + 1)) \ast \text{lh}(s)$$

where $k = \min(n, m)$. It is easy to check that this defines a strategy for $\Pi$.

Claim 18.5.1. $\forall x \in \mathbb{R} ((x \ast \tau)_\Pi \notin \mathbb{R}^2)$.

Proof. Deny. Let $x \in \mathbb{R}$ be such that $(x \ast \tau)_\Pi \notin \mathbb{R}^2$, that is

(1) $\exists^\infty n \tau_0(x \upharpoonright n) = 0$

and

(2) $\exists^\infty n \tau_1(x \upharpoonright n) = 0$.

Then by (1) $(x \ast \tau_0)_\Pi \notin A^\omega$ hence $x \notin B$ and by (2) $(x \ast \tau_1)_\Pi \in A^\omega$ hence $x \in B$: a contradiction.

Let us check that $\tau$ is winning for $\Pi$ in $G_W(B, A^\omega)$. Suppose $I$ plays $x$ and let $y = (x \ast \tau)_\Pi$ be $\Pi$’s response. Then either

(A) $\tau$ settles on $\tau_0$, that is

$$y = (u_0 + 1)^\ast (0)^\ast (v_0 + 1)^\ast (0)^\ast \cdots (0)^\ast (u_n + 1)^\ast (0)^\ast (v_n + 1)^\ast (0)^\ast (z + 1)$$

where $(u_0 + 1)^\ast (0)^\ast \cdots (0)^\ast (u_n + 1)^\ast (0)^\ast (z + 1) = (x \ast \tau_0)_\Pi$, or else

(B) $\tau$ settles on $\tau_1$, that is

$$y = (u_0 + 1)^\ast (0)^\ast (v_0 + 1)^\ast (0)^\ast \cdots (0)^\ast (u_n + 1)^\ast (0)^\ast (w + 1)$$

where $(v_0 + 1)^\ast (0)^\ast \cdots (0)^\ast (v_n - 1 + 1)^\ast (0)^\ast (w + 1) = (x \ast \tau_1)_\Pi$. 

\[\square\]
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Assume (A) holds, i.e., \( \tau \) settles on \( \tau_0 \). Then
\[
x \in B \iff (x * \tau_0)_{\Pi} \in A^k \quad \text{(by definition of } \tau_0) \\
\iff z \in A^k \quad \text{(by 22.15(i))} \\
\iff y \in A^k \quad \text{(by 22.15(i) again)}
\]

Case (B) when \( \tau \) settles on \( \tau_1 \) is completely analogous. \( \square \)

Let’s go back to the proof of 22.16. For any \( s \in <\omega \omega \) let
\[
n(s) = \begin{cases} 
\text{the largest } i < \lh(s) \text{ such that } s(i) = 0, \\
0, \text{ if } \forall i < \lh(s) \ (s(i) \neq 0),
\end{cases}
\]
and let \( < \) be the strict partial order on \( <\omega \omega \) defined by
\[
s < t \iff s \upharpoonright n(s) \subset t \upharpoonright n(t).
\]

Fix a strategy \( \tau \) as in the Lemma and set
\[
s < t \iff (s * \tau)_{\Pi} < (t * \tau)_{\Pi}.
\]

Then \( < \) is a strict partial order on \( <\omega \omega \), and it is well-founded, since otherwise there would be an \( x \in \mathbb{R} \) such that \( y \notin \mathbb{R}^k \), where \( y \) is \( \Pi \)'s response via \( \tau \) to \( \text{I} \) playing \( x \). We will prove by induction on \( \alpha \) that \( B \in [A]_W \cdot (\alpha + 1) \), where \( \alpha = \|\emptyset\|_\omega \).

Suppose \( \alpha = 0 \). This corresponds to the case where \( \forall s \in <\omega \omega \ (\tau(s) \) does not contain any \( 0 \).

Then \( \Pi \) wins \( G_W(B, A) \) by playing \( \tau(s) - 1 \), for any \( s \) played by \( \text{I} \): this is indeed a winning strategy since if \( x \) and \( y \) are the reals played this way by \( \text{I} \) and \( \Pi \) then
\[
x \in B \iff y + 1 \in A^k \iff y \in A.
\]

This proves that \( B \leq_W A \).

Suppose \( \alpha > 1 \) and let \( C \in [A]_W \cdot \alpha \). Consider the following strategy for \( \Pi \) in the game \( G_W(B, C + A) \):

As long as \( \text{I} \) stays in a position \( p \) such that \( \|p\|_\alpha = \alpha \) then \( \Pi \) plays \( \tau(p) \). Suppose at some round \( \text{I} \) reaches a position \( p \) such that \( \|p\|_\alpha < \alpha \). Then \( \Pi \) answers 0 and from now on, to any further move he answers \( \tau_p(p \upharpoonright q) + 1 \), where \( \tau_p \) is a strategy witnessing \( B_{|p|} \leq_W C \). Such a \( \tau_p \) exists by inductive hypothesis.

Let us check that this strategy is indeed winning for \( \Pi \). Let \( x \) and \( y \) be the reals produced by a complete play. If \( \|x \upharpoonright n\|_\alpha < \alpha \) for some \( n \), then let \( x' \) and
18. JUMPING \( \omega_1 \)-MANY STEPS

\( y' \) be such that \( x \upharpoonright n \upharpoonright x' = x \) and \( ((x \upharpoonright n) \ast \tau) \upharpoonright (0) \upharpoonright (y' + 1) = y. \) (Note that \( \text{lh}((x \upharpoonright n) \ast \tau) < n \) is possible since \( \tau \) may pass.) Then \( x' \in B_{[\tau]} \iff y' \in C \) since \( \tau \) witnesses \( B_{[\tau]} \leq_W C \) and therefore \( x \in B \iff y \in C + A. \) If instead \( \forall n (\| x \upharpoonright n \|_q = \alpha) \) then \( \forall \alpha (\tau(x \upharpoonright n) \neq 0) \) so \( y \in \mathbb{R}^2 \) hence

\[
x \in B \iff y \in A^k \iff y - 1 \in A \iff y \in C + A.
\]

Therefore \( [B]_w \leq [A]_w \cdot (\alpha + 1). \)

Finally, let us show that \( A^2 \not\equiv_W A^3. \) Otherwise if \( A^2 \leq_W A^3 \) then by the argument above, since \( A^2 \leq_W A^3, \) we should have that \( [A^2]_w \leq [A]_w \cdot \alpha \) for some \( \alpha < \omega_1. \) Then

\[
[A]_w \cdot \alpha \leq [A]_w \cdot (\alpha + 1) \leq [A]_w \cdot \alpha^2
\]

by 22.8(i) and 22.15(iii)

hence \( [A^2]_w \leq [A^3]_w, \) a contradiction.

**Corollary 18.6.** Assume \( \neg \text{FS} + \text{DC}(\mathbb{R}) + \text{AD}^L. \) Let \( A \) be self-dual. For every \( B \) such that \( \| A \|_w < \| B \|_w < \| A \|_w \cdot \omega_1, \)

\[
[B^2]_w = [A^2]_w \quad \text{and} \quad [B^3]_w = [A^3]_w.
\]

**Proof.** Let \( \alpha \) be least such that \( \| B \|_w \leq \| A \|_w \cdot \alpha \) and let \( C \in [A]_w \cdot \alpha. \) By monotonicity (22.14) \( A^2 \leq_W B^3 \leq_W C^3. \) The argument for \( A^2 \) is similar.

**Exercise 18.7.** (i) Show that

\[
\mathbb{R}^3 = \{ x \in \mathbb{R} \mid \forall n x(n) \neq 0 \} \in \Sigma^0_2 \setminus \Delta^0_2
\]

and that

\[
\emptyset^3 = \{ x \in \mathbb{R} \mid \exists n x(n) = 0 \} \in \Pi^0_2 \setminus \Delta^0_2.
\]

(ii) Show that \( \mathbb{R}^3 = (\mathbb{R} + \emptyset)^3 \) and \( \emptyset^3 = (\mathbb{R} + \emptyset)^3, \emptyset^3 = \emptyset, \emptyset^3 = \mathbb{R}. \)

(iii) (\( \text{DC}(\mathbb{R}) \)) Show that \( \Sigma^0_2 \setminus \Delta^0_2 \) and \( \Pi^0_2 \setminus \Delta^0_2 \) occupy the \( \omega_1 \)-th level of the Wadge hierarchy.

(iv) Show that \( \mathbb{R}^3 + \mathbb{R}^3 \equiv_W \mathbb{R}^3 \) and therefore \( \forall B \leq_W \mathbb{R}^3 (\mathbb{R}^3 + B \equiv_W \mathbb{R}^3). \)

(v) Show that \( ([\mathbb{R}^3]_w, [\emptyset^3]_w, ([\mathbb{R}^2]_w, [\emptyset^2]_w), \ldots \) is an increasing sequence of non-self-dual pairs of degrees whose Wadge rank (here we assume \( \text{DC}(\mathbb{R}) \) so that we can talk about Wadge ranks) are \( \omega^2_1, \omega^3_1, \ldots \).
Therefore the map $[A]_W \mapsto [A]_W^\uparrow$ acts like this

![Diagram of the map $[A]_W \mapsto [A]_W^\uparrow$.]

and the map $[A]_W \mapsto [A]_W^\downarrow$ acts like this

![Diagram of the map $[A]_W \mapsto [A]_W^\downarrow$.]
Chapter III

Borel reducibilities

In order to prove more results on the $\mathcal{F}$-degrees we must impose further restrictions on the set $\mathcal{F}$.

**Definition 18.8.** $\mathcal{F} \subseteq \mathbb{R}\mathbb{R}$ is **amenable** iff either $\mathcal{F} = \text{Lip}$, or else:

1. there is a surjection $\mathbb{R} \to \mathcal{F}$,
2. $\mathcal{F} \supseteq \text{Lip}$,
3. $\mathcal{F}$ is closed under composition,
4. if each $f_n \in \mathcal{F}$ then $\bigoplus_n f_n \in \mathcal{F}$, where
   \[ \bigoplus_n f_n(x) = f_{x(0)}(x^-), \]
   and $x^- = \langle x(n + 1) \mid n \in \omega \rangle$.

Typical examples of amenable $\mathcal{F}$ are the collections of all Lipschitz functions, all continuous functions, and all Borel functions. The “$\mathcal{F}$” in $\leq_{\mathcal{F}}$, $[A]_{\mathcal{F}}$, $\text{SLO}^\mathcal{F}$ etc., will be replaced by “$\mathcal{L}$” in the Lipschitz case, by “$\mathcal{W}$” in the continuous case (after Wadge), and by “$\mathcal{B}$” in the Borel case. Notice that Lip satisfies (1), (2), and (3), but not (4).

**Lemma 18.9.** Let $\mathcal{F} \neq \text{Lip}$ be amenable and let $A \subseteq \mathbb{R}$.

1. $A \oplus \neg A$ is $\mathcal{F}$-self-dual and $A, \neg A \leq_{\mathcal{F}} A \oplus \neg A$. Moreover if $A, \neg A \leq_{\mathcal{F}} C$, then $A \oplus \neg A \leq_{\mathcal{F}} C$. In particular $[A \oplus \neg A]_{\mathcal{F}}$ is the $\leq_{\mathcal{F}}$-least degree above $[A]_{\mathcal{F}}$ and $[\neg A]_{\mathcal{F}}$.

2. Assume $\text{SLO}^\mathcal{F}$ and suppose $[A]_{\mathcal{F}}$ is limit of countable cofinality. Then $[A]_{\mathcal{F}}$ is self-dual.
Proof. (a) The first part is trivial by (2) of Definition 18.8. For the second part, notice that if $f^{-1}C = A$ and $g^{-1}C = \neg A$, then $\bigoplus f_n$ witnesses $A \oplus \neg A \leq_F C$, where $f_{2n} = f$ and $f_{2n+1} = g$.

(b) Let $A_0 <_F A_1 <_F \ldots$ witness that $[A]_F$ is limit of countable cofinality. Since $A_{n+1} \not<_F A_n$ then there is $f'_n \in F$ witnessing $\neg A_n \leq_F A_{n+1}$ by SLO$^F$. The functions $f_n(x) = (n + 1) \cdot f'_n(x)$ belong to $F$ by (2) and (3) in Definition 18.8. Therefore $\bigoplus f_n$ witnesses that $\bigoplus \neg A_n \leq_F \bigoplus A_n$, that is $\bigoplus A_n$ is $F$-self-dual. Clearly (2) implies that $A_i \leq_F \bigoplus A_n$ for each $i$, and if $g_n$ witnesses $A_n \leq_F C$ then $\bigoplus g_n$ witnesses $\bigoplus A_n \leq_F C$. In other words $\bigoplus A_n$ is a least upper bound of the $A_n$’s. Therefore $\bigoplus A_n \equiv_F A$.

The Lemma is still true if $F = \text{Lip}$ (and hence it is true for all amenable $F$) but the argument is more involved and SLO$^L$ must be assumed also for case (a)—see [?].

The Lipschitz game on $A, B \subseteq \mathbb{R}, G_L(A, B)$, introduced by Wadge in [Wad83] is the game on $\omega$ where $I$ plays a real $a$, $II$ plays a real $b$, and $II$ wins iff $a \in A \Leftrightarrow b \in B$. Wadge’s Lemma is the simple, but fundamental observation that a winning strategy for $II$ yields a Lipschitz map witnessing $A \leq_L B$, while a winning strategy for $I$ yields a Lipschitz map (in fact: a contraction) witnessing $\neg B \leq_L A$. Therefore AD implies SLO$^L$, and since the smaller the $F$ the stronger the SLO$^F$,

$$\text{AD} \Rightarrow \text{SLO}^L \Rightarrow \text{SLO}^W \Rightarrow \text{SLO}^B.$$ 

We do not know whether any of these implications can be reversed—see [?] for more on this. In fact, a well-known open problem (probably first formulated by R. Solovay) asks whether SLO$^L$ or even SLO$^W$ implies AD, assuming $V = \text{L}(\mathbb{R})$. A similar question can be asked for SLO$^B$ or, more boldly, for SLO$^F$:

**Open problem 18.10.** Assume $V = \text{L}(\mathbb{R})$. Does SLO$^B \Rightarrow \text{AD}$? Does SLO$^F \Rightarrow \text{AD}$, if $F$ is amenable?

A less ambitious goal would be to prove some of the standard consequences of AD (like BP and LM, the assertion that all sets of reals are Lebesgue measurable) from some form of semi-linear ordering principle. This would yield some evidence for positive solutions to these open problems. For example it is known that the perfect set property [Wad83] and the axiom of countable choices for sets of reals [?] follow from SLO$^W$, but the following seems to be open:

**Open problem 18.11.** Assume $V = \text{L}(\mathbb{R})$ and let $F$ be amenable. Does SLO$^F \Rightarrow \text{BP}$? Does SLO$^F \Rightarrow \text{LM}$?
This is open even when $\mathcal{F}$ is the smallest amenable set of functions (and hence $\text{SLO}^\mathcal{F}$ is strongest semi-linear ordering principle), that is when $\mathcal{F} = \text{Lip}$, the collection of all Lipschitz functions.

Another partial evidence for the truth of the Open Problem 18.10 would be to prove the equivalence between the various semi-linear ordering principles, say between $\text{SLO}^L$, $\text{SLO}^W$, and $\text{SLO}^B$—again see [?].

18.A The Wadge and Lipschitz hierarchies

Assuming $\text{AD} + \text{DC}^* (\mathbb{R})$ then the following properties hold of the Wadge degrees:

1. $\leq_W$ is well-founded,
2. immediately above a self-dual degree there is a non-self-dual pair of degrees, and immediately above a non-self-dual pair of degrees there is a self-dual degree,
3. at limit levels of countable cofinality there is a single self-dual degree, and at uncountable cofinality there is a non-self-dual pair,

while for the Lipschitz degrees we have the following:

4. $\leq_L$ is well-founded,
5. every self-dual Wadge degree is the union of $\omega_1$ consecutive Lipschitz self-dual degrees, while the non-self-dual pairs of Wadge degrees coincide exactly with the non-self-dual pairs of Lipschitz degrees,
6. at limit levels of countable cofinality there is a single self-dual degree, and at uncountable cofinality there is a non-self-dual pair.

Therefore the Wadge hierarchy looks like this:

\[
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with each $\omega_1$-block of self-dual Lipschitz degrees collapsing to a single self-dual Wadge degree. In [?] it is shown that (1)–(3) follow from $\text{SLO}^W + \text{BP}$, and that (4)–(6) follow from $\text{SLO}^L + \text{BP}$. Therefore, if $\mathcal{F}$ is amenable—so that the $\mathcal{F}$ degrees are coarser than (or equal to) the Lipschitz degrees—and if $\text{AD}$ (or even just $\text{SLO}^L + \text{BP}$) is assumed, then $\leq_{\mathcal{F}}$ is well-founded and every non-self-dual $\mathcal{F}$-degree is a non-self-dual Lipschitz degree, i.e., $A \not\equiv_{\mathcal{F}} \neg A \Rightarrow [A]_{\mathcal{F}} = [A]_{\text{L}}$.

19 The Borel-Wadge hierarchy

We now focus on Borel-Wadge degrees. Our first goal is to prove that the well-foundedness of $\leq_{\mathcal{B}}$ follows from $\text{SLO}^B + \text{BP}$. The standard proof of the non-existence of an infinite $<_L$-descending sequence $\langle A_n \mid n \in \omega \rangle$ uses $\text{AD}$ to pick winning strategies for $I$ in $G_L(A_n, A_{n+1})$ and in $G_L(A_n, \neg A_{n+1})$. By pitting them against each other, a flip-set is constructed, contradicting $\text{BP}$. If we start from an infinite $<_B$-descending sequence $\langle A_n \mid n \in \omega \rangle$ we would like to argue, assuming $\text{SLO}^B$, that $I$ wins $G_L(A_n, A_{n+1})$ and $G_L(A_n, \neg A_{n+1})$ and proceed as before. In order to do this we need a few preliminary results.

A topological space is 0-dimensional if its topology is generated by the clopen sets. A metric space $(X, d)$ is Polish if it is separable and $d$ is complete. The collection of Borel subsets of $(X, d)$ is denoted by $B(X, d)$.

**Lemma 19.1.** Suppose $(X, d)$ is a Polish space and $\langle A_n \mid n \in \omega \rangle$ is a sequence of Borel subsets of $(X, d)$. Then there is metric $d'$ on $X$ such that

1. $(X, d')$ is Polish and 0-dimensional;
2. the new topology is finer than the old one, i.e., every $d$-open set is also $d'$-open;
3. each $A_n$ is $d'$-clopen;
4. the two topologies give rise to the same Borel sets, that is $B(X, d) = B(X, d')$.

See [Kec95, Theorem 13.1 and Exercise 13.5] for a proof. An easy consequence of this is the following result—see [Kec95, Theorem 13.11].

**Lemma 19.2.** Let $(X, d)$ be a Polish space, let $B \in B(X, d)$, and let $f : B \to B$ be a Borel function. There is a metric $d'$ on $B$ such that

1. $(B, d')$ is Polish and 0-dimensional;
19. THE BOREL-WADGE HIERARCHY

(2) the topology \( \tau' \) generated by \( d' \) on \( B \) refines the topology that \( B \) inherits from \( X \), i.e., \( \tau' \supseteq \{ U \cap B \mid U \in \tau \} \), where \( \tau \) is the topology on \( X \);

(3) \((B, d')\) has the same Borel structure as \( B \), that is: for every \( C \subseteq B \),
\[
C \in B(X, d) \iff C \in B(B, d');
\]

(4) \( f : (B, d') \to (B, d) \) is continuous.

By [Kec95, Theorem 7.8] every 0-dimensional Polish space is homeomorphic to a closed subset of the Baire space, so using Lemmata 19.1 and 19.2 will not take us outside of \( \mathcal{P}(\mathbb{R}) \).

Lemma 19.3. (a) If \( A \leq B \) then there is \( A^* \equiv_B A \) such that \( A^* \leq_L A \) and \( A^* \leq_L B \).

(b) Assume SLO\(^B\) and \( A <_B B \). Then there is \( A^* \equiv_B A \) such that \( \text{I has a winning strategy in } G_L(\neg B, A^*) \) and in \( G_L(B, A^*) \).

Proof. (a) Let \( f : \mathbb{R} \to \mathbb{R} \) be Borel such that \( f^{-1}B = A \). By Lemma 19.2 there is a 0-dimensional Polish topology \( \tau \) on \( \mathbb{R} \) that is finer than the standard one and makes \( f \) continuous. Let \( G : C \to (\mathbb{R}, \tau) \) be a homeomorphism with \( C \subseteq \mathbb{R} \) a closed set, and by [Kec95, Proposition 2.8] let \( \pi : \mathbb{R} \to C \) be Lipschitz and such that \( \pi \upharpoonright C \) is the identity. Let
\[
A' = (G \circ \pi)^{-1}A.
\]

Then \( A' \leq_W A \) via \( G \circ \pi \), and \( A \leq_B A' \) via \( G^{-1} : \mathbb{R} \to C \subseteq \mathbb{R} \). Since \( f \circ G \circ \pi : \mathbb{R} \to \mathbb{R} \) is continuous and
\[
x \in A' \iff G(\pi(x)) \in A \iff f(G(\pi(x))) \in B
\]
then \( A' \leq_W B \). We need the following result from [?, Lemma 19, part (a)]:

Lemma 19.4. If \( A' \leq_W A \) then there is \( A'' \equiv_W A' \) such that \( A'' \leq_L A' \) and \( A'' \leq_L A \).

Let \( A'' \) be as in Lemma 19.4. Since \( A'' \leq_L A' \leq_W B \), then \( A'' \leq_W B \), so by Lemma 19.4 again there is \( A^* \) such that \( A^* \leq_L A'' \), \( A'' \leq_W A^* \), and \( A^* \leq_L B \), which is what we had to prove.

(b) If \( \forall n (A \not\leq_B B_{[n]} \) or \( \forall n (A \not\leq_B \neg B_{[n]} \)), then by SLO\(^B\) we would have that \( \forall n (B_{[n]} \leq_B \neg A) \) or \( \forall n (B_{[n]} \leq_B A) \), hence \( B \leq_B \neg A \) or \( B \leq_B A \), contradicting our assumption in either case. Therefore there are \( n_0, m_0 \in \omega \) such that \( A \leq_B B_{[n_0]} \) and \( A \leq_B \neg B_{[m_0]} \) via Borel functions \( f \) and \( g \). By successive
applications of Lemma 19.2 there is a 0-dimensional Polish topology on $\mathbb{R}$ that is finer than the standard one and makes $f$ and $g$ continuous. Arguing as in part (a) there is an $A' \leq_L A$ such that $A \leq_B A'$ and $A' \leq_L B_{[m_0]}$, and since $A' \leq_B \neg B_{[m_0]}$, there is $A^* \leq_L A'$ such that $A' \leq_B A^*$ and $A^* \leq_L \neg B_{[m_0]}$. By playing $m_0$ and then following the reduction witnessing $A^* \leq_L \neg B_{[m_0]}$, $I$ wins $G_L(B, A^*)$; similarly $I$ has a winning strategy in $G_L(\neg B, A^*)$. \hfill $\square$

**Corollary 19.5.** Assume SLO$^B$ + BP. Then $\leq_B$ is a well-founded relation on $\mathcal{P}(\mathbb{R})$.

**Proof.** Suppose $\langle A_n \mid n \in \omega \rangle$ is a $<_B$-descending sequence of sets. Then $A_{n+1} \leq_B A_n$ and $A_n \not\leq_B A_{n+1}$, and hence, by SLO$^B$, $A_{n+1} \prec_B A_n$ and $\neg A_{n+1} \prec_B A_n$. By Lemma 19.3 we can construct inductively $A^*_0 = A_0$ and $A^*_n \equiv_B A_n$ such that $I$ has a winning strategy $\sigma^1_n$ in $G_L(A^*_n, A^*_{n+1})$ and $\sigma^0_n$ in $G_L(\neg A^*_n, A^*_{n+1})$. For any $z \in \omega^2$ let $x_n = x^z_n$ be the real in the $n$-th row of the following diagram where $I$ uses $\sigma^z(n)$ on the $n$-th row against his opponent on the $(n+1)$-st row:

$$
\begin{align*}
\sigma^z_0(0) &\quad x_0(0) \quad x_0(1) \quad \cdots &= x_0 \\
\sigma^z_1(0) &\quad x_1(0) \quad x_1(1) \quad \cdots &= x_1 \\
\vdots &\quad \vdots &\quad \vdots &\quad \vdots 
\end{align*}
$$

Thus $x^z_n$ is the result of applying $\sigma^z(n)$ to $x^z_{n+1}$. Then $\{z \in \omega^2 \mid x^z_0 \in A^*_0\}$ is a flip set, contradicting BP.

Lastly, to show that $\prec_B$ is well-founded on $\mathcal{P}(\mathbb{R})$, it is enough to show that $\prec_B$ is well-founded on $\{B \in \mathcal{P}(\mathbb{R}) \mid B \leq_B A\}$, for any $A \subseteq \mathbb{R}$. So fix $A \subseteq \mathbb{R}$. Since there is a surjection $\mathbb{R} \rightarrow \{f \in \mathcal{P}(\mathbb{R}) \mid f \text{ is Borel}\}$, $x \mapsto f_x$, consider the pre-order on $\mathbb{R}$ defined by

$$x \prec y \Leftrightarrow f_x^{-1}uA \prec_B f_y^{-1}uA.$$ 

Then $\prec_B$ is well-founded on $\{B \in \mathcal{P}(\mathbb{R}) \mid B \leq_B A\}$ iff $\prec$ is well-founded on $\mathbb{R}$, which, by DC($\mathbb{R}$) is equivalent to the non-existence of an infinite $\prec$-descending sequence. But any $\prec$-descending sequence in $\mathbb{R}$ yields a $<_B$-descending sequence in $\{B \in \mathcal{P}(\mathbb{R}) \mid B \leq_B A\}$, hence we are done by the first part of the proof. \hfill $\square$

Thus, assuming SLO$^B$ + BP, the canonical rank function for the well-founded relation $<_B$ on $\mathcal{P}(\mathbb{R})$ can be defined. It is called the Borel-Wadge rank and it is denoted by $A \mapsto \|A\|_B$. It is immediate that $[A]_B$ is a limit degree iff $\|A\|_B$ is a limit ordinal, and that $[A]_B$ is of countable cofinality iff $\text{cof}(\|A\|_B) = \omega$. For technical reasons (see [?, Proposition 13]) it is convenient
to assume that the Wadge rank $|A|_W$ of a set is a non-zero ordinal, and hence, by analogy, we make the same assumption of the Borel-Wadge rank. Thus $|\emptyset|_B = |\mathbb{R}|_B = 1$.

The tree $T(A) = \{ s \in \omega | A_{|s|} \equiv_W A \}$ is a standard tool to investigate the structure of the Wadge degrees. For example $[A]_W$, the Wadge degree of $A$, is self-dual iff $T(A)$ is well-founded i.e., if the converse of the extension relation on $T(A)$ is well-founded. Notice that if $T(A)$ is well-founded, then

$$\{ N_s | s \notin T(A) & s \uparrow \text{lh}(s) - 1 \in T(A) \}$$

is a partition of $\mathbb{R}$ into countably many clopen sets $D$ such that $D \cap A <_W A$. This suggests the correct generalization of $T(A)$ to the Borel context.

**Definition 19.6.** Let $B \subseteq \mathbb{R}$. A **Borel partition** of $B$ is a family $\{ B_n | n < N \}$ of non-empty pairwise disjoint Borel sets such that $B = \bigcup_{n < N} B_n$ and $2 \leq N \leq \omega$.

First a trivial but useful fact:

**Lemma 19.7.** Let $B \subseteq B'$ be Borel. If $A \cap B' \neq \emptyset$, then $A \cap B \leq_B A \cap B'$. In particular if $B$ is Borel and $A \neq \mathbb{R}$, then $A \cap B \leq_B A$.

Then:

**Lemma 19.8.** Let $\{ B_n | n < N \}$ be a Borel partition of $\mathbb{R}$, and let $A \neq \mathbb{R}$.

(a) $\forall n < N (A \cap B_n \leq_B A)$, and if $C$ is such that $\forall n < N (A \cap B_n \leq_B C)$, then $A \leq_B C$. In other words: $[A]_B$ is the $\leq_B$-least upper bound of $\{ [B_n \cap A]_B | n < N \}$.

(b) Assume SLO$^B$. If $\forall n < N (A \cap B_n \leq_B A)$ then $A \leq_B \neg A$. Moreover, if $N < \omega$ then $[A]_B$ is a successor degree.

**Proof.** (a) The first part follows from Lemma 19.7. If $g_n$ witnesses $B_n \cap A \leq_B C$, then $g = \bigcup_n g_n \upharpoonright B_n$ is Borel and witnesses $A \leq_B C$.

(b) $A \cap B_n \leq_B A$ implies, by SLO$^B$, that there are Borel functions $f_n : \mathbb{R} \to \mathbb{R}$ witnessing $A \cap B_n \leq_B \neg A$. Then $f = \bigcup_n f_n \upharpoonright B_n$ is Borel and $f^{-1} \circ \neg A = A$. Suppose now, towards a contradiction, that $[A]_B$ is a limit degree and $N < \omega$. Let $C_0 = B_0 \cap A$ and, for $n + 1 < N$, let $C_{n+1} = C_n \oplus (B_{n+1} \cap A)$. By induction, using Lemma 18.9(a) and that $[A]_B$ is limit, $\forall n < N (C_n \leq_B A)$. But $B_n \cap A \leq_B C_{N-1}$ for $n < N$, hence $A \leq_B C_{N-1}$ by part (a): a contradiction. \qed
Definition 19.9. For $A \subseteq \mathbb{R}$, let

$$\mathcal{I}(A) = \{ B \mid B \text{ is Borel and } \exists \langle B_n \mid n \in \omega \rangle \text{ Borel sets such that } B = \bigcup_n B_n \text{ and } B_n \cap A \leq B A \}.$$  

By Lemma 19.7, the sets $B_n$ in the definition can be taken to form a partition of $B$, since $\mathcal{I}(A)$ is empty when $A = \mathbb{R}$ or $A = \emptyset$. The following result can be easily verified.

Lemma 19.10. Assume SLO$^B$.

(a) If $B \in \mathcal{I}(A)$ and $C \subseteq B$ is Borel, then $C \in \mathcal{I}(A)$.

(b) If $B_n \in \mathcal{I}(A)$, then $\bigcup_n B_n \in \mathcal{I}(A)$.

Recall that a $\sigma$-ideal of Borel sets is a non-empty collection $\mathcal{J}$ of Borel subsets of $\mathbb{R}$, closed under Borel subsets and countable unions. A $\sigma$-ideal of Borel sets $\mathcal{J}$ is proper if $\mathbb{R} \notin \mathcal{J}$. Then Lemma 19.10 says that $\mathcal{I}(A)$ is a $\sigma$-ideal of Borel sets, and Lemma 19.8(b) says that if $\mathcal{I}(A)$ is not proper, then $[A]_B$ is self-dual.

Theorem 19.11. Assume BP and suppose $A \leq B \neg A$. Then there is a Borel partition $\{ B_n \mid n \in \omega \} \text{ of } \mathbb{R}$ such that $\forall n < N (B_n \cap A \leq B A)$.

Proof. Towards a contradiction, suppose that for every Borel partition $\{ B_n \mid n < N \}$ of $\mathbb{R}$ there is $n_0 < N$ such that $B_{n_0} \cap A \leq B A$. For ease of notation, let $\mathcal{I} = \mathcal{I}(A)$.

Claim 19.11.1. If $B$ is Borel and $B \notin \mathcal{I}$, then there is a Borel function $f : B \to B$ witnessing

$$\forall x \in B (x \in A \cap B \iff f(x) \in \neg A \cap B).$$

Proof of Claim. By case assumption $B \neq \emptyset$, and if $B = \mathbb{R}$ the result follows at once, so we may assume $B \neq \emptyset, \mathbb{R}$. By Lemma 19.7, $A \cap B \leq B A$ and $\neg A \cap B \leq B \neg A$. If $A \cap B \leq B A$, then taking $B_n = B$ in Definition 19.9, we would have $B \in \mathcal{I}$: a contradiction. Therefore $\neg A \cap B \leq B \neg A \leq B A \equiv B A \cap B$. Let $h : \mathbb{R} \to \mathbb{R}$ be a Borel function witnessing that $\neg A \cap B \leq B A \cap B$, and let $k : \mathbb{R} \to B$ be defined as

$$k(x) = \begin{cases} x & \text{if } x \in B, \\ b & \text{otherwise,} \end{cases}$$

where $b$ is some fixed element of $A \cap B$. Then $f = (k \circ h) \upharpoonright B$ is the required function. \qed
We will construct a sequence of Borel sets
\[ \mathbb{R} = B_0 \supseteq B_1 \supseteq \cdots \]
such that \( B_n \notin \mathcal{I} \). Using the Claim, let \( f_n : B_n \to B_n \) be Borel and such that
\[ \forall x \in B_n \,(x \in A \cap B_n \iff f_n(x) \in \neg A \cap B_n). \]
We will also choose a separable complete metric \( d_n \) on \( B_n \) such that \( d_0 \) is the usual metric on \( \mathbb{R} \), and the topologies \( \tau_n \) generated by the metrics \( d_n \) are all 0-dimensional and \( \tau_{n+1} \) refines \( \tau_n \), that is, \( \{U \cap B_{n+1} \mid U \in \tau_n\} \subseteq \tau_{n+1} \). We also require that \( f_n \upharpoonright B_{n+1} : (B_{n+1}, d_{n+1}) \to (B_n, d_n) \) be continuous, and that for any \( m \leq n \) and every \( a, b \in B_{n+1} \)
\[ d_m(g_m \circ \cdots \circ g_n(a), g_m \circ \cdots \circ g_n(b)) < 2^{-n}, \]
where each \( g_i \) is either \( f_i \upharpoonright B_{i+1} \) or the identity on \( B_{i+1} \). Then we can apply the Martin-Monk method as follows:

Fix \( z \in \omega^2 \) and let
\[ g_n = \begin{cases} f_n \upharpoonright B_{n+1} & \text{if } z(n) = 1, \\ \text{id} \upharpoonright B_{n+1} & \text{if } z(n) = 0. \end{cases} \]

For each \( n \), pick \( y_{n+1} \in B_{n+1} \) and let
\[ x^n_m = g_m \circ \cdots \circ g_n(y_{n+1}) \in B_m, \]
for all \( m \leq n \). By construction, for any fixed \( m \)
\[ \forall n > m \,(g_m(x^n_{m+1}) = x^n_m) \]
and \( \{x^n_m \mid n \geq m\} \subseteq B_m \) is a Cauchy sequence with respect to \( d_m \), since \( d_m(x^n_m, x^k_m) < 2^{-\min(n,k)} \) by (1). Therefore we get an
\[ x_m = \lim_{n \to \infty} x^n_m \in B_m \]
and by continuity of \( g_m : (B_{m+1}, d_{m+1}) \to (B_m, d_m) \) and by (2)
\[ g_m(x_{m+1}) = x_m. \]

Naturally \( x_m \) really depends on \( z \in \omega^2 \), so we should write \( x_m = x_m(z) \). By construction, if \( \forall n > n_0 \,(z(n) = w(n)) \) then
\[ \forall n > n_0 \,(x_n(z) = x_n(w)) \]
and if \( z(n_0) \neq w(n_0) \) then
\[
x_{n_0}(z) \in A \cap B_{n_0} \Leftrightarrow x_{n_0}(w) \notin A \cap B_{n_0}.
\]
The usual argument yields that \( \{ z \in \omega^2 \mid x_0(z) \in A \} \) is a flip-set, contradicting the property of Baire.

Therefore it is enough to construct the \( B_n \)'s and \( d_n \)'s. As required, set \( B_0 = \mathbb{R}, d_0 \) the usual distance on \( \mathbb{R} \), and let \( f_0 : \mathbb{R} \rightarrow \mathbb{R} \) be a Borel function witnessing \( A \leq_{B} \lnot A \).

Suppose \( B_m, f_m, \) and \( d_m \) have been defined for all \( m \leq n \). Fix an \( s \in n^{+1}2 \), and, for \( i \leq n \), let \( g_i^s = g_i \) or the identity, depending on whether \( s(i) = 1 \) or \( s(i) = 0 \). For each \( m \leq n \) let \( \{ C_m^i \mid i \in \omega \} \) be a Borel partition of \( B_m \) such that \( \text{diam}(C_m^i) < 2^{-n} \). We now inductively construct \( B_n \supseteq B^0 \supseteq B^1 \supseteq \cdots \supseteq B^n \) as follows. By the \( \sigma \)-additivity of \( \mathcal{I} \) there is \( i_0 \in \omega \) such that
\[
B^0 = (g_0 \circ \cdots \circ g_n)^{-1} \lnot C_{i_0}^0 \notin \mathcal{I},
\]
and by \( \sigma \)-additivity again, inductively choose \( i_m \in \omega \) such that
\[
B^{m+1} = B^m \cap (g_m \circ \cdots \circ g_n)^{-1} \lnot C_{i_m}^m \notin \mathcal{I},
\]
for \( m < n \). Since the construction above depends on the chosen \( s \in n^{+1}2 \), let \( B(s) = B^n \). Now we can repeat the construction above for each element of \( n^{+1}2 \): let \( \langle s_i \mid 1 \leq i \leq 2^{n+1} \rangle \) be an enumeration of \( n^{+1}2 \), and construct \( B(s_1) \) as above, then construct \( B(s_2) \) using \( B(s_1) \) instead of \( B_n \), and so on. This gives a sequence of Borel sets not in \( \mathcal{I} \)
\[
B_n \supseteq B(s_1) \supseteq \cdots \supseteq B(s_{2^n}) = B_{n+1}
\]
and by construction, for any \( a, b \in B_{n+1} \), any \( m \leq n \), and any \( s \in n^{+1}2 \)
\[
d_m (g_m^s \circ \cdots \circ g_n^s(a), g_m^s \circ \cdots \circ g_n^s(b)) < 2^{-n}.
\]
Since \( B_{n+1} \notin \mathcal{I} \) then \( A \cap B_{n+1} \leq_{B} \lnot A \cap B_{n+1} \) and let \( f_{n+1} : B_{n+1} \rightarrow B_{n+1} \) witness this. In order to complete the construction we need to prove the existence of \( d_{n+1} \) on \( B_{n+1} \). This follows at once from Lemma 19.2 taking \( (X, d) = (B_n, d_n) \), \( f = f_{n+1} \), and \( B = B_{n+1} \).

Notice that the Property of Baire is used in a “local way” in the proof of Theorem 19.11: if \( \prod_n B_n \subset \omega \mathbb{R} \) is endowed with the product topology of the \( \tau_n \)'s, rather than with the topology induced by \( \omega \mathbb{R} \), then the map
\[
\omega^2 \rightarrow \prod_n B_n, \quad z \mapsto (x_m(z) \mid m \in \omega)
\]
The Borel-Wadge hierarchy

is continuous by (1), hence the map \( \omega^2 \to B_0 = \mathbb{R}, \ z \mapsto x_0(z) \), is continuous, and hence the flip-set is the continuous pre-image of \( A \). Therefore the proof only requires the property of Baire for sets which are Wadge reducible to \( A \).

Suppose \([A]_B\) is limit and self-dual, and let \( \{B_n \mid n < \omega\} \) be a Borel partition of \( \mathbb{R} \) as in Theorem 19.11. If \( C <_B A \) were an upper bound for the \( A \cap B_n \)'s, i.e., \( \forall n (A \cap B_n \leq_B C) \), then Lemma 19.8 implies that \( A \leq_B C \), a contradiction. Since by Lemma 19.7

\[ A \cap B_0 \leq_B A \cap (B_0 \cup B_1) \leq_B A \cap (B_0 \cup B_1 \cup B_2) \leq_B \ldots \leq_B A, \]

and \( A \cap B_n \leq_B A \cap \bigcup_{i \leq n} B_i \), then \( \|A\|_B = \sup_n \|A \cap \bigcup_{i \leq n} B_i\|_B \). Therefore if \([A]_B\) is limit and \( \operatorname{cof}(\|A\|_B) > \omega \), then \([A]_B\) is non-self-dual.

We have already seen that immediately above a non-self-dual pair \(([A]_B, [\neg A]_B)\) there is a self-dual degree \([A \oplus \neg A]_B\). We will now argue that immediately above a self-dual degree there is a non-self-dual pair. This amounts to proving that if \([A]_B < [B]_B\) are both self-dual then \( A <_B C <_B B \) for some \( C \). Let \( \{D_n \mid n \in \omega\} \) be a Borel partition of \( \mathbb{R} \) such that \( \forall n (B \cap D_n <_B B) \). If \( B \cap D_n \leq_B A \) for all \( n \) then \( B \leq_B A \), which is absurd, so let \( n_0 \in \omega \) be such that \( B \cap D_{n_0} \not\leq_B A \). By SLO\(^B\), \( \neg A <_B B \cap D_{n_0} \) and since \( A \equiv_B \neg A \), then \( A <_B B \cap D_{n_0} <_B B \). Thus we have proved:

**Corollary 19.12.** Assume SLO\(^B + \text{BP} \).

(a) A limit Borel-Wadge degree of uncountable cofinality is non-self-dual.

(b) Immediately above a self-dual Borel-Wadge degree there is a non-self-dual pair.

Therefore the structure of the Borel degrees is isomorphic to the structure of the Wadge degrees:

\[ \cdots \cdots \cdots \cdots \cdots \cdots \]

\[ \uparrow \quad \uparrow \]

\[ \text{cof} = \omega \quad \text{cof} > \omega \]

At the bottom of the hierarchy there is the non-self-dual pair \(([\mathbb{R}]_B, [\emptyset]_B)\) which—as already pointed-out in section ??—is \(([\mathbb{R}], \{\emptyset\})\). Immediately above it there is the least self-dual degree, \( \Delta^+_1 \setminus \{0, \mathbb{R}\} \). We call these three degrees \([\mathbb{R}]_B, [\emptyset]_B, \) and \( \Delta^+_1 \setminus \{0, \mathbb{R}\} \) trivial: in other words, \([A]_B\) is non-trivial just in case \( \|A\|_B \geq 3 \). Lastly we prove the converse to the second half of Lemma 19.8(b).
Proposition 19.13. Assume \( \text{SLO}^B + \text{BP} \). If \([A]_B^B\) is a non-trivial self-dual successor degree, then there is a Borel partition of \( \mathbb{R} \), \( \{B_0, B_1\} \), such that \( B_i \cap A <_B A \), for \( i = 0, 1 \).

Proof. Let \([C]_B\) be the immediate predecessor of \([A]_B\). Then \([C]_B\) is non-self-dual and \([C \oplus \neg C]_B\) is its immediate successor, that is \( A \equiv_B C \oplus \neg C \).

Let \( f : \mathbb{R} \to \mathbb{R} \) be Borel witnessing \( A \leq_B C \oplus \neg C \), let \( D_0 = \{ x \in \mathbb{R} \mid x(0) \text{ is even} \} \) and \( D_1 = \{ x \in \mathbb{R} \mid x(0) \text{ is odd} \} \), and let \( B_i = f^{-1}(D_i) \). Then \( \{B_0, B_1\} \) is a Borel partition of \( \mathbb{R} \), and \( f \) witnesses \( B_i \cap A \leq_B D_i \cap (C \oplus \neg C) \). Since \( C \neq \mathbb{R}, \emptyset \), then \( D_0 \cap (C \oplus \neg C) \equiv_B C \), and since \( C <_B A \) we have that \( B_0 \cap A <_B A \). Similarly, \( B_1 \cap A \leq_B D_1 \cap (C \oplus \neg C) \equiv_B C <_B A \). \( \square \)

Corollary 19.14. Assume \( \text{SLO}^B + \text{BP} \). \([A]_B\) is a non-trivial self-dual successor degree iff there is a Borel partition of \( \mathbb{R} \), \( \{B_n \mid n < N\} \) such that \( \forall n < N (B_n \cap A <_B A) \). Moreover \([A]_B\) is a successor degree iff \( N \) can be taken to be finite, and in fact \( N \) can be taken to be 2.

20 Non-self-dual pointclasses

Assuming \( \text{AD} \), Steel and Van Wesep independently showed that \( A \leq_W \neg A \Rightarrow A \leq_L \neg A \), hence \([A]_W = [A]_L\) for every non-self-dual \([A]_L\). In fact both statements are provable assuming \( \text{SLO}^L + \text{BP} \)—see [?]. The analogue of the first statement for \( \leq_B \) and \( \leq_W \) is clearly false: if \( U \) is open but not closed, then \( U \leq_B \neg U \) but \( U \not< W \neg U \). Nevertheless the second statement can be generalized to the Borel case.

Proposition 20.1. Assume \( \text{SLO}^W + \text{BP} \). If \( A \not= B \neg A \), then \([A]_B = [A]_W\).

Proof. If \( B \in [A]_B \) and \( B \not= W A \), then, since \( A \equiv_W \neg B \) cannot hold, either \( B <_W A \) or \( A <_W B \). For the sake of definitiveness, assume the former. Then \( C \in [A]_B \) for any \( B \leq_W C <_W A \). Since by \( \text{SLO}^W + \text{BP} \) we certainly can find such a \( C \) which is Wadge self-dual, we have that \( A \equiv_B C \equiv_W \neg C \equiv_B \neg A \), a contradiction. \( \square \)

Each non-self-dual pair of Wadge-Borel degrees is a non-self-dual pair of Wadge degrees, but not vice versa. Notice that assuming \( \text{SLO}^L + \text{BP} \) the conclusion for non-self-dual \([A]_B\) can be strengthened to \([A]_B = [A]_L\). On the other hand, the self-dual Borel degrees are obtained by glueing together many Wadge degrees. For example, the first self-dual degree is the collection of all Borel sets except for \( \mathbb{R} \) and \( \emptyset \). For \( A \equiv_B \neg A \) let \( h([A]_B) \) be the
length of the interval of Wadge degrees used to construct \([A]_B\). Thus, if 
\(A \in \Delta^1_1 \setminus \{\emptyset, \mathbb{R}\}\), then 
\([A]_B = \Delta^1_1 \setminus \{\emptyset, \mathbb{R}\}\) and

\[
\begin{align*}
  h([A]_B) &= \delta \\
  &= \text{the length of the Wadge hierarchy restricted to } \Delta^1_1 \\
  &= \|B\|_W, \text{ where } B \in \Sigma^1_1 \cup \Pi^1_1 \setminus \Delta^1_1.
\end{align*}
\]

In analogy with the case of the Lipschitz-vs-Wadge hierarchies, where each self-dual Wadge degree is the union of \(\omega_1\) consecutive Lipschitz degrees, it is tempting to conjecture that 
\(h([A]_B) = \delta\) for any self-dual \([A]_B\). However this is not true. In fact, 
\(h([A]_B) > \|A\|_W\) for all self-dual \([A]_B\), and therefore the Borel-Wadge hierarchy is obtained by collapsing to a single self-dual Borel-Wadge degree larger and larger blocks of the Wadge hierarchy. To see this we need to recall the definition—due to Wadge—of addition of sets of reals.

For \(A, B \subseteq \mathbb{R}\) let

\[
A + B = \{(s + 1) \wedge (0)^\omega (x + 1) \mid s \in \omega^\omega \& x \in A\} \cup \{x + 1 \mid x \in B\},
\]

where \(y + 1 = (y(n) + 1 \mid n < \text{lh}(y))\), for any sequence (finite or infinite) \(y\). If \(A \equiv_W \neg A\) and assuming \(\text{SLO}^{W} + \text{BP}\), we have that 
\(\|A + B\|_W = \|A\|_W + \|B\|_W\) (see [Wad83] or [?] for more on this). In particular, if \(B \leq_B A\) and \(f\) witnesses this, then

\[
g(x) = \begin{cases} 
  f(x - 1) & \text{if } \forall n \; x(n) \neq 0, \\
  y & \text{if } x = (s + 1) \wedge (0)^\omega, \text{ for some } s \in \omega^\omega.
\end{cases}
\]

is Borel and witnesses \(A + B \leq_B A\). Therefore 
\(h([A]_B) \geq \|A\|_W + \|A\|_W > \|A\|_W\).

Assuming \(\text{SLO}^W\) we can now describe the first few Borel degrees. Immediately above the trivial degrees \([\mathbb{R}]_B, [\emptyset]_B\), and \(\Delta^1_1 \setminus \{\mathbb{R}, \emptyset\}\) there is, by Proposition 20.1, the non-self-dual pair \((\Sigma^1_1 \setminus \Delta^1_1, \Pi^1_1 \setminus \Delta^1_1)\). At the next level we have a self-dual degree: it is the collection of all Borel-separated-unions of a true \(\Sigma^1_1\) and a true \(\Pi^1_1\),

\[
\{A \cup B \mid A \in \Sigma^1_1 \setminus \Delta^1_1 \& B \in \Pi^1_1 \setminus \Delta^1_1 \& \exists C \in \Delta^1_1 (A \subseteq C \& B \cap C = \emptyset)\}.
\]

In order to compute the next non-self-dual pair of degrees it is more convenient to work with pointclasses rather than with degrees. Recall that a collection of sets \(\Gamma \subseteq \mathcal{P}(\mathbb{R})\) is a **boldface pointclass** if it is non-empty and closed under continuous pre-images. It is self-dual if it is closed under complements, otherwise it is non-self-dual. The **dual** of \(\Gamma\) is the pointclass 
\(\bar{\Gamma} = \{\neg A \mid A \in \Gamma\}\), and let \(\Delta = \Delta_{\Gamma}\) be the pointclass \(\Gamma \cap \bar{\Gamma}\). Under \(\text{SLO}^W\),
the non-self-dual boldface pointclasses are of the form \( \{ X \subseteq \mathbb{R} \mid X \leq_W A \} \) with \([A]_W \) non-self-dual, while the self-dual ones are of the form \( \{ X \subseteq \mathbb{R} \mid X <_W A \} \), with \( A \neq \mathbb{R}, \emptyset \). Conversely, if \( \Gamma \) is non-self-dual, then \( \Gamma \setminus \bar{\Gamma} \) is a non-self-dual Wadge degree by \( \text{SL}^\omega \). Therefore \( \text{SL}^\omega \) yields that boldface pointclasses are (essentially) well-ordered under inclusion: either \( \Gamma \subseteq \Lambda \) or \( \Lambda \subseteq \Gamma \). By Proposition 20.1 and the discussion following its proof, \([A]_B \) is a non-self-dual degree iff \( \{ X \mid X \leq_W A \} \) is closed under Borel pre-images. A set \( U \) is \( \Gamma \)-universal if it \( \mathbb{R} \)-parametrizes \( \Gamma \) and belongs to \( \Gamma \), i.e., \( U \subseteq \mathbb{R}^2 \), \( \Gamma = \{ U_x \mid x \in \mathbb{R} \} \) where \( U_x = \{ y \mid (x, y) \in U \} \), and \( U \) (or better: its image under the standard homeomorphism \( \mathbb{R}^2 \approx \mathbb{R} \)) is in \( \Gamma \). If \( \Gamma \) has a universal set then it is non-self-dual. Conversely, \( \text{SL}^\omega + \text{BP} \) implies every non-self-dual \( \Gamma \) has a universal set: choose an \( \mathbb{R} \)-parametrization \( \langle g_x \mid x \in \mathbb{R} \rangle \) of all Lipschitz functions such that \( (x, y) \mapsto g_x(y) \) is continuous; by the theorem of Steel and Van Wesep mentioned at the beginning of this section, \( \Gamma = \{ X \mid X \leq_L A \} \) for some \( A \neq \mathbb{R} \), and let \( U = \{ (x, y) \mid g_x(y) \in A \} \). Similarly, by choosing a parametrization \( \langle f_x \mid x \in \mathbb{R} \rangle \) of all continuous functions such that \( (x, y) \mapsto f_x(y) \) is Borel, \( \text{SL}^\omega \) implies that if \([A]_B \) is non-self-dual then \( \Gamma = \{ X \mid X \leq_W A \} \) has a universal set.

Wadge gave a concrete description of the next non-self-dual pair of boldface pointclasses above a non-self-dual \( \Gamma \): Suppose \( \Gamma = \{ X \mid X \leq_W A \} \) with \( A \) non-self-dual and let

\[
\Gamma^\vee = \{ (U \cap X) \cup (U' \setminus X') \mid U, U' \in \Sigma_1^0 \text{ and } U \cap U' = \emptyset \}.
\]

Then \( \Gamma \cup \bar{\Gamma} \subseteq \Gamma^\vee \), and \( \Gamma^\vee \) and its dual \((\Gamma^\vee)^\wedge\) are the least non-self-dual pair of pointclasses above \( \Gamma \cup \bar{\Gamma} \). Notice that the self-dual pointclass \( \Delta_{\Gamma^\vee} = \{ X \mid X \leq_W B \oplus \neg B \} \) is made-up of those \((U \cap X) \cup (U' \setminus X') \in \Gamma^\vee \) such that \( U \subseteq C \subseteq \neg U' \) for some clopen set \( C \). This suggests the following definition.

For \( \Gamma \) a boldface pointclass closed under Borel pre-images let

\[
\Gamma^* = \{ (P \cap X) \cup (P' \setminus X') \mid P, P' \in \Pi_1^1 \text{ and } P \cap P' = \emptyset \}.
\]

and let \( \Delta^* = \Delta_{\Gamma^*} \). Taking \( P = \mathbb{R} \) or \( P' = \mathbb{R} \), we have that \( \Gamma \subseteq \Gamma^* \) and \( \bar{\Gamma} \subseteq \Gamma^* \).

**Lemma 20.2.** Assume \( \text{SL}^\omega \) and let \( \Gamma \) be a non-self-dual pointclass closed under Borel pre-images. Then \( \Gamma^* \) is non-self-dual and is closed under Borel pre-images.

**Proof.** As both \( \Gamma \) and \( \Pi_1^1 \) have universal sets, \( \Gamma^* \) also has a universal set, hence it is non-self-dual. Since both \( \Gamma \) and \( \Pi_1^1 \) are closed under Borel pre-images, it follows that \( \Gamma^* \) is closed under Borel pre-images. \( \square \)
Lemma 20.3. Let $[A]_B$ be non-self-dual and let $\Gamma = \{X \mid X \leq W A\}$. Then any set in $\Delta^*$ is Borel reducible to $A \oplus \neg A$.

Proof. Let $Y \in \Delta^*$ and let $P_1, P'_1 \in \Pi^1$ and $X_1, X'_1 \in \Gamma$ witness that

$$Y = (P_1 \cap X_1) \cup (P'_1 \setminus X'_1) \in \Gamma^*$$

and let $P_2, P'_2 \in \Pi^1$, $X_2, X'_2 \in \Gamma$ witness that

$$\neg Y = (P_2 \cap X_2) \cup (P'_2 \setminus X'_2) \in \Gamma^*.$$  

Then $P_1 \cup P'_1 \cup P_2 \cup P'_2 = \mathbb{R}$. By Reduction for $\Pi^1$, let $\{B_1, B'_1, B_2, B'_2\}$ be a Borel partition of $\mathbb{R}$ such that $B_1 \subseteq P_1$, $B'_1 \subseteq P'_1$, $B_2 \subseteq P_2$, and $B'_2 \subseteq P'_2$. Then

$$x \in B_1 \Rightarrow (x \in Y \Leftrightarrow x \in X_1)$$

$$x \in B'_1 \Rightarrow (x \in Y \Leftrightarrow x \notin X'_1)$$

$$x \in B_2 \Rightarrow (x \in Y \Leftrightarrow x \notin X_2)$$

$$x \in B'_2 \Rightarrow (x \in Y \Leftrightarrow x \in X'_2).$$

This implies the desired conclusion. \hfill \Box

Theorem 20.4. Assume $\text{SLO}^W$. Let $[A]_B$ be non-self-dual and let $\Gamma = \{X \mid X \leq W A\}$. Then

$$\Delta^* = \{X \mid X \leq_B A \oplus \neg A\} = \{X \cup \neg X' \mid \exists B \in \Pi^1 (X \subseteq B \subseteq \neg X') \land X, X' \in \Gamma\}$$

and $\Delta^* \setminus (\Gamma \cup \tilde{\Gamma}) = [A \oplus \neg A]_B$, i.e., it is the self-dual degree immediately above $([A]_B, [\neg A]_B)$ and $(\Gamma^* \setminus \Delta^*, (\Gamma^*)^* \setminus \Delta^*)$ is the next non-self-dual pair above it.

Proof. It is easy to check that $A \oplus \neg A \in \Delta^*$ and that the sets which are Borel-reducible to $A \oplus \neg A$ are of the form $X \cup \neg X'$ with $X, X' \in \Gamma$ Borel separated. Therefore we are done by Lemmata 20.2 and 20.3. \hfill \Box

Wadge’s analysis shows that if $\Gamma_n$ is an increasing sequence of boldface pointclasses, then

$$\Gamma = \{\bigcup_n (U_n \cap X_n) \mid U_n \in \Sigma^0_n \text{ are pairwise disjoint and } X_n \in \Gamma_n\}$$

is non-self-dual, and $\Gamma$ and its dual are the least non-self-dual pointclasses above the $\Gamma_n$’s.
Similarly, if \( \langle \Gamma_n \mid n \in \omega \rangle \) is a strictly increasing sequence of pointclasses closed under Borel pre-images, then let

\[
\Delta = \{ \bigcup_n (B_n \cap X_n) \mid B_n \in \Delta^1 \text{ are pairwise disjoint and } X_n \in \Gamma_n \}
\]

and let

\[
\Lambda = \{ \bigcup_n (P_n \cap X_n) \mid P_n \in \Pi^1 \text{ are pairwise disjoint and } X_n \in \Gamma_n \}.
\]

If \( A_{n+1} \subseteq \Gamma_{n+1} \setminus \Gamma_n \) and there are pairwise disjoint Borel sets \( B_n \) such that \( A_n \subseteq B_n \), then it is not hard to see that \( \bigcup_n A_n \) is Wadge self-dual, that

\[
\Delta = \{ X \mid X \leq_w \bigcup_n A_n \}
\]

is self-dual, that \( \bigcup_n \Gamma_n \subseteq \Delta \), and that there is no boldface pointclass in between. Arguing as above we get:

**Theorem 20.5.** Assume \( \text{SLO}^W + \text{BP} \) and suppose \( \Gamma_n, \Delta \) and \( \Lambda \) are as above. Then \( \Lambda \) and \( \Lambda \) are closed under Borel pre-images and are the least non-self-dual pair of boldface pointclasses above the \( \Gamma_n \)'s, and \( \Delta = \Delta_\Lambda \).

We can now give a complete description of the first \( \omega_1 \) levels of the \( \leq_B \) hierarchy. By Theorem 20.4 the least non-self-dual pair of pointclasses closed under \( \leq_B \) and above \( \Sigma^1_1 \) and \( \Pi^1_1 \) is \( (\Gamma, \tilde{\Gamma}) \), where \( Y \in \Gamma \) iff \( Y = P_1 \cup P_2 \setminus P_3 \) with \( P_1, P_2, P_3 \in \Pi^1_1 \) and \( P_1 \cap P_2 = \emptyset \). Without loss of generality we may assume \( P_3 \subseteq P_2 \) hence \( Y = (P_1 \cup P_2) \setminus P_3 \), hence \( \Gamma \) is the collection \( \text{Diff}(2; \Pi^1_1) \) of all differences of \( \Pi^1_1 \) sets. Inductively, using Theorem 20.5, one can show that the \( \alpha \)-th pair of non-self-dual pointclasses closed under Borel reducibility is \( (\text{Diff}(\alpha; \Pi^1_1), \text{Diff}(\alpha; \Pi^1_1)) \).
Chapter IV

Wadge determinacy
CHAPTER IV. WADGE DETERMINACY
Chapter V

The Semi-Linear Ordering Principle

21 $\text{SLO}^W$ and the Structure of the Wadge Hierarchy

We briefly recall the basic facts about the Wadge hierarchy. Unless otherwise indicated, $\mathbb{R}$ denotes the Baire space

$$\mathbb{R} = \omega^\omega.$$  

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** if there is a $\varphi : <\omega \omega \to <\omega \omega$ which is

1. **monotone**, that is $s \subseteq t \Rightarrow \varphi(s) \subseteq \varphi(t)$, and
2. $\forall x \in \mathbb{R} \left( \lim_{n \to \infty} \operatorname{lh}(\varphi(x \upharpoonright n)) = \infty \right)$,

and such that $f(x) = \bigcup_n \varphi(x \upharpoonright n)$. We will say that $f$ is induced by $\varphi$ and write $f = f_\varphi$. With a minor abuse of notation, a $\varphi$ as above, satisfying (1) and (2) is also said to be continuous. A function $\varphi : <\omega \omega \to <\omega \omega$ is **tame** if it is continuous and satisfies

3. $\forall s < \omega \omega \forall n \left( \varphi(s) \subseteq \varphi(s' n) \Rightarrow \operatorname{lh}(\varphi(s' n)) = \operatorname{lh}(\varphi(s)) + 1 \right)$.

Every continuous $f : \mathbb{R} \to \mathbb{R}$ is induced by a tame $\varphi$. If condition (2) is strengthened to

3'. $\operatorname{lh}(\varphi(s)) = \operatorname{lh}(s)$,
and \( \varphi \) satisfies (1)+(2') then it is said to be \textbf{Lipschitz}, since
\[
x \upharpoonright n = y \upharpoonright n \Rightarrow f_\varphi(x) \upharpoonright n = f_\varphi(y) \upharpoonright n,
\]
i.e., \( f_\varphi \) is Lipschitz with constant \( \geq 1 \) with respect to the usual metric on \( \mathbb{R} \)
\[
d(x, y) = \begin{cases} 2^{-n} & \text{if } x \upharpoonright n = y \upharpoonright n \& x(n) = y(n), \\ 0 & \text{if } x = y. \end{cases}
\]
Clearly every Lipschitz function \( f : \mathbb{R} \rightarrow \mathbb{R} \) of constant \( \geq 1 \) is of the form \( f = f_\varphi \) with \( \varphi \) Lipschitz. A function \( \varphi : <\omega, \omega \rightarrow <\omega, \omega \) is a \textbf{contraction} if it is monotone and satisfies
\[
(4) \quad \text{lh}(\varphi(s)) = \text{lh}(s) + 1.
\]
Thus a contraction is continuous and the induced map \( f_\varphi \) has Lipschitz constant \( \leq 1/2 \).

**Definition 21.1.** For \( A, B \subseteq \mathbb{R} \), say that \( A \) is \textbf{Wadge reducible} to \( B \), in symbols \( A \leq_W B \) just in case \( A = f^{-1} \downarrow B \) for some continuous \( f : \mathbb{R} \rightarrow \mathbb{R} \). If \( f^{-1} \downarrow B \) and \( f \) is Lipschitz, then \( A \) is said to be \textbf{Lipschitz reducible} to \( B \), in symbols \( A \leq_L B \). \( A <_W B \) stands for \( A \leq_W B \) and \( \not\leq_W A \), and similarly for \( A <_L B \).

It is clear that \( \leq_W \) and \( \leq_L \) are reflexive and transitive, so the relations
\[
A \equiv_W B \iff A \leq_W B \& B \leq_W A \\
A \equiv_L B \iff A \leq_L B \& B \leq_L A
\]
are equivalence relations, and the equivalence classes are called, respectively, \textbf{Wadge degrees} and \textbf{Lipschitz degrees}. \([A]_W\) and \([A]_L\) are, respectively, the Wadge degree and the Lipschitz degree of \( A \). It is immediate to check that the relations \( \leq_W \) and \( \leq_L \) can be defined on the set of all Wadge degrees, and similarly for \( \leq_L \) and \( <_L \). The \textbf{dual} of the Wadge/Lipschitz degree of a set \( A \) is the degree of \( \neg A \); a degree is \textbf{self-dual} if it coincides with its dual; otherwise it is called \textbf{non-self-dual}.

In order to prove non-trivial results about the ordering \( A \leq_W B \) we need two games invented by Wadge. The first one, called the \textbf{Lipschitz game} for \( (A, B) \), in symbols \( G_L(A, B) \), is the game:

\[
\begin{array}{c|cc}
I & a_0 & a_1 \\
II & b_0 & b_1 & \ldots
\end{array}
\]
where I and II alternate and play \( a_i, b_i \in \omega \). Let \( a = \langle a_i \mid i \in \omega \rangle \) and \( b = \langle b_i \mid i \in \omega \rangle \). Then II wins iff \( \langle a \in A \iff b \in B \rangle \).

The second game \( G_W(A, B) \), which we will call the Wadge game, is similar to \( G_L(A, B) \), but II has the option of passing, with the proviso that if he does not play infinitely often then he loses:

\[
\begin{align*}
I & \quad a_0 \quad a_1 \quad \ldots \quad a_n \quad a_{n+1} \quad a_{n+2} \quad \ldots \quad a_m \quad \ldots \ldots \\
II & \quad p \quad p \ldots p \quad b_0 \quad p \quad p \ldots p \quad b_1
\end{align*}
\]

As before II wins iff \( a \in A \iff b \in B \).

For \( s < \omega \), let \( A_{[s]} = \{ x \in \mathbb{R} \mid s \upharpoonright x \in A \} \).

(When \( s = \langle n \rangle \) we will write \( A_{[n]} \) rather than \( A_{\langle(n)\rangle} \).)

**Exercise 21.2.** Show that

(i) II wins \( G_W(A, B) \) iff \( A \leq_W B \);
(ii) II wins \( G_L(A, B) \) iff \( A \leq_L B \);
(iii) I wins \( G_L(A, B) \) iff \( \exists n \left( \neg B \leq_L A_{[n]} \right) \);
(iv) if I wins \( G_W(A, B) \) then I wins \( G_L(A, B) \);
(v) if II wins \( G_L(A, B) \) then I wins \( G_W(A, B) \).

Since \( A_{[n]} \leq_L A \) for all \( n \in \omega \), (iii) implies that I wins \( G_L(A, B) \) \( \Rightarrow \neg B \leq_L A \), hence we have shown:

**Theorem 21.3** (Wadge’s Lemma). If \( G_L(A, B) \) is determined, then \( A \leq_L B \) or \( \neg B \leq_L A \). If \( G_W(A, B) \) is determined, then \( A \leq_W B \) or \( \neg B \leq_W A \).

The **Semi-Linear-Ordering Principle**, \( \text{SLO}^W \) for short, is the statement

\[ \forall A, B \subseteq \mathbb{R} \left( A \leq_W B \vee \neg B \leq_W A \right). \]

\( \text{SLO}^L \) is the analogous statement where the maps are taken to be Lipschitz rather than continuous. Therefore

\[ \text{AD} \Rightarrow \text{SLO}^L \Rightarrow \text{SLO}^W. \]

It is open whether these implications can be reversed. It is known that \( \text{SLO}^W \) implies certain simple consequences of \( \text{AD} \), for example \( \text{AC}_\omega(\mathbb{R}) \) and the perfect set property—see [?, 11.16, 11.10]. \( \text{SLO}^W \) implies that if \( [A]_W \) and \( [B]_W \) are not \( \leq_W \)-comparable, then \( A \equiv_W \neg B \). Hence the antichains
in the $\leq_W$ order on degrees have size at most 2, and every antichain of size 2 is of the form \{[$A]_W, [\neg A]_W\} with $A \not\equiv_W \neg A$. We will call such a pair \{[$A]_W, [\neg A]_W\} a non-self-dual pair. If $A_n \subseteq \mathbb{R}$ let
\[
\bigoplus_n A_n = \bigcup_n \langle n \rangle \uparrow A_n,
\]
and let $A \oplus \neg A = \bigoplus_n A_n$ with $A_{2n} = A$ and $A_{2n+1} = \neg A$. Clearly

- $A \leq_L A \oplus \neg A$ and $\neg A \leq_L A \oplus \neg A$,
- $\neg (A \oplus \neg A) = \neg A \oplus A \leq_L A \oplus \neg A$,
- $A, \neg A \leq_W C \Rightarrow A \oplus \neg A \leq_W C$.

Therefore $[A \oplus \neg A]_W$ is self-dual, and it is the least degree above $[A]_W$ and $[\neg A]_W$. Here is a characterization of successor self-dual degrees.

**Exercise 21.4.** Assume SLO$^W$. A successor degree $[B]_W$ is self-dual iff there is a clopen partition $C \cup D = \mathbb{R}$, $C \cap D = \emptyset$, such that $B \cap C, B \cap D <_W B$.

If every degree is coalesced with its dual, then SLO$^W$ implies that $\leq_W$ is a linear order. In fact Martin, extending a partial result of Leonard Monk, showed that $\leq_W$ is well-founded, assuming AD. The technique used in this proof (the so-called Martin–Monk method) is a standard tool in the theory of Wadge degrees.

**Theorem 21.5** (Martin). AD implies there is no infinite $<_W$-descending sequence

$$\ldots A_2 <_W A_1 <_W A_0.$$  

**Proof.** Suppose otherwise and let $\langle A_n \mid n \in \omega \rangle$ be the offending sequence. By SLO$^W$, $\neg A_{n+1} <_W A_n$, hence $A_n \not\equiv_W \neg A_{n+1}$, so II does not win $G_W(A_n, \neg A_{n+1})$. Since II does not win $G_W(A_n, A_{n+1})$ as well, then, by AD, fix winning strategies $\sigma_n^0$ and $\sigma_n^1$ for I in $G_W(A_n, A_{n+1})$ and $G_W(A_n, \neg A_{n+1})$, respectively. For any fixed $z \in \omega^2$ we pit the strategies $\sigma_n^{(n)}$ against each other, with
\( \sigma_{n+1} \) playing as player I in \( G_W(A_{n+1}, A_{n+2}) \) and as player II in \( G_W(A_n, A_{n+1}) \),

\[
\begin{align*}
\sigma_0^{(0)} & \quad a_0^{(0)} & \quad a_0^{(1)} & \quad a_0^{(2)} & \quad a_0^{(3)} & \quad \ldots & \quad a_0^{(n)} \\
\sigma_1^{(1)} & \quad a_1^{(0)} & \quad a_1^{(1)} & \quad a_1^{(2)} & \quad \ldots & \quad a_1^{(n)} \\
\sigma_2^{(1)} & \quad a_2^{(0)} & \quad a_2^{(1)} & \quad \ldots & \quad a_2^{(n)} \\
\sigma_3^{(1)} & \quad a_3^{(0)} & \quad \ldots & \quad a_3^{(n)} \\
\vdots & & & & & & \\
\end{align*}
\]

\( a_n \) is the real constructed in the \( n \)-th line of the diagram above. If \( z, w \) differ exactly in one coordinate \( n_0 \), that is, \( \forall n \neq n_0 (z(n) = w(n)) \) and \( z(n_0) \neq w(n_0) \), then the two diagrams obtained using \( z \) and \( w \) agree below the \( n_0 \)-th line, hence \( n > n_0 \Rightarrow a_n = a_n^w \). On the other hand

\[ a_{n_0} \in A_{n_0} \iff a_{n_0}^w \notin A_{n_0} \]

and by induction on \( n_0 \) it is easy to check that \( a_0^z \in A_0 \iff a_0^w \notin A_0 \). We need the following

**Definition 21.6.** A set \( Z \subseteq \omega^2 \) is a **flip set** if for all \( z, w \in \omega^2 \) if \( \{ n \in \omega \mid z(n) \neq w(n) \} \) is finite, then

\[
(z \in Z \iff w \in Z) \iff |\{ n \in \omega \mid z(n) \neq w(n) \}| \text{ is even.}
\]

A flip set cannot be measurable nor can have the Baire property (see [5, 5.23, 6.41]). In particular, \( \text{AD} \) implies that flip sets do not exist. Going back to Martin’s proof, it is easy to check that

\[ Z = \{ z \in \omega^2 \mid a_0^z \in A_0 \} \]

is a flip set, contradicting \( \text{AD} \).

Note that \( \text{AD} \) was used twice, once to get winning strategies for I in the Wadge games, and the second time to obtain a contradiction from the existence of a flip set. We will now re-prove Theorem 21.5 under the weaker assumption \( \text{SLO}^W \oplus \text{BP} \), where \( \text{BP} \) stands for “All sets have the Baire property”. \( \text{BP} \) is needed to obtain the contradiction from the existence of a flip set. In order to avoid the first use of determinacy in Martin’s proof we need to prove a preliminary lemma.
Lemma 21.7. (a) If $A \leq_W B$ then there is $A' \equiv_W A$ such that $A' \leq_L A$ and $A' \leq_L B$.

(b) Assume SLO$^W$ and $A <_W B$. Then there is $A' \equiv_W A$ such that $A' \leq_L B$ and $A' \leq_L \neg B$ via a contraction, i.e., $\mathbf{I}$ wins $G_L(\neg B, A')$ and $G_L(B, A')$.

Proof. (a) Let $\varphi : \omega \omega \rightarrow \omega \omega$ be a tame map whose induced function witnesses $A \leq_W B$. The set

$$S = \{s \in \omega \omega \mid \forall t \in s \ \text{lh}(\varphi(t)) < \text{lh}(\varphi(s))\}$$

is the set of all sequences where $\varphi$ properly extends its previous values. The idea is to replace $S$ with $\omega \omega$ via the standard enumeration $\langle s_i \mid i \in \omega \rangle$ of $\omega \omega$, so that the resulting map will be Lipschitz. Let $\gamma : \omega \omega \rightarrow \omega \omega$ be the continuous map

$$\gamma(t) = \begin{cases} s_{k_0+1} \cdots \cdots s_{k_n+1} & \text{if } t = \langle k_0, \ldots, k_n \rangle, \\ \emptyset & \text{if } t = \emptyset, \end{cases}$$

and let $G = f_\gamma : \mathbb{R} \rightarrow \mathbb{R}$ be its induced function. Let

$$C = \{z \in \mathbb{R} \mid \forall n \ \text{lh}(\varphi(\gamma(z \upharpoonright n))) = n\} = \{z \in \mathbb{R} \mid \forall n \ \gamma(z \upharpoonright n) \in S\}.$$

Given a real $x \in \mathbb{R}$ we can construct, continuously in $x$, a real $F(x) = z \in C$ such that $x = s_{z(0)+1} \cdots \cdots s_{z(1)+1} \cdots$: if $z \upharpoonright n$ has been defined, let $s$ be of least length (which exists since $\varphi$ is continuous) such that $s_{z(0)+1} \cdots \cdots s_{z(n-1)+1} \cdots s \subset x$ and $\text{lh}(\varphi(s_{z(0)+1} \cdots \cdots s_{z(n-1)+1} \cdots s)) = n + 1$, and set $z(n) = s$ such that $s_{k_n} = s$. Clearly $C$ is closed and $G \upharpoonright C : C \rightarrow \mathbb{R}$, $z \mapsto s_{z(0)+1} \cdots \cdots s_{z(1)+1} \cdots$ is Lipschitz. It is easy to check (see e.g. [?, 2.23]) that there is a Lipschitz $\pi : \mathbb{R} \rightarrow C$ such that $\pi \upharpoonright C$ is the identity. Let

$$A' = (G \circ \pi)^{-1}A.'$$

Then $A' \leq_L A$, and $A \leq_W A'$ since $A = F^{-1}A'$. We are only left to check that $A' \leq_L B$. Since the map $t \mapsto \varphi(\gamma(\pi(t)))$ is Lipschitz, its induced function witnesses $A' \leq_L B$.

(b) If $\forall n (A \not\leq_W B_{[n]})$ then, by SLO$^W$, $\forall n (B_{[n]} \leq_W A)$ hence $B \leq_W \neg A <_W \neg B$: a contradiction. Similarly $\forall m (A \not\in W B_{[m]})$ does not hold, so fix $n_0, m_0 \in \omega$ such that $A \leq_W B_{[n_0]}, \neg B_{[m_0]}$. Fix tame $\varphi$ and $\psi$ such that $f_{\varphi}^{-1}B_{[n_0]} = A$ and $f_{\psi}^{-1}B_{[m_0]} = A$. Define

$$C = \{z \in \mathbb{R} \mid \forall n \ \min\{\text{lh}(\varphi(z \upharpoonright n)), \text{lh}(\psi(z \upharpoonright n))\} = n\}.$$
Arguing as in part (a), $C$ is closed and fix $\pi : \mathbb{R} \to C$, $F : \mathbb{R} \to C$ and $A' = (G \circ \pi)^{-1} A$ as above. Then $A' \leq_L A$ via $G \circ \pi$ and $A \leq_W A'$ via $F$, and $t \mapsto \varphi(\gamma(\pi(t)))$ and $t \mapsto \psi(\gamma(\pi(t)))$ witness $A' \leq_L B_{[m]}$ and $A' \leq_L \neg B_{[m]}$, respectively. Therefore $A' \leq_L B, \neg B$ via contractions. 

**Theorem 21.8.** Assume $\text{BP} + \text{SLO}^W$. There is no sequence $\langle A_n \mid n \in \omega \rangle$ such that 

$$ \cdots <_W A_2 <_W A_1 <_W A_0. $$

**Proof.** Using 21.7(b), by induction on $n$ define $A'_n \in [A_n]_W$ such that $I$ wins $G_L(A'_n, A'_{n+1})$ and $G_L(\neg A'_n, A'_{n+1})$. We can now apply the Martin–Monk method of Theorem 21.5 and reach the desired contradiction:

Let $\sigma^0_n$ be $I$’s winning strategy in $G_L(A'_n, A'_{n+1})$, and $\sigma^1_n$ be $I$’s winning strategy in $G_L(\neg A'_n, A'_{n+1})$. For any $z \in \omega^2$ let $x_n = x^z_n$ be the real in the $n$-th row of the following diagram where $I$ uses $\sigma^z_n(n)$ on the $n$-th row against his opponent on the $(n+1)$-st row:

$$
\begin{array}{cccc}
\sigma^{(0)}_0 & x_0(0) & x_0(1) & \cdots \\
\sigma^{(1)}_1 & x_1(0) & x_1(1) & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
$$

Thus $x^z_n$ is the result of applying $\sigma^z_n(n)$ to $x^z_{n+1}$. Then $\{z \in \omega^2 \mid x^z_0 \in A'_0\}$ is a flip set, contradicting BP. 

**Exercise 21.9.** Assume $\text{BP} + \text{SLO}^W + \text{DC}(\mathbb{R})$. Show that $\leq_W$ is a well-founded pre-order on $\mathcal{P}(\mathbb{R})$.

Therefore $\text{BP} + \text{SLO}^W + \text{DC}(\mathbb{R})$ implies that the Wadge degrees are essentially well-ordered, each level consisting of a single self-dual degree or of a non-self-dual pair. Let $\|A\|_W$ be the rank of $A$ in the well-founded pre-order $<_W$. A degree $[A]_W$ is a successor/limit degree iff $\|A\|_W$ is a successor/limit ordinal.

In order to show that above a self-dual degree there is a non-self-dual pair we need the following tree introduced by Wadge:

**Definition 21.10.** For $A \subseteq \mathbb{R}$ let

$$ T(A) = \{ s \in \omega^\omega \mid A_{[s]} \equiv_W A \}. $$

**Lemma 21.11.** Assume $\text{AC}_\omega(\mathbb{R}) + \text{BP}$. If $A \leq_W \neg A$ then $T(A)$ is well-founded.
**Proof.** The proof uses the Martin–Monk method and an idea as in [VW78, 3.1]—see also [?, 11.31].

Towards a contradiction, let \( x \in \mathcal{T}(A) \) and \( A \leq_W \neg A \). Clearly for every \( n \in \omega \), \( \Pi \) wins \( G_W(A_{[x|n]}, \neg A_{[x|n]}) \) via some strategy \( \tau_n \), and \( I \) wins \( G_L(A_{[x|n]}, \neg A_{[x|n+1]}) \) via the strategy \( \sigma_n \) that plays \( x(n) \) in the first round and then copies \( \Pi \)'s moves. Let \( \tau \) be \( \Pi \)'s copying strategy, i.e., \( \tau(s) = \) the last element of \( s \). We are now going to apply the Martin–Monk method. The basic idea is to pit \( \tau \) against the \( \tau_n \)'s so that a flip set is constructed, contradicting \( \text{BP} \). The problem is that \( \tau \) and \( \tau_n \) are strategies for \( \Pi \), while we need strategies for \( I \) in order to start the filling-in procedure of the Martin–Monk diagram. This is where the \( \sigma_n \)'s come into the picture: an increasing sequence of natural numbers \( 0 = M_0 < M_1 < M_2 < \cdots \) is defined so that \( M_k + 1 < M_{k+1} \) and on the \( M_k \)-th row \( \Pi \) plays using either \( \tau \) or \( \tau_n \) for some \( n \), and on the \( m \)-th row with \( m \notin \{ M_k \mid k \in \omega \} \) \( I \) plays using \( \sigma_n \) for an appropriate \( n \). More precisely, letting \( \langle N_j \mid j \in \omega \rangle \) be the enumeration of \( \omega \setminus \{ M_k \mid k \in \omega \} \)

- on the \( M_k \)-th row \( \Pi \) is playing \( G_W(A_{[x|i]}, \neg A_{[x|i]}) \) using \( \tau_i \) or \( \tau \), where \( N_i = M_k + 1 \);

- on the \( N_j \)-th row \( I \) is playing \( G_L(A_{[x|i]}, \neg A_{[x|i+1]}) \) using \( \sigma_j \).

E.g., if the first few \( M_k \)'s are \( \langle 0, 3, 5, \ldots \rangle \), then the table below summarizes the games and strategies used.

<table>
<thead>
<tr>
<th>Row</th>
<th>Game</th>
<th>Strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_0 = 0 )</td>
<td>( G_W(A, \neg A) )</td>
<td>( \tau ) or ( \tau_0 )</td>
</tr>
<tr>
<td>( N_0 = 1 )</td>
<td>( G_L(A, \neg A_{[x</td>
<td>1]}) )</td>
</tr>
<tr>
<td>( N_1 = 2 )</td>
<td>( G_L(A_{[x</td>
<td>1]}, \neg A_{[x</td>
</tr>
<tr>
<td>( M_1 = 3 )</td>
<td>( G_W(A_{[x</td>
<td>2]}, \neg A_{[x</td>
</tr>
<tr>
<td>( N_2 = 4 )</td>
<td>( G_L(A_{[x</td>
<td>2]}, \neg A_{[x</td>
</tr>
<tr>
<td>( M_2 = 5 )</td>
<td>( G_W(A_{[x</td>
<td>3]}, \neg A_{[x</td>
</tr>
<tr>
<td>( N_3 = 6 )</td>
<td>( G_L(A_{[x</td>
<td>3]}, \neg A_{[x</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
</tbody>
</table>
Lemma 21.12. Assume $\text{SLO}^W$. If $A$ is Wadge non-self-dual then $T(A)$ is pruned and therefore ill-founded.
Proof. Suppose \( s \in \text{tn}(T(A)) \). Then

\[
A_{[s^-(m)]} \leq_W A_{[s]} = \bigoplus_n A_{[s^-(n)]}
\]

and clearly \( A_{[s]} \) is the \( \leq_W \)-least upper bound of the \( A_{[s^-(m)]} \)'s. Therefore \( A \equiv_W A_{[s]} \) is Wadge self-dual.

\[ \Box \]

**Corollary 21.13.** Assume \( \text{SLO}^W \). \( A \) is Wadge self-dual iff there is a partition of \( \mathbb{R} \) into clopen sets \( \mathbb{R} = \bigcup_n D_n \), such that \( \forall n \ (A \cap D_n \leq_W A) \).

**Proof.** If \( T(A) \) is well-founded then

\[
A = \bigoplus_{s \in \partial T(A)} A_{[s]} = \bigcup_{s \in \partial T(A)} N_s \cap A.
\]

Since \( \langle N_s \mid s \in \partial T(A) \rangle \) is a clopen partition of \( \mathbb{R} \) we are done.

Conversely, suppose \( \langle D_n \mid n \in \omega \rangle \) be a partition of \( \mathbb{R} \) into clopen sets such that \( A \cap D_n \leq_W A \). By \( \text{SLO}^W \) \( A \cap D_n \leq_W \neg A \) via some \( f_n \), hence \( A \leq_W \neg A \) via \( \bigcup_n f_n \upharpoonright D_n \).

\[ \Box \]

We can now show that immediately above a self-dual degree there is a non-self-dual pair. Suppose \( [A]_W \leq_W [B]_W \) are both self-dual and let \( s \in \text{tn}(T(B)) \), which exists by the well-foundedness of \( T(B) \). If \( \forall n \ (B_{[s^-(n)]} \leq_W A) \) then \( B_{[s]} \leq_W A \), which is absurd, so \( \exists n_0 \ (A \leq_W B_{[s^-(n_0)]} \leq_W B_{[s]}) \). Thus \( [B]_W \) cannot be the immediate successor of \( [A]_W \), and therefore immediately above a self-dual degree there is a non-self-dual pair.

We have already seen that if \( [A]_W \) is the least upper bound of an increasing \( \omega \) sequence of degrees then it is self-dual. Conversely, if \( [A]_W \) is limit and self-dual, then \( A = \bigoplus_{s \in \partial T(A)} A_{[s]} \), and \( [A]_W \) is the \( \leq_W \)-least upper bound of \( \{A_{[s]} \mid s \in \partial T(A)\} \). Since \( T(A) \) is countable, \( \text{cof}([A]_W) = \omega \).

Therefore \( \text{BP} + \text{SLO}^W + \text{DC}(\mathbb{R}) \) yield the usual picture for the Wadge hierarchy: at levels of countable cofinality there is a single self-dual degree, at levels of uncountable cofinality there is a non-self-dual pair.

Exercise 21.14. If \( s^A \leq_L B \) for some \( s \in <\omega \setminus \{0\} \), then \( I \) wins \( G_L(\neg B, A) \), i.e., \( A \) is reducible to \( B \) via a contraction.
Exercise 21.15. Show that if $\text{Res}_2(\neg B) \neq \emptyset$ then $\Pi$ wins $G_W(A, B)$ for any $A \in \text{Diff}(2; 0_0)$.  

Conclude that $\Delta(\text{Diff}(2; 0_0)) \setminus (0_0 \cup \sigma')$ is the $\leq_W$ successor of $(0_0 \cup \sigma')$ and the $\leq_W$ predecessor of $(\text{Diff}(2; 0_0) \setminus \text{Diff}(2; 0_0))$. 


(i) Show that if $[A]_W$ is self-dual then $[A \oplus \neg A]_W$ is the immediate successor of $([A]_W, [-A]_W)$. 

(ii) Show that $[A_0 \oplus (A_1 \oplus \cdots (A_n \oplus \cdots)]_W$ is the least upper bound of $[A_0]_W, \ldots, [A_n]_W$. 


(a) If $T(A)$ is well-founded then $A \equiv_W \neg A$. 

(b) If $[A]_W$ is self-dual and limit then it is a join-degree, i.e., it is the least upper bound of an $\omega$-sequence $[B_0]_W <_W [B_1]_W <_W \ldots$ 

Proof. (a) Replacing $A$ with $A[s]$ with $s \in \text{tn}T(A)$, we may assume that $T(A) = \{\emptyset\}$. Then $\forall n \ (A[n] <_W A)$ hence by SLO$^W$, $\forall n \ (A[n] \leq_W \neg A)$ and therefore $A = \bigoplus_n A[n] \leq_W \neg A$. 

(b) As before we may assume $T(A) = \{\emptyset\}$ so that $\forall n \ (A[n] <_W A)$ and we can take $B_n = A_0 \oplus \cdots \oplus A_n$. $B_n <_W A$, since $[A]_W$ is limit, and if the $[B_n]_W$’s were bounded by $[C]_W <_W [A]_W$, then $A = \bigoplus_n A_n \leq_W \bigoplus_n B_n \leq_W C$, a contradiction. Lastly, by thinning-down the sequence of the $B_n$’s we may assume that they are strictly increasing. 

Exercise 21.18. For $A \subseteq \mathbb{R}$ let $\langle \text{Res}_\alpha(A) \mid \alpha < \omega_1 \rangle$ be the sequence of sets defined by: 

$$
\text{Res}_0(A) = A \\
\text{Res}_\alpha(A) = A \cap \bigcap_{\beta < \alpha} \text{Cl}(\text{Res}_\beta(\neg A)).
$$

(i) Show that 

$$
\alpha < \beta \Rightarrow \text{Res}_\alpha(A) \supseteq \text{Res}_\beta(A), \\
\text{Res}_{\alpha+1}(A) = X \cap \text{Cl}(\text{Res}_\alpha(\neg A)), \\
\text{Res}_{\lambda}(A) = \bigcap_{\alpha < \lambda} \text{Res}_\alpha(A) \quad (\lambda \text{ limit}), \\
\text{Res}_\alpha(A) = \emptyset \Rightarrow \text{Res}_{\alpha+1}(\neg A) = \emptyset.
$$
Thus, for example,
\[
\text{Res}_1(A) = A \cap \text{Cl} (\neg A)
\]
\[
= \{a \in A \mid \exists \langle b_n \mid n \in \omega \rangle \in \omega^n (\neg A) \ \lim_n b_n = a\}
\]
is the set of points in \(A\) which are limit of points in \(\neg A\), and \(\text{Res}_2(A)\) is the set of points in \(A\) which are limit of points in \(\neg A\) which are limit of points in \(A\). \(\text{Res}_2(A)\) is sometimes called, after Hausdorff, the first residue of \(A\).

(ii) Show that
\[
A \in \Sigma^0_1 \iff \text{Res}_1(A) = \emptyset
\]
and that if \(\text{Res}_2(A) = \emptyset\) then \(\neg A \cap \text{Cl}(A)\) is closed, and therefore \(A \in \text{Diff}(2; \Pi^0_1)\). Conversely show that for \(U, V\) open sets \(\text{Res}(U \setminus V) = \emptyset\) and therefore
\[
A \in \text{Diff}(2; \Pi^0_1) \iff \text{Res}_2(A) = \emptyset .
\]
Conclude that if \(A \in \Delta(\text{Diff}(2; \Pi^0_1))\) then
\[
\forall x \in \mathbb{R} \ \exists k \in \omega \ (A_{x[k]} \text{ is open or closed}) .
\]
(The converse implication will be proved in 23.7.)

(iii) Show that if \(\text{Res}_{2\alpha}(A) = \emptyset\) then
\[
A = \bigcup_{\beta < \alpha} \text{Cl}(\text{Res}_{2\beta}(A)) \setminus \text{Cl}(\text{Res}_{2\beta+1}(\neg A)) \in \text{Diff}(2\alpha; \Pi^0_1),
\]
and that if \(\text{Res}_{2\alpha+1}(A) = \emptyset\) then
\[
\neg A = \bigcup_{\beta \leq \alpha} \text{Cl}(\text{Res}_{2\beta}(\neg A)) \setminus \text{Cl}(\text{Res}_{2\beta+1}(A)) \in \text{Diff}(2\alpha + 1; \Pi^0_1),
\]
Therefore
\[
\{ A \subseteq \mathbb{R} \mid \exists \alpha < \omega_1 \ \text{Res}_{\alpha}(A) = \emptyset \} \subseteq \Delta^0_2 .
\]
(The other inclusion will be proved in 23.3.)

(iv) Show that \(x \in \text{Res}_n(A)\) implies \(\langle x(n+i) \mid i \in \omega \rangle \in \text{Res}_n(A_{x[n]} \neq \emptyset).

22 Operations on the Degrees

This and the next section are devoted to the detailed analysis of the Wadge hierarchy. They safely can (and probably should) be skipped on first reading. We will develop the theory under \(\text{BP} + \text{AD}^L\). The assumption that all sets have the property of Baire (or even that there are no flip sets) comes from the Martin–Monk theorem ?? and the Steel–Van Wesep theorem ??.
22. OPERATIONS ON THE DEGREES

22.A The Next Non-Self-Dual Pair

We have seen how the maps

\[ A \mapsto A \oplus \neg A \]

\[ \langle A_n \mid n \in \omega \rangle \mapsto \bigoplus_n A_n \]

allow us to construct the immediate successor of a non-self-dual degree, and the supremum of an \( \omega \)-sequence of degrees. There is a uniform way to construct from a self-dual \( A \) a pair of non-self-dual sets immediately above \( A \).

**Definition 22.1.** Let \( T^\forall \) be the tree \( \{0^{(n)} \mid n \in \omega \} \) given by the initial segments of \( 0 \). For \( A \subseteq \mathbb{R} \) let

\[ A^\forall = S(A; T^\forall) \]

\[ = \bigcup_{n,m} 0^{(n)} \setminus (m+1)^\forall A \]

and

\[ A^\circ = S(A; T^\forall) \cup [T^\forall] \]

\[ = A^\forall \cup \{\emptyset\} \]

**Exercise 22.2.** Show that:

(i) \( A \subseteq_w A^\forall, A^\circ \),

(ii) if \( A \subseteq_w B \) (or \( A \leq_L B \)) then \( A^\forall \leq_w B^\forall \) and \( A^\circ \leq_w B^\circ \) (respectively: \( A^\forall \leq_L B^\forall \) and \( A^\circ \leq_L B^\circ \)),

(iii) \( A^\forall \equiv_w A^\forall, A^\circ \equiv_w A^\circ \).

By (ii) we can define the operations \( \forall \) and \( \circ \) on the Wadge degrees by letting \( [A]^\forall_w = [A^\forall]^w \) and \( [A]^\circ_w = [A^\circ]^w \), and similarly for the Lipschitz degrees.

**Proposition 22.3.** Assume \( \text{BP} + \text{AD}^L \). Suppose \( A \) is self-dual. Then \( ([A]^\forall_w, [A]^\circ_w) \) is the least non-self-dual pair above \([A]^w\).

**Proof.** We first show that \( A \leq_w A^\forall \) and that \( A^\circ \nleq_w A^\forall \) and \( A^\forall \nleq_w A^\circ \). By 22.2(i) \( \Pi \) has a winning strategy in \( G_w(A, A^\forall) \) so it is enough to show that \( \Pi \) does not have a winning strategy in \( G_w(A^\forall, A) \). Towards a contradiction
let \( \tau \) be a winning strategy for \( \text{II} \) and let \( n \) be least such that \( \tau \)'s response to \( 0^{(n)} \) is in \( \partial T(A) \), i.e.,

\[
(0^{(n)} \ast \tau)_{\text{II}} = s \in \partial T(A).
\]

Thus

\[
A^\uparrow = A^\uparrow_{[0^{(n)}]} \leq_w A_{[s]} <_w A,
\]
a contradiction. Therefore \( A <_w A^\uparrow \).

The proof that no strategy \( \tau \) for \( \text{II} \) in \( G^w(A^\uparrow, A^\circ) \) can be winning is analogous. If \( 0 \ast \tau = 0 \) then \( \text{II} \) would lose since \( 0 \in A^\circ \setminus A^\uparrow \), so let \( n \) be least such that \( \tau(0^{(n)}) = m \neq 0 \). Then \( \text{I}'s \) position is \( 0^{(n)} \) while \( \text{II} \)'s position is \( 0^{(k)} \sim \langle m \rangle \) with \( k \leq n \). Then

\[
A^\uparrow = A^\uparrow_{[0^{(n)}]} \leq_w A^\circ_{[0^{(k)} \sim \langle m \rangle]} = A
\]

which is impossible.

Similarly \( A^\circ \nleq_w A^\uparrow \).

Therefore \( ([A^\uparrow]_W, [A^\circ]_W) \) is a non-self-dual pair above \( [A]_W \). We now show it is the least such.

Let \( B \) be non-self-dual so that by ?? \( T(B) \) is pruned, and such that \( A <_w B \).

Claim 22.3.1.

\[
[T(B)] \cap \neg B \neq \emptyset \quad \Rightarrow \quad A^\uparrow \leq_L B
\]

\[
[T(B)] \cap B \neq \emptyset \quad \Rightarrow \quad A^\circ \leq_L B
\]

Proof. Suppose \( b \in [T(B)] \setminus B \). Then \( \text{II} \) wins \( G_L(A^\uparrow, B) \) as follows:

As long as \( \text{I} \) plays \( 0^{(n)} \) then \( \text{II} \) plays \( b \upharpoonright n \), so that if \( \tilde{0}, b \) are the final outcome then \( \text{II} \) has won. If at some stage \( \text{I} \) first plays \( m \neq 0 \) then \( \text{II} \) still follows \( b \) so that after this round \( \text{I} \) and \( \text{II} \) have reached positions \( 0^{(n-1)} \sim \langle m \rangle \) and \( b \upharpoonright n \), respectively. Then

\[
A^\uparrow_{[0^{(n-1)} \sim \langle m \rangle]} = A \quad \text{and} \quad B_{[b \upharpoonright n]} \equiv_L B, \quad \text{since} \quad b \upharpoonright n \in T(B) \quad \text{and by ??(c), so} \quad \text{II} \quad \text{can now use his winning strategy in} \quad G_L(A, B).
\]

The proof that \( \text{II} \) wins \( G_L(A^\circ, B_{[b \upharpoonright n]}) \) is analogous.

This proves that \( ([A^\uparrow]_W, [A^\circ]_W) \) is the immediate successor of \( [A]_W \).

We can now give a characterization of the limit non-self-dual sets in terms of \( T \).
Proposition 22.4. Assume $\text{BP} + \text{AD}^L$. Let $A$ be non-self-dual. Then

$$[A]_W \text{ is limit } \iff [T(A)] \cap A \neq \emptyset \& [T(A)] \cap \neg A \neq \emptyset.$$ 

Proof. $(\iff)$ It is clear that $A \neq \emptyset, \mathbb{R}$, so it is enough to show that $[A]_W$ is not a successor. Let $C <_W A$ be self-dual. Then by Claim 22.3.1 $C^\circ, C^\circ \leq_W A$ and since $C^\circ \neq W C^\circ$, then $C^\circ, C^\circ <_W A$.

$(\implies)$ Suppose instead $[A]_W$ is limit and, say, that $[T(A)] \subseteq A$. Fix an enumeration $\langle A_n \mid n < \omega \rangle$ of $\{A_{s[n]} \mid s \notin T(A)\}$. Pick $A_n <_W B_n <_W A$ such that the $B_n$’s are self-dual. This can be done as follows: if $A_n$ is non-self-dual, let $B_n = A_n \oplus \neg A_n$, if $A_n$ is self-dual let $B_n = A_n^\circ \oplus A_n^\circ$. Then $B = \bigoplus_n B_n \leq_W A$, $\neg A$ is self-dual by ??(iv), and therefore $B <_W A$. Let $C = B^\circ$ so that $C$ is non-self-dual, $C <_W A$ and $[T(C)] \subseteq C$. (Had we assumed $[T(A)] \cap A = \emptyset$ we would let $C$ be $B^\circ$.) Then II wins $G_L(A, C)$ as follows:

As long as I’s position is in $T(A)$ then II enumerates some fixed $c \in [T(C)]$. (Although irrelevant for the proof, notice that $c$ must be 0.) If at some round $n$ I reaches a position $s \in \partial T(A)$ then II plays $c(n)$: at this point the two positions of the players are $s$ and $c \upharpoonright n+1$ and $A_{[s]} <_W A$ and $C_{[c \upharpoonright n+1]} =_W C$. Since $A_{[s]} = A_k$ for some $k$, and $A_k <_W B_k \leq_W B <_W C$, then $A_{[s]} <_W C$, so II can now use his winning strategy in $G_L(A_{[s]}, C)$.

Therefore $A \leq_W C$, hence $A$ cannot be limit: a contradiction. \hfill \Box

Exercise 22.5. Assume $\text{BP} + \text{AD}^L$.

(i) Show that if $A$ is successor and non-self-dual then $A \equiv_W A^\circ$ or $A \equiv_W A^\circ$ depending on whether $[T(A)] \subseteq \neg A$ or $[T(A)] \subseteq A$. Conclude that for $B$ self-dual $B^\circ \equiv_W B^\circ$ and $B^\circ \equiv_W B^\circ$.

(ii) Show that for $A$ limit and non-self-dual then $A \equiv_W A^\circ \equiv_W A^\circ$.

The following diagram summarizes the situation when $[B]_W$ is self-dual.
and \( \text{cof}(\|A\|_W) > \omega \):

\[
\begin{array}{c}
\bigcirc \quad \bigcirc \\
[B^\omega]_W \\
\bigcirc \\
[B]_W \\
\bigcirc \\
[B^\omega \oplus B^\omega]_W \\
\bigcirc \\
[-A]_W \\
\end{array}
\]

\[
[\cdots] \\
[\cdots]
\]

22.B Addition of Degrees

Using the operations \( A \mapsto A^\omega \), \( A^\omega \) and \( \langle A_n \mid n \in \omega \rangle \mapsto \bigoplus_n A_n \) it is easy to construct the next \( \omega \) degrees above any \([A]_W\). We will now define the \textbf{sum of two sets} \( A + B \) in such a way that \( \|A + B\|_W = \|A\|_W + 1 + \|B\|_W \), whenever \( A \) is self-dual.

\textbf{Definition 22.6.} (i) For \( x \in \omega \omega \) or \( x \in \omega \omega \) let

\[ x + 1 = \langle x(n) + 1 \mid n \in \text{lh}(x) \rangle \]

and for \( X \subseteq \mathbb{R} \) or \( X \subseteq \omega \omega \) let

\[ X^+ = \{ x + 1 \mid x \in X \} . \]

(ii) For \( A, B \subseteq \mathbb{R} \) let

\[ A + B = \{ (s + 1)^\omega (0)^\omega x \mid s \in \omega \omega \& x \in A \} \cup B^+ . \]

Therefore \( B^+ = \emptyset + B \).

\textbf{Exercise 22.7.} Show that \( A_0 \leq_L A_1 \& B_0 \leq_L B_1 \Rightarrow A_0 + B_0 \leq_L A_1 + B_1 \), and similarly for \( \leq_W \).

Therefore we can define the \textbf{addition of two Lipschitz/Wadge degrees} by

\[ [A] + [B] = [A + B] . \]

Note that the sum of two degrees is far from being commutative, that is \( A + B \not\equiv_W B + A \). On the other hand part (v) below shows it is associative.
Exercise 22.8. (i) Show that $A^\triangledown \equiv L A + \emptyset$ and $A^\circ \equiv L A + \mathbb{R}$. Conclude that (assuming $\text{BP} + \text{AD}^L$) if $A$ is self-dual and $B$ is arbitrary, then $A \ll W A + B$.

(ii) Show that

$$(A + B)_{[s+1]} = A + B_{[s]}$$

and

$$(A + B)_{[(s+1)\upharpoonright(0)]} = A.$$ 

(iii) Show that $\neg(A + B) = \neg A + \neg B$. In particular, for $A$ self-dual, $\neg(A + B) \equiv_w A + \neg B$.

(iv) Show that $\bigoplus_n (A + B_n) \equiv_w A + (\bigoplus_n B_n)$.

(v) Show that $(A + B) + C \equiv_w A + (B + C)$.

(vi) Show that $\emptyset + \emptyset = \emptyset$ and $\mathbb{R} + \mathbb{R} = \mathbb{R}$.

(vii) Let $C$ be closed and $U$ be open. Show that $\emptyset + C$ is closed and $\mathbb{R} + U$ is open. Show that if $C \neq \mathbb{R}$ and $U \neq \emptyset$ then $\emptyset + U$ is not open and $\mathbb{R} + C$ is not closed.

Lemma 22.9. Assume $\text{BP} + \text{AD}^L$. For $A, B, B' \subseteq \mathbb{R}$ and $A$ self-dual

$$A + B \leq_w A + B' \iff B \leq_w B'.$$

Proof. $(\iff)$ follows from 22.7 so it is enough to prove $(\implies)$.

Suppose $\tau$ is a winning strategy for II in $G_W(A + B, A + B')$. It is enough to show that II is not the first player to play 0, i.e.,

$$\neg \exists s, t \in ^{\omega \omega} \{\tau(s + 1) = (t + 1) \upharpoonright (0)\}$$

since then II wins $G_W(B, B')$ using $\tau'$ defined by

$$\tau'(s) = t \iff \tau(s + 1) = t + 1.$$ 

Towards a contradiction suppose $\tau(s + 1) = (t + 1) \upharpoonright (0)$, for some $s, t \in ^{\omega \omega}$.

Then

$$A \ll W A + B_{[s]} \quad \text{(by 22.8(i))}$$

$$= (A + B)_{[s+1]} \quad \text{(by 22.8(ii))}$$

$$\leq_w (A + B')_{[(t+1)\upharpoonright (0)]} \quad \text{(since $\tau$ is winning)}$$

$$= A \quad \text{(by 22.8(ii))}$$

a contradiction. \qed
The lemma fails if $A$ is non-self-dual—see 22.19. Therefore for $A$ self-dual the map $A \mapsto A + B$ is $\leq_W$-preserving. The next result shows that it induces an isomorphism between the whole collection of Wadge degrees and the ones above $[A]_W$. Again 22.19 shows that the assumption that $A$ is self-dual is crucial.

**Lemma 22.10.** Assume $\mathsf{BP} + \mathsf{AD}^L$. Suppose $A$ is self-dual and $A <_W B$. Then

$$\exists C \subseteq \mathbb{R} (A + C \equiv_W B).$$

**Proof.** Let

$$T = \{ s \in {}^\omega \omega \mid A <_W B_{[s]} \}.$$

$T$ is a tree since $\emptyset \in T$ by hypothesis and $t \subset s \Rightarrow B_{[s]} \leq_W B_{[t]}$. Also if $\forall n (B_{[s \upharpoonright n]} \leq_W A)$ then $B_{[s]} \leq_W A$, hence $T$ is pruned. Let $\varphi : {}^\omega \omega \rightarrow T$ be Lipschitz and such that $\varphi \upharpoonright T = \text{id}$, and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the map induced by $\varphi$ (see ??). Let $C = f^{-1} \{ B \}$ so that $B \cap [T] = C \cap [T]$. We claim that

$$B \equiv_W A + C.$$

$\square$ wins $G_W(B, A + C)$ as follows:

As long as $I$ plays $s \in T$ then $\square$ plays $s + 1$: if $I$'s play $x$ belongs to $[T]$ then

$$x \in B \iff x \in B \cap [T] = C \cap [T] \iff x + 1 \in A + C$$

so $\square$ wins. If at some point $I$ reaches a position $s \notin T$, by $\mathsf{SLO}^W$, then $B_{[s]} \leq_W A$ so $\square$ plays 0 and then uses his strategy in $G_W(B_{[s]}, A)$: in this case $I$'s play will be $s \upharpoonright x \notin [T]$ and $\square$'s play will be $(s + 1) \upharpoonright (0)^\infty y$ so that

$$s \upharpoonright x \in B \iff x \in B_{[s]} \iff y \in A \iff (s + 1) \upharpoonright (0)^\infty y \in A + C,$$

hence $\square$ wins.

And here is a winning strategy for $\square$ in $G_W(A + C, B)$:

As long as $I$ plays $s + 1$, then $\square$ plays $\varphi(s)$: if $x + 1, y$ are the reals constructed by $I$ and $\square$ then

$$x + 1 \in A + C \iff x \in C \iff y = f(x) \in B.$$

Suppose instead there is a first stage when $I$ plays 0. Then $\square$ plays an extension $t \in T$ of $\varphi(s)$ (which exists since $T$ is pruned) so that the positions of the two players are $(s + 1) \upharpoonright (0)$ and $t \in T$. Thus $A <_W B_{[t]}$, so that $\square$ has a winning strategy in $G_W(A, B_{[t]})$. Let $\square$ use this strategy.
The following diagram summarizes the behavior of the map $C \mapsto A + C$ with $A$ self-dual. (For the sake of definiteness $C$ is taken to be non-self-dual.)

Corollary 22.11. Assume BP + AD$^L$ + DC$(\mathbb{R})$ so that the Wadge rank is defined, and let $A$ be self-dual. Then

$$\|A + B\|_W = \|A\|_W + 1 + \|B\|_W.$$  

Exercise 22.12. Assume BP + AD$^L$ + DC$(\mathbb{R})$. Show that if $\|C\|_W \geq \omega$ or $\|C\|_W = 0$ then

$$[A]^\omega_W + [C]_W = [A]^\omega_W + [C]_W = [A]_W + [C]_W.$$  

Notice that DC$(\mathbb{R})$ was used only to state 22.12 in a more compact form, using the Wadge rank, but can easily be eliminated.

What about $B + C$ when $B$ is non-self-dual? Exercise 22.19(iii) shows that for $B$ of limit Wadge rank and small $C$, $B + C \equiv_W B$ can hold. By 22.12 if $B \equiv_W A^\omega$ or $B \equiv_W A^\circ$ and $\|C\|_W \geq \omega$ or $\|C\|_W = 0$ then $A + C \equiv_W B + C$. The assumption that $\|C\|_W \geq \omega$ cannot be removed: if $U \neq \mathbb{R}$ is open then $A + U \equiv_W B + U$, since otherwise by 22.8(v) and 22.9

$$A + U \equiv_W (A + \emptyset) + U \equiv_W A + (\emptyset + U) \Rightarrow U \equiv_W \emptyset + U$$

contradicting 22.8(viii). Yet we will see in 23.20 that if $C$ is non-self-dual and $0 < \|C\|_W < \omega$ and $A$ is self-dual, then either $A^\omega + C \equiv_W A + C$ or $A^\circ + C \equiv_W A + C$.

### 22.C Jumping $\omega_1$-many Steps

Iterating the $+$ operation one can construct

$$A \cdot n = \underbrace{A + \cdots + A}_{n}$$
for any \( n > 0 \) and therefore by 22.9 we can define \([A]_W \cdot n = [A \cdot n]_W\). We can also set
\[
A \cdot \omega = \bigoplus_n A \cdot (n+1) = \bigcup_n (n) \cap (A \cdot (n+1)),
\]
and \([A]_W \cdot \omega = [A \cdot \omega]_W\). The definition of multiplication of a degree by an ordinal can be easily extended to all countable ordinals by letting
\[
[A]_W \cdot 1 = [A]_W
\]
and for \( 1 \leq \alpha, \lambda < \omega_1 \)
\[
[A]_W \cdot (\alpha + 1) = [A]_W \cdot \alpha + [A]_W
\]
\[
[A]_W \cdot \lambda = \sup_n [A]_W \cdot \alpha_n
\]
where \( \lambda \) is limit and \( \alpha_n \to \lambda \). By ??(iv) the definition of \([A]_W \cdot \lambda \) does not depend on the choice of the \( \alpha_n \)'s.

By an argument as in Remark ?? it can be shown that there is no definable map \( (A, \alpha) \mapsto A \cdot \alpha \) such that \([A \cdot \alpha]_W = [A]_W \cdot \alpha\).

We will now introduce two new operations on sets, \( A^\sharp \) and \( A^\flat \) such that \( ([A^\sharp]_W, [A^\flat]_W) \) is the least non-self-dual pair above the \([A]_W \cdot \alpha \)'s, \( (\alpha < \omega_1) \), when \( A \) is self-dual.

**Definition 22.13.** For \( A \subseteq \mathbb{R} \)
\[
A^\sharp = \{ s \cap (0) \cap (x+1) \mid s \in \langle \omega \rangle \omega \land x \in A \} \cup A^\dagger
\]
and
\[
A^\flat = A^\sharp \cup \{ x \in \mathbb{R} \mid \exists n x(n) = 0 \}.
\]

\( A^\sharp \) can be seen as some sort of infinite sum of copies of \( A \), and since \( A^\sharp = A^\dagger + A \), then it is natural to think of it as \( A^\sharp = \cdots + A + A \).

**Exercise 22.14.** Show that for any \( A_0, A_1 \subseteq \mathbb{R} \)
\[
A_0 \leq_W A_0^\sharp, A_0^\flat
\]
and
\[
A_0 \leq_W A_1 \Rightarrow A_0^\sharp \leq_W A_1^\sharp \land A_0^\flat \leq_W A_1^\flat.
\]
Thus we can define the $\mathcal{i}$ and $\mathcal{b}$ operations on degrees by setting
\[
[A]_{\mathcal{i}}^2 = [A^2]_W \quad \text{and} \quad [A]_{\mathcal{b}}^2 = [A^b]_W.
\]

**Exercise 22.15.** (i) Show that for any $s \in <\omega \omega$:
\[
x \in A^2 \iff s \cdot 0 \cdot x \in A^2
\]
\[
x \in A^b \iff s \cdot 0 \cdot x \in A^b
\]
\[
A_{[s \cdot 0]}^i = A^2
\]
\[
A_{[s \cdot 0]}^b = A^b.
\]

(ii) Show that $A_{[h]}^2 =_W A^2$ and $A_{[h]}^b =_W A^b$.

(iii) Show that $[A]_W \cdot \alpha =_W [A^2]_W$, [A^b]_W$, for any self-dual $A \subseteq \mathbb{R}$ and any $1 \leq \alpha < \omega_1$.

**Theorem 22.16.** Assume $\text{BP} + \text{AD}^L$. For a self-dual $([A^2]_W, [A^b]_W)$ is the least non-self-dual pair above the $[A]_W \cdot \alpha$’s, ($1 \leq \alpha < \omega_1$), and therefore, assuming $\text{DC}(\mathbb{R})$,
\[
\|A^i\|_W = \|A^b\|_W = \|A\|_W \cdot \omega_1.
\]

**Proof.** We first show that $B \in \bigcup_{1 \leq \alpha < \omega_1} A^2 \cdot \alpha \iff B \leq_W A^2, A^b$. One direction follows from 22.15(iii) so it is enough to prove that $B \leq_W A^2, A^b$ implies that $B \in \bigcup_{1 \leq \alpha < \omega_1} A^2 \cdot \alpha$. We need the following

**Lemma 22.17.** Assume $\text{BP} + \text{AD}^L$. For any $A, B \subseteq \mathbb{R}$, if $B \leq_W A^2, A^b$ then there is a winning strategy $\tau$ for $\mathbb{II}$ in $G_W(B, A^2)$ such that any play for $\mathbb{II}$ according to $\tau$ belongs to $\mathbb{R}^2 = \{ x \in \mathbb{R} \mid \forall^\infty n \ x(n) \neq 0 \}$.

**Proof.** Let $\tau_0, \tau_1$ be $\mathbb{II}$’s winning strategies for $G_W(B, A^2)$ and $G_W(B, A^b)$, respectively. The plan is to use alternatively $\tau_0$ or $\tau_1$, switching each time the strategy being used requires to play a 0. We dove-tail $\tau_0$ and $\tau_1$ as follows: for any $s \in <\omega \omega$ let $u_0, \ldots, u_n, v_0, \ldots, v_m \in <\omega \omega$ be such that
\[
(s * \tau_0)_\mathbb{II} = (u_0 + 1) \cdot 0 \cdot \cdots \cdot 0 \cdot (u_n + 1)
\]
and
\[
(s * \tau_1)_\mathbb{II} = (v_0 + 1) \cdot 0 \cdot \cdots \cdot 0 \cdot (v_m + 1)
\]
then let
\[
(s \tau)_\mathbb{II} = ((u_0 + 1) \cdot 0 \cdot (v_0 + 1) \cdot 0 \cdot \cdots \cdot 0 \cdot (u_k + 1) \cdot 0 \cdot (v_k + 1)) \uparrow \text{lh}(s)
\]
where $k = \min(n, m)$. It is easy to check that this defines a strategy for $\mathbb{II}$. 

Claim 22.17.1. \( \forall x \in \mathbb{R} \left( (x \ast \tau)_\mathcal{H} \notin \mathbb{R}^k \right) \).

Proof. Deny. Let \( x \in \mathbb{R} \) be such that \( (x \ast \tau)_\mathcal{H} \notin \mathbb{R}^k \), that is

(1) \[ \exists \infty n \tau_0(x \upharpoonright n) = 0 \]

and

(2) \[ \exists \infty n \tau_1(x \upharpoonright n) = 0 \].

Then by (1) \( (x \ast \tau_0)_\mathcal{H} \notin A^k \) hence \( x \notin B \) and by (2) \( (x \ast \tau_1)_\mathcal{H} \in A^k \) hence \( x \in B \): a contradiction. \( \square \)

Let us check that \( \tau \) is winning for \( \mathcal{H} \) in \( G_{W}(B,A^k) \). Suppose \( I \) plays \( x \) and let \( y = (x \ast \tau)_\mathcal{H} \) be \( \mathcal{H} \)'s response. Then either

(A) \( \tau \) settles on \( \tau_0 \), that is

\[ y = (u_0 + 1) \cdot (0) \cdot (v_0 + 1) \cdot (0) \cdot \cdots \cdot (0) \cdot (u_n + 1) \cdot (0) \cdot (v_n + 1) \cdot (0) \cdot (z + 1) \]

where \( (u_0 + 1) \cdot (0) \cdot \cdots \cdot (0) \cdot (u_n + 1) \cdot (0) \cdot (z + 1) = (x \ast \tau_0)_\mathcal{H} \), or else

(B) \( \tau \) settles on \( \tau_1 \), that is

\[ y = (u_0 + 1) \cdot (0) \cdot (v_0 + 1) \cdot (0) \cdot \cdots \cdot (0) \cdot (v_n + 1) \cdot (0) \cdot (w + 1) \]

where \( (v_0 + 1) \cdot (0) \cdot \cdots \cdot (0) \cdot (v_n + 1) \cdot (0) \cdot (w + 1) = (x \ast \tau_1)_\mathcal{H} \).

Assume (A) holds, i.e., \( \tau \) settles on \( \tau_0 \). Then

\[ x \in B \iff (x \ast \tau_0)_\mathcal{H} \in A^k \] (by definition of \( \tau_0 \))

\[ \iff z \in A^k \] (by 22.15(i))

\[ \iff y \in A^k \] (by 22.15(i) again)

Case (B) when \( \tau \) settles on \( \tau_1 \) is completely analogous. \( \square \)

Let’s go back to the proof of 22.16. For any \( s \in \prec \omega \omega \) let

\[ n(s) = \begin{cases} 
\text{the largest } i < \text{lh}(s) \text{ such that } s(i) = 0, \\
0, \text{ if } \forall i < \text{lh}(s) \ (s(i) \neq 0),
\end{cases} \]

and let \( < \) be the strict partial order on \( \prec \omega \omega \) defined by

\[ s < t \iff s \upharpoonright n(s) \subseteq t \upharpoonright n(t). \]
Fix a strategy $\tau$ as in the Lemma and set

$$s \triangleleft t \iff (s * \tau)_\Pi \triangleleft (t * \tau)_\Pi .$$

Then $\triangleleft$ is a strict partial order on $<^\omega \omega$, and it is well-founded, since otherwise there would be an $x \in \mathbb{R}$ such that $y \notin \mathbb{R}^2$, where $y$ is $\Pi$’s response via $\tau$ to $\I$ playing $x$. We will prove by induction on $\alpha$ that $B \in [A]_W \cdot (\alpha + 1)$, where $\alpha = \|0\|_\omega$.

Suppose $\alpha = 0$. This corresponds to the case where $\forall s \in <^\omega \omega (\tau(s) \text{ does not contain any } 0)$. Then $\Pi$ wins $G_W(B, A)$ by playing $\tau(s) - 1$, for any $s$ played by $\I$: this is indeed a winning strategy since if $x$ and $y$ are the reals played this way by $\I$ and $\Pi$ then

$$x \in B \iff y + 1 \in A \iff y \in A .$$

This proves that $B \leq_W A$.

Suppose $\alpha > 1$ and let $C \in [A]_W \cdot \alpha$. Consider the following strategy for $\Pi$ in the game $G_W(B, C + A)$:

As long as $\I$ stays in a position $p$ such that $\|p\|_\omega = \alpha$ then $\Pi$ plays $\tau(p)$. Suppose at some round $\I$ reaches a position $p$ such that $\|p\|_\omega < \alpha$. Then $\Pi$ answers 0 and from now on, to any further move he answers $\tau_p(p \triangleleft q) + 1$, where $\tau_p$ is a strategy witnessing $B_{[p]} \leq_W C$. Such a $\tau_p$ exists by inductive hypothesis.

Let us check that this strategy is indeed winning for $\Pi$. Let $x$ and $y$ be the reals produced by a complete play. If $\|x \upharpoonright n\|_\omega < \alpha$ for some $n$, then let $x'$ and $y'$ be such that $x \upharpoonright n \triangleleft x' = x$ and $((x \upharpoonright n) * \tau)^- (0) \triangleleft (y' + 1) = y$. (Note that \text{Lh}(x \upharpoonright n * \tau) < n$ is possible since $\tau$ may pass.) Then $x' \in B_{[p]} \iff y' \in C$ since $\tau_p$ witnesses $B_{[p]} \leq_W C$ and therefore $x \in B \iff y \in C + A$. If instead $\forall n (\|x \upharpoonright n\|_\omega = \alpha)$ then $\forall n (\tau(x \upharpoonright n) \neq 0)$ so $y \in \mathbb{R}^2$ hence

$$x \in B \iff y \in A \iff y - 1 \in A \iff y \in C + A .$$

Therefore $[B]_W \leq [A]_W \cdot (\alpha + 1)$.

Finally, let us show that $A^2 \not\equiv_W A^b$. Otherwise if $A^2 \leq_W A^b$ then by the argument above, since $A^2 \leq_W A^3$, we should have that $[A^2]_W \leq_W [A]_W \cdot \alpha$ for some $\alpha < \omega_1$. Then

$$[A]_W \cdot \alpha \leq_W [A^2]_W \cdot (\alpha + 1) \quad \text{by } 22.8(\text{i})$$

$$\leq_W [A^3]_W \quad \text{by } 22.15(\text{iii})$$

hence $[A^3]_W \leq_W [A]_W$; a contradiction. 

\qed
Corollary 22.18. Assume BP + DC(\(\mathbb{R}\)) + AD. Let \(A\) be self-dual. For every \(B\) such that \(\|A\|_W < \|B\|_W < \|A\|_W \cdot \omega_1\),
\[
[B^2]_W = [A^2]_W \quad \text{and} \quad [B^3]_W = [A^3]_W.
\]

Proof. Let \(\alpha\) be least such that \(\|B\|_W \leq \|A\|_W \cdot \alpha\) and let \(C \in [A]_W \cdot \alpha\). By monotonicity (22.14) \(A^2 \leq_W B^2 \leq_W C^2\). The argument for \(A^3\) is similar. \(\square\)

Exercise 22.19. (i) Show that
\[
\mathbb{R}^2 = \{x \in \mathbb{R} \mid \forall^\infty n \ x(n) \neq 0\} \in \Sigma^0_2 \setminus \Delta^0_2
\]
and that
\[
\emptyset^2 = \{x \in \mathbb{R} \mid \exists^\infty n \ x(n) = 0\} \in \Pi^0_2 \setminus \Delta^0_2.
\]

(ii) Show that \(\mathbb{R}^2 = (\mathbb{R} \oplus \emptyset)^2\) and \(\emptyset^2 = (\mathbb{R} \oplus \emptyset)^2\), \(\emptyset^2 = \emptyset\), \(\mathbb{R}^2 = \mathbb{R}\).

(iii) (DC(\(\mathbb{R}\))) Show that \(\Sigma^0_2 \setminus \Delta^0_2\) and \(\Pi^0_2 \setminus \Delta^0_2\) occupy the \(\omega_1\)-th level of the Wadge hierarchy.

(iv) Show that \(\mathbb{R}^2 + \mathbb{R}^2 \equiv_W \mathbb{R} \mathbb{R}^2\) and therefore \(\forall B \leq_W \mathbb{R}^2 \ (\mathbb{R}^2 + B \equiv_W \mathbb{R}^2)\).

(v) Show that \(([\mathbb{R}^\emptyset]_W, [\emptyset^\emptyset]_W), ([\mathbb{R}^2]_W, [\emptyset^2]_W), \ldots\) is an increasing sequence of non-self-dual pairs of degrees whose Wadge rank (here we assume DC(\(\mathbb{R}\)) so that we can talk about Wadge ranks) are \(\omega^2_1, \omega^3_1, \ldots\).

Therefore the map \([A]_W \mapsto [A]^3_2\) acts like this

\[
\begin{array}{cccc}
[\emptyset]_W & \bullet & \cdot & \cdots \\
\mathbb{R}^2 & \bullet & \cdot & \cdots \\
\emptyset^2 & \bullet & \cdot & \cdots
\end{array}
\]

and the map \([A]_W \mapsto [A]^\emptyset_2\) acts like this

\[
\begin{array}{cccc}
[\emptyset]_W & \cdot & \cdot & \cdots \\
\mathbb{R}^2 & \cdot & \cdot & \cdots \\
\emptyset^2 & \cdot & \cdot & \cdots
\end{array}
\]
22.D Back-track Reducibility

There is a coarser notion of reducibility which is quite useful. It is defined in terms of games.

Definition 22.20 (Van Wesep). $G_{bt}(A, B)$ is a game like $G_W$ where I and II alternate in playing integers. Just as before II has the option of passing at any round (with the proviso that he must play infinitely often, otherwise he loses), but moreover II has a further advantage: at any round he can back-track (whence the subscript in $G_{bt}$) and erase his moves (but not changing I's moves) starting his board anew. The only restriction is that he cannot backtrack infinitely often, otherwise he loses. Let $x$ be I's play and let $y$ be the sequence of moves that II plays after his final back-tracking (if he ever used this option). Then II wins $G_{bt}(A, B)$ if

$$x \in A \Leftrightarrow y \in B.$$ 

In Exercise 22.50 we will look at similar games where various forms of back-tracking is allowed.

For $A, B \subseteq \mathbb{R}$, $A$ is **back-track reducible to** $B$, in symbols $A \leq_{bt} B$ iff II has a winning strategy in $G_{bt}(A, B)$.

**Exercise 22.21.** (i) Show that $\leq_{bt}$ is reflexive and transitive so that

$$A \equiv_{bt} B \Leftrightarrow A \leq_{bt} B \& B \leq_{bt} A$$

is an equivalence relation. Let $[A]_{bt}$ be the $\equiv_{bt}$ equivalence class of $A$, called the **bt-degree** of $A$.

(ii) Show that $A \leq_W B \Rightarrow A \leq_{bt} B$. Conclude that $\leq_{bt}$ is well-founded on the bt-degrees.

(iii) Show that $A \leq_{bt} B \Leftrightarrow A \leq_W B^\sharp, B^\sharp$.

(iv) Show that if $A \not\equiv_{bt} \neg A$ and then $[A]_{bt} = [A]_W$.

(v) Show that if $A \equiv_W \bigoplus_n A_n$ and $\forall n (A_n \equiv_{bt} \neg A_n)$ then $A \equiv_{bt} \neg A$.

By (ii) each bt-degree is union of Wadge degrees and $[A]_{bt} \supseteq [A]_W$, and (iii) implies that $[A]_{bt} \supseteq [A]_W$ for $A$ self-dual. As with the Wadge degrees $\{\emptyset\} = [\emptyset]_{bt}$ and $\{\mathbb{R}\} = [\mathbb{R}]_{bt}$ are at the bottom of the bt-hierarchy. If $[A]_{bt}$ self-dual i.e., $[A]_{bt} = [\neg A]_{bt}$ then for any $B \in [A]_{bt}$ such that $B \equiv_W \neg B$ the degrees $[B^\sharp]_{bt}$ and $[B^\sharp]_{bt}$ are dual to each other and are minimal above $[A]_{bt}$. Therefore above each self-dual bt-degree there is a non-self-dual pair of bt-degrees, and above each non-self-dual pair of bt-degrees there is a single
self-dual bt-degree. By (v) at limit levels of cofinality \( \omega \) there is a single self-dual bt-degree. Proposition 22.24 will imply that at levels of uncountable cofinality there is a non-self-dual pair of bt-degrees, just like in the Wadge hierarchy. Therefore the bt-hierarchy looks like this:

\[
\begin{array}{cccccccc}
\square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square \\
\end{array}
\]

\[\text{cof} = \omega \quad \text{cof} > \omega\]

In order to analyze the bt-degrees we need the following result. It can be used to characterize those Wadge degrees that contain homogeneous sets i.e., sets \( B \) which are Wadge-equivalent to any of their localizations, i.e., \( T(B) = \omega^0 \).

**Proposition 22.22** (Van Wesep). Assume BP + AD\(^L\). For \( A \notin \Pi_1^0 \) the following are equivalent.

1. \( A \not\sqsubseteq_{bt} \neg A \),
2. \( \exists B \in [A]_L \ (T(B) = \omega) \),
3. \( \exists B \in [A]_L \ (B \cap [T(B)] \equiv L B) \),
4. \( \forall B \in [A]_L \ (B \cap [T(B)] \equiv L B) \).

**Proof.** (1) \(\Rightarrow\) (2): By 22.21(ii) \( A \not\sqsubseteq W \neg A \) so \( \forall B \in [A]_L = [A]_W \ (T(B) \text{ is pruned}) \) by \( \Box \). Let \( \sigma \) be a winning strategy for \( \mathbf{I} \) in \( G_{bt}(A, \neg A) \). Then \( T = \text{ran}(\sigma) \cup \{\emptyset\} \) is a pruned tree so by \( \Box \) there is a Lipschitz map \( f : \mathbb{R} \rightarrow \{\} \) such that \( f \upharpoonright \{\} = \text{id} \). Let also \( \varphi : \omega^0 \rightarrow T \) be the Lipschitz map inducing \( f \). Let \( B = f^{-1}A \). Since \( f \) is Lipschitz \( B \leq L A \), hence in order to prove that \( A \leq L B \) it is enough to show (by AD\(^L\)) that \( B \not\sqsubseteq L \neg A \), i.e., that \( \mathbf{I} \) wins \( G_L(B, \neg A) \). As \( \text{ran}(\sigma) \subseteq \{\} \) and \( \{\} \cap A \subseteq B \) it is easy to see that \( \sigma \) is winning for \( \mathbf{I} \) in \( G_L(B, \neg A) \). Therefore \( B \in [A]_L \). We will show that this \( B \) works, that is that \( B_{[s]} \equiv W B \) for every \( s \in \omega \).

Fix \( s \in \omega \). As \( B_{[\emptyset]} = B \) we may assume that \( \text{lh}(s) > 0 \). Since \( B_{[s]} \leq W B \) comes for free, by AD\(^L\) it is enough to show that \( \mathbf{I} \) has a winning strategy \( \check{\sigma} \) in \( G_W(B_{[s]}, \neg A) \):

Since \( \varphi(s) \in \text{ran}(\sigma) \) let \( p \) be a position for \( \mathbf{II} \) in \( G_{bt}(A, \neg A) \) such that \( \varphi(s) \) is \( \sigma \)'s answer to \( p \). (Notice that \( p \) may contain some back-tracking or passing.) It is \( \mathbf{II} \)'s turn to move in this game
and let’s have him back-track.

\[ \begin{array}{cccc}
I & \quad \varphi(s) \quad \varphi(s) \quad \varphi(s) \\
G_{bt}(A, \neg A) & \quad \ldots \\
II & \quad p \quad \ldots \\
bt & \quad \text{bt} \\
\end{array} \]

Then I using \( \sigma \) plays some \( x_0 \in \omega \) in \( G_{bt}(A, \neg A) \) and let \( z_0 \in \omega \) be such that \( \varphi(s^\omega(z_0)) = \varphi(s^\omega(x_0)) \) — such a \( z_0 \) certainly exists since \( \varphi(s^\omega(x_0)) \in T \). Define \( \hat{\sigma}(\emptyset) = z_0 \), i.e., this is I’s first move in \( G_W(B, \neg A) \). From now on we simply copy II’s moves from \( G_W(B, \neg A) \) to \( G_{bt}(A, \neg A) \) and transfer-back I’s moves via \( \varphi \): if II plays \( y_0 \) in \( G_W(B, \neg A) \) let \( x_1 \) be \( \sigma \)’s response to this move in this game and let \( z_1 \) be such that \( \varphi(s^\omega(z_0, z_1)) = \varphi(s^\omega(x_0, x_1)) \), and so on:

\[ \begin{array}{cccc}
I & \quad \varphi(s) \quad \varphi(s) \quad \varphi(s) \\
G_{bt}(A, \neg A) & \quad \ldots \\
II & \quad p^\langle bt \rangle \quad \ldots \\
G_W(B_{[s]}, \neg A) & \quad \text{bt} \\
II & \quad \text{bt} \\
\end{array} \]

Thus we end up with reals \( \varphi(s^\omega x \setminus y) \) in \( G_{bt}(A, \neg A) \), \( z \) and \( y \) in \( G_W(B_{[s]}, \neg A) \). Notice that \( \varphi(s^\omega x) \in [T] \) and since \( \sigma \) is winning

\[ \varphi(s^\omega x) \in A \cap [T] \iff y \notin \neg A. \]

Since \( f(s^\omega x) = x \) then

\[ s^\omega z \in B \iff y \notin \neg A \]

that is \( \hat{\sigma} \) is winning for \( I \) in \( G_W(B_{[s]}, \neg A) \).

(2) \( \Rightarrow \) (3) is obvious.

(3) \( \Rightarrow \) (4) follows from ??.

(4) \( \Rightarrow \) (1). Assume (4) so that in particular \( [T(A)] \cap A \equiv_L A \). Since \( [T(A)] \neq \emptyset \) then \( A \) is non-self-dual by ???. If \( [A]_W \) were successor then either
\[ T(A) \cap A = \emptyset \text{ or else } [T(A)] \subseteq A \text{ by 22.4 and therefore either } A \equiv L \emptyset \text{ or else } A \equiv L [T(A)], \text{ contradicting the assumption that } A \text{ is not closed. Since} \]

\[
\forall p \in T(A) \left( T(A_{[p]}) = T(A)_{[p]} \text{ and } A_{[p]} \equiv L A \right)
\]

the argument above shows that

\[
(*) \quad \forall p \in T(A) \left( [T(A)_{[p]}] \not\subseteq A_{[p]} \right).
\]

Since \( \neg A_{[p]} \equiv L \neg A \) for any \( p \in T(A) \), fix \( \sigma_p \) a winning strategy for \( I \) in \( G_W(A_{[p]} \cap [T(A)_{[p]}], \neg A) \). Consider the following strategy for \( I \) in \( G_{bt}(A, \neg A) \):

As the play develops we inductively define a sequence of positions for the two Players \((p_0, q_0), (p_1, q_1), \ldots, (p_n, q_n)\) with \( \emptyset = p_0 \subset \cdots \subset p_n \) and \( \emptyset = q_0 \subset \cdots \subset q_n \), such that \( (q_1, \ldots, q_n) \) lists all the positions where \( II \) back-tracks (i.e., if \( q \) is a position of the current play such that \( q(lh(q) - 1) = bt \) then \( q = q_i \) for some \( 1 \leq i \leq n \). We will maintain that \( p_i \in T(A) \) so that \( A_{[p_i]} \equiv W A \). Let \( I \) follow \( \sigma = \sigma_{p_0} = \sigma_{\emptyset} \) as long as \( II \) does not back-track or \( \sigma \) does not force \( I \) out of \( T(A) \). Suppose at some round we are at positions \( p \in T(A) \) for \( I \) and \( q \) for \( II \), but \( p \prec (\sigma(q)) \notin T(A) \). Notice that \( N_q \cap A = \emptyset \) since otherwise \( I \) could defeat \( \sigma \) in \( G_W(A \cap [T(A)], \neg A) \) by playing the elements of some \( y \in N_q \cap A \) from this point on. Then \( I \) picks a branch \( b \in [T(A)_{[p]}] \setminus A_{[p]} \) which exists by (*) above, and follows \( b \) from now on. Suppose we reach a round when \( II \) decides to back-track and let \( p_1, q_1 \) be the positions of the players at this round: then in order for \( I \) to win it is enough to show that he has a winning strategy for \( G_{bt}(A_{[p_1]}, \neg A) \). Since \( A \equiv W A_{[p_1]} \) we can repeat the argument above: \( I \) follows \( \sigma = \sigma_{p_1} \) until—if ever—either \( II \) back-tracks or \( \sigma \) forces us out of \( T(A) \). And so on.

The following flow-chart gives a dynamic description of \( \sigma \):

\[
\text{[Flow-chart description]}
\]
Let us check that this strategy is winning: consider a play according to it and let \( n \) be the latest round when \( \text{II} \) did back-track, if this ever happened, or \( n = 0 \) otherwise. Let \( p = p_n \) be \( \text{I} \)'s position at round \( n \). If \( \sigma_p \) never takes us out of \( T(A)_{|p|} \), then we kept on using it forever. Let \( x \) and \( y \) be the reals played by \( \text{I} \) and \( \text{II} \) after round \( n \). Since \( \sigma_p \) is winning for \( \text{I} \) in \( G_W(A_{|p|}, \neg A) \) then \( x \in A_{|p|} \Leftrightarrow y \in A \) so \( p \prec x \in A \Leftrightarrow y \in A \), hence \( \text{I} \) wins \( G_{bt}(A, \neg A) \). If at some round \( n' \geq n \), \( \sigma_p \) forces us out of \( T(A) \) then we follow a \( b \in [T(A)] \setminus A \) passing through our (i.e., Player \( \text{I} \)'s) current position \( p' \supseteq p \). Since \( \text{II} \) does not back-track any more, we stick to \( b \) from now until the end of the game. Since \( b \notin A \), in order for \( \text{I} \) to win \( G_{bt}(A, \neg A) \) we must check that \( y \), the real played by \( \text{II} \) after round \( n \), belongs to \( \neg A \). Let \( q' \) be \( \text{II} \)'s position at round \( n' \): since \( \sigma_p \) is winning for \( \text{I} \) in \( G_W(A_{|p|} \cap [T(A)_{|p|}], \neg A) \) and as \( \sigma_p(q') \notin T(A) \) then \( N_{q'} \cap A = \emptyset \), since otherwise \( \text{II} \) could defeat \( \sigma_p \) by playing any \( y' \in N_{q'} \cap A \). Therefore \( y \notin A \).

Exercise 22.23. Assume BP + ADL.

(i) Show that if \( A \notin \Pi_0^0 \) and \( A \equiv_{bt} \neg A \) then for any \( C \in [A]_W \)

\[
\forall s \in \omega \exists t \supset s \ (t \notin T(C))
\]
that is \( \lessdot \omega_1 \setminus T(C) \) is dense in \( \lessdot \omega_1 \).

(ii) Let \( A \not\sqsubseteq_{bt} \neg A \) with \( A \not\in \Pi_1^b \) and let \( [B]_{bt} \) be the \( \sqsubseteq_{bt} \)-successor of \( [A]_{bt} \) and \( [\neg A]_{bt} \). Show that \( \{ s \in \lessdot \omega_1 \mid B_{[s]} \lessdot W A \lor B_{[s]} \lessdot W \neg A \} \) is dense in \( \lessdot \omega_1 \).

**Proposition 22.24** (Steel). Assume \( BP + AD^L \). If \( A \) is non-self-dual and \( \forall B \lessdot W A (B^2 \lessdot W A) \) then \( A \not\sqsubseteq_{bt} \neg A \).

**Proof.** It is enough to show that \( A \cap [T(A)] \equiv W A \) and apply 22.22. Since \( A \cap [T(A)] \leq W A \) follows from \( A \) being non-self-dual, we may assume, towards a contradiction, that \( A \cap [T(A)] < W A \). If \( [A]_W \) were the successor of a self-dual \( [B]_W \) then \( A < W B^2 \), contrarily to our assumption, so \( [A]_W \) is a limit degree. Also \( [A]_W \) is not a join degree, since otherwise it would be self-dual. Therefore \( \bigoplus_n A_n < W A \) where \( \langle A_n \mid n \in \omega \rangle \) is an enumeration of \( \{ A_{[s]} \mid s \notin T(A) \} \). Thus we can find a \( B < W A \) such that \( \forall s \notin T(A) (A_{[s]} \leq W B) \) and \( A \cap [T(A)] \leq W B \). Since \( B \leq W B^2 \) and \( A_{[s]} \leq W B \) for \( s \notin T(A) \), we can fix winning strategies for \( \Pi \): \( \tau_0 \) in \( G_W (A \cap [T(A)], B^2) \) and \( \tau_s \) in \( G_W (A_{[s]}, B) \). Then \( \Pi \) wins \( G_W (A, B^2) \) as follows:

As long as \( I \)'s position is inside \( T(A) \) then \( \Pi \) uses \( \tau_0 \). If \( I \) ever moves into a position \( s \notin T(A) \) then \( \Pi \) plays 0 and then uses \( \tau_s + 1 \).

Therefore \( A \leq W B^3 \) and since \( B^2 \leq W \neg A \) by hypothesis, \( A \leq W \neg A \): a contradiction.

**Corollary 22.25.** Assume \( BP + AD^L + DC(\mathbb{R}) \).

\[
A \not\sqsubseteq_{bt} \neg A \iff \left( \|A\|_W = 0 \lor \exists \xi > 0 \left( \|A\|_W = \omega^\xi \land \text{cof}(\xi) \neq \omega \right) \right).
\]

Below is the picture of the bt-hierarchy versus the Wadge hierarchy (a dot \( \bullet \) denotes a Wadge degree and a box \( \square \) denotes a bt-degree):
Using Wadge’s analysis of the Borel hierarchy (see §23???), it can be shown that the sets of Wadge rank $\omega^\xi$ with $\xi < \omega_1$ are exactly the sets in $\Delta^\xi \subset \Delta^\xi \setminus \text{Diff}(\xi; \Pi^0_2) \setminus \text{Diff}(\xi; \Pi^0_2)$. Therefore the first $\omega_1$ levels of the $\leq_{bt}$-hierarchy are:

$$
\begin{array}{ccccccc}
[\emptyset]_W & \Delta_{D_1} & \Delta_{D_2} & \cdots & \Delta_{D_\xi} & \Delta_{D_{\xi+1}} & \cdots \\
[\mathbb{R}]_W & \bar{\Delta}_{D_1} & \bar{\Delta}_{D_2} & \cdots & \bar{\Delta}_{D_\xi} & \bar{\Delta}_{D_{\xi+1}} & \cdots \\
\end{array}
$$

22.E The Stretch Operation

Here is another useful operation on sets and degrees.

**Definition 22.26.** (i) For $x, y \in \mathbb{R}$ $y$ is the **stretch** of $x$ iff there is a strictly increasing $f : \omega \to \omega$ such that for all $m \in \omega$

$$
y(m) = \begin{cases} 
x(n) + 1 & \text{if } f(n) = m \\
0 & \text{otherwise.} 
\end{cases}
$$

Notice that $y$ can be the stretch of at most one real. In this case we say that $x$ is the **unstretch** of $y$.

(ii) For $A \subseteq \mathbb{R}$ the stretch of $A$ is the set

$$
A_{\text{stretch}} = \{ y \in \mathbb{R} \mid \exists x \in A \, (y \text{ is a stretch of } x) \}
$$

Here are the basic properties of the stretch operation.

**Exercise 22.27.** Assume $\text{BP} + \text{AD}^L$.

(i) $A \leq_L A_{\text{stretch}}$;

(ii) $A_{\text{stretch}} \subseteq_L (A_{\text{stretch}})_{\text{stretch}}$;

(iii) $\forall s \in \omega \omega \left( (A_{[s]}_{\text{stretch}} = (A_{\text{stretch}})_{[s+1]} \right)$;

(iv) $A_{\text{stretch}}$ is non-self-dual;

(v) $A \leq_L B \Rightarrow A_{\text{stretch}} \leq_L B_{\text{stretch}}$.

Therefore we can define the stretch operation on the Lipschitz/Wadge degrees by

$$
[A]_{\text{stretch}} \leq_L [A]_{\text{stretch}} \leq_W [A]_{\text{stretch}},
$$

the middle equality being true by (iv).
(vi) Show that $\mathbb{R}^{\text{stretch}} = \{ x \in \mathbb{R} \mid \exists n \ x(n) \neq 0 \}$ is $\Pi^0_2$-complete and

$$\forall A \in \Delta^0_2 \left( A \neq \emptyset \Rightarrow A^{\text{stretch}} \text{ is } \Pi^0_2\text{-complete} \right).$$

(vii) $[T(A)] \cap A \neq \emptyset \Rightarrow [A]^{\text{stretch}}_W$ is limit non-self-dual;

(viii) $(A + B)^{\text{stretch}} \equiv_L A^{\text{stretch}} + B^{\text{stretch}}.$

22.27 yields a complete description of the stretch operation on degrees of rank $< \omega_1^n$. If $A = \{ x \in \mathbb{R} \mid \forall n \ x(n) = 0 \} \in \Sigma^0_2 \setminus \Pi^0_2$ then $A^{\text{stretch}} = \{ x \in \mathbb{R}^{\text{stretch}} \mid \forall n \ (x(n) \neq 0 \Rightarrow x(n) = 1) \} \in \text{Diff}(2; \Pi^0_2)$. More generally, if $A \in \text{Diff}(n; \Pi^0_2) \setminus \text{Diff}(n; \Pi^0_2)$, then $A^{\text{stretch}} \in \text{Diff}(n + 1; \Pi^0_2)$ and $\neg A \leq_W A^{\text{stretch}}$ (see ??). Since $D_n = \text{Diff}(n; \Pi^0_2) \setminus \text{Diff}(n; \Pi^0_2)$ occupies the $\omega_1^n$-th level of the Wadge hierarchy (see §23??), then by 22.4 and part (vii) above, $A \in \check{D}_n$ implies $\text{cof}(||A^{\text{stretch}}||_W) > \omega$ and therefore $A^{\text{stretch}} \in D_{n+1}$.

In other words the first $\omega$ degrees of the form $[A^{\text{stretch}}]_W$ are $[\emptyset]_W, D_1, D_2, \ldots$ and their Wadge ranks are $0, \omega_1, \omega_1^1, \ldots$. And this is how the stretch operator acts on the first $\omega_1^n$ degrees:

\[
\begin{array}{c c c}
\circ \uparrow & \emptyset \cdots & \circ \downarrow \\
\circ \downarrow & \cdots & \circ \uparrow \\
[\emptyset]_W & \emptyset & [\emptyset]_W \\
[\mathbb{R}]_W & D_1 & \check{D}_1 \\
D_1 & \cdots & \check{D}_1 \\
\cdots & \cdots & \cdots \\
D_2 & \cdots & \check{D}_2 \\
\check{D}_2 & \cdots & \cdots \\
\end{array}
\]

Next we want to see how the stretch operates on a degree $[A]_W$ which is the upper bound of stretched degrees. If $\text{cof}(||A||_W) = \omega$ then $A <_W A^{\text{stretch}}$ by 22.27(iv) and in fact $||A^{\text{stretch}}||_W$ is an immediate successor of $||A||_W$.

**Proposition 22.28** (Van Wesep). Assume BP + ADL. Suppose $A$ is self-dual and $\forall B (B <_W A \Rightarrow B^{\text{stretch}} <_W A)$.

Then

(a) $[A]^{\text{stretch}}_W = [A]^\vee_W = [A]_W + [\emptyset]_W$

and

(b) $\forall C \subseteq \mathbb{R} \left( [A + C]^{\text{stretch}}_W = [A]_W + [C]^{\text{stretch}}_W \right)$

**Proof.** We start proving part (a). $A <_W A^{\text{stretch}}$ since $A$ is self-dual and 22.27(iv), hence either $A^\vee \leq_W A^{\text{stretch}}$, or else $A^0 \leq_W A^{\text{stretch}}$. We prove that the former holds by showing that $\Pi$ wins $G_L(A^\vee, A^{\text{stretch}})$:

$\Pi$ plays 0 until, if ever, $I$ plays a non-zero integer: then $\Pi$ plays 0 again and from now on he plays $i + 1$, where $i$ is $I$'s move.
In order to show that II wins $G_W(\text{A}^{\text{stretch}}, A^\uparrow)$ keep in mind that $T(A)$ is well-founded (??):

II plays 0 until, if ever, I reaches a position $p$ of the form

$$p = 0^{(k_0)} \cdots 0^{(k_1)} \cdots 0^{(k_n)} \cdots$$

and $A_{[s]} <_W A$, where $s = \langle m_0, \ldots, m_n \rangle \in \omega \omega$. At this round II plays 1, and since $(A^{\text{stretch}})[p] = (A_{[s]})^{\text{stretch}}$ by 22.27(iii), from now on he uses the strategy to reduce $(A_{[s]})^{\text{stretch}}$ to $A$.

Therefore $[A]_W^{\text{stretch}} = [A^\uparrow]_W$.

Now for (b): If $C = \emptyset$ then by (a) $[A + C]_W = [A]_W^{\text{stretch}}$ hence by 22.27(ii) and by $\emptyset^{\text{stretch}} = \emptyset$, then

$$[A + C]_W^{\text{stretch}} = [A]_W^{\text{stretch}} = [A]_W + [C]_W^{\text{stretch}}.$$ 

If $C \neq \emptyset$ then $[C]_W^{\text{stretch}} = [\emptyset]_W + [C]_W^{\text{stretch}}$ by 22.27 (v) and (vii), hence

$$[A]_W + [C]_W^{\text{stretch}} \leq [A]_W^{\text{stretch}} + [C]_W^{\text{stretch}} \quad \text{by 22.7}$$

$$= ([A]_W + [\emptyset]_W) + [C]_W^{\text{stretch}} \quad \text{by (a)}$$

$$= [A]_W + [C]_W^{\text{stretch}} \quad \text{by 22.12}.$$ 

Since the stretch operation commutes with addition the result follows at once.

Notice that the appeal to DC$(\mathbb{R})$ in the proof of part (b) could be easily eliminated at the expense of clarity. Assuming the results in 23?? the least $A$ as in 22.28 is of rank $\omega^\omega$, e.g. $A = \bigoplus_n A_n$ with $A_n \in D_{n+1}$.

**Exercise 22.29.** Assume $\text{AD}^L$ and $A \not\leq_L B^{\text{stretch}}$. Then I wins $G_W(A, B)$.

22.28 shows that the stretch operation is periodic with period $\omega^\omega$ and enables us to compute $A^{\text{stretch}}$ when $\|A\|_W < \omega_1^\omega \cdot \omega_1 = \omega_1^{\omega+1}$. The next result settles what happens at later stages: if $[A]_W$ is non-self-dual and has infinitely many $\leq_{\text{bst}}$-predecessors, then $A \equiv W A^{\text{stretch}}$.

**Proposition 22.30** (Steel). Assume $BP + \text{AD}^L$. If $A \not\equiv_{\text{bst}} \neg A$ and $A \not\in \text{Diff}(\leq \omega; \Pi^0_2)$ = the smallest Boolean algebra containing $\Pi^0_2$, then $[A]_W = [A]_W^{\text{stretch}}$.

**Proof.** In order to motivate the technicalities that follow, let us see where the naive attempt to prove the result breaks down. Towards a contradiction, suppose $A^{\text{stretch}} \not\leq_L A$ and let $\sigma$ be a winning strategy for I in $G_L(A^{\text{stretch}}, A)$.
We now apply the Martin–Monk method to \( \sigma \) and \( \tau \), the copying strategy for \( \Pi \). For any \( z \in \omega^2 \) let \( \Sigma_n^\omega = \sigma \) if \( z(n) = 0 \) and \( \Sigma_n^\omega = \tau \) if \( z(n) = 1 \), and consider the diagram generated by \( \langle \Sigma_n^\omega \mid n \in \omega \rangle \) and constructed according to the following rule: take \( x^{n+1}(z) \), the real in the \( n+1 \)-st row, unstretch it and feed it into \( \Sigma_n^\omega \) to construct the \( n \)-th row \( x^n(z) \). If \( z \) and \( w \) differ for exactly one entry, then \( H(z) \in A^\text{stretch} \iff H(w) \notin A^\text{stretch} \), where \( H : \omega^2 \to \mathbb{R} \) is the map \( z \mapsto x^0(z) \). Therefore \( \{ z \in \omega^2 \mid H(z) \in A^\text{stretch} \} \) is a flip-set, contradicting the property of Baire. The only problem is that for some \( z \) and \( n \) the real \( x^{n+1}(z) \) may not be the stretch of anything, that is \( x^{n+1}(z) = 0 \) for all sufficiently large \( i \), so that the diagram does not admit a global play. As in the proof of ?? we need some “padding” in the Martin–Monk diagram, i.e., we need to introduce enough well-behaved strategies that guarantee the existence of a global play for every \( z \in \omega^2 \). Here are the details:

Recall from 22.6 that \( X^+ = \{ x+1 \mid x \in X \} \). Rather than working with \( A \) we shall work with \( A^+ \) since it is technically more convenient not to subtract 1 when unstretching a real. Let \( \pi : <\omega \omega \to <\omega \omega \) be the map that erases all 0’s from a sequence, i.e., \( \pi(\emptyset) = \emptyset \) and \( \pi(s \upharpoonright \{0\}) = \pi(s) \). Therefore \( \pi \) induces a continuous function \( g : \mathbb{R}^\text{stretch} \to \mathbb{R}^+ \).

By 22.22 we may assume that \( T(A) = <\omega \omega \) so that \( T(A^+) = \{ t+1 \mid t \in <\omega \omega \} \). \( \Pi \) wins \( G_L(A^+, A) \) as follows:

As long as \( I \) plays integers of the form \( n+1 \) then \( \Pi \) answers \( n \).

Suppose at some round \( I \) plays 0, and let \( p \) be \( \Pi \)'s position before this round. Since \( A[p] \equiv_W A \neq \mathbb{R} \), let \( x \notin A[p] \). Then \( \Pi \) follows \( x \) from now on.

Since \( A \preceq_L A^+ \) is immediate, we have that \( A \equiv_L A^+ \) and since \( A \) is non-self-dual, so is \( A^+ \). In other words \( I \) wins \( G_L(A^+, \neg A^+) \).

Towards a contradiction suppose that \( A^\text{stretch} \not\preceq_L A \equiv_L A^+ \), so fix \( \sigma \) a winning strategy for \( I \) in \( G_L(A^\text{stretch}, A^+) \).

Exercise 22.31. (i) Show that for any \( x \in \mathbb{R} \)

\[
\sigma \ast (x+1) \in \mathbb{R}^\text{stretch} = \{ y \mid \exists m (y(m) \neq 0) \}.
\]

(ii) Show that \( \forall s \in T(A^+) = \{ t+1 \mid t \in <\omega \omega \} \) there is a winning strategy \( \sigma_s \) for \( I \) in \( G_L(A^+, \neg A^+) \) that always plays non-zero natural numbers (i.e., \( \text{ran}(\sigma_s) \subseteq \omega \setminus 1 \) and such that \( s \) in the first \( \text{lh}(s) \) moves, regardless of \( \Pi \)'s moves.

Also let \( \tau \) be the copying strategy for \( \Pi \). We now apply the Martin–Monk method. Let \( \langle \Sigma_n \mid n < \omega \rangle \) be a sequence of strategies such that

\[
\forall n \left( \Sigma_{2n} \in \{ \sigma_s \mid s \in T(A^+) \} \text{ & } \Sigma_{2n+1} \in \{ \sigma, \tau \} \right).
\]
Let also
\[ G_n = \begin{cases} 
G_L(A^+, \neg A^+) & \text{if } n \text{ is even,} \\
G_L(A^{\text{stretch}}, A^+) & \text{if } \Sigma_n = \sigma, \\
G_L(A^+, A^{\text{stretch}}) & \text{if } \Sigma_n = \tau, 
\end{cases} \]
so that \( \Sigma_n \) is winning in \( G_n \). The \( \Sigma_n \)'s construct together with \( \pi \) the following diagram:

\[
\begin{array}{cccccc}
x_0^0 & x_1^0 & x_2^0 & \cdots & = x^0 \\
x_0^1 & x_1^1 & x_2^1 & \cdots & = x^1 \\
\uparrow & \uparrow & \uparrow & & \\
x_0^2 & x_1^2 & x_2^2 & \cdots & = x^2 \\
\end{array}
\]

\[
\begin{array}{cccccc}
x_0^3 & x_1^3 & x_2^3 & \cdots & = x^3 \\
\uparrow & \uparrow & \uparrow & & \\
x_0^4 & x_1^4 & x_2^4 & \cdots & = x^4 \\
\end{array}
\]

\[
\begin{array}{cccccc}
x_0^5 & x_1^5 & x_2^5 & \cdots & = x^5 \\
\vdots & 
\end{array}
\]

where the vertical arrows represent the application of the \( \pi \) map: if in the \((2n)\)-th row we have filled-in a sequence \( \langle x_0^{2n}, \ldots, x_k^{2n} \rangle \) then in the \((2n+1)\)-st row we copy \( \pi(\langle x_0^{2n}, \ldots, x_k^{2n} \rangle) = \langle x_0^{2n-1}, \ldots, x_{k-1}^{2n-1} \rangle \), the subsequence of all non-zero elements of \( \langle x_0^{2n}, \ldots, x_k^{2n} \rangle \). Notice that if \( \langle x_0^{2n}, \ldots, x_k^{2n} \rangle = 0^{(k+1)} \) then \( \pi(\langle x_0^{2n}, \ldots, x_k^{2n} \rangle) = 0 \), i.e., we do not copy anything at this point in the \((2n+1)\)-st row. Thus if for some \( n > 0 \), \( x^{2n} \) is definitively equal to 0 then we cannot complete the \((2n+1)\)-st row and hence \( \langle \Sigma_n \mid n \in \omega \rangle \) does not admit a global play. If instead \( \exists^m \langle x_n^{2m} \mid n \in \omega \rangle \neq 0 \) then \( x^{2n+1} = g(x^{2n}) \). Since \( \tau \) and the \( \sigma_x \)'s always yield elements of \( \mathbb{R}^+ \), problems can only occur between rows with indices of the form \( 4n+2 \) and \( 4n+1 \) and such that \( \Sigma_{2n+1} = \sigma \).

We will construct \( \langle s_n \mid n \in \omega \rangle \in \omega \mathcal{T}(A^+) \) such that for any \( z \in \omega^2 \) the sequence \( \langle \Sigma_n^z \mid n \in \omega \rangle \) defined by

\[
\Sigma_n^z = \begin{cases} 
\sigma & \text{if } n = 2m, \\
\sigma & \text{if } n = 2m+1 \text{ and } z(m) = 0, \\
\tau & \text{if } n = 2m+1 \text{ and } z(m) = 1, 
\end{cases} 
\]

admits a global play, which amounts to say that \( \forall n > 0 \exists^m \langle x_n^{2m} \mid n \in \omega \rangle \neq 0 \), or equivalently (by the exercise), that \( \exists^m \langle x_0^0 \mid n \in \omega \rangle \neq 0 \). Notice that if \( n \) is odd
and $\Sigma_n = \sigma$ then the $(2n)$-th and $(2n + 1)$-st rows in the diagram are

$$
\begin{array}{ccc}
I & x_0^{2n} & x_1^{2n} & \cdots & = x^{2n} \\
II & x_0^{2n+1} & x_1^{2n+1} & \cdots & = x^{2n+1} 
\end{array}
$$

and $x^{2n} \in A^\text{stretch} \iff x^{2n+1} \notin A^+$. Vice versa is $n$ is odd but $\Sigma_n = \tau$ then the $(2n)$-th and $(2n + 1)$-st rows are

$$
\begin{array}{ccc}
II & x_0^{2n} & x_1^{2n} & \cdots & = x^{2n} \\
I & x_0^{2n+1} & x_1^{2n+1} & \cdots & = x^{2n+1} 
\end{array}
$$

and $x^{2n} \in A^\text{stretch} \iff x^{2n+1} \in A^+$, since $x^{2n} = x^{2n+1}$ and $x^{2n+1} = g(x^{2n+2}) \in \mathbb{R}^+$. 

Given $\langle s_n \mid n \in \omega \rangle$ as above we reach a contradiction as in ??: let $H : \omega \rightarrow \mathbb{R}$, $z \mapsto x^0(z) = \text{the real in the 0th row of the Martin–Monk diagram when } \langle \Sigma_n^z \mid n \in \omega \rangle$ is used. As before $H$ is continuous and $H^{-1}(A^+)$ is a flip-set, i.e.,

$$
\exists! n \left( z(n) \neq w(n) \right) \Rightarrow \left( H(z) \in A^+ \iff H(w) \notin A^+ \right),
$$

contradicting ???. Therefore we are only left to construct the $s_n$’s.

We need the following

**Lemma 22.32.** Let $\langle \Sigma_i \mid i < 2n \rangle$ be such that $\Sigma_i \in \{ \sigma_s \mid s \in T(A^+) \}$ if $i$ is even, and $\Sigma_i \in \{ \sigma, \tau \}$ if $i$ is odd. Then the set of all $t$ such that $\langle \Sigma_i \mid i < 2n \rangle \upharpoonright \langle \sigma_i \rangle$ admits a finite play with its 0-th row of length at least $n$ is dense in $T(A^+)$, that is to say

$$
\forall s \in T(A^+) \exists u \in T(A^+) \left( s \subseteq u \wedge \forall t \in T(A^+) \exists (D, f) \text{ finite play for } \langle \Sigma_i \mid i < 2n \rangle \upharpoonright \langle \sigma_i \rangle \text{ and } |D \cap \{0\} \times \omega| \geq n \right).
$$

**Proof.** Let $\langle \Sigma_i \mid i < 2n \rangle$ and $s \in T(A^+)$ be as in the hypothesis. It is enough to show that for some $\bar{x} \in \mathbb{R}^+ \cap N_s$, letting $x^{4n-1} = \bar{x}$, then the reals in the first $4n - 1$ rows $x^{4n-2}, x^{4n-3}, \ldots, x^0$ are defined, since, by continuity, we can take $u = \bar{x} \upharpoonright k$ for $k$ sufficiently large. How do we get such an $\bar{x}$? Since the only possible source of troubles is when $\Sigma_i = \sigma$, we may assume that $\{ i < 2n \mid \Sigma_i = \sigma \} \neq \emptyset$ and since the $x^m$’s with larger labels are constructed first, let $\langle J_0, \ldots, J_M \rangle$ be the enumeration of $\{ i < 2n \mid \Sigma_i = \sigma \}$ in decreasing order. We must find an $\bar{x} \in \mathbb{R}^+ \cap N_s$ such that the rows $x^{2J_0}, x^{2J_1}, \ldots, x^{2J_M}$
are in $\mathbb{R}^{\text{stretch}}$ so that the filling-in procedure of the diagram does not come to a grinding halt. For any $z \in A^{\text{stretch}}$ then $g(z) \in A^+$ and

\[
\begin{align*}
(\sigma \ast g(z))_I & \in A^+, \\
(g(z) \ast \tau)_I & \in A^+, \\
(\sigma \ast g(z))_I & \notin A^{\text{stretch}}.
\end{align*}
\]

In other words, applying $\sigma$'s or $\tau$'s do not change membership in $A^{\text{stretch}}$ while $\sigma$ does change membership in $A^{\text{stretch}}$. Using this

$$\bar{x} \notin A^+ \iff x^{2J_0+1} \notin A^+ \iff x^{2J_0} = \sigma \ast x^{2J_0+1} \in A^{\text{stretch}}.$$ 

So if $x \notin A^+ \Rightarrow x^{2J_0} \in \mathbb{R}^{\text{stretch}}$. However the converse implication does not hold since by continuity of $\bar{x} \mapsto x^{2J_0}$ it would imply that $A^+$ (and hence $A$) is $\Pi^0_2$, contrarily to our hypothesis. Thus we can find $Y \subseteq \mathcal{N}_s \cap A^+$ such that $x^{2J_0} \in \mathbb{R}^{\text{stretch}}$ for all $\bar{x} \in Y$, and therefore $x^{2J_0-1}, \ldots, x^{2J_1}$ are defined. By (†) again, for $\bar{x} \in Y$,

$$\bar{x} \in A^+ \iff x^{2J_0} \notin A^{\text{stretch}} \iff x^{2J_1} \in A^{\text{stretch}},$$

so that $x^{2J_1-1}$ (and hence $x^{2J_1}$) is defined. Trying to enforce that $g(x^{2J_2})$ is defined presents the same problem again, so that we must find a $Y' \subseteq \mathcal{N}_s \setminus A^+$ such that for every $\bar{x} \in Y'$ all the rows up to $x^{2J_2}$ are defined. By iterating the argument $M$-many times, each time changing our minds on whether or not $\bar{x} \in A^+$, we can determine $\bar{x}$.

The informal argument above can be given a sleeker presentation so that the real $\bar{x}$ will be obtained as a play in a game which we now describe.

For every $m \in \omega$ we say that $x \in \mathbb{R}$ codes a real in the $m$-th coordinate iff

$$\forall i \in \omega \exists j \in \omega \exists k \in \omega (x(k) = \langle m, \langle i, j \rangle \rangle),$$

that is $\{(n)_0, (n)_1 \mid (m, n) \in \text{ran}(x)\}$ is the graph of a function $\omega \to \omega$. In this case let $c_{(x,m)}(i)$ be the unique $j$ such that $\langle m, \langle i, j \rangle \rangle \in \text{ran}(x)$. If $x$ codes $c_{(x,m)}$ in the $m$-th coordinate and $c_{(x,m)} \in \mathbb{R}^+$ then $g(c_{(x,m)})$ is the real obtained from $c_{(x,m)}$ by deleting all 0 entries. It is easy to check that

$$\{x \in \mathbb{R} \mid g(c_{(x,m)}) \text{ is defined} \} = \{x \in \mathbb{R} \mid c_{(x,m)} \in \mathbb{R}^{\text{stretch}} \} \in \Pi^0_2.$$

As above, let $\langle J_0, \ldots, J_M \rangle$ be the enumeration in decreasing order of $\{i < 2n \mid \Sigma_i = \sigma \}$ and let

$$B = \{y \in \mathbb{R} \mid \exists k \leq M (g(c_{(y,J_k)}) \text{ is undefined and } k \text{ is odd } \& \forall i < k (g(c_{(y,J_i)}) \text{ is defined})) \}.$$ 

Then $B$ is a Boolean combination of $\Pi^0_2$ sets and hence, by our assumption, $B \preceq \omega A$. Therefore $I$ wins $G_L(A^+, \neg B)$. In fact by an argument as before
we may pick a winning strategy \( \hat{\sigma} \) for \( I \) in \( G_L(A^+, \neg B) \) that plays \( s \) in the first \( \ell(s) \) moves and such that never plays a 0. The real \( \bar{x} \) we are looking for will be given by \( \hat{\sigma} \) played against the strategy for \( II \) \( \hat{\tau} \) defined as follows:

Suppose \( p \) and \( q \) represent the positions of \( I \) and \( II \) at some round in \( G_L(A^+, \neg B) \). If for some \( i \leq M \) there are \( k, h \) such that \( \langle J_i, \langle k, h \rangle \rangle \notin \text{ran}(q) \) and \( h \) is in the \( m \)-th row and the \( k \)-th column of the partial play of \( \langle \Sigma_i \mid i < 2n \rangle \) generated by \( p \), where \( m = 2J_i + 1 \), i.e., \( x_{m}^{n} = h \), then \( \hat{\tau} \) plays the least such \( \langle J_i, \langle k, h \rangle \rangle \).

Otherwise \( \hat{\tau} \) passes, if there is no such a triple.

Let \( \bar{x} \) and \( \bar{y} \) be the reals played by \( \hat{\sigma} \) and \( \hat{\tau} \). The following Claim shows that \( \bar{y} \) is indeed defined.

**Claim 22.32.1.** \( \hat{\tau} \) does not pass forever.

**Proof.** Since \( 2n > k > J_0 \Rightarrow \Sigma_k \neq \sigma \) then \( x^{4n-1}, x^{4n-2}, \ldots, x^{2J_0} \) are defined, and so is \( x^{2J_0} = \sigma \ast x^{2J_0+1} \). Therefore by waiting long enough \( II \) can always find triples of the form \( \langle J_0, \langle k, h \rangle \rangle \) which he has not yet played. \( \Box \)

As \( \hat{\sigma} \) is winning in \( G_L(A^+, \neg B) \), then \( \bar{x} \in A^+ \iff \bar{y} \in B \).

**Claim 22.32.2.** \( \bar{x} \notin A^+ \).

**Proof.** Deny. Then \( \bar{y} \in B \), and if \( k \) is least such that \( g(c(\bar{y}, J_k)) \) is undefined, that is \( c(\bar{y}, J_k) \notin \mathbb{R}^{\text{stretch}} \), then \( k \) is odd. As \( \bar{y} \) encodes the rows \( x^{2J_0+1}, x^{2J_1+1}, \ldots, x^{2J_k+1} \) in the sense that

\[
\forall i < k \, (c(\bar{y}, J_i) = x^{2J_i+1}),
\]

then, by (†) and since \( k \) is odd,

\[
\bar{x} = x^{4n-1} \in A^+ \iff x^{2J_0+1} = c(\bar{y}, J_0) \notin A^{\text{stretch}}
\]

\[
\iff x^{2J_1+1} = c(\bar{y}, J_1) \in A^{\text{stretch}}
\]

\[
\vdots
\]

\[
\iff x^{2J_{k-1}+1} = c(\bar{y}, J_{k-1}) \notin A^{\text{stretch}}.
\]

Therefore \( g(c(\bar{y}, J_{k-1})) \), which is defined by the minimality of \( k \), does not belong to \( A^+ \). As the \( \sigma \)'s and \( \tau \) always yield elements of \( \mathbb{R}^+ \), then \( x^{2J_k+1} = c(\bar{y}, J_k) \) is defined and since \( x^{2J_k+1} = g(c(\bar{y}, J_{k-1})) \notin A^+ \), then \( x^{2J_k+1} = c(\bar{y}, J_k) \in A^{\text{stretch}} \) and therefore \( g(c(\bar{y}, J_k)) \) is defined: a contradiction. \( \Box \)
Thus $\tilde{x} \in A^+$ and $\bar{y} \notin B$. Suppose $g(c(\bar{y}, J_k))$ is undefined for some $k \leq M$. If $k$ is the least such, $k$ must be even, since otherwise $\bar{y} \in B$. Arguing as above

$$\tilde{x} \in A^+ \iff c(\bar{y}, J_0) \notin A^{\text{stretch}} \iff c(\bar{y}, J_1) \notin A^{\text{stretch}} \iff \cdots \iff c(\bar{y}, J_{k-1}) \notin A^{\text{stretch}},$$

hence $g(c(\bar{y}, J_k))$ is defined: a contradiction. Therefore $g(c(\bar{y}, J_k))$ is defined for all $k \leq M$, that is

$$\forall k \leq 4n - 1 \ (x^k \text{ is defined})$$

and in particular the 0th row $x^0$ is defined, and this is what we had to prove.

Let’s go back to the proof of 22.30. Suppose the $\langle s_i \mid i < n \rangle$ has been defined: considering the $2^n$ many $\langle \Sigma_i \mid i < 2n \rangle$’s such that $\Sigma_{2i} = s_i$ and $\Sigma_{2i+1} \in \{\sigma, \tau\}$, and applying consecutively $2^n$ many times the Lemma we get a $s_n$ such that $\langle \Sigma_i \mid i < 2n \rangle^{\sim} \langle s_n \rangle$ admits a finite play with at least $2n$ non-zero entries in the 0th row.

This completes the proof of 22.30.

Using 22.25 and the fact that any set in $\text{Diff}(< \omega; \Pi^0_2)$ has Wadge rank $< \omega^\omega$ (see §23??) we obtain at once that

$$\|A\|_W = \omega^\xi \land \text{cof}(\xi) > \omega \Rightarrow A^{\text{stretch}} \equiv_L A.$$

This result will be generalized in 22.41 to the case when $\|A\|_W = \omega^\omega \cdot \alpha$ and cof($\alpha$) > $\omega$.

### 22.F Degree Multiplication

We have seen several ways to construct new sets of higher Wadge rank from old sets. For example $\|A^n\|_W = \|A^p\|_W = \|A\|_W \cdot \omega_1$, for $A$ non-self-dual. We now introduce a new operation on sets that will allow to take even larger leaps on the Wadge hierarchy.

**Definition 22.33** (Steel). For $A, B \subseteq \mathbb{R}$ the multiplication of $A$ and $B$,
in symbols $A \cdot B \subseteq \mathbb{R}$ is defined by

\[ x \in A \cdot B \iff \exists n \exists s_0, \ldots, s_{2n} \in \omega \setminus \{0\} \exists a \in A \\
\left( x = (s_0 + 1) \cdot (s_1 + 1) \cdot (s_2 + 1) \cdots (s_{2n} + 1) \cdot (a + 1) \right) \lor \\
\exists n \exists s_0, \ldots, s_{2n+1} \in \omega \setminus \{0\} \exists y \in \mathbb{R} \\
\left( x = (s_0 + 1) \cdot (s_1 + 1) \cdot (s_2 + 1) \cdots (s_{2n+1} + 1) \cdot (y + 1) \& \\
\quad s_0 \cdot s_2 \cdot \ldots \cdot s_{2n} \cdot y \in B \right) \lor \\
\exists \langle s_i \mid i \in \omega \rangle \in \omega \setminus \{0\} \\
\left( x = (s_0 + 1) \cdot (s_1 + 1) \cdot (s_2 + 1) \cdots (s_{n+1} + 1) \cdot (y + 1) \& \\
\quad s_0 \cdot s_2 \cdot \ldots \cdot s_{2n} \cdot y \in B \right) \right].
\]

The intuition behind this definition is that for any real $x$ not containing two consecutive 0’s and not starting with 0, i.e.,

\[ x = (s_0 + 1) \cdot (s_1 + 1) \cdot (s_2 + 1) \cdots \]

we can think of $x$ as being entered on two different rows, the $s_{2n}$’s on the $B$-row, the $s_{2n+1}$’s on the $A$-row:

\[
\begin{array}{cccc}
\text{B-row} & s_0 & s_2 & \ldots \\
\text{A-row} & s_1 & s_3 & \ldots
\end{array}
\]

with the understanding that if $x(m)$ is definitively non-zero, that is if

\[ x = (s_0 + 1) \cdot (s_1 + 1) \cdot (s_2 + 1) \cdots (s_n + 1) \cdot (y + 1), \]

then $y$ is recorded on the $A$-row, if $n$ is even, or on the $B$-row, if $n$ is odd. Call an $x$ such that it does not contain two consecutive 0s, a product real. Thus $x \in A \cdot B$ just in case $x$ is a product real and either we write infinitely often on the $B$-row and $b \in B$, where $b$ is the real recorded there, or else from some point on we only write on the $A$-row and $a \in A$, where $a$ is the real recorded there from that point on.

Just like with $A + B$ or $A^2$, the game $G_W(C; A \cdot B)$ can be given an alternate description.

**Exercise 22.34.** For $A, B, C \subseteq \mathbb{R}$ the game $G(C; A, B)$ is a Wadge-style game where I and II play natural numbers and II has the option of passing at any stage, but must play infinitely many times. Player II’s moves are recorded on two different rows, the $A$-row and the $B$-row. His first move must be on the $B$-row, but after that he can flip between the two rows. Let $x$ be the real played by I. If II plays infinitely often on the $B$-row (i.e., if he
flips infinitely often between the two rows or if he eventually settles on the
B-row) let \( y \) be the real recorded on it:

\[
\begin{array}{ccc}
\text{II} & \text{B-row} & y_0, \ldots, y_{n_0}, \ldots, y_{n_1}, \\
& \text{A-row} & z_0, \ldots, z_{m_0}, \ldots, z_{m_1}, \\
\end{array}
\]

Then II wins \( G(C; A, B) \) iff

\[ x \in C \iff y \in B. \]

If instead after a certain stage II always plays on the A-row let \( y \) be the real
on the A-row recorded after this stage:

\[
\begin{array}{ccc}
\text{II} & \text{B-row} & s_0, \ldots, s_{2m}, \\
& \text{A-row} & s_1, \ldots, y_0, y_1, \ldots \\
\end{array}
\]

Then II wins \( G(C; A, B) \) iff

\[ x \in C \iff y \in A. \]

Show that \( G(C; A, B) \) is equivalent to \( G_W(C, A \cdot B) \).

The following exercise summarizes the main properties of multiplication
which will be used in the sequel.

**Exercise 22.35.** (i) \( A_0 \leq_W A_1 \) & \( B_0 \leq_L B_1 \) \( \Rightarrow \) \( A_0 \cdot B_0 \leq_W A_1 \cdot B_1 \).

Thus if \( B \) is non-self-dual we can define

\[ [A]_W \cdot [B]_W = [A \cdot B]_W. \]

(ii) \( A \cdot \emptyset \equiv_W A^c \) and \( A \cdot \mathbb{R} \equiv_W A^c \).

(iii) If \( B \neq \emptyset \), then \( A \cdot ((0)^c \cap B) \equiv_W A \cdot B + A \).

Therefore \( B_0 \leq_L B_1 \) in (i) cannot be replaced by \( B_0 \leq_W B_1 \).

(iv) \( A \cdot (\bigoplus_n B_n) \equiv_W \bigoplus_n (A \cdot B_n + A) \).

(v) \( A \cdot B^c \equiv_W (A \cdot B)^c \) and \( A \cdot B^c \equiv_W (A \cdot B)^c \).

(vi) \( A \cdot (B + C) \equiv_W (A \cdot B) + (A \cdot C) \).

(vii) If \( A \) and \( B \) are self-dual, so is \( A \cdot B \).

(viii) \( A^{stretch} \leq_L \emptyset \cdot ((0)^c \cap A) \).

The main result on multiplication of sets is the following
Theorem 22.36 (Steel). Assume BP + AD\(^L\) + DC(\(\mathbb{R}\)). Let A be self-dual. If \(\text{cof}(\|B\|_W) > \omega\) then
\[
\|A \cdot B\|_W = \|A\|_W \cdot \omega \cdot \|B\|_W.
\]

This result will be proved through a series of lemmata.

Lemma 22.37. Assume BP + AD\(^L\). For \(A, B, C \subseteq \mathbb{R}\) with A self-dual
\[
A \cdot B \leq_L A \cdot C \Rightarrow B \leq_L C
\]
and
\[
A \cdot B \leq_W A \cdot C \Rightarrow B \leq_W C.
\]

Proof. We shall prove the second implication only (i.e., the case of \(\leq_W\)) since the other one is similar. Fix a winning strategy \(\tau\) for II in \(G_W(A \cdot B, A \cdot C)\). We will define a winning strategy \(\tilde{\tau}\) for II in \(G_W(B, C)\) such that if \(b\) and \(c = b \ast \tilde{\tau}\) are the reals played by I and II in this game, then there are reals \(x\) and \(y = x \ast \tau\) played by I and II in \(G_W(A \cdot B, A \cdot C)\) so that \(b\) is the component of \(x\) on the B-row and \(c\) is the component of \(y\) on the C-row. Here is the definition of \(\tilde{\tau}\):

Suppose I plays \(b_0, b_1, \ldots\) in \(\hat{G} = G_W(B, C)\). These values are copied on the B-row of I in \(G = G_W(A \cdot B, A \cdot C)\). As long as \(\tau\) passes or plays \(c_0, c_1, \ldots\) in the C-row, the \(\hat{\tau}\) passes or copies \(c_0, c_1, \ldots\) in \(G_W(B, C)\). Suppose we reach a round \(n - 1\) when II first decides to play in the A-row: then \(\tau\) responds with 0 and \(a_1\) to \(b_n\) and \(b_{n+1}\)

\[
\hat{G}
\]
\[
\tilde{\tau} \text{ passes at round } n: \text{ this is denoted by the “—” in the diagram above. If } \{a_1\} \in T(A) \text{ then } \hat{\tau}\text{ passes at this round. Suppose I plays } b_{n+2}. If \tau\text{ announces}
\]
that he is going to play in the C-row, i.e., $\tau$ plays 0 then $\tilde{\tau}$ passes this turn and then copies $\tau$’s moves:

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>$b_0, \ldots, b_{n-1}, b_n, b_{n+1}, b_{n+2}, b_{n+3}, \ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(G)</td>
<td>$c_0, \ldots, c_{n'-1}, \ldots, c_{n'}$</td>
</tr>
</tbody>
</table>

If instead $\tau$ passes or plays $a_2$ in the A-row such that \(\langle a_1, a_2 \rangle \in T(A)\) then $\tilde{\tau}$ passes again. And so on. Since $A$ is self-dual $T(A)$ is well-founded, hence there is a least $i \geq 0$ such that for every $j < i - 1$, at round $n + j$ the strategy $\tau$ has either passed or has played $a_j$ in the A-row (with $j' \leq j$ since $\tau$ might have taken naps) such that \(\langle a_1, \ldots, a_{j'} \rangle \in T(A)\), but at round $n + i$ either

1. $\tau$ decides to play in the C-row, or else
2. $\tau$ plays $a_{j'}$ in the A-row such that $t = \langle a_1, \ldots, a_{j'} \rangle \notin T(A)$.

If (1) holds then $\tilde{\tau}$ first passes and then copies $\tau$.

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>$b_0, \ldots, b_{n-1}, b_n, b_{n+1}, b_{n+i}, b_{n+i+1}, \ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(G)</td>
<td>$c_0, \ldots, c_{n'-1}, \ldots, c_{n'}$</td>
</tr>
</tbody>
</table>

Otherwise (2) holds: since $A_{\{i\}} \prec_W A$, let $\sigma$ be a winning strategy for I in $G_W(A, A_{\{i\}})$ and let Player I use $\sigma$ to obtain values in his A-row of $G_W(A \cdot B, A \cdot C)$ by pitting $\sigma$ against $\tau$. (In the diagram
below the output of \( \sigma \) is \( \bar{a} = \langle \bar{a}_0, \bar{a}_1, \ldots \rangle \) and \( a_{\nu+1} \) is \( \tau \)'s response to \( I \) playing 0, i.e., having announced his entering the \( A \)-row.)

\[
\begin{array}{llll}
I & B\text{-row} & b_0, \ldots, b_{n-1}, b_n, b_{n+1}, \ldots, b_{n+i}, & a_0, \bar{a}_1, \ldots \\
& A\text{-row} & & \\
G & C\text{-row} & c_0, \ldots, c_{n'-1}, & a_1, \ldots, a_{\nu}, a_{\nu+1}, a_{\nu+2}, a_{\nu+3}, \ldots \\
& A\text{-row} & & \\
II & C\text{-row} & c_0, \ldots, c_{n'-1}, & a_1, \ldots, a_{\nu}, a_{\nu+1}, a_{\nu+2}, a_{\nu+3}, \ldots \\
& A\text{-row} & & \\
\end{array}
\]

As \( \sigma \) is winning, \( \tau \)'s response cannot always be on the \( A \)-row, since otherwise

\[
\tilde{a} \in A \iff \langle a_m \mid m > i \rangle \notin A_{[i]} \iff a \notin A
\]

which would imply that \( \Pi \) loses this run of \( G_W(A \cdot B, A \cdot C) \), contrarily to the assumption that \( \tau \) is winning. Therefore at some round \( \tau \) must decide to switch-back to the \( C \)-row; then \( \tilde{\tau} \) passes until then, and after that copies \( \tau \)'s moves on his \( C \)-row:

\[
\begin{array}{llll}
\tilde{\tau} & b_0, \ldots, b_{n-1}, b_n, b_{n+1}, \ldots, b_{n+i} & & \\
II & c_0, \ldots, c_{n'-1}, & - \ldots \ldots - & c_n' \\
& & & \\
\end{array}
\]

The argument can now be repeated.

By construction we have insured that \( \tilde{\tau} \) plays infinitely often so we end up with two reals \( b, c \) which are a play of \( G_W(B, C) \) and \( x, y \) which are a play of \( G_W(A \cdot B, A \cdot C) \). Since \( b \) and \( c \) are the components of \( x \) and \( y \) on the \( B \)-row of \( I \) and on the \( C \)-row of \( \Pi \), respectively, and since \( \tau \) is winning

\[
b \in B \iff x \in A \cdot B \iff y \in A \cdot C \iff c \in C,
\]

hence \( \tau \) is a winning strategy for \( \Pi \) in \( G_W(B, C) \).

\[\square\]
Exercise 22.38. Assume $BP + AD^L$. Show that if $A$ is self-dual and $B$ is non-self-dual then

(i) $A \cdot B$ is non-self-dual and $\neg(A \cdot B) \equiv_w A \cdot (\neg B)$, and

(ii) $A \cdot (B \oplus \neg B) \equiv_w A \cdot B + A$.

The heart of the proof of 22.36 is the fact that the map $B \mapsto A \cdot B$ is—in some sense—continuous when $A$ is self-dual.

Lemma 22.39. ($AD^L + DC(\mathbb{R})$) If $A$ is self-dual and $B$ non-self-dual, $B \neq \emptyset$, then

$$\|A \cdot B\|_w = \sup\{\|A \cdot C\|_w \mid C <_w B\}.$$  

Proof. $\|A \cdot B\|_w \geq \sup\{\|A \cdot C\|_w \mid C <_w B\}$ follows from 22.35 i ) so it is enough to prove by induction on $\|B\|_w$ the other inequality.

Suppose first $\|B\|_w$ is a successor ordinal. Pick $C$ self-dual such that $[C]_w$ is the immediate successor of $[B]_w$. Then either $B \equiv_w C + \emptyset$ or else $B \equiv_w C + \mathbb{R}$. If the former holds, using 22.35

$$A \cdot B \equiv_w A \cdot (C + \emptyset) \quad \text{(by (i))}$$

$$\equiv_w A \cdot C + A \cdot \emptyset \quad \text{(by (vi))}$$

$$\equiv_w A \cdot C + A^\emptyset. \quad \text{(by (ii))}$$

Analogously, if $B \equiv_w C + \mathbb{R}$ then $A \cdot B \equiv_w A \cdot C + A^\mathbb{R}$. In both cases, using 22.35(vii) and 22.16, $\|A \cdot B\|_w = \|A \cdot C\|_w + \|A\|_w \cdot \omega_1$. If $\|C'\|_L = \|C\|_L + \alpha$ for some $\alpha < \omega_1$, then 22.35 (iii) and (iv) imply that $\|A \cdot C'\|_w = \|A \cdot C\|_w + \|A\|_w \cdot \alpha$, and since $[C]_w$ is the union of $\omega_1$ Lipschitz degrees,

$$\sup\{\|A \cdot C'\|_w \mid C' \in [C]_w\} = \|A \cdot C\|_w + \|A\|_w \cdot \omega_1.$$  

Therefore $\sup\{\|A \cdot C'\|_w \mid C' <_w C\} = \|A \cdot B\|_w$.

Suppose now $\|B\|_w$ is limit. We distinguish three cases.

- $\exists \nu < \|B\|_w (\|B\|_w = \nu \cdot \omega_1)$.

  Then by changing $\nu$ to $\nu + 1$ if needed we may assume that there is a self-dual $C$ of Wadge rank $\nu$ so that $B \equiv_w C^\omega$ or $B \equiv_w C^\rho$. Then using 22.35 (vii) and (v) we see that $A \cdot C$ is self-dual hence, in either case,

$$\|A \cdot B\|_w = \|A \cdot C\|_w \cdot \omega_1.$$  

- $\exists \nu < \|B\|_w (\|B\|_w < \nu \cdot \omega_1)$.

  Fix $\nu$ as above and let $\xi < \omega_1$ be least such that $\|B\|_w \leq \nu \cdot \xi$. Since $B$ is non-self-dual then $\text{cof}(\|B\|_w) > \omega$ hence $\xi$ must be a successor, $\xi = \alpha + 1$. 


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Thus \( \nu \cdot \alpha + \beta \) with \( \beta \) limit, \( \text{cof}(\beta) > \omega \), and \( \beta < \nu < \|B\|_W \). Let \( C \) be self-dual of Wadge rank \( \nu \cdot \alpha \) or \( \nu \cdot \alpha + 1 \), and let \( D \) be such that \( \|D\|_W = \beta \). Then \( B \equiv_W C + D \) and \( A \cdot B \equiv_W A \cdot C + A \cdot D \). Since \( \|D\|_W < \|B\|_W \), by inductive hypothesis

\[
\|A \cdot B\|_W = \|A \cdot C\|_W + \|A \cdot D\|_W
\]

\[
= \|A \cdot C\|_W + \sup\{\|A \cdot D'\|_W \mid D' <_W D\}
\]

\[
= \sup\\{\|A \cdot (C + D')\|_W \mid D' <_W D\}
\]

\[
\leq \sup\\{\|A \cdot B'\|_W \mid B' <_W B\}.
\]

- \( \forall \nu < \|B\|_W \) \( (\nu \cdot \omega_1 < \|B\|_W) \).

Then \( \|B\|_W = \omega_1^x \) with \( \text{cof}(\xi) > \omega \), so by 22.30 and the remarks after its proof on page 137, \( B^{\text{stretch}} \equiv_W B \) and \( (\neg B)^{\text{stretch}} \equiv_W \neg B \). Let \( C <_W A \cdot B \).

We must show that \( C \leq_W A \cdot D \) for some \( D <_W B \). Since \( C <_W \neg(\neg B) \equiv_W A \cdot (\neg B) \) by 22.38, fix \( \tau_0, \tau_1 \) winning strategies for \( \text{II} \) in \( G_W(C, A \cdot B) \) and \( G_W(C, A \cdot (\neg B)) \), respectively. Let \( Z = \{ x \in \mathbb{R} \mid \exists \infty j \ ((x * \tau_0)_{II}(j) \text{ is entered in the } B\text{-row}) \& \exists \infty j \ ((x * \tau_1)_{II}(j) \text{ is entered in the } \neg B\text{-row}) \}$

**Exercise 22.40.** \( C \cap Z \leq_W B^{\text{stretch}} \) and \( C \cap Z \leq_W (\neg B)^{\text{stretch}} \).

Since \( (\neg B)^{\text{stretch}} \equiv_W \neg(\neg B) \equiv_W \neg B \), it follows that \( C \cap Z <_W B \), so it is enough to show that \( C \leq_W A \cdot (C \cap Z) \). We will define a winning strategy \( \tau \) for \( \text{II} \) in \( G_W(C, A \cdot (C \cap Z)) \) such that any play of \( \text{II} \) according to \( \tau \) is a product real, and if \( y = (c \ast \tau)_{II} \) then either

1. \( y \) was entered infinitely often on the \( C \cap Z \)-row, and \( c \) is the real recorded there, i.e., \( y = (s_0 + 1) \cdot (0) \cdot (s_1 + 1) \cdot (0) \cdot \cdots \) and \( c = s_0 \cdot s_2 \cdot \cdots \); or else

2. from some point on \( y \) was only entered on the \( A \)-row, i.e.,

\[
y = (s_0 + 1) \cdot (0) \cdot \cdots \cdot (0) \cdot (s_{2n} + 1) \cdot (a + 1)
\]

and for some \( i \in \{0, 1\} \), \( \exists m \leq n \ \exists t_0, \ldots, t_{2m} \in \omega \setminus \{\emptyset\} \) such that \( (c \ast \tau_i)_{II} = (t_0 + 1) \cdot (0) \cdot \cdots \cdot (0) \cdot (t_{2m} + 1) \cdot (a + 1) \).

Here is the definition of \( \tau \):

Let \( I \) play the real \( c \) in the three games

\[
G = G_W(C, A \cdot (C \cap Z)),
\]

\[
G_0 = G_W(C, A \cdot B),
\]

\[
G_1 = G_W(C, A \cdot (\neg B)).
\]
As long as \( \tau_0 \) and \( \tau_1 \) pass or enter elements in the \( B \)-row and the \( \neg B \)-row, respectively, when presented with \( c_0, c_1, \ldots \) then \( \tau \) simply copies them on the \( C \cap Z \)-row. If this happens forever then \( c \in Z \) so \( c \in C \iff c \in C \cap Z \). So suppose \( n \) is least such that, say, \( \tau_0 \) responds to \( c_n \) with entering the \( A \)-row. We temporarily forget about \( \tau_1 \) and \( G_1 \) and concentrate on \( \tau_0 \) and \( G_0 \). As long as \( \tau_0 \) keeps on playing on the \( A \)-row of \( G_0 \) then \( \tau \) simply copies these values in the \( A \)-row of \( G \):

\[
\begin{array}{c}
\text{I} \\
G_0 \\
\text{II} \\
\begin{array}{c}
B \text{-row} \\
A \text{-row}
\end{array}
\end{array}
\begin{array}{c}
e_0, \ldots, e_{n-1}, \quad e_n, e_{n+1}, \ldots \\
0, 0, 1, \ldots
\end{array}
\]

(As usual \( n' < n \) can happen since \( \tau_0 \) may pass.)

Suppose \( \tau_0 \) stays in the \( A \)-row forever. If \( \langle b_0, \ldots, b_{n'-1} \rangle \neq \emptyset \) then

\[
c \in C \iff a^0 \in A \quad \text{(since } \tau_0 \text{ is winning)}
\]

\[
\iff ((c \upharpoonright n) + 1)^\wedge \langle 0 \rangle \wedge (a^0 + 1) \in A \cdot (C \cap Z).
\]

If \( \langle b_0, \ldots, b_{n'-1} \rangle = \emptyset \) then \( (c * \tau_0)_B = \text{II}'s \) play in \( G_0 \) is not a product real, hence it does not belong to \( A \cdot B \), hence \( c \notin C \) and \( c \notin C \cap Z \). Therefore in either case \text{II} wins \( G \).

Suppose instead \( \tau_0 \) comes back to the \( B \)-row at some round \( m > n \). Let \( s_0 = c \upharpoonright n \) and \( s_1 = \langle a^0_0, \ldots \rangle \) be the values entered by \( \tau_0 \) in the \( A \)-row between rounds \( n \) and \( m - 1 \). First \( \tau \) goes back to the \( C \cap Z \) row and there he plays \( c_n \)

\[
\begin{array}{c}
\text{I} \\
G \\
\text{II} \\
\begin{array}{c}
C \cap Z \text{-row} \\
A \text{-row}
\end{array}
\end{array}
\begin{array}{c}
s_0 \\
0 \\
c_n, c_{n+1}, \ldots, c_m, \quad c_{m+1} \\
s_1
\end{array}
\]

and now he turns his attention to \( \tau_1 \).

If at round \( m + 1 \) the strategy \( \tau_1 \) is acting (i.e., playing or passing) in the \( A \)-row, then we set \( s_2 = \langle c_n \rangle \) and have \( \tau \) enter the \( A \)-row
of $\mathcal{G}$ and copy $\tau_1$'s values from round $m + 1$ on:

\[
\begin{array}{c}
\mathcal{G}_1 \\
G & c_0, \ldots, c_m, c_{m+1}, c_{m+2}, \ldots \\
\neg B\text{-row} & b'_0, \ldots \\
A\text{-row} & a_0^1, a_1^1, \ldots \\
\end{array}
\]

\[
\begin{array}{c}
\mathcal{G} \\
G & c_0, \ldots, c_m, c_{m+1}, c_{m+2}, \ldots \\
C \cap Z\text{-row} & s_0, s_2 \\
A\text{-row} & s_1, a_0^1, a_1^1, \ldots \\
\end{array}
\]

where $s_3 = \langle a_0^1, \ldots \rangle$ are the values played by $\tau_1$ after round $m + 1$. Thus we now turn the attention again to $\tau_0$ and repeat the argument.

If instead $\tau_1$ at round $m + 1$ was already in the $\neg B$-row then we skip the paragraph above and focus the attention on $\tau_0$ right away.

If both $\tau_0$ and $\tau_1$ are in the $B$-row and $\neg B$-row, then we are in the same situation as in the beginning of the game, so the arguments above can be repeated.

And so on: we keep dove-tailing $\tau_0$ and $\tau_1$ unless one of the two settles on its $A$-row.

It is immediate to check that $\tau$ is winning for II in $G_W(C, A \cdot (C \cap Z))$.  \(\square\)

We are now ready to show that if $A$ is non-self-dual then $\text{cof}(\|B\|_W) > \omega \Rightarrow \|A \cdot B\|_W = \|A\|_W \cdot \omega_1 \cdot \|B\|_W$.

**Proof.** 22.36 By induction on $\|B\|_W$.

If $\|B\|_W = \omega_1$ then either $B \equiv_W \mathbb{R}^2$ or $B \equiv_W \emptyset$. Assume the former. Then keeping in mind that $\|A^{\mathbb{R}^2}\|_W = \|A\|_W \cdot \omega_1 \cdot \omega_1$ and using 22.35

\[
\|A \cdot B\|_W = \|A \cdot \mathbb{R}^2\|_W = \|(A \cdot \mathbb{R})^2\|_W = \|A^{\mathbb{R}^2}\|_W = \|A\|_W \cdot \omega_1 \cdot \|B\|_W .
\]

Similarly if $B \equiv_W \emptyset$.  \(\square\)
So we may assume that \( \|B\|_W > \omega_1 \). By Cantor’s normal form theorem
\[
\|B\|_W = \omega_{\xi_0} \cdot \eta_0 + \cdots + \omega_{\xi_k} \cdot \eta_k
\]
with \( \xi_0 > \cdots > \xi_k \) and \( 0 < \eta_0, \ldots, \eta_k < \omega_1 \).

Suppose first \( k > 0 \). Then pick \( C \) of least possible Lipschitz rank and such that \( \|C\|_W = \omega_{\xi_0} \cdot \eta_0 \). We distinguish two cases:

**C is self-dual.** Then by 22.10 there is a \( D \) such that \( C + D \equiv_W B \) and \( \|D\|_W = \omega_{\xi_1} \cdot \eta_1 + \cdots + \omega_{\xi_k} \cdot \eta_k \) is of cofinality > \( \omega \). By inductive hypothesis \( \|A \cdot D\|_W = \|A\|_W \cdot \omega_1 \cdot \|D\|_W \). Fix \( \langle \nu_n \mid n < \omega \rangle \) increasing and converging to \( \omega_{\xi_0} \cdot \eta_0 \) such that \( \text{cof}(\nu_n) > \omega_1 \), and let \( \|C_n\|_W = \nu_n \). Then by inductive hypothesis \( \|A \cdot C_n\|_W = \|A\|_W \cdot \omega_1 \cdot \|C_n\|_W \), and \( \bigoplus_n C_n \equiv_L C \). Thus by 22.35
\[
\|A \cdot C\|_W = \left\| A \cdot \bigoplus_n C_n \right\|_W \quad \text{(by (i))}
\]
\[
= \left\| \bigoplus_n (A \cdot C_n + A) \right\|_W \quad \text{(by (iv))}
\]
\[
= \sup \{ \|A \cdot C_n + A\|_W \mid n < \omega \}.
\]
Since \( A \cdot C_n \) is non-self-dual we cannot apply 22.11 and we only have
\[
\|A\|_W \cdot \omega_1 \cdot \|C_n\|_W \leq \|A \cdot C_n + A\|_W
\]
\[
\leq \|A \cdot C_n\|_W + \|A\|_W
\]
\[
= \|A\|_W \cdot \omega_1 \cdot \|C_n\|_W + \|A\|_W
\]
\[
< \|A\|_W \cdot \omega_1 \cdot \|C_{n+1}\|_W
\]
hence
\[
\sup \{ \|A \cdot C_n + A\|_W \mid n < \omega \} = \sup \{ \|A\|_W \cdot \omega_1 \cdot \|C_n\|_W \mid n < \omega \}
\]
\[
= \|A\|_W \cdot \omega_1 \cdot \|C\|_W ,
\]
and therefore
\[
\|A \cdot B\|_W = \|A \cdot (C + D)\|_W \quad \text{(by (i))}
\]
\[
= \|A \cdot C\|_W + \|A \cdot D\|_W \quad \text{(by (vi))}
\]
\[
= \|A\|_W \cdot \omega_1 \cdot \|C\|_W + \|A\|_W \cdot \omega_1 \cdot \|D\|_W
\]
\[
= \|A\|_W \cdot \omega_1 \cdot \|B\|_W.
\]
Corollary 22.41. Assume BP + AD^L + DC(R).

(a) If \( \|A\|_W = \omega_1 \cdot \|A\|_W \) is of uncountable cofinality, then \( A^{\text{stretch}} \equiv W A \).

(b) If \( \|A\|_W = \omega_1^\delta \cdot \alpha \) and \( B <_W A \) then \( B^{\text{stretch}} <_W A \).
Proof. (a) Let $D$ be clopen and non-trivial, i.e., $D \neq \emptyset, \mathbb{R}$. By 22.35

$$A^\text{stretch} \leq_W \emptyset \cdot ((0)^\cdot A) \equiv_W \emptyset \cdot A + \emptyset \leq_W D \cdot A + \emptyset.$$ 

Then by 22.36, $\|D \cdot A + \emptyset\|_W \leq \|A\|_W + 1$ and since $A^\text{stretch}$ is non-self-dual (22.27) then either $A^\text{stretch} \leq_W A$ or else $A^\text{stretch} \leq_W A^\text{stretch}$. The latter is impossible, since $A \leq_W A^\text{stretch}$, hence $A \equiv_W A^\text{stretch}$.

(b) If $\alpha$ is limit, then $\|B\|_W < \omega_1^\cdot \gamma$ for some $\gamma < \alpha$. Let $C$ be of least Lipschitz degree and such that $B <_W C <_W A$ and $\|C\|_W = \omega_1^\cdot \gamma$. If $\text{cof}(\gamma) > \omega$ then by part (a) $B^\text{stretch} \leq_W C^\text{stretch} \equiv_W C <_W A$, so we may assume $\text{cof}(\gamma) = \omega$. Let $\langle \nu_n \mid n \in \omega \rangle$ be increasing and cofinal in $\omega_1^\cdot \gamma$, and such that $\text{cof}(\nu_n) > \omega$. Choose $C_n$ such that $\|C_n\|_W = \nu_n$. Then $\bigoplus_n C_n \equiv_C C$ and, as before,

$$B^\text{stretch} \leq_W C^\text{stretch} \leq_W D \cdot C + \emptyset \equiv_W \bigoplus_n (D \cdot C_n + D) + \emptyset$$

where $D \in \Delta^0_1 \setminus \{\emptyset, \mathbb{R}\}$. Arguing as in the proof of 22.36

$$\left\| \bigoplus_n (D \cdot C_n + D) \right\|_W = \sup\{\|D \cdot C_n\|_W \mid n \in \omega\} = \sup\{\omega_1 \cdot \nu_n \mid n \in \omega\} < \omega_1^\cdot \alpha$$

and therefore $B^\text{stretch} <_W A$. 

22.G Wadge Determinacy and the Semi-Linear-Ordering Principle

We are now going to prove the equivalence between $\text{AD}^W$ and $\text{AD}^L$.

**Theorem 22.42.** $\text{BP} + \text{DC}(\mathbb{R}) + \text{AD}^L \Rightarrow \text{AD}^W$.

**Proof.** We will show that

(a) $\neg A \equiv_W B$ and $A$ is non-self-dual, or

(b) $B <_W \neg A$.

then $I$ wins $G_W(A, B)$. This implies that every game $G_W(A, B)$ is determined: if $A \leq_W B$ then $II$ wins and if $A \not<_W B$ then by SLO$^W$ (which follows from $\text{AD}^L$) either (a) or (b) holds, hence $I$ wins. The result will be proved by induction on $(\|A\|_W, B\|_W)$ using Gödel’s well-ordering of $\text{Ord} \times \text{Ord}$

$$(\alpha, \beta) < (\gamma, \delta) \iff \max(\alpha, \beta) < \max(\gamma, \delta) \lor$$

$$\left( \max(\alpha, \beta) = \max(\gamma, \delta) \& (\alpha, \beta) <_{\text{lex}} (\gamma, \delta) \right)$$
• Suppose (a) holds.

We will show that $I$ wins $G_W(A, B)$. Let’s agree on the following notation: If $q$ is a sequence of non-zero integers and $p$’s let $q - 1$ be the sequence obtained from $q$ by subtracting 1 whenever possible, i.e., for all $i < \text{lh}(q)$

$$(q - 1)(i) = \begin{cases} q(i) - 1 & \text{if } q(i) \in \omega, \\ p & \text{if } q(i) = p. \end{cases}$$

There are four cases:

1. $[A]_W$ is a successor degree.

Then either $[T(A)] \subseteq A$ or else $[T(A)] \cap A = \emptyset$. For the sake of definitiveness suppose the former, so that $[T(B)] \cap B = \emptyset$. Let $I$ enumerate some fixed real $a \in [T(A)]$ as long as $\Pi$ plays inside $T(B)$ or passes. If $\Pi$ never plays outside $T(B)$ then he loses so let’s assume that at some round we have positions $p \in T(A)$ for $I$ and $q \in \partial T(B)$ for $\Pi$. Then $B_{1[1]} B \equiv_W A$ and $A_{[1]} \equiv_W A$ so $B_{1[1]} <_W A_{[1]}$. By inductive hypothesis $I$ wins $G_W(A_{[1]}, B_{[1]})$.

2. $[A]_W$ is a limit degree and $\|A\|_W = \alpha \cdot \omega_1$, with $\alpha < \|A\|_W$.

Pick a self-dual $C <_W A$ such that $\{[C^\alpha]_W, [C^b]_W\} = \{[A]_W, [B]_W\}$, for the sake of definitiveness say $C^\alpha \equiv_W A$ and $C^b \equiv_W B$. Since by $??(c)$ there are Lipschitz reductions $C^\alpha \leq_L A$ and $C^b \leq_L B$, any winning strategy for $I$ in $G_W(C^\alpha, C^b)$ induces a winning strategy for $I$ in $G_W(A, B)$. Since $C <_W \neg C^b$ by 22.16, then $(\|C^\alpha\|_W, \|C\|_W) < (\|C^\alpha\|_W, \|C^b\|_W)$ hence by inductive hypothesis there is a winning strategy $\sigma$ for $I$ in $G_W(C^\alpha, \neg C^b)$. Then $I$ wins $G_W(C^\alpha, C^b)$ as follows:

At round 0 $I$ plays $\sigma(\emptyset) + 1$.

If at any round $n \Pi$ plays 0 then $I$ plays 0 at round $n + 1$.

If after having played a 0 $\Pi$ plays non-zero integers or passes then $I$ applies $\sigma$ to $\Pi$’s inputs $-1$ as if the game had just restarted after the 0. In other words if $\Pi$’s true play is $q_0 \cdot (0) \cdot q_1 \cdot (0) \cdot \ldots$ with $q_i \in \omega(\{p\} \cup \omega \setminus \{0\})$ then

$I$ \hspace{1cm} $\sigma(\emptyset)$ \hspace{0.5cm} $p_0$ \hspace{0.5cm} 0 \hspace{0.5cm} $p_1$ \hspace{0.5cm} 0 \hspace{0.5cm} \ldots$

$\Pi$ \hspace{1cm} $q_0$ \hspace{0.5cm} 0 \hspace{0.5cm} $q_1$ \hspace{0.5cm} 0

where

$$p_i = \begin{cases} \sigma \ast (q_i - 1) & \text{if } q_i \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$
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If \( \Pi \) plays 0 infinitely often then so does \( \Gamma \), and \( \Gamma \) wins by definition of \( C^g \) and \( C^h \). Thus we may assume that the play of \( \Gamma \) is of the form \( s \cdot (0)^{\omega} \cdot (x + 1) \) and that the true play of \( \Pi \) is of the form \( t \cdot (0)^{\omega} \cdot z \), with \( s \in \omega, t \in \omega(\{p\} \cup \omega), \, lh(s) = lh(t), \, x \in \mathbb{R}, \, z \in \omega(\{p\} \cup \omega \setminus \{0\}) \), and \( x + 1 = \sigma \cdot (z - 1) \). If \( z \) is definitively = \( p \) then \( \Pi \) loses so we may assume otherwise. Let \( y + 1 \) be the real obtained from \( z \) by dropping all \( p \)'s and, similarly, let \( \ell \) be the sequence obtained from \( t \) by deleting \( p \). Then

\[
s \cdot (0)^{\omega} \cdot (x + 1) \in C^g \iff x + 1 \in C^g \iff y \notin C \iff \ell \cdot (0) \cdot (y + 1) \in C^h.
\]

Therefore \( \Gamma \) wins \( G_W(C^g, C^h) \).

(3) \( [A]_W \) is a limit degree and \( \|A\|_W = \alpha + \omega_1 \) and (2) does not hold.

Choose \( A', B', C \) with \( C \) is non-self-dual, \( A' \equiv_L \neg B', A \equiv_L C + A', B' \equiv_L C + B', \) and \( \omega_1 = \|A'\|_W = \|B'\|_W < \|C\|_W < \|A\|_W \). By (2) and by inductive hypothesis \( \Gamma \) has winning strategies \( \sigma \) and \( \tau \) for \( G_W(A', B') \) and \( G_W(C, \neg C) \) respectively. It is enough to show that \( \Gamma \) wins \( G_W(C + A', C + B') \).

As long as \( \Pi \) does not play 0 \( \Gamma \) can use \( \sigma \) against \( q - 1 \), where \( q \) is the true position of player \( \Pi \). If at some round \( n \) \( \Pi \) plays 0 then at round \( n + 1 \) \( \Gamma \) plays 0 and then switches to \( \tau \).

It is immediate to check that this is a winning strategy for \( \Gamma \) in \( G_W(C + A', C + B') \).

(4) \( [A]_W \) is a limit degree and (2) and (3) fail.

Hence \( \|A\|_W = \omega^{\omega} \cdot \alpha \) with \( \text{cof}(\alpha) > \omega \). Then by 22.41 \( B \equiv_L B^{\text{stretch}} \notin_L A \) and we are done by 22.29.

\( \bullet \) Suppose now (b) holds. Again we must examine various cases.

(5) \( [A]_W \) is self-dual.

Then \( \Gamma \) plays along \( T(A) \) until at some point he hits a terminal node \( p \in \text{tn}(T(A)) \). Then he can play \( i \) such that \( B \leq_W \neg A_{[p \cdot (i)]} \). Let \( q \) be \( \Pi \)'s position at this round. (Note that \( q = \emptyset \) is possible, since he can pass.) In any case \( B_{[q]} \leq_W B \), so by inductive hypothesis \( \Gamma \) wins \( G_W(A_{[p \cdot (i)]}, B_{[q]}) \), and hence \( \Gamma \) wins \( G_W(A, B) \).

(6) \( [A]_W \) is non-self-dual and \( [B]_W \) is self-dual.

Then \( \Gamma \) plays in \( T(A) \) as long as \( \Pi \) plays in \( T(B) \). Since \( T(B) \) is well-founded and \( \Pi \) cannot pass forever there is a stage after which the
positions are $p \in T(A)$ and $q \in \partial T(B)$. Since $B_{[q]} <_W B$ then, by inductive hypothesis, there is a winning strategy $\sigma$ in $G_W(A_{[p]}, B_{[q]})$, hence $I$ can follow $\sigma$ from now on.

Therefore we may assume

$A$ and $B$ are non-self-dual.

(7) $[A]_W$ is a successor degree.

Then either $A \equiv_L C^{\overline{\nu}}$ or else $A \equiv_L C^\emptyset$ where $C$ is self-dual and $\|C\|_W + 1 = \|A\|_W$, say $A \equiv_L C^{\overline{\nu}}$. Then it is enough to show that $I$ wins $G_W(C^{\overline{\nu}}, B)$: $I$ plays 1 (or any non-zero integer, for that matter) so that after the first round the two Players are facing the sets $C_{[q]} = C$ and $B_{[q]}$ where $lh(q) \leq 1$. In any case, since $B \leq_W \neg C$, then $B_{[q]} \leq_W \neg C$ so by inductive hypothesis $I$ wins.

(8) $[A]_W$ is a limit degree.

Let $C$ be non-self-dual and such that $\|C\|_W = \|B\|_W + 2$. By (7) $I$ wins $G_W(C, B)$ via some $\sigma$, and since $C \leq_L A$ $\sigma$’s moves can be copied, obtaining a winning strategy of $I$ in $G_W(A, B)$.

We now turn to the other implications, that is $AD^W \Rightarrow AD^L$ and $SLO^L \Rightarrow AD^L$. These are somewhat trickier than 22.42 since the theory in the previous sections was developed under the assumption of $AD^L$, and hence many of the basic constructions are not immediately available. For example, special care must be paid when dealing with (non)-self-dual sets, since the equivalence of the Lipschitz and Wadge version depends on $??$ which used $AD^L$ in an essential way. We will develop the theory as much as we can under the weakest assumption, namely $SLO^W$.

Since the proof of 22.3 used only that $T(A)$ is well-founded iff $[A]_W$ is self-dual, then $SLO^W$ implies that immediately above each self-dual $[A]_W$ there is a non-self-dual pair $([A^\emptyset]_W, [A^{\overline{\nu}}]_W)$.

**Lemma 22.43.** Assume $BP + DC(\mathbb{R})$ and either $SLO^L$ or $AD^W$.

(a) $\|A\|_W < \|B\|_W \Rightarrow II$ wins $G_L(A, B)$ and $\|A\|_W > \|B\|_W \Rightarrow I$ wins $G_L(A, B)$.

(b) If $A \equiv_W B$ are non-self-dual then $A \equiv_L B$. 
Proof. (a) Assume first SLO$^L$. If $||A||_W < ||B||_W$ then $\neg B \not\leq L A$ so, by SLO$^L$, $A \leq L B$ hence II wins $G_L(A,B)$. Vice versa, if $||A||_W > ||B||_W$ then $(0)^{\neg B} \equiv W B < W \equiv L A$ so by 21.14 I wins $G_L(A,B)$. The argument under AD$^W$ is simpler, since a winning strategy for I in $G_W$ is also winning in the Lipschitz game.

(b) Since $A \not\leq W \neg B$ and $B \not\leq W \neg A$, then $A \not\leq L \neg B$ and $B \not\leq L \neg A$. Hence $A \equiv L B$ by SLO$^L$. The simpler case under AD$^W$ is left to the reader.

We can now re-prove the Steel–Van Wesep theorem.

**Theorem 22.44** (same as ??). Assume BP + DC($\mathbb{R}$) and either SLO$^L$ or AD$^W$. Then

\[ A \leq W \neg A \Rightarrow A \leq L \neg A. \]

Proof. By induction on $||A||_W \cdot ||T(A)||$ let us show that II wins $G_L(A,\neg A)$. As long as I plays inside $T(A)$ then II copies I's moves. When I reaches a position $p^{\neg \langle n \rangle} \in \partial T(A)$ (and this is bound to happen since $T(A)$ is well-founded) then $A[p^{\neg \langle n \rangle}] < W A \equiv W A[p]$ hence by 22.43(a) I wins $G_L(A[p], A[p^{\neg \langle n \rangle}])$ via some $\sigma$. Then II can use $\sigma$ to answer I's moves in $G_L(A,\neg A)$. \[ \square \]

Since the proofs of 22.3 and 22.4 used only that $T(A)$ is well-founded when $A \equiv W \neg A$, we can conclude, under BP + DC($\mathbb{R}$) and either SLO$^L$ or AD$^W$, that for $A$ non-self-dual

- $[A]_W$ is a successor degree iff either $[T(A)] \subseteq A$ or else $[T(A)] \cap A = \emptyset$, but not both; and
- $[A]_W$ is a limit degree iff $[T(A)] \cap A \neq \emptyset$ and $[T(A)] \cap \neg A \neq \emptyset$.

Finally:

**Theorem 22.45.** Assume BP + DC($\mathbb{R}$) and either SLO$^L$ or AD$^W$. Then AD$^L$ holds.

Proof. By induction on $(||A||_W, ||B||_W)$ we show that if $[A]_W = [B]_W$ is self-dual then

\[
||T(A)|| \leq ||T(B)|| \Rightarrow \text{II wins } G_L(A,B) \\
||T(A)|| > ||T(B)|| \Rightarrow \text{I wins } G_L(A,B).
\]

Since 22.43(a) takes care of the cases when $||A||_W \neq ||B||_W$, and 22.43(b) takes care of the cases when $A \equiv W \neg B$ are non-self-dual, this will prove the theorem.

Let $\alpha = ||T(A)||$ and $\beta = ||T(B)||$. If $\alpha \leq \beta$ then II wins $G_L(A,B)$ as follows.
As long as I plays so that his position \( p \) is inside \( T(A) \) then II plays so that his position \( q \) is inside \( T(B) \) and \( \| p \|_{T(A)} \leq \| q \|_{T(B)} \). Suppose at some round I plays outside \( T(A) \) (and this is going to happen sooner or later, since \( T(A) \) is well-founded), say I plays \( i \) so that \( p^i \langle i \rangle \in \partial T(A) \). Let \( \langle p \rangle \) be \( \Pi \)'s position before starting this round, i.e., \( \text{lh}(p) = \text{lh}(q) \) and \( q \in T(B) \). If \( \| q \|_{T(B)} > \| p \|_{T(A)} = 0 \), then II can play \( j \) such that \( q^j \langle j \rangle \in T(B) \). Then \( A_{\langle p \rangle} \prec_w B_{\langle q \rangle} \), so \( (\| A_{\langle p \rangle} \|_W, \| B_{\langle q \rangle} \|_W) < (\| A \|_W, \| B \|_W) \), hence we are done by inductive hypothesis.

If instead \( \| q \|_{T(B)} = \| p \|_{T(A)} = 0 \) we distinguish two cases.

- \([A]_W\) is a limit degree. Then \([A]_W\) is a join degree and since \( A_{\langle p \rangle} = \bigoplus_n A_{\langle p^{-n} \rangle} \) and \( B_{\langle q \rangle} = \bigoplus_n B_{\langle q^{-n} \rangle} \), then II can choose \( j \) such that \( A_{\langle p \rangle} \leq_w B_{\langle q \rangle} \). Since \( (\| A_{\langle p \rangle} \|_W, \| B_{\langle q \rangle} \|_W) < (\| A \|_W, \| B \|_W) \) again we are done by inductive hypothesis.

- \([A]_W\) is a successor degree. Then \( A \equiv_w C \oplus \neg C \) for some non-self-dual \( C \). If \( A_{\langle p \rangle} \leq_w C \), since \( B_{\langle q \rangle} \equiv_w C \oplus \neg C \) then II can choose \( j \) such that \( B_{\langle q \rangle} \equiv_w C \), and similarly if \( A_{\langle p \rangle} \leq_w \neg C \). Again we are done by inductive hypothesis.

Suppose now \( \alpha > \beta \). Then I plays \( a_0 \) such that \( \langle a_0 \rangle \in T(A) \) and \( \| a_0 \| \geq \beta \). Then both players can switch roles and since \( T(\neg(A_{\langle a_0 \rangle})) = T(A_{\langle a_0 \rangle}) \) has rank \( \geq \beta \) we can apply the argument above and conclude that II wins \( G_L(B, \neg(A_{\langle a_0 \rangle})) \), hence I wins \( G_L(A, B) \). \( \square \)

Additional Exercises

Exercise 22.46. Assume BP + AD^L and let \( A \subseteq \mathbb{R} \) be non-self-dual.

(i) Suppose \( T \) is an ill-founded tree on \( \omega \) such that \( \prec_{\omega} \omega \setminus T \) is dense, i.e.,

\[
\forall s \in \prec \omega \exists t \geq s (t \notin T).
\]

Show that \( S(A; T) \equiv_w A^\gamma \) and \( S(A; T) \equiv_w A^\gamma \cup [T(A)] \equiv_w A^\circ \).

(ii) Show that

\[
B \in [A^\gamma]_W \Rightarrow [T(B)] \cap B = \emptyset,
\]

\[
B \in [A^\circ]_W \Rightarrow [T(B)] \subseteq B.
\]
Exercise 22.47. For \( \Sigma \) and \( \Lambda \) pointclasses let
\[
\bigvee\limits_\Lambda (\Gamma) = \{ \bigcup_n (U_n \cap A_n) \mid \langle U_n \mid n < \omega \rangle \in \omega \Lambda \land \langle A_n \mid n < \omega \rangle \in \omega \Gamma \land \forall n < m (U_n \cap U_m = \emptyset) \}
\]
and let
\[
\bigwedge\limits_\Lambda (\Gamma) = (\bigvee\limits_\Lambda (\Gamma))^\complement.
\]
Similarly
\[
\bigcup\limits_\Lambda (\Gamma) = \{ \bigcup_n (U_n \cap A_n) \mid \langle U_n \mid n < \omega \rangle \in \omega \Lambda \land \langle A_n \mid n < \omega \rangle \in \omega \Gamma \land \forall n < m (U_n \cap U_m = \emptyset) \land \bigcup_n U_n = \mathbb{R} \}.
\]
Thus, for example, the elements of \( \bigvee_{\Sigma_0^0}(\Gamma) \) are countable disjoint unions of sets in \( \Gamma \) each of which is contained in an open set so that these open sets are pairwise disjoint. If the \( U_n \)'s form a partition, i.e., \( \bigcup_n U_n = \mathbb{R} \), then they must be clopen so we simply obtain the pointclass \( \bigcup_{\Lambda_0}(\Gamma) = \bigcup_{\Sigma_0}(\Gamma) \) described in ??.

(i) Show that if \( \Sigma \) and \( \Lambda \) are boldface pointclasses then \( \bigvee_{\Lambda}(\Gamma) \) is a boldface pointclass.

(ii) Show that for any \( A, B \subseteq \mathbb{R} \)
\[
B \leq_w A^\complement \iff B \in \bigvee_{\Sigma_1^0}\langle \{ C \subseteq \mathbb{R} \mid C \leq_w A \} \rangle
\]
and
\[
B \leq_w A^\complement \iff B \in \bigwedge_{\Sigma_1^0}\langle \{ C \subseteq \mathbb{R} \mid C \leq_w A \} \rangle
\]

Fix \( A \subseteq \mathbb{R} \).

(iii) Let \( F_n = s_n \setminus (0, \infty) \), and let \( A_n = \{ u \mid \text{lh}(u) = \text{lh}(s_n) + 1 \land s_n \setminus (0, \infty) \subseteq \langle x \mid x < n \rangle \in A \} \), where \( n \setminus m = n - m \) if \( n \geq m \) and \( = 0 \) otherwise, and \( \langle s_n \mid n < \omega \rangle \) was defined on page ??.
Show that \( A_n \leq_w A \) and that \( A_n \cap F_n = A^2 \cap F_n \), for any \( n \).

(iv) Show that \( \{ B \mid B \leq_w A^2 \} \subseteq \bigvee_{\Pi_1^0}\langle \{ B \mid B \leq_w A \} \rangle \subseteq \bigvee_{\Sigma_1^0}\langle \{ B \mid B \leq_w A \} \rangle \).

(v)
(vi) Conclude that for any $A \subseteq \mathbb{R}$

$$\{ B \mid B \leq W A^2 \} = \bigvee_{\Pi_1^0} (\{ B \mid B \leq W A \}) = \bigvee_{\Sigma_2^0} (\{ B \mid B \leq W A \})$$

and

$$\{ B \mid B \leq W A^p \} = \bigwedge_{\Pi_1^0} (\{ B \mid B \leq W A \}).$$

**Exercise 22.48.** Show that

(i) $1 \leq \alpha \leq \beta < \omega_1 \Rightarrow [A]_W \cdot \alpha \leq [A]_W \cdot \beta$, and if $A \neq \mathbb{R}, \emptyset$ then

$$\alpha < \beta \Rightarrow [A]_W \cdot \alpha < [A]_W \cdot \beta.$$

(ii) $[A]_W \cdot (\sup_{n} \alpha_n) = \sup_{n} [A]_W \cdot \alpha_n$.

(iii) $[A]_W \cdot (\alpha + \beta) = [A]_W \cdot \alpha + [A]_W \cdot \beta$.

(iv) $[A]_W \cdot (\alpha \cdot \beta) = ([A]_W \cdot \alpha) \cdot \beta$.

**Exercise 22.49.** We now describe a few Wadge-style games (like $G'_{bt}$) that model the relations $A \leq W B + C$, $A \leq W B \cdot n$, $A \leq W B^2$, etc. In all of them I and II alternatively play natural numbers $x(0), x(1), \ldots$ and $y(0), y(1), \ldots$, and, just like in $G_W$, II has the option of passing at any round, with the proviso that if he does not play infinitely often then he loses by default. Player II will also have the opportunity at any round to erase his moves and start his play all over again.

- In $G_{+}(A, B, C)$ Player II can take back his moves at any round, but can do so at most once. If II never took back his moves then he wins iff

$$x \in A \Leftrightarrow y \in B.$$ 

Suppose instead II erased his moves at round $n$, that is first he plays $y(0), \ldots, y(n-1)$ and then he plays $y'(0), y'(1), \ldots$; then II wins iff

$$x \in A \Leftrightarrow y' \in C.$$ 

- In $G_{\cdot n}(A, B)$ Player II can erase his board at most $n$ times, so that a typical play looks like $(s_0, \ldots, s_k, y)$ where $k < n$ and $s_0, \ldots, s_k \in ^{<\omega} \omega$ are his initial plays before he changed his mind. II wins iff

$$x \in A \Leftrightarrow y \in B.$$
22. OPERATIONS ON THE DEGREES

• $G_\omega(A, B)$ is similar, but $\text{II}$ must play some $n \in \omega$ together with his first (real, non-passing) move, and then he can take-back his moves at most $n$ times. Therefore a play of $\text{II}$ looks like $(n, s_0, \ldots, s_k, y)$ where $k < n$, $s_0, \ldots, s_k \in \omega^\omega$ and $y \in \mathbb{R}$. The winning condition is as before.

• In $G_\omega(A, B)$ $\text{II}$ can take-back his moves as many times as he wants—even infinitely often. If $\text{II}$ erases the board $n$ many times, i.e., his play is $(s_0, \ldots, s_{n-1}, y)$ then $\text{II}$ wins iff

$$x \in A \Leftrightarrow y \in B.$$  

If instead he erases his back infinitely often, then $\text{II}$ wins iff

$$x \notin A.$$  

• $G_\omega(A, B)$ is similar, but when $\text{II}$ takes back his moves infinitely often then he wins iff

$$x \in A.$$  

Show that

$\text{II}$ wins $G_+(A, B, C) \iff A \leq_W B + C$,  
$\text{II}$ wins $G_n(A, B) \iff A \leq_W B \cdot n$,  
$\text{II}$ wins $G_\omega(A, B) \iff A \leq_W B : \omega$,  
$\text{II}$ wins $G_\delta(A, B) \iff A \leq_W B^\delta$,  
$\text{II}$ wins $G_\delta(A, B) \iff A \leq_W B^\delta$.  

The following exercise asks to develop a division algorithm for degrees below $A^\delta$.

Exercise 22.50. Assume $\text{AD}^\delta$. Show that if $A$ is self-dual and $B <_W A^\delta$ then exactly one of the following holds:

1. $[B]_W < [A]_W$,  
2. $[B]_W = [A]_W \cdot \alpha$, for some $1 \leq \alpha < \omega_1$,  
3. $[B]_W = [A]_W \cdot \alpha + [C]_W$, for some $1 \leq \alpha < \omega_1$ and some $C <_W A$.

Notes and References

Everything in §§A–D and exercises 22.46–22.50 are from [Wad83]. The material in §§E–G is from the union of [Ste77] and [VW77]. The material in §22.G is, to the best of our knowledge, due to the author.
23 Borel-Wadge degrees

23.A The Structure of $\Delta^0_2$

Recall from 21.18 the definition of $\text{Res}_\alpha(A)$, and from ?? that $\Delta^0_2 = \bigcup_{\alpha < \omega_1} \text{Diff}(\alpha; \Pi^0_1)$. Also for simplicity of notation let

$$D_\alpha = \begin{cases} 
\text{Diff}(\alpha; \Pi^0_1) & \text{if } \alpha \text{ is even,} \\
\text{Diff}(\alpha; \Pi^0_1) & \text{if } \alpha \text{ is odd.}
\end{cases}$$

Note that $D_\alpha$ is a non-self-dual pointclass.

**Lemma 23.1.** $(\text{AC}_\omega(\mathbb{R}))$ Suppose $\alpha < \omega_1$ and $A, B \subseteq \mathbb{R}$. Then

$$A \in D_\alpha \& \text{ Res}_\alpha(\neg B) \neq \emptyset \Rightarrow A \leq_L B.$$  

**Proof.** By induction on $\alpha$. If $\alpha = 0$ then $D_\alpha = \text{Diff}(0; \Pi^0_1) = \{\emptyset\}$ so $A = \emptyset$. Since $\neg B = \text{Res}_0(\neg B) \neq \emptyset$ then $B \neq \mathbb{R}$ so $A \leq_L B$.

Suppose $\alpha > 0$ and assume the result for all $\beta < \alpha$. We assume first that $\alpha$ is even and hence $D_\alpha = \text{Diff}(\alpha; \Pi^0_1)$. Then

$$A = \bigcup_{2^\nu < \alpha} (C_{2^\nu} \setminus C_{2^\nu+1})$$

with $C_0 \supseteq C_1 \supseteq \cdots$ closed sets, and let

$$U = \bigcup_{2^\nu < \alpha} \neg C_{2^\nu+1} \in \Sigma^0_1.$$  

Pick $b \in \text{Res}_\alpha(\neg B) \neq \emptyset$. Then $\Pi$ wins $G_L(A, B)$ via the following strategy:

$\Pi$ plays $b(0), b(1), \ldots$ while $I$ plays $a(0), a(1), \ldots$ until, if ever, $N_{a[n]} \subseteq U$. If this never happens then $\Pi$'s play will be $b$. If instead $n_0$ is least such that $N_{a[n_0]} \subseteq U$ let $\nu$ be least such that $N_s \subseteq \neg C_{2\nu+1}$, were $s = a \upharpoonright n$. Then

$$A_{\{s\}} = \bigcup_{2^\gamma < 2\nu+1} C'_{2\gamma} \setminus C'_{2\gamma+1} \in \text{Diff}(2\nu+1; \Pi^0_1)$$

where $C'_\gamma = (C_\gamma)_{\{s\}}$. Thus $\neg A_{\{s\}} \in D_{\alpha'}$ where $\alpha' = 2\nu + 1 < \alpha$.

Let $t = b \upharpoonright n_0$. By definition of $\text{Res}_\alpha(\neg B)$ there is a $b' \in \text{Res}_{\alpha'}(B)$ such that $t \subseteq b'$, and since the “Res” operation is defined in terms of Boolean and topological operations by ??

$$\text{Res}_{\alpha'}(B_{\{t\}}) = (\text{Res}_{\alpha'}(B))_{\{t\}} \neq \emptyset.$$  

Therefore by inductive hypothesis $\Pi$ has a winning strategy in $G_L(\neg A_{\{s\}}, \neg B_{\{t\}})$. Let $\Pi$ follow this strategy from now on in the game $G_L(A, B)$.  

It is easy to check that this is indeed a winning strategy for \( \Pi \).
The case when \( \alpha \) is odd is similar: since
\[
\neg A = \bigcup_{2\nu<\alpha} C_{2\nu} \setminus C_{2\nu+1},
\]
then set
\[
U = \bigcup_{2\nu<\alpha} \neg C_{2\nu}
\]
and proceed as before.

This implies

**Proposition 23.2.** \((AC_\omega(\mathbb{R}))\) For all \( \alpha < \omega_1 \) and \( B \subseteq \mathbb{R} \) the following conditions are equivalent:

1. \( \text{Res}_\alpha(\neg B) \neq \emptyset \),
2. \( \neg B \notin D_\alpha \),
3. \( B \) is \( D_\alpha \)-hard.

**Proof.** \((2) \Rightarrow (1)\) follows from 21.18(iii), and \((1) \Rightarrow (3)\) follows from the lemma, so it is enough to prove \((3) \Rightarrow (2)\).

Since \( \text{Diff}(\alpha; \Pi^0_1) \) is non-self-dual \((??(iv))\) pick \( A \in D_\alpha \setminus \bar{D}_\alpha \). Then \( A \leq_W B \) by hypothesis hence \( \neg B \in D_\alpha \) would imply \( \neg A \in D_\alpha \): a contradiction.

**Corollary 23.3.** \( \{ A \subseteq \mathbb{R} \mid \exists \alpha < \omega_1 \text{ Res}_\alpha(A) = \emptyset \} = \Delta^0_2 \).

**Proof.** From 21.18(iii) we already know \( \subseteq \). If \( \forall \alpha < \omega_1 (\text{Res}_\alpha(A) \neq \emptyset) \) then \( A \notin \bigcup_{\alpha<\omega_1} D_\alpha \), and by \( ??(iii) \) and \( ?? \bigcup_{\alpha<\omega_1} D_\alpha = \Delta^0_2 \), so \( A \notin \Delta^0_2 \).

**Proposition 23.4.** For any \( 1 \leq \alpha < \omega_1 \) and any \( B \subseteq \mathbb{R} \) the following are equivalent:

1. \( \forall \beta < \alpha (\text{Res}_\beta(B) \neq \emptyset \& \text{Res}_\beta(\neg B) \neq \emptyset) \),
2. \( B \notin \bigcup_{\beta<\alpha} D_\beta \) and \( \neg B \notin \bigcup_{\beta<\alpha} D_\beta \),
3. \( B \) is \( \Delta(D_\alpha) \)-hard.

**Proof.** \((1) \Leftrightarrow (2)\) follows by 23.2.

Since \( D_\beta \neq D_\alpha \) for all \( \beta < \alpha \) by \( ?? \) then \((3) \Rightarrow (2)\).

Suppose \( B, \neg B \notin \bigcup_{\beta<\alpha} D_\beta \). Let \( A \in D_\beta \cup \bar{D}_\beta \) for some \( \beta < \alpha \). By 23.2

\[
\emptyset = \text{Res}_\alpha(A) = A \cap \bigcap_{\beta<\alpha} \text{Cl}(\text{Res}_\alpha(\neg A))
\]
and
\[ \emptyset = \text{Res}_\alpha(-A) = A \cap \bigcap_{\beta < \alpha} \text{Cl}(\text{Res}_\alpha(A)) \]

Fix \(a \in \mathbb{R}\): since either \(a \in A\) or \(a \in -A\), there is a \(\beta < \alpha\) such that
\[ a \notin \text{Cl}(\text{Res}_\beta(-A)) \quad \text{or} \quad a \notin \text{Cl}(\text{Res}_\beta(A)) \]
hence there is \(n\) such that
\[ \mathcal{N}_{a|n} \cap \text{Res}_\beta(-A) = \emptyset \quad \text{or} \quad \mathcal{N}_{a|n} \cap \text{Res}_\beta(A) = \emptyset. \]

Then \(\Pi\) wins \(G_W(A, B)\) as follows:

\(\Pi\) passes until \(I\) reaches a position \(a \upharpoonright n\) such that either \(\mathcal{N}_{a|n} \cap \text{Res}_\beta(-A) = \emptyset\) or \(\mathcal{N}_{a|n} \cap \text{Res}_\beta(A) = \emptyset\) for some \(\beta < \alpha\). This means that either \(\text{Res}_\beta(-A_{a|n}) = \emptyset\) or \(\text{Res}_\beta(A_{a|n}) = \emptyset\), which by 23.2 means that
\[ A_{a|n} \in D_\beta \cup \tilde{D}_\beta. \]

Since \(B, -B \notin \bigcup_{\gamma < \alpha} D_\gamma\) then by 23.2 \(B\) is \(D_\beta \cup \tilde{D}_\beta\)-hard hence \(\Pi\) has a winning strategy in \(G_W(A_{a|n}, B)\). Let \(\Pi\) use this strategy in \(G_W(A, B)\) from now on.

It is easy to verify that this is a winning strategy for \(\Pi\), hence \(A \leq_W B\). Since \(A \in \Delta(D_a)\) was arbitrary this establishes (3).

**Corollary 23.5.** (\(AC_\omega(\mathbb{R})\)) For any \(B \subseteq \mathbb{R}\), either \(B \in \Delta^0_2\) or \(\forall A \in \Delta^0_2 (A \leq_W B)\).

**Proof.** If \(\text{Res}_\alpha(B) = \emptyset\) for some \(\alpha < \omega_1\) then \(B \in \Delta^0_2\), so by 23.2 we may assume that \(B\) is \(D_\alpha\)-hard for any \(\alpha < \omega_1\), which is the same as saying that \(\forall A \in \Delta^0_2 (A \leq_W B)\).

**Exercise 23.6.** (\(AC_\omega(\mathbb{R})\)) Show that the only Wadge degrees below \(\Delta^0_2\) are
\[ \text{Diff}(\alpha; \Pi^0_1) \setminus \Delta(\text{Diff}(\alpha; \Pi^0_1)), \quad \tilde{\text{Diff}}(\alpha; \Pi^0_1) \setminus \Delta(\text{Diff}(\alpha; \Pi^0_1)), \quad \Delta(\text{Diff}(\alpha; \Pi^0_1)) \].

**Exercise 23.7.** (\(AC_\omega(\mathbb{R})\)) Show that
\[ \Delta(\text{Diff}(2; \Pi^0_1)) = \{A \subseteq \mathbb{R} \mid \forall a \in \mathbb{R} \exists k (A \cap N_a[k] \text{ is open or closed})\} \]
shows that if $A \in \Pi_2^1 \setminus \Delta_2^0$ then we cannot prove in ZFC that $A$ is $\Pi_2^0$-hard. We shall see shortly that the pair $\Pi_2^0, \Pi_1^1$ is—in a sense—the least potential counterexample to $\text{AD}_L$, in that if $A \in \Pi_2^1$ and $B \in \Sigma_1^1$ then either

$$A \leq_W B \lor \neg B \leq_W A$$

or

$$A \leq_W \neg B \lor B \leq_W A.$$  

Before we prove this we turn to a generalization (due again to W. Wadge) of the reducibility $\leq_W$ to pairs of sets:

**Definition 23.8.** For $A_0, A_1, B_0, B_1 \subseteq \mathbb{R}$

$$(A_0, A_1) \leq_{\text{sep}} (B_0, B_1)$$

just in case $f^{-1}[B_0] \supseteq A_0$ and $f^{-1}[B_1] \supseteq A_1$, for some continuous $f : \mathbb{R} \to \mathbb{R}$.

The intuition behind $\leq_{\text{sep}}$ is that $A_0$ and $A_1$ are no harder to separate (i.e., to tell-apart) than $B_0$ and $B_1$ are. The reason why we do not require $f^{-1}[B_i] = A_i$ ($i = 0, 1$) is that even if $A_0 \cap A_1 = B_0 \cap B_1 = \emptyset$ and $A_0$ and $A_1$ may be very complex but lying far apart (and hence easy to separate) while $B_0$ and $B_1$ are simpler but tightly entwined (and hence hard to separate). We shall see that $\text{AD}$ implies an $\text{AD}_L$ principle for pairs of sets

$$(A_0, A_1) \leq_{\text{sep}} (B_0, B_1).$$

**Exercise 23.9.** For $A_0, A_1, B_0, B_1 \subseteq \mathbb{R}$ show that

(i) $A_0 \leq_W B_0 \iff (A_0, \neg A_0) \leq_{\text{sep}} (B_0, B_1)$.

(ii) $(A_0, A_1) \leq_{\text{sep}} (B_0, B_1) \iff (A_1, A_0) \leq_{\text{sep}} (B_1, B_0)$.

(iii) $(A_0, A_1) \leq_{\text{sep}} (A_0, \neg A_0)$ and $(A_0, A_1) \leq_{\text{sep}} (\neg A_1, A_1)$.

(iv) $A_0 \subseteq B_0$ and $A_1 \subseteq B_1$ implies $(A_0, A_1) \leq_{\text{sep}} (B_0, B_1)$.

(v) $A_0 = A_1 = \emptyset \Rightarrow (A_0, A_1) \leq_{\text{sep}} (B_0, B_1)$.

(vi) $B_0 = B_1 = \emptyset \Rightarrow (A_0, A_1) \leq_{\text{sep}} (B_0, B_1)$.

(vii) $\leq_{\text{sep}}$ is reflexive and transitive.

Thus every pair of non-disjoint sets is $\leq_{\text{sep}}$-maximal and $\equiv_{\text{sep}}$-equivalent to one another. (Here $\equiv_{\text{sep}}$ is the equivalence relation induced by $\leq_{\text{sep}}$.)

**Exercise 23.10.** For $A_0, A_1, B_0, B_1 \subseteq \mathbb{R}$ and $\Gamma$ a boldface pointclass, if $(A_0, A_1) \leq_{\text{sep}} (B_0, B_1)$ and $B_0, B_1$ can be separated by a set in $\Gamma$ (i.e., $\exists C \in \Gamma (B_0 \subseteq C \land B_1 \cap C = \emptyset)$) then $A_0, A_1$ can be separated by a set in $\Gamma$. 
CHAPTER V. THE SEMI-LINEAR ORDERING PRINCIPLE

Just like in the case of $\leq_W$ the preorder $\leq_{\text{sep}}$ can be characterized in terms of games.

**Definition 23.11.** Let $G_{\text{sep}}(A_0, A_1, B_0, B_1)$ be the game where I plays $a$, II plays $b$ with the understanding that he can “pass” infinitely often as long as he plays infinitely often, and II wins iff

$$ (a \in A_0 \& b \in B_0) \lor (a \in A_1 \& b \in B_1) \lor (a \notin A_0 \cup A_1) \, . $$

If $A_0 \cap A_1 = B_0 \cap B_1 = \emptyset$ then the winning conditions of $G_{\text{sep}}$ can be expressed via the following table

<table>
<thead>
<tr>
<th>$b \in B_0$</th>
<th>$b \in B_1$</th>
<th>$b \notin B_0 \cup B_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a \in A_0$</td>
<td>II</td>
<td>I</td>
</tr>
<tr>
<td>$a \in A_1$</td>
<td>I</td>
<td>II</td>
</tr>
<tr>
<td>$a \notin A_0 \cup A_1$</td>
<td>II</td>
<td>I</td>
</tr>
</tbody>
</table>

**Exercise 23.12.** Show that in the game $G_{\text{sep}}(A_0, A_1, B_0, B_1)$ Player II wins $(A_0, A_1) \leq_{\text{sep}} (B_0, B_1)$, and Player I wins $(B_1, B_0) \leq_{\text{sep}} (A_0, A_1)$.

Therefore AD implies that

$$ (A_0, A_1) \leq_{\text{sep}} (B_0, B_1) \lor (B_1, B_0) \leq_{\text{sep}} (A_0, A_1) \, . $$

As a simple application let us show the following lemma.

**Lemma 23.13.** ($AC_\omega(\mathbb{R})$) For $A, B \in \Sigma^0_2$ disjoint sets

$$(A, B) \leq_{\text{sep}} (E^*, O^*)$$

where

$E^* = \{ x \in \mathbb{R} \mid \forall^\infty n \, x(n) \text{ is even} \}$

$O^* = \{ x \in \mathbb{R} \mid \forall^\infty n \, x(n) \text{ is odd} \}$.

**Proof.** Let $A = \bigcup_n [T_{2n}]$ and $B = \bigcup_n [T_{2n+1}]$, with $T_n \subseteq T_{n+2}$ pruned trees. The II wins $G_{\text{sep}}(A, B, E^*, O^*)$ by playing at round $n$ the least $m$ such that $a \upharpoonright m + 1 \in T_m$. \hfill $\Box$

**Definition 23.14.** A set $A \subseteq \mathbb{R}$ is **guessable** iff there are disjoint subsets $Y$ (for yes) and $N$ (for no) of $\preceq_\omega$ such that for all $a \in \mathbb{R}$

- $a \in A \iff \exists n_0 \forall m \geq n_0 \, a \upharpoonright m \in Y$
- $a \notin A \iff \exists n_0 \forall m \geq n_0 \, a \upharpoonright m \in N$

The pair $(Y, N)$ is called a pair of guessing sets for $A$. 

Definition 23.15 (Hausdorff). For $A \subseteq \mathbb{R}$ let $\text{Rm}(A)$ the remainder of $A$ be the set

$$\text{Rm}(A) = \left( A \cap \bigcap_{\alpha < \omega_1} \text{Cl} \left( \text{Res}(\neg A) \right) \right) \cup \left( \neg A \cap \bigcap_{\alpha < \omega_1} \text{Cl} \left( \text{Res}(A) \right) \right)$$

Lemma 23.16. $A \cap \text{Rm}(A)$ and $\neg A \cap \text{Rm}(A)$ are dense in $\text{Rm}(A)$.

Proposition 23.17. If $A$ is guessable and $\text{Rm}(A) \neq \emptyset$ then $I$ wins $G_{\text{G}}(B, \neg A)$.

Proof. Let $(Y, N)$ be guessing sets for $A$ and let $z \in \text{Rm}(B) \cap B$. $I$ wins by playing $z(0), z(1), \ldots$ until $II$’s sequence is guessed to be in $\neg A$. If we reach such a place then since $z \upharpoonright k$ has an extension in $\text{Rm}(B) \cap B$ (i.e., $z$) there is an extension in $\text{Rm}(B) \cap \neg B$, thus let $z'$ be such an extension and let $I$ play $z'$ until $II$ guesses $A$. And so on. \qed

Proposition 23.18. $A$ is guessable $\iff A \in \Delta^0_2$.

Proof. Suppose $A$ is guessable and let $Y, N$ be guessing sets for $A$. Notice that $N, Y$ are guessing sets for $\neg A$. Then

$$(A, \neg A) \leq_{\text{sep}} (E^*, O^*)$$

where $E^*$ and $O^*$ be as in 23.13, since $II$ wins $G_{\text{sep}}(\neg A, A, E^*, O^*)$ by playing at stage $n$

$$
\begin{align*}
0 & \quad \text{if } a \upharpoonright n + 1 \in Y \\
1 & \quad \text{if } a \upharpoonright n + 1 \in N
\end{align*}
$$

where $a \upharpoonright n + 1$ is $I$’s play at stage $n$. Since $E^*, O^* \in \Sigma^0_2$ and $\neg A = f^{-1}[E^*]$ and $A = f^{-1}[O^*]$ then $A \in \Delta^0_2$.

Conversely suppose $A \in \Delta^0_2$. Since by 23.13 $(E^*, O^*)$ is complete for pairs of disjoint $\Sigma^0_2$ sets, then $(A, \neg A) \leq_{\text{sep}} (E^*, O^*)$. Let $\tau$ be $II$’s winning strategy in $G_{\text{sep}}(A, \neg A, E^*, O^*)$. Then

$$
Y = \{ s \mid \tau(s) \text{ ends with an even number} \} \\
N = \{ s \mid \tau(s) \text{ ends with an odd number} \}
$$

are guessing sets for $A$. \qed

Proposition 23.19. For any $A \subseteq \mathbb{R}$

$$A \in \Delta^0_2 \iff \text{Rm}(A) = \emptyset.$$
CHAPTER V. THE SEMI-LINEAR ORDERING PRINCIPLE

Proof. If $A \in \Delta_2^0$ and $\text{Rm}(A) \neq \emptyset$ then $I$ wins $G_L(A, A)$: a contradiction.

Suppose $\text{Rm}(A) = 0$. Let

$$Y = \{s \mid \exists \nu < \omega_1 (s \text{ has an extension in } \text{Res}_\nu(A), \text{ but none in } \text{Res}_\nu(\neg A))\}$$

$$N = \{s \mid \exists \nu < \omega_1 (s \text{ has an extension in } \text{Res}_\nu(\neg A), \text{ but none in } \text{Res}_\nu(A))\}$$

Proposition 23.20 (Van Wesep). Assume $\text{AD}^L$ and $\text{DC}(\mathbb{R})$. Suppose $A$ is self-dual and $C$ is non-self-dual and $0 < \|C\|_W < \omega_1$. Then

$$C \in \text{Diff}(n; \Pi_1^0) \setminus \text{Diff}(n; \Pi_1^0) \Rightarrow A^\varphi + C \equiv_W A + C$$

and

$$C \in \text{Diff}(n; \Pi_1^0) \setminus \text{Diff}(n; \Pi_1^0) \Rightarrow A^\varphi + C \equiv_W A + C.$$  

Proof. By 22.8(iii) it is enough to prove the result when $C \notin \text{Diff}(n; \Pi_1^0) \setminus \text{Diff}(n; \Pi_1^0)$. Let $C = C_0 \setminus (C_1 \setminus \cdots (C_{n-1} \setminus C_n) \cdots)$, with $C_0 \supseteq C_1 \supseteq \cdots \supseteq C_n = \emptyset$ closed and let

$$D_0 = \neg C$$
$$D_{2j+1} = \text{Cl}(D_{2j}) \cap C$$
$$D_{2j+2} = \text{Cl}(D_{2j+1}) \setminus C.$$  

since $C \notin \text{Diff}(n; \Pi_1^0)$ then $D_n \neq \emptyset$. $I$ wins $G_W((A^\varphi + \emptyset) + C, A + C)$ as follows:

Pick $x_n \in D_n$ and let $I$ play $x_n + 1$ until, if ever, $I$ plays outside $C_{n-1} + 1$, i.e., reaches a position $p \notin T_{C_{n-1} + 1}$, where $T_{C_{n-1}}$ is the tree associated to $C_{n-1}$, i.e., $[T_{C_{n-1}}] = C_{n-1}$. If this happens at some round $k$ then $II$ chooses $x_{n-1} \in D_{n-1}$ such that $x_{n-1} \supset x_n \upharpoonright k$ and now follows $x_{n-1} + 1$ until, if ever, $I$ plays out of $C_{n-2} + 1$. And so on. If at some stage $I$ plays 0 then $II$ starts playing (if he is not already doing so) some $x_0 + 1$, with $x_0 \in D_0 = \neg C$: $II$ follows $x_0 + 1$ as long as $I$ stays in $T(A^\varphi + \emptyset)$. If at some round $I$ reaches a position $p \notin T(A^\varphi + \emptyset)$ then $(A^\varphi)_{[p]} \leq_W A$ so $II$ plays 0 and follows a winning strategy for the game $G_W((A^\varphi)_{[p]}, A)$.

\qed
23. BOREL-WADGE DEGREES

23.B Generalized Boolean Operations

For every natural boldface pointclasses \( \Gamma \) there is a “natural construction principle” that builds elements of \( \Gamma \) from countable sequences of open sets: for example the map \( \langle U_n \mid n < \omega \rangle \mapsto \bigcap_n U_n \) constructs all \( \Pi^0_2 \) sets, the map \( \langle U_n \mid n < \omega \rangle \mapsto \bigcup_i \bigcap_j U_{(i,j)} \) constructs all \( \Sigma^0_3 \) sets, while Suslin’s operation \( A \) (see ??) yields (modulo a bijection between \( <\omega \omega \) and \( \omega \)) a map \( \omega \Sigma^0_1 \rightarrow \Sigma^1_1 \).

We need a general definition of “natural construction principle.”

**Definition 23.21.** (i) Let \( F : I^P(\mathbb{R}) \rightarrow P(\mathbb{R}) \) and \( T \subseteq P(I) \). Then \( T \) is a **truth-table for** \( F \) iff

\[
\forall A \in I^P(\mathbb{R}) \forall x \in \mathbb{R} \ (x \in F(A) \iff \{ i \in I \mid x \in A_i \} \subseteq T).
\]

(ii) \( F : I^P(\mathbb{R}) \rightarrow P(\mathbb{R}) \) is a **generalized Boolean I-ary operation** iff it admits a truth-table \( T \subseteq P(I) \).

(iii) Let \( \Gamma, \Lambda \) be pointclasses. Then \( \Gamma \) is a \( \omega \)-ary Boolean iff \( \exists F (F : \omega^P(\mathbb{R}) \rightarrow P(\mathbb{R})) \) is a generalized \( \omega \)-ary Boolean operation and \( \Gamma = F^\omega \Lambda \). If \( \Lambda = \Sigma^0_1 \) we will simply say that \( \Gamma \) is open-Boolean.

Therefore \( F \) is a generalized \( I \)-ary Boolean operation just in case the answer to “\( x \in F(A) \)” depends only on the answers to “\( x \in A_i \)” for \( i \in I \). In other words: \( F : I^P(\mathbb{R}) \rightarrow P(\mathbb{R}) \) is a generalized Boolean \( I \)-ary operation iff \( \forall A, B \in I^P(\mathbb{R}), \forall x, y \in \mathbb{R} \)

\[
\forall i \in I \ (x \in A(i) \iff y \in B(i) \Rightarrow (x \in F(A) \iff y \in F(B))).
\]

**Exercise 23.22.** Show that \( \bigcap_n A_n, \bigcup_n A_n, \bigcup_n \bigcap_m A_{(n,m)} \) are examples of generalized \( \omega \)-ary Boolean operations. Show that operation \( A \) is generalized \( \leq_\omega \)-ary Boolean.

If \( f : I \rightarrow J \) is a bijection, then we can copy any generalized \( I \)-ary Boolean operation \( F : I^P(\mathbb{R}) \rightarrow P(\mathbb{R}) \) to a generalized \( J \)-ary Boolean operation \( G : J^P(\mathbb{R}) \rightarrow P(\mathbb{R}) \) by letting

\[
G(\langle A_j \mid j \in J \rangle) = F(\langle A_{f(i)} \mid i \in I \rangle).
\]

Therefore, with a minor blurring of vision, any generalized \( \leq_\omega \)-ary Boolean operation (like Suslin’s operation \( A \)) will be considered to be \( \omega \)-ary.

**Exercise 23.23.** (i) Show that if \( \Gamma \) is open \( \omega \)-ary Boolean, then so is \( \Gamma \).

(ii) Show that if \( \Gamma \) is open \( \omega \)-ary Boolean, then \( \Gamma \) is a non-self-dual boldface pointclass.
The main point of this section is to prove the converse of this namely:

**Theorem 23.24.** Assume AD + DC(\(\mathbb{R}\)). Every non-self-dual boldface point-class \(\Gamma\) is open \(\omega\)-ary Boolean.

Let us prove first a particular case of the theorem.

**Lemma 23.25.** (DC(\(\mathbb{R}\))) Suppose \(\Gamma\) is non-self-dual of rank < \(\omega_1^\omega\), that is \(\forall A \in \Gamma (\|A\|_w < \omega_1^\omega)\). Then \(\Gamma\) is open \(\omega\)-ary Boolean.

**Proof.** Later.

We also need the following lemma due to Lon Radin: it says that the theorem is true for many \(\Gamma\)'s.

**Lemma 23.26** (Radin). (AD\(^L\)) If \([A]_w = [A]_{\text{stretch}}^w\) then \(\{B \subseteq \mathbb{R} \mid B \leq_w A\}\) is open \(\omega\)-ary Boolean.

**Proof.** Let \(\mathcal{F} : \mathcal{P}(\mathbb{R}) \rightarrow \mathcal{P}(\mathbb{R})\) be defined by

\[
\mathcal{F}(\langle U_s \mid s \in \omega^\omega \rangle) = \{x \in \mathbb{R} \mid \forall n \in \omega \exists s \in n^\omega (x \in U_s) \& \exists z \in A \forall n \in \omega (x \in U_{z[n]})\}.
\]

It is easy to check that \(\mathcal{F}\) is \(\omega\)-ary (and hence \(\omega\)-ary) generalized Boolean. Suppose \(\forall s \in \omega^\omega (U_s \in \Sigma^0_1)\) and let \(B = \mathcal{F}(\langle U_s \mid s \in \omega^\omega \rangle)\). We will show that \(B \leq_\omega A^{\text{stretch}}\) by exhibiting a winning strategy for \(\Pi\) in \(G_L(B, A^{\text{stretch}})\):

Let \(\Pi\) play \(x(0), x(1), \ldots\). \(\Pi\) plays 0 until, if ever, a round \(m_0\) is reached when \(N_{x|m_0} \subseteq U_0\). If this happens then \(\Pi\) still keeps playing 0 until, if ever, a round \(m_1 > m_0\) is reached when \(N_{x|m_1} \subseteq U_{s_1}\) for some unique \(s_1 \in \text{1}^\omega\). Then \(\Pi\) plays \(s_1(0) + 1\). This takes care of the sequences of length 1. Now \(\Pi\) plays 0 until, if ever, a round \(m_2 > m_1\) is reached when \(N_{x|m_2} \subseteq U_{s_2}\) for some unique \(s_2 \in \text{2}^\omega\). Then \(\Pi\) plays \(s_2(1) + 1\). And so on.

If \(x \in B\) then \(x \in \bigcap_n U_{z[n]}\) for some \(z \in A\), and since the \(U\)'s are open, \(\forall n \exists m_n (N_{x|m_n} \subseteq U_{z[n]}\), so that \(y\), the real constructed by the strategy, is a stretch of \(z\). Vice versa suppose \(x \notin B\). If \(|\{s \in n^\omega \mid x \in U_s\}| \neq 1\) for some \(n \in \omega\), then \(\Pi\) plays a sequence eventually equal to 0 hence not in \(A^{\text{stretch}}\). If \(\forall n \exists s \in n^\omega (x \in U_s)\) then \(\neg \exists z \in A (x \in \bigcap_n U_{z[n]}\), hence \(y \notin A^{\text{stretch}}\), where \(y\) is the real played by \(\Pi\). Therefore \(\mathcal{F}^{\omega \omega}(\Sigma^0_1) \subseteq \{B \subseteq \mathbb{R} \mid B \leq_w A^{\text{stretch}} \equiv_w A\}\).

On the other hand suppose \(B = f^{-1}[A]\) for some continuous \(f\). For \(s \in \omega^\omega\) let \(U_s = f^{-1}[N_s]\). Then

\[
\forall x \in \omega^\omega \forall n \in \omega \exists s \in n^\omega (x \in U_s)
\]

and thus clearly \(B = \mathcal{F}(\langle U_s \mid s \in \omega^\omega \rangle)\). Therefore \(\{B \mid B \leq_w A\} = \mathcal{F}^{\omega \omega}(\Sigma^0_1)\).
Proof. 23.25 Let $B \in \Gamma \setminus \tilde{\Gamma}$ and let $\alpha$ be largest such that $\omega_1^\omega \cdot \alpha \leq \|B\|_W$. If equality holds then $\text{cof}(\alpha) > \omega$ so by 22.41(a) and 23.26 we are done. Hence we may assume $\omega_1^\omega \cdot \alpha < \|B\|_W$. Choose $A$ such that
\[
\|A\|_W = \begin{cases} 
\omega_1^\omega \cdot \alpha + 1 & \text{if } \text{cof}(\alpha) > \omega, \\
\omega_1^\omega \cdot \alpha & \text{otherwise}.
\end{cases}
\]
Then $A$ is self-dual and by 22.10 there is $C$ such that $A + C \equiv_W B$ and $\|C\|_W < \omega_1^\omega$.

Suppose first $\|C\|_W \geq \omega$. (In fact, since $C$ is non-self-dual, $\|C\|_W > \omega$.) Then we have the following:

Claim 23.26.1. $\Gamma = \{(F \cap U) \cup (G \setminus U) \mid F \leq_W A + \emptyset \& \ G \leq_W C \& \ U \in \Sigma_1^0\}$.

Proof. Let $F \leq_W A + \emptyset$, $G \leq_W C$, and $U$ be open. Fix $\tau_0$ and $\tau_1$ winning strategies for $\text{II}$ in $G_W(G, C)$ and $G_W(F, A + \emptyset)$, respectively. Then $\text{II}$ wins $G_W((F \cap U) \cup (G \setminus U), (A + \emptyset) + C)$ as follows:

As long as $I$ has not reached a position $p$ such that $N_p \subseteq U$ (in other words: as long as $I$ has not entered $U$), then $\text{II}$ follows $\tau_0 + 1$, that is for any position $q$ of $I$ he plays $\tau_0(q) + 1$. If $I$ ever enters $U$ at some round $n$ then $\text{II}$ plays $0$ and then uses $\tau_1$ as if the game just started now:

\[
\begin{array}{l}
I \quad x_0, \ldots, x_{n-1} \quad x_n \quad x_{n+1} \quad x_{n+2} \quad \ldots \\
\text{II} \quad \tau_0(\langle x_0, \ldots, x_{n-1} \rangle) + 1 \quad 0 \quad \tau_1(\langle x_0 \rangle) \quad \tau_1(\langle x_0, x_1 \rangle) \quad \ldots
\end{array}
\]

Since $\|C\|_W \geq \omega$ then $(A + \emptyset) + C \equiv_W A + C \equiv_W B$ and therefore $(F \cap U) \cup (G \setminus U) \in \Gamma$.

For the other inclusion it is enough to show that
\[A + C \in \{(F \cap U) \cup (G \setminus U) \mid F \leq_W A + \emptyset \& \ G \leq_W C \& \ U \in \Sigma_1^0\},\]

since the collection of sets on the right is obviously a boldface pointclass. Let $F = A + \emptyset$, let $G = \{x \mid x \prec 1 \in C\}$, and let $U = \{x \mid \exists n \ x(n) = 0\}$. Then $F = F \cap U$ and

\[A + C = A + \emptyset \cup C^+ = (F \cap U) \cup (G \setminus U),\]

which is what we had to prove. □
Since

\[(A + \emptyset) \text{stretch} \equiv_W (A^{\text{stretch}}) \text{stretch} \equiv_W A^{\text{stretch}} \equiv_W A + \emptyset\]

by 22.27(ii), 22.28(a), and 22.41(b), then by Radin’s lemma \(\{F \mid F \leq_W A + \emptyset\}\) is open \(\omega\)-ary Boolean, and let \(\mathcal{F}\) witness this. Using 23.25 then \(\{G \mid G \leq_W C\}\) is also open \(\omega\)-ary Boolean via some \(\mathcal{G}\). Then \(\mathcal{G}\) is open \(\omega\)-ary Boolean via \(\mathcal{H}\), where

\[
\mathcal{H}(\langle U_n \mid n \in \omega \rangle) = (\mathcal{F}(\langle U_{2n} \mid n \geq 1 \rangle) \cap U_0) \cup (\mathcal{G}(\langle U_{2n+1} \mid n \geq 0 \rangle) \setminus U_0).
\]

Suppose now \(\|C\|_W = n < \omega\). Then by §2323.A either \(C \in \text{Diff}(n; \Pi^0_0) \setminus \text{Diff}(n; \Pi^0_1)\), or else \(C \in \text{Diff}(n; \Pi^0_1) \setminus \text{Diff}(n; \Pi^0_0)\). By 23.23(i) we may assume the former. If \(n = 0\) then \(C = \emptyset\) and by 22.28(a)

\[B \equiv_W A + \emptyset \equiv_W A^{\text{stretch}}\]

hence we are done by Radin’s lemma, so we may assume \(n \geq 1\). Then by 23.20 \((A + \emptyset) + C \equiv_W A + C\) hence the Claim holds also in this case and we are done by the same argument.

\[\square\]

Additional Exercises

Notes and References

23.23(ii) is due to Addison.

24 Boldface Pointclasses

Recall that a non-empty \(\Gamma \subseteq \mathcal{P}(\mathbb{R})\) is a boldface pointclass if it is downward closed under \(\leq_W\). Under \(\text{AD}\) any non-self-dual boldface pointclass is of the form \(\{B \subseteq \mathbb{R} \mid B \leq_W A\}\) for some non-self-dual \([A]_W\), and under \(\text{AD} + \text{DC}(\mathbb{R})\) any self-dual boldface pointclass is of the form \(\mathcal{P}^{(\alpha)}(\mathbb{R})\), for some \(0 < \alpha < \Theta\).

**Exercise 24.1** (Wadge’s Lemma for boldface pointclasses). (\(\text{AD}\)) Show that if \(\Gamma, \Lambda\) are boldface pointclasses then \(\Gamma \subseteq \Lambda\) or \(\Lambda \subseteq \Gamma\).

We now investigate some structural properties of pointclasses.
24. BOLDFACE POINTCLASSES

24.A Separation

Definition 24.2. A pointclass $\Gamma$ has the separation property, $\text{Sep}(\Gamma)$, iff

$$\forall A, B \in \Gamma \ (A \cap B = \emptyset \Rightarrow \exists C \in \Delta_\Gamma (A \subseteq C \& C \cap B = \emptyset)) \ .$$

By ??? $\text{Sep}(\Sigma_1^1)$ and by ??? $\neg \text{Sep}(\Pi_1^1)$. We will show

Theorem 24.3 (Steel–Van Wesep). (AD) Let $\Gamma$ be a non-self-dual boldface pointclass. Then exactly one of $\text{Sep}(\Gamma)$ and $\text{Sep}(\bar{\Gamma})$ holds.

The theorem is a corollary of

Theorem 24.4 (Steel). (AD) Let $\Gamma$ be non-self-dual. Then $\text{Sep}(\Gamma)$ or $\text{Sep}(\bar{\Gamma})$.

and

Theorem 24.5 (Van Wesep). (AD) Let $\Gamma$ be non-self-dual. Then $\neg \text{Sep}(\Gamma)$ or $\neg \text{Sep}(\bar{\Gamma})$.

Given $A_0, A_1, B_0, B_1 \subseteq \mathbb{R}$, the separation game $G_{\text{sep}}^*(A_0, A_1, B_0, B_1)$ is the game where I plays $x$, II plays $y$, and I wins iff

$$(y \in A_0 \setminus A_1 \Rightarrow x \in B_0) \& (y \in A_1 \setminus A_0 \Rightarrow x \in B_1) \ .$$

The next exercise explains the reason for the name $G_{\text{sep}}^*$.

Exercise 24.6. Let $A_0, A_1, B_0, B_1 \subseteq \mathbb{R}$.

(i) Show that if I wins $G_{\text{sep}}^*(A_0, A_1, B_0, B_1)$ then there is a continuous $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f[A_0 \setminus A_1] \subseteq B_0 \text{ and } f[A_1 \setminus A_0] \subseteq B_1 .$$

In particular if $A_0 \cap A_1 = \emptyset$ then $f[A_i] \subseteq B_i$, for $i = 0, 1$.

(ii) Show that if II wins $G_{\text{sep}}^*(A_0, A_1, B_0, B_1)$ then there is a continuous $f : \mathbb{R} \rightarrow A_0 \Delta A_1 \subseteq \mathbb{R}$. Moreover if $B_0 \cap B_1 = \emptyset$ and $A_0, A_1 \in \Gamma$ or $A_0, A_1 \in \bar{\Gamma}$, where $\Gamma$ is a non-self-dual boldface pointclass, then $B_0, B_1$ are $\Delta_\Gamma$-separable, that is

$$\exists C \in \Delta_\Gamma (B_0 \subseteq C \& B_1 \cap C = \emptyset) .$$

Proof. 24.4 Suppose, towards a contradiction, that $A_0, A_1 \in \Gamma$ and $C_0, C_1 \in \bar{\Gamma}$ are $\Delta_\Gamma$-inseparable, i.e.,

$$A_0 \cap A_1 = \emptyset \& \neg \exists D \in \Delta_\Gamma (A_0 \subseteq D \& A_1 \cap D = \emptyset) ,$$

$$C_0 \cap C_1 = \emptyset \& \neg \exists D \in \Delta_\Gamma (C_0 \subseteq D \& C_1 \cap D = \emptyset) .$$

By 24.6 there is a continuous $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f[A_i] \subseteq C_i$, $i \in 2$. Let $B_i = f^{-1}[C_i]$. Therefore
Consider the games
\[ G_0 = G_{\text{sep}}(A_0, A_1, A_0), \]
\[ G_1 = G_{\text{sep}}(A_0, \neg B_0, A_0, A_1), \]
\[ G_2 = G_{\text{sep}}(A_1, \neg B_1, A_1, A_0). \]

By 24.6 again, let \( \sigma_i \) be \( \Gamma \)'s winning strategy in \( G_i \), for \( i \in \mathbb{N} \). Let also \( f_i \) be the continuous function given by \( \sigma_i \). Thus
\[
\begin{align*}
  f_0[A_0] &\subseteq A_1 & f_0[A_1] &\subseteq A_0, \\
  f_1[A_0] &\subseteq A_0 & f_1[\neg B_0] &\subseteq A_1, \\
  f_2[A_1] &\subseteq A_1 & f_2[\neg B_1] &\subseteq A_0. 
\end{align*}
\]

We will apply the Martin–Monk technique.

For \( z \in {}^\omega 3 \), \( \langle (G_{z(i)}, \sigma_{z(i)}) \mid i \in \omega \rangle \) admits a global play \( x(z) \) as each \( \text{Player}(\sigma_{z(i)}) = \Gamma \). Let \( x(z)_i \) be the \( i \)-th row of \( x(z) \), i.e., the real on the \( i \)-th row of the global board so that
\[ x(z)_i = f_{z(i)}(x(z)_{i+1}). \]

Let
\[ X = \{ z \in {}^\omega 3 \mid x(z)_0 \notin A_0 \cup A_1 \}. \]

By the definition of the \( f_i \)'s, it is easy to see that
\[ z \in X \Rightarrow \forall n \in \omega \ (x(z)_n \notin A_0 \cup A_1) \]

hence if \( s \in {}^n 3 \)
\[ s \triangledown z \in X \Rightarrow x(s \triangledown z)_n = x(z)_0 \notin A_0 \cup A_1 \Rightarrow z \in X. \]

By AD the set \( X \) has the property of Baire.

Suppose first \( X \) is non-meager. Let \( s \in {}^{<\omega} 3 \) be such that \( X \) is comeager in \( N_s \), that is: \( \{ z \in {}^\omega 3 \mid s \triangledown z \in X \} \) is comeager. By the remark above
\[ X \supseteq \{ z \in {}^\omega 3 \mid s \triangledown z \in X \} \]
so $X$ is comeager. As $\neg B_0 \cup \neg B_1 = \mathbb{R}$ pick $i \in 2$ such that

$$Y = \{ z \in {}^\omega 3 \mid x(z)_0 \notin B_i \}$$

is non-meager. Let $z \in Y$. If $i = 0$ then $f_1(x(z)_0) \in A_1$, and if $i = 1$ then $f_2(x(z)_0) \in A_0$. Therefore

$$z \in Y \Rightarrow x((i + 1)\ulcorner) \in A_0 \cup A_1.$$ Let $Y' = \{(i + 1)\ulcorner \mid z \in Y\}$. Then $Y'$ is non-meager in $\mathbb{N}_{i+1}$, hence in $\omega^\omega$, and $Y' \cap X = \emptyset$, contradicting the fact that $X$ is comeager.

Suppose now $X$ is meager, i.e.,

$$\neg X = \{ z \in {}^\omega 3 \mid x(z)_0 \in A_0 \cup A_1 \}$$

is comeager. Pick $i \in 2$ such that

$$Y = \{ z \in {}^\omega 3 \mid x(z)_0 \in A_i \}$$

is non-meager. Then $Y$ is comeager in $\mathbb{N}_s$, for some $s \in {}^\omega 3$. By extending $s$, if needed, we can assume that $s$ has an even number of 0’s. As $f_1$ and $f_2$ map $A_0$ into $A_0$ and $A_1$ into $A_1$, while $f_0[A_0] \subseteq A_1$ and $f_0[A_1] \subseteq A_0$, it is easy to check that

$$x(z)_0 \in A_i \Rightarrow x(s\ulcorner) = A_i .$$ For $z \in Y$

$$x(z)_0 \in A_i \Rightarrow x((0)\ulcorner) = f_0(x(z)_0) \in A_{1-i}$$

$$\Rightarrow x(s\ulcorner(0)\ulcorner) \in A_{1-i}$$

$$\Rightarrow s\ulcorner(0)\ulcorner \notin Y .$$

therefore $\{ s\ulcorner(0)\ulcorner \mid z \in Y \}$ is non-meager and disjoint from $Y$ which is comeager: a contradiction.

We now turn to Theorem 24.5. Let $\text{Sep}^*(\Gamma)$ be the following assertion:

$$\forall A, B \in \Gamma \exists A', B' \in \Gamma ( A \setminus B \subseteq A' \& B \setminus A \subseteq B' \& A' \cap B' = \emptyset ).$$

Theorem 24.5 will follow from the following two lemmata:

**Lemma 24.7.** (AD) Let $\Gamma$ be a non-self-dual pointclass. Then

$$\text{Sep}(\Gamma) \leftrightarrow \neg \text{Sep}^*(\Gamma) .$$
Lemma 24.8. (AD) Let $\Gamma$ be a non-self-dual pointclass. Then

$$\text{Sep}^*(\Gamma) \lor \text{Sep}(\check{\Gamma}).$$

Assume, towards a contradiction, that $\text{Sep}(\Gamma)$ and $\text{Sep}(\check{\Gamma})$. The by 24.7 $\neg\text{Sep}^*(\check{\Gamma})$ and $\neg\text{Sep}^*(\Gamma)$, contradicting 24.7. Therefore we will be done once we prove 24.7 and 24.7.

Proof. 24.7 ($\Leftarrow$) Suppose $\neg\text{Sep}(\Gamma)$ and let $B_0, B_1$ witness this, i.e., $B_0, B_1 \in \Gamma$, $B_0 \cap B_1 = \emptyset$, and $\neg\exists C \in \Delta_{\Gamma} (B_0 \subseteq C \land B_1 \cap C = \emptyset)$. We must show that

$$\forall A_0, A_1 \in \check{\Gamma} \exists A'_0, A'_1 \in \Gamma (A_0 \setminus A_1 \subseteq A'_0 \land A_1 \setminus A_0 \subseteq A'_1 \land A'_0 \cap A'_1 = \emptyset).$$

Fix $A_0, A_1 \in \check{\Gamma}$ and consider the game $G^*_{\text{sep}}(A_0, A_1, B_0, B_1)$ defined on page 169. By Exercise 24.6 $\Pi$ cannot have a winning strategy, so by determinacy, there is a continuous $f : \mathbb{R} \to \mathbb{R}$ such that

$$f[A_0 \setminus A_1] \subseteq B_0 \quad \text{and} \quad f[A_1 \setminus A_0] \subseteq B_1.$$ 

Take $A'_0 = f^{-1}[B_0]$ and $A'_1 = f^{-1}[B_1]$.

($\Rightarrow$) Suppose $\text{Sep}^*(\check{\Gamma})$. We must find a pair of inseparable $\Gamma$-sets. Let $A \in \Gamma \setminus \check{\Gamma}$ and let

$$U = \{ (x \oplus y, z) \mid \ell_x(z) \in A \}, \quad V = \{ (x \oplus y, z) \mid \ell_y(z) \in A \}.$$ 

Then $(U, V)$ is a universal pair for $\Gamma$, in the sense of ??.

By $\text{Sep}^*(\check{\Gamma})$ there are $B_0, B_1 \in \Gamma$ such that

- $B_0, B_1 \subseteq \mathbb{R} \times \mathbb{R}$, $B_0 \cap B_1 = \emptyset$,
- $U \setminus V = \neg V \setminus \neg U \subseteq B_0$,
- $V \setminus U = \neg U \setminus \neg V \subseteq B_1$.

Let $A_i = \{ z \mid (z, z) \in B_i \}$ with $i \in 2$. As $B_0$ and $B_1$ are disjoint, so are $A_0$ and $A_1$, and as $z \mapsto (z, z)$ is continuous, then $A_0, A_1 \in \Gamma$. We will show that they cannot be separated by a set in $\Delta_{\Gamma}$. Suppose, towards a contradiction, that $A_0 \subseteq C$ and $A_1 \cap C = \emptyset$ for some $C \in \Delta_{\Gamma}$. As $\neg C, C \in \Gamma$ let $x, y \in \mathbb{R}$ be such that $\neg C = \ell_x^{-1}[A]$ and $C = \ell_y^{-1}[A]$. Let $w = x \oplus y$. Then $\neg C = U_w$ and $C = V_w$. 


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\[ w \in C \Rightarrow (w, w) \in V \setminus U \subseteq B_1 \]
\[ \Rightarrow w \in A_1 \]
\[ \Rightarrow w \notin C \]

and

\[ w \in \neg C \Rightarrow (w, w) \in U \setminus V \subseteq B_0 \]
\[ \Rightarrow w \in A_0 \]
\[ \Rightarrow w \in C \]

in both cases reaching a contradiction. 

We now turn to 24.7. Let \( U \) and \( V \) be a universal pair for \( \Gamma \) as in the proof of 24.7 on page 172. Define the following two games \( G_0 \) and \( G_1 \) on \( \omega^* \):

- \( \mathbf{II} \) wins iff \( G_0 \)
  \[ (y \notin U \cap V) \& (x \in U \setminus V \Rightarrow y \in V \setminus U) \& (x \in V \setminus U \Rightarrow y \in U \setminus V) \].

- \( \mathbf{II} \) wins iff \( G_1 \)
  \[ (y \in U \cup V) \& (x \in U \setminus V \Rightarrow y \in U \setminus V) \& (x \in V \setminus U \Rightarrow y \in V \setminus U) \].

**Lemma 24.9.** (a) If \( \mathbf{II} \) wins \( G_0 \) then \( \text{Sep}^*(\Gamma) \).

(b) If \( \mathbf{II} \) wins \( G_1 \) then \( \text{Sep}^*(\Gamma) \).

**Proof.** Suppose \( \tau \) is a winning strategy in \( G_0 \) and let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be the induced function. Let \( U' = f^{-1}[V] \) and \( V' = f^{-1}[U] \).

**Claim 24.9.1.** \( U \setminus V \subseteq U' \), \( V \setminus U \subseteq V' \) and \( U' \cap V' = \emptyset \).

**Proof.** Suppose \( x \in U \setminus V \). Then \( f(x) \in V \setminus U \subseteq V \) so \( x \in U' = f^{-1}[V] \). Similarly \( V \setminus U \subseteq V' \). If \( x \in U' \cap V' \) then \( f(x) \in V \cap U \), contradicting the fact that \( \tau \) is winning.

Let \( A, B \in \tilde{\Gamma} \) and let \( a, b \in \mathbb{R} \) such that \( A = \neg (U_{a \sqcup b}) = (\neg U)_{a \sqcup b} \) and \( B = \neg (V_{a \sqcup b}) = (\neg V)_{a \sqcup b} \). Then

\[
A \setminus B = (\neg U \setminus \neg V)_{a \sqcup b} = (V \setminus U)_{a \sqcup b} \subseteq (U')_{a \sqcup b}
\]
\[
B \setminus A = (\neg V \setminus \neg U)_{a \sqcup b} = (U \setminus V)_{a \sqcup b} \subseteq (V')_{a \sqcup b}
\]
\[
(U')_{a \sqcup b} \cap (V')_{a \sqcup b} = \emptyset
\]
and \((U')_{a \in b}, (V')_{a \in b} \in \Gamma\). Therefore \(\text{Sep}^*(\Gamma)\).

(b) As in (a) let \(\tau\) be a winning strategy for \(\Pi\) in \(G_i\) and let \(f : \mathbb{R} \to \mathbb{R}\) be the induced function. Let \(U' = f^{-1}[-V]\) and \(V' = f^{-1}[-U]\). The proof proceeds as before.

\[\text{Proof.} \quad 24.8\] By the Lemma and by determinacy it is enough to show that \(I\) cannot win both \(G_0\) and \(G_1\). Suppose, towards a contradiction, that \(\sigma_0\) and \(\sigma_1\) are winning strategies for \(I\) in \(G_0\) and \(G_1\), and let \(f_0\) and \(f_1\) be the induced functions. Let \(y \in \mathbb{R}\) and \(x = f_0(y) = \sigma_0 \ast y\). As \(\sigma_0\) is winning then

\[(x \in U \setminus V \& y \notin V \setminus U) \lor (x \in V \setminus U \& y \notin U \setminus V) \lor (y \in U \cap V)\]

hence, by a tedious but straightforward computation,

\[(y \in U \setminus V \Rightarrow f(y) \in U \setminus V) \& (y \in U \setminus V \Rightarrow f(y) \in V \setminus U) \& (y \in U \setminus V \Rightarrow f(y) \in U \Delta V)\, .\]

Similarly for every \(y \in \mathbb{R}\),

\[(y \in V \setminus U \Rightarrow f(y) \in V \setminus U) \& (y \in V \setminus U \Rightarrow f(y) \in U \setminus V) \& (y \in U \cap V \Rightarrow f(y) \in U \Delta V)\, .\]

To summarize:

1. \(f_0[U \setminus V] \subseteq U \setminus V\) and \(f_0[V \setminus U] \subseteq V \setminus U\),
2. \(f_0[U \setminus V] \subseteq U \setminus V\) and \(f_0[V \setminus U] \subseteq V \setminus U\),
3. \(f_i[U \Delta V] \subseteq U \Delta V\), for \(i = 0, 1\),
4. \(f_0[-(U \cap V)] \subseteq U \Delta V\),
5. \(f_1[U \cap V] \subseteq U \Delta V\).

We now apply the Martin–Monk technique. For any \(z \in \omega \), \(\langle (G_{z(i)}, \sigma_{z(i)}) \mid i \in \omega \rangle\) admits a global play \(x(z)\), as \(\text{Player}(\sigma_{z(i)}) = I\) for all \(i \in \omega\). As usual \(x(z)_i\) is the real played on the \(i\)-th row and \(x(z)_i = f_{z(i)}(x(z)_{i+1})\).

\[\text{Claim 24.9.2.} \quad \{z \in \omega \mid x(z)_0 \in \neg(U \Delta V)\} \text{ is meager.}\]

\[\text{Proof.}\] Suppose the set in question is non-meager. Then it is comeager in some \(\mathcal{N}_s\), with \(s \in <\omega 2\), hence

\[\{z \in \omega \mid x(s \cdot z)_0 \in \neg(U \Delta V)\}\]
is comeager. Notice that
\[ x(s \cdot z)_0 = x(z)_{lh(s)} = f_s(0) \circ \cdots \circ f_{s(lh(s)-1)}(x(z)_0) \]
so by (3)
\[ x(s \cdot z)_0 \in \neg (U \triangle V) \Rightarrow x(z)_0 \in \neg (U \triangle V) \]
hence \( Z = \{ z \in 2^\omega \mid x(z)_0 \in \neg (U \triangle V) \} \) is comeager, and therefore \( \{ z \in 2^\omega \mid x(z)_0 \in U \triangle V \} \) is meager. Then \( Z = A \cup B \), where
\[ A = \{ z \in Z \mid x(z)_0 \in U \cap V \}, \quad B = \{ z \in Z \mid x(z)_0 \in \neg (U \cup V) \}, \]
so \( A \) or \( B \) (or both) is non-meager. We will finish the proof of the claim by showing that both \( A \) and \( B \) are meager, reaching thus a contradiction.

Assume \( A \) is non-meager. By (5)
\[ z \in A \Rightarrow x((1)^{\cdot}z)_0 = f_1(x(z)_0) \in U \triangle V \]
hence \( \{ z \mid x(z)_0 \in U \triangle V \} \) is non-meager, as it contains \((1)^{\cdot}A\): a contradiction.

Similarly if \( B \) is non-meager then (4) implies
\[ z \in B \Rightarrow x((0)^{\cdot}z)_0 = f_0(x(z)_0) \in U \triangle V \]
hence \( \{ z \mid x(z)_0 \in U \triangle V \} \) is non-meager, as it contains \((0)^{\cdot}B\): again a contradiction.

Therefore \( C = \{ z \mid x(z) \in U \triangle V \} \) is comeager. Suppose without loss of generality that \( \{ z \mid x(z) \in U \setminus V \} \) is non-meager, hence comeager in some \( N_s \). Then
\[ D = \{ z \mid x(s \cdot z)_0 \in U \setminus V \} \]
is comeager. Let \( k = \text{\#}\{ i < lh(s) \mid s(i) = 1 \} \) and recall that \( x(s \cdot z)_0 = f_{s(0)} \circ \cdots \circ f_{s(n-1)}(x(z)_0) \).

If \( k \) is odd then by (1) and (2)
\[ x(z)_0 \in U \setminus V \Rightarrow x(s \cdot z)_0 \in V \setminus U \]
\[ x(z)_0 \in V \setminus U \Rightarrow x(s \cdot z)_0 \in U \setminus V \]
so \( E = \{ z \mid x(z)_0 \notin U \setminus V \} \) is comeager as \( E \cap C \supseteq \{ z \mid x(z)_0 \in V \setminus U \} \supseteq D \). But this contradicts our assumption that \( \neg E \) is non-meager.

Suppose now \( k \) is even. By (1) and (2) again
\[ x(z)_0 \in U \setminus V \Rightarrow x(s \cdot z)_0 \in U \setminus V \]
\[ x(z)_0 \in V \setminus U \Rightarrow x(s \cdot z)_0 \in V \setminus U \]
so \( F = \{ z \mid x(z)_0 \notin V \setminus U \} \) is comeager. By (1) and (2)
\[
z \in F \cap C \Rightarrow x((1)^\sim z)_0 \in V \setminus U.
\]
As the map \( \omega \rightarrow N_{(1)} \), \( z \mapsto (1)^\sim z \), is a homeomorphism, then \( \{ z \mid x(z)_0 \in V \setminus U \} \) is comeager in \( N_{(1)} \), hence it is non-meager, contradicting the comeagerness of \( F \). This concludes the proof of 24.8. \( \square \)

The separation property allows to “transfer” closure property of point-classes from \( \Delta_\Gamma \) to \( \Gamma \).

**Theorem 24.10** (Kechris–Steel). *(AD)* Suppose \( \Gamma \) is non-self-dual and \( \text{Sep}(\Gamma) \) holds. Let \( \Delta = \Delta_\Gamma \).

(a) If \( \Delta \) is closed under finite (countable) unions then so is \( \Gamma \), that is
\[
\bigcup (\alpha; \Delta) \subseteq \Delta \Rightarrow \bigcup (\alpha; \Gamma) \subseteq \Gamma
\]
for \( \alpha = 2, \omega \).

(b) If \( \Delta \) is closed under projections then so is \( \Gamma \), and hence \( \hat{\Gamma} \) is closed under co-projections, that is
\[
\exists^R \Delta \subseteq \Delta \quad \Rightarrow \quad \left( \exists^R \Gamma \subseteq \Gamma \quad \& \quad \forall^R \hat{\Gamma} \subseteq \Gamma \right).
\]

**Proof.** (a) Let us assume \( \Delta \) is closed under countable unions, the finite case being similar. Towards a contradiction suppose \( A_n \in \Gamma \) and \( \bigcup_n A_n \notin \Gamma \). Let \( B \in \hat{\Gamma} \setminus \Gamma \). Then \( \bigcup_n A_n \not\leq_w -B \) so by Wadge’s Lemma \( B \leq_w \bigcup_n A_n \) via some \( f \). Then \( B = f^{-1} \left[ \bigcup_n A_n \right] = \bigcup_n f^{-1} [A_n] \). Each \( f^{-1} [A_n] \in \Gamma \) and it is disjoint from \( -B \in \Gamma \), hence there are \( C_n \in \Delta \) such that \( B \supseteq C_n \supseteq f^{-1} [A_n] \). Therefore \( B = \bigcup_n C_n \in \Delta \), contradicting our choice of \( B \).

(b) Suppose, towards a contradiction, that \( \exists^R \Gamma \not\subseteq \Gamma \). By Wadge’s Lemma \( \hat{\Gamma} \subseteq \exists^R \Gamma \). Therefore it is enough to show that any pair of disjoint sets in \( \exists^R \Gamma \) can be separated by a set in \( \Delta \), as this would imply \( \text{Sep}(\hat{\Gamma}) \), contradicting \( \text{Sep}(\Gamma) \) and 24.4. So let \( A, B \in \exists^R \Gamma \) with \( A \cap B = \emptyset \). Then
\[
A = \{ x \in \mathbb{R} \mid \exists y \in \mathbb{R} (x, y) \in P \}
B = \{ x \in \mathbb{R} \mid \exists z \in \mathbb{R} (x, z) \in Q \}
\]
for some \( P, Q \in \Gamma \). Let \( P', Q' \subseteq \mathbb{R}^3 \),
\[
P' = \{ (x, y, z) \mid (x, y) \in P \}
Q' = \{ (x, y, z) \mid (x, z) \in Q \}.
\]
Then $P', Q' \in \Gamma$ and $P' \cap Q' = \emptyset$, by the fact that $A \cap B = \emptyset$. Let $D \subseteq \mathbb{R}^3$, $D \in \Delta$, such that $P' \subseteq D$ & $D \cap Q' = \emptyset$, and let

$$C = \exists^R \forall^R D = \{x \mid \exists y \forall z (x, y, z) \in D\}.$$ 

Then $C \in \Delta$ since for self-dual pointclasses closure under $\exists^R$ is equivalent to closure under $\forall^R$. If $x \in A$ then $(x, y) \in P$ for some $y$ and hence $(x, y, z) \in P' \subseteq D$ for every $z$, and therefore $x \in C$. Vice versa if $x_0 \in B \cap C$ then pick $y_0, z_0 \in \mathbb{R}$ such that $(x_0, z_0) \in Q$ and $(x_0, y_0) \in \forall^R D = \{(x, y) \mid \forall z (x, y, z) \in D\}$: then $(x_0, y_0, z_0) \in D \cap Q' = \emptyset$, a contradiction. Therefore $B \cap C = \emptyset$. □

**Theorem 24.11** (Steel). (AD) Let $\Gamma$ be non-self-dual and suppose $\neg$Sep($\Gamma$). If $\Gamma$ is closed under finite unions then $\Gamma$ is closed under countable unions, that is

$$\bigcup(2; \Gamma) \subseteq \Gamma \Rightarrow \bigcup(\omega; \Gamma) \subseteq \Gamma.$$ 

**Proof.** Towards a contradiction, suppose $\bigcup(\omega; \Gamma) \not\subseteq \Gamma$. By Wadge’s Lemma 24.1 $\Gamma \subseteq \bigcup(\omega; \Gamma)$. Let $A_0, A_1 \in \Gamma$, $A_0 \cup A_1 = \emptyset$ and $\Delta_{\Gamma}$-inseparable. As $\Gamma$ is closed under finite unions $\neg(A_0 \cup A_1) \in \Gamma$ and therefore $\neg(A_0 \cup A_1) = \bigcup_n C_n$, for some $C_n \in \Gamma$. Notice that $A_0 \cup C_n, A \in \Gamma$ are also $\Delta_{\Gamma}$-inseparable so by 24.6 let

$$\sigma_{2n} \text{ a winning strategy for } I \text{ in } \mathcal{G}_{2n} = G_{sep}^n(A_0 \cup C_n, A_1, A_0, A_1)$$

$$\sigma_{2n+1} \text{ a winning strategy for } I \text{ in } \mathcal{G}_{2n+1} = G_{sep}^n(A_0 \cup C_n, A_1, A_1, A_0).$$

Now apply the Martin–Monk method: as Player($\sigma_n$) = I for all $n$, $\langle (G_{z(n)}, \sigma_{z(n)}) \mid n \in \omega \rangle$ admits a global play for any $z \in {}^\omega \omega = \mathbb{R}$. Let $x(z)_n$ be the real on the $n$-th row of the Martin–Monk diagram, so that $x(z)_n = f_{\sigma(n)}(x(z)_{n+1})$, where $f_i$ is the function induced by $\sigma_i$. By 24.6

$$f_{2n}[A_0 \cup C_n] \subseteq A_0 \quad f_{2n}[A_1] \subseteq A_1$$

$$f_{2n+1}[A_0 \cup C_n] \subseteq A_1 \quad f_{2n+1}[A_1] \subseteq A_0$$

hence $f_{s_0} \circ \cdots \circ f_{s_{k-1}}(x) \in A_0 \cup A_1$ for any $s \in {}^k \omega$, any $i < 2$ and any $x \in A_i$, and

$$f_{s_0} \circ \cdots \circ f_{s_{k-1}}(x) = A_i \Leftrightarrow \{|j < k \mid s_j \text{ is odd }\} \text{ is even}.$$ 

As $\mathbb{R} = A_0 \cup A_1 \cup \bigcup_n C_n$, to reach a contradiction it is enough to show that

$$\{z \in \mathbb{R} \mid x(z)_0 \in A_0 \cup A_1\}$$
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and

\[ \{ z \in \mathbb{R} \mid x(z)_0 \in C_n \} \quad (n < \omega) \]

are meager.

Suppose \( \{ z \in \mathbb{R} \mid x(z)_0 \in A_0 \cup A_1 \} \) is non-meager and pick \( i < 2 \) and \( s \in \omega^- \omega \) such that \( \{ z \in \mathbb{R} \mid x(z)_0 \in A_i \} \) is comeager in \( N_s \). By extending \( s \), if needed, we may assume that \( |\{ j < k = \text{lh}(s) \mid s_j \text{ is odd} \}| \) is even. Notice that

\[ x(s^\frown z)_0 = f_{s_0} \circ \cdots \circ f_{s_{k-1}}(x(s^\frown z)_k) = f_{s_0} \circ \cdots \circ f_{s_{k-1}}(x(z)_0) \]

so \( x(s^\frown z)_0 \in A_i \iff x(z)_0 \in A_i \) and therefore \( X = \{ z \in \mathbb{R} \mid x(z)_0 \in A_i \} \) is comeager. But

\[ z \in X \Rightarrow x(1^\frown z)_0 = f_1(x(z)_0) \in A_{1-i}, \]

hence

\[ \{ z \in \mathbb{R} \mid x(z)_0 \in A_{1-i} \} \supseteq 1^\frown X \]

is non-meager: a contradiction.

Suppose now \( M_n = \{ z \in \mathbb{R} \mid x(z)_0 \in C_n \} \) is non-meager and hence comeager in some \( N_s \), with \( s \in \omega^- \omega \). If \( x(z)_0 \in C_n \) then \( f_{2n}(x(z)_0) \in A_0 \) hence

\[ x(s^\frown (2n)^\frown z)_0 = f_{s_0} \circ \cdots \circ f_{s_{k-1}}(f_{2n}(x(z)_0)) \in A_0 \cup A_1 \]

so

\[ \{ z \in \mathbb{R} \mid x(z)_0 \notin C_n \} \supseteq s^\frown (2n)^\frown M_n \]

hence it is comeager in \( N_s \cap (2n)^\frown s \), and therefore non-meager in \( N_s \): a contradiction. \( \square \)

Open problem 24.12 (Steel). Assume AD and suppose \( \Gamma \) is a non-self-dual pointclass closed under countable unions and intersections. Does \( \mathcal{A} \Gamma \subseteq \Gamma \) or \( \mathcal{A} \Gamma \not\subseteq \Gamma \)?

24.B Reduction

Definition 24.13. A pointclass \( \Gamma \) has the reduction property, \( \text{Red}(\Gamma) \), iff

\[ \forall A, B \in \Gamma \exists A', B' \in \Gamma (A' \subseteq A \& B' \subseteq B \& A \cup B = A' \cup B' \& A' \cap B' = \emptyset) . \]

We say that the pair \((A', B')\) reduces the pair \((A, B)\).
Exercise 24.14. (i) Show that $\text{Unif}(\Gamma) \Rightarrow \text{Red}(\Gamma)$.

(ii) Show that $\text{Red}(\Gamma) \Rightarrow \text{Sep}(\bar{\Gamma})$. Therefore under $\text{AD}$, if $\Gamma$ is a non-self-dual pointclass then $\text{Red}(\Gamma) \Rightarrow \neg \text{Sep}(\Gamma)$.

The second part of (ii) can also be proved by quoting 24.1 and using Proposition 24.15 (Novikov). Suppose $\Gamma$ has a universal set. Then $\text{Red}(\Gamma) \Rightarrow \neg \text{Sep}(\Gamma)$.

Proof. Deny. Let $U \subseteq \mathbb{R}^2$ be $\Gamma$-universal and let

\[ A = \{(x, y) \in \mathbb{R}^2 \mid (x, y_1) \in U\} \]
\[ B = \{(x, y) \in \mathbb{R}^2 \mid (x, y_\bar{1}) \in U\}. \]

Then $A, B \in \Gamma$ so there are $A', B' \in \Gamma$ such that $A' \subseteq A$, $B' \subseteq B$, and $A' \cap B' = \emptyset$. Suppose $A' \subseteq C$ and $B' \cap C = \emptyset$ for some $C \in \Delta_\Gamma$. Then $C$ is $\Delta_\Gamma$-universal, contradicting ??(iii): if $D \in \Delta_\Gamma$ let $a, b \in \mathbb{R}$ such that $D = U_a$ and $\neg D = U_b$, and let $d = a \oplus b$. Then

\[ (x, d) \in A \iff (x, a) \in U \]
\[ \iff x \in D \]
\[ \iff x \notin \neg D \]
\[ \iff (x, b) \notin U \]
\[ \iff (x, d) \notin B \]

so $(x, d) \in A' \iff (x, d) \notin B'$ hence $x \in D \iff (x, d) \in C$. \hfill \square

Corollary 24.16. ($\text{AD}$) $\neg \text{Red}(\Gamma) \lor \neg \text{Red}(\bar{\Gamma})$, for any non-self-dual $\Gamma$.

By Kondo’s theorem ?? then $\text{Red}(\Pi^1_1)$, but one can give a direct proof of this fact.

Theorem 24.17 (Kuratowski). $\text{Red}(\Pi^1_1)$.

Proof. Blackwell Let $A, B \in \Pi^1_1$ and let $T, U$ be trees on $\omega \times \omega$ such that $\neg A = p[T]$ and $\neg B = p[U]$. For $z \in \mathbb{R}$ consider the game $G_z$ on $\omega$ where $I$ plays $x$, $\Pi$ plays $y$ and $I$ wins iff $\exists n (x \upharpoonright n \in U_z \land y \upharpoonright n \notin T_z)$.

Exercise 24.18. Show that $G_z$ is a clopen game and hence determined.

Let

\[ A' = \{z \in A \mid \Pi \text{ does not win } G_z\} \]
\[ B' = \{z \in B \mid I \text{ does not win } G_z\}. \]
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Claim 24.18.1. \( A \setminus B \subseteq A' \) and \( B \setminus A \subseteq B' \).

Proof. Let \( z \in A \setminus B = p[U] \setminus p[T] \). Then \( U_z \) is ill-founded while \( T_z \) is well-founded so \( I \) wins \( G_z \) by playing any \( x \in [U_z] \). Therefore \( z \in A' \).

The other inclusion is similar. \( \square \)

If \( z \in A \cap B \) then \( z \in A' \cup B' \) as otherwise both \( I \) and \( II \) would have been winning strategies for \( G_z \). As all \( G_z \)'s are determined, \( A' \) and \( B' \) are disjoint.

Summarizing \( A_0 \cap B_0 = A' \cup B' \) and \( A_0 \cap B_0 = A' \cap B' = \emptyset \).

Moreover

\[
\begin{align*}
  z \in A' & \iff z \in A_0 & \forall \tau \exists n \exists s \in \omega ((z \upharpoonright n, s) \in U & \& (z \upharpoonright n, (s \uparrow n) \not\in T) \\
  & \text{by } 24.10 \text{ } U & \text{closed under finite unions} & \text{and therefore } A_0 \subseteq A, B_0 \subseteq B, A' \cap B' = \emptyset.
\end{align*}
\]

shows that \( A_0 \in \Pi^1_1 \) and similarly \( B_0 \in \Pi^1_1 \). \( \square \)

Theorem 24.19 (Steel–Van Wesep). Let \( \Gamma \) be non-self-dual such that \( \Delta = \Delta_F \) is closed under finite unions. Then \( \text{Red}(\Gamma) \lor \text{Red}(\Gamma) \).

Proof. By 24.3 \( \text{Sep}(\Gamma) \) or \( \text{Sep}(\bar{\Gamma}) \) but not both. Assume for the sake of definiteness that \( \text{Sep}(\bar{\Gamma}) \). By 24.14 or 24.15 \( \neg \text{Red}(\Gamma) \), as otherwise \( \text{Sep}(\Gamma) \), so we are left to prove \( \text{Red}(\Gamma) \). Suppose that \( \neg \text{Red}(\Gamma) \). We will reach a contradiction by proving \( \neg \text{Sep}(\Gamma) \).

Let \( A, B \in \Gamma \) witness \( \neg \text{Red}(\Gamma) \) and let \( C, D \in \Gamma \) be disjoint. Consider the game where \( I \) and \( II \) play \( x \) and \( y \) and \( II \) wins iff

\[
y \in A \cup B \& \text{ (} x \in C \Rightarrow y \in A \setminus B \text{) } \& \text{ (} x \in D \Rightarrow y \in B \setminus A \text{).}
\]

Claim 24.19.1. \( I \) does not win this game.

Proof. Let \( \sigma \) be a winning strategy for \( I \) and let \( f \) be the induced function. Let \( A' = f^{-1}[D] \cap A \) and \( B' = f^{-1}[C] \cap B \). By 24.10 \( \bigcup (2; \bar{\Gamma}) \subseteq \bar{\Gamma} \) hence \( \bigcap (2; \Gamma) \subseteq \Gamma \) and therefore \( A', B' \in \Gamma \). By construction \( A' \subseteq A, B' \subseteq B \) and \( A' \cap B' = \emptyset \). Let \( y \in A \cup B \) and let \( x = f(y) \): as \( \sigma \) is winning then

\[
(x \in C \& y \in B) \lor (x \in D \& y \in A)
\]

so that \( y \in f^{-1}[C] \cap B = B' \lor (y \in f^{-1}[D] \cap A = A') \). In other words \( (A', B') \) reduces \( (A, B) \) contrarily to our assumption. \( \square \)
Therefore \( \Pi \) wins and let \( f \) be the map induced by the winning strategy. Then \( \text{ran}(f) \subseteq A \cup B, f[C] \subseteq A \setminus B, \) and \( f[D] \subseteq B \setminus A. \) Let \( C' = f^{-1}[\neg B] = f^{-1}[A \setminus B] \) and \( D' = f^{-1}[\neg A] = f^{-1}[B \setminus A]. \) Then \( C', D' \in \Gamma, C \subseteq C', \) \( D \subseteq D' \) and \( C' \cap D' = \emptyset. \) By Sep(\( \Gamma \)) there is an \( E \in \Delta \) which separates \( C' \) from \( D' \) and hence \( C \) from \( D. \) As \( C, D \) are arbitrary disjoint sets in \( \Gamma, \) this proves Sep(\( \Gamma \)).

**Exercise 24.20.** (AD) Show that if \( \Gamma \) is non-self-dual then

\[
\bigcap (2; \Gamma) \subseteq \Gamma \& \text{Sep}(\tilde{\Gamma}) \Rightarrow \text{Red}(\Gamma).
\]

The next result shows that there are plenty of non-self-dual boldface pointclasses \( \Gamma \) for which reduction fails for \( \Gamma \) and \( \tilde{\Gamma}. \) In order to state it in a convenient form we need the notion of backtrack reducibility—see 22.20.

**Theorem 24.21** (Van Wesep). Assume AD\(^L\). Suppose \( A \not\in_{bt} \neg A, A \not\in \{0, \mathbb{R}\}, \) and let \( B \) be non-self-dual such that \( [B]_{bt} \) is the \( \leq_{bt} \)-successor of \( [A]_{bt} \) and \( [\neg A]_{bt}. \) Let \( \Gamma = \{ C \subseteq \mathbb{R} \mid C \leq_w \neg B \} \)

and

\( \tilde{\Gamma} = \{ C \subseteq \mathbb{R} \mid C \leq_w B \}. \)

Then reduction fails for \( \Gamma \) and \( \tilde{\Gamma}, \) that is \( \neg \text{Red}(\Gamma) \& \neg \text{Red}(\tilde{\Gamma}). \)

**Proof.** Let \( \Lambda = \{ C \subseteq \mathbb{R} \mid C \leq_w A \}. \) By 24.14 reduction must fail for \( \Lambda \) or \( \tilde{\Lambda}, \) say \( \neg \text{Red}(\tilde{\Lambda}). \) Let

\( C = \{ x \oplus y \mid x \in A \} \)

and

\( D = \{ x \oplus y \mid y \in A \}. \)

We shall prove that the pair \( (C, D) \) cannot be reduced by a pair of sets in \( \Gamma \) nor by a pair of sets in \( \tilde{\Gamma}. \)

**Claim 24.21.1.** \( (C, D) \) cannot be reduced by a pair of sets in \( \Lambda \cup \tilde{\Lambda}. \)
Proof. Towards a contradiction, let \((C^*, D^*)\) be a pair of sets in \(\Lambda \cup \tilde{\Lambda}\) reducing \((C, D)\). If \(C^*, D^* \leq_W A\) then for any \(C', D' \leq_W A\) pick continuous functions \(f\) and \(g\) such that \(C' = f^{-1}[A]\) and \(D' = g^{-1}[A]\) and let \(h(x) = (f(x), g(x))\). Then \((h^{-1}[C^*], h^{-1}[D^*])\) reduces \((C', D')\), contradicting \(\neg \text{Red}(\Lambda)\). Therefore at least one among \(C^*\) and \(D^*\) is not in \(\Lambda\), say \(C^* \in \Lambda \setminus \Lambda\). Pick \(y_0 \notin A\) —here is where \(\emptyset \subset A \subset \mathbb{R}\) is used. Then

\[
x \in A \Rightarrow x \oplus y_0 \in C \ & \ x \oplus y_0 \notin D \Rightarrow x \oplus y_0 \in C^*
\]

that is to say: the map \(x \mapsto x \oplus y_0\) witnesses \(A \leq_W C^*\) and therefore \(\Lambda \subseteq \tilde{\Lambda}\): a contradiction.

By 22.22 we may assume \(T(A) = \langle \omega \omega \rangle\) and thus \(\forall s \in \langle \omega \omega \rangle (C_{[s]} \equiv_W C \ & \ D_{[s]} \equiv_W D)\).

Therefore by the arguments above

\[
\forall s \in \langle \omega \omega \rangle (C_{[s]}; D_{[s]}) \text{ cannot be reduced by a pair of sets in } \Lambda \cup \tilde{\Lambda}.
\]

So suppose \((C^*, D^*)\) is a pair of sets in \(\Gamma\) reducing \((C, D)\) —the case when \(C^*, D^* \in \Gamma\) is completely analogous and it is left to the reader. By the arguments above we may assume that \(\{C^*, D^*\} \not\subseteq \Lambda \cup \tilde{\Lambda}\), say \(C^* \in [B]_{bt}\).

Since \([B]_{bt}\) is the \(\leq_{bt}\)-successor of \([A]_{bt}\), by 22.23(ii) there is an \(s_0 \in \langle \omega \omega \rangle\) such that \(C^*_{[s_0]} \in \Lambda \cup \tilde{\Lambda}\). If \(D^* \in \Lambda \cup \tilde{\Lambda}\) then set \(s = s_0\); otherwise \(D^* \equiv_{bt} B\) so again by 22.23(ii) there is an \(s \supseteq s_0\) such that \(D^*_{[s]} \in \Lambda \cup \tilde{\Lambda}\). In both cases \((C^*_{[s]}, D^*_{[s]})\) is a pair of sets in \(\Lambda \cup \tilde{\Lambda}\) reducing \((C, D)\): a contradiction. \(\Box\)

**Exercise 24.22.** Show in \(ZF + AC_\omega(\mathbb{R})\) that the least non-self-dual \(\Gamma \supset \Sigma^0_2 \cup \Pi^0_2\) witnesses \(\neg \text{Red}(\Gamma) \& \neg \text{Red}(\Gamma^*)\)

**Notes and References**

### 25 Scattered results

Suppose \(\mu\) is a countably complete ultrafilter on some \(\kappa > \Theta^L(\mathbb{R})\) and let \(\text{Ult}(L(\mathbb{R}), \mu) = L(\mathbb{R})/\mu\) be the ultrapower formed using functions in \(L(\mathbb{R})\).

**Lemma 25.1.** \(\text{Ult}(L(\mathbb{R}), \mu)\) satisfies Los’ theorem.

**Proof.** It is enough to show that if

\[
S = \{ \alpha < \kappa \mid L(\mathbb{R}) \models \exists v_0 \varphi[v_0, f_1(\alpha), \ldots, f_n(\alpha)] \} \in \mu
\]

where \(f_1, \ldots, f_n \in L(\mathbb{R})\), then there is an \(f_0 \in L(\mathbb{R})\) such that

\[
\{ \alpha \in S \mid L(\mathbb{R}) \models \varphi[f_0(\alpha), f_1(\alpha), \ldots, f_n(\alpha)] \} \in \mu.
\]
Let $\Phi : \mathbb{R} \times \text{Ord} \to L(\mathbb{R})$ and let $g : S \to \text{Ord}$,

$$g(\alpha) = \text{the least } \beta \left( \exists r \in \mathbb{R} \ L(\mathbb{R}) \models \varphi[\Phi(r, \beta), f_1(\alpha), \ldots f_n(\alpha)] \right)$$

For $\alpha \in S$ let

$$A_\alpha = \{ r \in \mathbb{R} \mid L(\mathbb{R}) \models \varphi[F(r, g(\alpha)), f_1(\alpha), \ldots f_n(\alpha)] \} \neq \emptyset$$

so there is $S' \subset S$, $S' \in \mu$ such that

$$\forall \alpha, \beta \in S' (A_\alpha = A_\beta).$$

Fix $\bar{r} \in A_\alpha$, with $\alpha \in S'$. Set $f_0 : S \to L(\mathbb{R})$, $f_0(\alpha) = F(\bar{r}, g(\alpha))$. As $g \in L(\mathbb{R})$ by definability, $f_0 \in L(\mathbb{R})$ too. Hence

$$S' = \{ \alpha < \kappa \mid L(\mathbb{R}) \models \varphi[f_0(\alpha), f_1(\alpha), \ldots f_n(\alpha)] \} \in \mu.$$

Notice that $\text{Ult}(L(\mathbb{R}), \mu)$ is well-founded, otherwise, by DC in $L(\mathbb{R})$, there would be $f_\eta \in L(\mathbb{R})$ such that $[f_{\eta+1}]_\mu \in^{\text{Ult}(L(\mathbb{R}), \mu)} [f_\eta]_\mu$. Therefore

$$S_\eta = \{ \alpha < \kappa \mid f_{\eta+1}(\alpha) \in f_\eta(\alpha) \} \in \mu$$

hence $S = \bigcap \eta S_\eta \in \mu$ and therefore $\bar{\alpha} \in S \Rightarrow \forall n (f_{n+1}(\alpha) \in f_n(\alpha))$: a contradiction.

Also $j : L(\mathbb{R}) \to \text{Ult}(L(\mathbb{R}), \mu)$, the canonical embedding is elementary. Thus $\text{Ult}(L(\mathbb{R}), \mu) \models V = L(j(\mathbb{R}))$. As $j(\mathbb{R}) = \mathbb{R}$, $\text{Ult}(L(\mathbb{R}), \mu) = L(\mathbb{R})$, hence $\mathbb{R}^\sharp$ exists, Therefore we have shown that

**Lemma 25.2.** In $L(\mathbb{R})$ there is no measurable cardinal above $\Theta$.

$\Theta$ is the supremum of measurable cardinals, so if $\mathcal{P} \not\subset L(\mathbb{R})$, then $\Theta^{L(\mathbb{R})} < \Theta$ and there is a measurable $\Theta^{L(\mathbb{R})} < \kappa < \Theta$.

Some remarks.

Suppose $M$ and $N$ are proper class models with divergent Wadge hierarchies. Let $\mathbb{R} = \mathbb{R}^M$. Then $(\mathcal{P}(\mathbb{R}))^M \not\subset L(\mathbb{R})$ so $\mathbb{R}^\sharp$ exists in $M$. By absoluteness of sharps, $\mathbb{R}^\sharp$ exists in $V$.

**Proposition 25.3.** Assume $\text{AD} + \Theta$ singular. Then $\mathbb{R}^\sharp$ exists and hence $\text{Con}(\text{ZF} + \text{AD})$.

**Proof.** In $L(\mathbb{R})$ there is a $\Theta$-sequence cofinal in $\mathcal{P}(\mathbb{R})$ and this is equivalent to the regularity of $\Theta$. Therefore $\Theta^{L(\mathbb{R})} < \Theta$. Now proceed as before. \qed
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Recall the following conjecture of Martin’s in recursion theory:

Open problem 25.4 (Martin’s Conjecture). Assume $AD + DC(\mathbb{R})$. Suppose $f : D \to D$ is such that $\mu_M(\{d \in D \mid f(d) \not\leq d\}) = 1$. Then

$$\exists c \in D \mu_M(\{d \in D \mid f(d) = c\}) = 1$$

i.e. $f$ is constant on a cone.

Proposition 25.5. Assume $AD + DC(\mathbb{R})$. Then Martin’s conjecture implies that $|D \times 2| \neq |D|$.

Proof. Let $f : D \times 2 \to D$ be a bijection. For $i = 0, 1$ let $f_i(d) = f(d, i)$ so that $f_i : D \to D$. By Martin’s conjecture $f_i(d) \geq_T d$ on a cone of $d$’s hence $\text{ran}(f_i)$ cannot be disjoint from a cone. But $\text{ran}(f_0) \cup \text{ran}(f_1) = D$ and $\text{ran}(f_0) \cap \text{ran}(f_1) = \emptyset$, so one the two must be of $\mu_M$-measure 0, hence disjoint from a cone. $\square$

Theorem 25.6. The perfect set property for $\Pi^1_1$ implies $\omega_1$ is inaccessible in $L$. 

Bibliography


[Sol78] Robert M. Solovay. The independence of DC from AD. In Kechris and Moschovakis [KM78]. 2.C, 10


