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Passive Portfolio Management over a Finite Horizon with a Target Liquidation Value under Transaction Costs and Solvency Constraints

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We consider a passive investor who divides his capital between two assets: a risk-free money market instrument and an index fund, or ETF, tracking a broad market index. We model the evolution of the market index by a lognormal diffusion. The agent faces both fixed and proportional transaction costs and solvency constraints. The objective is to maximize the expected utility from the portfolio liquidation at a fixed horizon but if the portfolio reaches a pre-set target value then the position in the risky asset is liquidated. The model is formulated as a parabolic impulse control problem and we characterize the value function as the unique constrained viscosity solution of the associated quasi-variational inequality. We show the existence of an impulse policy which is arbitrarily close to the optimal one by reducing the model to a sequence of iterated optimal stopping problems. The value function and the quasi-optimal policy are computed numerically by an iterative finite element discretization technique. We present extended numerical results in the case of a CRRA utility function, showing the non-stationary shape of the optimal strategy and how it varies with respect to the model parameters. The numerical experiments reveal that, even with small transaction costs and distant horizons, the optimal strategy is essentially a buy-and-hold trading strategy where the agent recalibrates his portfolio very few times.

Keywords: Dynamic Trading Strategies, Passive Portfolio Management, Quasi-variational Inequalities, Solvency Constraints, Transaction Costs, Viscosity Solutions

1. Introduction

We consider a portfolio problem for an investor who pursues a passive investment strategy making no attempt to "beat" the market. Our agent chooses a broad market index, such as the S&P 500, as his risky portfolio and divides his capital between it and a risk-free security. This strategy is easy to implement trading only two financial securities: an index fund, or ETF, representing all the stocks in the index, and a money market instrument, such as Treasury Bills. We will model the evolution of the market index, and therefore of the risky security, by a geometric Brownian motion. The investor's objective is to maximize the expected utility from the portfolio liquidation at a given terminal horizon. However our investor has a prudent attitude and if the portfolio's value reaches a pre-set upper bound then he liquidates the risky asset, bearing no more risk up to the final date.

In his seminal article, Merton (1969) first developed a continuous time model to find the dynamic optimal strategy for an investor managing a portfolio of risky assets, whose prices evolve according to geometric Brownian motions. Since then, research in this area has focused on different aspects, aiming to make the mathematical model closer to the real market. It is well known that, in the real economy, investors face nontrivial transaction costs, which influence their trading policies. It is not possible to rebalance a portfolio in a continuous way, as assumed by Merton, and margin requirements and bounds on the open short positions are commonly present. Most of the literature on portfolio optimization with

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transaction costs considers the problem of maximizing the cumulative expected utility of consumption over a infinite horizon, with proportional transaction costs. See for instance Davis and Norman (1990), Shreve and Soner (1994), Akian et al. (1996), Kumar and Muthuraman (2006) and, in case of small transaction costs, Mokkhavesa and Atkinson (2002) where the optimization problem is solved for arbitrary utility functions using perturbation theory. The same infinite horizon problem but with fixed and proportional costs has been studied in Oksendal and Sulem (2002) and Liu (2004). A second class of articles studies the problem of maximizing the long-term growth rate of portfolio value. See Morton and Pliska (1995) for a problem with transaction costs equal to a fixed fraction of the portfolio value ("portfolio management fee"), Assaf et al. (1988), Dumas and Luciano (1991), Akian et al. (2001) for models with proportional transaction costs, and Bielecki and Pliska (2000) in the more general framework of risk-sensitive impulse control. In Gashi and Date (2012) a completely different approach is adopted: in order to reduce the variation in the asset holdings and the consequent proportional transaction costs, the trading strategies to build optimal portfolios are constrained to be of finite variation. Fewer papers consider a portfolio optimization problem with transaction costs over a finite horizon. Liu and Loewenstein (2002) consider proportional transaction costs and approximate the value function by a sequence of problems with exponentially distributed horizons. However for a given terminal date the optimal trading strategy is described by a stationary policy. In Eastham and Hastings (1988), Korn (1998) both fixed and variable transaction costs are considered and the model is solved by using impulse control techniques. These last articles use verification theorems to characterize the value function and the optimal policy and, apart from some simple cases, only approximate the solution by an asymptotic analysis. In the recent paper by Ly Vath et al. (2007) the authors consider a portfolio optimization problem over a finite horizon with a permanent price impact and a fixed transaction cost. The main result in Ly Vath et al. (2007) is a viscosity characterization of the value function, but neither a characterization of the optimal policy nor a numerical solution of the problem is given.
To deal with the fixed component in the transaction costs we formulate our model as an impulse control problem, associated by the dynamic programming principle to a Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQVI). The features of our stochastic control problem lead to consider a parabolic HJBQVI in two variables and time, and to impose state constraints on the space variables. To characterize the value function we consider, as in Akian et al. (2001), Oksendal and Sulem (2002), Ly Vath et al. (2007), the very general notion of discontinuous constrained viscosity solutions. In Section 3 of this paper, by means of a weak comparison principle, we show that the value function is the unique constrained viscosity solution of the HJBQVI verifying certain boundary conditions, and that it is (almost everywhere) continuous. These results are summarized in Theorem 3.3.

To show the existence of an optimal trading strategy and to describe its structure, in Section 4 we decompose our impulse control problem into a sequence of iterated optimal stopping problems (as in Chancelier et al. (2002), Baccarin (2009)). This reduction, first introduced in Bensoussan and Lions (1984), has both a theoretical and a computational interest. It allows to represent the value function by the limit of a sequence of solutions of variational inequalities. Moreover it makes possible to characterize a Markovian quasi-optimal policy which is arbitrarily close to the optimal one. We propose an iterative finite element discretization technique to solve numerically this sequence of variational inequalities, and therefore to compute the value function and the optimal policy for arbitrary utility functions.

In Section 5 we present extended numerical results for our model in the case of a constant relative risk aversion (CRRA) utility, which is the most commonly used utility function in expected utility maximization problems. We analyze the transaction regions, the target portfolios, i.e., the portfolios where it is optimal to move from the transaction regions, and how the agent's optimal strategy varies as time goes on and for different time horizons. To the best of our knowledge this is the first paper

where a non-stationary optimal policy is fully described for a portfolio selection problem in continuous time. We show explicitly how the transaction regions and the target portfolios change, in a asymmetrical way, as time passes up to the finite horizon. Sensitivity analysis with respect to the market and agent's parameters and a comparison between our optimal strategy and others suggested in literature is also provided. Our numerical results show that the transaction costs have a dramatic impact on the frequency of trading of an optimal policy. This phenomenon has already been noted, in a qualitative way, in Dumas and Luciano (1991), Morton and Pliska (1995), Liu and Loewenstein (2002) and Liu (2004). The optimal strategy is essentially a buy-and-hold trading strategy where the agent recalibrates his portfolio very few times, in contrast with the continuous interventions of the Merton's model without transaction costs. In the Appendix we collect the proofs of the main results.

2. The model formulation

We denote by $S(t)$ the value invested by the agent in the stock market index at time t, and by $B(t)$ his amount of money invested in a risk-free asset, such as Treasury Bills. The initial wealth in $t=0$ is given by (B_0, S_0). The value $S(t)$ evolves as a geometric Brownian motion

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \;\; S(0) = S_0,$$

where W_t is an adapted Wiener process on the filtered probability space $(\Omega, F, P, \mathbb{F}_t)$, verifying the usual conditions. The risk-free asset grows in a certain way at the fixed rate r

$$dB(t) = rB(t)dt, \;\; B(0) = B_0.$$

At any time the investor can buy ($\xi > 0$) or sell ($\xi < 0$) the value $\xi \in \mathbb{R}$ of stocks, reducing (or increasing) correspondingly the the investment in the risk-free asset. However to make a transaction it is necessary to bear the associated transaction costs $C(\xi)$, which we assume of a fixed plus proportional type

$$C(\xi) = K + c\,|\xi|\,, \;\; K > 0,\; 0 \leqslant c < 1\,.$$

These costs are drawn immediately from the risk-free asset: if the value ξ of stocks is bought (or sold) the variation in the risk-free asset is given by $-\xi - K - c\,|\xi|$.
A portfolio control policy p is a sequence $\{(\tau_i, \xi_i)\}\,, i = 1, 2, \cdots$, of stopping times τ_i and corresponding random variables ξ_i, which represent the value of stocks bought (or sold) in τ_i. We define a policy as feasible if it verifies the following conditions:

$$\begin{cases} \tau_i \text{ is a } \mathbb{F}_t \text{ stopping time} \\ \tau_i \leqslant \tau_{i+1} \qquad \forall i \\ \lim_{i \to +\infty} \tau_i = +\infty \text{ almost surely} \\ \xi_i \text{ is } F_{\tau_i} \text{ measurable .} \end{cases} \tag{1}$$

Note that condition $\tau_i \to \infty$ a.s. implies that the number of stopping times in any bounded time interval is almost surely finite ($\tau_i = +\infty$ for some $i < \infty$ is possible, it means a policy which consists of at most $i-1$ transactions). Starting from the initial amounts (B_0, S_0) of the two assets in $t=0$, the dynamics of the portfolio $(B^p(t), S^p(t))$, controlled by policy p, is given by the following set of stochastic differential equations:

$$\begin{cases} dS^0(t) = \mu S^0(t)dt + \sigma S^0(t)dW(t)\,,\; S^0(0) = S_0 \\ dB^0(t) = rB^0(t)dt\;,\;\; B^0(0) = B_0, \end{cases} \quad \text{for } t \in [0, \tau_1] \tag{2}$$

and, for $t \in [\tau_i, \tau_{i+1}]$, $i \geqslant 1$,

$$\begin{cases} dS^i(t) = \mu S^i(t)dt + \sigma S^i(t)dW(t)\ ,\ S^i(\tau_i) = S^{i-1}(\tau_i) + \xi_i \\ dB^i(t) = rB^i(t)dt\ ,\ B^i(\tau_i) = B^{i-1}(\tau_i) - \xi_i - K - c\,|\xi_i|\,. \end{cases} \tag{3}$$

When $\tau_i < \tau_{i+1}$, we define $(B^p(t), S^p(t)) = (B^i(t), S^i(t))$ for $t \in [\tau_i\ ,\ \tau_{i+1})$. If we have, for example, $\tau_{i-1} < \tau_i = \tau_{i+1} = ... = \tau_{i+n} < \tau_{i+n+1}$, then we set

$$\begin{cases} (B^p(\tau_{i+n}^-), S^p(\tau_{i+n}^-)) = (B^{i-1}(\tau_{i+n}), S^{i-1}(\tau_{i+n})) \\ (B^p(\tau_{i+n}), S^p(\tau_{i+n})) = (B^{i+n}(\tau_{i+n}), S^{i+n}(\tau_{i+n})) \end{cases}$$

where $(B^p(\tau_{i+n}^-), S^p(\tau_{i+n}^-))$ are the left limits in $t = \tau_i = ... = \tau_{i+n}$. The resulting controlled process $(B^p(t), S^p(t))$ is cadlag and adapted to the filtration $\mathbb{F}_t$.

A fundamental notion in our model is the liquidation value of the assets. We define the liquidation value $L(B,S)$ of the portfolio (B,S) as

$$L(B,S) = \begin{cases} \max\{S + B - K - c\,|S|\,, B\} & \text{if } S \geqslant 0 \\ S + B - K - c\,|S| & \text{if } S < 0\,. \end{cases}$$

It represents the value when the long or short position in stocks is cleared out (if $S - cS < K$ a long position in stocks is closed without any income or additional payment). Note that $L(B,S) < B + S$, except for $S = 0$, and that every transaction cannot increase the liquidation value of the portfolio, that is $L(B,S) \geqslant L(B - \xi - K - c\,|\xi|\,, S + \xi)$, $\forall \xi \in \mathbb{R}$ (the equality holds only if $L(B,S) = S + B - K - c\,|S|$ and $\xi = -S$). Besides the transaction costs, our investor must face another kind of constraints. We assume that there are bounds on the open short positions and that the portfolio liquidation value must be greater than zero. This kind of solvency constraints correspond to the margin requirements required by brokers to allow an investor to buy stocks on margin or to shortsell securities. Therefore the set of admissible portfolios is given by the closed region $\overline{Adr} \subset \mathbb{R}^2$,

$$\overline{Adr} = \left\{(B,S) \in \mathbb{R}^2 : (L(B,S) \geqslant 0) \wedge (B \geqslant B_{\min}) \wedge (S \geqslant S_{\min})\right\}\ .$$

Here $B_{\min} < 0$ and $S_{\min} < 0$ are the bounds in the short position in the risk-free asset and in the risky security, respectively. We assume $B_0 \geqslant B_{\min}$, $S_0 \geqslant S_{\min}$ and $L(B_0, S_0) \geqslant 0$. The admissible region is depicted in Figure 1. The investor's preferences are represented by a continuous, increasing, utility function $U : \mathbb{R}_+ \to \mathbb{R}_+$, with $U(0) = 0$. We assume that U satisfies, for some $C > 0$ and $0 < \gamma < 1$, the upper bound

$$U(L) \leqslant CL^\gamma\,. \tag{4}$$

Note that we are not assuming a particular form or the concavity of the utility function and that (4) is only an upper bound. The objective of our investor is to liquidate his portfolio at a fixed time horizon $T > 0$. This means that the problem is to maximize the expected utility of the portfolio liquidation value at the terminal date T. However if the portfolio reaches a threshold liquidation value $L_{\max}$, at a time $t < T$, then the agent is satisfied. He liquidates the risky asset in t and invests all $L_{\max}$ in the risk-free asset up to the finite horizon T.

REMARK 2.1 *This assumption has a natural financial meaning: the investor has a target value for his portfolio and if this value is reached before T, then he is not interested in holding the risky asset any longer and in bearing the associated risk up to the final date. Of course if $L_{\max}$ is set to a too large value with respect to the initial wealth (B_0, S_0) and the horizon T, the target portfolio will never be*

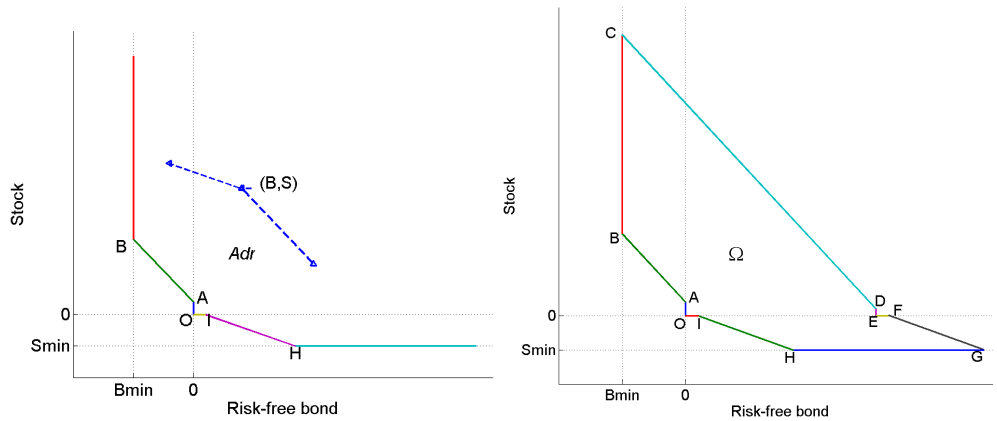

FIG. 1. Admissible region (left) and bounded value function domain (right). The points in the figures have coordinates A= $(0, \frac{K}{1-c})$, B= $(B_{\min}, \frac{K-B_{\min}}{1-c})$, C= $\left(B_{\min}, \frac{-B_{\min}+K+L_{\max}}{1-c}\right)$, D= $(L_{\max}, \frac{K}{1-c})$, E= $(L_{\max}, 0)$, F= $(L_{\max}+K, 0)$, G= $(-(1+c)S_{\min}+K+L_{\max}, S_{\min})$, H= $(-(1+c)S_{\min}+K, S_{\min})$ and I= $(K, 0)$.

reached and the investor maximizes the expected utility in T using both assets. Indeed, for theoretical and computational purposes, the problem without a target liquidation value can be treated as the limit case, letting $L_{\max} \to +\infty$.

We define the open control region Cor

$$Cor = \left\{(B,S) \in \mathbb{R}^2 : L(B,S) < L_{\max}\right\}$$

and by $\overline{Cor}$ its closure. Cor is the region where it may be useful to rebalance the portfolio because the threshold value $L_{\max}$ has not been reached yet. Let θ^p be the first exit time of the controlled process from the control region

$$\theta^p = \inf\{t : (B^p(t), S^p(t)) \notin Cor\}.$$

We set $\vartheta^p \equiv \theta^p \wedge T$, and we define a policy p admissible if the corresponding controlled process verifies $(B^p(t), S^p(t)) \in \overline{Adr}$, $\forall t \in [0, \vartheta^p]$. The payoff functional J^p associated to policy p is then given by

$$J^p = E\left[U(L(B^p(\vartheta^p), S^p(\vartheta^p))\, e^{r(T-\vartheta^p)})\right].$$

Note that the behavior of $(B^p(t), S^p(t))$ for $t > \vartheta^p$ is irrelevant in our formulation: that is to say $(B^p(t), S^p(t))$ represents the financial position of our investor only up to ϑ^p. If we denote by A the set of admissible policies, the control problem can be formulated as

$$\max_{p \in A} J^p \ .$$

It is a stochastic impulse control problem over a finite horizon where the system is controlled only in the Cor region, and with the state constraint $(B^p(t), S^p(t)) \in \overline{Adr}$, $\forall t \in [0, \vartheta^p]$. We will solve this problem by using a dynamic programming approach. We consider the compact set $\overline{\Omega} = \overline{Adr} \cap \overline{Cor}$ which is depicted in Figure 1 and we denote by Ω its interior. We define the set $\overline{Q} \equiv [0,T] \times \overline{\Omega}$ and we will denote by Q the subset $[0,T) \times \Omega$. Now we can introduce the value function $V(t,B,S) : \overline{Q} \subset \mathbb{R}^3 \to \mathbb{R}$

defined by

$$V(t,B,S) = \sup_{p\in A(t,B,S)} J^p(t,B,S)\ .$$

Here $A(t,B,S)$ is the set of admissible policies when the controlled process starts in t with values (B,S) and

$$J^p(t,B,S) = E_{t,B,S}\left[U\left(L(B^p(\vartheta^p),S^p(\vartheta^p))\,e^{r(T-\vartheta^p)}\right)\right].$$

REMARK 2.2 *We have $A(t,B,S)\neq\emptyset$ for any initial condition $(t,B,S)\in\overline{Q}$ since the policy*

$$\left\{\begin{array}{l} \tau_1 = \left\{\begin{array}{ll} +\infty & \text{if } S\geqslant 0 \text{ and } B\geqslant 0 \\ t & \text{otherwise} \end{array}\right. \\ \xi_1 = \left\{\begin{array}{ll} \text{arbitrary} & \text{if } S\geqslant 0 \text{ and } B\geqslant 0 \\ -S & \text{otherwise} \end{array}\right. \end{array}\right. , \qquad \left\{\begin{array}{l} \tau_i = +\infty \\ \xi_i \text{ arbitrary} \end{array}\right. \text{for } i>1$$

is clearly always admissible. Note that $V(t,0,0)=0$, $\forall t\in[0,T]$, because the only admissible policy is doing nothing, and $U(0)=0$ by assumption. Moreover $V(t,B,S)>U(L_{\max}e^{r(T-t)})$ only if $(B,S)\in\{EF\backslash E\}$, because the points in the set $\{EF\backslash E\}$ (see Figure 1(right)) can be reached by an admissible policy only after ϑ^p, if the initial position $(B,S)\notin\{EF\backslash E\}$.

The value function V of our problem verifies the following dynamic programming property (see Fleming and Soner (1993), Section V.2, or Ly Vath et al. (2007)).

Dynamic Programming Property:
(a) For any $(t,B,S)\in\overline{Q}$, $p\in A(t,B,S)$ and $\{\mathbb{F}_s\}$-stopping time $\alpha\geqslant t$ we have

$$V(t,B,S)\geqslant E_{t,B,S}\left[V(\vartheta^p\wedge\alpha,B^p(\vartheta^p\wedge\alpha),S^p(\vartheta^p\wedge\alpha))\right]; \tag{5}$$

(b) For any $(t,B,S)\in\overline{Q}$, and $\delta>0$, there exists $p'(\delta)\in A(t,B,S)$ such that for all $\{\mathbb{F}_s\}$-stopping time $\alpha\geqslant t$ we have

$$V(t,B,S)\leqslant E_{t,B,S}[V(\vartheta^{p'}\wedge\alpha,B^{p'}(\vartheta^{p'}\wedge\alpha),S^{p'}(\vartheta^{p'}\wedge\alpha))]+\delta. \tag{6}$$

Combining (a) and (b) we obtain the subsequent version of the dynamic programming principle, which holds for any $(t,B,S)\in\overline{Q}$ and $\{\mathbb{F}_s\}$-stopping time $\alpha\geqslant t$:

$$V(t,B,S) = \sup_{p\in A(t,B,S)} E_{t,B,S}\left[V(\vartheta^p\wedge\alpha,B^p(\vartheta^p\wedge\alpha),S^p(\vartheta^p\wedge\alpha))\right]\ .$$

Now, we denote by $F(B,S)$ the set of admissible transactions from $(B,S)\in\overline{\Omega}$

$$F(B,S) = \left\{\xi\in\mathbb{R} : (B-\xi-K-c\,|\xi|\,,S+\xi)\in\overline{\Omega}\right\}$$

and by $\digamma$ the subset of $\overline{\Omega}$ where $F(B,S)\neq\emptyset$.

REMARK 2.3 *The set $F(B,S)$ can be empty. For example it is always empty when $B+S<K$, but if $F(B,S)\neq\emptyset$, then it is a compact subset of $\mathbb{R}$. Moreover let $(B_n,S_n)\in\overline{Adr}$ be a sequence converging to $(B',S')\in\overline{Adr}$ with $F(B_n,S_n)\neq\emptyset$. Since the function L is upper semicontinuous we have $L(B',S')\geqslant 0$ and $F(B',S')\neq\emptyset$. Any sequence $\xi_n\in F(B_n,S_n)$ is bounded and therefore contains a subsequence ξ_n' converging to some $\xi'\in\mathbb{R}$. As $L(B_n-\xi_n'-K-c\,|\xi_n'|\,,S_n+\xi_n')\geqslant 0$ and L is upper semicontinuous, it also holds that $\xi'\in F(B',S')$.*

For any given function $Z : \overline{Q} \to \mathbb{R}$ we define the intervention (non local) operator $\mathscr{M}$ by

$$\mathscr{M}Z(t,B,S) = \begin{cases} \sup\limits_{\xi \in F(B,S)} Z(t,B-\xi-K-c\,|\xi|\,,S+\xi) & \text{if } (B,S) \in F \\ -1 & \text{if } (B,S) \notin F. \end{cases} \tag{7}$$

Considering any $p \in A(t,B,S)$ with an immediate transaction in t of arbitrary size $\xi \in F(B,S)$ and setting $\alpha = t$ in (5), we can see as a direct consequence of dynamic programming property that $V(t,B,S) \geqslant \mathscr{M}V(t,B,S)$, for any $(t,B,S) \in \overline{Q}$ (this is obvious if $F(B,S) = \emptyset$ because V is non-negative).

It is well known that we can associate to the value function of an impulse control problem a Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQVI) which plays the same role of the HJB equation in continuous optimization. We introduce the second order differential operator $\mathscr{L}$

$$\mathscr{L}V(t,B,S) = rB\frac{\partial V}{\partial B} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

which corresponds to the infinitesimal generator of the uncontrolled process $(B(t),S(t))$. We will show that the value function of our problem is a weak solution of the following HJBQVI in $\overline{Q}$

$$\min\left\{-\frac{\partial V}{\partial t} - \mathscr{L}V, V - \mathscr{M}V\right\} = 0. \tag{8}$$

One cannot hope to show that V is a classical solution of (8). It is easy to see that the value function is not even continuous in some points of ∂Q, such as, for instance, points A and I in Figure 1, for any $t \in [0,T]$, or line $S = 0$ in $t = T$. In these points V is only upper-semicontinuous. In the next section we will characterize V as the unique constrained viscosity solution of (8) verifying certain boundary conditions.

3. Boundary properties, bounds, and viscosity characterization of the value function

By $\partial^* Q$ we denote the subset of ∂Q given by $\partial^* Q \equiv ([0,t) \times \partial\Omega) \cup (T \times \overline{\Omega})$. The boundary $\partial\Omega$ of Ω is divided in two parts:

$$\partial_1\Omega \equiv \{(B,S) \in \partial\Omega : L(B,S) < L_{\max}\}$$

and its complement $\partial_2\Omega \equiv \partial\Omega \backslash \partial_1\Omega$. In $\partial_2\Omega$ the threshold liquidation value $L_{\max}$ has already been reached. It is also useful to define $\partial_2^* Q \equiv ([0,t) \times \partial_2\Omega) \cup (T \times \overline{\Omega})$, which is the part of $\partial^* Q$ where $t = T$ or $L(B,S) \geqslant L_{\max}$. We now investigate the behavior of V at the boundary $\partial^* Q$.

For $t = T$ we have obviously $V(T,B,S) = U(L(B,S))$ for any $(B,S) \in \overline{\Omega}$. It is always optimal not to intervene in T because any intervention cannot increase the portfolio liquidation value. However one single transaction $\xi = -S$ is also optimal if $S < 0$ or if $S > 0$ and $S + B - K - c\,|S| \geqslant B$. In this case we have $V = \mathscr{M}V$, otherwise $V > \mathscr{M}V$. Note that V is upper-semicontinuous but not continuous for any point $(T,B,0) \in \overline{Q}$.

For $t \in [0,T)$ the behavior of V depends on which part of $\partial\Omega$ we are considering:

(a) Along the segments OA and OI in Figure 1 it is not possible to intervene because this will bring the process (B,S) outside the admissible region $\overline{Adr}$. Actually in the points A and I there is one admissible transaction which leads us to the origin O, but this is certainly unprofitable. Therefore we have $V > \mathscr{M}V$. Apart from $V(t,0,0) = 0$, the value of V is not known a priori in this part of $\partial^* Q$.

(b) Except for the points A and I, along the segments AB and IH it is necessary to make a transaction, otherwise the process could leave $\overline{Adr}$ with a positive probability. Moreover the only admissible intervention brings the process to O. Consequently it holds $V = \mathscr{M}V = 0$. Note that V is upper-semicontinuous but not continuous in A and I.

(c) In the interior points of the segments BC and HG it is necessary to make a transaction because one of the bounds in the short position is reached. The value of V is not known a priori. We have $V = \mathscr{M}V$.

(d) In the upper part of $\partial^* Q$, that is along the segments CD, DE, EF and FG, the threshold liquidation value $L_{\max}$ has already been reached. The value of V is known. If $(B,0) \in$ EF then $V(t,B,0) = U(B\ e^{r(T-t)})$. If $(B,S) \in$CD$\cup$DE$\cup\{$FG$\backslash$F$\}$ then $V(t,B,S) = U(L_{\max}\ e^{r(T-t)})$. It is always optimal not to intervene, but we also have $V = \mathscr{M}V$, with $\xi = -S$ in (7), if $S < 0$ or if $S > 0$ and $S + B - K - c\,|S| \geqslant B$. Note that V is upper-semicontinuous but not continuous in the point F, for any $t \in [0,T)$.

We give now some bounds on the value function. Since $J^p \geqslant 0$, for any $p \in A(t,B,S)$, it is obvious that $V(t,B,S)$ is nonnegative in $[0,T] \times \overline{\Omega}$. By the problem definition we also have $V(t,B,S) \leqslant U((L_{\max} + K)e^{r(T-t)})$, that is the value function is bounded. Moreover, as it holds $U(L(B,S)e^{r(T-t)}) \leqslant V(t,B,S) \leqslant U(L_{\max}e^{r(T-t)})$ when $(B,S) \notin$EF, the value function is also continuous in the segments CD, $\{$DE$\backslash$E$\}$, $\{$FG$\backslash$F$\}$. It is not difficult to show that V is also bounded from above by the value function of the same problem with $U(L) = CL^\gamma$ and without transaction costs and solvency constraints, i.e., a Merton problem over a finite horizon without consumption and a CRRA utility function, see Merton (1969).

PROPOSITION 3.1 We have

$$V(t,B,S) \leqslant Ce^{\delta(T-t)}\,(B+S)^\gamma \tag{9}$$

in $[0,T] \times \overline{\Omega}$, where

$$\delta = \gamma\left(r + \frac{(\mu - r)^2}{2\sigma^2(1-\gamma)}\right)\ .$$

Proof. See Appendix A.1. □

The bound (9) shows in particular that $V(t,B,S)$ is continuous in $(t,0,0)$, where $V(t,0,0) = 0$, $\forall t \in [0,T]$. Now we give the precise characterization of the value function as a viscosity solution of (8). Since V is not even continuous at some points in ∂Q it is necessary to consider the notion of discontinuous viscosity solution. Moreover the state constraint $(B^p(t), S^p(t)) \in \overline{Adr}$, $\forall t \in [0, \vartheta^p]$, requires a particular treatment of the lateral boundary conditions when $(t,B,S) \in [0,T) \times \partial_1\Omega$ and the use of constrained viscosity solutions. Let $USC(\overline{Q})$ and $LSC(\overline{Q})$ be respectively the sets of upper-semicontinuous (usc) and lower-semicontinuous (lsc) functions defined on $\overline{Q}$. Given a locally bounded function $u : \overline{Q} \to \mathbb{R}_+$ we will denote by u^* and u_* respectively the usc envelope and the lsc envelope of u

$$\begin{aligned} u^*(t,B,S) &= \limsup_{\substack{(t',B',S')\in\overline{Q} \\ (t',B',S')\to(t,B,S)}} u(t',B',S') \qquad \forall\,(t,B,S) \in \overline{Q} \\ u_*(t,B,S) &= \liminf_{\substack{(t',B',S')\in\overline{Q} \\ (t',B',S')\to(t,B,S)}} u(t',B',S') \qquad \forall\,(t,B,S) \in \overline{Q}\ . \end{aligned}$$

We have $u_* \leqslant u \leqslant u^*$ and u is usc (lsc) if and only if $u = u^*$ ($u = u_*$). In the following, unless otherwise specified, we set $x \equiv (B,S) \in \bar{\Omega}$ to simplify the notation.

DEFINITION 3.1 Given $\mathscr{O} \subset \overline{\Omega}$, a locally bounded function $u : \overline{Q} \to \mathbb{R}_+$ is called a viscosity subsolution (resp. supersolution) of (8) in $[0,T) \times \mathscr{O}$ if for all $(\bar{t},\bar{x}) \in [0,T) \times \mathscr{O}$ and $\varphi(t,x) \in C^{1,2}(\overline{Q})$ such that $(u^* - \varphi)(\bar{t},\bar{x}) = 0$ (resp. $(u_* - \varphi)(\bar{t},\bar{x}) = 0$) and $(\bar{t},\bar{x})$ is a maximum of $u^* - \varphi$ (resp. a minimum of $u_* - \varphi$) on $[0,T) \times \mathscr{O}$, we have

$$\min\left\{-\frac{\partial \varphi}{\partial t}(\bar{t},\bar{x}) - \mathscr{L}\varphi(\bar{t},\bar{x}), u^*(t,x) - \mathscr{M}u^*(t,x)\right\} \leqslant 0 \tag{10}$$

$$\text{(resp. } u_* \text{ and } \geqslant 0) \tag{11}$$

On $[0,T) \times \partial_2\Omega$ the value function V verifies the Dirichlet boundary condition

$$V(t,B,S) = U(L(B,S)e^{r(T-t)}).$$

To deal properly with the state constraint $(B^p(t), S^p(t)) \in \overline{Adr}$, $\forall t \in [0, \vartheta^p]$, it will be necessary to require that V satisfies the subsolution property also on the $[0,T) \times \partial_1\Omega$ part of the lateral boundary $[0,T) \times \partial\Omega$ (see Crandall et al. (1992), section 7C, Oksendal and Sulem (2002), Ly Vath et al. (2007)).

DEFINITION 3.2 We say that a locally bounded function $u : \overline{Q} \to \mathbb{R}_+$ is a $\partial_1\Omega$ constrained viscosity solution of (8) in $Q = [0,T) \times \Omega$ if it is a viscosity supersolution of (8) in Q and a viscosity subsolution of (8) in $[0,T) \times \{\Omega \cup \partial_1\Omega\}$.

The next Lemma shows some properties of the non-local operator $\mathscr{M}$.

LEMMA 3.1 Given a locally bounded function $u : \overline{Q} \to \mathbb{R}_+$ we have:
(a) if u is lower-semicontinuous (resp. usc) then $\mathscr{M}u$ is lower-semicontinuous (resp. usc)
(b) $\mathscr{M}u_* \leqslant (\mathscr{M}u)_*$ and $\mathscr{M}u^* \geqslant (\mathscr{M}u)^*$
(c) if u is upper-semicontinuous then there exists a Borel measurable function $\xi_u^* : \mathcal{F} \to \mathbb{R}$ such that for any $(B,S) \in \mathcal{F}$

$$\mathscr{M}u(t,B,S) = u(t, B - \xi_u^*(B,S) - K - c\,|\xi_u^*(B,S)|, S + \xi_u^*(B,S)). \tag{12}$$

Proof. (a) and (b) can be proven in the same way as in Ly Vath et al. (2007), Lemma 5.5. As u is upper-semicontinuous and for $(B,S) \in \mathcal{F}$ the set $F(B,S)$ is compact the sup in (7) is reached for some values of ξ, $\forall (B,S) \in \mathcal{F}$. Moreover, as $\mathcal{F}$ is σ-compact, we can select a Borel measurable function $\xi_u^* : \mathcal{F} \to \mathbb{R}$ such that (c) holds true (see Fleming and Rishel (1975), Appendix B, Lemma B). □

It is now possible to prove the viscosity property of the value function.

THEOREM 3.1 The value function $V(t,B,S)$ is a $\partial_1\Omega$ constrained viscosity solution of (8) in Q.

Proof. Using the dynamic programming property (5-6), and properties (a) and (b) of Lemma 3.1, the proof can be done in the same way as the proof of Theorem 5.3 in Ly Vath et al. (2007). The only difference is that in our problem it is possible to prove the subsolution property only in Q and in the $[0,T) \times \partial_1\Omega$ part of the lateral boundary. The reason is that an admissible policy can now allow the controlled process to leave $\overline{Q}$ from the subset $[0,T) \times \partial_2\Omega$ of ∂^*Q. On $[0,T) \times \partial_2\Omega$ the value function will be determined by the Dirichlet type condition $V(t,B,S) = L(B,S)e^{r(T-t)}$. □

As there can be many viscosity solutions of (8) the next step is to determine the right boundary conditions on ∂^*Q which are sufficient to uniquely determine the value function. The usual way to show uniqueness of viscosity solutions is to prove a comparison theorem between viscosity sub and

supersolution. The purpose is to show that a subsolution is lower than a supersolution on the whole domain if it assumes the same or a lower value at the boundary $\partial^* Q$. However in our problem the value of V is not known in some part of $[0,T)\times\partial_1\Omega$, such as the segments BC and HG in Figure 1. Thus on $[0,T)\times\partial_1\Omega$ we will need the viscosity boundary condition given by the subsolution property. Moreover if we look at V^* as a subsolution and at V_* as a supersolution, along the segments $I_T F_T \equiv \left\{(t,B,S)\in\overline{Q} : t=T,\, S=0, B\geqslant K\right\}$ and $F_0 F_T \equiv \left\{(t,B,S)\in\overline{Q} : B=L_{\max}+K,\, S=0\right\}$ the subsolution V^* is greater than the supersolution V_*. Therefore on the rectangular region

$$R\equiv\left\{(t,B,S)\in\overline{Q} : S=0, B\geqslant K\right\}$$

we cannot hope to show that $V^*\leqslant V_*$, and consequently that V is continuous in R (because by definition $V^*\geqslant V_*$, and thus $V^*=V_*$). This will induce us to prove only a weaker comparison principle between viscosity sub and supersolutions, which holds on $Q\backslash R$. Thus we will distinguish the cases $S>0$ and $S<0$. We denote by Ω^+, Q^+, the sets

$$\Omega^+\equiv\{(B,S)\in\Omega : S>0\}, \quad Q^+\equiv[0,T)\times\Omega^+,$$

and by $\overline{\Omega}^+$, $\overline{Q}^+$, their closures. We also define the boundaries

$$\begin{aligned}
&\partial^* Q^+\equiv[0,T)\times\partial\Omega^+\cup T\times\overline{\Omega}^+,\\
&\partial_1\Omega^+\equiv\left\{(B,S)\in\partial\Omega^+ : L(B,S)<L_{\max}\right\},\ \partial_2\Omega^+=\partial\Omega^+\backslash\partial_1\Omega^+,\\
&\partial_2^* Q^+\equiv[0,T)\times\partial\Omega_2^+\cup T\times\overline{\Omega}^+.
\end{aligned}$$

The sets Ω^-, Q^-, $\overline{\Omega}^-$, $\overline{Q}^-$, $\partial^* Q^-$, $\partial_2\Omega^-$, $\partial_2^* Q^-$ are defined similarly by setting $S<0$.

THEOREM 3.2 (Weak Comparison Principle) Assume that $u\in USC(\overline{Q})$ is a viscosity subsolution of (8) in $[0,T)\times\{\Omega\cup\partial_1\Omega\}$ and $v\in LSC(\overline{Q})$ is a viscosity supersolution of (8) in $Q=[0,T)\times\Omega$ Furthermore assume that

$$\begin{cases}
\displaystyle\limsup_{\substack{(t',B',S')\in Q^+\\(t',B',S')\to(t,B,S)}} u(t',B',S')\leqslant \liminf_{\substack{(t',B',S')\in Q^+\\(t',B',S')\to(t,B,S)}} v(t',B',S') \quad \forall\,(t,B,S)\in\partial_2^* Q^+\\[2ex]
\displaystyle\limsup_{\substack{(t',B',S')\in Q^-\\(t',B',S')\to(t,B,S)}} u(t',B',S')\leqslant \liminf_{\substack{(t',B',S')\in Q^-\\(t',B',S')\to(t,B,S)}} v(t',B',S') \quad \forall\,(t,B,S)\in\partial_2^* Q^-\\[2ex]
\displaystyle\limsup_{\substack{(t',B',S')\in Q\\(t',B',S')\to(t,0,0)}} u(t',0,0)\leqslant \liminf_{\substack{(t',B',S')\in Q\\(t',B',S')\to(t,0,0)}} v(t',0,0) \quad \forall t\in[0,T)\,.
\end{cases} \tag{13}$$

Then $u\leqslant v$ on $Q\backslash R$.

Proof. See Appendix A.2. □

In order to use the comparison principle to identify the only viscosity solution which represents the value function we need to describe the behavior of V approaching the boundary $\partial_2^* Q$ and taking into account of the discontinuity in $\partial_2^* Q\cap R$.

LEMMA 3.2 The value function V verifies the following limit conditions near the boundary $\partial_2^* Q$:

$$\begin{cases} \lim\limits_{\substack{(t',B',S')\in Q \\ (t',B',S')\to(t,B,S)}} V(t',B',S') = U(L(B,S)e^{r(T-t)}) \quad \forall (t,B,S) \in \partial_2^* Q \backslash R \\ \\ \lim\limits_{\substack{(t',B',S')\in Q^+ \\ (t',B',S')\to(t,B,0)}} V(t',B',S') = U(B\,e^{r(T-t)}) \qquad \forall (t,B,0) \in \partial_2^* Q \cap R \\ \\ \lim\limits_{\substack{(t',B',S')\in Q^- \\ (t',B',S')\to(t,B,0)}} V(t',B',S') = U((B-K)e^{r(T-t)}) \quad \forall (t,B,0) \in \partial_2^* Q \cap R \end{cases} \tag{14}$$

Proof. See Appendix A.3. □

Now we are able to give the complete viscosity characterization of the value function.

THEOREM 3.3 The value function $V(t,B,S)$ is continuous in $Q\backslash R$ and it is the unique $\partial_1\Omega$ constrained viscosity solution in $Q\backslash R$ of (8) which verifies the limit conditions (14) and

$$\lim_{\substack{(t',B',S')\in \bar{Q} \\ (t',B',S')\to(t,0,0)}} V(t',B',S') = V(t,0,0) = 0 \qquad \forall t \in [0,T]\,. \tag{15}$$

Proof. We apply the comparison principle theorem, using V^* as a subsolution and V_* as a supersolution. In particular the boundary conditions (13) are verified as equalities since (14) and (15) hold true. We derive that $V^* \leqslant V_*$ on $Q\backslash R$ and since by definition $V^* \geqslant V_*$ we obtain immediately that V is continuous in $Q\backslash R$. Now suppose $\overline{V}$ is another $\partial_1\Omega$ constrained viscosity solution of (8) in Q which verifies the boundary conditions (14), (15). By the comparison principle it follows that $\overline{V}^* \leqslant V_* \leqslant V^* \leqslant \overline{V}_*$ and therefore $\overline{V} = V$ in $Q\backslash R$. □

4. Existence and structure of the optimal trading strategy

To show the existence of a quasi-optimal impulse policy and to characterize its form we reduce our impulse control problem to a sequence of optimal stopping time problems. By this reduction, first introduced in Bensoussan and Lions (1984), it is possible to reduce the solution of a HJBQVI to the solution of an iterative sequence of variational inequalities, where the obstacles are explicit (see Korn (1998), Chancelier et al. (2002), Oksendal and Sulem (2007), Baccarin (2009)). We denote by A_n the set of admissible policies with at most $n \geqslant 1$ interventions, that is

$$A_n(t,B,S) = \{p \in A(t,B,S) : \tau_{n+1} = +\infty\}$$

and by $V_n(t,B,S) : \overline{Q} \subset \mathbb{R}^3 \to \mathbb{R}$ the value function of the corresponding problem with a bounded number of transactions

$$V_n(t,B,S) = \sup_{p\in A_n(t,B,S)} J^p(t,B,S)\,.$$

It is not difficult to show that increasing the number of interventions V_n converges to V.

THEOREM 4.1 We have $\lim_{n\to\infty} V_n = V$ for all $(t,B,S) \in \overline{Q}$.

Proof. As $A_1(t,B,S) \subset A_2(t,B,S) \subset .... \subset A(t,B,S)$, it holds $V_1(t,B,S) \leqslant V_2(t,B,S) \leqslant .... \leqslant V(t,B,S)$ and $\lim_{n\to\infty} V_n \leqslant V$ for all $(t,B,S) \in \overline{Q}$. To obtain the reverse inequality consider an ε-optimal policy $p_\varepsilon \in A(t,B,S)$ such that

$$V(t,B,S) \leqslant J^{p_\varepsilon}(t,B,S) + \varepsilon \ . \tag{16}$$

Setting $\overline{\tau}_i \equiv \tau_i^{p_\varepsilon} \wedge \vartheta^{p_\varepsilon}$, by (1) for a.a. ω there exists $n(\omega)$ such that $\overline{\tau}_n(\omega) = \vartheta^{p_\varepsilon}(\omega)$. If we define

$$J_n^{\overline{p}}(t,B,S) = E_{t,B,S}\left[U\left(L(B^{p_\varepsilon}(\overline{\tau}_n), S^{p_\varepsilon}(\overline{\tau}_n))\, e^{r(T-\overline{\tau}_n)}\right)\right]$$

by the dominated convergence theorem it follows that

$$J^{p_\varepsilon}(t,B,S) = \lim_{n\to\infty} J_n^{\overline{p}}(t,B,S),$$

and we can choose $\overline{n}$ such that

$$J^{p_\varepsilon}(t,B,S) \leqslant J_{\overline{n}}^{\overline{p}}(t,B,S) + \varepsilon \ . \tag{17}$$

Consider now the policy $\overline{p}_{\overline{n}} = \left\{(\overline{\tau}_i, \xi_i^{p_\varepsilon})\right\}$, $i = 1,2,\cdots,\overline{n}$, setting $\overline{\tau}_{\overline{n}+1} = \infty$ a.s.. We have $\overline{p}_{\overline{n}} \in A_{\overline{n}}(t,B,S)$ and combining (16) and (17) we obtain

$$V(t,B,S) \leqslant J^{\overline{p}_{\overline{n}}}(t,B,S) + 2\varepsilon \ .$$

Since ε is arbitrary, it follows $V \leqslant \lim_{n\to\infty} V_n$ for all $(t,B,S) \in \overline{Q}$. □

We consider now the following iterative sequence of optimal stopping problems. Let $(B(s),S(s))$ be the uncontrolled process. We set

$$\theta \equiv \{\inf s \geqslant t : (B(s),S(s)) \notin Cor\}, \quad \text{and} \quad \vartheta \equiv \theta \wedge T.$$

and we define on $\overline{Q} \cap \mathbb{R}^3_+$

$$P_0(t,B,S) = E_{t,B,S}\left[U\left(L(B(\vartheta),S(\vartheta))\, e^{r(T-\vartheta)}\right)\right]$$

that is the expected utility without interventions, starting with nonnegative B and S (to be sure the process does not exit from $\overline{Adr}$ before ϑ). Then we define, recursively, for $n \geqslant 1$

$$\begin{aligned} P_n(t,B,S) \quad = \quad & \sup_{\tau \in A_1(t,B,S)} E_{t,B,S}\Big[\mathscr{M}P_{n-1}(\tau,B(\tau),S(\tau))\chi_{\tau<\vartheta} \\ & \qquad + U\left(L(B(\vartheta),S(\vartheta))\, e^{r(T-\vartheta)}\chi_{\tau\geqslant\vartheta}\right)\Big] \end{aligned} \tag{18}$$

for all $(t,B,S) \in \overline{Q}$, where we denote with χ the indicator function. Here $\mathscr{M}P_{n-1}$ is defined by (7) and it is a given function at step n (note that P_0 is defined in $\overline{Q} \cap \mathbb{R}^3_+$ but all P_n and $(\mathscr{M}P_{n-1})$, $n \geqslant 1$, are defined in $\overline{Q}$). To the optimal stopping problem (18) it is associated the variational inequality

$$\min\left\{-\frac{\partial P_n}{\partial t} - \mathscr{L}P_n, P_n - \mathscr{M}P_{n-1}\right\} = 0 \ . \tag{19}$$

Using the same techniques as in the previous section, it is not difficult to show that P_n is the unique constrained viscosity solution of (19) verifying the same boundary conditions of (8), where $\mathscr{M}P_n$ is replaced by $\mathscr{M}P_{n-1}$. By the following theorem we can reduce the impulse control problem to the sequence (18) of optimal stopping problems.

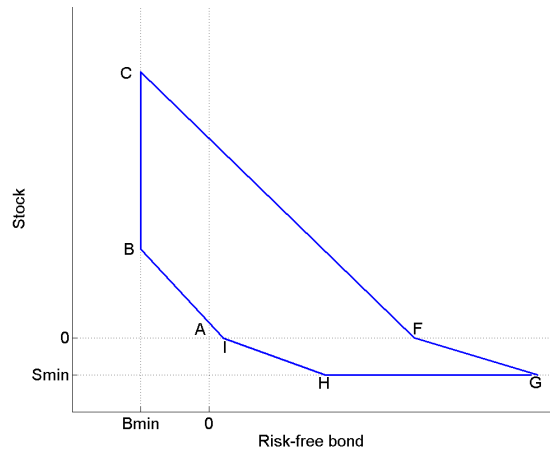

FIG. 2. Simplified value function domain for numerical discretization. Coordinates of vertices as in Figure 1.

THEOREM 4.2 For all $(t,B,S) \in \overline{Q}$ and $n \geqslant 1$ it holds $P_n(t,B,S) = V_n(t,B,S)$. Moreover for each $(t,B,S) \in \overline{Q}$ there exists $p^* \in A_n(t,B,S)$ such that

$$V_n(t,B,S) = J^{p^*}(t,B,S) \ .$$

Proof. See Appendix A.4. □

Therefore, as $\lim_{n\to\infty} V_n = \lim_{n\to\infty} P_n = V$, we can compute the value function by solving the sequence (19) of variational inequalities. Each solution $V_n = P_n$ has the meaning of the value function of the same problem with at most n transactions. Moreover the optimal trading strategy p^* described in Theorem 4.2 gives us, for n large enough, a payoff which is arbitrarily close to the optimal one.

4.1 *Numerical computation of the value function and the optimal policy*

In our numerical experiments we have simplified the domain in Figure 1 as in Figure 2, i.e., we have prolonged the segments AB and CD respectively up to the points I and F considering $V(t,B,S)$ continuous at the boundary ∂Q, and therefore in all the domain. This corresponds to impose a transaction whenever the investor liquidates his position, i.e., to assume $L(B,S) = S + B - K - c\,|S|$. We are quite confident that, for the small values of K we used in our numerical experiments, assuming V continuous everywhere is irrelevant for the numerical results. We denote by $\mathscr{D}$ the numerical domain and we set $\overline{Q}' = [0,T] \times \mathscr{D}$. Thus we have slightly modified the boundary conditions stated in the previous section, setting:

- $V(t,B,S) = U(L_{\max} e^{r(T-t)})$, $\forall t \in [0,T]$, along the entire edges CF and FG;
- $V(t,B,S) = 0 \ \forall t \in [0,T]$ along the entire edges BI and HI.

Moreover, we compute the function $P_0(t,B,S)$, that is the expected utility without interventions, solving the PDE

$$-\frac{\partial P_0}{\partial t} - \mathscr{L} P_0 = 0$$

in $\overline{Q}' \cap \mathbb{R}^3_+$ with the additional boundary conditions, when $S = 0$ or $B = 0$:

- $P(t,B,0) = U(Be^{r(T-t)})$;
- $\mu S \frac{\partial P}{\partial S}(t,0,S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2}(t,0,S) + \frac{\partial P}{\partial t}(t,0,S) = 0.$

Thus we are now ready to deal with the numerical discretization of our iterative optimal stopping problem. Each variational inequality (19) can be solved by a discrete approximation using the finite element method. See, for instance, Achdou and Pironneau (2005), Ballestra and Sgarra (2010) and Marazzina et al. (2012) for the finite element method in finance, and Barucci and Marazzina (2012) and Federico and Gassiat (2012) for applications of finite element and finite difference techniques to financial optimization problems. Setting $\mathscr{L}_t V = -\frac{\partial V}{\partial t} - \mathscr{L} V$, we discretize the PDE $\mathscr{L}_t V = 0$ with a finite element technique based on polynomial of degree 1, coupled with a Crank-Nicholson scheme. We consider a triangular mesh onto the space $\mathscr{D}$ with N nodes and a equally-spaced time grid $0 = t_0 < t_1 < \cdots < t_W = T$, of W time steps. Denoting by v^i the discrete approximation of V_i, if (B_n, S_n) is a vertex of the mesh, $n = 1, ..., N$, we set $v^i_{j,n} = v^i(t_j, B_n, S_n)$. As proved in Wilmott et al. (1993) with reference to American options, the i-th discrete variational inequality can be solved backward-in-time ($j = W-1, \cdots, 0$) by the following algebraic systems in the unknown vectors $\mathbf{v}^i_j$:

$$\mathbf{v}^i_j \geqslant \mathbf{M}^i_j, \quad \mathbf{A}\mathbf{v}^i_j \geqslant \mathbf{b}^i_{j+1}, \quad \left(\mathbf{v}^i_j - \mathbf{M}^i_j\right)\left(\mathbf{A}\mathbf{v}^i_j - \mathbf{b}^i_{j+1}\right) = 0 \tag{20}$$

Here $\mathbf{v}^i_j$ is the N dimensional vector $v^i(t_j, ., .)$, the obstacle $\mathbf{M}_j$, depending on the solution v^{i-1}, is defined by $M^i_{j,n} = M v^{i-1}(t_j, B_n, S_n)$, $\mathbf{A}$ is the Crank-Nicholson finite element matrix associated to the operator $\mathscr{L}_t$, and the vector $\mathbf{b}^i_{j+1}$ is constructed using vector $\mathbf{v}^i_{j+1}$.

Problem (20) can be solved using a Projected SOR (PSOR) algorithm, see Wilmott et al. (1993). To compute $\mathbf{v}^i_j$, we used as first guess solution $\mathbf{v}^i_{j+1}$ and we stopped the PSOR iterations when the L^∞ distance between two consecutive solutions falls under a given tolerance (TOL). Similarly, we considered v^i a good approximation of $V(t,B,S)$, the value function of our problem, when the distance between v^i and v^{i-1} falls under another given tolerance (TOL2). In the next section we show numerically the convergence of our numerical scheme when we increase the number of mesh nodes and time steps.

5. Numerical results for CRRA utility functions

In this section we present extended numerical results in the case of the CRRA utility function

$$U(L) = \frac{L^\gamma}{\gamma}$$

with $0 < \gamma < 1$. This utility, which is the most commonly used in the literature, belongs to the class of hyperbolic absolute risk aversion (HARA) utility functions. Using these functions the Merton's portfolio problem without transaction costs admits closed form solutions. Therefore it is possible to compare these exact solutions with the numerical results in the presence of transaction costs and solvency constraints. The main alternative would be to consider the exponential utility which implies a constant absolute risk aversion (see, for instance, Liu (2004)). However, if we consider our portfolio problem without transaction costs and exponential utility, the optimal strategy would be to maintain constant the discounted amount of money invested in the risky asset, independently of investor's wealth, which appears to be a rather unrealistic policy (see Korn (1997), chapter 3, and Merton (1969)). In all the case studies we set the values $B_{\min} = S_{\min} = -20$, $L_{\max} = 100$, TOL=10^{-5} (the tolerance threshold in the PSOR algorithm), TOL2=0.001 (the tolerance threshold to exit from the iterated optimal stopping cycle).

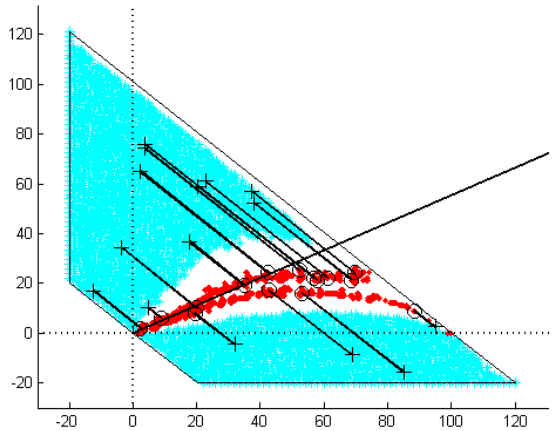

FIG. 3. Transaction region in the plane (B,S). Time $t = 0$, $N = 5000$, $W = 250$, $K = 0.1$ and $c = 0.01$.

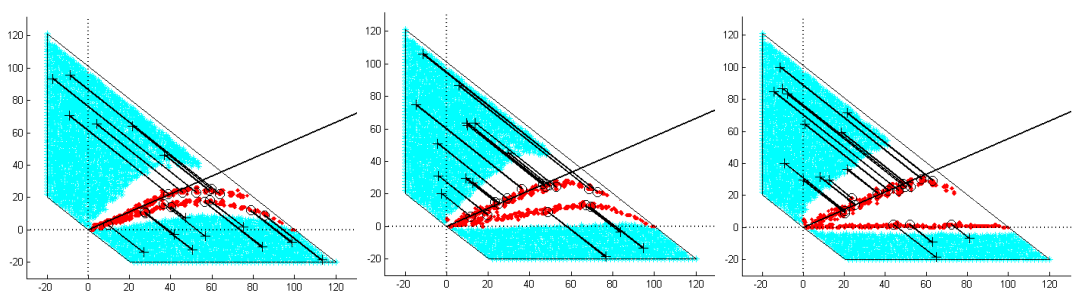

FIG. 4. Transaction region in the plane (B,S). Time $t = 2$ (left) - $t = 4$ (middle) - $t = 4.5$ (right), $N = 5000$, $W = 250$, $K = 0.1$ and $c = 0.01$.

The structure of the section is the following. First of all, we investigate the form of the optimal transaction strategy and we describe how the transaction regions, the no-trade region and the target portfolios vary as time approaches the final horizon. Moreover we show the convergence of our numerical scheme. Subsection 5.1 contains a comparative static analysis to show how the optimal policy is influenced by the different model parameters. Finally, in Subsection 5.2, the impact of transaction costs on the value of the final portfolio and on the frequency of trading is analyzed.

In our first numerical experiment, which we use as base case, we set the following values of the model parameters: $K = 0.1$, $c = 0.01$, $r = 0.02$, $\mu = 0.06$, $\sigma = 0.4$, $\gamma = 0.3$ and $T = 5$.

Figures 3 and 4 show the corresponding (dark) transaction regions and (white) no-trade region, at different time instants. The lines inside the no-trade region represent the re-calibrated portfolios, i.e. the portfolios where it is optimal to move when the investor's position falls in the intervention area. After a possible first transaction, made if the initial portfolio is in the intervention region, the investor will

maintain his position inside the white region re-calibrating his portfolio only if it reaches the boundary of the dark areas. In these figures some optimal transactions have been depicted: these are represented by the straight lines connecting the threshold portfolios in the transaction region to the corresponding target portfolios inside the no-trade area. The target portfolios are always inside the continuation region because the intervention costs make two consecutive transactions unprofitable. Moreover the upper (lower) line of target portfolios corresponds to the upper (lower) part of the transaction area. Unlikely the infinite horizon case (see Davis and Norman (1990), see Dumas and Luciano (1991), see Shreve and Soner (1994)) the optimal policy is not stationary: the transaction regions, as well as the target portfolios, change as time goes by. As expected the size of the intervention regions decreases as the time increases because, approaching the finite horizon, only a large change in the portfolio composition can compensate the transaction costs. However the evolution of the two parts of the transaction region is not symmetric. The size of the lower part decreases faster than the upper one. This reveals that the finite horizon and the bounded liquidation region induce a bias, as time goes on, in favor of the riskless asset. For example in $t = 4.5$ the lower transaction region is already below the axis $S = 0$. This implies that if in $t = 4.5$ the investor has a long position in stock he will never buy again the stock up to $T = 5$. Similarly the lower target line decreases with time towards the axis $S = 0$, and it is already equal to the axis $S = 0$ in $t = 4.5$. The same kind of liquidity preference in case of shorter investment horizons can be noted if we fix $t = 0$ and we consider a variable terminal date T, as we will do in Section 5.2. The more distant is the horizon T, the lower is the no-transaction region and the percentange of cash which is allowed to remain in the portfolio. This result is consistent with the common life-cycle investment advice that a young investor should hold a greater share of stocks in the portfolio than an old investor (see Liu and Loewenstein (2002)). In the graphs the Merton straight line is also depicted, which is constant in time and it represents the optimal portfolios for the same problem but without transaction costs and solvency constraints. It is interesting to note that the upper line of target portfolios remains approximately equal to the Merton line. In both Figures 3-4 the two optimal lines move down approaching to the edges CF and FG of the liquidation region (see Figure 2). This is due to the fact that the portfolio liquidation value is already near to $L_{\max} = 100$, the value considered satisfactory by the investor. Probably he will liquidate his position in short time and before T, this induces again a bias in favor of cash. For all t and most of the value function domain, the shape of the continuation region closely resembles a cone, enlarging with time, containing two moving straight lines of optimal portfolios. We conjecture that this would be the exact shape if we considered the same problem with an unbounded domain ($L_{\max} = +\infty$, i.e. the investor is never satisfied before T). Figure 5 shows the value and the shape of the value function at time $t = 0$.

In Table 1 and 2 we illustrate the convergence of our numerical scheme. We consider the solutions at $t = 0$ increasing the number of sub-intervals of the time-grid (W), and the number of mesh-points (N). In Table 1 we compute the L^2-norm error assuming as exact solution the one computed with $W = 250$ and $N = 5000$. More specifically in the upper part of the table we fix $N = 5000$ and we show the convergence increasing the time grid. Conversely, in the lower part we fix $W = 250$ and we make the space grid more dense. As expected, in both cases the solutions converge. Moreover, in Table 2 we show the convergence when we increase W and N at the same time; we do not assume an exact solution (an analytical solution is not available) but we list the distances, increasing both W and N, between two consecutive solutions in the numerical sequence. We consider both the L^2 and L^∞ relative errors. To show the convergence of the optimal control regions, we have also computed the Hausdorff distances (normalized by the length of the domain L) between the transaction regions and between the target portfolios of the consecutive solutions (the Hausdorff distance is the supremum of the distances of the points in one region to the

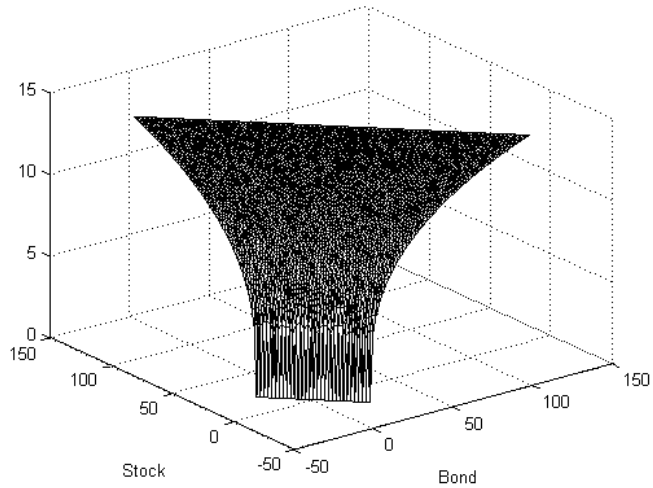

FIG. 5. Value function in the plane (B,S): Solution at $t = 0$. $N = 5000$, $W = 250$, $K = 0.1$ and $c = 0.01$.

Table 1. L^2 distance from the solution with $W = 250$ and $N = 5000$ at time $t = 0$, increasing W (above) and N (below).

Distance from the $W = 250$ solution, setting $N = 5000$				
$W =$	25	50	100	200
	0.0051	0.0026	0.0012	0.0008

Distance from the $N = 5000$ solution, setting $W = 250$				
$N =$	1000	2000	3000	4000
	0.0032	0.0014	0.0011	0.0005

Table 2. L^∞ and L^2 errors and Hausdorff distances between the transaction regions (HD1) and the optimal lines (HD2) of two consecutive solutions in the sequence, at time $t = 0$.

W	N	L^∞	L^2	HD1	HD2	Iterations	CPU Time (s)
25	1000	-	-	-	-	6	283
50	2000	0.0324	0.0023	0.0734	0.1562	6	1578
75	3000	0.0219	0.0014	0.0565	0.0720	5	3715
100	4000	0.0163	0.0011	0.0350	0.0348	5	8435
250	5000	0.0098	0.0005	0.0291	0.0296	4	23635

other region, and vice versa). Both Tables 1 and 2 indicate a rapid convergence of the solutions and of the optimal regions. Finally in Table 2 we also list the number of variational inequalities (number of iterations above the obstacle) which were necessary to achieve the TOL2 convergence and the CPU time necessary for the computation. All the computation have been performed in Matlab R2011a and on a personal computer equipped with a Pentium Dual-Core 2.70 GHz and 4 GB RAM.

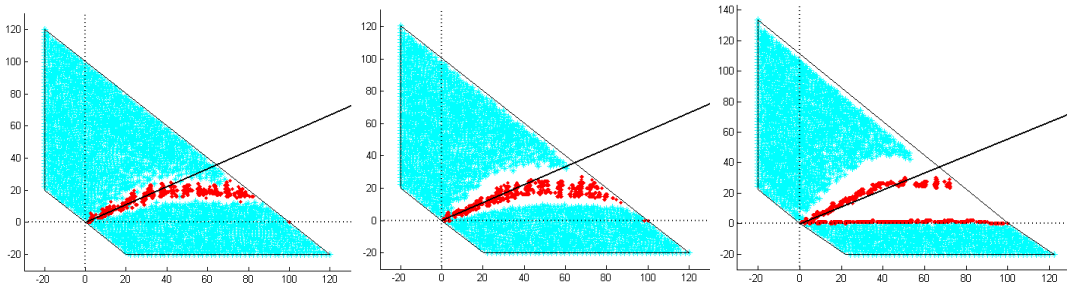

FIG. 6. Transaction area in the plane (B,S). Time $t=0$, $K=0.01$, $c=0.001$ (left) - $K=0.05$, $c=0.005$ (middle) - $K=0.1$, $c=0.1$.

5.1 *Sensitivity analysis*

Except for the parameters under investigation, in this subsection the values of the other parameters are the same as in the base case. The numerical results have been obtained setting $W=100$ and $N=4000$.

5.1.1 *Sensitivity with respect to the transaction costs* Naturally enough, increasing the transaction costs, the size of the intervention regions decreases. Due to the finite horizon T, if we increase K and c only fewer large transactions can be profitable. Figure 6 depicts the optimal regions for different values of K and c. Increasing the transaction costs produces a variation in the optimal policy which is similar to that caused by approaching the finite horizon T. The lower part of the intervention region decreases faster than the upper one, indicating a shift towards the riskless asset which is not present without transaction costs and solvency constraints. For $K=c=0.1$ the lower target portfolios are made only of the riskless asset while the upper optimal line stays close to the Merton one. It is interesting to observe how the optimal policy varies when we change the size of the variable cost c with respect to the fixed component K. In Figure 7 we set $K=0.1$ and we consider different values of c. See also Figure 3. For vanishing c the lower optimal line converge to the upper one, i.e., the Merton line. In fact, in all our experiments with $c=0$ we have only one line of target portfolios. Conversely, an increase in c pull the lines apart and closer to the intervention region. Vanishing K the solution tends towards the solution of a singular control problem where the optimal policy is an instantaneous reflection at the boundary of the intervention region (see Davis and Norman (1990), Shreve and Soner (1994). This behavior of the optimal control sets, varying the relative size of the variable and of the fixed part of the intervention costs, has already been noted, for a cash management problem (see, for instance, Baccarin (2009)).

5.1.2 *Sensitivity with respect to the other model parameters* When we modify the market parameters σ and r or the relative risk aversion coefficient $(1-\gamma)$ the Merton line varies its position and the no transaction area follows it in the same direction. If σ or $(1-\gamma)$ increase the investor will hold more of the riskless asset because he is risk averse, and, if r increases, he will hold more cash since the stock becomes less attractive. Consequently the Merton's line moves down towards the axis $S=0$. In Table 3 we show the percentage of the transaction area on the overall domain increasing r, σ and $(1-\gamma)$. This percentage grows in all cases, essentially because the upper optimal line follows closely the Merton's one and the upper part of the transaction region becomes bigger.

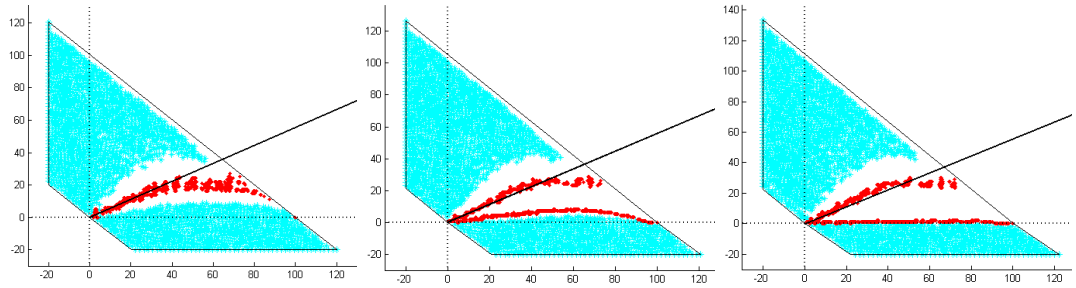

FIG. 7. Transaction area in the plane (B,S). Time $t=0$, $K=0.1$, $c=0.005$ (left) - $K=0.1$, $c=0.05$ (middle) - $K=0.1$, $c=0.1$ (right).

Table 3. Transaction Region in the plane (B,S) (percentage) for different values of interest rate (r), volatility (σ) and risk aversion coefficient (γ). $K=0.1$, $c=0.01$ and $T=5$.

	r			σ			$1-\gamma$		
t	0.01	0.02	0.04	0.3	0.4	0.5	0.6	0.7	0.8
0	69.09	69.14	74.21	62.77	69.14	74.21	66.75	69.14	70.59
2	68.29	68.52	73.38	60.86	68.52	73.95	66.18	68.52	70.34
4	60.83	62.03	68.43	48.56	62.03	69.97	58.47	62.03	64.79

5.2 *The impact of transaction costs on the frequency of trading and on the value of the final portfolio*

In order to get an estimate of the number of transactions made by an investor who follows our optimal policy we have coupled our numerical solution to a Monte Carlo simulation. Precisely we have considered an agent with initial portfolio made only of cash, $B_0=20$, $S_0=0$, who behaves according to the optimal intervention and continuation regions that we have computed numerically. When the investor's position is in the continuation region, which changes dynamically according to our numerical solution, we simulate the evolution of the stock value $S(t)$ by a computer generated random walk (the risk-free asset value $B(t)$ grows in a deterministic way). Whenever the simulated portfolio falls into the transaction region the agent re-calibrates its portfolio moving to the corresponding (at that time instant) optimal target portfolio and paying the necessary transaction costs. The Monte Carlo simulations were performed with 100 time steps, according to the time grid of the numerical solution computed with $W=100$ and $N=4000$.

In Table 4 we show the average and the standard deviation of the number of transactions, computed using 500 000 simulations, considering increasing values of T, up to forty years and different values for the transaction costs. It is surprising to observe that, even with the smallest transaction costs ($K=0.01$ and $c=0$), on average more than three years are necessary to have three transactions and that less than five interventions are made every ten years on the overall period ($T=40$). Note that if we consider ten thousand euros as the unit of measure, $K=0.01$ means a cost of 100 euros for each transaction, to rebalance a portfolio of initial value 200,000 euros. If we think of this cost, not only as a fixed commission, but also as the opportunity cost for the investor to collect information and submit an order to his broker, this value does not seem large.

It is also interesting to compare some alternative policies with the optimal one. In Table 5 we have considered the following trading strategies:

- the risk-free strategy (RF): the agent only invests his wealth in the risk-free asset

Table 4. Average (up) and standard deviation (down) of the number of transactions. Parameters: $r = 0.02$, $\mu = 0.06$, $\sigma = 0.4$, and $\gamma = 0.3$. Initial portfolio: $B_0 = 20,\ S_0 = 0$. Number of simulations equal to 500000.

		T					
K	c	2	3	5	10	20	40
		Average					
0.1	0.01	2.0007	2.0167	2.0952	2.6626	3.8775	6.1432
0.1	0.005	2.0014	2.0223	2.1233	2.7154	3.9962	6.3354
0.1	0.001	2.0030	2.0283	2.1454	2.7573	4.1726	6.4378
0.1	0	2.0052	2.0322	2.2523	2.8319	4.2174	6.8480
0.01	0.01	2.0414	2.2146	2.9961	4.6006	7.6706	12.5226
0.01	0.005	2.2142	2.5109	3.0501	4.8450	8.0899	13.5611
0.01	0.001	2.2150	2.6285	3.2504	5.5152	8.9542	15.1048
0.01	0	2.2629	2.7840	3.5118	5.6233	9.9649	17.1312
		Standard Deviation					
0.1	0.01	0.0257	0.1361	0.3072	0.7238	1.1943	1.8453
0.1	0.005	0.0379	0.1543	0.3481	0.8113	1.2589	1.8552
0.1	0.001	0.0545	0.1664	0.3761	0.8438	1.2771	1.8743
0.1	0	0.0649	0.1988	0.4733	0.8692	1.4450	1.9174
0.01	0.01	0.2027	0.4427	0.9364	1.6249	2.3478	3.4454
0.01	0.005	0.4256	0.6821	0.9717	1.6285	2.3940	3.5861
0.01	0.001	0.5263	0.7030	1.0277	1.8462	3.0124	5.8325
0.01	0	0.5287	0.8204	1.1833	1.9236	4.9104	8.7569

- the Merton strategy (Mer): it is the optimal strategy without transaction costs. The expected utility of the final position is given by the closed formula

$$E\left[U(L(B(T),S(T)))\right] = \frac{B_0^{\gamma}}{\gamma}\exp\left(\gamma(r+\frac{(\mu-r)^2}{2\sigma^2(1-\gamma)})T\right)$$

- the optimal strategy (Opt): in this case we have $E\ \left[U(L(B(T),S(T)))\right] = V(B_0,0,T)$. To obtain the average number of transactions we couple the Monte Carlo simulation with the numerical solution, as described above

- the Merton strategy with transaction costs: the agent recalibrates his portfolio moving to the Merton's line when the distance between his portfolio and the line itself is bigger than 5% (MTC(5%)) or 10% (MTC(10%)) of his wealth

- the barrier strategy (Bar(1%)): here we assume that the no-transaction region is a time-independent region delimited by two fixed barriers. The agent recalibrates his portfolio only when his position touches one of the two barriers and he makes the minimal transactions necessary to stay inside the no-trade region. We define the fixed barriers as the borders of the transaction regions that we have computed numerically at $t = 0$. To avoid unbounded transaction costs, due the fixed component K, we assumed that the portfolio is recalibrated towards the lower/upper barrier only if it falls below/above the barrier by more than the 1% of the agent's wealth.

For each of the last three strategies we have simulated 500000 possible scenarios, and thus 500000 possible values of $B(T)$ and $S(T)$, computing the mean value of $U(L(B(T),S(T))$ and the average

Table 5. Comparison of different strategies: certainty equivalent and average number of transactions. Parameters: $K = 0.01$, $r = 0.02$, $\mu = 0.06$, $\sigma = 0.4$, and $\gamma = 0.3$. Initial portfolio: $B_0 = 20$, $S_0 = 0$. Number of simulations equal to 500000.

c	T	RF	Mer	Opt	MTC(5%)	MTC(10%)	Bar(1%)
		Certainty Equivalent					
0.01	3	21.236	21.696	21.482	21.393	21.473	21.445
0.01	5	22.103	22.906	22.747	22.539	22.651	22.520
0.01	10	24.427	26.236	26.014	25.687	25.889	25.481
0.005	3	21.236	21.696	21.531	21.480	21.511	21.435
0.005	5	22.103	22.906	22.789	22.626	22.722	22.506
0.005	10	24.427	26.236	26.058	25.778	25.979	25.470
0	3	21.236	21.696	21.629	21.553	21.624	21.431
0	5	22.103	22.906	22.816	22.685	22.801	22.487
0	10	24.427	26.236	26.099	25.885	26.081	25.439
		Average Number of transactions					
0.01	3	0	∞	2.21	13.42	5.44	5.23
0.01	5	0	∞	3.00	18.87	7.44	6.14
0.01	10	0	∞	4.60	29.14	11.76	7.78
0.005	3	0	∞	2.51	13.43	5.44	5.33
0.005	5	0	∞	3.05	18.87	7.45	6.25
0.005	10	0	∞	4.84	29.14	11.77	7.83
0	3	0	∞	2.78	13.43	5.45	5.44
0	5	0	∞	3.51	18.87	7.46	6.37
0	10	0	∞	5.62	29.15	11.81	8.13

number of transactions. To make a more readable comparison among the different policies, in Table 5, besides the average number of transactions, we have shown the certainty equivalent of the utility of the final position, that is $U^{-1}(E\,[U(L(B(T),S(T)))])$. In this numerical experiment, where $B_0 = 20$, $S_0 = 0$ and at most $T = 10$, practically no simulated path reached the threshold liquidation value $L_{\max} = 100$, as it is also shown by the certainty equivalents which never exceed 30. This is an instance where the agent never liquidates the risky asset before T because $L_{\max}$ is too large compared to the initial wealth and T. As expected, if we do not consider the Merton (Mer) strategy, without transaction costs, the optimal strategy is the best one, i.e., the one with the highest certainty equivalent. It is also the policy with the lowest average number of interventions. We also notice that the optimal strategy and the MTC(10%) one are close, while the Bar(1%) strategy is the worst one, despite a low number of transactions. Notice that the Bar(1%) strategy is similar to the trading strategy which has been proven optimal for portfolio optimization problems with only proportional transaction costs (see Davis and Norman (1990), Dumas and Luciano (1991), Fleming and Soner (1993), Liu and Loewenstein (2002)). Thus trying to use this kind of policy in the presence of a fixed cost K different from zero clearly becomes unprofitable (and the results are even worse if we decrease the 1% level). We also notice that this strategy results in a lower utility when the proportional cost c approaches to zero. This rather surprising effect depends on the increased number of transactions due to a smaller no-transaction region, and thus on the increased fixed transaction costs.

Finally, in order to understand how the risk aversion index $1-\gamma$ influences the agent's behavior, in Table 6 we report the average number of transactions for agents who use our optimal policy with

Table 6. Sensitivity with respect to γ considering a fixed transaction cost $K = 0.01$: transaction region at $t = 0$ (TR), average (Av) and standard deviation (Std) of the number of transactions, and certainty equivalent for the optimal strategy with transaction costs (CE). Other parameters: $r = 0.02$, $\mu = 0.06$, and $\sigma = 0.4$. Initial portfolio: $B_0 = 20$, $S_0 = 0$. Number of simulations equal to 500 000.

	$c = 0.1$				$c = 0.001$			
γ	TR (%)	Av	Std	CE	TR (%)	Av	Std	CE
	$T = 1$							
0.1	0.704	0	0	20.404	0.820	2.022	0.068	20.434
0.2	0.702	0	0	20.404	0.810	2.016	0.056	20.454
0.3	0.670	0	0	20.404	0.806	2.004	0.070	20.476
0.4	0.639	0	0	20.404	0.789	2.002	0.039	20.506
0.5	0.586	0	0	20.404	0.758	2.002	0.047	20.543
0.6	0.517	0	0	20.404	0.728	2.001	0.037	20.590
0.7	0.434	0	0	20.404	0.685	2.000	0.006	20.677
	$T = 5$							
0.1	0.715	2.003	0.037	22.182	0.837	4.923	4.292	22.590
0.2	0.708	2.003	0.034	22.191	0.832	3.342	1.056	22.665
0.3	0.687	2.002	0.046	22.194	0.822	3.250	1.027	22.791
0.4	0.658	2.003	0.042	22.195	0.811	3.141	1.007	22.924
0.5	0.611	2.002	0.048	22.195	0.794	3.038	1.000	23.116
0.6	0.565	2.002	0.055	22.200	0.770	2.945	0.926	23.404
0.7	0.499	2.002	0.059	22.203	0.745	2.366	0.942	23.892
	$T = 10$							
0.1	0.727	4.059	1.073	24.643	0.839	6.578	4.191	25.638
0.2	0.723	3.705	1.124	24.664	0.836	5.980	2.003	25.775
0.3	0.705	3.025	0.957	24.683	0.826	5.515	1.846	26.074
0.4	0.673	2.553	0.986	24.718	0.815	5.012	1.998	26.345
0.5	0.632	2.519	0.837	24.790	0.803	4.596	1.790	26.774
0.6	0.591	2.444	0.774	24.932	0.785	4.211	1.674	27.400
0.7	0.544	2.207	0.854	25.139	0.764	4.023	1.948	28.335

different values of γ. We have considered $K = 0.01$, and $c = 0.1$ or $c = 0.001$ (the other parameters are the same considered in this section). We notice that, decreasing γ, i.e., considering more risk-averse investors, both the percentage of the transaction region and the average number of transactions increase. This is due to the fact that a more risk-averse agent prefers to pay higher transaction costs to maintain his portfolio into a less risky position.

6. Conclusions

In this paper we have investigated a portfolio selection problem for a passive investor who divides his capital between a risk-free money market instrument and a financial security representing a broad market index. At every time instant the agent must decide the proportion of his wealth to invest in the risky asset but the presence of transaction costs makes it unprofitable to trade continuously. Under general assumptions we have characterized the value function as a constrained viscosity solution of the related HJBQVI and we have proven the existence and structure of an optimal impulse trading policy. Moreover we have proposed an iterative finite element method to numerically solve the model

for arbitrary utility functions. In the case of a power utility we have shown that the no-transaction region closely resembles a cone which is the same shape of similar problems with infinite horizon and proportional transaction costs. However the presence of a fixed cost leads to a big structural difference in the optimal strategy: in our model there are two lines of target portfolios where to move from the borders of the no-trade region, while in the proportional case the agent makes the minimal transactions to stay inside the no-trade region. By a numerical example we have shown that trying to implement this second policy in the presence of a fixed cost is clearly sub-optimal. Unlike most of the literature on portfolio selection in continuous time, our optimal trading strategy is not stationary. This is due to the finite horizon because approaching the final date the intervention costs tend to be greater than the benefits from rebalancing. Consequently, as time goes on, the no-trade region increases and the target lines move apart. Furthermore the evolution of the two parts of the transaction region and of the two lines of target portfolio is not symmetric, showing a liquidity preference in favor of the riskless asset as time goes on up to the final date. Finally we have computed the average number of transactions made by an agent using the optimal trading strategy. Our numerical simulations show that, even with small transaction costs, the investor rebalances his portfolio very few times. This contrast sharply with the continuous interventions of the Merton's model without transaction costs.

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A. Proofs

A.1 *Proof of Proposition 3.1*

We set $Z(t,B,S) := Ce^{\delta(T-t)}(B+S)^{\gamma}$. The inequality (9) holds true in $T\times\overline{\Omega}$ as $V(T,B,S) = U(L(B,S)) \leqslant CL(B,S)^{\gamma} \leqslant C(B+S)^{\gamma}$ and in $[0,T)\times(0,0)$ because here we have $V(t,0,0)=0$. Moreover, in $\overline{\Omega}\setminus\{\mathbf{0}\}$, Z verifies $Z > \mathscr{M}Z$ and $-\frac{\partial Z}{\partial t} - \mathscr{L}Z \geqslant 0$. Indeed $\mathscr{M}Z = -1$ if $(B,S)\notin \digamma$, $\mathscr{M}Z \leqslant Ce^{\delta(T-t)}(B+S-K)^{\gamma} < Z$ if $(B,S)\in\digamma$, and, differentiating Z, it is easy to verify that $\frac{\partial Z}{\partial t} + \mathscr{L}Z \leqslant 0$ in $\overline{\Omega}\setminus\{\mathbf{0}\}$. Now consider an admissible policy $p\in A(t,B,S)$, for the controlled process starting in $t\in[0,T)$ with values $(B,S)\in\overline{\Omega}\setminus\{\mathbf{0}\}$. We define $\tau_0^p = t$ and, almost surely, $n^p(\omega) = \max\left\{i\geqslant 0 : \tau_i^p(\omega)\leqslant \vartheta^p(\omega)\right\}$. Applying the generalized Itô's formula to the function Z, from t to ϑ^p, we have:

$$Z(\vartheta^p, B^p(\vartheta^p), S^p(\vartheta^p)) = Z(t,B,S) + \int_t^{\vartheta^p}\left(\tfrac{\partial Z}{\partial t} + \mathscr{L}Z\right)ds + \int_t^{\vartheta^p}\sigma S\tfrac{\partial Z}{\partial S}\,dW(s) +$$
$$+\sum_{i=1}^{n^p}\left(Z(\tau_i, B(\tau_i^-)-\xi_i-K-c\,|\xi_i|, S(\tau_i^-)+\xi_i) - Z(\tau_i, B(\tau_i^-), S(\tau_i^-))\right).$$

Since $\frac{\partial Z}{\partial t} + \mathscr{L}Z \leqslant 0$ and $Z > \mathscr{M}Z$ it follows that, a.s.,

$$Z(\vartheta^p, B^p(\vartheta^p), S^p(\vartheta^p)) < Z(t,B,S) + \int_t^{\vartheta^p}\sigma S\frac{\partial Z}{\partial S}\,dW(s)\,.$$

Taking expectations, the stochastic integral vanishes, since $\frac{\partial Z}{\partial S}$ is bounded, and we obtain

$$Z(t,B,S) > E[Z(\vartheta^p, B^p(\vartheta^p), S^p(\vartheta^p))] \qquad \forall p\in A(t,B,S)\,.$$

Therefore

$$\begin{aligned} Z(t,B,S) &\geqslant \sup_{p\in A(t,B,S)} E[Z(\vartheta^p, B^p(\vartheta^p), S^p(\vartheta^p))] \\ &= \sup_{p\in A(t,B,S)} E[C(B^p(\vartheta^p)+S^p(\vartheta^p))^{\gamma}e^{\delta(T-\vartheta^p)}] \\ &\geqslant \sup_{p\in A(t,B,S)} J^p = V(t,B,S)\,. \end{aligned}$$

A.2 *Proof of Theorem 3.2*

To prove the weak comparison principle, we adapt our problem to the techniques in Akian et al. (2001), Barles (1994), Ly Vath et al. (2007), Oksendal and Sulem (2002), giving all the necessary preliminary definitions and results. To prove comparison results for second-order equations is useful to give equivalent definitions of viscosity solutions in terms of parabolic second order super and subdifferentials, see Crandall et al. (1992). We will denote by $\mathscr{S}^2$ the set of all 2×2 symmetric matrices and, when it is convenient, by x the couple $(B,S)\in\overline{\Omega}$.

DEFINITION A.1 1) The set of parabolic second order superdifferentials of a function $u:\overline{Q}\to\mathbb{R}$ at the point $(t,x)\in\overline{Q}$ is defined by

$$\begin{aligned} D^{+(1,2)}u(t,x) \quad = \quad & \Big\{(q,p,A)\in\mathbb{R}\times\mathbb{R}^2\times\mathscr{S}^2: \\ & \limsup_{\substack{(h,y)\to 0\\ (t+h,x+y)\in\overline{Q}}} \frac{u(t+h,x+y)-u(t,x)-qh-py-\frac{1}{2}Ay\cdot y}{|h|+|y|^2}\leqslant 0\Big\} \end{aligned} \tag{21}$$

2) A triplet $(q,p,A)\in\mathbb{R}\times\mathbb{R}^2\times\mathscr{S}^2$ belongs to $\overline{D}^{+(1,2)}u(t,x)$, the closure of $D^{+(1,2)}u(t,x)$, if there exists a sequence (t_m,x_m) converging to (t,x), and another sequence

$$(q_m,p_m,A_m)\in D^{+(1,2)}u(t_m,x_m)$$

converging to (q,p,A) as m tends to infinity.

The set $D^{-(1,2)}u(t,x)$ of parabolic second order subdifferentials of $u:\overline{Q}\to\mathbb{R}$ at $(t,x)\in\overline{Q}$ is defined in a symmetric way using the $\liminf$ and the $\geqslant$ inequality in (21) and the definition of its closure $\overline{D}^{-(1,2)}u(t,x)$ is analogous to the definition of $\overline{D}^{+(1,2)}u(t,x)$.

DEFINITION A.2 Given $\mathscr{O}\subset\overline{\Omega}$, a locally bounded function $u:\overline{Q}\to\mathbb{R}_+$ is called a viscosity subsolution (resp. supersolution) of (8) in $[0,T)\times\mathscr{O}$ if

$$\begin{aligned} \min\Big\{-q-rBp_1-\mu Sp_2-\frac{1}{2}\sigma^2S^2A_{22},u^*(t,x)-\mathscr{M}u^*(t,x)\Big\} \quad &\leqslant \quad 0 \\ \text{(resp. } u_* \text{ and} \quad &\geqslant \quad 0) \end{aligned}$$

for all $(t,x)\in[0,T)\times\mathscr{O}$, $(q,p,A=\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix})\in\overline{D}^{+(1,2)}u^*(t,x)$ (resp. $\overline{D}^{-(1,2)}u_*(t,x)$).

In order to prove the weak comparison principle it is useful to obtain strict viscosity supersolutions of (8) in $Q=[0,T)\times\Omega$.

LEMMA A.1 Fix $\delta'>\delta=\gamma\Big(r+\frac{(\mu-r)^2}{2\sigma^2(1-\gamma)}\Big)$ and consider the smooth perturbation function $g(t,B,S)=e^{\delta'(T-t)}(B+S)^\gamma$. Let $v\in LSC(\overline{Q})$ be a viscosity supersolution of (8) in Q. Then for any $\varepsilon>0$ the lsc function $v_\varepsilon=v+\varepsilon g$ is a strict viscosity supersolution of (8) in any compact set $G\subset Q$. This means that for any compact $G\subset Q$ there exists a constant $\rho>0$, depending on G, such that

$$\min\Big\{-q-rBp_1-\mu Sp_2-\frac{1}{2}\sigma^2S^2A_{22},v_\varepsilon-\mathscr{M}v_\varepsilon\Big\}\geqslant\varepsilon\rho$$

for all $(t,B,S) \in G$, $\varepsilon > 0$ and $(q,p,A) \in \overline{D}^{-(1,2)} v_\varepsilon(t,B,S)$.

Proof. From the definition (7) we have, for $\varepsilon > 0$,

$$\mathscr{M} v + \varepsilon \mathscr{M} g \geqslant \mathscr{M} v_\varepsilon$$

and thus

$$v_\varepsilon - \mathscr{M} v_\varepsilon \geqslant v - \mathscr{M} v + \varepsilon (g - \mathscr{M} g) \ . \tag{22}$$

Since v is a supersolution it holds $v - \mathscr{M} v \geqslant 0$. Moreover from (7) and the definition of g it follows

$$g(t,B,S) - \mathscr{M} g\,(t,B,S) \geqslant \begin{cases} e^{\delta'(T-t)}\left[(B+S)^\gamma - (B+S-k)^\gamma\right] & \text{if } (B,S) \in F \\ 1 & \text{if } (B,S) \notin F \ . \end{cases}$$

Hence for any compact $G \subset Q$ there exists $\rho_1 > 0$ such that $g - \mathscr{M} g \geqslant \rho_1$ for $(t,B,S) \in G$. Combining this with (22) we obtain $v_\varepsilon - \mathscr{M} v_\varepsilon \geqslant \varepsilon \rho_1$ in G. We consider now $-\frac{\partial g}{\partial t} - \mathscr{L} g$. We have

$$-\frac{\partial g}{\partial t} - \mathscr{L} g = e^{\delta'(T-t)} (B+S)^\gamma \left[\delta' - \gamma \frac{rB + \mu S}{B+S} - \frac{1}{2}\gamma(\gamma-1)\sigma^2 \frac{S^2}{(B+S)^2} \right] \tag{23}$$

and, setting $\frac{S}{B+S} = \alpha$, $\frac{B}{B+S} = (1-\alpha)$, it is not difficult to see that $\delta' > \gamma \left(r + \frac{(\mu - r)^2}{2\sigma^2(1-\gamma)} \right)$ is sufficient to get $-\frac{\partial g}{\partial t} - \mathscr{L} g > 0$, when $B + S > 0$. Therefore for any compact $G \subset Q$ there exists $\rho_2 > 0$ such that $-\frac{\partial g}{\partial t} - \mathscr{L} g \geqslant \rho_2$ for all $(t,B,S) \in G$. Since v is already a supersolution of (8), we obtain that

$$-q - rBp_1 - \mu S p_2 - \frac{1}{2}\sigma^2 S^2 A_{22} \geqslant \varepsilon \rho_2$$

for all $(t,B,S) \in G$ and $(q,p,A) \in \overline{D}^{-(1,2)} v_\varepsilon(t,B,S)$. Therefore v_ε is a strict viscosity supersolution of (8) in any compact set $G \subset Q$. □

Now it is sufficient to prove the weak comparison principle between a viscosity subsolution u and a strict viscosity supersolution $v_\varepsilon = v + \varepsilon\, f$, for all $\varepsilon > 0$, because $u \leqslant v$ in $Q \backslash R$ will follow in the limit $\varepsilon \downarrow 0$. We show the result first reasoning in $\overline{Q}^+$. Let u and v be as in theorem 3.2. We redefine the supersolution v on $\partial^* Q^+$ by

$$v(t,B,S) = \liminf_{\substack{(t',B',S') \in Q^+ \\ (t',B',S') \to (t,B,S)}} v(t',B',S') \qquad \forall\, (t,B,S) \in \partial^* Q^+, \tag{24}$$

and we still denote v this function. Now we consider the difference $u - v_\varepsilon$ in $\overline{Q}^+$, and we argue by contradiction supposing that

$$m \equiv \sup_{(t,B,S) \in \overline{Q}^+} u - v_\varepsilon > 0 \ . \tag{25}$$

Since $u - v_\varepsilon$ is u.s.c, $\overline{Q}^+$ is compact and the boundary conditions (13) hold true, the maximum m is attained in some point $(t_0, x_0) \in \{[0,T) \times \{\Omega^+ \cup \partial_1 \Omega^+\}\} \setminus \{\mathbf{0}\}$. To obtain a contradiction we apply the Ishii's technique redoubling the variables and penalizing this doubling, see Crandall et al. (1992) and Barles (1994). First suppose $(t_0, x_0) \in Q^+$ and consider the test functions for $i \geqslant 1$

$$\Phi_i(t,x,x') = u(t,x) - v_\varepsilon(t,x') - \varphi_i(t,x,x'),$$

where

$$\varphi_i(t,x,x') = |t-t_0|^2 + |x-x_0|^4 + \frac{i}{2}\left|x-x'\right|.$$

As $\Phi_i(t,x,x')$ is usc in $\overline{Q}^+$, there exists $(\hat{t}_i,\hat{x}_i,\hat{x}'_i) \in \overline{Q}^+$ such that

$$m_i = \sup_{(t,x,x')\in[0,T]\times\overline{\Omega}^+\times\overline{\Omega}^+} \Phi_i(t,x,x') = \Phi_i(\hat{t}_i,\hat{x}_i,\hat{x}'_i),$$

and, at least for a subsequence, $(\hat{t}_i,\hat{x}_i,\hat{x}'_i)$ converges to some $(\hat{t}_0,\hat{x}_0,\hat{x}'_0) \in \overline{Q}^+$. By definition we have

$$m \leqslant m_i \leqslant u(\hat{t}_i,\hat{x}_i) - v_\varepsilon(\hat{t}_i,\hat{x}'_i),$$

and it is not difficult to show that, approaching i to infinity, we obtain

$$\begin{cases} \hat{t}_0 = t_0,\ x_0 = \hat{x}_0 = \hat{x}'_0 \\ m_i \longrightarrow m \\ \frac{i}{2}\left|\hat{x}_i - \hat{x}'_i\right| \to 0. \end{cases} \tag{26}$$

Therefore we can apply Ishii's lemma to the interior maximum $(\hat{t}_i,\hat{x}_i,\hat{x}'_i) \in [0,T)\times\Omega^+\times\Omega^+$ of Φ_i, see Theorem 8.3 in Crandall et al. (1992). There exist $q,q' \in \mathbb{R}$, $p,p' \in \mathbb{R}^2$ and $A,A' \in \mathscr{S}^2$ such that

$$(q,p,A) \in \overline{D}^{+(1,2)}u(\hat{t}_i,\hat{x}_i) \ \text{ and } \ (q',p',A') \in \overline{D}^{-(1,2)}v_\varepsilon(\hat{t}_i,\hat{x}_i),$$

where

$$\begin{cases} q - q' = \frac{\partial\varphi_i}{\partial t}(\hat{t}_i,\hat{x}_i,\hat{x}'_i) = 2(\hat{t}_i - t_0) \\ p = \frac{\partial\varphi_i}{\partial x}(\hat{t}_i,\hat{x}_i,\hat{x}'_i) = 4(\hat{x}_i - x_0)\left|\hat{x}_i - x_0\right|^2 + i(\hat{x}_i - \hat{x}'_i) \\ p' = -\frac{\partial\varphi_i}{\partial x'}(\hat{t}_i,\hat{x}_i,\hat{x}'_i) = i(\hat{x}_i - \hat{x}'_i), \end{cases} \tag{27}$$

and A, A' are such that

$$\begin{bmatrix} A & 0 \\ 0 & A' \end{bmatrix} \leqslant \frac{\partial^2\varphi_i}{\partial x\partial x'}(\hat{t}_i,\hat{x}_i,\hat{x}'_i) + \frac{1}{i}\left(\frac{\partial^2\varphi_i}{\partial x\partial x'}(\hat{t}_i,\hat{x}_i,\hat{x}'_i)\right)^2. \tag{28}$$

The subsolution property of u in $(\hat{t}_i,\hat{x}_i)$ and the strict supersolution property of v_ε in $(\hat{t}_i,\hat{x}'_i)$, imply that

$$\min\left\{-q - r\hat{B}_i p_1 - \mu\hat{S}_i p_2 - \frac{1}{2}\sigma^2\hat{S}_i^2 A_{22},\, u(\hat{t}_i,\hat{x}_i) - \mathscr{M}u(\hat{t}_i,\hat{x}_i)\right\} \leqslant 0 \tag{29}$$

$$\min\left\{-q' - r\hat{B}'_i p'_1 - \mu\hat{S}'_i p'_2 - \frac{1}{2}\sigma^2\hat{S}'^2_i A'_{22},\, z_\varepsilon(\hat{t}_i,\hat{x}'_i) - \mathscr{M}z_\varepsilon(\hat{t}_i,\hat{x}'_i)\right\} \geqslant \varepsilon\rho. \tag{30}$$

If $u(\hat{t}_i,\hat{x}_i) - \mathscr{M}u(\hat{t}_i,\hat{x}_i) \leqslant 0$ in (29), then, combining with $z_\varepsilon(\hat{t}_i,\hat{x}'_i) - \mathscr{M}z_\varepsilon(\hat{t}_i,\hat{x}'_i) \geqslant \varepsilon\rho$ due to (30), we obtain

$$m_i \leqslant u(\hat{t}_i,\hat{x}_i) - v_\varepsilon(\hat{t}_i,\hat{x}'_i) \leqslant \mathscr{M}u(\hat{t}_i,\hat{x}_i) - \mathscr{M}z_\varepsilon(\hat{t}_i,\hat{x}'_i) - \varepsilon\rho\ .$$

Using Lemma 3.1 and (26), when i goes to infinity, we have

$$m \leqslant \mathscr{M}u(t_0,x_0) - \mathscr{M}z_\varepsilon(t_0,x_0) - \varepsilon\rho\ .$$

Since by Remark 2.2, $F(x_0)$ is compact, if it is not empty, and u is usc, then there exists x_0' such that $\mathscr{M}u(t_0,x_0) = u(t_0,x_0')$ and we obtain a contradiction using the definitions of m and $\mathscr{M}$

$$m \leqslant u(t_0,x_0') - z_\varepsilon(t_0,x_0') - \varepsilon\rho \leqslant m - \varepsilon\rho \ .$$

Therefore it must be $-q - r\hat{B}_i p_1 - \mu\hat{S}_i p_2 - \frac{1}{2}\sigma^2\hat{S}_i^2 A_{22} \leqslant 0$ in (29), and, combining with $-q' - r\hat{B}_i' p_1' - \mu\hat{S}_i' p_2' - \frac{1}{2}\sigma^2\hat{S}_i'^2 A_{22}' \geqslant \varepsilon\rho$ of (30), we obtain

$$-(q-q') - r(\hat{B}_i p_1 - \hat{B}_i' p_1') - \mu(\hat{S}_i p_2 - \hat{S}_i' p_2') - \frac{1}{2}\sigma^2(\hat{S}_i^2 A_{22} - \hat{S}_i'^2 A_{22}') \leqslant -\varepsilon\rho \ . \tag{31}$$

By (26) and (27), as i goes to infinity, $(q-q')$, $(\hat{B}_i p_1 - \hat{B}_i' p_1')$, $(\hat{S}_i p_2 - \hat{S}_i' p_2')$ converge to zero. Moreover by (28) it follows

$$(\hat{S}_i^2 A_{22} - \hat{S}_i'^2 A_{22}') \leqslant \beta_i, \tag{32}$$

where

$$\beta_i = s_i \left[\frac{\partial^2\varphi_i}{\partial x \partial x'}(\hat{t}_i,\hat{x}_i,\hat{x}_i') + \frac{1}{i}\left(\frac{\partial^2\varphi_i}{\partial x \partial x'}(\hat{t}_i,\hat{x}_i,\hat{x}_i')\right)^2 \right] s_i^T, \tag{33}$$

with $s_i = [0,\hat{S}_i,0,\hat{S}_i']$. We have

$$\frac{\partial^2\varphi_i}{\partial x \partial x'}(\hat{t}_i,\hat{x}_i,\hat{x}_i') = \begin{bmatrix} iI_2 + Q_i & -iI_2 \\ -iI_2 & iI_2 \end{bmatrix}, \tag{34}$$

where $Q_i = 8\,|\hat{x}_i - x_0|^2 I_2 + 8(\hat{x}_i - x_0)(\hat{x}_i - x_0)^T$ and I_2 is the (2×2) identity matrix. Substituting (34) into (33), after some computations we obtain

$$\beta_i = 3i(\hat{S}_i - \hat{S}_i')^2 + s_i\left(\begin{bmatrix} 3Q_i & -Q_i \\ -Q_i & 0 \end{bmatrix} + \frac{1}{i}\begin{bmatrix} Q_i^2 & 0 \\ 0 & 0 \end{bmatrix}\right) s_i^T \ . \tag{35}$$

By (26) and (35), β_i also converges to zero as i goes to infinity and therefore (31) and (32) lead to another contradiction when $i \to \infty$. Therefore we have shown that the maximizer (t_0,x_0) of (25) cannot belong to Q^+. The more difficult case ,when we suppose the maximizer (t_0,x_0) is on the border $\{[0,T)\times\partial_1\Omega^+\}\setminus\{\mathbf{0}\}$, can be faced as in Oksendal and Sulem (2002) and Ly Vath et al. (2007) using a technique proposed in Barles (1994) which assumes some regularity of the boundary. Specifically if we denote by $d(x)$ the distance from x to $\partial\Omega^+$, this distance must be twice continuously differentiable in a neighborhood of x_0. It can be shown as in Ly Vath et al. (2007) that this regularity is satisfied on the border $\{[0,T)\times\partial_1\Omega^+\}\setminus\{\mathbf{0}\}$. By (24) there exists a sequence (t_i,x_i) in Q^+ converging to (t_0,x_0). Define $\alpha_i = |t_i - t_0|$, $\gamma_i = |x_i - x_0|$ and consider, as in Ly Vath et al. (2007), the test functions for $i \geqslant 1$

$$\Phi_i(t,t',x,x') = u(t,x) - v_\varepsilon(t',x') - \varphi_i(t,t',x,x'), \tag{36}$$

where

$$\varphi_i(t,t',x,x') = |t-t_0|^2 + |x-x_0|^4 + \frac{|t-t'|^2}{2\alpha_i} + \frac{|x-x'|^2}{2\gamma_i} + \left(\frac{d(x')}{d(x_i)} - 1\right)^4 .$$

It is not difficult to show that in the maximizer $(\hat{t}_i,\hat{t}_i',\hat{x}_i,\hat{x}_i')$ of Φ_i, the point $\hat{x}_i'$ always verifies $d(\hat{x}_i') > 0$. Therefore we can still use the strict supersolution property of v_ε in $(\hat{t}_i',\hat{x}_i')$. Applying Ishii's lemma to the point $(\hat{t}_i,\hat{t}_i',\hat{x}_i,\hat{x}_i')$ and repeating the preceding arguments with the test functions (36) we obtain again,

by contradiction, that it must be $m \geqslant 0$. Finally to get $u \leqslant v$ also in $Q^- \backslash R$ it is sufficient to redefine the subsolution u on $\partial^* Q^-$ by

$$v(t,B,S) = \limsup_{\substack{(t',B',S') \in Q^- \\ (t',B',S') \to (t,B,S)}} v(t',B',S') \qquad \forall\, (t,B,S) \in \partial^* Q^-$$

and to repeat the same proof of $\overline{Q}^+$ in $\overline{Q}^-$.

A.3 *Proof of Lemma 3.2*

First of all we consider $(t,B,S) \in \partial_2^* Q \backslash R$. Since $U(L(B,S)e^{r(T-t)})$ is continuous in $(t,B,S) \in \partial_2^* Q \backslash R$ and, by construction, it always holds $V(t',B',S') \geqslant U(L(B',S')e^{r(T-t')})$, it holds for any $(t,B,S) \in \partial_2^* Q \backslash R$

$$V_*(t,B,S) \equiv \liminf_{\substack{(t',B',S') \in Q \\ (t',B',S') \to (t,B,S)}} V(t',B',S') \geqslant U(L(B,S)e^{r(T-t)}) \,. \tag{37}$$

Now let

$$V^*(t,B,S) = \limsup_{\substack{(t',B',S') \in Q \\ (t',B',S') \to (t,B,S)}} V(t',B',S')$$

and (t_m, B_m, S_m) be a sequence in Q such that

$$\lim_{(t_m,B_m,S_m) \to (t,B,S)} V(t',B',S') = V^*(t,B,S).$$

By (6), for any m there exists a quasi-optimal policy $p^m = \{(\tau_i^m, \xi_i^m)\}$ such that $p^m \in A(t_m,B_m,S_m)$ and $V(t_m,B_m,S_m) \leqslant J^{p^m}(t_m,B_m,S_m) + \frac{1}{m}$. Denoting the controlled process (B^{p^m}, S^{p^m}) by X^m it follows (here $\vartheta^m = T \wedge \theta^{p^m}$)

$$V(t_m,B_m,S_m) \leqslant E_{t_m,B_m,S_m}\left[U\left(L(X^m(\vartheta^m))\, e^{r(T-\vartheta^m)}\right)\right] + \frac{1}{m} \,.$$

As it is always optimal not to intervene in ϑ^m we can assume $\tau_i^m \neq \vartheta^m$, $\forall i$. Defining $\Delta X_s^m \equiv X^m(s) - X^m(s^-)$, where $s \geqslant t_m$ and $X^m(t_m^-) \equiv (B_m, S_m)$, we have

$$X^m(\vartheta^m) = X^m(t_m^-) + \Delta X_{t_m}^m + \int_{t_m}^{\vartheta^m} \alpha(X^m(s))ds + \int_{t_m}^{\vartheta^m} \beta(X^m(s))dW_s + \sum_{t_m < s < \vartheta^m} \Delta X_s^m \tag{38}$$

where $\alpha(X^m) = [rB^{p^m}, \mu S^{p^m}]$ and $\beta(X^m) = [0, \sigma S^{p^m}]$. Since $(t_m,B_m,S_m) \to (t,B,S) \in \partial_2^* Q$ it follows that $\vartheta^m - t_m$ converges a.s. to zero when $m \to \infty$. Thus the two integrals in (38) vanish because $X^m \in \overline{\Omega}$ is bounded. Moreover the last summation also vanishes because the jump sizes are uniformly bounded and the number of interventions after t_m and before ϑ^m, converges to zero as $t_m \to \vartheta^m$. The first difference $\Delta X_{t_m}^m$, at least for a subsequence, converges to some ΔX_1 when $m \to \infty$. Finally, approaching m to infinity in (A.3), by the dominated convergence theorem we obtain

$$V^*(t,B,S) \leqslant U(L((B,S) + \Delta X_1)e^{r(T-t)}) \leqslant U(L(B,S)e^{r(T-t)}), \tag{39}$$

and therefore the first condition in (14) is true. If $(t,B,0) \in \partial_2^* Q \cap R$ and $(t',B',S') \in Q^+$ converges to $(t,B,0)$ from above R, we have

$$\lim_{\substack{(t',B',S')\in Q^+ \\ (t',B',S')\to(t,B,0)}} U(L(B',S')e^{r(T-t')}) = U(Be^{r(T-t)}),$$

and we can repeat the same reasoning as for $(t,B,S) \in \partial_2^* Q \backslash R$. However, if $(t',B',S') \in Q^-$ converges to $(t,B,0)$ from below R, we have

$$\lim_{\substack{(t',B',S')\in Q^- \\ (t',B',S')\to(t,B,0)}} U(L(B',S')e^{r(T-t')}) = U((B-K)e^{r(T-t)})$$

and, by the same procedure used before to obtain (37) and (39), we get

$$\begin{aligned} V^*_{Q^-}(t,B,0) &\equiv \limsup_{\substack{(t',B',S')\in Q^- \\ (t',B',S')\to(t,B,0)}} V(t',B',S') \leqslant U((B-K)e^{r(T-t)}) \\ &\leqslant V_{*\,Q^-}(t,B,0) \equiv \liminf_{\substack{(t',B',S')\in Q^- \\ (t',B',S')\to(t,B,0)}} V(t',B',S') \ . \end{aligned}$$

A.4 *Proof of Theorem 4.2*

We first show that $P_n \geqslant V_n$ for any $(t,B,S) \in \overline{Q}$. Let $p \in A_n(t,B,S)$, with $p = \{\tau_i^p, \xi_i^p\}_{i=1,\cdots,n}$, and (B^p, S^p) the corresponding controlled process. Since $\mathscr{M}P_{n-1}$ is given at step n, the function P_n is, for any n, the value function of an optimal stopping problem. By using the dynamic programming principle for the value functions of optimal stopping problems - see Chapter 3, Section 1, in Krylov (1980) - it can be shown, as in Corollary 3.7 in Chancelier et al. (2002), that the process

$$Z_n(s) = P_n(s \wedge \vartheta^p, B^p(s \wedge \vartheta^p), S^p(s \wedge \vartheta^p)), \quad s \geqslant t$$

is a supermartingale, for any n and any given stopping time $\alpha \geqslant t$. From the optional sampling theorem it follows that if $t \leqslant \alpha_1 \leqslant \alpha_2$ are stopping times then we have

$$\begin{aligned} & E_{t,B,S}\left[P_n(\alpha_1 \wedge \vartheta^p, B^p(\alpha_1 \wedge \vartheta^p), S^p(\alpha_1 \wedge \vartheta^p))\right] \\ \geqslant\ & E_{t,B,S}\left[P_n(\alpha_2 \wedge \vartheta^p, B^p(\alpha_2 \wedge \vartheta^p), S^p(\alpha_2 \wedge \vartheta^p))\right] \ . \end{aligned} \tag{40}$$

Define $\overline{\tau}_0 \equiv 0$, $\overline{\tau}_i \equiv \tau_i^p \wedge \vartheta^p$ and let $(B^p(s), S^p(s)) = (B(s), S(s))$ in any interval $[\overline{\tau}_j, \overline{\tau}_{j+1})$. By (40) and the definitions (7) and (18) we obtain for $j = 0, ..., n-1$

$$\begin{aligned} & E_{t,B,S}\left[P_{n-j}(\overline{\tau}_j, B(\overline{\tau}_j), S(\overline{\tau}_j))\right] \geqslant E_{t,B,S}\left[P_{n-j}(\overline{\tau}_{j+1}, B(\overline{\tau}_{j+1}^-), S(\overline{\tau}_{j+1}^-))\right] \\ & = E_{t,B,S}\left[P_{n-j}(\overline{\tau}_{j+1}, B(\overline{\tau}_{j+1}^-), S(\overline{\tau}_{j+1}^-))\right] \chi_{\tau^p_{j+1i} \leqslant \vartheta^p} \\ & + E_{t,B,S}\left[P_{n-j}(\overline{\tau}_{j+1}, B(\overline{\tau}_{j+1}^-), S(\overline{\tau}_{j+1}^-))\right] \chi_{\tau^p_{j+1i} > \vartheta^p} \\ & \geqslant E_{t,B,S}\left[\mathscr{M}P_{n-j-1}(\overline{\tau}_{j+1}, B(\overline{\tau}_{j+1}^-), S(\overline{\tau}_{j+1}^-))\right] \chi_{\tau^p_{j+1i} \leqslant \vartheta^p} \\ & + E_{t,B,S}\left[P_{n-j-1}(\overline{\tau}_{j+1}, B(\overline{\tau}_{j+1}^-), S(\overline{\tau}_{j+1}^-))\right] \chi_{\tau^p_{j+1i} > \vartheta^p} \\ & \geqslant E_{t,B,S}\left[P_{n-j-1}(\overline{\tau}_{j+1}, B(\overline{\tau}_{j+1}), S(\overline{\tau}_{j+1}))\right] \ . \end{aligned}$$

Summing up all these inequalities from $j=0$ to $j=n-1$ we obtain

$$P_n(t,B,S) \geqslant E_{t,B,S}\left[P_0(\overline{\tau}_n, B(\overline{\tau}_n), S(\overline{\tau}_n))\right].$$

By property (40) we also have

$$\begin{aligned} & E_{t,B,S}\left[P_0(\overline{\tau}_n, B(\overline{\tau}_n), S(\overline{\tau}_n))\right] \geqslant E_{t,B,S}\left[P_0(\vartheta^p, B(\vartheta^p), S(\vartheta^p))\right] \\ & = E_{t,B,S}\left[U\left(L(B(\vartheta^p), S(\vartheta^p))\, e^{r(T-\vartheta^p)}\right)\right] = J^p(t,B,S)\,. \end{aligned}$$

Thus we have shown that $P_n(t,B,S) \geqslant J^p(t,B,S)$, $\forall p \in A_n(t,B,S)$ and $P_n \geqslant V_n$, $\forall (t,B,S) \in \overline{Q}$.

To obtain the reverse inequality we build an optimal policy $p^* \in A_n(t,B,S)$ such that $J^{p^*}(t,B,S) = P_n(t,B,S)$. First of all, let us define the control sets

$$C_i \equiv \left\{(t,B,S) \in \overline{Q} : P_i(t,B,S) = \mathscr{M} P_{i-1}(t,B,S)\right\}, \quad i = 1, \cdots, n.$$

Moreover, let I_1 be the set

$$I_1 \equiv \left\{\vartheta \geqslant s \geqslant t : (s, B(s), S(s)) \in C_n\right\}.$$

We choose τ_1^* such that

$$\tau_1^* = \begin{cases} \inf I_1 & \text{if } I_1 \neq \emptyset \\ +\infty & \text{if } I_1 = \emptyset \end{cases}$$

and ξ_1^* is given by

$$\xi_1^* = \begin{cases} \xi^*_{P_{n-1}}(B(\tau_1^{*-}), S(\tau_1^{*-})) & \text{if } \tau_1^* < \infty \\ \text{arbitrary if } \tau_1^* = \infty \end{cases}$$

where $\xi^*_{P_{n-1}}(B,S)$ is defined in Lemma 3.1 (c). If α_1, α_2 are stopping times such that $t \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \tau_1^*$, it follows by the dynamic programming principle that (40) becomes an equality. See Corollary 3.7b in Chancelier et al. (2002) and Chapter 7 in Oksendal and Sulem (2007). From this fact and the choice of (τ_1^*, ξ_1^*), all the inequalities in (A.4) become equalities and, being $\overline{\tau}_1^* \equiv \tau_1^* \wedge \vartheta$, we obtain

$$P_n(t,B,S) = E_{t,B,S}\left[P_{n-1}(\overline{\tau}_1^*, B(\overline{\tau}_1^*), S(\overline{\tau}_1^*))\right]. \tag{41}$$

Now we define the policy p^* recursively by

$$\begin{cases} \tau_i^* = \begin{cases} \inf I_i & \text{if } I_i \neq \emptyset \\ +\infty & \text{if } I_i = \emptyset \end{cases} \\ \xi_i^* = \begin{cases} \xi^*_{P_{n-i}}(B(\tau_i^{*-}), S(\tau_i^{*-})) & \text{if } \tau_i^* < \infty \\ \text{arbitrary if } \tau_i^* = \infty \end{cases} \end{cases}$$

for $i = 1, \ldots, n$, with $\tau_0^* \equiv 0$ and where I_i is the set

$$I_i \equiv \left\{\vartheta \geqslant s \geqslant \tau_{i-1}^* : (s, B^{p^*}(s^-), S^{p^*}(s^-)) \in C_{n+1-i}\right\}.$$

By the same argument of (41) we have

$$E_{t,B,S}\left[P_{n-i}(\overline{\tau}_i^*, B^{p^*}(\overline{\tau}_i^*), S^{p^*}(\overline{\tau}_i^*))\right] = E_{t,B,S}\left[P_{n-i-1}(\overline{\tau}_{i+1}^*, B^{p^*}(\overline{\tau}_{i+1}^*), S^{p^*}(\overline{\tau}_{i+1}^*))\right], \tag{42}$$

with $\overline{\tau}_i^* \equiv \tau_i^* \wedge \vartheta^{p^*}$. Considering all the n equalities (42) we conclude the proof since

$$\begin{aligned} P_n(t,B,S) &= E_{t,B,S}\left[P_0(\overline{\tau}_n^*, B^{p^*}(\overline{\tau}_n^*), S^{p^*}(\overline{\tau}_n^*))\right] \\ &= E_{t,B,S}\left[P_0(\vartheta^{p^*}, B(\vartheta^{p^*}), S(\vartheta^{p^*}))\right] \\ &= E_{t,B,S}\left[U\left(L(B(\vartheta^{p^*}), S(\vartheta^{p^*}))\, e^{r(T-\vartheta^{p^*})}\right)\right] = J^{p^*}(t,B,S)\,. \end{aligned}$$