Experimental approaches to theoretical thinking the mathematics classroom: artefacts and proofs

Experimental approaches to theoretical thinking in the mathematics classroom: artefacts and proofs

F. Arzarello, M.G. Bartolini Bussi, A. Leung, M.A. Mariotti, I. Stevenson

Introduction

From straight-edge and compass to a variety of computational and drawing tools created in the course of history, instruments are deeply intertwined with the genesis and development of many abstract concepts and ideas in mathematics. Their use introduces an "experimental" dimension in mathematics, and a dynamic tension between the *empirical nature* of the activities with them – which encompasses perceptual and operational components– and the *deductive nature* of the discipline –which entails a rigorous and sophisticated formalization. As Pierce points out, this singular peculiarity:

> (It) has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science.
>
> (Peirce, C.P., 3.363: quoted in Dörfler, 2005, p. 57)

Arsac (1987) shows that the observational component was also present in the masterpiece of Euclid the *Elements*. Euclid was aware of the dialectic between the decontextualized aspects of pure geometry and the phenomenology of our perception of objects in the space and the way we represent them in the plane. In the *Optics* (Euclide, 1996) another masterpiece, he gives a rationale of this tension.

In the nineteenth century, mathematicians from Pasch to Peano to Hilbert, tried to eliminate these "intuitive" hidden hypotheses from Geometry. In a similar manner the school of Wierstrass eliminated any reference to space or motion in the definition of limits with the masterpiece foundation through the epsilon-delta machinery: see Lakoff & Núñez (2000). It seemed that such observational components had completely disappeared from the scene until recently. The philosophical criticism of many scholars (for a survey see Tymoczko, 1998) and the development of computational techniques have produced a fresh approach to mathematical discoveries by supporting renewed epistemological stances which underlie the observational and experimental aspects of mathematical inquiry (see Lovasz, 2006). We recall here two important examples, *experimental mathematics* and *visual theorems*:

> Experimental mathematics is the use of a computer to run computations —sometimes non more than trial-and-error tests— to look for patterns, to identify particular numbers and sequences, to gather evidence in support of specific mathematical assertions that may themselves arise by computational means, including search. Like contemporary chemists —and before them the alchemists of old— who mix various substances together in a crucible and heat them to a high temperature to see what happens, today's experimental mathematicians puts a hopefully potent mix of numbers, formulas, and algorithms into a computer in the hope that something of interest emerges.
>
> (Borwein & Devlin, 2008, p. 1)

> Briefly, a visual theorem is the graphical or visual output from a computer program - usually one of a family of such outputs - which the eye organizes into a coherent, identifiable whole and which is able to inspire mathematical questions of a traditional nature or which contributes in some way to our understanding or enrichment of some mathematical or real world situation.
>
> (Davis, 1993, p. 333)

Such considerations throw a fresh light on mathematical epistemology as well as on the processes of mathematical discoveries and, as a consequence, the nature of mathematical learning processes must be rethought. In particular, the phenomenology of learning proof has been challenged and reconsidered from the new epistemological and cognitive points of view (see Balacheff, 1988, 1999; the volume edited by Boero, 2007; de Villers, 2010; and the plenary lecture of G. Longo in this volume). An outcome is that not only *deductive* but also *abductive* and *inductive* processes have been scrutinized and revealed as crucial in all mathematical activities, emphasising in teaching the experimental component of proofs. The related didactical phenomena become particularly interesting when proving activities are planned within a technological environment (see Jones et al., 2000; Arzarello et al., 2007), where teacher's interventions are carefully designed. By "technological environment" we do not mean just digital technologies but any environment where there are instruments used to learn mathematics (for a non computer-like such environment, see Bartolini-Bussi, 2010).

The main goal of the chapter is entering into the aforementioned dynamic tension between the empirical and the theoretical nature of mathematics. Our purpose is to underline the elements of historical continuity in the stream of thought that is called experimental mathematics, and show the concrete possibility so offered to today's teachers for pursuing the learning of proof in the classroom.

Specifically we examine how such a tension regulates the processes of students, who are asked to solve mathematical problems, first making explorations with technological tools, then formulating suitable conjectures and finally proving them. We discuss this issue framing it from different linked perspectives: historical, epistemological, didactical and pedagogical.

First (Part 1: *From straight-edge and compass to dynamic geometry software*) we consider some emblematic events from the history of mathematics in different cultures and epochs, where instruments have played a crucial role in generating mathematical concepts.

Next (Part 2: *A student-centred analysis*) we move from this inquiry to analyse some didactical episodes from the classroom life, where the use of instruments in proving activities makes palpable the aforementioned dynamic tension. We develop a careful analysis of students' processes while interacting with tools. The result consists in some theoretical frameworks that give reason how such a tension can be positively used to design suitable didactical situations. Within them students can learn practices with technological tools that support them in a route from the empirical to the theoretical side of mathematics. In particular we discuss the complex interactions between inductive, abductive and deductive modalities in such a transition. By analysing the roles for technologies within our framework we show that the cultural aspects of experimental mathematics, founded upon these historical continuity stances, can and should be made visible to students.

Finally (Part 3: *What is the role of technologies in geometrical proof?*) we show how a general pedagogical frame (Activity Theory) is suitable for giving a sense to the previous microanalyses within a general unitary educational standpoint.

Part 1
From straight-edge and compass to dynamic geometry software

The classical age in Europe

Euclidean Geometry is often referred to as 'straight-edge and compass geometry', because of the centrality of construction problems in Euclid's work. Since antiquity geometrical constructions have had a fundamental theoretical importance in the Greek tradition (Heath, 1956, p. 124). This issue is clearly illustrated by the history of the classic *impossible* problems, which so much puzzled the Greek geometers (Henry, 1993). Despite the apparent practical objective, i.e. the drawing which can be produced on a sheet of parchment or papyrus, geometrical constructions do have a theoretical meaning. In Euclid's Elements, no real, material tools are envisaged; rather their use is objectified into the geometrical objects defined by definitions and axioms. By contrast Giusti writes: “the mathematical objects are not generated through abstraction from real objects […] but they formalize human operations”[1](Giusti, 1999). We would add that they are shaped by the tools with which people accompany such operations.

As a consequence, the tools, and rules of their use, have a counterpart in the axioms and theorems of a theoretical system, so that any construction may be conceived as a theoretical problem stated inside a specific theoretical system. The solution of a problem is correct, therefore, in so far as it can be validated within such theoretical system,. Any successful construction corresponds to a specific theorem, and *validates* the specific relationship between properties of the geometrical figure represented by the drawing, obtained after the construction.

However, the practical meaning of a construction, related to the possibility of its concrete realization, may critically interfere with the theoretical perspective within which a geometrical construction must be solved. In fact, any geometrical construction may be utilized to obtain a drawing with a certain guaranty of efficiency, including ‘impossible’ constructions, which, despite their theoretical impossibility, can be realized with arbitrarily chosen precision (take for instance, the case of the trisection of an angle, Henry, 1993, p. 104).

From the perspective of classic geometry, drawing tools, despite their empirical counterpart, may be conceived as theoretical tools defining a particular geometry. In this sense, classic Euclidean geometry has been traditionally called “straight-edge and compass geometry”, referring to both the origin and limitations of its objects.

The classical age in Far East: China and India

The difference between a practical and a theoretical attitude towards drawing tools may be evident if we refer to the use of the same tools in Far East (e. g. China). We have evidence that straight-edge compass and other geometrical tools were known also many centuries BC and used in the construction of buildings. Certainly drawing instruments such as compasses and tri-squares were used in ancient China, as depicted in some ancient murals in tombs dating back to the Han Dynasty. An example that is often referred to is a mural depicting two mythical figures, Nu Wa and Fu Xi, the former holding a set of compasses (gui) and the latter a set of tri-square (ju). In some ancient texts four types of instruments for use in building and engineering are mentioned, namely, gui (compasses), ju (tri-square), zhun (an instrument to determine a horizontal line), sheng (a rope to determine a straight line). Indeed, the Chinese term "gui ju" means "abiding by the common code and discipline", while the Chinese term "zhun sheng" means "accurate". Both terms derive their origin from ancient instruments.[2] Historians wonder why, in spite of the availability of similar drawing tools, Chinese culture has nothing similar to the idea of geometrical constructions, and there is no documentation of theoretical validations for geometrical constructions. Although the issue is still debated, there is some consensus about the position of Bloom (1981), who connected the lack of Euclid-style arguments in Chinese ancient mathematics to the structure of classical Chinese language, where counter-factual reasoning has no easy way of expression (see also Dauben, 1998).

In China, however, a very refined mathematical knowledge has been created (Chemla & Guo, 2005). Historians wonder whether the reasoning contained in this ancient text (and in other texts from different cultural traditions) is related to mathematical proofs. Joseph (1991) puts things in a very provocative way, contrasting the dominant (Eurocentric) tendency to give value only to the ways of reasoning connected to the Greek tradition with other non-European perspectives. In very ancient ages (e.g. Egypt and Babylonia) most problems books contains “algorithms” that, although

[1] “gli oggetti matematici provengono non dall’astrazione da oggetti reali […] ma formalizzano l’operare umano”.

[2] We thank Siu Man Keung for these informations.

not presented in a deductive form, are nevertheless general, being applicable to a large class of situations. He claims that:

> From the practical concerns of society have arisen a number of rules which should be judged for both their effectiveness and their intrinsic qualities. A 'good' algorithm should have three properties:
> a) It should be clear and simple, laying out step by step the procedure to be followed,
> b) It should emphasize the general character of its applications by pointing out its appropriateness, not to a single problem but to a group of similar problems, and
> c) It should show how clearly the answer obtained after the prescribed set of operations is completed.
> (Joseph, 1991, p. 127)

He contends "a modern proof is a procedure, based on axiomatic deduction, which follows a chain of reasoning from the initial assumptions to the final conclusion. But is this not taking a highly restrictive view of what is a proof?" (*ibidem*). We may conclude by stressing the fact that beyond evident concerns in generalization and acceptability of solution (procedures) answering to practical issues, other cultures do not elaborate a specific theoretical approach based on an axiomatic system. By contrast this emphasis on theoretical reasoning remains a defining characteristic of some Greek mathematicians and of the subsequent western culture.

The modern age in Europe

In the Euclidean tradition, interesting European developments about the theoretical status of drawing tools and curves are found in the seventeen century. In the '*Géométrie*' (Descartes), two methods of representing curves are clearly stated: the representation by a continuous motion and the representation by an equation (Bos, 1981). Descartes deals with the following question: *'Which are the curved lines that can be accepted in geometry?* (*ibidem,* p. 315), and gives an answer (or, better, two answers) different from that of classical geometers :

> 1) [...] we can imagine them as described by a continuous motion, or by several motions following each other, the last of which are completely regulated by those which precede. For in this way one can always have an exact knowledge of their measure (*ibidem,* p. 316);
> …
> 2) [...] those which admit some precise and exact measure, necessarily have some relation to all points of a straight line, which can be expressed by some equation, the same equation for all points (*ibidem,* p. 319).

He invented a tool, his compass (see Fig. 1), to make evident what he meant: moving the different t-squares YBC, DCE, EDF, …a point like H moves and generates a curve with the features described under 1) and 2).

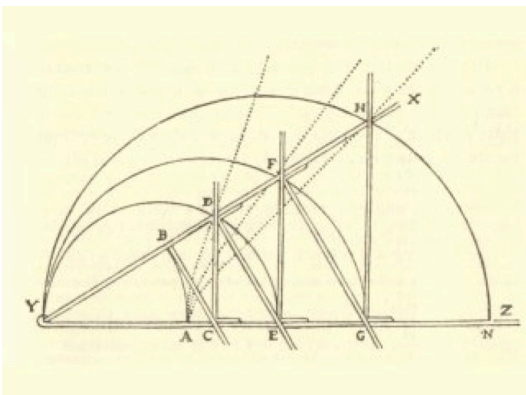

Figure 1

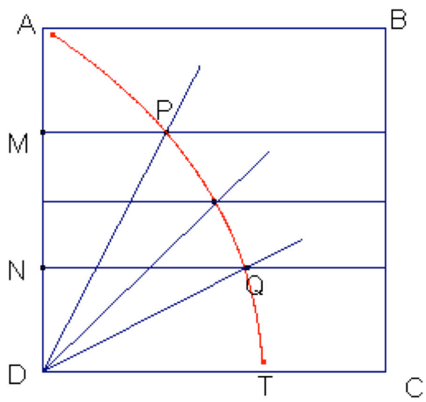

Figure 2

This was not the case for example for the Hippias trisectrix (Fig. 2), where the points of the curve APQT are generated through a continuous synchronous motion of the ray DP (which rotates uniformly around D like the hand of a clock) and of the ray MP (which moves uniformly and horizontally, so that while the ray rotates from DA to DC the ray MP moves from AB to DC): in fact, it impossible to get effectively the position of all the points of the curve APQT since one can only imagine them through the description of the movement and not concretely get them as in the case of a figure drawn using straight-edge and compass.

The goal of Descartes was related to the very foundations of geometry: if a curve (e. g. a conic or a conchoid) is to be accepted as a tool to solve geometrical problems, one must be sure that, under certain conditions, the intersection points of two such curves exist. Hence, point-wise generation is not sufficient and the continuum problem is called into play. As Lebesgue (1950) claims, when a curve is traced pointwise, it is obtained by approximation: it is only a graphic solution. But if one designs a tracing instrument, the graphic solution becomes a mechanical solution. By the standards of the seventeenth century mathematicians, the mechanical solution is acceptable because it refers to one of the basic intuitions about the continuum, i.e. the movement of an object. Descartes did not confront the question whether the two given criteria - the mechanical and the algebraic - are equivalent or not. This problem actually requires constructing more advanced algebraic tools and, what is more important, changing the status of the new drawing instruments from tools for solving geometric problems to objects of a theory.

In the classical age and in the seventeen century as well, it was clear that changing the drawing tools would have changed the set of solvable problems in a rigorous way. So, if only the straight-edge and compass are accepted (i. e. only straight lines and circles) it is not possible to solve the problem of cube duplication and of angle trisection in a rigorous way. If, on the contrary, other tools are admitted (e.g. the Nicomedes compass that draws a conchoids, see Heath 1956), the above problems can be solved rigorously.

Constructions: practical versus theoretical

The previous discussion has shown a critical/dialectical relationship between practical and theoretical issues. In the specific domain of classic Euclidean Geometry, the core of such relationship seems to reside in the notion of *construction* to be related to the specific tools that are assumed available. The practical realization of any graphical element has a counterpart in a theoretical element, therefore, in either an axiom that states how to use a tool, or a theorem that validates the construction procedure according to the stated axioms. In these terms, a geometrical construction can be considered archetypal for a theoretical approach to Geometry. As a consequence, construction problems not only have a critical place in the development of Geometry as a discipline, but also keep a central place in traditional Geometry textbooks that form educational programs.

In the recent past, in spite of their long tradition, geometrical constructions have lost their centrality and almost disappeared from the Geometry curriculum at least in the western world. One can rarely find any reference to 'drawing tools' when geometrical axioms are stated, and geometrical constructions no longer belong to the set of problems proposed in the textbooks. This is a pity, because, as we show below, geometric constructions are very rich in meaning and perfectly suitable to the implementation in today's classrooms.

The relationship between a geometrical construction and the theorem which validates it, is very complex and certainly not immediate for students, as clearly described and discussed by Schoenfeld (1985). As he explains, "many of the counterproductive behaviors we see in students are learned as unintended by-products of their mathematics instruction" (p. 374). It seems that the very nature of the construction problem makes it difficult to take a theoretical perspective, as shown by Mariotti (1996) in a completely different school context.

Constructions in a DGS

The interest for constructions has been renewed by the appearance of Dynamic Geometry Systems (DGS), where the basic role played by construction has been brought on the scene by the use of graphic tools available in a dynamic system, like Cabri-géomètre, Sketchpad, Geogebra, etc. Any DGS-figure is the result of a construction process, since it is obtained after the repeated use of tools, chosen among those available in the "tool bar". However, what makes DGS so interesting is not just the construction facility; but the direct manipulation of its figures, conceived in terms of the embedded logic system (Laborde & Straesser, 1990; Straesser 2001). DGS-figures possess an intrinsic logic, as a result of their construction, placing the elements of a figure in a hierarchy of relationships, which correspond to the procedure of construction according to the chosen tools. This relationship is made evident in the dragging mode, where, by varying the basic points (elements) from which a figure has been built, what cannot be dragged constitutes the results of the construction. The DGS-figure is the complex of these elements, incorporating various relationships which can be differently referred to definitions and theorems of geometry.

Compared to the classic world of paper and pencil figures, the novelty of a dynamic geometry environment consists in the possibility of direct manipulation of its figures, and such manipulation is conceived in terms of the logic system of Euclidean Geometry. The dynamics of the DGS figures, realized by the dragging function, preserves its intrinsic logic, i.e. the logic of its construction. The elements of a figure are situated in a hierarchy of properties, and this hierarchy corresponds to a relationship of logical conditionality.

The presence of the dragging mode introduces in the DGS environment a specific criterion of validation for the solution of the construction problems: a solution is valid if and only if the figure on the screen is stable under the dragging test. However the system of DGS-figures embodies a system of relationships consistent in the broad system of a geometrical theory. Thus, solving construction problems in DGS means not only accepting all the facilities of the software, but also accepting a logic system within which to make sense of them.

DGS's intrinsic relation to Euclidean geometry make it is possibile to interpret the control 'by dragging' as corresponding to theoretical control – 'by proof and definition' - within the system of Euclidean Geometry or of another geometry that allows the recourse to a larger set of tools. Consider for instance the history of *neusis* problems and their connections with the trisection of an angle in antiquity, as discussed by Heath (1921, 235-41). In other words, it is possible to state a correspondence between the world of DGS constructions and the theoretical world of Euclidean Geometry.

Constructions with straight-edge and compass in the mathematics classroom

According to the previous analysis, a specific hypothesis can be stated concerning the potentiality of the notion of *geometrical construction* as a *key of accessing* to the meaning of proof. Different research groups have undertaken different directions with different tools and different mathematical theories can be considered according to a common approach. We will concentrate on one of these routes because the authors have guided some experiences along this stream.

A recent teaching experiment carried out in Italy has shown the potential of straight-edge and compass for developing an experimental approach with theoretical aims (Bartolini Bussi et al. In print). The project was developed with a group of 80 mathematics teachers (only 6 from primary school and the others equally divided between junior secondary and high school, see Martignone, 2010) and nearly 2000 students (scattered all over a large region of Northern Italy). Straight-edge and compass were put in the larger context of mathematical machines (Bartolini Bussi, 2000 p. 343), which are tools that force a point to follow a trajectory or to be transformed according to a given law. A common theoretical framework (that is presented shortly below, see Bartolini Bussi & Mariotti, 2008) structured the exploration of the tools, on the one hand, and of the functions played by the tools in the solution of geometrical problems by construction on the other hand. Similar processes were implemented, first with teachers through in-service course of 6 meetings, and then later with students by the participating teachers. A total of 79 teaching experiments, with detailed documentations, have been collected, 25% of which concerned straight-edge and compass.

The general structure of the approach was the following:

A) Exploration and analysis of the tool (shorter for teachers, yet longer for students, in order to make them aware of the relationship between the physical structure of the compass and Euclid's definition of circle).
B) Production of very simple constructions of geometrical figures (e. g. "To draw an equilateral triangle with a given side"), in open form, in order to allow a variety of constructions based on different known properties.
C) Comparison of the different constructions in large group discussion, to show that the "same" drawing may be based on very different processes, each drawing on either implicit or explicit assumptions and on the technical features of the tool.
D) Production of proofs of the constructions exploiting each times the assumptions.

The processes were structured by the following key questions, concerning the compass as a tool:

1) How is it made?
2) What does it do?
3) Why does it do that?

The third question, which is dependent on the others, aims at connecting the practical use of the tool to the theoretical aspects. In fact, the justification of a construction draws on the geometrical properties of the compass, as it is clearly shown in the proof of the Proposition 1 of the Book1 of Euclid's elements (Heath, 1956, p. 241), where the construction of an equilateral triangle is carried out.

Constructions within DGS environment

As presented in previous work (Mariotti, 2000, 2001), some teaching experiments were carried out with 10^{th} grade students, attending the first year in a scientific oriented school (Liceo Scientifico). The design of the teaching sequence was based on the development of the field of experience (Boero et al., 1995) of Geometrical Constructions in a Dynamic Geometry System (Cabri-Géomètre). The educational aim was that of introducing students to a theoretical perspective and the achievement of such aim is based on the potential correspondence between DGS constructions and geometric theorems.

The activity started by revisiting drawings and artefacts which belong to the pupils' experience. Such objects were part of physical experience. For example the compass is a concrete object, whose use was more or less familiar to the pupils. In any case, the students were familiar with the constraints and relationships which determine possible actions and expected results; for instance, the intrinsic properties of a compass directly affect the properties of the graphic trace produced.

Revisitation was accomplished by transferring the drawing activity into the Cabri environment, moving the external context from the world of straight-edge and compass drawings to the virtual world of DGS figures and commands.

In a DGS environment the new 'objects' available are as Evocative Computational Objects (Hoyles, 1993; Hoyles & Noss, 1996, p. 68),. These are characterised by their computational nature and their power to evoke geometrical knowledge. For Cabri:

- the Cabri-figures realising geometrical figures;
- the Cabri-commands (primitives and macros), realising the geometrical relationships which characterise geometrical figures;

- the dragging function which provides a perceptual control of the correctness of the construction, corresponding to the theoretical control consistent with geometry theory.

The development of the field of experience occurs through activities carried out within the world of Cabri such asconstruction tasks, interpretation and prediction tasks and mathematical discussions. Such development, however, also concerned the practice of straight-edge and compass constructions, which became both concrete referents and signs of the Cabri figures. The relationship between drawings (carried out on paper, using straight-edge & compass), and Cabri figures constitutes a peculiar aspect of students' experience, which presented a double face, one physical and the other virtual.

In the DGS environment, a construction activity, such as drawing figures through the available commands on the menu, is integrated with the dragging function. Thus a construction task is accomplished if the figure on the screen passes the dragging test.

In this case, the necessity of justifying the solution comes from the need of validating one's own construction, in order to explain why it works and/or to foresee that it will work. Of course, dragging the figure may be sufficient to convince one of the correctness of the solution, but at this point the second component of the teaching/learning activities come into play. Namely construction problems become part of a social interchange, where different solutions are reported and compared, and this represents a crucial element.

Experiments and proofs with the computer

A typical aspect of experimental mathematics is accomplished by making computations with a computer. Crucially validating numerical solutions, which may have been found, requires producing suitable proofs: the book by Borwein and Devlin (2009) contains many nice examples. We illustrate with an example this relationship between found numerical solutions and proofs; more precisely, we show how CAS can be used as a tool for promoting the production of proofs, hence becoming a useful companion to empirical numerical explorations.

The example is taken from Arzarello (2009) and concerns 9-th grade students attending the first year in a scientifically oriented higher secondary school (Liceo Scientifico), who were studying functions through their tables of differences. The students had already learnt that , for first degree functions, the first differences are constant. In the example below, they are asked to make conjectures on which functions have the first differences that change linearly. Their conjecture is that quadratic functions have this property, and arrange a spreadsheet like in Fig. 3a, where they utilise:

- columns A, B, C, D to indicate respectively the values of the variable x, of the function $f(x)$ (in B_i there is the value of $f(A_i)$) and of its related first and second differences (namely in C_i there is the value $f(A_{i+1})$-$f(A_i)$ and D_j there is the value C_{j+1}- C_j);
- variable numbers in cells E2, F2, …,I2 to indicate respectively: the values x_0 (the first value for the variable x to put in A2); a, b, c for the coefficients of the second degree function $ax^2+bx + c$; the step h of which the variable in column A is incremented each time for passing to A_i to A_{i+1}.

	A	B	C	D	E	F	G	H	I
◆									
1	x	f(x)	df(x)	ddf(x)	x0	a	b	c	h
2	0	3	-1	-4	0	-2	1	3	1
3	1	2	-5	-4					
4	2	-3	-9	-4					
5	3	-12	-13	-4					
6	4	-25	-17	-4					
7	5	-42	-21	-4					
8	6	-63	-25	-4					
9	7	-88	-29	-4					
10	8	-117	-33	-4					
11	9	-150	-37	-4					
12	10	-187	-41	-4					
13	11	-228	-45	-4					
14	12	-273	-49	-4					
15	13	-322	-53	-4					
16	14	-375	-57						
17	15	-432							

Figure 3a

	A	B	C	D	E	F	G	H
◆								
1	x	f(x)	df(x)	ddf(x)	x0	a	b	c	h
2	x0	a*x0^2+b*x0…	a*(h^2+2*h*x0)+…	2*a*h^2	x0	a	b	c	h
3	h+x0	a*(h+x0)^2+…	a*(3*h^2+2*h*x0…	2*a*h^2					
4	2*h+x0	a*(2*h+x0)^2…	a*(5*h^2+2*h*x0…	2*a*h^2					
5	3*h+x0	a*(3*h+x0)^2…	a*(7*h^2+2*h*x0…	2*a*h^2					
6	4*h+x0	a*(4*h+x0)^2…	a*(9*h^2+2*h*x0…	2*a*h^2					
7	5*h+x0	a*(5*h+x0)^2…	a*(11*h^2+2*h*x…	2*a*h^2					
8	6*h+x0	a*(6*h+x0)^2…	a*(13*h^2+2*h*x…	2*a*h^2					
9	7*h+x0	a*(7*h+x0)^2…	a*(15*h^2+2*h*x…	2*a*h^2					
10	8*h+x0	a*(8*h+x0)^2…	a*(17*h^2+2*h*x…	2*a*h^2					
11	9*h+x0	a*(9*h+x0)^2…	a*(19*h^2+2*h*x…	2*a*h^2					
12	10*h…	a*(10*h+x0)^…	a*(21*h^2+2*h*x…	2*a*h^2					
13	11*h…	a*(11*h+x0)^…	a*(23*h^2+2*h*x…	2*a*h^2					
14	12*h…	a*(12*h+x0)^…	a*(25*h^2+2*h*x…	2*a*h^2					
15	13*h…	a*(13*h+x0)^…	a*(27*h^2+2*h*x…	2*a*h^2					
16	14*h…	a*(14*h+x0)^…	a*(29*h^2+2*h*x…						
17	15*h…	a*(15*h+x0)^…							

Figure 3b

Modifying the values of E2, F2, …,I2 the students could easily do their explorations. This was a practice that they learnt which gradually became shared in the classroom, through the interventions of the teacher. In fact the class teacher stressed its value as an instrumented action (in the sense of Rabardel, 2002), to support explorations in the numerical environment. Students realised that:

- if they changed only the value of *c*, column B changed, while the columns C and D of the first and second differences did not change; hence they argued that the fact that the way in which a function increases/decreases did not depend on the coefficient *c*;
- if they changed the coefficient *b*, then columns B and C changed but column D did not; many students conjectured that the coefficient *b* determines whether a function increases or decreases, but not its concavity;
- if they changed the coefficient *a*, then columns B, C and D changed; hence it was the coefficient *a* that is responsible for the concavity of the function.

A difficult point here is to understand why such relationships hold, and to produce at least an argument or even a proof of such conjectures. The tables of numbers do not suggest anything about a possible justification. It is the symbolic power of the spreadsheet which is useful in this case[3]. Students' instrumented actions in this case are very interesting, and consist in substituting letters to the numbers: see Figure 3b. The teacher in most cases had suggested this practice, but a couple of students in the classroom had used it autonomously. The spreadsheet shows clearly in this case that the value of the second difference is $2ah^2$. The letters condense the symbolic meaning of the explorations developed in the numerical environment. A proof can so be produced with the help of the teacher, because of the symbolic support provided by the spreadsheet. In the following lesson, with the teacher stressing the power of the symbolic spreadsheet, a fresh practice had entered the classroom.

A final comment on the nature of the proof, produced through the use of a spreadsheet, is needed. A typical algebraic proof, where the main steps are computations, is apparently different from more discursive proofs produced in elementary geometry environments. Such algebraic proofs are a result of the "algebrisation" of geometry, started with Descartes and improved further in the successive development of mathematics, e.g. in the Erlangen program by F. Klein (1872; see also Baez). It is well known that, because of this, so-called synthetic proofs have been surrogated by the computations developed in linear algebra environments. It is also well known what a big obstacle students find in learning algebra as a meaningful topic (e.g. see Dorier, 2000). It is interesting that

[3] They were using the TI-Nspire software of Texas Instruments.

CAS environments can support students in conceiving and producing such types of proofs: a result that typically is not so easy to reach in paper and pencil environments

The implementation in mathematics classrooms

In all cases described above (straight-edge and compass as well as DGS or CAS tools) the teacher's role is crucial. The teacher not only selects suitable tasks to be solved through constructions and visual, numerical or symbolic explorations, but also orchestrates the transition from practical actions to theoretical argumentations. This transition is quite complex. Argumentations produced by students rest on their experimental experience (drawing, dragging, computing and so on) and the transition to a validation within a theoretical system requires a delicate mediation by the expert. We represent the situation with the diagram of Fig.4.

The upper part of the scheme represents the student's space. The students are given a task (left upper vertex of the scheme) to be solved with an artefact or with a set of artefacts. The presence of the artefact(s) calls into play experimental activities: e. g. drawing by means of the pair straight-edge and compass; creating DGS-figures by means of the DGS-tools; using numbers and letters in the symbolic spreadsheet. An observer, the teacher for instance, may monitor the process going on: students gesticulate, point, and tell themselves or schoolfellows something that accompanies their actions; from this observable behavior it is possible to get insight of their cognitive processes. If the task requires giving a final report (either oral or written), traces of this experience are likely to remain in the text produced by the students. Of course, such reports may differ from decontextualized texts typical of mathematics. Nevertheless, though maintaining clear traces of the experimental work, they have the potential of evoking specific mathematical meanings.

The lower part of the scheme represents exactly the mathematical counterpart of the students' experience. There is place for mathematics, as a cultural product of humankind, and for mathematics to be taught, according to curricula. The link between the students' productions and mathematics to be taught is under the responsibility of the teacher, who has to construct a suitable process that connects the personal students' productions with the statements and proofs expected in the mathematics to be taught.

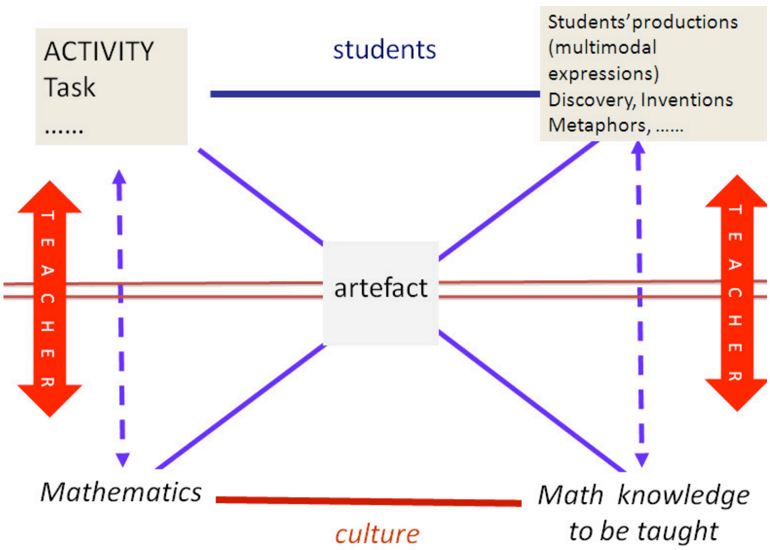

Figure 4

Hence the scheme highlights two important roles which are the teacher's responsibility:

- the choice of suitable tasks (left side of the scheme);
- the monitoring and managing of the process from students' productions to mathematical statements and proofs (right side of the scheme?)

This last point constitutes the core of the semiotic mediation process, when the teacher is expected to foster and guide the evolution towards what is recognizable as mathematics. The teacher acts both at the cognitive and the metacognitive level, by fostering the evolution of meanings and guiding pupils to be aware of their mathematical status (see the idea of mathematical norms, as stated in Cobb, Wood, & Yackel, 1993; see also chapter 5 in this volume). From a socio-cultural perspective this may be interpreted as the process of relating students' "personal senses" (Leont'ev, 1964/1976, p. 244 ff.) and mathematical meanings, or that of relating "spontaneous" and "scientific" concepts (Vygotsky, 1934/1990, p. 286 ff.). The teacher, as an expert representative of mathematical culture, participates in the collective discourse to help it proceed towards sense-making within mathematics.

Within this perspective, investigations were carried out. The analysis focused on the teacher's contribution to the development of a mathematical discourse in the classroom: in the specific case of school activities, centred on the use of an artefact (Mariotti & Bartolini Bussi 1998; Mariotti, 2001; Bartolini Bussi, Mariotti & Ferri, 2005). The authors aimed at identifying specific semiotic games (Mariotti & Bartolini Bussi, 1998; Arzarello & Paola, 2007) played by the teacher when intervening in the discourse to make the students' personal senses emerge from the common experience with the artefact, and develop towards shared meanings, consistent with the mathematical meanings that are the object of the educational project. The analysis of the data highlighted a recurrent pattern of interventions, encompassing a sequence of operations belonging to different categories (Bartolini Bussi & Mariotti, 2008). The description and discussion of teachers' actions is out of the scope of this contribution; we invite the interested reader to see Mariotti (2009), and Mariotti & Maracci (2010).

Up to now we have seen how the artefacts have historically been a fruitful germ for generating the idea of proof, and consequently they can be a strong didactical support for tackling proofs in the classroom. Specifically, we have highlighted the role of the teacher as a semiotic mediator in the task of making proofs accessible to students. In the next part we illustrate this issue from the point of view of the students, discussing some specific examples from different research projects and from different perspectives.

Part 2
A student-centred analysis

In this part we investigate when and how the technology, suitably designed, can help the students to face and possibly to overcome some of the obstacles related to the tension between the empirical nature of the activities and the theoretical nature of the discipline.

When artefacts are integrated in the teaching of proofs, they trigger a network of interactive activities among different components, which can be categorised at two different epistemological levels:

(i) The convincing linguistic logical arguments that explain WHY according to the specific theory of reference ;

(ii) The artefact-dependent convincing arguments that explain WHY according to the mathematical experimentation facilitated by an artefact.

Appoaching proof in school consists in promoting a *network of interactive activities* in order to connect these different components. For example, as we discuss below, abductive processes can support the interactions between (i) and (ii). Other interactive activities concern the multimodal behaviour[4] of students while interacting within technological environments. Such activities feature the transition to proof within experimental mathematics. A provide a micro-level analysis of concrete examples that exploit the novelty and specific features of such a transition with respect to the transition to the proof within more traditional approaches. More precisely, we scrutinize *when and how the distance between arguments and formal proofs* (Balacheff, 1999; Pedemonte, 2007) *produced by students can diminish because of the use of technologies within a precise pedagogical design*.

To focus the didactical and epistemological aspects of this claim, we recall four theoretical constructs taken from the current literature:

i. *almost-empiricism* and experimental mathematics;
ii. *abductive* vs. *deductive* activities in mathematics learning;
iii. *cognitive unity* between arguments and proofs;
iv. *negation* from a mathematical and cognitive point of view.

We use these theoretical constructs to scrutinize some protocols of students who are asked to explore different mathematical situations with different artefacts and within different pedagogical designs in order to prove some claims, based on conjectures formulated by themselves or given by the teacher. Specifically, we show that a suitable use of technologies may improve the *almost-empirical* aspects in students' mathematical activities through a specific production of *abductive arguments*, which generate a *cognitive unity* in the transition from arguments to proofs. We also focus on some reasons why such a unity may possibly not be achieved. The scrutiny is particularly useful in the case of arguments and proofs by contradiction, where typically the *logic of negation* represents a major difficulty for students.

Almost-empiricism and experimental mathematics

We use the notion of *almost-empirical actions*, introduced by Arzarello (2009) to describe some instrumented actions[5] within DGS and CAS environments. It refines further the usual dyadic structure epistemic-pragmatic of the instrumental approach. To give an idea of what is meant by almost-empirical, we provide a brief emblematic example (a wider discussion is in Arzarello, 2009).

A simple problem, originated by the PISA test is the following:

> *The students A and B attend the same school, which is 3 Km far from A's home and 6 Km far from B's home. What are the possible distances between the two houses?*

A solution produced by students of the 10-th grade using TI-Nspire software is illustrated in Fig. 5. They drew two circles, whose centre is the school, and they represent the possible positions of the

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[4] The notion of *multimodality* has evolved within the paradigm of *embodiment*, which has been developed in these last years (Wilson, 2002). Embodiment is a movement afoot in cognitive science that grants the body a central role in shaping the mind. It concerns different disciplines, e.g. cognitive science and neuroscience, interested with how the body is involved in thinking and learning. The new stance emphasizes sensory and motor functions, as well as their importance for successful interaction with the environment. This is particularly palpable when observing the interactions human-computer. A major consequence is that the boundaries among perception, action and cognition become *porous* (Seitz, 2000). Concepts are so analysed not on the basis of "formal abstract models, totally unrelated to the life of the body, and of the brain regions governing the body's functioning in the world" (Gallese & Lakoff, 2005, p.455), but considering the *multimodality* of our cognitive performances. We shall give an example of multimodal behaviours of students when discussing the multivariate language of students who work in DGE. For a more elaborated discussion see Arzarello & Robutti (2008).

[5] The so called *instrumentation approach* has been described by Vérillon & Rabardel (1995) and others (see: Rabardel & Samurçay, 2001; Rabardel, 2002; Trouche, 2005). In our case particular ways of using an artefact, e.g. specific dragging practices in DGS or data capture in TI-Nspire, may be considered an *artefact* that is used to solve a particular *task* (e.g. for formulating a conjecture). When the user has developed particular *utilization schemes* for the artefact, we say that it has become an *instrument* for the user.

two houses with respect to the school. They then created two points, say a and b, moving on each circle, constructed the segment ab and measured it using a command of the software. Successively they created a sequence of the natural numbers in column A of the spreadsheet (Fig. 5a) and through two animations (moving respectively a and b) they collected the corresponding lengths of ab in columns B, and C of the spreadsheet. In the end they built the “scattered plot” A vs. B and A vs. C (Fig. 5b), and drew their comments about the possible distances of A’s and B’s houses by considering the regularities of the obtained scattered graph and discussing the reasons why it is so.

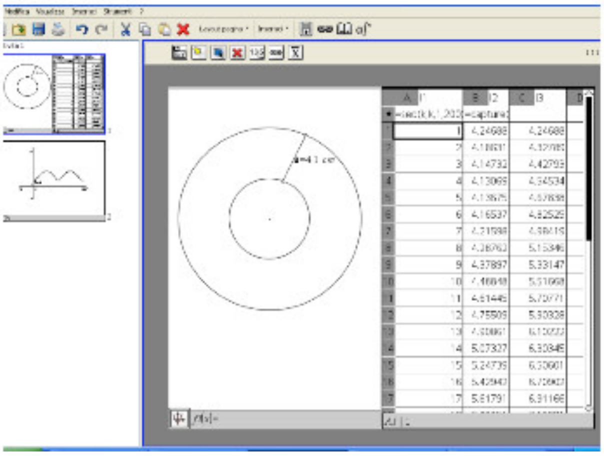

Figure 5a

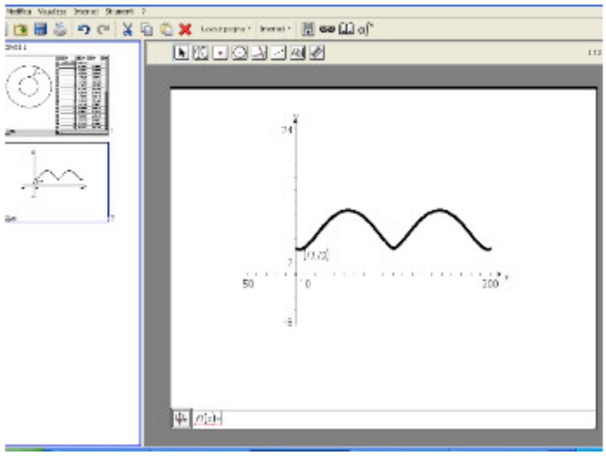

Figure 5b

Let us now consider the role of the variable points and the ways they are manipulated in the example, which is emblematic within this environment. The software allows a collection of data very similar to those accomplished in empirical sciences. The variables involved are first picked out, then through the sequence A, one gets a device to reckon the time in the animation in a conventional way: namely the variable time is made explicit. Of course this subtle point is not so explicit for students but it is a practice induced by the instrumented actions of TI-Nspire software which students find natural to do so that makes it work. It is interesting to observe that the scattered plot combines the time variable A vs. the length variable B or C. The reason is in the possibility for TI-Nspire software of making the time variable explicit within mathematics itself. This procedure is very similar to the way Newton introduced his idea of scientific time in science, distinguishing it from the fuzzy idea of time, about which hundreds of philosophers had (and would have) speculated. Newton writes:

> …for these reasons in what follows, I do not look at the time formally considered, but from proposed quantities, which are of the same genus, I suppose that one of them grows up with equable flux, to which all the others can be referred as if it were the time; so by analogy the name of time could be given to it with some reason. Hence, each time the word time will occur in what follows…by that word one will not mean the time formally considered, but that other quantity, through whose growth or equable flux the time is exposed and can be measured.” [6]
>
> (Newton, CW, III, p.72; Translation by the authors).

Given a mathematical problem (like our problem) one can “do an experiment” very similar to those made in empirical sciences. One picks out the variables that are important for the problem and makes a concrete experiment which involves such variables (collecting the data in a spreadsheet

[6] “…eapropter ad tempus formaliter spectatum in sequentibus haud respiciam, sed e propositis quantitatibus quae sunt ejusdem generis aliquam aequabili fluxione augeri fingam cui caeterae tanquam tempori referantur, adeoque cui nomen temporis analogicè tribui mereatur. Siquando itaque vocabulum temporis in sequentibus occurrat…eo nomine non tempus formaliter spectatum subintellegi debet sed illa alia quantitas cujus aequabili incremento sive fluxione tempus exponitur et mensuratur.”

through the data-capture command). Mutual inter-relationships between variables are studied (using the scattered plot) and a mathematical model is conjectured and validated, possibly with new "experiments". In the end, possibly the reasons why such a model is obtained are investigated, and a proof of a mathematical sentence can be produced. All these steps develop according to a very precise protocol: picking out of variables, designing the experiment, collecting data, producing the mathematical model, and validating it. The protocol is made palpable by different specific commands of TI-Nspire software, such as giving a name to the variables, animation or dragging, data capture, and using a scatter plot.

Such practices within TI-Nspire are as crucial as the dragging practices within DGS. Both incorporate almost-empirical features that can support the transition from the empirical to the deductive side of mathematics. This is a delicate point. Following an idea of Baccaglini-Frank (in print), we discuss below how this can happen: namely when the students are able to internalize such practices and to use them as *psychological tools* (Vygotsky, 1978, p. 52 ff; Kozulin, 1998) for solving conjecture-generation problems.

In this sense the practices with the software introduce new methods in mathematics. Of course this has to be integrated into a careful didactical design developed by the teacher, who must be aware of these potentialities of the software. Such practices do not only consist in the possibility of making explorations but also in the precise protocols that students learn to follow according to the teaching project designed by the teacher. They are very similar to the way external data concerning certain quantities are passed to a computer through the use of probes. In our case the measures are collected through the "data capture" from the "internal experiment" made in the TI-Nspire mathematical world, by connecting the three environments illustrated in Figures 5a-5b (Geometrical, Numerical, Cartesian) through suitable commands of the software. From the one side, these methods are empirical, but from the other side they concern mathematical objects and computations or simulations with the computer and not physical quantities and experiments. Hence the name of *almost-empirical* (Arzarello, 2009). The term recalls the vocabulary used by some scholars in the foundation of mathematics: for example, Lakatos (1976) and Putnam (1998) claim that mathematics has a *quasi-empirical* status (for a survey of this issue, see Tymoczko, 1998). The word "almost-empirical" stresses a different meaning. The main feature of the almost-empirical methods is the list of the precise protocol that its users follow to make their experiments, in the same way that experimental scientists follow their own precise protocols in using machines for their experiments. In such a sense this method is somehow different from the more general description of quasi-empirical methods given by Lakatos or Putnam.

We see below another example of almost-empirical methods within DGS environments. In fact, there are strong similarities between instrumented actions produced in TI-Ns and DGS environments. Almost-empirical actions made by students are not exclusively pragmatic but have also an epistemic nature. As we discuss below, they can support the production of abductions and, hence, the transition from an inductive, empirical modality to a deductive more formal one.

Abductions in mathematics learning

Abduction is a way of reasoning pointed out by Peirce; it has revealed a very fruitful analysis tool. Peirce observed that abductive reasoning is essential for every human inquiry since it is intertwined both with perception and with the general process of invention: "It [abduction] is the only logical operation which introduces any new ideas" (C.P. 5.171)[7]. In short, abduction becomes part of a *process of inquiry* in which abduction, as well as induction, and deduction play relevant roles.

In his scientific life Peirce gave different definitions of abduction. Two of them are particularly fruitful in the field of mathematical education (Arzarello, 1998; Arzarello & Sabena, in print;

[7] Peirce's work is usually referred to in the form C.P. n.m., with the following meaning. C.P. = Collected Papers; n = number of volume; m = number of paragraph.

Baccaglini-Frank, 2010a; Antonini, & Mariotti, 2009), particularly when technological tools are considered. We now briefly recall each.

The first one is the so-called *syllogistic abduction* (C.P. 2.623), according to which a *Case* is drawn from a *Rule* and a *Result*. It is a well known Peirce example about beans:

Rule: All the beans from this bag are white

Result: These beans are white

Case: These beans are from this bag

As such, an abduction is different from a *Deduction* that would have the form: the *Result* is drawn from the *Rule* and the *Case*, and it is obviously different from an *Induction*, which has the form: from a *Case* and many *Results* a *Rule* is drawn. Of course the conclusion of an abduction holds only with a certain probability (in fact Polya, 1954 called this abductive argument an *heuristic syllogism)*.

Peirce defined also a second form of abduction, as "the process of forming an *explanatory hypothesis*" (Peirce, CP 5.171; our emphasis). Along this stream of thought, Magnani (2001, pp. 17-18) proposed the following conception of abduction: the process of inferring certain facts and/or laws and hypotheses that render some sentences plausible, that explain or discover some (eventually new) phenomenon or observation. As such it is the process of reasoning in which explanatory hypotheses are formed and evaluated. A typical example is when a logical or causal dependence of two observed properties is captured during the exploration of a situation. The dependence is by all means an "explanatory hypothesis" developed to explain a situation as a whole.

According to Peirce (C.P. 5.14-212), three aspects determine whether an abduction (in one of the two forms) is promising: it must be *explanatory*, *testable*, and *economic*. It is an *explanation* if it accounts for the facts, but its status is that of a suggestion until it is verified, which explains the need for the *testability* criterion. The motivation for the *economic* criterion is twofold: it is a response to the practical problem of having innumerable explanatory hypotheses to test, and it satisfies the need for a criterion to select the best explanation amongst the testable ones.

As pointed out by Baccaglini-Frank (2010a, p.46-50), the two types of abductions correspond to two different logics of producing a hypothesis: the logic of *selecting a hypothesis* from among many possible ones (first type) versus the logic of *constructing a hypothesis* (second type).

We now illustrate how abductions can be produced within DGS environments. We show that abductive processes can bridge the gap between perceptual facts and their theoretical transposition through supporting a *structural cognitive unity* (see below) between the explorative and the proving phase, though this happens provided there is a suitable didactic design.

Let us consider the following problem from Arzarello (2000): it has been given to students of 17-18 years (grade 11-12), who knew Cabri-géomètre very well and had already had a course in Euclidean Geometry; moreover they were acquainted with exploring situations when presented with open problems (see Arsac et al., 1992) and were able to make the constructions of the main geometrical figures. The students were already at least beyond the third of van Hiele levels and were entering in the fourth-fifth one (for the use of van Hiele levels in DGS environments see Govender & de Villers, 2002)

> *Let ABCD be a quadrangle. Consider the perpendicular bisectors of its sides and their intersection points H, K, L, M of pairwise consecutive bisectors. Drag ABChD, considering all its different configurations: what happens to the quadrangle HKLM? What kind of figure does it become?*

Here we summarize what typically happens with many pairs of expert[8] students while they solve the problem in five "phases" (1-5 below):

1. The students start to shape ABCD into standard figures (parallelogram, rectangle, trapezium) and check what kind of figures they get for HKLM. In some cases the see that all the bisectors pass through the same point.

2. As soon as they see that HKLM becomes a point when ABCD is a square, they consider this an interesting fact: therefore they drag a vertex of ABCD (starting from the case when ABCD is a square) so that H, K, L, M keep on being coincident.

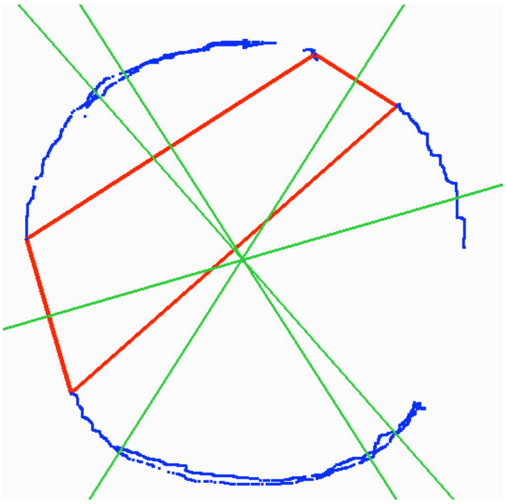

Figure 6

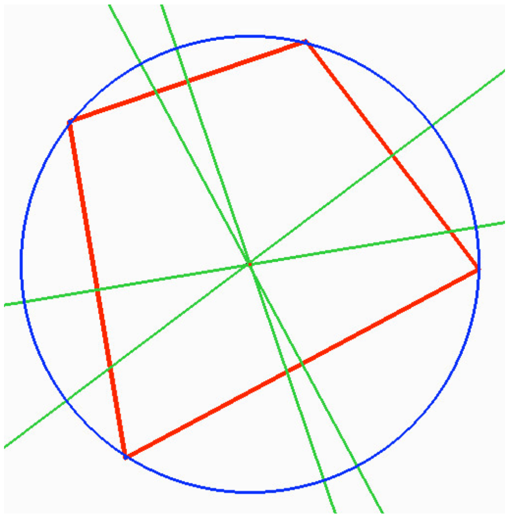

Figure 7

3. They realise that this kind of configuration is also true with quadrilaterals that apparently have no special property. Using the trace command they realise that dragging a vertex along a curve that resembles a circle, they can keep the four points together (Fig.6). Hence they formulate the conjecture: *If the quadrilateral ABCD can be inscribed in a circle, then its perpendicular bisectors meet in one point, centre of the circle.*

4. They validate their conjecture constructing a circle, a quadrilateral inscribed to this circle, its perpendicular bisectors and observing that all of them meet in the same point (Fig. 7).

5. They write a proof of the conjecture. This process mainly consists in transforming (or eliminating) parts of the discussion made in the previous phases into a linear discourse, which is essentially developed according the formal rules of proof.

Two major phenomena feature the development above and these are emblematic of these types of open tasks:

a) The production of an abduction: it typically marks a crucial understanding point in the process of solution;

b) The structural continuity between the conjecturing phases 1-4 and the transforming-eliminating activities of the last phase.

a) Production of an abduction. Students first see a perceptual invariant, namely the coincidence of the four points in some cases (phases 1 and 2). So they start an exploration in order to see what conditions make the four points H, K, L, M to coincide (phase 2 and 3). A particular kind of dragging (Baccaglini-Frank, 2010b, calls it *maintaining dragging*: see below) supports them in this exploration: using the trace command they carefully move the vertexes of ABCD so that the other

[8] That is students who have acquired a suffcient instrumented knowledge of dragging practices according to a precise didactical design. See below. The word is taken from Baccaglini-Frank (2010): see below.

four points remain together; at the end they realize to have so produced a curve that resembles a circle (phase 3), namely e second invariant. At this point, they conjecture a link between the two invariants and see the second as a possible “cause” of the first; namely they produce an abduction in the form of an “explanatory hypothesis” (phase 4).

b) Production of a proof. The written proof produced in the last phase exhibits a strong continuity with the discussion made by the students during their previous explorations: more precisely, it is achieved through linguistic eliminations and transformations of their aforementioned utterances.

Constraints of space do not allow us to describe it here (see Arzarello, 2000 for a wider discussion). We limit ourselves to sketch the structure of the whole process in Fig.8, where the main transformations are shown, namely:

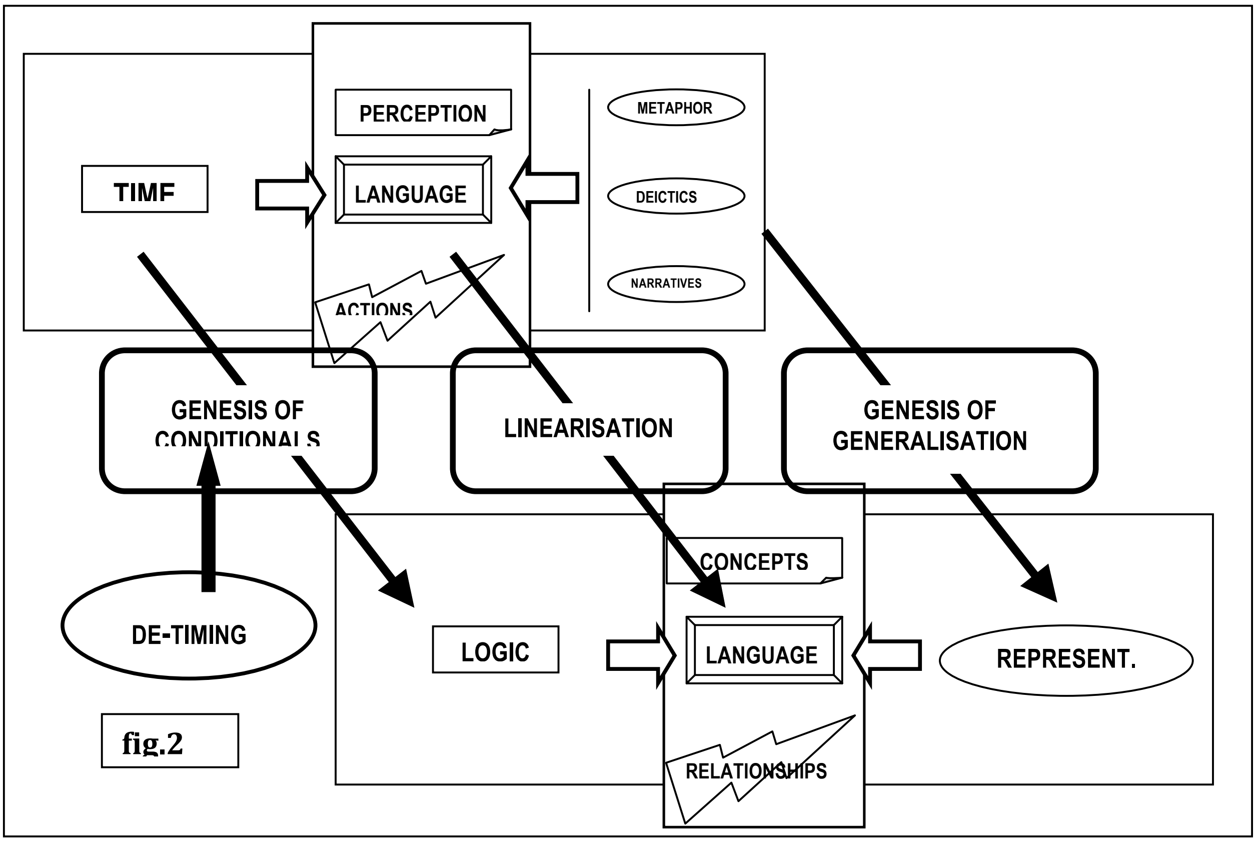

Figure 8

- The genesis of conditional statements through *de-timing.* That is, while in the discussions between the students during the exploration phases the time variable is generally present as a glue for the narratives that they produce to give sense to what they do and see at the screen, time disappears from their final productions. In this process generally the words that mark the time, like *when*, are transformed into implications, like *if…then*.
- The *linearisation* of sentences. Generally during the discussion in the first 4 phases the students use a non linear multivariate language (Simone, 2000). It is made of non-verbal components, like gestures, gazes, and broken sentences and utterances which tries to mimic the complex order of the experienced facts. Here is a typical example of multimodal language (Arzarello, 2000; see Note 4): pointing at different places on the screen, a girl says: «Hence if...when...and hmm, yes, that is natural, because when there are two...the two sides of the external one...the two sides parallel two by two, it is natural...that is it should always be that the perpendicular bisectors are...», and her mate answers : «It is so». Namely this multimodal language assures communication between the students. The non-linear features of their language constitute a common basis, upon which students

can share some ideas about the mathematical knowledge they are building. The following example of linearization, shows how a multimodal language is pruned of its nonlinear components, and is transformed into a more canonical mathematical sentence. A girl says: «No, no! I know it! It is the circumcentre..., why it must be equidistant from the sides, isn't it? [she indicates with fingers on the screen] This point [the meeting point of the bisectors] is the perpendicular bisector of this [AD] hence it is equidistant [from A and D] ». After some minutes the same girl says: «...because the bisector is the locus of points which are equidistant from the extremes...hence it is equidistant from this and from this [A and B]. But from this [A], these two are equal…[she repeats the reasoning and gestures with respect to all vertices] ...hence in the end they are all equal and it is the ray, isn't it?».

Maintaining dragging as an acquired instrumented action

The results illustrated above are acquired through a suitable design of the teaching interventions, where the instrumented practices with the software are carefully considered. The main aim is that students can interiorise such practices as psychological tools (in the sense of Vygotsky) they can use to solve mathematical problems. We elaborate on this issue by discussing an approach to the instrumentation of dragging in DGS according to this aim. More precisely, we sketch the configuration of the so-called *Maintaining Dragging* Conjecturing Model, for describing a specific process of conjecture-generation, as developed by A. Baccaglini-Frank in her Dissertation (2010; supervisor: M.A. Mariotti) and in Baccaglini-Frank & Mariotti (in print). Enhancing the analysis in Arzarello et al. (2002) of dragging modalities (and of the consequent abductive processes), Baccaglini-Frank has developed a finer analysis of dragging, and has advanced hypotheses on the potential of dragging practices, which can be introduced in the classroom, of becoming a psychological tool and not only a list of automatic practices learnt by root.

Since according to the literature (Olivero, 2002), spontaneous use of some typologies of dragging does not seem to occur frequently, first, she explicitly introduces the students to some of the different dragging modalities, elaborated from Arzarello et al.'s (2002) classification[9], then asks the students to solve open tasks, like the problem on quadrilateral in the previous section. In such problem solving activities a specific modality of dragging shows to be particularly useful to students: *maintaining dragging* (MD). Maintaining dragging consists of trying to drag a base point and maintaining some interesting property observed. In our example, the solvers noticed that the quadrilateral HKLM, part of the Cabri-figure, could ''become'' a single point, and thus attempt to drag a base point trying to keep the four points together. In other words, maintaining dragging (MD) involves the recognition of a particular configuration as interesting, and the user's attempt to induce the particular property to become an invariant during dragging. Using Healy's terminology such an invariant would be denoted as a soft invariant, as opposed to a robust invariant, which on the other hand derives directly from the construction steps (Healy 2000). MD is an elaboration from *dummy locus dragging* but differs slightly from it: dummy locus dragging can be described as ''wandering dragging that has found its path,'' a dummy locus that is not yet visible to the subject (Arzarello et al. 2002, p. 68), while MD is "the mode in which a base point is dragged, not necessarily along a

[9] Arzarello and his collaborators distinguish between the following typologies of dragging:
- *Wandering dragging*: moving the basic points on the screen randomly, without a plan, in order to discover interesting configurations or regularities in the figures.
- *Dummy locus* dragging: moving a basic point so that the figure keeps a discovered property; that means you are following a hidden path, even without being aware of this.
- *Line* dragging: moving a basic point along a fixed line (e.g. a geometrical curve seen during the dummy locus dragging).
- *Dragging test*: moving dragable or semi-dragable points in order to see whether the figure keeps the initial properties. If so, then the figure passes the test; if not, then the figure was not constructed according to the geometric properties you wanted it to have.

pre- conceived path, with the specific intention of the user to maintain a particular property." (Baccaglini-Frank & Mariotti, in print).

In our example above MD happened in phases 2 and 3, when the students dragged the vertexes of the quadrilateral in order to keep together the four points H, K, L, M. As we have seen in phases 3-4 of the example above, when MD is possible, the invariant observed during dragging may automatically become "the regular movement of the dragged-base-point along the curve" recognized through the trace mark, and this can be interpreted geometrically as the property "dragged-base-point belongs to the curve" (Baccaglini-Frank, in print). As pointed out by Baccaglini-Frank (2010b), the *expert* solvers proceed smoothly through the perception of the invariants and immediately interpret them appropriately, as conclusion and premise in the final conjecture. Of course, becoming expert is not so immediate since it requires a careful didactical design that pushes the students towards a suitable instrumented use of the MD-artefact. In fact, "from the perspective of the instrumental approach, MD practices may be considered a utilization scheme for expert users of the *MD- artefact* thus making MD an *instrument* (the *MD-instrument*) for the solver with respect to the task for producing a conjecture" (Baccaglini-Frank, 2010b).

This development of instrumented maintaining dragging (MD) is organized by Baccaglini-Frank in a Model that operatively becomes a precise protocol for students, who follow it in order to produce suitable conjectures, when asked to tackle open problems situations (Arsac, 1999). This protocol is structurally similar to the one we illustrated above for data-capture within TI-Nspire software. It is divided into 3 main parts:

1 (see phases 1-2 in our example). Determine a configuration to be explored by inducing it as a (soft) invariant. Through wandering dragging the solver can look for interesting configurations and conceive them as potential invariants to be intentionally induced.

2 (see phases 2-3 in our example). Searching for a Condition through MD: students look for a condition that makes the intentionally induced invariant be visually verified through maintaining dragging from path to the geometric interpretation of the path. Genesis of a Conditional Link through the production of an abduction.

3 (see phases 3-4 in our example). Checking the Conditional Link between the Invariants and verifying it through the dragging test.

After a conjecture has been generated through this process, the students (try to) prove their conjecture (see phase 5).

The MD-conjecturing Model puts dragging and the perception of invariants in relationship with the development of a conjecture, especially with the emergence of the premise and the conclusion. There does seem to be a common process, and it seems to be well illustrated by the MD-conjecturing protocol, as a sequence of tasks a solver can engage in. Baccaglini-Frank's model allows us to "unravel" the abductive process that supports both the formulation of a conjecture, and the transition from an explorative phase to one in which the conjecture is checked[10]. The *path* (in our example the circle created by the students in phase 4) plays a central role by incorporating an answer to the

[10] Arzarello et al. (1998a, 1998b, 2000, 2002) showed that the transition from the inductive to the deductive level is generally marked by an abduction, accompanied by a cognitive shift from *ascending to descending* modalities (see Saada-Robert, 1989), according to which the figures on the screen are looked at. The modality is ascending (from the environment to the subject) when the subject explores the situation, e.g. a graph on the screen, with an open mind and to see if the situation itself can show her/him something interesting (like in phases 1, 2, 3 of our example); the situation is descending (from the subject to the environment) when the subject explores the situation with a conjecture in her/his mind (like in phase 4 of our example). In the first case the instrumented actions have an explorative nature (to see if something happen), in the second case they have a checking nature (to see if the conjecture is corroborated or refuted). Epistemologically, the cognitive shift is marked by the production of an *abduction*, which also determines the transition from an inductive to a deductive approach.

solver's "search for a cause" for the intentionally-induced invariant (III) to be visually verified, and thus leading to the premise of a potential conjecture. Its figure-specific component (the actual curve that can be represented on the screen) contains geometrical properties that may be used as a bridge to proof.

When MD is used expertly, abduction seems to reside at a meta-level with respect to the dynamic exploration. Abduction at the level of the dynamic explorations seems to occur provided MD is used as a psychological tool (Vygotsky, 1978, p. 52 ff.; Kozulin, 1998). According to Baccaglini-Frank analysis, it seems that

> if solvers who have appropriated the MD- instrument *also internalize it* transforming it into a psychological tool, or a fruitful "mathematical habit of mind" (Cuoco, 2008) that may be exploited in various mathematical explorations leading to the generation of conjectures, a greater cognitive unity (Pedemonte, 2007) might be fostered. In other words, it may be the case that when the MD instrument is used as a psychological tool the conjecturing phase is characterized by the emergence of arguments that the solver can set in chain in a deductive way when constructing a proof (Boero et al., 1996).
>
> (Baccaglini-Frank, in print)

Something similar is true for the protocol of data-capture with TI-Nspire software. In both cases we have almost-empirical actions, in the sense discussed above. What is interesting is that such almost-empirical methods seem to be fruitful for supporting the transition to the theoretical side of mathematics, provided their instrumentation can produce their internalization as psychological tools in the long run. On the contrary, when such protocols are used "automatically" they tend to lead to conjectures with no theoretical elements to "bridge the gap" between the premise and the conclusion of the conditional link.

Since cognitive unity is a crucial issue in the transition from arguments to proofs, from the empirical to the theoretical side, in the next section we discuss such a concept in the form that the latest research has elaborated it.

Cognitive unity

Recently Boero and his collaborators (Boero, Douek, Morselli, & Pedemonte, 2010,) have integrated their analysis of *cognitive unity* with Habermas' elaboration of *rational behaviour in discursive practices*. We recall briefly these two notions.

Boero calls *cognitive unity* the continuity that may exist between argumentation as a process of producing a conjecture and constructing its proof (Boero et al., 1996). The hypothesis of cognitive unity is that, in some cases, "this argumentation can be exploited by the student in the construction of a proof by organizing some of the previously produced arguments into a logical chain" (Boero et al., 2010, p. 183). Pedemonte (2007) has further refined this concept introducing the notion of *structural continuity* between argumentation and proof: that is when inferences in argumentation and proof are connected through the same structure (abduction, induction, or deduction). For example, there is structural continuity between argumentation and proof if some abductive steps used in the argumentation are also present in the proof. That was the case in our example for the problem of the distances of the houses from the school. For a wider discussion of this point see Boero et al. (2010).

As to Habermas' construct, Boero and his collaborators have adapted the three components of rational behaviour according to Habermas (teleologic, epistemic, communicative) to the discursive practice of proving, and have identified:

> A) An *epistemic aspect***,** consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning (see the definition of "theorem" by Mariotti & al., 1997) as the system consisting of a statement, a proof, derived according to shared inference rules from axioms and other theorems, and a reference theory);

B) A *teleological aspect*, inherent in the problem-solving character of proving, and the conscious choices to be made in order to obtain the desired product;

C) A *communicative aspect*, consisting in the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning and the conformity of the products (proofs) to standards in a given mathematical culture.

(Boero et al., 2010, pp. 188)

In this model, the expert's behavior in proving processes can be described in terms of (more or less) conscious constraints upon the three components of rationality: "constraints of epistemic validity, efficiency related to the goal to achieve, and communication according to shared rules" (*ibid.*, p. 192). As the authors point out, such constraints result in *two levels of argumentation*:

- a level (that we call *ground level*) inherent in the specific nature of the three components of rational behaviour in proving;
- a *meta-level*, "inherent in the awareness of the constraints on the three components" (*ibid.*, p. 192).

The two notions of cognitive unity and of levels of argumentations are important lenses for analysing students' processes in the transition from argumentations to proofs within technological environments, especially in DGS. In particular, we show in the next section how this lens is very useful for analysing indirect proofs.

Indirect proofs

Antonini and Mariotti (2008) have developed a careful analysis of indirect proofs and related argumentations both from a mathematical and a cognitive point of view, and have elaborated a model appropriate for interpreting students' difficulties. Essentially, the model splits any indirect proof of a sentence S (principal statement) into a pair (*s,m*), where *s* is a direct proof (within a theory T, for example Euclidean Geometry) of a secondary statement S* and *m* is a meta-proof (within a meta-theory MT, generally coinciding with classical logic) of the statement S* → S. As an example, given by the authors, consider the (principal) statement S: "Let a and b be two real numbers. If ab = 0 then a = 0 or b = 0" and the following indirect proof: "Assume that ab = 0, a ≠ 0, and b ≠ 0. Since a ≠ 0 and b ≠ 0 one can divide both sides of the equality ab = 0 by a and by b, obtaining 1 = 0". In this proof, the secondary statement S* is: 'let a and b be two real numbers; if ab = 0, a ≠ 0, and b ≠ 0 then 1 = 0'. A direct proof is given. The hypothesis of this new statement is the negation of the original statement and the thesis is a false proposition (''1 = 0'').

The two authors use their model to point out that the main difficulties for students who face indirect proof consists in switching from *s* to *m*. Yet the difficulties seem less strong for statements that require a proof by contraposition: that is to prove ¬B → ¬A (secondary statement) in order to prove A → B (principal statement). Let us observe that the meta-proof *m* does not coincide with the meta-level we considered above; rather, it is at the meta-mathematical level (based on logic). Integrating the two models, we can say that switching from *s* to *m* requires a well-established epistemic and teleological rationality in the students and in this respect needs the activation of the meta-level of argumentation.

To better disentangle this issue, we introduce the notion of *meta-cognitive unity* between argumentative and proving processes. The distinction between the ground level and the meta-level drawn from the Boero-Habermas model sketched above may be very useful to investigate the proving processes related to indirect proof. Basing on such a distinction, we introduce the notion of *meta-cognitive unity*, as *a cognitive unity between the two levels of argumentation described above, specifically between the teleological component at the meta-level and the epistemic component at the ground level*.

Differently from the structural and referential cognitive unit (Garuti, Boero, Lemut, & Mariotti, 1996; Pedemonte, 2007), meta-cognitive unity is not concerned with two diachronic moments in the discursive activities of students (namely the argumentation and the proving processes, which are produced in subsequent moments), rather it refers two a synchronic integration between the two levels of argumentations. Our hypothesis is that the existence of such a meta-cognitive unity is an important condition for producing indirect proofs. In other words, the missing integration between the two levels of argumentations can block students' proving processes, or produce cognitive breaks as those described in the literature on indirect proofs.

Meta-cognitive unity may possibly entail structural cognitive unity at the ground level. We shall illustrate how this unity can be realised with an emblematic example, in which meta-cognitive unity is accompanied by structural-cognitive unity and develops through what we call 'the logic of not' (see also Arzarello & Sabena, in print).

The logic of not

The *'logic of not'* is an interesting epistemological and cognitive aspect of argumentation that sometimes is produced by students who tackle a problem where a direct argument is revealed as not viable.

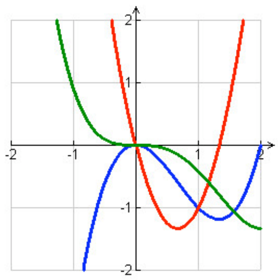

Figure 9

Their strategy is similar to the one of a chemist, who in the laboratory has to detect the nature of some substance. For example he knows that the substance must belong to one of three different categories (a, b, c) and uses suitable reagents to accomplish his task: he knows that if a substance reacts in a certain way to a certain reagent it may be of type a or b but *not* c, and so on. In such practices abductive processes are usually used: if, as a *Rule*, the substance S makes blue the reagent r and if the *Result* of the experiment shows that the unknown substance X makes blue the reagent r, one gets that X=S (*Case* of the abduction).

To make concrete the point we summarise an emblematic example, comprehensively discussed in Arzarello & Sabena (in print). The student S (grade 9 in a scientifically oriented school) is solving the following task:

> *The drawing (Fig. 9) shows the graphs of: a function f, its derivative, one of its antiderivatives. Identify the graph of each function, and justify your answer.*

The functions are differently coloured: the parabola red (indicated with R); the cubic (with a maximum point in the origin) is blue (B); the last one (a quartic with an inflection point in the origin) is green (G). In what follows we report the students' words, which refer to the colours. The students do not know the analytic representations of the functions but have only their graphs.
In the first part of his protocol S checks which of the three functions can be *f*. He does this by looking for possible abductions, which involve the features of the given graphs.
For example, he starts supposing that *f* is the red function, probably because it is the simplest graph, and wonders whether he can apply an abductive argument with the following form to conclude that its derivative possibly is the green function:

(1) Rule: "any derivative of a decreasing function is negative"
(2) Result: "the green function is negative";
(3) Case: "the green function is the derivative of f"

(ARG. 1)

Recalling the metaphor with the chemist, S is able first to find a reagent that discriminates between the substances he is analysing, and then to validate his hypothesis with a further discriminating experiment. The experiments of the chemists are here his learnt practices with the graphs of functions. Arzarello & Sabena use the Boero-Habermas model to show that some of S arguments are *teleological* and at the *meta-level*: they address the successive actions of the subject and his control of what is happening. The teleological component at the meta-level intertwines with the epistemological component at the ground level in a deep unity between the two. Let us now see how this complex form of unity allows S to produce a proof by contraposition (it is the form of reasoning that logicians call "modus tollens": from "A implies B" to "not B implies not A"). Through this transition to a new epistemological status of his statements, S is able to lighten the cognitive load of the task. Arzarello and Sabena call this kind of process "the logic of not". Let us illustrate it through what is written in a part of S protocol:

> "Then I compared the "red" with the "green" function: but, the "green" function cannot be a derivative of the "red" one, because in the first part, when the "red" function is decreasing, its derivative should have a negative sign, but the "green" function has a positive sign".

Here the structure of the sentence is more complex than before: S is thinking to a possible argument of the following form:

(1) "any derivative of a decreasing function is negative"
(4) "the "green" function has a positive sign"
(5) "the "green" function cannot be a derivative of an increasing function"
(ARG. 2)

In terms of the structure of the possible abduction ARG. 1, ARG. 2 has the form: (1) and not (2); hence not (3). It is crucial here to observe that also the refutation of the usual Deduction (Rule, Case; hence Result) has the same structure, because of the converse of an implication ("A implies B" is equivalent to "not B implies not A"). In other words, the refutation of an argument drawn through an abduction coincides with the refutation of an argument drawn through a deduction. While abductions and deductions are structurally and cognitively different, their refutations formally are exactly the same argument. So S is able to produce this form of deductive argument in a very "natural" way, namely within an abductive modality. This is remarkable from an epistemological and cognitive point of view. Whereas the abductive approach may appear "natural" for students in conjecturing phases (Arzarello et al., 1998), there is often a break with the deductive approach of the proving phase (Pedemonte, 2007). Actually the transition from an abductive to a deductive modality requires a sort of "somersault", namely an inversion in the way things are seen and structured in the argument (the Case- and the Result-functions in the argument are exchanged) and this may be a cognitive load for the students. This inversion is not present in case of refutation of an abduction: indeed it coincides with the refutation of a deduction. An "impossible" abductive argument has already the structure of a deduction, namely it is an argument by contraposition. Of course a greater cognitive load is required to manage the refutation of an abduction compared with that required developing a simple direct abduction. But the coincidence between abduction and deduction in case they are refuted allows avoiding the somersault.

It is interesting to recall another possible use of the logic of not, pointed out by de Villers (who does not use this terminology). He observes that in DGS environments it is important

> to sensitize students to the fact that although *Sketchpad* is very accurate and extremely useful for exploring the validity of conjectures, one could still make false conjectures with it if one is not very careful. Generally, even if one is measuring and calculating to 3 decimal accuracy, which is the maximum capacity of *Sketchpad 3*, one cannot have absolute certainty that there are no changes to the fourth, fifth or sixth decimals (or the 100th decimal!) that are just not displayed when rounding off to three decimals. This is why a logical explanation/proof, even in such a convincing environment as *Sketchpad*, is necessary for absolute certainty.

(de Villers, 2002, p. 9)

One way of promoting the sensibility of students is to create some cognitive conflict to counter-act students' natural inclination for just accepting the empirical evidence provided by the software. A typical example is an activity where students are 'led' to make a false conjecture, and though they are convinced it is true, it turns out false: in these cases the logic of not can drive them towards the production of a proof.

What has been discussed above reveals the potential of DGS in introducing students to indirect arguments and proofs. Specifically, the support that the use of maintaining dragging provides for producing abductions can be fruitfully analysed in terms of the "logic of not". This is what we consider in the following section.

Indirect proof within DGS

Theorem acquisition and justification in a DGS environment is a "cognitive-visual dual process potent with structured conjecture-forming activities, in which dynamic visual explorations through different dragging modalities are applied on geometrical entities." (Leung and Lopez-Real, 2002, p.149)

In this duality, the role of visualization is pivotal in the development of epistemic behaviour like the MD Dragging Model (Baccaglini-Frank and Mariotti, in print). In another vein, DGS facilitates experimental identification of geometrical invariants through functions of variation induced by dragging modalities which serves as a cognitive-visual lens to conceptualize conjectures and DGS-situated argumentative discourse (Leung, 2008). With respect to indirect proof within DGS, Leung proposed a visualization scheme to "see a proof by contradiction" in a DGS environment (Leung and Lopez-Real, 2002). The key elements in this scheme were the DGS constructs of pseudo object and locus of validity. They serve together as the main cognitive-visual bridge to connect the semiotic controls and the theoretical controls in the argumentative process. The background for the development of this scheme was a Cabri problem-solving workshop conducted for a group of Grade 9 and Grade 10 students in Hong Kong. The following problem was given to students to explore in the Cabri environment:

> *Let ABCD be a quadrilateral such that each pair of interior opposite angles adds up to 180°. Find a way to prove that ABCD must be a cyclic quadrilateral.*

The Proof.

After exploration, a pair of students wrote down the following "Cabri-proof" (Figure 10):

PROOF:

Assume that for a quadrilateral with each pair of interior opposite angles adding up to 180°, the four vertices can be on different circles.

From the diagram we see that it has a contradiction as the sum of the opposite angles of the blue quadrilateral (EBFD) is 360°, which is impossible.

Therefore, for a quadrilateral with each pair of interior opposite angles adding up to 180°, the four vertices must be on the same circle.

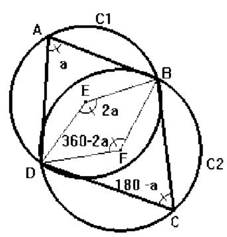

Figure 10

Take note that the labelling of the angles in the diagram was not part of the actual Cabri figure. The key idea in the proof was the construction of an impossible quadrilateral EBFD. However, the written proof did not reflect the dynamic variation of the impossible quadrilateral that promoted the argumentative activity in the Cabri environment. After an in-depth interview with the two students on how they used Cabri to arrive at the proof, a cognitive-visual scheme emerged.

The Argumentation

The impossible quadrilateral EBFD, henceforth called a *pseudo-quadrilateral*, in Figure 10 plays a critical role in organizing the cognitive-visual process that leads to the acquisition and justification of a theorem. EBFD is a visual object that measures the degree of anomaly of a biased Cabri world with respect to the different positions of the vertices A, B, C and D. There are positions where the pseudo-quadrilateral EBFD vanishes when a vertex of ABCD is being dragged. Figure 11 depicts a sequence of snapshots in a dragging episode when C is being dragged until EBFD vanishes.

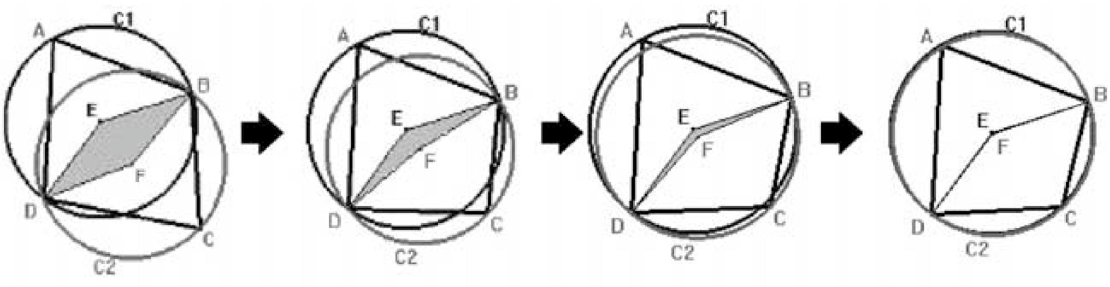

Figure 11

The last picture in the sequence shows that when C lies on the circumcircle C1 of the quadrilateral ABCD, E and F coincide, and at this instance, $\angle DEB + \angle DFB = 360^\circ$ (which has been a contradiction arising from the pseudo-quadrilateral EBFD). Furthermore, this condition holds only when C lies on C1. That is, when A, B, C and D are concyclic. The pseudo-quadrilateral EBFD and the circumcircle C1 play a dual role in an argumentative process. On the one hand, they restrict the quadrilateral ABCD to a special configuration that leads to a discovery of the quadrilateral ABCD possessing certain properties (through an inference of an abductive nature). On the other hand, they generate a convincing argument that collapses onto a Reductio ad Absurdum proof (Figure 10), which is acceptable in Euclidean Geometry.

The Scheme

Suppose A is a figure (quadrilateral ABCD in the Cabri-proof) in a DGS environment. Assume that A satisfies a certain condition C(A) (interior opposite angles are supplementary) and impose it on all figures of type A in the DGS environment. This *forced presupposition* evokes a 'mental labelling' (the arbitrary labelling of $\angle DAB = 2a$ and $\angle DCB = 180\text{-}2a$ in the Cabri-proof) in our minds which *acts cognitively* on the DGS environment. Thus C(A) makes an object of type A biased with extra meaning that might not necessarily be true in the actual DGS environment. This *biased DGS environment* exists as a kind of hybrid state between the *visual-true* DGS (a virtual representation of the Euclidean world) and a *pseudo-true* interpretation, C(A), insisted on by us. In this pseudo world, an object associated with A can be constructed which inherits a local property that is not necessarily consistent with the Euclidean world because of C(A) (the impossible quadrilateral EBFD in the Cabri-proof). We call such an object associated with A in the biased DGS environment a *pseudo object* and denote it by O(A).When part of A (the point C) is being dragged to different positions, O(A) might vanish (or degenerate, i.e., a plane figure to a line, a line to a point). The path or locus on which this happens gives a constraint (both semiotic and theoretical) under which the forced presupposition C(A) is Euclidean valid; that is, where the biased microworld is being realized in the Euclidean world. This path is called the *locus of validity* of C(A) associated with O(A) (the circle C1).

In the context of Indirect proof (Antonini and Mariotti, 2008), this scheme can be interpreted as follows:
S is the principle statement "If the interior opposite angles of a quadrilateral add up to 180^o, then it is a cyclic quadrilateral"; S^* is the secondary statement "If the interior opposite angles of a quadrilateral add up to 180^o and its vertices can lie on two circles, then there exists a quadrilateral

$$\begin{tikzcd} T \arrow[r, "s"] \arrow[dr, dashed] & S^* \arrow[d, "m"] \\ & S \end{tikzcd}$$

with the property that a pair of interior opposite angles add up to 360^o"; T is Euclidean Geometry; s is a direct Euclidean proof. In a DGS environment, the meta-proof m could be a kind of drag-based visual logic. In the case discussed above, when a pseudo object and a locus of validity arise, m could be a drag-to-vanish MD visual logic. Thus the *composite proof*

$$m \circ s : T \dashrightarrow S$$

is an indirect proof that is both theoretical and DGS-mediated.

Relating to the Boero-Habermas model, the theoretical part (s) is the epistemological component (theoretical control) at the ground level where the existence of the pseudo-quadrilateral EBFD was deduced. The DGS-mediated part (m) is the teleological component (semiotic control) at the meta-level where the drag-based argumentation took place. Hence the intertwined composite proof ($m \circ s$) can be seen as a kind of meta-cognitive unity in which argumentation crystallizes into a Reductio ad Absurdum proof.
Within the "logic of not", the DGS-mediated part (m) allows a link from the abductive modality to the deductive modality. In the example of S, illustrated previously, the distance between the abductive and the deductive modality was annihilated because of the coincidence between the negation of the abduction and the deduction; here the distance is shortened through m: in both cases the cognitive load is lightened.

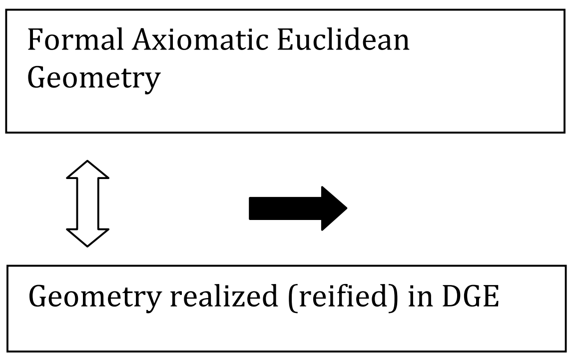

Figure 12

In Lopez-Real and Leung (2006), it was suggested that Formal Axiomatic Euclidean Geometry (FAEG) and Dynamic Geometry Environment (DGE) are 'parallel' systems that are "situated in different semiotic phenomena” instead of two systems having a hierarchical relationship (Figure 12):

> The vertical two-way arrow denotes the connection (networking) that enables an exchange of meaning between the systems. The horizontal arrow stands for a concurrent mediation process that signifies some kind of mathematical reality. This perspective embraces the idea that dragging in DGE is a semiotic tool (or a conceptual tool) that helps learners to form mathematical concepts, rather than just a tool for experimentation and conjecture making that doesn’t seem to match the ‘logical rigour’ in FAEG.
>
> (Lopez-Real and Leung, 2006, p.667)

In this connection, the MD dragging scheme, together with the construct of pseudo object and locus of validity, and with the associated reasoning carried out, on the one hand, in the context of the DGS and, on the other hand, in T by the solver may serve as channels to enable an exchange of meaning between the two systems (Figure 13).

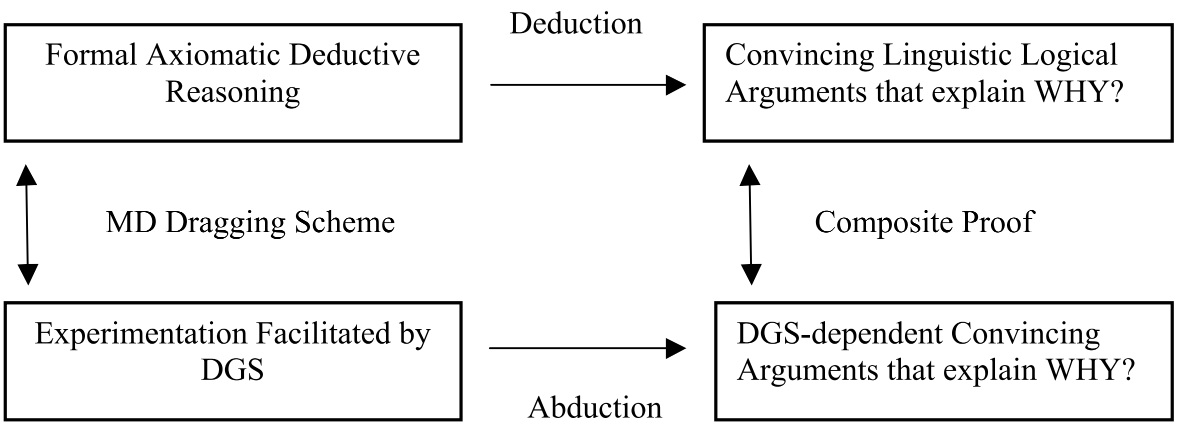

Figure 13

The idea of composite proof in DGS environment could possibly be expanded to a wider scope where there is a hybrid of Euclidean and DGS registers. Such a case was presented in (Leung, 2009) where a student produced a written proof that was intertwined with Euclidean and DGS registers. The first results of this analysis are very promising, opening new perspectives of investigation that the researchers have just undertaken.

Part 3
Towards a framework for understanding the role of technologies in geometrical proof

As Part 1 of this chapter indicates, geometry may be split into what Einstein (1921) called “practical geometry”, obtained from physical experiment and experience, and “purely axiomatic geometry” containing its logical structure. Central to learning geometry is an understanding of the relationship between the technologies of geometry and its epistemology. Technology in this context is the range of artefacts-objects created by humans-and the associated techniques, which, together, are needed to achieve a desired outcome. Part 2 of this chapter sets out in detail how the processes of coordinating technologies with the development of geometric reasoning combines artefactual “know-how” with cognitive issues. Part 3 has two aims. First, to provide a model using Activity Theory, which both highlights the role of technology in the process, and examine the consequent tensions that arise from it. Second, this part discusses the mediational role of digital technology in learning geometry, and the implications for developing proofs.

Modelling Proof in a technological context

To analyse proof in a technological context, it is useful to consider a framework derived from Activity Theory, shown in Figure 14 (Stevenson, 2008). The framework provides a way of describing the use of artefacts, (e.g. digital devices, straight-edges etc.) in processes of proving. It should be stressed that Activity Theory is a framework for analysing artefact-based social activity, and it is a "theory" in the sense that it claims that such systems came be described using the categories shown in Figure 14. A principle aim of this section is to "flesh out" the epistemological, cultural and psychological dimensions of the system in relation to technology and proof.

Referring to Figure 14, the "object" of the system is the formulation of a problem which motivates and drives the proof process, with the "outcome" being the proof created from this system. "Artefacts" are any material objects used in the processes of achieving the objectives of the system, which, in this case, include straight-edges, compasses and DGS. A "subject" is an individual who is part of a "community" in which the activity is set, "managed" in particular ways through power structures that assign roles and status within a given context (e.g. classroom, professional mathematics community). "Rules" gathers together the relationships that define different aspects of the system.

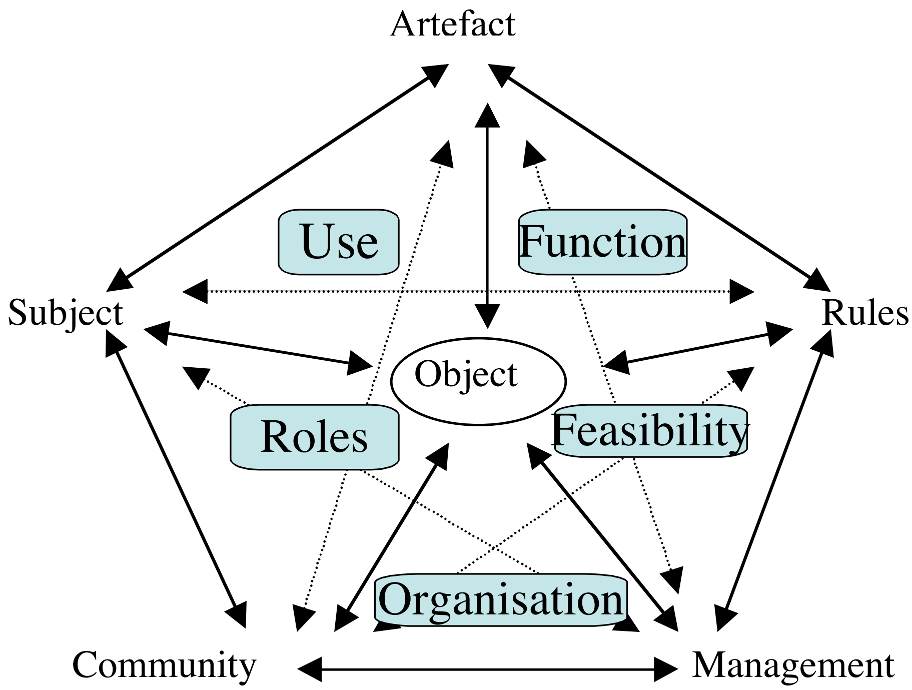

Figure 14

A "technology" in this system consists of the artefacts used, objects to which they can be applied, who is permitted to use the artefacts, and the sets of rules appropriate to each of the relationships which make up the system. In Figure 14, five aspects of the interactions with technology are identified which are useful for the analysis. From a functional point of view, a technology is the set techniques that are possible with the artefacts on a specific object, within a particular setting and social grouping. Using technology relates to the ways in which individuals or groups actually behave with the artefacts within the social context, governed by the norms of community organisation. The roles adopted by the participants in the system are the types of linguistic interactions occurring between those participating, and the organisation refers to the groupings of those participating in the technology-based activities. Finally, feasibility is related to the practical constraints placed on an activity by its physical and temporal location.

Figure 4 (Part 1 of the Chapter) and its associated description highlights the cultural dimension of a mathematics classroom engaging in proof activities with the technology of straight-edge and compass or DGS. By linking together the five aspects of the model, (Use, Function, Roles, Feasibility and Organisation) this cultural dimension can be expressed in detail (Stevenson, 2008). The model brings together the selection of tasks related to the object and outcomes of activities, and teachers' use of artefacts. In particular, it expresses how their use of artefacts is tailored to a specific classroom in order to mediate ideas about proof and its forms to their students. As a result, the model expresses how specific forms of activity and styles of linguistic interaction between pupil and teacher in a given physical location, provide the context for studying proof.

Epistemologically, the system is defined by the rules governing the formal object of mathematical knowledge that is the context for the proof process (e.g. geometry). For "standard" proof, the technology consists of the artefacts needed to create the proof (e,g. paper and pencil) and the rules of inference which govern how statements are organised on paper. In the standard account of creating proofs, rules of inference describe the syntactical dimension of the process. Semantics deal with the meanings given to geometric statements, for example, and are concerned with the truth and knowledge claims which a proof contains. A major claim for using proof as a method for obtaining knowledge is that if the rules of inference are "applied correctly", they preserve the semantic integrity of the argument forming the proof. Technology, therefore, plays a key role in the relationship between semantics and syntactical structures of a proof. Are syntax and semantics separable, or are they intertwined with the technology chosen for the proof?

As the process described in 5a and b of the section "Abductions in mathematics learning" of Part 2 of this chapter proposes, conjectures become proofs by applying the technology of logical inference. Abduction is, epistemologically, a non-linear process which develops over a period of time, and involves iterations between facts and the conjectured rules that gradually come to explain those facts. Constructing a proof involves restructuring a conjecture to suit the linear form of logical inference so that the technique can be applied to organise the argument "on the page". Such linearization re-interprets (or removes) the diachronic aspect of abduction as an epistemological structure. In the process of translating conjecture into a proof, however, constructions, false starts, and strings of informal calculations, are removed. Reference to sensori-motor processes, in geometry, are suppressed by talking about "ideal" points and lines, with the paper surface acting as a kind of window on the "real" geometry. (Livingstone, 2006). In terms of the model in Figure 14, the role of the teacher is critical in helping learners make this linguistic transition. The extracts of dialogue presented in Part 2 related to this process of linearization indicate how the cues and leads given by the teacher aids the learner in filtering her conjecture, so that the techniques of inference can be used to organise it appropriately.

Adding DGS to this situation does not change the essential dynamic tension that arises as a result of the need to translate from one technological setting (straight-edge and compass or DGS) to another (rules of inference). The discussion in Part 2 of the "logic of not" and "indirect proof" places the focus on the Use aspect of the model in Figure 14. These detailed analysis of types of reasoning imply that abduction arises as a strategy to deal with the dynamic tensions which result from translating between the different technologies.

Digital technology as a mediational artefact

Learning how to use geometrical equipment, whether physical or digital is part of the instrumentation of geometry. (Verillon and Rabardel, 1995). This describes the interplay between facility with artefacts and the development of psychological concepts. Physically we have the experience of using a straight edge to draw a "straight line" and compasses to make a "curve". In

Lakoff's (1988) framework, this can be interpreted as developing a "prototype". The motor-sensory action of using a straight-edge and pencil, combined with the linguistic sign "straight" and the Gestalt perception of the resulting mark on a surface, embodies the concept of "straightness".

A question raised by this account is the extent to which technology, in general, mediates understanding of geometrical concepts, and how that relates to proof in the formal sense. For example, rather than simply motivating proofs of results, does/can/should DGS play an integral part in forming the conceptual structures that constitute what we understand as geometry? Much of the interest in DGS lies in their representation of Euclidean Geometry, but as Part 2 implies, DGS provide a different kind of geometry to that obtained by the use of paper and pencil constructions, or the axiomatic version of the Euclidean geometry. As a consequence, different versions of geometry emerge with these different technologies: "pencil geometry"; "digital geometry"; "axiomatic geometry". The question is not whether technology mediates knowledge, but in what ways do different technologies mediate different kinds of knowledge. Proof, as a means of establishing knowledge claims, should, therefore, take account of the mediational role which artefacts play in epistemology.

Learning non-euclidean geometry, for example, has a number of complexities when compared to the Euclidean case. On the one hand, spherical geometry is relatively straightforward since learners have may opportunities to develop visual intuitions in their everyday experience. Being "smaller" than Euclidean space, spherical surfaces are both closed and bounded, and capable of both physical and digital manipulation. On the other hand, hyperbolic space poses a different problem since it is "larger" than Euclidean space, and finding a complete physical surface to manipulate is difficult. (Coxeter, 1969). The process is complicated by the lack of visual intuition, and there are difficulties in finding appropriate artefacts to support instrumentation. Digital technologies, however, offer possibilities for engaging with hyperbolic geometries that cannot be found in other ways (Jones, Mackrell, and Stevenson, 2010). Figure 15 shows how a two-dimensional Euclidean model can be obtained by projecting a hyperbolic surface, and illustrates the geometry associated with the projection. The central image in Figure 15 is an Escher tessellation obtained using the projective model, and the right-hand image shows the underlying geometry of the projection.

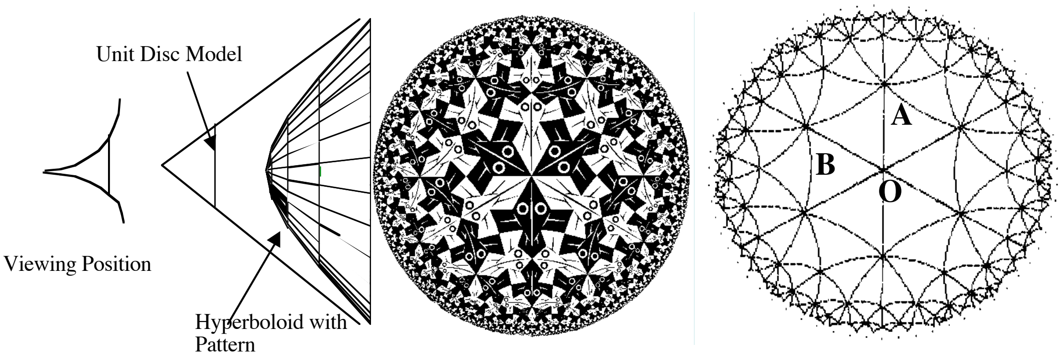

Figure 15

Imagining that the Escher tessellation is spread isometrically across the hyperboloid on the left-hand side of Figure 15, and viewing it as a projection onto a flat disc gives the image in the centre of Figure 15. The grid for the tessellation is shown on the right-hand side of Figure 15, together with the basic hyperbolic triangle OAB used to tessellate the disc. Triangle OAB shows one of the key differences of hyperbolic geometry compared to Euclidean: the angle sum of triangles is less than 180 degrees. Infinity is represented by the edge of the circle, which can be approached but never reached, as is indicated by the "bunching" of the tessellations at the circumference in the central image. Further, "straight" lines can be either Euclidean straight lines or circular arcs.

Using Turtle geometry, it is possible to animate the two-dimensional projective model of the hyperbolic surface to provide an artefact for exploring the geometry. (Jones et al. *ibid*). Taken from Stevenson (2000), the following snippet shows the work of two adults (S and P) using the non-euclidean turtle microworld to illustrate the role of artefacts in mediating understanding of the geometry. S and P first draw lines OA and OB, shown in Figure 16, and are attempting to find the line (AB) to close the triangle. Starting with the Turtle at B, pointing to right of the screen, they turn it through 135 degrees, and use a built in procedure called "Path" which indicates how the Turtle would travel if it was moved with that heading. They see that it does not close the triangle and so they turn the turtle by a further 5 degrees to the left. This time the path goes through A, and they reflect on the screen results. S picks up a hyperbolic surface provided for them and reminds himself about the projection process (shown on the left-hand side of Figure 15).

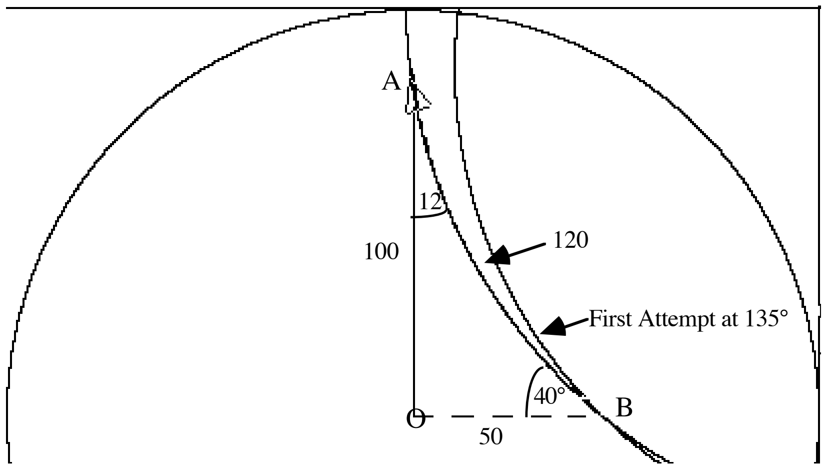

Figure 16

S comments on the diagram in Figure 16:

S: We haven't got 180, but it's walking a straight-line path.(reflectively)

S is referring to the metaphor that in order for the Turtle to draw straight line on a curved surface, in this case the hyperboloid, it must take equal strides, hence the phrase "walking a straight-line path".

P: Yeah, you've probably got to turn.

Instinctively P thinks that a turtle walking on the curved surface must turn to compensate for the curvature. S is clear that this is not happening.

S: No, you don't have to turn. It's actually drawing a triangle on the surface.

Looking at the hyperboloid, S imagines the Turtle marking out the triangle on the surface. He adds:

S: The projection defies Pythagoras. No! Hang on, walking on the surface is defying it, isn't it! Because we walk straight lines on the surface we just see them as curves on the projection.

For S it is clear that the projection preserves the geometric properties of the turtle's path on the surface. What they are seeing, he believes, is a fact about the geometry, and not a result of the software or the projection. Two points are significant. First, the insight would not be possible without the material artefacts (physical surface and digital application), and the metaphor that turtles walk straight lines on curved surfaces by taking "equal-strides without turning". (Abelson and diSessa, 1980:204). Second, S is convinced about what he is seeing, and provides an explanation about why the image is preserving something about the geometry of the hyperbolic surface.

What might constitute a proof in this context that the angle sum of any hyperbolic triangle is always less than 180 degrees? Given that the artefacts mediate the object of study (hyperbolic geometry), can the insight in this snippet be converted into the technology of logical inference with its associated linearization and the re-interpretation of the diachronic aspect? Proofs of the angle sum do exist, but they rely on the axiomatic approach described by Einstein and the technology of inference (e.g. Coxeter, 1969: p. 296 ff.). However the proofs are abstract and lack visual intuition, both of which are qualities that the digital and physical artefacts in this snippet offer to learners.

It is possible to speculate that in the not too distant future, learners may be using electronic media, rather than paper, to develop their work, which would enable them to embed digital applications. Effectively this will separate the technology of inference from the need to layout arguments on paper as some kind of final statement. As for the model presented in Figure 14, its value lies in being able to provide the cultural and pedagogic context for these activities, which embeds technologies in social relationships and human motivation. It also shows how dynamic tensions arise in reasoning due to conflicts between technologies. Coupled with Balacheff's analysis of proof types (2008), the model is fruitful in identifying how the assumptions and expectations of those engaging in proof generate contradictions in their practices (Stevenson, in preparation).

Closure

In this chapter we have discussed some strands of experimental mathematics both from an epistemological and from a didactical point of view. Precisely, we have introduced some past and recent historical examples in western and eastern cultures in order to illustrate how the use of tools has driven the genesis of many abstract mathematical concepts.

We have shown how the intertwining between concrete tools and abstract ideas introduces an "experimental" dimension in mathematics, and a dynamic tension between the *empirical nature* of the activities with the tools –which encompass perceptual and operational components– and the *deductive nature* of the discipline –which entails a rigorous and sophisticated formalization. This *almost empirical* aspect of mathematics has been hidden in the years across the second half of the XIX and the first half of the XX century because of a prevailing formalistic attitude, but in more recent times the perceptual and empirical aspects of the discipline has come again on the scene. This is mainly due to the heavy use of the new technology, which is deeply and fast changing both research and teaching in mathematics (Lovasz, 2006).

We have illustrated how this happens in proving activities within the classroom and have introduced some theoretical frameworks, which highlight the dynamics of students' cognitive processes while working in CAS and DGS environments to explore problematic situations, to formulate conjectures and finally to prove them. We have so pointed out the complex interplay between inductive, abductive, and deductive modalities in the delicate transition from the empirical to the theoretical side in the production of proofs. We have stressed that such a dynamics is strongly supported by a suitable use of technologies, provided the students are taught to learn some practices in their use, for example the *maintaining dragging* scheme in DGS. We have shown how the induced instrumental genesis can help them also in producing indirect proofs.

Finally we have used Activity Theory to model the above tension as a consequence of *translating between different technologies*, understood in the broadest sense as something that can mediate between different ontologies and epistemologies.

Aknowledgments
We thank all the people who participated to the Working Group 3 in Tai Pei for the useful discussions we had there. Particular thanks to Anna Baccaglini-Frank for the fruitful discussions that at least four of the authors had with her during and after the preparation of her dissertation and whose contribution to this chapter is apparent. Many thanks also to the referees for their helpful suggestions.

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List of the participants to WG3 (<u>DGS/experimentation/Geometry):</u>

1. Ferdinando Arzarello, Università di Torino, Italy;
2. Liping Ding, Massey University, New Zealand;
3. Maria Giuseppina Bartolini Bussi, Università di Modena e Reggio Emilia, Italy;
4. Víctor Larios-Osorio, Universidad Autónoma de Querétaro, México;
5. Ian Stevenson, King's College, London, UK;
6. Maria Alessandra Mariotti, Università di Siena, Italy;
7. Allen Leung, Hong Kong Institute of Education;
8. Nurit Zehavi, Weizmann Institute of Science, Israel;
9. Giora Mann, Beit Chanan, Israel.

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