An analysis of the Podelski–Rybalchenko termination theorem via bar recursion

(Article begins on next page)

21 July 2024

Effective Bounds on the Podelski-Rybalchenko Termination Theorem

September 30, 2014

1 Introduction

We report here on current work towards an effective proof (with explicit bounds) of Podelski and Rybalchenko Termination Theorem [7]. Our long-term project is to obtain a priori-bounds for the termination of computer programs, and compare these with bounds obtained via other intuitionistic proofs of the Termination Theorem.

Theorem 1.1 (Termination Theorem). *A program P is terminating iff there exists a well-founded transition invariant for P.*

The authors proved this result by using Ramsey Theorem, which is a purely classical results [1]. In [9] and [2] two proofs of an intuitionistic versions of this theorem have been proposed, where the notion of a program being terminating is replaced by intuitionistically weaker (but classically equivalent) notion. Let us call this "inductively terminating".

Theorem 1.2. *A program P is (inductively) terminating iff there exists a (inductively) well-founded transition invariant for P.*

In this result both the hypothesis and the thesis are intuitionistically weaker than the ones in Theorem 1.1, since, intuitionistically, classical well-foundedness is strictly stronger than intuitionistic well-foundedness. Co-quand's proof is based on almost-full relations (see also [4]), while the second one is based on H-well-founded relations.

In this work we want to intuitionistically prove Podelski and Rybalchenko Termination Theorem considering the classical definition of well-foundness. In the result by Podelski they define a program to be terminating iff

$$R \cap (\mathrm{Acc} \times \mathrm{Acc}) \text{ is well-founded,}$$

1

where R is the transition relation of P and Acc is the set of accessible states. In this work instead we consider the following definition, which is classically equivalent to the Podelski's one. Assume I is any set, R is a binary relation over I, and S (the set of initial states) is included in Acc which is included in I (the set of states). We call a computation over I any sequence $\alpha : \mathbb{N} \to I$, and for all $n \in \mathbb{N}$ we write α_n for $\alpha(n)$ in I.

Definition 1.3. P is terminating iff $\forall \alpha (\alpha_0 \in S \implies \exists n \neg (\alpha_n R \alpha_{n+1}))$.

Lemma 1.4. *Classically, P is termimating by Podelski definition iff P is terminating by Definition 1.3.*

Proof. $\Rightarrow$ Assume that $R \cap (\text{Acc} \times \text{Acc})$ is well founded. Then each chain in the relation $R \cap (\text{Acc} \times \text{Acc})$ is finite. Moreover every chain which starts with some initial state is finite.

$\Leftarrow$ Assume that
$$\forall \alpha (\alpha_0 \in S \implies \exists n \neg (\alpha_n R \alpha_{n+1})),$$
we need to prove that each $R \cap (\text{Acc} \times \text{Acc})$-chain is finite. Let c be a chain in $R \cap (\text{Acc} \times \text{Acc})$, then in particular the first element of c is in Acc. Hence we may extend c to a chain c^* which start from some initial state. By applying the hypothesis we are done.

$\square$

Remark 1.5. Note that $\alpha_0 \in S$ is a decidable formula, whereas $\alpha_0 \in \text{Acc}$ is potentially Σ_1 when I is infinite. Hence, P being terminating is a Π_2 formula, which would not be the case had we used the definition $\forall \alpha (\alpha_0 \in \text{Acc} \implies \exists n \neg (\alpha_n R \alpha_{n+1}))$.

From now on we will say "P is terminating" meaning it is terminating as in Definition 1.3. We will prove the following result:

Theorem 1.6. *Given a program P, with transition relation R, if*

$$\exists n \; \exists T_1, \ldots, T_n \; \exists \omega_1, \ldots, \omega_n \; (T_1 \cup \cdots \cup T_n \supseteq R^+ \cap (\text{Acc} \times \text{Acc})$$
$$\wedge \; \forall i \in [1, n] \; (\forall \alpha \; \exists j < \omega_i(\alpha) \; \neg (\alpha_j T_i \alpha_{j+1})))$$

then there exists Φ such that

$$\forall \alpha \; (\alpha_0 \in S \implies \exists m < \Phi(\bar{T}, \bar{\omega}) \; \neg (\alpha_m R \alpha_{m+1})).$$

The functional Φ is definable in $\mathsf{T} + $ Bar recursion. Thanks to the results of Schwitchtenberg [8] and Kohlenbach [5] that give some closure properties for bar recursion, we can study the complexity of the bound Φ relative to the complexity of the relations T_i and the functions ω_i.

The basic ideas behind the proof are those presented in [6] for the classical Ramsey theorem. Our proof of Theorem 1.6 requires the Transitive Ramsey Theorem for pairs, a corollary of the Ramsey Theorem for pairs which can be stated as follows:

Theorem 1.7 (Transitive Ramsey Theorem for Pairs)**.** *For any $c : \mathbb{N}^2 \to n$ there exists an infinite homogeneous chain.*

The structure of the proof is the following. Given a computation α, we consider all initial finite subsequences of α. We associate to each initial finite subsequence of α a finite tree which follows the idea of the Blackwell proof of Erdős-Szerkeres Theorem (see [3]). Then from each of these trees we obtain a set of monochromatic sequences whose length is bounded by the ω_i. Therefore if α is infinite by applying bar recursion we obtain a contradiction, which gives us the bound we use to define Φ.

2   Proof for 2 colors

The goal of this section is to prove Theorem 1.6 for two colors. We first generalize Blackwell proof of Erdős-Szerkeres Theorem in [3] for Transitive Ramsey Theorem. If the coloring is not transitive then we adapt this proof and we build either an infinite homogeneous chain in color 0 or an infinite homogeneous set in color 1 (instead of infinite homogeneous set in both the cases).

Theorem 2.1 (Transitive Ramsey Theorem)**.** *Let $c : \mathbb{N}^2 \to 2$, then there exists an infinite chain in color 0 or an infinite homogeneous set in color 1.*

Proof. Given a well ordered set X we say that s is a *leftmost* sequence of X iff all $s_i \in X$ and

- $s_0 = \min X$;

- $s_{i+1} > s_i$

- $c(\{s_i, s_{i+1}\}) = 0$

- $\forall x^X (s_i < x < s_{i+1} \implies c(s_i, x) = 1)$

3

Then we construct a set of sequences, as Blackwell does, as follows.

- w_0 is the *leftmost* sequence of $\mathbb{N}$.

- w_i is the *leftmost* sequence of $\mathbb{N} \setminus \bigcup \{w_j : j < i\}$.

Since $\mathbb{N}$ is infinite we have either an infinite sequence w_i or infinitely many finite non-empty sequences w_i. In the first case we will have an infinite 0-chain. In the second case let $\{l_i : i \in \mathbb{N}\}$ be the set of the last element of each such sequence. Since it is an infinite sequence of natural numbers, it should contain an infinite increasing subsequence. This is an infinite 1-homogeneous set by construction, since we only ever stop a sequence w_i if there are no more 0-edges from the last element of w_i. In fact the last element l_i of each sequence w_i is by construction related by a color 1 edge to each element of $\{x \in \mathbb{N} \mid x > y \wedge x \notin \bigcup \{w_j \mid j \le i\}\}$. In particular, for all $j > i$ if $l_j > l_i$, then they are related by one color 1 edge. $\qquad \square$

Now we can prove Theorem 1.6 for two colors. Let P be a program with its transition relation R. Assume α is such that $\alpha_0 \in S$, where S is the set of the initial states of P and that there exists T_0, T_1, ω_0, ω_1 as in the hypothesis of Theorem 1.6. We have to prove that there exists Φ such that

$$\exists m < \Phi(\alpha) \ \neg(\alpha_m R \alpha_{m+1}).$$

The idea is to consider finite approximations of the tree given by the leftmost sequences of the Blackwell proof, and work with these approximations.

In order to do this we need to define the following functions. The first one which we call β is the function which gives us the successor (in the sense of the leftmost sequences) of a node with respect to a finite sequence. If a successor does not exists in the given finite sequence β returns the empty set.

Definition 2.2. Let $x \in \mathbb{N}$ and let $x_1 < \cdots < x_n \in \mathbb{N}$.

$$\beta(x, \langle x_1, \ldots, x_n \rangle) = \begin{cases} x_i & \text{if for some } i \in [1, n] \\ & \quad x_i > x \ \wedge \\ & \quad c(\{x, x_i\}) = 0 \ \wedge \\ & \quad \forall j < i (c(\{x, x_j\}) = 1 \vee x_j < x) \\ \emptyset & \text{otherwise.} \end{cases}$$

Observe that β is primitive recursive. Given a non-empty list l, we define $\mathrm{hd}(l)$ as the first element of l, $\mathrm{tl}(l)$ as the tail of l and $\mathrm{last}(l)$ as the last element of l. Now we define a function φ which provides the finite approximation of the tree of leftmost sequences.

4

Definition 2.3. Let s be a finite list of natural numbers, $\langle w_i \rangle_{i \in \mathbb{N}}$ be a list of finite non-empty lists of natural numbers and let $k \in \mathbb{N}$. Define the function φ as

$$\varphi(\emptyset, s) = \begin{cases} \emptyset & \text{if } s = \emptyset \\ \varphi(\langle \mathrm{hd}(s) \rangle, \mathrm{tl}(s)) & \text{otherwise} \end{cases}$$

$$\varphi(\langle w_i \rangle_{i \leq k}, s) = \begin{cases} \langle w_i \rangle_{i \leq k} & \text{if } s = \emptyset; \\ \varphi(\langle w_i \rangle_{i < k} * (w_k * \langle x \rangle), s \setminus x) & \text{if } \beta(\mathrm{last}(w_k), s) = x \\ \varphi(\langle w_i \rangle_{i \leq k} * \langle \mathrm{hd}(s) \rangle, \mathrm{tl}(s)) & \text{if } \beta(\mathrm{last}(w_k), s) = \emptyset \end{cases}$$

Define $x <_i y$ iff $x < y$ and $c(\{x, y\}) = i$. Observe that the following are invariants for φ:

- for all $i \leq k$, w_i is finite, not empty and is a chain for $<_0$.

- for all $i < k$ and $y \in s$, $\mathrm{last}(w_i) < y$ implies $\mathrm{last}(w_i) <_1 y$.

- for all $i < j \leq k$, $\mathrm{last}(w_i) < \mathrm{last}(w_j)$ implies $\mathrm{last}(w_i) <_1 \mathrm{last}(w_j)$.

The proof that the first and the second ones are invariant follows from the construction, while the third one holds since if $x > \mathrm{last}(w_j) \wedge x >_0 \mathrm{last}(w_j)$ then we should have added it in a previous step of the construction, at the end of the j-line. So x does not belong to s. Moreover, since s is a finite list and its length decreases during the computation of φ, φ terminates.

The last definition we need is the definition of σ. Given a sequence of natural numbers b, the map σ produces an increasing subsequence.

Definition 2.4. Let $b = \langle x_1, \ldots, x_n \rangle$ be a list of natural numbers, define $\sigma(b)$ as

$$\sigma(\langle x_1, \ldots, x_n \rangle) = \begin{cases} \emptyset & \text{if } n = 0 \\ \langle x_1 \rangle * \sigma(\langle x_j, \ldots, x_n \rangle) & \text{if } (x_j > x_1) \wedge \forall h < j (x_h < x_1) \end{cases}$$

Given a list of non-empty finite lists of natural number $\langle w_i \rangle_{i \leq k}$, let us denote by b the list of the last elements of these lists, i.e.

$$b = \langle \mathrm{last}(w_i) \mid i \leq k \rangle.$$

Recall that we are assuming we are giving "moduli of termination" ω_i for each of the relations T_i. The main property of the modulus is

$$\forall \alpha \exists j < \omega_i(\alpha) \neg (\alpha_j T_i \alpha_{j+1}),$$

5

i.e. for any given infinite sequence the modulus of termination gives us an upper bound on the point where termination is guaranteed to have happened. Therefore, we must consider a large enough approximation of the Blackwell tree, so as to make sure that we obtain a counter-example on one of the two colours. The following function ξ will check whether it is enough to consider the first n elements of a given sequence α, but notice that this function as defined is not primitive recursively. It is essentially performing an unbounded search on n. We claim that this can be defined using Spector's bar recursion.

Definition 2.5. Given α and n let $\varphi(\emptyset, \langle 0, \dots, n \rangle) = \langle w_i \rangle_{i \leq k}$. Define

$$\xi(\alpha, n) = \begin{cases} n & \text{if } \exists i \leq k (\omega_0(w_i) < |w_i|) \vee \omega_1(\sigma(b)) < |\sigma(b)| \\ \xi(\alpha, n+1) & \text{otherwise} \end{cases}$$

$$\Phi(\alpha) = \xi(\alpha, 0).$$

Observe that it depends from α since the coloring (fixed from now on) is defined as follows: $c(i, j) = d$ iff $i < j$ and $\alpha_i T_d \alpha_j$, for each $d \in \{0, 1\}$.

3 Φ is in T

In this section we prove that $\Phi(\alpha)$ is in T for any computation α. Observe that $\Phi(\alpha)$ builds what we call a finite structure M which corresponds to a sequence $\langle w_i \rangle_{i \leq k}$, an approximation of the Blackwell's tree. Firstly we prove that given a sequence γ satisfying certain conditions, we can compute the structure M by primitive recursion. Then we will approximate γ via bar recursion, and we will prove that such γ is in T. Therefore also $\Phi(\alpha)$ is.

3.1   The construction of the structure M given an oracle γ

Let $A(i, k, X)$ abbreviate the formula $k > i \wedge c(i, k) = 0 \wedge k \notin X$. First, let us assume that we have a sequence $\gamma(X)(i)$ such that for all i

$$\exists k A(i, k, X) \Leftrightarrow A(i, \gamma(X)(i), X) \wedge \forall j < \gamma(X)(i) \neg A(i, j, X)$$

where X is a finite set of integers. The sequence $\gamma(X)$ is obviously non-computable as it finds a least witness for $\exists k A(i, k, X)$ whenever such witness exists.

Nevertheless, we now show that given such sequence the construction of the structure $\Phi(\alpha)$ is effective. We will then find an effective approximation to γ which will be good enough for ours purposes.

Let us denote by $\mu_{\mathrm{Sp}}(\omega)(\alpha)$ the primitive recursive functional that finds the first point n such that $\omega(\overline{\alpha,n}) < n$. Also, let $\gamma^k(X)(i)$ be defined as

$$\gamma^0(X)(i) = i;$$
$$\gamma^{k+1}(X)(i) = \gamma(X \cup \bigcup_{h \leq k} \left\{\gamma^h(X)(i)\right\})(\gamma^k(X)(i)).$$

Definition 3.1. Define by simultaneous primitive recursion the following sequences:

$$\alpha_0 = \lambda k.\gamma^k(\emptyset)(0) \qquad\qquad \alpha_{i+1} = \lambda k.\gamma^k(X_i)(0)$$
$$n_0 = \mu_{\mathrm{Sp}}(\omega_0)(\alpha_0) \qquad\qquad n_{i+1} = \mu_{\mathrm{Sp}}(\omega_0)(\alpha_{i+1})$$
$$w_0 = \text{maximal 0-prefix of } \alpha_0 \quad w_{i+1} = \text{maximal 0-prefix of } \alpha_{i+1}$$
$$X_0 = \{w_0(k)\}_{k<|w_0|} \qquad\qquad X_{i+1} = X_i \cup \{w_{i+1}(k)\}_{k<|w_{i+1}|}$$
$$a_0 = \mu k(k \notin X_0) \qquad\qquad a_{i+1} = \mu k(k \notin X_{i+1})$$
$$j_0 = \max\{\alpha_0(k)\}_{k\leq n_0} \qquad\qquad j_{i+1} = \max\{\alpha_{i+1}(k)\}_{k\leq n_{i+1}}$$

We explain these as follows: α_i is the chain generated by γ starting with the smallest number not yet used, i.e. a_{i-1}, while avoiding all numbers already used, i.e. X_{i-1}. The point n_i is the first such that $\omega_0(\overline{\alpha_i, n_i}) < n_i$. By the assumption on ω_0, we know that before point n_i we will find a 1-link in the sequence α_i, i.e a point k such that $c(\alpha_i(k), \alpha_i(k+1)) = 1$. We then take w_i to be the maximal 0-chain, and n_i is clearly a bound on the length of w_i. The finite sequences w_i will form the rows of the Blackwell matrix. We also keep a record in j_i of the largest index used in building w_i. We will see how this is important later.

Definition 3.2. Recall that σ is a functional that computes an increasing subsequence of a given sequence together with the indices filtered out. Hence, given the above define the sequences

$$b(i) = \mathrm{last}(w_i)$$
$$\langle \rho_0, \rho_1 \rangle = \sigma(b)$$
$$m = \mu_{\mathrm{Sp}}(\omega_1)(\rho_1)$$
$$t = \text{maximal 1-prefix of } \rho_1 \text{ (note that } |t| \leq m)$$

Hence, we see that the following finite Blackwell matrix (parametrised by γ) is sufficient to obtain a contradiction:

$$M(\gamma)(i) = w_i$$

where $|M(\gamma)| = \rho_0(|t|)$.

3.2 Approximating γ via bar recursion

Recall that $A(i, k, X) \equiv k > i \wedge c(i, k) = 0 \wedge k \notin X$. Let $q, \omega : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ be such that $q(\gamma) = \omega(\gamma)$ is the greatest element in the finite matrix $M(\gamma)$. Observe that j_i, as define in the previous section, is needed to define the greatest element of $M(\gamma)$:

$$q(\gamma) = \omega(\gamma) = \max \left\{ j_i : i \leq \rho_0(|t|) \right\}.$$

Our goal is to build a finite approximation to γ, in the sense that it will only satisfy

$$\exists k \leq q\gamma A(i, k, X) \iff A(i, \gamma(X)(i), X) \wedge \forall j < \gamma(X)(i) \neg A(i, j, X)$$

for all $i \leq \omega\gamma$. But this approximation is clearly enough to give us a proper bound, since we only ever use γ up to the point $\omega\gamma$ when building the structure $M(\gamma)$. Let us define the following sequence of selection functions:

$$\varepsilon_i(X)(p) = \begin{cases} k & \exists k \in (i, p(i+1)] \setminus X(c(i, k) = 0) \\ i + 1 & \forall k \in (i, p(i+1)] \setminus X(c(i, k) \neq 0). \end{cases}$$

where k in the first case is taken to be the least one witnessing the formula $\exists k \in (i, p(i+1)] \setminus X(c(i, k) = 0)$.

Lemma 3.3. $\varepsilon_i(X)$ *satisfies, for all p*

$$\exists k \leq p(a) A(i, k, X) \implies A(i, a, X) \wedge \forall j < a \neg A(i, j, X).$$

where $a = \varepsilon_i(X)(p)$.

Proof. Assume that $\exists k \in (i, p(\varepsilon_i(X)(p))](c(i, k) = 0 \wedge k \notin X)$, then we cannot have $\varepsilon_i(X)(p) = i + 1$, then $\exists k \in (i, p(i+1)](c(i, k) = 0 \wedge k \notin X)$, therefore $c(i, \varepsilon_i(X)(p)) = 0$. $\qquad\square$

Then we define

$$\gamma = \mathrm{EPS}^{\omega}_{\langle\rangle}(\varepsilon)(q);$$
$$p_s(x) = \overline{\mathrm{EPS}^{\omega}_{s * x}(\varepsilon)}(q_{s*x});$$

hence thanks to the main theorem for any $n < \omega(\gamma)$

$$\gamma(X)(n) = \varepsilon_{[\gamma](n)}(X)(p_{[\gamma](n)});$$
$$q\gamma(X) = \overline{\varepsilon_{[\gamma](n)}}(X)(p_{[\gamma](n)}) = p_{[\gamma](n)}(\varepsilon_{[\gamma](n)}(X)(p_{[\gamma](n)}));$$

Since the set $(\varepsilon_i)_{i \in \mathbb{N}}$ is indexed in the natural we have that for all $n \in \mathbb{N}$:

$$\varepsilon_{[\gamma](X)(n)} = \varepsilon_n.$$

Therefore γ satisfies

$$\forall i < \omega\gamma \ (\exists k \in (i, q\gamma] \ (c(i,k) = 0 \ \wedge \ k \notin X) \iff$$
$$(\gamma(X)(i) > i \ \wedge \ \gamma(X)(i) \notin X \ \wedge \ c(i, \gamma(X)(i)) = 0)).$$

In fact let $i < \omega(\gamma)$; if $\exists k \in (i, q_\gamma](c(i,k) = 0 \wedge k \notin X)$ holds, then we have

$$\exists k \in (i, p_{[\gamma](i)}(\varepsilon_i(X)(p_{[\gamma](i)})](c(i,k) = 0 \wedge k \notin X)$$

and this implies (by construction of the selection functions)

$$\varepsilon_i(X)(p_{[\gamma](i)}) > i \wedge \varepsilon_i(X)(p_{[\gamma](i)}) \notin X \wedge c(i, \varepsilon_i(X)(p_{[\gamma](i)})) = 0.$$

and so

$$\gamma(X)(i) > i \wedge \gamma(X)(i) \notin X \wedge c(i, \gamma(X)(i)) = 0.$$

To conclude observe that $\mathrm{EPS}^\omega_{\langle\rangle}(\varepsilon)(q)$ is defined by Bar Recursion of type 1 (since $q : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ and $s \in \mathbb{N}^*$) then, thanks to Schwichtenberg's result [8] and by assuming that ω_0 and ω_1 are in T, we can conclude that γ is in T and so also $\Phi(\alpha)$ is.

4 Proof for n colors

In this section we prove Theorem 1.6 for n colors, following the argument we use in the case with two colors. Firstly we want to generalize Blackwell proof Theorem 2.1 for n-many colors. We may observe that it can be done easily by induction.

Theorem 4.1 (Transitive Ramsey Theorem). *Let $c : \mathbb{N}^2 \to n$, then there exists an infinite homogeneous chain.*

Proof. Proof by induction on n. Assume it holds for n, then consider a coloring $c : \mathbb{N}^2 \to n + 1$. Given a well ordered set X we say that s is a *leftmost* sequence of X iff all $s_i \in X$ and

- $s_0 = \min X$;

- $s_{i+1} > s_i$

- $c(\{s_i, s_{i+1}\}) = 0$

- $\forall x^X (s_i < x < s_{i+1} \implies c(s_i, x) \neq 0)$

Then we construct a set of sequences, as Blackwell does, as follows.

- w_0 is the *leftmost* sequence of $\mathbb{N}$.

- w_i is the *leftmost* sequence of $\mathbb{N} \setminus \bigcup \{w_j : j < i\}$.

Since $\mathbb{N}$ is infinite we have either an infinite sequence or infinitely many finite sequences. In the first case we will have an infinite 0-chain and we are done. In the second case let $\{l_i : i \in \mathbb{N}\}$ be the set of the last element of each such sequence. Since it is an infinite sequence of natural numbers, it should contain an infinite increasing subsequence. This is an infinite set in n colors, by construction. Then by induction hypothesis there exists an infinite homogeneous chain.

$\square$

The idea now is to prove Theorem 1.6 for n, by using the induction on n and the schema of the proof for two colors. Assume that the Theorem 1.6 holds for n, we will prove it for $n + 1$.

Let P be a program with its transition relation R. Assume α is such that $\alpha_0 \in S$, where S is the set of the initial states of P and that there exist $T_0, \ldots, T_n, \omega_0, \ldots, \omega_n$ as in the hypothesis of Theorem 1.6.

We have to prove that there exists Φ_n such that

$$\exists m < \Phi_n(\alpha) \ (\neg(\alpha_m R \alpha_{m+1})).$$

In order to do this we need to define the following functions as we did in the case for 2 colors.

Definition 4.2. Let $x \in \mathbb{N}$, $x_1 < \cdots < x_m \in \mathbb{N}$.

$$\beta_n(x, \langle x_1, \ldots, x_m \rangle) = \begin{cases} x_i & \text{if for some } i \in [1, m] \\ & \quad x_i > x \ \wedge \\ & \quad c(\{x, x_i\}) = 0 \ \wedge \\ & \quad \forall j < i(c(\{x, x_j\}) \neq 0 \vee x_j < x) \\ \emptyset & \text{otherwise.} \end{cases}$$

Now we define a function φ_n (as φ for the case with 2 colors) which provides the finite approximation of the tree of leftmost sequences.

Definition 4.3. Let s be a list of natural numbers, $\langle w_i \rangle_{i < \mathbb{N}}$ be a list of lists of finite non-empty lists of natural numbers and let $k \in \mathbb{N}$. Define the function φ_n as

$$\varphi_n(\emptyset, s) = \begin{cases} \emptyset & \text{if } s = \emptyset; \\ \varphi_n(\langle \mathrm{hd}(s) \rangle, \mathrm{tl}(s)) & \text{otherwise.} \end{cases}$$

$$\varphi_n(\langle w_i \rangle_{i \leq k}, s) = \begin{cases} \langle w_i \rangle_{i \leq k} & \text{if } s = \emptyset; \\ \varphi_n(\langle w_i \rangle_{i < k} * (w_k * \langle x \rangle), s \setminus x) & \text{if } \beta_n(\mathrm{last}(w_k), s) = x; \\ \varphi_n(\langle w_i \rangle_{i \leq k} * \langle \mathrm{hd}(s) \rangle, \mathrm{tl}(s)) & \text{if } \beta_n(\mathrm{last}(w_k), s) = \emptyset. \end{cases}$$

While the σ is exactly the function defined for the case 2. As in the case for two colors, given a sequence of natural numbers b, the map σ produces an increasing subsequence.

Definition 4.4. Given α and n let $\varphi(\emptyset, \langle 0, \ldots, m \rangle) = \langle w_i \rangle_{i \leq k}$. Define

$$\xi_n(\alpha, m) = \begin{cases} m & \text{if } \exists i \leq k(\omega_0(w_i) < |w_i|) \\ & \quad \vee \; \Phi_{n-1}(\sigma(b)) < |\sigma(b)| \\ \xi_n(\alpha, m+1) & \text{otherwise} \end{cases}$$

$$\Phi_n(\alpha) = \xi_n(\alpha, 0).$$

In order to justify the previous definition, observe that in this construction the sequence $\sigma(b)$ is an homogeneous set in n colors, then we are in the following case:

$$T_1 \cup \cdots \cup T_n \supseteq (R \restriction \sigma(b))^+ \cap (\mathrm{Acc} \times \mathrm{Acc}) \; \wedge \; T_1, \ldots, T_n \text{ well founded}$$

Then $R \restriction \sigma(b) = \sigma(b)$ is well founded (with modulus Φ_{n-1}).

Observe that again the definition of Φ_n is not primitive recursive. But we can prove it is in T, by using the same argument we used in the case with two relations.

Proposition 4.5. Φ_n *is in* T.

Proof. Again by induction. For $n = 2$ we proved it in the previous section. Assume that Φ_{n-1} is in T. Then, by applying the same procedure we used for the case $n = 2$ providing to put $\omega_1 = \Phi_{n-1}$, we obtain that also Φ_n is in T. $\qquad\square$

References

[1] Stefano Berardi and Silvia Steila. Ramsey Theorem for pairs as a classical principle in Intuitionistic Arithmetic. submitted, 2013.

[2] Stefano Berardi and Silvia Steila. Ramsey Theorem as an intuitionistic property of well founded relations. submitted, 2014.

[3] Paul Blackwell. An Alternative Proof of a Theorem of Erdos and Szekeres. *The American Mathematical Monthly*, 78(3):p. 273, 1971.

[4] Thierry Coquand. A direct proof of Ramsey's Theorem. Author's website, revised in 2011, 1994.

[5] Ulrich Kohlenbach. On the no-counterexample interpretation. *J. Symb. Log.*, 64(4):1491–1511, 1999.

[6] Paulo Oliva and Thomas Powell. A Constructive Interpretation of Ramsey's Theorem via the Product of Selection Functions. *Mathematical Structures in Computer Science*. to appear.

[7] Andreas Podelski and Andrey Rybalchenko. Transition invariants. In *LICS*, pages 32–41, 2004.

[8] Helmut Schwichtenberg. On bar recursion of types 0 and 1. *J. Symb. Log.*, 44(3):325–329, 1979.

[9] Dimitrios Vytiniotis, Thierry Coquand, and David Wahlstedt. Stop when you are almost-full - adventures in constructive termination. In *ITP*, pages 250–265, 2012.