Polymorphic Functions with Set-Theoretic Types - Part 1: Syntax, Semantics, and Evaluation

This is a pre print version of the following article:

Original Citation:

Availability:
This version is available http://hdl.handle.net/2318/153599 since 2017-11-17T20:42:54Z

Published version:
DOI:10.1145/2578855.2535840

Terms of use:
Open Access
Anyone can freely access the full text of works made available as "Open Access". Works made available under a Creative Commons license can be used according to the terms and conditions of said license. Use of all other works requires consent of the right holder (author or publisher) if not exempted from copyright protection by the applicable law.

(Article begins on next page)
Polymorphic Functions with Set-Theoretic Types
Part 1: Syntax, Semantics, and Evaluation

Giuseppe Castagna1  Kim Nguyễn2  Zhiwu Xu1,3  Hyeonseung Im2  Serguei Lenglet4  Luca Padovani5
1CNRS, PPS, Univ Paris Diderot, Sorbonne Paris Cité, Paris, France  2LRI, Université Paris-Sud, Orsay, France
3State Key Laboratory of Computer Science, Institute of Software, Chinese Academy of Sciences, Beijing, China
4LORIA, Université de Lorraine, Nancy, France  5Dipartimento di Informatica, Università di Torino, Italy

Abstract. This article is the first part of a two articles series about a calculus with higher-order polymorphic functions, recursive types with arrow and product type constructor and set-theoretic type connectives (union, intersection, and negation). In this first part we define and study the explicitly-typed version of the calculus in which type instantiation is driven by explicit instantiation annotations. In particular, we define an explicitly-typed λ-calculus with intersection types and an efficient evaluation model for it. In the second part, presented in a companion paper, we define a local type inference system that allows the programmer to omit explicit instantiation annotations, and a type reconstruction system that allows the programmer to omit explicit type annotations. The work presented in the two articles provides the theoretical foundations and technical machinery needed to design and implement higher-order polymorphic functional languages for semi-structured data.

Categories and Subject Descriptors D.3.3 [Programming Languages]: Language Constructs and Features—Polymorphism

Keywords Types, polymorphism, XML, intersection types

1. Introduction

The extensible markup language XML is a current standard format for exchanging structured data. Many recent XML processing languages, such as XDuce [14], CDuce [2], XQuery [3], Ocaml-Duce [11], XHaskell [17], XACT [15], are statically-typed functional languages. However, parametric polymorphism, an essential feature of such languages, is still missing, or when present it is in a limited form (no higher-order functions, no polymorphism for XML types, and so on). Polymorphism for XML has repeatedly been requested to and discussed in various working groups of standards (eg, RELAX NG [7]) and higher-order functions have been recently proposed in the W3C draft for XQuery 3.0 [10]. Despite all this interest, spurs, and motivations, a comprehensive polymorphic type system for XML was still missing for the simple reason that, until recently, it was deemed unfeasible. A major stumbling block to this research —ie, the definition of a subtyping relation for regular tree types with type variables— has been recently lifted by Castagna and Xu [6], who defined and studied a polymorphic subtyping relation for a type system with recursive, product, and arrow types and set-theoretic type connectives (union, intersection, and negation).

In this work we present the next logical step of that research, that is, the definition of a higher-order functional language that takes full advantage of the new capabilities of Castagna and Xu’s system. In other words, we define and study a calculus with higher-order polymorphic functions and recursive types with union, intersection, and negation connectives. The approach is thus general and, as such, goes well beyond the simple application to XML processing languages. As a matter of facts, our motivating example developed all along this paper does not involve XML, but looks like a rather classic display of functional programming specimens:

map :: (α -> β) -> [α] -> [β]
map f l = case l of
| [] -> []
| (x : xs) -> (f x : map f xs)

even :: (Int -> Bool) ∧ ((α \ Int) -> (α \ Int))
even x = case x of
| Int -> (x ‘mod’ 2) == 0
| _ -> x

The first function is the classic map function defined in Haskell (we just used Greek letters to denote type variables). The second would be an Haskell function were it not for two oddities: its type contains type connectives (type intersection “∧” and type difference “\”); and the pattern in the case expression is a type, meaning that it matches all values returned by the matched expression that have that type. So what does the even function do? It checks whether its argument is an integer, if it is so it returns whether the integer is even or not, otherwise it returns its argument as it received it (although the definition of even may be considered weird, it’s a perfect minimal example to illustrate all the aspects of our system).

The goal of this work is to define a calculus and a type system that can pass three tests. The first test is that it can define the two functions above. The second, harder, test is that the type system must be able to verify that these functions have the types declared in their signatures. That map has the declared type will come as no surprise (in practice, in the second part of this work we show that in the absence of a signature given by the programmer the system can reconstruct a type slightly more precise than this [5]). That even was given an intersection type means that it must have all the types that form the intersection. So it must be a function that when applied to an integer it returns a Boolean and that when applied to a result of the same type. In other terms, even is a polymorphic (dynamically bounded) overloaded function.

The third test, the hardest one, is that the type system must be able to infer the type of the partial application of map to even, and the inferred type must be equivalent to the following one

map even :: ([Int] -> [Bool]) ∧ ((α \ Int) -> (α \ Int)) ∧ ((α \ Int) -> [(α \ Int) \ vBool])

1 This type is redundant since the first type of the intersection is an instance (eg, for α=Int) of the third. We included it for the sake of the presentation.
since map even returns a function that when applied to a list of integers it returns a list of Booleans; when applied to a list that does not contain any integer then it returns a list of the same type (actually, the same list); and when it is applied to a list that may contain some integers (e.g., a list of reals), then it returns a list of the same type, without the integers but with some Booleans instead (in the case of reals, a list with Booleans and reals that are not integers).

Technically speaking, the definition of such a calculus and its type system is difficult for two distinct reasons. First, for the reasons we explain in the next section, it demands to define an explicitly typed \( \lambda \)-calculus with intersection types, a task that, despite many attempts in the last 20 years, still lacked a satisfactory definition. Second, even if working with an explicitly typed setting may seem simpler, the system needs to solve “local type inference”\(^5\); namely, the problem of checking whether the types of a function and of its argument can be made compatible and, if so, of inferring the type of their result as we did for (1). The difficulty, once more, mainly resides in the presence of the intersection types: a term can be given different types either by subsumption (the term is coerced into a super-type of its type) or by instantiation (the term is used as a particular instance of its polymorphic type) and it is typed by the intersection of all these types. Therefore, in this setting, the problem is not just to find a substitution that unifies the domain type of the function with the type of its argument but, rather, a set of substitutions that produce instances whose intersections are in the right subtyping relation: our map even example should already have given a rough idea of how difficult this is.

The presentation of our work is split in two parts, accordingly: in the first part (this paper) we show how to solve the problem of defining an explicitly-typed \( \lambda \)-calculus with intersection types and how to efficiently evaluate it; in the second part (the companion paper [5]) we will show how to solve the problem of “local type inference” for a calculus with intersection types. In the next section we outline the various problems we met (focusing on those that concern the part of the work presented in this paper) and how they were solved.

2. Problems and overview of the solution

The driver of this work is the definition of an XML processing functional language with high-order polymorphic functions, that is, in particular, a polymorphic version of the language CDuce [2].

CDuce in a nutshell. The essence of CDuce is a \( \lambda \)-calculus with pairs, explicitly-typed recursive functions, and a type-case expression. Types can be recursively defined and include the arrow and product type constructors and the intersection, union, and negation type connectives. In summary, they are the regular trees coinductively generated by the following productions:

\[
t ::= b \mid t \rightarrow t \mid t \times t \mid t \mathbin{\&} t \mid t \mathbin{\|} t \mid \neg t \mid 0 \mid 1
\] (2)

where \( b \) ranges over basic types (e.g., Int, Bool) and 0 and 1 respectively denote the empty (that types no value) and top (that types all values) types. We use possibly indexed meta-variables \( s \) and \( t \) to range over types. Coinduction accounts for recursive types. We use the standard convention that infix connectives have a priority higher than constructors and lower than prefix connectives.

From a strictly practical viewpoint, recursive types, products, and type connectives are used to encode regular tree types, which subsume existing XML schema/types while, for what concerns expressions, the type-case is an abstraction of CDuce pattern match-

\(^5\)There are different definitions for local type inference. Here we use it with the meaning of finding the type of an expression in which not all type annotations are specified. This is the acceptance used in Scala where type parameters for polymorphic methods can be omitted. In our specific problem, we will omit—and, thus, infer—the annotations that specify how function and argument types can be made compatible.

ing (this uses regular expression patterns on types to define powerful and highly optimized capture primitives for XML data). We initially focus on the functional core and disregard products and recursive functions since the results presented here can be easily extended to them (we show it in the Appendix), though we will freely use them for our examples. So we initially consider the following “CoreCDuce” terms:

\[
e ::= e \mid x \mid e \mathbin{\&} t \mathbin{\rightarrow} (t \times t) \mathbin{\&} x \mathbin{\&} e \mid e \mathbin{\&} t \mathbin{\rightarrow} e
\] (3)

where \( e \) ranges over constants (e.g., true, false, 1, 2, ...) which are values of basic types (we use \( b \) to denote the basic type of the constant \( e \)); \( x \) ranges over expression variables; \( e \mathbin{\&} t \mathbin{\rightarrow} e \) denotes the type-case expression that evaluates either \( e_1 \) or \( e_2 \) according to whether the value returned by \( e \) (if any) is of type \( t \) or not; \( \lambda x.\mathbin{\&} t \mathbin{\rightarrow} t \mathbin{\&} x \mathbin{\&} e \) is a value of type \( \lambda x.\mathbin{\&} t \mathbin{\rightarrow} t \mathbin{\&} e \) that denotes the function of parameter \( x \) and body \( e \).

In this work we show how to define the polymorphic extension of this calculus, which can then be easily extended to a full-fledged polymorphic functional language for processing XML documents. But before let us explain the two specificities of the terms in (3), namely, why a type-case expression is included and why we explicitly annotate whole \( \lambda \)-abstractions (with an intersection of arrow types) rather than just their parameters.

The reason of inclusion of a type-case is twofold. First, a natural application of intersection types is to type overloaded functions, and without a type-case only “coherent overloading” à la Forsythe [21] can be defined (which, for example, precludes in our setting the definition of a —non diverging— function of type, say, \( (\text{Int} \rightarrow \text{Bool}) \mathbin{\&} (\text{Bool} \rightarrow \text{Int}) \)). The second motivation derives from the way arrow types are interpreted in [6, 12]. In particular, for any types \( s_1, s_2, t_1, t_2 \) the following containment is, in general, strict:

\[
s_1 \mathbin{\&} s_2 \rightarrow t_1 \mathbin{\&} t_2 \subset (s_1 \mathbin{\&} s_2 \rightarrow t_1 \mathbin{\&} t_2)
\] (4)

so there is a function in the type on the right that is not in the type of the left. Notice that from a typing viewpoint the functions on the left do not distinguish inputs of \( s_1 \) and \( s_2 \) types, while the ones on the right do. So the interpretation of types naturally induces the definition of functions that can distinguish inputs of different types \( s_1 \) and \( s_2 \), whatever \( s_1 \) and \( s_2 \) are. Actually this second motivation is just a different facet of the full-fledged vs. only “coherent” overloading motivation, since the functions that are in the difference of the two types in (4) are also those that make the difference between coherent and non coherent overloading. Both arguments, thus, advocate for “real” overloaded functions, that execute different code according to the type of their input, whence the need of type-case. Therefore our terms include a type-case.

The need of explicitly typed functions is a direct consequence of the introduction of the type-case, because without explicit typing we can run into paradoxes such as the following recursively defined (constant) function

\[
\mu f.\lambda x.f((1 \rightarrow \text{Int})? \mathbin{\&} \text{true} : 42)
\] (5)

This function has type \( 1 \rightarrow \text{Int} \) if and only if it does not have type \( 1 \rightarrow \text{Int} \). In order to decide whether the function above is well-typed or not, we must explicitly give a type to it. For instance, the function in (5) is well-typed if it is explicitly assigned the type \( 1 \rightarrow \text{Int} \mathbin{\&} \text{Bool} \). This shows both that functions must be explicitly typed and that specifying not only the type of parameters but also the type of the result is strictly more expressive, as more terms can be typed. As a matter of fact, if we provide just the type of the parameter \( x \) (not used in the body), then there is no type (apart from the useless \( 0 \) type) that makes (5) typeable.

In summary, we need to define an explicitly typed language with intersection types. This is a difficult problem for which no full-fledged solution existed, yet: there exist only few intersection type systems with explicitly typed terms, and none of them is completely...
that when applied to an \( \text{Int} \) returns \text{true} and returns 42 otherwise. Intuitively, we want to assign to this function the type \((\text{Int} \to \text{Bool}) \land (\neg \text{Int} \to \text{Int})\), the type of a function that when applied to an \text{Int}, returns a \text{Bool}, and when applied to a value which is not an \text{Int}, returns an \text{Int}. For the sake of presentation, let us say that we are happy to deduce for the function above the less precise type \((\text{Int} \to \text{Bool}) \land (\text{Bool} \to \text{Int})\) (which is a super-type of the former since if a function maps anything that is not an \text{Int} into an \text{Int}—it has type \(\neg \text{Int} \to \text{Int}—\), then in particular it maps \text{Booleans} to integers—i.e., it has also type \text{Bool} \to \text{Int}). The problem is to determine which type we should use for \(t\) in equation (6). If we use, say, \(\text{Int} \lor \text{Bool}\), then under the hypothesis that \(x: \text{Int} \lor \text{Bool}\) the type deduced for the body of the function is \(\text{Int} \lor \text{Bool}\). So the best type we can give to the function in (6) is \(\text{Int} \lor \text{Bool} \to \text{Int} \lor \text{Bool}\) which is far less precise than the sought intersection type, insofar as it does not make any distinction between arguments of type \text{Int} and those of type \text{Bool}.

The solution, which was introduced by Coquand, is to explicitly type—by an intersection type—whole \(\lambda\)-abstractions instead of just their parameters:

\[
\lambda (\text{Int} \lor \text{Bool}) \land (\text{Bool} \to \text{Int}). \quad (x \in \text{Int} \lor \text{true} : 42)
\]

In doing so we also explicitly define the result type of functions which, as we have just seen, increases the expressiveness of the calculus. Thus the general form of \(\lambda\)-abstractions is, as stated by the grammar in (3), \(\lambda^{\alpha} \epsilon_{\Gamma} x^\beta : \gamma. e\). Such a term is well typed if for all \(i \in I\) from the hypothesis that \(x\) has type \(s_i\), it is possible to deduce that \(e\) has type \(t_i\). Unfortunately, with polymorphic types, this simple solution introduced by Coquand no longer suffices.

**Polymorphic extension.** The novelty of this work is to allow type variables (ranged over by lower-case Greek letters: \(\alpha, \beta, \ldots\)) to occur in the types in (2) and, thus, in the types labeling \(\lambda\)-abstractions in (3). It becomes thus possible to define the polymorphic identity function as \(\lambda^{\alpha \rightarrow \alpha} x. x\), while classic "auto-application" term can be written as \(\lambda^{(\alpha \rightarrow \beta) \rightarrow \gamma} x. x\). The intended meaning of using a type variable, such as \(\alpha\), is that a (well-typed) \(\lambda\)-abstraction not only has the type specified in its label (and by subsumption all its super-types) but also all the types obtained by instantiating the type variables occurring in the label. So \(\lambda^{\alpha} x. x\) has not only type \(\alpha \rightarrow \alpha\) but also, for example, by subsumption the types \(\emptyset \rightarrow \emptyset\) (the type of all functions, which is a super-type of \(\alpha \rightarrow \alpha\)) and \(\neg \text{Int}\) (since every well-typed \(\lambda\)-abstraction is not an integer, then \(\neg \text{Int}\) contains \(-\text{true}\) or is a super-type of \(-\text{all function types}\)), and by instantiation the types \(\text{Int} \to \text{Int}, \text{Bool} \to \text{Int}\), etc.

The use of instantiation in combination with intersection types is complex and surprising, for if a term has two distinct types, then it has also their intersection type (eg. \(\lambda^{\alpha \rightarrow \alpha} x. x\) has type \((\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool}) \land \neg \text{Int}\)). In the monomorphic case a term can have distinct types only by subsumption and, thus, intersection types are transparently assigned to terms via subsumption. In the polymorphic case this is no longer possible: a term can be typed by the intersection of two distinct instances of its polymorphic type which, in general, are not in any subtyping relation with the latter: for instance, \(\alpha \rightarrow \alpha\) is neither a subtype of \(\text{Int} \to \text{Int}\) nor vice versa, since the subtyping relation must hold for all possible instantiations of \(\alpha\) and there are infinitely many instances of \(\alpha \rightarrow \alpha\) that are neither a subtype nor a super-type of \(\text{Int} \to \text{Int}\).

**Explicit instantiation.** Concretely, if we want to apply the polymorphic identity \(\lambda^{\alpha \rightarrow \alpha} x. x\) to, say, 42, then the particular instance obtained by the type-substitution \(\{\text{Int}\}_{\alpha}\) (denoting the replacement of every occurrence of \(\alpha\) by \text{Int}) must be used, that is \((\lambda^{\alpha \rightarrow \alpha} x. x)_{\text{Int}} 42\). We have thus to relabel the type decorations of \(\lambda\)-abstractions before applying them. In implicitly typed languages, such as ML, the relabeling is meaningless (no type decoration is used in terms) while in their explicitly-typed counterparts relabeling can be seen as a logically meaningful but computationally useless operation, insofar as execution takes place on type erasures (i.e., the terms obtained by erasing all type decorations). In the presence of type-case expressions, however, relabeling is necessary since the label of a \(\lambda\)-abstraction determines its type: testing whether an expression has type, say, \(\text{Int} \to \text{Int}\) should succeed for the application of \(\lambda^{\alpha \rightarrow \alpha} \epsilon_{\Gamma} x.y\) to 42 and fail for its application to \text{true}. In practice, we have that

\[
\lambda^{\alpha \rightarrow \alpha} x. \lambda^{\alpha \rightarrow \alpha} y.x \quad 42 \in \text{Int} \to \text{Int} \quad 0 : 1
\]

must reduce to \(\lambda^{\text{Int} \to \text{Int}} \epsilon_{\text{Int} \to \text{Int}} y. 42 \in \text{Int} \to \text{Int} \quad 0 : 1\) and thus to 0, while

\[
\lambda^{\alpha \rightarrow \alpha} x. \lambda^{\alpha \rightarrow \alpha} y.x \quad \text{true} \in \text{Int} \to \text{Int} \quad 0 : 1\]

must reduce to \(\lambda^{\text{Bool} \to \text{Bool}} \epsilon_{\text{Int} \to \text{Int}} y. \text{true} \in \text{Int} \to \text{Int} \quad 0 : 1\) and thus to 1. This means that, in Reynolds’s terminology, our terms have an *intrinsic* meaning [22], that is to say, the semantics of a term depends on its typing.

If we need to relabel some function, then it may be necessary to relabel also its body as witnessed by the following "daffy"—though well-typed—definition of the identity function:

\[
\lambda^{\alpha \rightarrow \alpha} x. (\lambda^{\alpha \rightarrow \alpha} y.x) x)
\]

While we have to apply this function to, say, 3, then we have first to relabel it by applying the substitution \(\{\text{Int}\}_{\alpha}\). However, applying the relabeling only to the outer "\(\lambda\)" does not suffice since the application of (7) to 3 reduces to \((\lambda^{\alpha \rightarrow \alpha} y.3)3\) which is not well-typed (it is not possible to deduce the type \(\alpha \rightarrow \alpha\) for \(\lambda^{\alpha \rightarrow \alpha} y.3\), which is the constant function that always returns 3) although it is the reductum of a well-typed application.

The solution is to apply the relabeling also to the body of the function. Here what "to relabel the body" means is straightforward: apply the same type-substitution \(\{\text{Int}\}_{\alpha}\) to the body. This yields a reductum \((\lambda^{\text{Int} \to \text{Int}} y.3) 3\) which is well typed. In general, however, the way to perform a relabeling of the body of a function is not so straightforward and clearly defined, since two different problems may arise: (i) it may be necessary to apply more than a single type-substitution and (ii) the relabeling of the body may depend on the dynamic type of the actual argument of the function (both problems are better known as—or are instances of—the problem of determining expansions for intersection type systems [8]). Next, we discuss each problem in detail.

**Multiple substitutions.** First of all, notice that we may need to relabel/instantiate functions not only when they are applied but also when they are used as arguments. For instance, consider a function that expects arguments of type \(\text{Int} \to \text{Int}\). It is clear that we can apply it to the identity function \(\lambda^{\alpha \rightarrow \alpha} x. x\), since the identity function has type \(\text{Int} \to \text{Int}\) (feed it by an integer and it will return an integer). Before, though, we have to relabel the term by the substitution \(\{\text{Int}\}_{\alpha}\) yielding \(\lambda^{\text{Int} \to \text{Int}} x.x\). As the identity \(\lambda^{\alpha \rightarrow \alpha} x.x\) has type \(\text{Int} \to \text{Int}\), so it has type \(\text{Bool} \to \text{Bool}\). Therefore, the intersection of the two: \((\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool})\). So we can apply a function that expects an argument of this intersection type to our identity function. The problem is now how to relabel \(\lambda^{\alpha \rightarrow \alpha} x.x\). Intuitively, we have to apply two distinct type-substitutions \(\{\text{Int}\}_{\alpha}\) and \(\{\text{Bool}\}_{\alpha}\) to the label of the \(\lambda\)-abstraction.
and replace it by the intersection of the two instances. This corresponds to relabel the polymorphic identity from $\lambda^\alpha\to\alpha.x.x$ into $(\lambda^{[\alpha/\alpha]}\int\to\int)\land(\lambda^{\beta/\beta}bool\to bool).x.x$. This is the solution adopted by this work, where we manipulate sets of type-substitutions — delimited by square brackets. The application of such a set (eg, in the previous example $\{[\alpha/\alpha]\int\to\int, [\beta/\beta]bool\to bool\}$) to a type $t$ returns the intersection of all types obtained by applying each substitution in the set to $t$ (eg, in the example $\{[\alpha/\alpha]\int\to\int \land [\beta/\beta]bool\to bool\}$). Thus the first problem has an easy solution.

**Relabeling of function bodies.** The second problem is much harder and concerns the relabeling of the body of a function. While the naive solution consists of propagating the application of type-substitutions to the bodies of functions works for single type-substitutions, in general, it fails for sets of type-substitutions. This can be seen by considering the relabeling via the set of type-substitutions $\{[\alpha/\alpha]\int\to\int, [\beta/\beta]bool\to bool\}$ of the daffy function in (7). If we apply the naive solution, this yields

$$\lambda^\alpha\to\alpha.(\lambda^\alpha\to\int \land (\lambda^\beta\to bool).y.x)y.x$$

(8)

which is not well typed. That this term is not well typed is clear if we try applying it to, say, $\lambda^\alpha\int\to\int$. The application of a function of type $(\lambda^\alpha\to\int\land(\lambda^\beta\to bool).y.x)y.x$ to an Int should have type $\text{Int}$, but here it reduces to $(\lambda^{[\alpha/\alpha]}/\lambda^{[\beta/\beta]}).y.x$, and there is no way to deduce the intersection type $(\lambda^\int\to\int)\land(\lambda^\beta\to bool)$ for the constant function $\lambda y.x$. But we can also directly verify that it is not well typed, by trying typing the function in (8). This corresponds to prove that under the hypothesis $x : \beta\text{Int}$ the term $(\lambda^\beta\to\int\land(\lambda^\beta\to bool).y.x)y.x$ has type $\beta\text{Int}$, and that under the hypothesis $x : \beta\text{Bool}$ this same term has type $\beta\text{Bool}$. Both checks fail because, in both cases, $(\lambda^\beta\to\int\land(\lambda^\beta\to bool).y.x)y.x$ is ill-typed (it neither has type $\beta\text{Int}$ when $x:\text{Bool}$, nor has it type $\beta\text{Bool}$ when $x:\int$). This example shows that in order to ensure that relabeling yields well-typed terms, the relabeling of the body must change according to the type of the value the parameter $x$ is bound to. More precisely, $(\lambda^\alpha\to\alpha.y.x)$ should be relabeled as $(\lambda^\alpha\to\int.y.x)$ when $x$ is of type $\int$, and as $(\lambda^\beta\to\int.y.x)$ when $x$ is of type $\beta\text{Bool}$. An example of this same problem less artificial than our daffy function is given by the classic apply function $\lambda f.x.f x$ which, with our polymorphic type annotations, is written as:

$$\lambda^{(\alpha\to\beta)}\to\beta\to\beta.\lambda^{\alpha\to\beta}f.(\lambda^{\alpha\to\beta}x.f x)$$

(9)

The apply function in (9) has type $(\text{Int}\to\text{Int})\to\text{Int}\to\text{Int}$, obtained by instantiating its type annotation by the substitution $\{[\alpha/\alpha]\int\to\int, [\beta/\beta]\text{Int}\to\int\}$, as well as type $(\text{Bool}\to\text{Bool})\to\text{Bool}\to\text{Bool}$, obtained by the substitution $\{[\alpha/\alpha]\text{Int}, [\beta/\beta]\text{Bool}\}$. If we want to feed this function to another function that expects arguments whose type is the intersections of these two types, then we have to relabel it by using the set of type-substitutions $\{[\alpha/\alpha]\int\to\int, [\beta/\beta]\text{Bool}\to\text{Bool}\}$. But, once more, it is easy to verify that the naive solution that consists in propagating the application of the set of type-substitutions down to the body of the function yields an ill-typed expression.

This second problem is the showstopper for the definition of an explicitly typed $\lambda$-calculus with intersection types. Most of the solutions found in the literature [4, 16, 23, 26] rely on the duplication of lambda terms and/or typing derivations, while other calculi such as [27] that aim at avoiding such duplication obtain it by adding new expressions and new syntax for types (see related work in Section 7); but none of them is able to produce an explicitly-typed $\lambda$-calculus with intersection types, as we do, by just adding annotations to $\lambda$-abstractions.

**Our solution.** Here we introduce a new technique that consists in performing a “lazy” relabeling of the bodies. This is obtained by decorating $\lambda$-abstractions by (sets of) type-substitutions. For example, in order to pass our daffy identity function (7) to a function that expects arguments of type $(\text{Int}\to\text{Int})\land(\text{Bool}\to\text{Bool})$, we first “lazily” relabel it as follows:

$$(\lambda^{[\alpha/\alpha]}\int\to\int, [\beta/\beta]\text{Bool}\to\text{Bool}).x.(\lambda^\alpha\to\beta.y.x)x)$$

(10)

The new annotation in the outer “$\lambda^\alpha$” indicates that the function must be relabeled and, therefore, that we are using the particular instance whose type is the one in the interface (ie, $\alpha\to\alpha$) to which we apply the set of type-substitutions. The relabeling will be actually propagated to the body of the function at the moment of the reduction, only if and when the function is applied (relabeling is thus lazy). However, the new annotation is statically used by the type system to check soundness. Notice that, unlike existing solutions, we preserve the structure of $\lambda$-terms (at the expenses of some extra annotation that is propagated during the reduction) which is of the utmost importance in a language-oriented study.

In this paper we focus on the study of the calculus with these “lazy” type-substitutions annotations. We temporarily avoid the problem of local type inference by defining a calculus with explicit sets of type substitutions: expressions will be explicitly annotated with appropriate sets of type-substitutions.

**Polymorphic Cduce.** From a practical point of view, however, it is important to stress that, at the end, these annotations will be invisible to the programmer and, as show in the second part presented in the companion paper [5], all the necessary type-substitutions will be inferred statically. In practice, the programmer will program in the language defined by grammar (3), but where the types that annotate $\lambda$’s may contain type variables, that is, the polymorphic version of Cduce. The problem of inferring explicit sets of type-substitutions to annotate the polymorphic version of the expressions in (3) is the topic of the second part of this work presented in the companion paper [5]. For the time being, simply notice that the language defined by (3) and extended with type variables passes our first test inasmuch as the $\text{even}$ function can be defined as follows (where $s.t$ is syntactic sugar for $s\land\lnot t$):

$$\lambda^{\alpha\to\beta\to\beta\to\beta}.x.x$$

(11)

while — with the products and recursive functions described in the Appendix — map is defined as (see also discussion in Appendix E)

$$\mu^{\alpha\to\beta\to\beta}.x.x$$

(12)

where the type $\text{nil}$ in the case-case denotes the singleton type that contains just the constant $\text{nil}$, and $[\alpha]$ denotes the regular type that is the (least) solution of $X = (\alpha, X) \lor \text{nil}$. When fed by any expression of this language, the type inference system defined in the companion paper [5] will infer sets of type-substitutions and insert them into the expression to make it well typed (if possible, of course). For example, for the application of the (terms defining) map to $\text{even}$, the inference system of the companion paper [5] infers the following set of type-substitutions $\{[\alpha\int\to\int, (\alpha\int\to\int)]\land(\alpha\int\to\int)/\text{Bool}\land\text{Bool}\}$ and textually inserts it between the two terms (so that the type-substitutions apply to the type variables of map) yielding the typing in (1). Finally, as we explain in Section 5.3 later on, the compiler will compile the expression into an expression of an intermediate language that can be evaluated as efficiently as the monomorphic calculus.

**Outline.** The rest of the presentation proceeds as follows. In Section 3 we define and study our calculus with explicit type-substitutions: we define its syntax, its operational semantics, and its type system; we prove that the type system is sound and subsumes classic intersection type systems. In Section 4 we define an algorithm for type inference and prove that it is sound, complete, and terminating. In Section 5 we show that the addition of type-substitutions has in practice no impact on the efficiency of the evaluation since the calculus can be compiled into an intermediate language that executes as efficiently as monomorphic Cduce.
Section 7 presents related work and in Section 8 we conclude our presentation.

In the rest of the presentation we will focus on the intuition and try to avoid as many technical details as possible. We dot the i’s and cross the t’s in the Appendix, where all formal definitions and complete proofs of properties can be found (n.b.: references in the text starting by capital letters —eg, Definition A.7— refer to this appendix). All these as well as other details can also be found in the third author’s PhD thesis manuscript [28].

Contributions. The overall contribution of our work is the definition of a statically-typed core language with (i) polymorphic higher-order functions for a type system with recursive types and union, intersection, and negation type connectives, (ii) an efficient evaluation model, (iii) local type inference for application, and (iv) a limited form of type reconstruction.

The main contribution of this first part of the work is the definition of an explicitly-typed λ-calculus (actually, a family of calculi) with intersection (and union and negation) types and of its efficient evaluation via the compilation into an intermediate language. From a syntactic viewpoint our solution is a minimal extension since it just requires to add annotations to λ-abstractions of the untyped λ-calculus (cf. Section 3.5). Although this problem has been studied for over 20 years, no existing solution proposes such a minimal extension, which is of paramount importance in a programming language-oriented study (see related works in Section 7).

The technical contributions are the definition of an explicitly typed calculus with intersection types; the proof that it subsumes existing intersection type systems; the soundness of its type system, the definition of a sound, complete and terminating algorithm for type inference (which as byproduct yields an intersection type proof term), the soundness of its type system, and the proof of the adequacy of the compilation. Local type inference for application and type reconstruction are studied in the second part of this work presented in the companion paper [5].

3. A calculus with explicit type-substitutions

The types of the calculus are those in the grammar (2) to which we add type variables ( ranged over by α) and, for the sake of presentation, stripped of product types. In summary, types are the regular trees coinductively generated by

\[ t ::= \alpha | \beta | \text{true} | \text{false} \]

and such that every infinite branch contains infinitely many occurrences of type constructors. We use \( \mathcal{F} \) to denote the set of all types.

The condition on infinite branches bars out ill-formed types such as \( t = \text{true} \lor t \) (which does not carry any information about the set denoted by the type) or \( t = \neg t \) (which cannot represent any set). It also ensures that the binary relation \( \rightarrow \subseteq \mathcal{F} \times \mathcal{F} \) defined by

\[ t_1 \lor t_2 \rightarrow t_1, t_1 \land t_2 \rightarrow t_1, \neg t \rightarrow t, t \rightarrow \text{Noetherian} \] (that is, strongly Noetherian). This gives an induction principle on \( \mathcal{F} \) that we will use without any further explicit reference to the relation. We use \( \text{var}(t) \) to denote the set of type variables occurring in a type \( t \) (see Definition A.2). A type \( t \) is said to be ground or closed if and only if \( \text{var}(t) \) is empty.

The subtyping relation for the types in \( \mathcal{F} \) is the one defined by Castagna and Xu [6]. For this work it suffices to consider that ground types are interpreted as sets of values (n.b., just values, not expressions) that have that type and subtyping is set containment (ie, a ground type \( s \) is a subtype of a ground type \( t \) if and only if \( t \) contains all the values of type \( s \)). In particular, \( s \rightarrow t \) contains all λ-abstractions that when applied to a value of type \( s \), if they return a result, then this result is of type \( t \) (so \( 0 \rightarrow 1 \) is the set of all functions and \( 1 \rightarrow 0 \) is the set of functions that diverge on all arguments). Type connectives (union, intersection, negation) are interpreted as the corresponding set-theoretic operators and subtyping is set containment. For what concerns non-ground types (ie, types with variables occurring in them) all the reader needs to know for this work is that the subtyping relation of Castagna and Xu is preserved by type-substitutions. Namely, if \( s \leq t \), then \( s \sigma \leq t \sigma \) for every type-substitution \( \sigma \) (the converse does not hold in general, while it holds for semantic type-substitutions in convex models: see [6]). Two types are equivalent if they denote the same set of values, that is, if they are subtype one of each other (type equivalence is denoted by \( \equiv \)). An important property of this system we will often use is that every type is equivalent to (and can be effectively transformed into) a type in disjunctive normal form, that is, a union of uniform intersections of literals. A literal is either an arrow, or a basic type, or a type variable, or their negations. An intersection is uniform if all the literals have the same constructor, that is, either it is an intersection of arrows, type variables, and their negations or it is an intersection of basic types, type variables, and their negations.

In summary, a disjunctive normal form is a union of summands whose form is either

\[ \bigwedge_{p \in P} \left( b_p \land \bigwedge_{w \in N} \neg b_w \land \bigwedge_{q \in P'} \alpha_q \land \bigwedge_{r \in N'} \neg \alpha_r \right) \]

or

\[ \bigwedge_{p \in P} \left( b_p \land \bigwedge_{w \in N} \neg b_w \land \bigwedge_{q \in P'} \alpha_q \land \bigwedge_{r \in N'} \neg \alpha_r \right) \]

When either \( P' \) or \( N' \) is not empty, we call the variables \( \alpha_q \)'s and \( \alpha_r \)'s the top-level variables of the normal form.

3.1 Expressions

Expressions are derived from those of CoreC:Duce (with type variables in types) with the addition that sets of explicit type-substitutions (ranged over by \( \sigma \)) may be applied to terms and decorate λ-abstractions

\[ e ::= c | e \mid e | \lambda^x^{\alpha} s.t. \rightarrow t.x.e | e \mid t ? e : e | e[\sigma] \]

and with the restriction that the type tested in type-case expressions is closed. Henceforth, given a λ-abstraction \( \lambda^{x} s.t. \rightarrow t.x.e \) we call the type \( \bigwedge_{i \in I} s_i \rightarrow t_i \) the interface of the function and the set of type-substitutions \( \sigma \) the decoration of the function. We write \( \lambda^{x} e \) for short when the decoration is a singleton containing the empty substitution. Let \( e \) be an expression. We use \( fe(e) \) and \( be(e) \) respectively to denote the sets of free expression variables and bound expression variables of the expression \( e \); we use \( tv(e) \) to denote the set of type variables occurring in \( e \) (see Definition A.8).

As customary, we assume bound expression variables to be pairwise distinct and distinct from any free expression variable occurring in the expressions under consideration. We equate expressions up to the α-renaming of their bound expression variables. In particular, when substituting an expression \( e \) for a variable \( y \) in an expression \( e' \) (see Definition A.10), we assume that the bound variables of \( e' \) are distinct from the bound and free variables of \( e \), to avoid unwanted captures. For example, \( (\lambda^{x} s.t. \rightarrow t.x.y) e \) is α-equivalent to \( (\lambda^{x} s.t. \rightarrow t.x.y) e \).

The situation is a bit more complex for type variables, as we do not have an explicit binder for them. Intuitively, a type variable can be α-converted if it is a polymorphic one, that is, if it can be instantiated. For example, \( (\lambda^{x} s.t. \rightarrow t.x.y) \) is α-equivalent to \( (\lambda^{x} s.t. \rightarrow t.x.y) \) and \( (\lambda^{x} s.t. \rightarrow t.x.y) \) is α-equivalent to \( (\lambda^{x} s.t. \rightarrow t.x.y) \). Polymorphic variables can be bound by interfaces, but also by decorations: for example, in \( \lambda^{x} (\text{of}(\alpha) \mid x) (\lambda^{x} s.t. \rightarrow t.x.y) x \), the α occurring in the interface of the inner abstraction is "bound" by the decoration \( [\alpha / \beta] \), and the whole expression is α-equivalent to...
\[ (\lambda^{\beta\to\gamma} \ y. y) x. \] If a type variable is bound by an outer abstraction, it cannot be instantiated; such a variable is called monomorphic. For example, the following expression
\[ \lambda^{\alpha\to\gamma} y. ((\lambda^{\beta\to\gamma} x. x)[[1\int\alpha \gamma]] y) \]
is not sound (\ie, it cannot be typed), because \( \alpha \) is bound at the level of the outer abstraction, not at level of the inner one. Consequently, in this expression, \( \alpha \) is monomorphic for the inner abstraction, but polymorphic for the outer one (strictly speaking, thus, the monomorphic and polymorphic adjectives apply to occurrences of variables rather than variables themselves). Monomorphic type variables cannot be \( \alpha \)-converted: \( \lambda^{\alpha\to\gamma} y. ((\lambda^{\alpha\to\gamma} x. x)[[1\int\alpha \gamma]] y) \)
is not equal to \( \lambda^{\alpha=\beta\to\gamma} y. ((\lambda^{\beta\to\gamma} x. x) y) \) (but it is\( \alpha \)-equivalent to \( \lambda^{\alpha=\beta\to\gamma} y. ((\lambda^{\beta=\gamma} x. x) y) \)). Note that the scope of polymorphic variables may include some type-substitutions \( \sigma_j \in \Sigma \): for example, \( (\lambda^{\alpha\to\gamma} x. x)[[\Sigma[\alpha]] y] \) is \( \alpha \)-equivalent to \( ((\lambda^{\beta\to\gamma} x. x) y)[[\Sigma[\beta/\gamma]] y] \). Finally, we have to be careful when performing expression substitutions and type-substitutions to avoid clashes of polymorphic variable namespaces. For example, substituting \( \lambda^{\alpha=\beta\to\gamma} z. z \) for \( y \) in \( \lambda^{\alpha=\beta\to\gamma} x. x \) would lead to an unwanted capture of \( \alpha \) (assuming \( \alpha \) is polymorphic, that is, not bound by a \( \lambda \)-abstraction placed above these two expressions), so we have to \( \alpha \)-convert one of them, so that the result of the substitution is, for instance, \( \lambda^{\alpha=\beta\to\gamma} x. x \) (\( \lambda^{\alpha=\beta\to\gamma} z. z \)).

To resume, we assume polymorphic variables to be pairwise distinct and distinct from any monomorphic variable in the expressions under consideration. We equate expressions up to \( \alpha \)-renaming of their polymorphic variables, and push \( \lambda \) abstractions to all outermost \( \lambda \)-abstractions occurring in \( e \) (and compiles and composes with the sets of type-substitutions it meets). Precisely, \( e[\sigma_j] \in \Sigma \) is defined for \( \lambda \)-abstractions (and (inductively) for applications of type-substitutions as:

\[ \Delta \cup \{ \lambda^{\alpha\to\gamma} e. x. e \in [\sigma_j] \in \Sigma \} \]

where \( \circ \) denotes the pairwise composition of all substitutions of the two sets (see Definition 3.1). It erases the set of type-substitutions when \( e \) is a variable and it is homomorphically applied on the remaining expressions (see Definition A.11).

Formally the composition of two sets of type-substitutions is defined as follows:

**Definition 3.1.** Given two sets of type-substitutions \( [\sigma_i] \in I \) and \( [\sigma_j] \in J \), we define their composition as

\[ [\sigma_i] \in I \circ [\sigma_j] \in J = [\sigma_i \circ \sigma_j] \in I, j \in J \]

where
\[ \sigma_i \circ \sigma_j (\alpha) = \begin{cases} (\sigma_i (\alpha)) & \text{if } \alpha \in \text{dom}(\sigma_i) \\ \sigma_j (\alpha) & \text{if } \alpha \notin \text{dom}(\sigma_i) \cap \text{dom}(\sigma_j) \\ \alpha & \text{otherwise} \end{cases} \]

Next, we formally define the relabeling of an expression \( e \) with a set of type-substitutions \( [\sigma_j] \in J \), which consists in propagating the \( \sigma_j \) to the \( \lambda \)-abstractions in \( e \) if needed. We suppose that the polymorphic type variables in \( e \) are distinct from the type variables in the range of \( \sigma_j \) (this is always possible by using \( \alpha \)-conversion).

**Definition 3.2 (Relabeling).** Given an expression \( e \) and a set of type-substitutions \( [\sigma_j] \in J \), we define the relabeling of \( e \) with \( [\sigma_j] \in J \), written \( e[\sigma_j] \in J \), as if \( e[\sigma_j] \in J \), and otherwise as follows:

\[ e[\sigma_j] \in J \]

where in (18) we have \( P \equiv \{ j \in J \mid \exists i \in I, t \vdash i : t \sigma_j \} \).

The first rule (17) performs relabeling, that is, it propagates the types of type-substitutions down into the decorations of the outermost \( \lambda \)-abstractions. The second rule (18) states the semantics of applications: this is standard call-by-value \( \beta \)-reduction, with the difference that the substitution of the argument for the parameter is performed on the relabeled body of the function. Notice that relabeling depends on the type of the argument and keeps only those substitutions that make the type of the argument \( v \) match (at least one of) the input types defined in the interface of the function (\ie, the set \( P \) which contains all substitutions \( \sigma_j \) such that the argument \( v \) has type \( t \sigma_j \), for some \( t \in I \); the system will ensure that \( P \) is never empty). For instance, take the daffy identity function, instantiate it as in (10) by both \( \int \) and \( \bool \), and apply it to \( 42 \), \( e \), \( \lambda^{\alpha=\beta\to\gamma} x. x \) \( e[\alpha=\beta\to\gamma] \), \( e[\alpha=\beta\to\gamma] \in J \), \( (\lambda^{\alpha=\beta\to\gamma} x. x)[[\Sigma[\alpha]] y. y] \), \( 42 \), which (observationally equivalent to \( (\lambda^{\alpha=\beta\to\gamma} y. y) 42 \)) since the reduction discards the \( \{ \bool \} \).
substitution. Finally, the third rule (19) checks whether the value returned by the expression in the type-case matches the specified type and selects the branch accordingly.

The reader may think that defining a functional language in which each \( \beta \)-reduction must perform an involved relabeling operation, theoretically interesting though it may be, will result in practice too costly and therefore unrealistic. This is not so. In Section 5 we show that this reduction can be implemented as efficiently as in C\( \text{C} \)Duce. By a smart definition of closures it is possible to compute relabeling in a lazy way and materialize it only in a very specific case for the reduction of the type-case (ie, to perform a type-case reduction (19) where the value \( v \) is a function whose interface contains monomorphic type variables and it is the result of the partial application of a polymorphic function) while all other reductions for applications can be implemented as plain classic \( \beta \)-reduction. For instance, to evaluate the expressions \( (\lambda y.x)x \) and \( (\lambda y.x)y \) as in (42) above, we can completely disregard all type annotations and decorations and perform a couple of standard \( \beta \) reductions that yield the result 42.

### 3.3 Type system

As expected in a calculus with a type-case expression, the dynamic semantics depends on the static semantics —precisely, on the typing of values. The static semantics of our calculus is defined in Figure 1. The judgments are of the form \( \Delta, \Gamma \vdash e : t \), where \( e \) is an expression, \( t \) a type, \( \Gamma \) a type environment (ie, a finite mapping from expression variables to types), and \( \Delta \) a finite set of type variables. The latter is the set of all monomorphic type variables; that is, the variables that occur in the type of some outer \( \lambda \)-abstraction and, as such, cannot be instantiated; it must contain all the type variables occurring in \( \Gamma \).

The rules for application and subsumption are standard. In the latter, the subtyping relation is the one defined in [6]. We just omitted the rule for constants (which states that \( e \) has type \( b \)).

The rule for abstractions applies each substitution specified in the decoration to each arrow type in the interface, adds all the variables occurring in these types to the set of monomorphic type variables \( \Delta \), and checks whether the function has all the resulting types. Namely, it checks that for every possible input type, the (relabeled) body has the corresponding output type. To that end, it applies each substitution \( \sigma_i \) in the decoration to each input type \( t_i \) of the interface and checks that, under the hypothesis that \( x \) has type \( t_1 \ldots t_r \sigma_i \), the function body relabeled with the substitution \( \sigma_i \) at issue has type \( s_i \sigma_i \). Notice that all these checks are performed under the same updated set of monomorphic type variables, that is, \( \Delta \cup \{ x \in J \mid t_i \sigma_i \rightarrow s_i \} \). If the test succeeds, then the rule infers for the function the type obtained by applying the set of substitutions of the decoration to the type of the interface. For example, in the case of the instance of the daffy identity function given in (10), the \( \Delta \) is always empty and the rule checks whether under the hypothesis \( x : \alpha \{ \text{Int} \} \) (ie, \( x : \text{Int} \)), it is possible to deduce that \( (\lambda x.\alpha x)y \text{Int}\{\text{Int}/\alpha\} \) has type \( \alpha \{ \text{Int}/\alpha \} \) (ie, that \( (\lambda x.\alpha x)y \text{Int} \) ). Notice that for the typing of the body \( \{ \text{Int}/\alpha \} \) this is necessary to the well-typing of the latter but not to that of the former. Indeed while under the hypothesis \( x : \alpha \{ \text{Int} \} \) we just showed that \( (\lambda x.\alpha x)y \{ \text{Int}/\alpha \} \) is well-typed, the term \( (\lambda x.\alpha x)y \{ \text{Int}/\alpha \} \) is not, for \( (\lambda x.\alpha x) \) does not have type \( \alpha \rightarrow \alpha \). The rule for abstractions also justifies the need for an explicit set \( \Delta \) for monomorphic type variables while, for instance, in ML it suffices to consider monomorphic type variables that occur in the image of \( \Gamma \) [20]; when checking an arrow of the interface of a function, the variables occurring in the other arrows must be considered monomorphic, too.

To type the applications of a set of type-substitutions to an expression, two different rules are used according to whether the set contains one or more than one substitution. When a single substitution is specified, the rule (inst) instantiates the type according to the specified substitution, provided that \( \sigma \) does not substitute variables in \( \Delta \) (ie, \( \text{dom}(\sigma) \cap \Delta = \emptyset \), noted \( \sigma \vdash \Delta \)). This condition is necessary to the soundness of the system. Without it an expression such as \( \lambda x.\beta .x.x \{ \beta/\alpha \} \) could be typed as follows:

\[
\begin{align*}
\{ \alpha, \beta \} & \vdash (x : \alpha) \rightarrow x : \alpha \\
\{ \alpha, \beta \} & \vdash \{x : \alpha \} \rightarrow \{ \{ \beta/\alpha \} \} : \beta \\
\vdash \lambda x.\beta .x.x \{ \{ \beta/\alpha \} \} : \alpha \rightarrow \beta
\end{align*}
\]

which is unsound (by applying the above functions to any value it is possible to create polymorphic values of type \( \beta \), for every \( \beta \)). If more than one substitution is specified, then the rule (inter) composes them by an intersection. Notice that the type system composes by intersection only different types of a same term obtained by instantiation. This is not restrictive since different types obtained by subsumption can be composed by intersection by applying just subsumption (see Lemma B.2).

Finally, the (case) rule first infers the type \( t \) of the expression whose type is tested. Then the type of each branch \( e_i \) is checked only if there is a chance that the branch can be selected. Here the use of “\( \not\in \)” is subtle but crucial (it allows us to existentially quantify over type-substitutions). The branch, say, \( e_1 \) can be selected (and therefore its well-typedness must be checked) only if \( e \) can return a value that is in \( t \). But in order to cover all possible cases we must also consider the case in which the type of \( e \) is instantiated as a consequence of an outer application. A typical usage pattern (followed also by our \( \text{even} \) function) is \( \lambda x.\beta .x.x \in \text{Int} ? e_1 : e_2 \); the branch \( e_1 \) is selected only if the function is applied to a value of type \( \text{Int} \), that is, if the type \( \alpha \) of \( e \) is instantiated to \( \text{Int} \) (notice that when typing the body of the function \( \Delta \) contains only \( \alpha \)). More generally, the branch \( e_1 \) in \( e \in \text{Int} ? e_1 : e_2 \) can be selected only if \( e \) can return a value in \( t \), that is to say, if there exists a substitution \( \sigma \) for any type variables even those in \( \Delta \) such as the intersection of \( t \) and \( t \) is not empty (\( t \) is a closed term, \( \sigma = t \)). Therefore, in order to achieve maximum precision must check \( \Delta \cup \{ \text{Int} \} \vdash e_1 : s \) only if there exists \( \sigma \) such that \( t \sigma \not\in \Delta \). Since \( t \not\in \Delta \) (strictly) implies that for all substitutions \( \sigma \), \( t \sigma \not\in \Delta \) (recall that \( t \) is a closed type), then by the contrapositive the existence of a substitution \( \sigma \) such that \( t \sigma \not\in \Delta \) implies \( t \not\in \Delta \). The latter is equivalent to \( t \not\in \Delta \): the intersection of \( t \) and \( t \) is not empty. So we slightly over-approximate the test of selection and check the type of \( e_1 \) under the weaker hypothesis \( t \not\in \Delta \) which ensures that the typing will hold also under the stronger (and sought) hypothesis that there exists \( \sigma \) such that \( t \sigma \not\in \Delta \) (the difference only matters with some specific cases involving \( \text{involutive} \) types: see [5]).

Notice that explicit type-substitutions are only needed to type applications of polymorphic functions. Since no such application occurs in the bodies of \( \text{map} \) and \( \text{even} \) as defined in Section 2 (the \( m \) and \( f \) inside the body of \( \text{map} \) are abstracted variables and, thus, have monomorphic types), then they can be typed by this system as they are (as long as they are not applied to one to the other there is no need to infer any set of type-substitutions). So we can already see that our language passes the second test, namely, that \( \text{map} \) and \( \text{even} \) have the types declared in their signatures. Let us detail just the most interesting case, that is, the typing of the term \( \text{even} \) defined in equation (11) (even though the typing of the type-case in (12), the term defining \( \text{map} \), is interesting, as well). According to
the rule (abstr) we have to check that under the hypothesis \( x \vdash \alpha \) the expression \( x \cdot \mathrm{Int} \ ? (x \mod 2) = 0 \) has type \( \text{Bool} \) and that under the hypothesis \( x \vdash \alpha \) the same expression has type \( \alpha \setminus \text{Int} \). So we have two distinct applications of the (case) rule. In one case, \( x \) is of type \( \text{Int} \), thus the check \( \text{Int} \nsubseteq \text{Int} \) fails, and therefore only the first branch, \( (x \mod 2) = 0 \), is type checked (the second is skipped). Since under the hypothesis \( x \vdash \text{Int} \) the expression \( (x \mod 2) = 0 \) has type \( \text{Bool} \), then so has the whole type-case expression. In the other application of (case), \( x \) is of type \( \alpha \setminus \text{Int} \), so the test \( \alpha \setminus \text{Int} \nsubseteq \text{Int} \) clearly fails, and only the second branch is checked (the first is skipped). Since this second branch is \( x \), then the whole type-case expression has type \( \alpha \setminus \text{Int} \), as expected. This example shows two important aspects of our typing rules. First, it shows the importance of \( \Delta \) to record monomorphic variables, since it may contain some variables that do not occur in \( \Gamma \). For instance, when typing the first branch of \( \text{even} \), the type environment contains only \( x \vdash \text{Int} \) but \( \Delta \) is \( \{ \alpha \} \) and this forbids to consider \( \alpha \) as polymorphic (if we allowed to instantiate any variable that does not occur in \( \Gamma \), then the term obtained from the \( \text{even} \) function (11) by replacing the first branch by \( (\lambda x'y' \cdot y')(\text{even} / \alpha) \) would be well-typed, which is wrong since \( \alpha \) is monomorphic in the body of \( \text{even} \)). Second, this example shows why in some application of the (case) rule a branch is not checked, then the type checking of the whole type-case expression must not necessarily fail: the well-typing of this branch may be checked under different hypothesis (typically when occurring in the body of an overloaded function).

The reader can refer to Section 3.3 of [12] for a more detailed discussion on this point.

We conclude the presentation of the type system with two observations.

First, we said at the beginning that we consider only judgments \( \Delta \vdash e \vdash t \) where the type variables occurring in \( \Gamma \) are contained in \( \Delta \). The intuition is that the types in \( \Gamma \) are the types of the formal parameters of some outer functions and therefore any type variable occurring in them must be considered monomorphic — that is, must be contained in \( \Delta \). Without such a requirement it would be possible to have a derivation such as the following one:

\[
\begin{align*}
\{ \alpha \} \vdash x : \alpha & \vdash \beta \\
\{ \gamma \} \nvdash x : \{ \gamma \} & \vdash \gamma
\end{align*}
\]

which states that by assuming that \( x \) has (any) type \( \beta \), we can infer that it has also (any other) type \( \gamma \), which is unsound. The condition \( \var(\Gamma) \subseteq \Delta \) is preserved by the typing rules (see Lemma B.1) and we always start with \( \Gamma \) and \( \Delta \) satisfying the condition (typically, when we type a closed expression we start by \( \Delta = \Gamma = \emptyset \)), therefore, henceforth, we implicitly assume the condition \( \var(\Gamma) \subseteq \Delta \) to hold in all judgments we consider.

Second, the rule (subsum) makes the type system dependent on the subtyping relation \( \leq \) defined in [6]. It is important not to confuse the subtyping relation \( \leq \) of our system, which denotes semantic subtyping (ie, set-theoretic inclusion of denotations), with the one typically used in the type reconstruction systems for ML, which stands for type variable instantiation. For example, in ML we have \( \alpha \rightarrow \beta \rightarrow \gamma \leq \alpha \rightarrow \beta \rightarrow \gamma \) (because \( \text{Int} \rightarrow \text{Int} \rightarrow \text{Int} \) is an instance of \( \alpha \rightarrow \beta \rightarrow \gamma \)). But this is not true in our system, as the relation must hold for every possible instantiation of \( \alpha \), thus in particular for \( \alpha \) equal to \( \text{Bool} \). In the companion paper [5] we define the preorder \( \sqsubseteq \), which includes the type variable instantiation of the preorder typically used for ML, so any direct comparison with constraint

\[\text{type soundness}\]

Subject reduction and progress properties hold for this system.

**Theorem 3.3 (Subject Reduction).** For every term \( e \) and type \( t \), if \( \Gamma \vdash e : t \) and \( e \leadsto e' \), then \( \Gamma \vdash e' : t \).

**Theorem 3.4 (Progress).** Let \( e \) be a well-typed closed term. If \( e \) is not a value, then there exists a term \( e' \) such that \( e \leadsto e' \).

The proofs of both theorems, though unsurprising, are rather long and technical and can be found in Appendix B.2. They allow us to conclude that the type system is sound.

**Corollary 3.5 (Type soundness).** Let \( e \) be a well-typed closed expression, that is, \( \Gamma \vdash e : t \) for some \( t \). Then either \( e \) diverges or it returns a value of type \( t \).

**3.5 Expressing intersection type systems**

We can now state the first stand-alone theoretical contribution of our work. Consider the sub-calculus of our calculus in which type-substitutions occur only in decorations and without constants and type-case expressions, that is,

\[
e := x \mid ee \mid \lambda x^\tau.\eta \mid \tau \mid \{\beta\} : \gamma \tag{20}
\]

and whose types are inductively produced by the grammar

\[
t := a \mid t \rightarrow t \mid t \land t
\]

This calculus is closed with respect to \( \beta \)-reduction as defined by the reduction rule (18) (without type-cases, union and negation types are not needed). It constitutes an explicitly-\( \lambda \)-calculus with intersection types whose expressive power subsumes that of classic intersection type systems (without an universal element \( \omega \), of course), as expressed by the following theorem.

**Theorem 3.6.** Let \( \vdash_{\text{BCD}} \) denote provability in the Barendregt, Coppo, and Dezani system [1], and \( \lceil e \rceil \) be the pure \( \lambda \)-calculus term obtained from \( e \) by erasing all types occurring in it.

If \( \vdash_{\text{BCD}} m : t \), then \( \exists e \) such that \( \vdash e : t \) and \( \lceil e \rceil = m \).

Therefore, this sub-calculus solves a longstanding open problem, that is the definition of explicit type annotations for \( \lambda \)-terms in intersection type systems, without any further syntactic modification. See Section 7 on related work for an extensive comparison.

The proof of Theorem 3.6 is constructive (cf, Appendix B.3). Therefore we can transpose decidability results of intersection type systems to our system. In particular, type reconstruction\(^2\) for the subcalculus (20) is undecidable and this implies the undecidability of type reconstruction for the whole calculus without recursive types (with recursive types type reconstruction is trivially decidable since every \( \lambda \)-term can be typed by the recursive type \( \mu X.X \rightarrow X \rightarrow X \nu^* \)). In Section 4 we prove that type inference for our system is decidable. The problem of reconstructing type-substitutions (ie, given a term of grammar (3), deciding whether it is possible to add sets of type-substitutions in it so that it becomes a well-typed term of our calculus) is dealt with in the companion paper [5].

**3.6 Elimination of sets of type-substitutions**

To compare with existing intersection type systems, the calculus in (20) includes neither type-cases nor expressions of the form

\[\text{type inference}\]

We recall that type reconstruction is the problem of finding whether there exists a type-annotation that makes a given expression well-typed; type inference is the problem of checking whether an expression is well-typed, and type checking is the problem of checking whether an expression has a given type.
For every well-typed expression $e$, the theorem

$$\Delta \vdash \Gamma \vdash e : t$$

is true if $\Delta$ is the identity for variables, and as its homomorphic propagation does not increase its expressive power. Consider the subcalculus whose terms are

$$e ::= x \mid e_1 \land e_2 \mid e_1 \lor e_2 \mid \bnot e \mid \btrue \mid \bfalse \mid \lambda x. e \mid e_1 e_2$$

that is, the calculus in which sets of type-substitutions appear only in type expressions. Consider the embedding $\llbracket \cdot \rrbracket$ of our calculus (16) into this subcalculus, defined as

$$\llbracket e \rrbracket_{\sigma_j \in J} = e[\sigma_j]_{j \in J}$$

as the identity for variables, and as its homomorphic propagation for all the other expressions. Then it is easy to prove the following theorem

**Theorem 3.7.** For every well-typed expression $e$:

1. $e \rightsquigarrow^* v \Rightarrow \llbracket e \rrbracket \rightsquigarrow^* \llbracket v \rrbracket$
2. $\llbracket e \rrbracket \rightsquigarrow^* v \Rightarrow e \rightsquigarrow^* v'$ and $v = \llbracket v' \rrbracket$

meaning that the subcalculus defined above is equivalent to the full calculus. Although expressions of the form $e[\sigma_j]_{j \in J}$ do not bring any further expressive power, they play a crucial role in local type inference, which is why we included them in our calculus. As we explain in details in the companion paper, for local type inference we need to reconstruct sets of type-substitutions that are applied to expressions but we must not reconstruct sets of type-substitutions that are decorations of $\lambda$-expressions. The reason is pragmatic and can be shown by considering the following two terms: $(\lambda_{\alpha \rightarrow \alpha} x.x) \beta$ and $(\lambda_{\alpha \rightarrow \alpha} x.x) \Delta$. Every functional programmer will agree that the first expression must be considered well-typed while the second must not, for the simple reason that the constant function $(\lambda_{\alpha \rightarrow \alpha} x.x)$ does not have type $\alpha \rightarrow \alpha$. Indeed in the first case it is possible to apply a set of type-substitutions that makes the term well typed, namely $(\lambda_{\alpha \rightarrow \alpha} x.x)[\llbracket \lambda_{\alpha \rightarrow \alpha} x.x \rrbracket] 3$, while no such application exists for the second term. However, if we allowed reconstruction also for decorations, then the second term could be made well typed by adding the decoration $(\lambda_{\alpha \rightarrow \alpha} x.x)[\llbracket \lambda_{\alpha \rightarrow \alpha} x.x \rrbracket] 4\Delta$. In conclusion, for type inference it is important to keep the expression $e[\sigma_j]_{j \in J}$, since well-typing of $e[\sigma_j]_{j \in J}$ does not imply that of $e\llbracket \sigma_j \rrbracket_{j \in J}$.

### 4. Typing algorithm

The rules in Figure 1 do not describe a typing algorithm since they do not syntactically directed. As customary the problem is the subsumption rule, and the way to go is to eliminate this rule by embedding appropriate checks of the subtyping relation into the rules that need it. This results in the system formed by the rules of Figure 2. This system is algorithmic (as stressed by $\vdash_{\Delta\Gamma}$): in every case at most one rule applies, either because of the syntax of the term or because of mutually exclusive side conditions. Subsumption is no longer present and, instead, subtype checking has been pushed in all the remaining rules.

The rule for type-cases has been split in three rules (plus a fourth uninteresting rule we omitted that states that when $e : \emptyset \rightarrow i\epsilon$, it is the provably diverging expression—then the whole type-case expression has type $\emptyset$) according to whether one or both branches can be selected. Here the only modification is in the case where both branches can be selected: in the rule (case) in Figure 1 the types of the two branches were subsumed to a common type $s$, while (ALG-CASE-BOTH) returns the least upper bound (ie, the union) of the two types.

The rule for abstractions underwent a minor modification with respect to the types returned for the body, which before were subsumed to the type declared in the interface while now the subtyping relation $s'_{ij} \leq s_{ij}$ is explicitly checked.

The elimination of the subsumption yields a simplification in typing the application of type-substitutions, since in the system of Figure 1 without subsumption every premise of an (inter) rule is the consequence of an (inst) rule. The two rules can thus be merged into a single one, yielding the (ALG-INST) rule (see Appendix C.1 and in particular Theorem C.1).

As expected, the core of the typing algorithm is the rule for application. In the system of Figure 1, in order to apply the (appl) rule, the type of the function had to be subsumed to an arrow type, and the type of the argument had to be subsumed to the domain of that arrow type: then the co-domain of the arrow is taken to type the application. In the algorithmic rule (ALG-APPL), this is done by the type meta-operator “$\Rightarrow$" which is formally defined as follows: $t \cdot s \equiv \min \{u \mid t \leq s \cdot u\}$. In words, if $t$ is the function and $s$ the type of the argument, this operator looks for the smallest arrow type larger than $t$ and with domain $s$, and it returns its co-domain. More precisely, when typing $e_1 e_2$, the rule (ALG-APPL) checks that the type $t$ of $e_1$ is a functional one ($ie$, $t \leq \emptyset \rightarrow \Delta$). It also checks that the type $s$ of $e_2$ is a subtype of the domain of $t$ (denoted by $\text{dom}(t)$). Because $t$ is not necessarily an arrow type (in general, it is equivalent to a disjunctive normal form like the one of equation (15) in Section 3), the definition of the domain is not immediate. The domain of a function whose type is an intersection of arrows and negation of arrows is the union of the domains of all positive literals. For instance the domain of a function of type $(\text{Int} \rightarrow \text{Int}) \land \text{Bool} \Rightarrow \text{Bool}$ is $\text{Int} \lor \text{Bool}$, since it can be equally applied to integer or Boolean arguments, while the domain of even as defined in (11) is $\text{Int} \lor \text{Odd}$, that is $\text{Int} \lor \text{Int}$. The domain of a union of functional types is the intersection of each domain. For instance an expression of type $(s_1 \rightarrow s_2) \land (t_1 \rightarrow t_2)$ will return either a function of type $s_1 \rightarrow s_2$ or a function of type $t_1 \rightarrow t_2$, so this expression can be applied only to arguments that fit both cases, that is, to arguments in $s_1 \land t_1$. Formally, if $t \leq \emptyset \rightarrow \Delta$, then $t \equiv \min_{s \in S} \{p | s_1 \rightarrow \text{dom}(t) \}$. With the union of all the $P_i$’s not empty, and therefore $\text{dom}(t) \equiv \bigcup_{s \in S} P_i$. Here type
variables do not count since they are intersected and universally quantified so the definition of the domain must hold also when their intersection is $t$). Finally, the type returned in (ALG-APP) is $t\cdot s$, which we recall is the smallest result type that can be obtained by subsuming $t$ to an arrow type compatible with $s$. We can prove that for every type $t$ such that $t \leq \Rightarrow 1$ and type $s$ such that $s \leq \text{dom}(t)$, the type $t\cdot s$ exists and can be effectively computed (see Appendix C.12).

The algorithmic system is sound and complete with respect to the type system of Figure 1 and satisfies the minimum typing property (see Appendix C for the proofs).

**Theorem 4.1** (Soundness). If $\Delta \cdot \Gamma \vdash e : t$, then $\Delta \cdot \Gamma \vdash e : t$.

**Theorem 4.2** (Completeness). If $\Delta \cdot \Gamma \vdash e : t$, then there exists a type $s$ such that $\Delta ; \Gamma \vdash e : t$.

**Corollary 4.3** (Minimum typing). If $\Delta \cdot \Gamma \vdash e : t$, then $t = \text{min}\{s \mid \Delta \cdot \Gamma \vdash e : s\}$.

Finally, it is quite easy to prove that type inference is decidable. It suffices to define the size of an expression as follows:

- \[ \text{size}(x) = 1 \]
- \[ \text{size}(e_1 \cdot e_2) = \text{size}(e_1) + \text{size}(e_2) + 1 \]
- \[ \text{size}(\lambda_{[j]} e_{[j]} : t \rightarrow s) x.e = \text{size}(e) + 1 \]
- \[ \text{size}(e_1 \cdot e_2) = \text{size}(e) + \text{size}(e_1) + \text{size}(e_2) + 1 \]
- \[ \text{size}(e_{[j]} : e_{[j]}) = \text{size}(e) + 1 \]

and show that the expressions occurring in the premises of every rule of the algorithm are strictly smaller than the expression in its conclusion. We can then deduce the termination of the type inference algorithm.

**Theorem 4.4** (Termination). Let $e$ be an expression. Then the type inference algorithm for $e$ terminates.

This system constitutes a further theoretical contribution of our work since with this type system the language defined by grammar (16), the one by grammar (21), and a fortiori the one by grammar (20) are intersection type systems that all satisfy the Curry-Howard isomorphism since there is a one-to-one correspondence between terms and proofs of the algorithmic system.

5. Evaluation

In this section we define an efficient execution model for the polymorphic calculus as a conservative extension of the execution model of the monomorphic calculus: by “efficient” we mean that monomorphic expressions will be evaluated as efficiently as in the original Cduce runtime. In fact, even polymorphic expressions will be evaluated as efficiently as well (as if type variables were basic monomorphic types) despite the fact that the formal reduction semantics of polymorphic expressions includes a run-time relabeling operation. The key observation that allows us to define an efficient execution model for the polymorphic calculus is that relabeling can be implemented lazily so that the only case in which relabeling is computed at run-time will correspond to testing the type of a partial application of a polymorphic function. In practice, this case is so rare —at least in the XML setting— that there is no difference between monomorphic and polymorphic evaluation.

5.1 Monomorphic Language

Let us start by recalling the execution model of monomorphic Cduce, which is a classic closure-based evaluation. Expressions and values are defined as:

- \[ e ::= c \mid x \mid \lambda^t x.e \mid ee \mid e \in s? e : e \]
- \[ v ::= c \mid (\lambda^r x.e, \delta) \]

where $t$ denotes an intersection of arrow types, $s$ denotes a closed type, and $\delta$ denotes an environment, that is, a substitution mapping expression variables into values. The big step semantics is:

\[
\begin{align*}
\text{(ME-CONST)} & \quad \delta \vdash_m c \Downarrow c \\
\text{(ME-VAR)} & \quad \delta \vdash_m \lambda^t x.e \Downarrow (\lambda^t x.e, \delta) \\
\text{(ME-CLOSURE)} & \quad \delta \vdash_m e_1 \Downarrow (\lambda^t x.e, \delta') \\
\text{(ME-APPLY)} & \quad \delta \vdash_m e_1 \Downarrow (\lambda^t x.e, \delta') \\
\text{(ME-TYPE CASE T)} & \quad \delta \vdash_m e_1 \Downarrow (\lambda^t x.e, \delta') \\
\text{(ME-TYPE CASE F)} & \quad \delta \vdash_m e_1 \Downarrow (\lambda^t x.e, \delta') \\
\end{align*}
\]

To complete the definition we define the relation $v \in_m t$, that is, membership of a (monomorphic) value to a (monomorphic) type:

- \[ c \in_m t \iff (\lambda^t x.e, \delta) \in_m t \iff s \leq t \text{ where } s \leq t \text{ is the subtyping relation of Cduce [12].} \]

5.2 Polymorphic Language

In the naive extension of this semantics to the explicitly-typed polymorphic calculus of Section 3, we deal with type-substitutions as we do for environments, that is, by storing them in closures. This is reflected by the following definition where, for brevity, we write $\sigma_j$ to denote the set of type-substitutions $[\sigma_1] : c_1$:

- \[ e ::= c \mid x \mid \lambda_{[j]} x.e \mid ee \mid e \in t? e : e \mid e \sigma_1 \]
- \[ v ::= c \mid (\lambda_{[j]} x.e, \delta, \sigma_j) \]

The big-step semantics is then defined as follows, where each expression is evaluated with respect to an environment $\delta$ determining the current value substitutions and a set of type-substitutions $\sigma_j$:

\[
\begin{align*}
\text{(PE-CONST)} & \quad \sigma_j; \delta; \Gamma \vdash_p c \Downarrow c \\
\text{(PE-VAR)} & \quad \sigma_j; \delta; \Gamma \vdash_p x \Downarrow \delta(x) \\
\text{(PE-CLOSURE)} & \quad \sigma_j; \delta; \Gamma \vdash_p \lambda_{[j]} x.e \Downarrow (\lambda_{[j]} x.e, \delta, \sigma_j) \\
\text{(PE-APPLY)} & \quad \sigma_j; \delta; \Gamma \vdash_p e_1 \Downarrow (\lambda_{[j]} x \rightarrow s_{[j]} e_{[j]} \sigma_j \sigma_1; \delta; \Gamma e \Downarrow v) \\
\text{(PE-TYPE CASE T)} & \quad \sigma_j; \delta; \Gamma \vdash_p e_1 \Downarrow v \quad \sigma_j; \Gamma \vdash_p e_2 \Downarrow v \\
\text{(PE-TYPE CASE F)} & \quad \sigma_j; \Gamma \vdash_p e_1 \Downarrow v \quad \Gamma \vdash_p e_2 \Downarrow v \\
\end{align*}
\]

The membership relation $v \in_m t$ for polymorphic values is inductively defined as:

\[ e \in_m t \iff (\lambda^t x.e, \delta, \sigma_j) \in_m t \]

where $\leq$ is the subtyping relation of Castagna and Xu [6]. It is not difficult to show that this big-step semantics is equivalent to the small-step one of Section 3. Let $[\cdot]$ be the transformation that maps values of the polymorphic language into corresponding values of the calculus, that is

\[ [c] = c \quad \text{and} \quad [(\lambda_{[j]} x.e, \delta, \sigma_1)] = \lambda_{[j]} \sigma_1 x.e(\delta) \]

where $[\cdot]$ applies to all the values in the range of $\delta$. Let $\{\cdot\}$ denote the singleton set containing the empty type-substitution $\{\cdot\}$, which is the neutral element of the composition of sets of type-substitutions. Then we have:
Theorem 5.1. Let \( e \) be a well-typed closed explicitly-typed expression \( \vdash e : t \). Then:
\[
\begin{align*}
\not\in & \quad \not\in
\end{align*}
\]
This implementation has a significant computational burden compared to that of the monomorphic language: first of all, each application of (PE-APPLY) computes the set \( P \), which requires to implement several type-substitutions and membership tests; second, each application of (PE-INSTANCE) computes the composition of two sets of type-substitutions. In the next section we describe a different solution consisting in the compilation of the explicitly typed calculus into an intermediate language so that these computations are postponed as much as possible and are performed only if and when they are really necessary.

5.3 Intermediate Language

The intermediate language in which we compile the explicitly-typed polymorphic language is very similar to the monomorphic version. The only difference is that \( \lambda \)-abstractions (both in expressions and closures) may contain type variables in their interface and have an extra decoration \( \Sigma \) which is a term denoting a set of type-substitutions.

\[
\begin{align*}
e & : = c \mid \lambda x.E & \Sigma & : = \sigma | \comp(\Sigma, \Sigma') \mid \sel(x, t, \Sigma)
\end{align*}
\]

Intuitively, a \( \comp(\Sigma, \Sigma') \) term corresponds to an application of the \( \circ \) composition operator to the sets of type substitutions denoted by \( \Sigma \) and \( \Sigma' \), while a \( \sel(x, t, \Sigma) \) term selects the subset of type substitutions \( \sigma \) denoted by \( \Sigma \) that are compatible with the fact that the (value instantiating) \( x \) belongs to the domain of \( \sigma \).

The big step semantics for this intermediate language is:

\[
\begin{align*}
\text{(OE-CONST)} & \quad \text{(OE-VAR)} & \quad \text{(OE-CLOSURE)} & \quad \text{(OE-APPLY)} \\
\text{\( \not\in \)} & \quad \text{\( \not\in \)} & \quad \text{\( \not\in \)} & \quad \text{\( \not\in \)} \\
\text{\( \not\in \)} & \quad \text{\( \not\in \)} & \quad \text{\( \not\in \)} & \quad \text{\( \not\in \)} \\
\text{\( \not\in \)} & \quad \text{\( \not\in \)} & \quad \text{\( \not\in \)} & \quad \text{\( \not\in \)} \\
\end{align*}
\]

Notice that this semantics is structurally the same as that of the monomorphic language. There are only two minor differences: (i) \( \lambda \)-abstractions have an extra decoration \( \Sigma \) which has no impact on efficiency since it corresponds in the implementation to manipulating descriptors with an extra field) and (ii) the corresponding \( \langle F \text{-TYPE CASE} \rangle \) rules use a slightly different relation: \( \not\in \) instead of \( \not\in \). It is thus easy to see that in terms of steps of reduction the two semantics have the same complexity. More precisely, if you take a term of the monomorphic calculus and a term of the intermediate language with the same erasure and that select the same branches of the typecases, then they perform exactly the same reduction. What changes is the test of the membership relation \( \not\in \) rather than \( \not\in \) since, when the value to be tested is a closure, we need to materialize relabelings. In other words, we have to evaluate the \( \Sigma \) expression decorating the function and apply the resulting set of substitutions to the interface of the function. Formally:

\[
\begin{align*}
c & \not\in t \quad \Leftrightarrow \quad \not\in \\
\lambda x.E & \not\in t \quad \Leftrightarrow \quad \not\in \\
\end{align*}
\]

where the evaluation of the symbolic set of type-substitutions is inductively defined as

\[
\begin{align*}
eval(\Sigma, \sigma) & = \sigma \\
eval(\Sigma, \comp(\Sigma, \Sigma')) & = \eval(\Sigma, \Sigma) \circ \eval(\Sigma', \Sigma') \\
eval(\Sigma, \sel(x, t, \Sigma)) & = \sigma \quad \text{if} \quad \sigma \in \eval(\Sigma, \Sigma) \quad \text{and} \quad \exists \epsilon \in I : \eval(\Sigma, \Sigma) \in t, \sigma \epsilon \]
\]

Notice in the last rule the crucial role played by \( x \) and \( \Sigma \) by using an expression variable \( x \) in the symbolic representation of type-substitutions and relying on its interpretation through \( \Sigma \), we have transposed to type-substitutions the same benefits that closures bring to value substitutions: just as closures allow value-substitutions to be materialized only when a formal parameter is used rather than at the moment of the reduction, so our technique allows type-substitutions to be materialized only when a type variable is effectively tested, rather than at the moment of the reduction.

It is easy to see that the only case in which the computation of \( \not\in \) is more expensive than that of \( \not\in \) is when the value whose type is tested is a closure \( \langle \lambda \Sigma x.e, \Sigma \rangle \) in which \( t \) is not closed and \( \Sigma \) is not \( t \). The \( \Sigma \) decoration is different from \( \not\in \) only if the closure is the result of a partial application of a curried function. The type \( t \) is not closed only if such partial application yielded a polymorphic function. In conclusion, the evaluation of an expression in the polymorphic language is more expensive than the evaluation of a similar expression of the monomorphic language only if it tests the type of a polymorphic function resulting from the partial application of a polymorphic curried function. The additional overhead is limited only to this particular test and in all the other cases the evaluation is as efficient as in the monomorphic case. Finally, it is important to stress that this holds true also if we add product types: the test of a pair of values in the polymorphic case is as expensive as in the monomorphic case and so is the rest of the evaluation. Since in the XML setting the vast majority of the computation time is spent in testing products (since they encode sequences, trees, and XML elements), then the overhead brought by adding polymorphism —ie, the overhead due to testing the type of a polymorphic partial application of a polymorphic curried function—is negligible in practice.

All that remains to do is to define the compilation of the explicitly-typed language into the intermediate language:

\[
\begin{align*}
\not\in & \quad \not\in \\
\not\in & \quad \not\in \\
\not\in & \quad \not\in \\
\not\in & \quad \not\in \\
\end{align*}
\]

Given a closed program \( e \) we compile it in the intermediate language as \( \langle \not\in \rangle \). In practice, the compilation will be even simpler since we apply it only to expressions generated by local type inference algorithm described in the companion paper where all \( \lambda \)'s are decorated by \( \not\in \) (cf. discussion at the end of Section 3.6). So the second case of the definition simplifies to:

\[
\not\in \langle \not\in \rangle = \lambda x.e, \not\in \langle \not\in \rangle
\]

The compilation is adequate:

Theorem 5.2. Let \( e \) be a well-typed closed explicitly-typed expression \( \vdash e : t \). Then:
\[
\begin{align*}
\not\in & \quad \not\in \\
\not\in & \quad \not\in \\
\end{align*}
\]

where \( \langle \not\in \rangle \) evaluates all the symbolic expressions and type-substitutions in the term (see (22)). By combining Theorems 5.1 and 5.2 we obtain the adequacy of the compilation:

6 To be more precise, when there exists a substitution \( \sigma \in \eval(\Sigma, \Sigma) \) such that \( \varnothing \cap \dom(\sigma) \neq \varnothing \). Notice that the tests of the subtyping relation for monomorphic and polymorphic types have the same complexity [6].

7 By similar we intend with the same syntax tree but only closed types.
Corollary 5.3. Let \( e \) be a well-typed closed explicitly-typed expression \((\Gamma \vdash e : t)\). Then
\[
\Gamma \vdash e \Downarrow v \iff e \Downarrow^* v
\]
Finally, let us come back to the membership relation for function types, namely:
\[
\lambda \delta . \mathcal{X} . e \in t \quad \iff \quad s(\mathcal{E}(\delta , \Sigma)) \leq t
\]
In Footnote 6 we signalled that the only case in which this test is more expensive than in the monomorphic case is when we have to evaluate \( e \) and that this may be necessary only if there exists \( \bar{\sigma} \in \mathcal{E}(\delta , \Sigma) \) that satisfies \( \bar{\sigma} \cap \mathsf{dom}(\bar{\sigma}) \neq \emptyset \). We may however overapproximate the condition \( \mathsf{var}(s) \cap \mathsf{dom}(\Sigma) \neq \emptyset \) by using the domain of \( \Sigma \) (the difference being that we consider also the substitutions \( \bar{\sigma} \) that would be discarded by a selection). Notice that all the expressions \( \Sigma \) denoting sets of substitutions are statically generated by our compilation and are not modified at run-time. This means that for every function value we can statically decide whether \( \mathsf{eval}(\delta , \Sigma) \) is defined as follows:
\[
\mathsf{dom}(\bar{\sigma} \circ \Sigma) = \bigcup_{\bar{J} \subseteq \bar{J}} \mathsf{dom}(\bar{\sigma} \circ \Sigma)
\]
In practice, we can modify our compilation technique to flag \((\Gamma \vdash \lambda \mathcal{X} . e : t)\) which may require the evaluation of \( \mathcal{E}(\delta , \Sigma) \), as follows:
\[
\mathcal{E}(\delta , \Sigma) = \left\{ \begin{array}{ll}
\lambda \delta . \mathcal{X} . e \in t & \text{if } \mathsf{var}(s) \cap \mathsf{dom}(\Sigma) = \emptyset \\
\lambda \delta . \mathcal{X} . e \in t & \text{otherwise}
\end{array} \right.
\]
and then evaluate the symbolic substitutions only for marked functions:
\[
\lambda \delta . \mathcal{X} . e \in t \quad \iff \quad \mathsf{eval}(\delta , \Sigma) \leq t
\]

### 5.4 Let-polymorphism

A function is polymorphic if it can be safely applied to arguments of different types. The calculus presented supports a varied palette of different forms of polymorphism: it uses subtype polymorphism (a function can be applied to arguments whose types are subtypes of its domain type), the combination of intersection types and typecase expressions yields “ad hoc” polymorphism (aka overloading), and finally the use of type variables in function interfaces provides parametric polymorphism. Polymorphism is interesting when used with bindings: instead of repeating the definition of a function every time we need to apply it, it is more convenient to define the function once, bind it to an expression variable, and use the variable every time we need to apply the function. In the current system it is possible to combine binding only with the first two kinds of polymorphism: different occurrences of a variable bound to a function can be given different types —thus, be applied to arguments of different types—, either by subsumption (\( \bar{\sigma} \), by assigning to the variable a super-type of the type of the function it denotes) or by intersection elimination (\( \bar{\sigma} \), by assigning to a variable one of the types that form the intersection type of the function it denotes). However, as it is well known in the languages of the ML-family, in the current setting it is not possible to combine binding and parametric polymorphism. Distinct occurrences of a variable cannot be given different types by instantiation (\( \bar{\sigma} \), by assigning to the variable a type which is an instance of the type of the function it denotes).

In other terms, all \( \lambda \)-abstracted variables have monomorphic types (with respect to parametric polymorphism), which is why in ML auto-application \( \lambda x . x x \) is not typeable.

The solution is well known and consists in introducing \( \text{let} \) bindings. This amounts to defining a new class of expression variables so that variables introduced by a \( \text{let} \) have polymorphic types, that is, types that have been generalized at the moment of the definition and can be instantiated in the body of the let. To sum up, \( \lambda \)-abstracted variables have monomorphic types, while \( \text{let} \)-bound variables (may) have polymorphic types and, thus, be given different types obtained by instantiation. For short we call them the \( \lambda \)-abstracted variables “monomorphic (expression) variables” and the latter \( \text{let} \)-bound variables “polymorphic (expression) variables”.

In the explicitly-typed calculus of the previous sections we had just \( \lambda \)-abstracted variables. That these variables have polymorphic types is clearly witnessed by the fact that, operationally, \( x \bar{\sigma}_I \) is equivalent to \((\bar{i} \text{reduce} \to x) \). Clearly this property must not hold for polymorphic type variables since
\[
\text{let } x = (\lambda x' \Rightarrow y . y) \in (x) \{ \bar{a} \to \bar{a}/\bar{a}_I \} \]  
(23)

is, intuitively, well typed, while the same term obtained by replacing \( x \{ \bar{a} \to \bar{a}_I \} \) with \( x \) is not (see the extension of the definition of relabeling for polymorphic variables later on).

To enable the definition of polymorphic functions to our calculus we add a \( \text{let} \) expression. To ease the presentation and to stress that the addition of \( \text{let} \)-bindings is a conservative extension of the previous system, we syntactically distinguish the current monomorphic variables (\( \bar{i} \), those abstracted by a \( \lambda \)) from polymorphic variables by underlining the latter ones.

\[
e \ ::= \cdots \mid \ul{x} \mid \text{let } \ul{x} = e \in e
\]

Reduction is as usual:
\[
\text{let } \ul{x} = v \in e \Downarrow e \Downarrow^* v \{ \bar{v} \}
\]
Relabeling is extended by the following definitions:
\[
\mathcal{R} \ul{e} / \{ \bar{e} / \bar{e}_I \} \mathcal{R} \bar{e} / \{ \bar{e} / \bar{e}_I \}
\]
and the (algorithmic) typing rule is as expected:
\[
\text{let } \ul{x} = e \in e_1 \Downarrow e_2 : t_2
\]
Type environments \( \Gamma \) now map also polymorphic expression variables into types. Notice that for a polymorphic expression variable \( \ul{e} \) it is no longer true that \( \mathsf{var}(\Gamma(\ul{e})) \subseteq \Delta \) (not adding \( \mathsf{var}(\Gamma(\ul{e})) \) to \( \Delta \) corresponds to generalizing the type of \( \ul{e} \) before typing \( e_2 \) as in the GEN rule of the Damas-Milner algorithm: cf. \([20]\)). As before we assume that polymorphic type variables of a \( \text{let} \)-expression (in particular those generalized for let-polymorphism) are distinct from monomorphic and polymorphic type variables of the context that the \( \text{let} \)-expression occurs in.

Likewise, environments now map both monomorphic and polymorphic expression variables into values so that the rules for evaluation in Section 5.2 are extended with the following ones:
\[
\text{(PE-PVAR)} \quad \mathcal{E}(\ul{x}) = e \\
\Gamma, \ul{x} : t \vdash e \Downarrow c \\
\ul{x} : t_1 \Gamma \vdash e \Downarrow c
\]
\[
\text{(PE-PVAR)} \quad \mathcal{E}(\ul{e}) = (\lambda x . e, \bar{e}') \\
\Gamma, \ul{x} : t_1 \vdash e \Downarrow c \\
\Gamma, \ul{e}' : t_2 \vdash e' \Downarrow c
\]
\[
\text{(PE-LET)} \quad \Gamma, \ul{x} : t_1 \vdash e_1 \Downarrow v_0 \\
\Gamma, \ul{e}' : t_2 \vdash e_2 \Downarrow v \\
\Gamma, \ul{e}' : t_2 \vdash \text{let } \ul{x} = e_1 \in e_2 \Downarrow v
\]

To compile \( \text{let} \)-expressions we have to extend the intermediate language likewise: we will distinguish polymorphic expression variables by decorating them with sets of type-substitution formule \( \Sigma \) that apply to that particular occurrence of the variable. So we add to the productions of Section 5.3 the following ones:
\[
e \ ::= \cdots \mid \ul{x} \mid \text{let } \ul{x} = e \in e\]
while the big-step semantics of the new added expressions is
\[
\delta'(x) = c \\
\delta'(y,e,e') = \langle \lambda^{\delta_e}_{y,e,e'} \rangle \\
\delta'(x,y,e) = \langle \lambda^{\delta_e}_{x,y,e} \rangle
\]

Notice how rule (OE-PVar \( y \)) uses the \( \Sigma \) decoration on the variable to construct the closure. The final step is the extension of the compiler for the newly added terms:
\[
\begin{align*}
\text{let } \lambda \in e_1 \text{ in } e_2 \Sigma & = \left[ \lambda(x) \right]^{\delta_1} \\
\text{let } \lambda \in e_1 \text{ in } e_2 \Sigma & = \left[ \lambda(x) \right]^{\delta_1} \Sigma \in e_2 \Sigma
\end{align*}
\]

As an example, the let-expression (23) is compiled into
\[
\text{let } \lambda \in e_1 \text{ in } e_2 \Sigma = \left[ \lambda(x) \right]^{\delta_1} \in x_1(x_1 o \rightarrow o_1):x_1
\]

where the substitution \( \{ o_1 o \rightarrow o_1 \} \) that is applied to the leftmost occurrence of \( x \) is recorded in the variable and will be used to instantiate the closure associated with \( x \) by the environment; the rightmost occurrence of \( x \) is decorated by 1 and therefore the value bound to it will not be instantiated. Theorem 5.1, 5.2, and Corollary 5.3 hold also for these extensions (see Appendix D for the proofs).

In an actual programming language there will not be any syntactic distinction between the two kinds of expression variables and compilation can be optimized by transforming variables that are let-bound to monomorphic values into monomorphic variables. So, whenever \( e_1 \) has a monomorphic type \( t_1 \), the let-expression should be compiled as
\[
\text{let } \lambda \in e_1 \text{ in } e_2 \Sigma = \left[ \lambda(x) \rightarrow t_2 \times e_2 \{ x/e \} \right]^{\delta_1} \Sigma
\]

where \( t_2 \) is the type deduced for \( e_2 \) under the hypothesis that \( x \) has type \( t_1 \). Notice that this optimization is compositional.

6. Design choices and extensions

For the sake of concision we omitted two key features in the presentation: recursive functions and pairs. Recursive functions can be straightforwardly added with minor modifications. In particular, for recursive functions, whose syntax is \( f \in \sigma \rightarrow \tau \rightarrow \tau \), \( f \) is added to the type environment \( \Gamma \) the recursion variable \( f \) associated with the type obtained by applying the decoration to the interface, that is, \( f : n \sigma \rightarrow \tau \rightarrow \tau \rightarrow \tau \); the reader can refer to Section 7.5 in [12] for a discussion on how and why recursion is restricted to functions.

The extension with product types, instead, is less straightforward but can be mostly done by using existing techniques. Syntactically, we add pairs \( (e,e) \) and projections \( \pi_i e \) (for \( i=1,2 \)) to terms and the product type constructor \( t \times t \) to types. Reduction semantics is standard: two notions of reduction \( \pi_i (v_1, v_2) \rightarrow v_i \) (for \( i=1,2 \)) plus the usual context reduction rules. Typing rules are standard, as well: a pair is typed by the product of the types of its components and if \( e \) is of type \( t_1 \times t_2 \), then \( i \)-th projection \( \pi_i e \) has type \( t_i \). The rule for pairs in the algorithmic system \( \Gamma \vdash \varphi \) is the same as in the static semantics, while the rules for projections \( \pi_i e \) become more difficult because the type inferred for \( e \) may not be of the form \( t_1 \times t_2 \) but, in general, is (equivalent to) a union of intersections of types. We already met the latter problem for application (where the function type may be different from an arrow) and there we checked that the type deduced for the function in an application is a functional type (ie, a subtype of \( \mathbb{O} \)). Similarly, for products we must check that the type of \( e \) is a product type (ie, a subtype of \( \mathbb{I} \)). If the constraint is satisfied, then it is possible to define the type of the projection (in a way akin to the definition of the domain \( \text{dom}() \) for function types) using standard techniques of semantic subtyping (see Section 6.11 in [12]). This is explained in details in Appendix C.

Another concession to the sake of concision is the use of the relabelling operation \( \sigma' \) in the premises of both \( \text{abstr} \) rules in Figures 1 and 2. A slightly different but better formulation would have been to use premises \( \ldots \vdash \sigma \rightarrow \sigma', \pi_i \sigma \) instead of \( \ldots \vdash \sigma \rightarrow \sigma', \pi_i \sigma \). Notice how this optimization is compositional.

For what concerns future extensions, in this work we dodged the problem of the negation of arrow types. Notice indeed that a value can have as type the negation of an arrow type just by substitution. This implies that the \( \tau \)-abstraction can have a negated arrow type. So while the type \( \neg (\text{Bool} \rightarrow \text{Bool}) \) can be inferred for, say, \( (3,4) \), it is not possible to infer it for \( \lambda^{\neg (\text{Int} \rightarrow \text{Int})} x. (x+1) \). This problem was dealt in CDue by deducing for a \( \lambda \)-abstraction the type in its type interface intersected with any negations of arrow types that did not make the type empty. Technically, this was dealt with "type schemas": a function such as \( \lambda^{\neg (\text{Int} \rightarrow \text{Int})} x. x + 1 \) has type schema \( \{ \text{Int} \rightarrow \text{Int} \} \), that is, it has every non empty type of the form \( \text{Int} \rightarrow \text{Int} \). A second choice, to restrict type-cases to closed types, was first one since it does not need the introduction of such technically-motivated new definitions.

For the semantics of the calculus we made few choices that restrict its generality. One of these, the use of a call-by-value reduction, is directly inherited from CDue and it is required to ensure subject reduction. If \( e \) is an expression of type \( \text{Int} \lor \text{Bool} \), then the application \( \lambda^{\text{Int} \lor \text{Int}} (x \rightarrow (\text{Bool} \lor \text{Bool}) (x,x)) e \) has type \( \text{Int} \lor \text{Int} \lor (\text{Bool} \lor \text{Bool}) \). If we use call-by-name, then this redex reduces to \( (e,c) \) whose type \( \text{Int} \lor \text{Bool} \lor \text{Int} \lor \text{Bool} \) is larger than the type of the redex. Although the use of call-by-name would not hinder the soundness of the type system (expressed in terms of progress) we preferred to ensure subject reduction since it greatly simplifies the theoretical development.

A second choice, to restrict type-cases to closed types, was made by practical considerations: using open types in a type-case would have been computationally prohibitive insofar as it demands to solve at run-time the problem whether for two given types \( s \) and \( t \) there exists a type-substitution \( \sigma \) such that \( s \sigma \leq t \sigma \) (we study this problem, that we call the tallying problems, in the companion paper [5]). Our choice, instead, is compatible with the highly optimized (and provably optimal) pattern matching compilation technique of CDue. We leave for future work the study of type-cases on types with monomorphic variables (ie, those in \( \Delta \)). This does not require dynamic type tallying resolution and would allow the
programmer to test capabilities of arguments bound to polymorphic type variables.

7. Related work
We focus on work related to this specific part of our work, namely, existing explicitly-typed calculi with intersection types, and functional languages to process XML data. Comparison with work on local type inference and type reconstruction is done in the second part of this work presented in the companion paper [5].

To compare the differences between the existing explicitly-typed calculi for intersection type systems, we discuss how the term of our existing explicitly-typed calculi with intersection types, and functions for a same term into a compact piece of syntax. To this end, XHaskell by Sulzmann and a typed representative of the same untyped term are joined together (calculi for intersection type systems, we discuss how the term of our existing explicitly-typed calculi with intersection types, and function for a same term into a compact piece of syntax. To this end, XHaskell by Sulzmann

In [23, 26], typing derivations are written as terms: different occurrences of a same mark can be paired with different types. In such systems, the function in (7) relabeled with \(\lambda x.\ (\lambda x.\ (\lambda y.x))\) is written \((\lambda x.\ (\lambda y.x))\). Type checking verifies that both \(\lambda x.\ (\lambda y.x)\) and \(\lambda y.x\) are well typed separately, which generates two very similar typing derivations. The proposal of [16] follows the same idea, except that a separation is kept between the computational and the logical contents of terms. A term consists in the association of a marked term and a proof term. The marked term is just an untyped term where term variables are marked with integers. The proof term encodes beta reduction (and other transformations on terms) more difficultly in parallel that the two copies are well typed.

Some problems are still open, notably the decidability of type-checking and, more generally, for functional languages with recursive types and set-theoretic unions, intersections, and negations. In particular, our results pave the way to the polymorphic extension of CDuce [2] and to the definition of a real type system for XQuery 3.0 [10] (not just one in which all higher-order functions have type “function(•)”). Thanks to local type inference and type reconstruction defined in the second part of this work, these languages can have a syntax and semantics very close to those of OCaml or Haskell, but will include primitives (in particular, complex patterns) to exploit the great expressive power of full-fledged set-theoretic types.

8. Conclusion
The work presented in this and in its companion paper [5] provides the theoretical basis and all the algorithmic tools needed to design and implement polymorphic functional languages for semi-structured data and, more generally, for functional languages with recursive types and set-theoretic unions, intersections, and negations. In particular, our results pave the way to the polymorphic extension of CDuce [2] and to the definition of a real type system for XQuery 3.0 [10] (not just one in which all higher-order functions have type “function(•)” but are defined so as not to break polymorphism, both have to give up something: higher-order functions for [13] and intersection, negation, and local type inference in [25] (the type of function arguments must be explicitly given). Furthermore, Vuillon’s work suffers from the original sin of starting from a subtyping relation that is given axiomatically by a deduction system. This makes the intuition underlying subtyping very difficult to grasp (at least, for us). Another route taken is the one of OCamlDuce [11], which juxtaposes OCaml and CDuce’s type systems in the same language, keeping them separated. This practical approach yields little theoretical problems but forces a value to be of one kind of type or another, preventing the programmer from writing polymorphic XML transformations. Lastly, XHaskell by Sulzmann et al. [23] mixes Haskell type classes with XDuce regular expression types but has two main drawbacks. First, every polymorphic variable must be annotated wherever it is instantiated with an XML type. Second, even without inference of explicit annotations (which they do not address), their system requires several restrictions to be decidable (while our system with explicit type-substitutions is decidable).
though the decidability of their type systems is still an open question). On the practical side, the most interesting directions of research is to couple the efficient compilation of the polymorphic calculus with techniques of static analysis that would perform partial evaluation of relabeling so as to improve the efficiency of type-case of functional values even in the rare cases in which it is more expensive than in the monomorphic version of CDuce.

Acknowledgments. This work was partially supported by the ANR TYPEX project n. ANR-11-BS02-007. Zhiwu Xu was also partially supported by an Eiffel scholarship of French Ministry of Foreign Affairs and by the grant n. 61070038 of the National Natural Science Foundation of China.

References


Appendix
A. Explicitly-Typed Calculus

In this section, we define our explicitly-typed λ-calculus with sets of type-substitutions that we outlined in Section 3.

A.1 Types

Definition A.1 (Types). Let \( \mathcal{Y} \) be a countable set of type variables ranged over by Greek letters \( \alpha, \beta, \gamma, \ldots \), and \( \mathcal{B} \) a finite set of basic (or constant) types ranged over by \( b \). A type is a term co-inductively produced by the following grammar

\[
\text{Types} \quad t ::= \alpha \quad \text{type variable} \\
| b \quad \text{basic} \\
| t \times t \quad \text{product} \\
| t \rightarrow t \quad \text{arrow} \\
| t \lor t \quad \text{union} \\
| \neg t \quad \text{negation} \\
| 0 \quad \text{empty}
\]

that satisfies two additional requirements:

- (regularity) the term must have a finite number of different sub-terms.
- (contractivity) every infinite branch must contain an infinite number of occurrences of atoms (i.e., either a type variable or the immediate application of a type constructor: basic, product, arrow).

We use \( \mathcal{T} \) to denote the set of all types.

We write \( t_1 \setminus t_2, t_1 \land t_2, \) and \( 1 \) respectively as an abbreviation for \( t_1 \land \neg t_2, \neg(t_1 \lor \neg t_2), \) and \( \neg 0 \). The condition on infinite branches bars out ill-formed types such as \( t = t \lor t \) (which does not carry any information about the set denoted by the type) or \( t = \neg t \) (which cannot represent any set). It also ensures that the binary relation \( \triangleright \subseteq \mathcal{T}^2 \) defined by \( t_1 \lor t_2 \triangleright t, \neg t \triangleright t \) is Noetherian (that is, strongly normalizing). This gives an induction principle on \( \mathcal{T} \) that we will use without any further explicit reference to the relation.

Since types are infinite, the accessory definitions on them will be given either by using memoization (e.g., the definition of \( \text{var} \)), the variables occurring in a type: Definition A.2, by co-inductive techniques (e.g., the definition or the application of type-substitutions: Definition A.5), or by induction on the relation \( \triangleright \), but only when induction does not traverse a type constructor (e.g., the definition of \( \text{tlv}() \)), the variables occurring at top-level of a type: Definition A.3).

Definition A.2 (Type variables). Let \( \text{var}_0 \) and \( \text{var}_1 \) be two functions from \( \mathcal{T} \times \mathcal{P}(\mathcal{T}) \) to \( \mathcal{P}(\mathcal{Y}) \) defined as:

\[
\begin{align*}
\text{var}_0(t, \emptyset) &= \begin{cases} 
\emptyset & \text{if } t \in \emptyset \\
\text{var}_1(t, \emptyset \cup \{t\}) & \text{otherwise}
\end{cases} \\
\text{var}_1(\alpha, \emptyset) &= \{\alpha\} \\
\text{var}_1(b, \emptyset) &= \emptyset \\
\text{var}_1(t_1 \times t_2, \emptyset) &= \text{var}_0(t_1, \emptyset) \cup \text{var}_0(t_2, \emptyset) \\
\text{var}_1(t_1 \rightarrow t_2, \emptyset) &= \text{var}_0(t_1, \emptyset) \cup \text{var}_0(t_2, \emptyset) \\
\text{var}_1(t_1 \lor t_2, \emptyset) &= \text{var}_1(t_1, \emptyset) \cup \text{var}_1(t_2, \emptyset) \\
\text{var}_1(\neg t_1, \emptyset) &= \text{var}_1(t_1, \emptyset) \\
\text{var}_1(0, \emptyset) &= \emptyset
\end{align*}
\]

The set of type variables occurring in a type \( t \), written \( \text{var}(t) \), is defined as \( \text{var}_0(t, \emptyset) \). A type \( t \) is said to be ground or closed if and only if \( \text{var}(t) \) is empty. We write \( \mathcal{T}_0 \) to denote the set of all the ground types.

Definition A.3 (Top-level variables). Let \( t \) be a type. The set \( \text{tlv}(t) \) of type variables that occur at top level in \( t \), that is, all the variables of \( t \) that have at least one occurrence not under a constructor, is defined as:

\[
\begin{align*}
\text{tlv}(\alpha) &= \{\alpha\} \\
\text{tlv}(b) &= \emptyset \\
\text{tlv}(t_1 \times t_2) &= \emptyset \\
\text{tlv}(t_1 \rightarrow t_2) &= \emptyset \\
\text{tlv}(t_1 \lor t_2) &= \text{tlv}(t_1) \cup \text{tlv}(t_2) \\
\text{tlv}(\neg t_1) &= \text{tlv}(t_1) \\
\text{tlv}(0) &= \emptyset
\end{align*}
\]

Definition A.4 (Type substitution). A type-substitution \( \sigma \) is a total mapping of type variables to types that is the identity everywhere but on a finite subset of \( \mathcal{Y} \), which is called the domain of \( \sigma \) and denoted by \( \text{dom}(\sigma) \). We use the notation \( \{\alpha_1, \ldots, \alpha_n\} \) to denote the type-substitution that maps \( \alpha_i \) to \( t_i \) for \( i = 1 \ldots n \). Given a substitution \( \sigma \), the range of \( \sigma \) is defined as the set of types \( \text{ran}(\sigma) = \{\sigma(\alpha) \mid \alpha \in \text{dom}(\sigma)\} \), and the set of type variables occurring in the range is defined as \( \text{tvran}(\sigma) = \bigcup_{\alpha \in \text{dom}(\sigma)} \text{var}(\sigma(\alpha)) \).
Definition A.5. Given a type $t \in \mathcal{T}$ and a type-substitution $\sigma$, the application of $\sigma$ to $t$ is co-inductively defined as follows:

\[
\begin{align*}
b \sigma &= b \\
(t_1 \times t_2) \sigma &= (t_1 \sigma) \times (t_2 \sigma) \\
(t_1 \rightarrow t_2) \sigma &= (t_1 \sigma) \rightarrow (t_2 \sigma) \\
(t_1 \lor t_2) \sigma &= (t_1 \sigma) \lor (t_2 \sigma) \\
¬t \sigma &= ¬(t \sigma)
\end{align*}
\]

Union Definition A.6. Let $\sigma_1$ and $\sigma_2$ be two substitutions such that $\text{dom}(\sigma_1) \cap \text{dom}(\sigma_2) = \emptyset$ (for short). Their union $\sigma_1 \cup \sigma_2$ is defined as:

\[
(\sigma_1 \cup \sigma_2)(\alpha) = \begin{cases} 
\sigma_1(\alpha) & \text{if } \alpha \in \text{dom}(\sigma_1) \\
\sigma_2(\alpha) & \text{if } \alpha \in \text{dom}(\sigma_2) \\
\alpha & \text{otherwise}
\end{cases}
\]

A.2 Expressions

Definition A.7 (Expressions). Let $\mathcal{C}$ be a set of constants ranged over by $c$ and $\mathcal{X}$ a countable set of expression variables ranging over $x, y, z, \ldots$. An expression $e$ is a term inductively generated by the following grammar:

\[
\text{Expressions } e ::= c \quad \text{constant} \\
\quad \langle e, e \rangle \quad \text{pair} \\
\quad \pi_i(e) \quad \text{projection} (i \in \{1, 2\}) \\
\quad \lambda^{\backslash \iota_i \rightarrow \sigma_i} x.e \quad \text{abstraction} \\
\quad e\ e \quad \text{application} \\
\quad e\ \?\ e\ :\ e \quad \text{type case} \\
\quad e;[\sigma_j]_{j \in J} \quad \text{instantiation}
\]

where $t_i, \sigma_i$ range over types, $t \in \mathcal{T}_0$ is a ground type and $\sigma_j$ ranges over type-substitutions. We write $\mathcal{E}$ to denote the set of all expressions.

A $\lambda$-abstraction comes with a non-empty sequence of arrow types (called its interface) and a possibly empty set of type-substitutions (called its decorations). When the decoration is an empty set, we write $\lambda^{\backslash \iota_i \rightarrow \sigma_i} x.e$ for short.

Since expressions are inductively generated, the accessor definitions on them can be given by induction. Given a set of type variables $\Delta$ and a set of type-substitutions $\sigma_j$, for simplicity, we use the notation $\Delta[\sigma_j]_{j \in J}$ to denote the set of type variables occurring in the applications $\alpha \sigma_j$ for all $\alpha \in \Delta, j \in J$, that is:

\[
\Delta[\sigma_j]_{j \in J} = \bigcup_{j \in J} (\bigcup_{\alpha \in \Delta} \text{var}(\sigma_j(\alpha)))
\]

Definition A.8. Let $e$ be an expression. The set $\text{fv}(e)$ of free variables of the expression $e$ is defined by induction as:

\[
\begin{align*}
\text{fv}(x) &= \{x\} \\
\text{fv}(c) &= \emptyset \\
\text{fv}(\langle e_1, e_2 \rangle) &= \text{fv}(e_1) \cup \text{fv}(e_2) \\
\text{fv}(\pi_i(e)) &= \text{fv}(c) \\
\text{fv}(\lambda^{\backslash \iota_i \rightarrow \sigma_i} x.e) &= \text{fv}(e) \setminus \{x\} \\
\text{fv}(e_1 e_2) &= \text{fv}(e_1) \cup \text{fv}(e_2) \\
\text{fv}(e_1 ? e_1 : e_2) &= \text{fv}(e) \cup \text{fv}(e_1) \cup \text{fv}(e_2) \\
\text{fv}(e;[\sigma_j]_{j \in J}) &= \text{fv}(e)
\end{align*}
\]

The set $\text{bv}(e)$ of bound variables of the expression $e$ is defined by induction as:

\[
\begin{align*}
\text{bv}(x) &= \emptyset \\
\text{bv}(c) &= \emptyset \\
\text{bv}(\langle e_1, e_2 \rangle) &= \text{bv}(e_1) \cup \text{bv}(e_2) \\
\text{bv}(\pi_i(e)) &= \text{bv}(c) \\
\text{bv}(\lambda^{\backslash \iota_i \rightarrow \sigma_i} x.e) &= \text{bv}(e) \cup \{x\} \\
\text{bv}(e_1 e_2) &= \text{bv}(e_1) \cup \text{bv}(e_2) \\
\text{bv}(e_1 ? e_1 : e_2) &= \text{bv}(e) \cup \text{bv}(e_1) \cup \text{bv}(e_2) \\
\text{bv}(e;[\sigma_j]_{j \in J}) &= \text{bv}(e)
\end{align*}
\]
The set $tv(e)$ of type variables occurring in $e$ is defined by induction as:

$$
tv(x) = \emptyset
$$
$$
tv(c) = \emptyset
$$
$$
tv((e_1, e_2)) = tv(e_1) \cup tv(e_2)
$$
$$
tv(\pi(e)) = tv(e)
$$
$$
tv(\lambda^{i_1, \ldots, i_N}_{e \sigma_j} \cdot x.e) = tv(e) \cap tv(\lambda^{i_1, \ldots, i_N}_{e} \cdot x.e)
$$
$$
tv(e_1 \ e_2) = tv(e_1) \cup tv(e_2)
$$
$$
\begin{align*}
&tv(e \ ? \ e_1 : e_2) = tv(e) \cup tv(e_1) \cap tv(e_2) \\
&tv(e_1[e_2]) = tv(e_1) \cap tv(e_2)
\end{align*}
$$
$$
\begin{align*}
&tv(e[\sigma_j]) = tv(e) \cap tv(\sigma_j)
\end{align*}
$$

An expression $e$ is closed if $fv(e)$ is empty.

Note that the set of type variables in $e[\sigma_j]_{i \in J}$ is the set returned by the “application” of $[\sigma_j]_{i \in J}$ to the set $tv(e)$.

Next, we define expression substitutions. Recall that, as stated in Section 3.1, when substituting an expression $e$ for a variable $y$ in an expression $e'$, we suppose the polymorphic type variables of $e'$ to be distinct from the monomorphic and polymorphic type variables of $e$ to avoid unwanted captures. In the discussion in Section 3.1, the definitions of the notions of polymorphic and monomorphic variables remain informal. To make them more formal, we would have to distinguish between the two by carrying around a set of type variables $\Delta$ which would contain the monomorphic variables that cannot be $\alpha$-converted. Then all definitions (such as expression substitutions, for example) would have to be parameterized with $\Delta$, making the definitions and technical developments difficult to read just because of $\alpha$-conversion. Therefore, for the sake of readability, we decided to keep the distinction between polymorphic and monomorphic variables informal.

**Definition A.9 (Expression substitution).** An expression substitution $\rho$ is a total mapping of expression variables to expressions that is the identity everywhere but on a finite subset of $\mathcal{X}$, which is called the domain of $\rho$ and denoted by $\text{dom}(\rho)$. We use the notation $\{e_1/x_1, \ldots, e_n/x_n\}$ to denote the expression substitution that maps $x_i$ into $e_i$ for $i = 1..n$.

The definitions of free variables, bound variables, and type variables are extended to expression substitutions as follows.

$$
fv(\rho) = \bigcup_{x \in \text{dom}(\rho)} fv(\rho(x)), \quad bv(\rho) = \bigcup_{x \in \text{dom}(\rho)} bv(\rho(x)), \quad tv(\rho) = \bigcup_{x \in \text{dom}(\rho)} tv(\rho(x))
$$

Next, we define the application of an expression substitution $\rho$ to an expression $e$. To avoid unwanted captures, we remind that we assume that the bound variables of $e$ do not occur in the domain of $\rho$ and that the polymorphic type variables of $e$ are distinct from the type variables occurring in $\rho$ (using $\alpha$-conversion if necessary).

**Definition A.10.** Given an expression $e \in \mathcal{E}$ and an expression substitution $\rho$, the application of $\rho$ to $e$ is defined as follows:

$$
\begin{align*}
&\sigma \rho = c \\
&(e_1, e_2) \rho = (e_1 \rho, e_2 \rho) \\
&(\pi_i(e)) \rho = \pi_i(e \rho) \\
&(\lambda^{i_1, \ldots, i_N}_{e \sigma_j} \cdot x.e) \rho = \lambda^{i_1, \ldots, i_N}_{(e \rho)} \cdot x.(e \rho) \\
&(e_1 \ e_2) \rho = (e_1 \rho) \ e_2 \rho \\
&(e ? \ e_1 : e_2) \rho = \begin{cases} e \rho & \text{if } x \in \text{dom}(\rho) \\
\rho(x) & \text{if } x \notin \text{dom}(\rho) \end{cases} \\
&(e[\sigma_j] \rho) = ([\sigma_j] \rho)(e \rho)
\end{align*}
$$

In the case for instantiation $(e[\sigma_j]_{i \in J}) \rho$, the $\sigma_j$ operate on the polymorphic type variables, which we assume distinct from the variables in $\rho$ (using $\alpha$-conversion if necessary). As a result, we have $tv(\rho) \cap \bigcup_{j \in J} \text{dom}(\sigma_j) = \emptyset$. Similarly, in the abstraction case, we have $x \notin \text{dom}(\rho)$.

Next, we define the relabeling of an expression $e$ with a set of type-substitutions $[\sigma_j]_{i \in J}$, which consists in propagating the $\sigma_j$ to the $\lambda$-abstractions in $e$ if needed. We suppose that the polymorphic type variables in $e$ are distinct from the type variables in the range of $\sigma_j$ (this is always possible using $\alpha$-conversion).

**Definition A.11 (Relabeling).** Given an expression $e \in \mathcal{E}$ and a set of type-substitutions $[\sigma_j]_{i \in J}$, we define the relabeling of $e$ with $[\sigma_j]_{i \in J}$, written $e @ [\sigma_j]_{i \in J}$, as $e$ if $tv(e) \cap \bigcup_{j \in J} \text{dom}(\sigma_j) = \emptyset$, and otherwise as follows:

$$
\begin{align*}
&(e_1, e_2) @ [\sigma_j]_{i \in J} = (e_1 @ [\sigma_j]_{i \in J}, e_2 @ [\sigma_j]_{i \in J}) \\
&(\pi_i(e)) @ [\sigma_j]_{i \in J} = \pi_i(e @ [\sigma_j]_{i \in J}) \\
&(e_1 \ e_2) @ [\sigma_j]_{i \in J} = (e_1 @ [\sigma_j]_{i \in J}) \ e_2 @ [\sigma_j]_{i \in J} \\
&(e ? \ e_1 : e_2) @ [\sigma_j]_{i \in J} = e @ [\sigma_j]_{i \in J} ? e_1 @ [\sigma_j]_{i \in J} : e_2 @ [\sigma_j]_{i \in J} \\
&(e[\sigma_j]_{i \in J}) @ [\sigma_j]_{i \in J} = e @ ([\sigma_j]_{i \in J} \circ [\sigma_j]_{i \in J}) \\
&(\lambda^{i_1, \ldots, i_N}_{e \sigma_j} \cdot x.e) @ [\sigma_j]_{i \in J} = \lambda^{i_1, \ldots, i_N}_{(e \rho)} \cdot x.(e \rho)
\end{align*}
$$
The substitutions are not propagated if they do not affect the variables of e (i.e., if \( tv(e) \cap \bigcup_{j \in J} \text{dom}(\sigma_j) = \emptyset \)). In particular, constants and variables are left unchanged, as they do not contain any type variable. Suppose now that \( tv(e) \cap \bigcup_{j \in J} \text{dom}(\sigma_j) \neq \emptyset \). In the abstraction case, the propagated substitutions are composed with the decorations of the abstraction, without propagating them further down in the body. Propagation in the body occurs, whenever is needed, that is, during either reduction (see (Rappl) in Section A.4) or type-checking (see (abstr) in Section A.3). In the instantiation case \( e[\sigma_j]_{j \in J} \), we propagate the result of the composition of \([\sigma_j]_{j \in J}\) with \([\sigma_j]_{j \in J}\) in e. The remaining cases are simple inductive cases. Finally notice that in a type case \( e \in t \neq e_1 : e_2 \), we do not apply \([\sigma_j]_{j \in J}\) to t, simply because t is ground.

### A.3 Type System

Because of the type directed nature of our calculus (i.e., the presence of the type-case expression), its dynamic semantics is defined only for well-typed expressions. Therefore, we introduce the type system before giving the reduction rules.

**Definition A.12 (Typing environment).** A typing environment \( \Gamma \) is a finite mapping from expression variables \( \mathcal{F} \) to types \( \mathcal{T} \), and written as a finite set of pairs \( \{(x_1 : t_1), \ldots, (x_n : t_n)\} \). The set of expression variables defined in \( \Gamma \) is called the domain of \( \Gamma \), denoted by \( \text{dom}(\Gamma) \). The set of type variables occurring in \( \Gamma \), that is, \( \bigcup_{(e : t) \in \Gamma} \text{var}(t) \), is denoted by \( \text{var}(\Gamma) \). If \( \Gamma \) is a type environment, then \( \Gamma, (x : t) \) is the type environment defined as

\[
(\Gamma, (x : t))(y) = \begin{cases} t & \text{if } y = x \\ \Gamma(y) & \text{otherwise} \end{cases}
\]

We extend the definition of type-substitution application to type environments by applying the type-substitution to each type in the type environment as follows:

\[
\Gamma\sigma = \{(x : t\sigma) \mid (x : t) \in \Gamma\}
\]

The typing judgment for expressions has the form \( \Delta ; \Gamma \vdash e : t \), which states that under the set \( \Delta \) of (monomorphic) type variables and the typing environment \( \Gamma \) the expression \( e \) has type \( t \). When \( \Delta \) and \( \Gamma \) are both empty, we write \( \vdash e : t \) for short. We assume that there is a basic type \( b_0 \) for each constant \( c \). We write \( \sigma \not\in \Delta \) as abbreviation for \( \text{dom}(\sigma) \cap \Delta = \emptyset \). The typing rules are given in Figure 3 which are the same in Section 3 except for the rules for products.

![Figure 3. Typing rules](image)

### A.4 Operational Semantics

**Definition A.13 (Values).** An expression \( e \) is a value if it is closed, well-typed (i.e., \( \vdash e : t \) for some type \( t \)), and produced by the following grammar:

\[
\text{Values} \quad v ::= e \mid (v, v) \mid \lambda^{(\bigwedge_{j \in J} t_j \rightarrow s_j)}_{j \in J} x.e
\]
We write \( \mathcal{V} \) to denote the set of all values.

**Definition A.14 (Context).** Let the symbol \([_]\) denote a hole. A context \(C[_]\) is an expression with a hole:

\[
\begin{align*}
\text{Contexts} & \quad C[_] \ ::= \ [_] \\
& \quad (C[_], e) \mid (e, C[_]) \\
& \quad C[\_] \mid e \ C[_] \\
& \quad C[_] \mid \epsilon \ C[_] \\
& \quad \pi_i(C[_])
\end{align*}
\]

An evaluation context \(E[_]\) is a context that implements outermost leftmost reduction:

\[
\begin{align*}
\text{Evaluation Contexts} & \quad E[_] \ ::= \ [_] \\
& \quad (E[_], e) \mid (v, E[_]) \\
& \quad E[\_] \mid v \ E[_] \\
& \quad E[\] \mid e : e \\
& \quad \pi_i(E[_])
\end{align*}
\]

We use \(C[e]\) and \(E[e]\) to denote the expressions obtained by replacing \(e\) for the hole in \(C[_]\) and \(E[_]\), respectively.

We define a small-step call-by-value operational semantics for the calculus. The semantics is given by the relation \(\Rightarrow\), which is shown in Figure 4. There are four notions of reduction: one for projections, one for applications, one for applications, and one for instantiation, plus context closure. Henceforth we will establish all the properties for the reduction using generic contexts but, of course, these holds also when the more restrictive evaluation contexts are used.

\[
\begin{align*}
\text{Notions of reduction:} & \quad \pi_i(v_1, v_2) \Rightarrow v_i \\
\text{(Rappl)} & \quad \lambda^{\neg \ni \in \mathcal{I} \mapsto \mathcal{S}}. x.e' \Rightarrow (e' \triangleright \pi_i_{\mathcal{I}} \{ v / x \}) \\
& \quad \text{where } \mathcal{P} = \{ j \in J \mid \ni \in \mathcal{I} \mapsto v : t, \mathcal{S}_j \} \\
\text{(Rcase)} & \quad (v \triangleright t ? e_1 : e_2) \Rightarrow \begin{cases} \\
& \quad e_1 \text{ if } v : t \\
& \quad e_2 \text{ otherwise} \\
\text{(Rinst)} & \quad e[\pi_i_{\mathcal{I}}] \Rightarrow e[\pi_i_{\mathcal{I}}]
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{Context closure:} & \quad e \Rightarrow e' \\
\text{(Rctx)} & \quad C[e] \Rightarrow C[e']
\end{align*}
\]

**Figure 4.** Operational semantics of the calculus

The (Rproj) rule is the standard projection rule while the other notions of reduction have already been explained in Section 3.3.

We used a call-by-value semantics for application to ensure the type soundness property: subject reduction (or type preservation) and progress (closed and well-typed expressions which are not values can be reduced), which are discussed in Section B.2. To understand why, consider each basic reduction rule in turn.

The requirement that the argument of a projection must be a value is imposed to ensure that the property of subject reduction holds. Consider the expression \(e = \pi_1(e_1, e_2)\) where \(e_1\) is an expression of type \(t_1\) (different from \(0\)) and \(e_2\) is a (diverging) expression of type \(0\). Clearly, the type system assigns the type \(t_1 \times 0\) to \((e_1, e_2)\). In our system, a product type with an empty component is itself empty, and thus \(e\) has type \(0\). Therefore the type of the projection as well has type \(0\) (since \(0 \leq 0 \times 0 \), then by subsumption \((e_1, e_2) : 0 \times 0 \), and the result follows from the (proj) typing rule). If it were possible to reduce a projection when the argument is not a value, then \(e\) could be reduced to \(e_1\), which has type \(t_1\) : type preservation would be violated.

Likewise, the reduction rule for applications requires the argument to be a value. Let us consider the application \(\lambda^{(t \rightarrow t \times t) \rightarrow (s \rightarrow s \times s)}.(x, z)(e)\), where \(\triangleright e : t \triangleright s \triangleright s\). The type system assigns to the abstraction the type \((t \rightarrow t \times t) \rightarrow (s \rightarrow s \times s)\), which is a subtype of \((t \triangleright s) \rightarrow ((t \times t) \triangleright (s \times s))\). By subsumption, the abstraction has type \((t \triangleright s) \rightarrow ((t \times t) \triangleright (s \times s))\), and thus the application has type \((t \times t) \triangleright (s \times s)\). If the semantics permits to reduce an application when the argument is not a value, then this application could be reduced to the expression \((e, e)\), which has type \((t \triangleright s) \times (t \triangleright s)\) but not \((t \times t) \triangleright (s \times s)\).
Finally, if we allowed \( e \in \Gamma \) to reduce to \( e_1 \) when \( \vdash e : t \) but \( e \) is not a value, we could break type preservation. For example, assume that \( \vdash e : \emptyset \). Then the type system would not check anything about the branches \( e_1 \) and \( e_2 \) (see the typing rule (case) in Figure 3) and so \( e_1 \) could be ill-typed.

Notice that in all these cases the usage of values ensures subject reduction but it is not a necessary condition: in some cases weaker constraints could be used. For instance, in order to check whether an expression is a list of integers, in general it is not necessary to fully evaluate the whole list: the head and the type of the tail are all that is needed. Studying weaker conditions for the reduction rules is an interesting topic, which we leave for future work, in particular, in the view of adapting our framework to lazy languages.

## B. Properties of the Type System

In this section we present some properties of our type system. First, we study its syntactic meta-theory: in particular, we prove admissibility of the intersection rule, a generation lemma for values, and that substitutions preserve typing. These properties are needed to prove soundness, the fundamental property which links every type system of a calculi with its operational counterpart: well-typed expressions do not go wrong [18]. Next, we prove that the explicitly-typed calculus is able to derive the same typing judgments as the BCD intersection type system defined by Barendregt, Coppo, and Dezani [1]. Finally, we prove that the expressions of the form \( e[\sigma]_{j \in e} \) are redundant insofar as their presence in the calculus does not increase its expressive power.

### B.1 Syntactic meta-theory

**Lemma B.1.** If \( \Delta \Gamma \vdash e : t \) and \( \text{var}(\Gamma) \subseteq \Delta \), then \( \text{var}(\Delta') \subseteq \Delta' \) holds for every judgment \( \Delta' \Gamma \vdash e' : t' \) in the derivation of \( \Delta \Gamma \vdash e : t \).

*Proof.* By induction on the derivation of \( \Delta \Gamma \vdash e : t \).

**Lemma B.2.** (Admissibility of intersection introduction). Let \( e \) be an expression. If \( \Delta \Gamma \vdash e : t \) and \( \Delta \Gamma \vdash e : t' \), then \( \Delta \Gamma \vdash e : t \land t' \).

*Proof.* The proof proceeds by induction on the two typing derivations. First, assume that these two derivations end with an instance of the same rule corresponding to the top-level constructor of \( e \).

- **(const):** both derivations end with an instance of (const):
  
  \[ \Delta \Gamma \vdash e : b_c \] \[ \Delta \Gamma \vdash e : b_c \]

  Trivially, we have \( b_c \land b_c \simeq b_c \), by subsumption, the result follows.

- **(var):** both derivations end with an instance of (var):
  
  \[ \Delta \Gamma \vdash x : \Gamma(x) \] \[ \Delta \Gamma \vdash x : \Gamma(x) \]

  Trivially, we have \( \Gamma(x) \land \Gamma(x) \simeq \Gamma(x) \), by subsumption, the result follows.

- **(pair):** both derivations end with an instance of (pair):
  
  \[ \Delta \Gamma \vdash e_1 : t_1 \] \[ \Delta \Gamma \vdash e_2 : t_2 \]

  By induction, we have \( \Delta \Gamma \vdash e_1 : (t_1 \land t'_1) \). Then the rule (pair) gives us \( \Delta \Gamma \vdash (e_1, e_2) : (t_1 \land t'_1) \times (t_2 \land t'_2) \). Moreover, because intersection distributes over products, we have \( (t_1 \land t'_1) \times (t_2 \land t'_2) \simeq (t_1 \times t'_1) \times (t_2 \times t'_2) \). Then by (subsum), we have \( \Delta \Gamma \vdash (e_1, e_2) : (t_1 \times t'_1) \land (t_2 \times t'_2) \).

- **(proj):** both derivations end with an instance of (proj):
  
  \[ \Delta \Gamma \vdash e' : t_1 \times t_2 \] \[ \Delta \Gamma \vdash \pi_i(e') : t_i \]

  By induction, we have \( \Delta \Gamma \vdash e' : (t_1 \times t_2) \land (t'_1 \times t'_2) \). Since \( (t_1 \times t'_1) \times (t_2 \times t'_2) \simeq (t_1 \times t_2) \land (t'_1 \times t'_2) \) (see the case of (pair)), by (subsum), we have \( \Delta \Gamma \vdash e' : (t_1 \land t'_1) \times (t_2 \land t'_2) \). Then the rule (proj) gives us \( \Delta \Gamma \vdash \pi_i(e') : t_i \land t'_i \).

- **(appl):** both derivations end with an instance of (appl):
  
  \[ \Delta \Gamma \vdash e_1 : t_1 \rightarrow t_2 \] \[ \Delta \Gamma \vdash e_2 : t_1 \]

  \[ \Delta \Gamma \vdash e_1 : t_2 \] \[ \Delta \Gamma \vdash e_2 : t_1 \]

  By induction, we have \( \Delta \Gamma \vdash e_1 : (t_1 \rightarrow t_2) \land (t'_1 \rightarrow t'_2) \) and \( \Delta \Gamma \vdash e_2 : t_1 \land t'_1 \). Because intersection distributes over arrows, we have \( (t_1 \land t'_1) \rightarrow (t_2 \land t'_2) \simeq (t_1 \lor t'_1) \rightarrow (t_2 \land t'_2) \). Then by the rule (subsum), we get \( \Delta \Gamma \vdash e_1 : (t_1 \land t'_1) \rightarrow (t_2 \land t'_2) \). Finally, by applying (appl), we get \( \Delta \Gamma \vdash e_1 e_2 : t_2 \land t'_2 \) as expected.
both derivations end with an instance of (abstr):
\[
\Delta' = \Delta \cup \text{var}(\bigwedge_{i \in I,j \in J} t_i \sigma_j \rightarrow s_i \sigma_j)
\]
\[
\Delta_1 \Gamma \vdash \lambda^{\Delta_1 \Gamma}_{[\sigma_1]} x. e': \bigwedge_{i \in I,j \in J} t_i \sigma_j \rightarrow s_i \sigma_j
\]
\[
\forall i \in I,j \in J, \Delta' \Gamma, (x : \tau_i) \vdash e'[\sigma_j] : s_i \sigma_j
\]

Proof.

\[
\Delta' \Gamma \vdash e'[\sigma_j] : s_i \sigma_j
\]
\[
\Delta \Gamma \vdash e'[\sigma_j] : t \sigma
\]

By induction, the result follows.

By subsumption, the result follows.

\[
\forall i \in I,j \in J, \Delta' \Gamma, (x : \tau_i) \vdash e'[\sigma_j] : s_i \sigma_j
\]

\[
(\bigwedge_{i \in I,j \in J} t_i \sigma_j \rightarrow s_i \sigma_j) \land (\bigwedge_{i \in I,j \in J} t_i \sigma_j \rightarrow s_i \sigma_j)
\]

\[
\Delta_3 \Gamma \vdash (e_0 \in t? e_1 : e_2) : s'
\]

\[
\Delta_4 \Gamma \vdash (e_0 \in t? e_1 : e_2) : s'
\]

\[
\Delta_3 \Gamma \vdash (e_0 \in t? e_1 : e_2) : s'
\]

\[
\Delta_4 \Gamma \vdash (e_0 \in t? e_1 : e_2) : s'
\]

\[
\Delta_3 \Gamma \vdash (e_0 \in t? e_1 : e_2) : s'
\]

\[
\Delta_4 \Gamma \vdash (e_0 \in t? e_1 : e_2) : s'
\]

\[
\Delta_3 \Gamma \vdash e': t \land t'. \text{ Since } \sigma \nsubseteq \Delta, \text{ the rule (inst) gives } \Delta_3 \Gamma \vdash e'[\sigma] : (t \land t')\sigma, \text{ that is } \Delta_3 \Gamma \vdash e'[\sigma] : (t \land t')\sigma.
\]

By induction, we have \( e : (t \land t') \) as expected.

Lemma B.3 (Generation for values). Let \( v \) be a value. Then

1. If \( \Delta_3 \Gamma \vdash v : b \), then \( v \) is a constant \( c \) and \( b \leq b \).

2. If \( \Delta_3 \Gamma \vdash v : t_1 \times t_2 \), then \( v \) has the form of \((v_1, v_2)\) with \( \Delta_3 \Gamma \vdash v_i : t_i \).

3. If \( \Delta_3 \Gamma \vdash v : s \) and \( \Delta_3 \Gamma \vdash v : t \rightarrow s \), then \( v \) has the form of \( \lambda^{\Delta_3 \Gamma}_{[\sigma_j]} x. e_0 \) with \( \bigwedge_{i \in I,j \in J} (t_i \sigma_j \rightarrow s_i \sigma_j) \leq t \rightarrow s \).

Proof.

By a simple examination of the rules it is easy to see that a derivation for \( \Delta_3 \Gamma \vdash v : t \) is always formed by an instance of the rule corresponding to the kind of \( v \) (i.e. \( \text{const} \) for constants, \( \text{pair} \) for pairs, and \( \text{abstr} \) for abstractions), followed by zero or more instances of \( \text{subsum} \). By induction on the depth of the derivation it is then easy to prove that if \( \Delta_3 \Gamma \vdash v : t \) is derivable, then \( t \neq 0 \). The lemma then follows.
by induction on the number of instances of the subsumption rule that end the derivation of \( \Delta; \Gamma \vdash v : t \).
The base cases are straightforward, while the inductive cases are:

\[
\Delta; \Gamma \vdash v : b; \quad v \text{ is by induction a constant } c \text{ such that } b_c \leq b.
\]
\[
\Delta; \Gamma \vdash v : t_1 \times t_2; \quad v \text{ is by induction a pair } (v_1, v_2) \text{ and } t' \text{ is of form } (t'_1 \times t'_2) \text{ such that } \Delta; \Gamma \vdash v_i : t'_i.
\]

Here we use the fact that the type of a value cannot be \( \emptyset \): since \( 0 \neq (t'_1 \times t'_2) \subseteq (t_1 \times t_2) \), then we have \( t'_i \leq t_i \). Finally, by (subsum), we have \( \Delta; \Gamma \vdash v_i : t_i \).

\[
\Delta; \Gamma \vdash v : t \rightarrow s; \quad v \text{ is by induction an abstraction } \lambda^{s \in t_0 \rightarrow s_0} x.e_0 \text{ such that } \bigwedge_{i \in I} j \in J (t_i \sigma_j \rightarrow s_i \sigma_j) \leq t \rightarrow s.
\]

\[
\square
\]

**Lemma B.4.** Let \( e \) be an expression and \( \sigma_1, \sigma_2 \in \mathcal{K} \) two sets of type substitutions. Then

\[
(e \circ [\sigma_1]) @ \sigma_2 \in \mathcal{K} = e \circ ([\sigma_2] \circ [\sigma_1])
\]

**Proof.** By induction on the structure of \( e \).

\[
e = c:
\]

\[
(e \circ [\sigma_1]) @ \sigma_2 = c
\]

\[
e = x:
\]

\[
(x \circ [\sigma_1]) @ \sigma_2 = x
\]

\[
e = (e_1, e_2):
\]

\[
((e_1, e_2) \circ [\sigma_1]) @ \sigma_2 = (e_1 \circ [\sigma_1]) @ \sigma_2, (e_2 \circ [\sigma_1]) @ \sigma_2
\]

\[
e = \pi(e):
\]

\[
(\pi(e) \circ [\sigma_1]) @ \sigma_2 = \pi((e \circ [\sigma_1]) @ \sigma_2)
\]

\[
e = e_1, e_2:
\]

\[
((e_1, e_2) \circ [\sigma_1]) @ \sigma_2 = (e_1 \circ [\sigma_1]) @ \sigma_2, (e_2 \circ [\sigma_1]) @ \sigma_2
\]

\[
e = e_0 \in t ? e_1, e_2:
\]

\[
(e_0 @ [\sigma_1]) @ \sigma_2 = (e_1 @ [\sigma_1]) @ \sigma_2, (e_2 @ [\sigma_1]) @ \sigma_2
\]

\[
e = e[\sigma_1]_i \in I:
\]

\[
((e'[\sigma_1]) _i @ [\sigma_1]) @ \sigma_2 = (e_1 \circ [\sigma_1]) @ \sigma_2, (e_2 \circ [\sigma_1]) @ \sigma_2
\]

\[
\square
\]

**Lemma B.5.** Let \( e \) be an expression, \( \sigma \) an expression substitution and \( \sigma_1 \in J \) a set of type substitutions such that \( tv(\sigma) \cap \bigcup_{j \in J} \text{dom}(\sigma_j) = \emptyset \). Then \( (e \circ \sigma) @ [\sigma_1] \in J = (e \circ [\sigma_1]) \cdot \sigma \).

**Proof.** By induction on the structure of \( e \).

\[
e = c:
\]

\[
(c \circ [\sigma_1]) @ [\sigma_1] = c @ [\sigma_1]
\]

\[
e = e:
\]

\[
(e \circ [\sigma_1]) @ [\sigma_1] = e @ [\sigma_1]
\]

\[
e = e_1, e_2:
\]

\[
((e_1, e_2) \circ [\sigma_1]) @ [\sigma_1] = (e_1 \circ [\sigma_1]) @ [\sigma_1], (e_2 \circ [\sigma_1]) @ [\sigma_1]
\]

\[
e = \pi(e):
\]

\[
(\pi(e) \circ [\sigma_1]) @ [\sigma_1] = \pi((e \circ [\sigma_1]) @ [\sigma_1])
\]

\[
e = e_1, e_2:
\]

\[
((e_1, e_2) \circ [\sigma_1]) @ [\sigma_1] = (e_1 \circ [\sigma_1]) @ [\sigma_1], (e_2 \circ [\sigma_1]) @ [\sigma_1]
\]

\[
e = e_0 \in t ? e_1, e_2:
\]

\[
(e_0 @ [\sigma_1]) @ [\sigma_1] = (e_1 @ [\sigma_1]) @ [\sigma_1], (e_2 @ [\sigma_1]) @ [\sigma_1]
\]

\[
e = e[\sigma_1]_i \in I:
\]

\[
((e'[\sigma_1]) _i @ [\sigma_1]) @ [\sigma_1] = (e_1 \circ [\sigma_1]) @ [\sigma_1], (e_2 \circ [\sigma_1]) @ [\sigma_1]
\]

\[
\square
\]
$e = x$: if $x \notin \text{dom}(\varrho)$, then

$$(x\varrho)@_{[\sigma_j]_{j \in J}} = x@_{[\sigma_j]_{j \in J}}$$

$$_{J=k \in K}$$

$$= x$$

$$= x\varrho$$

$$(x\varrho)@_{[\sigma_j]_{j \in J}} = (x@_{[\sigma_j]_{j \in J}})$$

$$= (x@_{[\sigma_j]_{j \in J}})$$

Otherwise, let $\varrho(x) = e'$. As $\text{tv}(\varrho) \cap \bigcup_{j \in J} \text{dom}(\sigma_j) = \emptyset$, we have $\text{tv}(e') \cap \bigcup_{j \in J} \text{dom}(\sigma_j) = \emptyset$. Then

$$(x\varrho)@_{[\sigma_j]_{j \in J}} = e'@_{[\sigma_j]_{j \in J}}$$

$$= e'$$

$$= x\varrho$$

$$(x\varrho)@_{[\sigma_j]_{j \in J}} = (x@_{[\sigma_j]_{j \in J}})$$

$e = (e_1, e_2)$:

$$((e_1, e_2)\varrho)@_{[\sigma_j]_{j \in J}} = ((e_1, e_2)\varrho)@_{[\sigma_j]_{j \in J}}$$

$$= (e_1, e_2)\varrho@_{[\sigma_j]_{j \in J}}$$

$$= (e_1, e_2)\varrho@_{[\sigma_j]_{j \in J}}$$

$e = \pi_i(e')$:

$$(\pi_i(e')\varrho)@_{[\sigma_j]_{j \in J}} = (\pi_i(e')\varrho)@_{[\sigma_j]_{j \in J}}$$

$$= \pi_i(e')@_{[\sigma_j]_{j \in J}}$$

$$= \pi_i(e')@_{[\sigma_j]_{j \in J}}$$

$$= (\pi_i(e'))@_{[\sigma_j]_{j \in J}}$$

$e = e_1 e_2$:

$$((e_1 e_2)\varrho)@_{[\sigma_j]_{j \in J}} = ((e_1 e_2)\varrho)@_{[\sigma_j]_{j \in J}}$$

$$= (e_1 e_2)\varrho@_{[\sigma_j]_{j \in J}}$$

$$= (e_1 e_2)\varrho@_{[\sigma_j]_{j \in J}}$$

$$= (e_1 e_2)\varrho@_{[\sigma_j]_{j \in J}}$$

$e = \lambda_{[\sigma_k]_{k \in K}}^\leftrightarrow: x.e'$:

$$((\lambda_{[\sigma_k]_{k \in K}}^\leftrightarrow x.e')\varrho)@_{[\sigma_j]_{j \in J}} = ((\lambda_{[\sigma_k]_{k \in K}}^\leftrightarrow x.e')\varrho)@_{[\sigma_j]_{j \in J}}$$

$$= \lambda_{[\sigma_k]_{k \in K}}^\leftrightarrow x.(e')@_{[\sigma_j]_{j \in J}}$$

$$= \lambda_{[\sigma_k]_{k \in K}}^\leftrightarrow x.(e')@_{[\sigma_j]_{j \in J}}$$

$$= ((\lambda_{[\sigma_k]_{k \in K}}^\leftrightarrow x.e')@_{[\sigma_j]_{j \in J}})\varrho$$

$e = e_0 e t ? e_1 : e_2$:

$$((e_0 e t ? e_1 : e_2)\varrho)@_{[\sigma_j]_{j \in J}}$$

$$= (e_0 e t ? e_1 : e_2)\varrho@_{[\sigma_j]_{j \in J}}$$

$$= (e_0 e t ? (e_1 : e_2)\varrho)@_{[\sigma_j]_{j \in J}}$$

$$= (e_0 e t ? (e_1 : e_2)\varrho)@_{[\sigma_j]_{j \in J}}$$

$$= (e_0 e t ? (e_1 : e_2)\varrho)@_{[\sigma_j]_{j \in J}}$$

$e = e'[\sigma_k]_{k \in K}$: using $\alpha$-conversion on the polymorphic type variables of $e$, we can assume $\text{tv}(\varrho) \cap \bigcup_{k \in K} \text{dom}(\sigma_k) = \emptyset$. Consequently, we have $\text{tv}(\varrho) \cap \bigcup_{k \in K} \text{dom}(\sigma_k) = \emptyset$, and we deduce

$$(e'[\sigma_k]_{k \in K})@_{[\sigma_j]_{j \in J}}$$

$$= ((e'[\sigma_k]_{k \in K})@_{[\sigma_j]_{j \in J}})$$

$$= ((e'[\sigma_k]_{k \in K})@_{[\sigma_j]_{j \in J}})$$

$$= ((e'[\sigma_k]_{k \in K})@_{[\sigma_j]_{j \in J}})$$

$\square$

**Lemma B.6 ([Expression] substitution lemma).** Let $e, e_1, \ldots, e_n$ be expressions, $x_1, \ldots, x_n$ distinct variables, and $t, t_1, \ldots, t_n$ types. If $\Delta; \Gamma, (x_1 : t_1), \ldots, (x_n : t_n) \vdash e : t$ and $\Delta; \Gamma, e : t_i$ for all $i$, then

$$\Delta; \Gamma \vdash e'[x_1/x_1, \ldots, x_n/x_n] : t.$$

**Proof.** By induction on the typing derivations for $\Delta; \Gamma, (x_1 : t_1), \ldots, (x_n : t_n) \vdash e : t$. We simply “plug” a copy of the derivation for $\Delta; \Gamma, e : t_i$ wherever the rule (var) is used for variable $x_i$. For simplicity, in what follows, we write $\Gamma'$ for $\Gamma, (x_1 : t_1), \ldots, (x_n : t_n)$ and $\varrho$ for $\{e_1/\Delta; \Gamma, e : t_1, \ldots, e_n/\Delta; \Gamma, e : t_n\}$. We proceed by a case analysis on the last applied rule.
(const): straightforward.
(var): \( e = x \) and \( \Delta \vdash \Gamma \vdash x : \Gamma'(x) \).

If \( x = x_i \), then \( \Gamma'(x) = t_i \) and \( x_i \sigma = e_i \). From the premise, we have \( \Delta \vdash e_i : t_i \). The result follows. Otherwise, \( \Gamma'(x) = \Gamma(x) \) and \( x \sigma = x \). Clearly, we have \( \Delta \vdash x : \Gamma(x) \). Thus the result follows as well.

(pair): consider the following derivation:

\[
\begin{array}{cc}
\Delta \vdash e_1 : t_1 & \Delta \vdash e_1 : t_1 \\
\hline
\Delta \vdash (e_1, e_2) : (t_1 \times t_2) & (pair)
\end{array}
\]

By applying the induction hypothesis twice, we have \( \Delta \vdash e_1 : t_i \). By (pair), we get \( \Delta \vdash (e_1, e_2) : (t_1 \times t_2) \), that is, \( \Delta \vdash (e_1, e_2) : (t_1 \times t_2) \).

(proj): consider the following derivation:

\[
\begin{array}{c}
\Delta \vdash e' : t_1 \times t_2 \\
\hline
\Delta \vdash \pi_i(e') : t_i & (proj)
\end{array}
\]

By induction, we have \( \Delta \vdash e' : t_1 \times t_2 \). Then the rule (proj) gives us \( \Delta \vdash \pi_i(e') : t_i \), that is \( \Delta \vdash \pi_i(e') : t_i \).

(abstr): consider the following derivation:

\[
\begin{array}{c}
\forall i \in I, j \in J, \Delta \vdash \Gamma, \langle x : t_i \sigma_j \rangle \vdash e' \circ \sigma_j : s_i \sigma_j \\
\Delta' = \Delta \cup \text{var}(\lambda_{i \in I, j \in J} t_i \sigma_j \rightarrow s_i \sigma_j) \\
\Delta \vdash \Gamma' \vdash \lambda_{i \in I, j \in J} t_i \sigma_j \rightarrow s_i \sigma_j \\
\end{array}
\]

By \( \alpha \)-conversion, we can ensure that \( \text{tv}(\sigma) \cap \bigcup_{j \in J} \text{dom}(\sigma_j) = \emptyset \). By induction, we have \( \Delta \vdash \Gamma, \langle x : t_i \sigma_j \rangle \vdash (e' \circ \sigma_j) : s_i \sigma_j \) for all \( i \in I \) and \( j \in J \). Because \( \text{tv}(\sigma) \cap \text{dom}(\sigma_j) = \emptyset \), by Lemma B.5, we get \( \Delta \vdash \Gamma \vdash \langle x : t_i \sigma_j \rangle \vdash (e' \circ \sigma_j) : s_i \sigma_j \). Then by applying (abstr), we obtain \( \Delta \vdash \Gamma \vdash \lambda_{i \in I, j \in J} t_i \sigma_j \rightarrow s_i \sigma_j \). That is, \( \Delta \vdash \Gamma \vdash \langle \lambda_{i \in I, j \in J} t_i \sigma_j \rightarrow s_i \sigma_j \rangle : \bigcup_{i \in I, j \in J} t_i \sigma_j \rightarrow s_i \sigma_j \) (because \( \text{tv}(\sigma) \cap \bigcup_{j \in J} \text{dom}(\sigma_j) = \emptyset \)).

(case): consider the following derivation:

\[
\begin{array}{c}
\Delta \vdash e_0 : t' \\
\hline
\Delta \vdash e_1 : s & (case)
\end{array}
\]

By induction, we have \( \Delta \vdash e_0 : t' \) and \( \Delta \vdash e_1 : s \) (for \( i \) such that \( \Delta \vdash e_i \vdash \text{type-checked in the original derivation} \)). Then the rule (case) gives us \( \Delta \vdash e_0 e_i t? e_1 e_2 \vdash s \).

(inst):

\[
\begin{array}{c}
\Delta \vdash e' : s \sigma \vdash \Delta \\
\hline
\Delta \vdash e'[\sigma] : s & (inst)
\end{array}
\]

Using \( \alpha \)-conversion on the polymorphic type variables of \( e \), we can assume \( \text{tv}(\sigma) \cap \text{dom}(\sigma) = \emptyset \). By induction, we have \( \Delta \vdash e' \vdash s \). Since \( \sigma \notin \Delta \), by applying (inst) we obtain \( \Delta \vdash (e'[\sigma]) : s \sigma \), that is, \( \Delta \vdash (e'[\sigma]) : s \sigma \) because \( \text{tv}(\sigma) \cap \text{dom}(\sigma) = \emptyset \).

(inter):

\[
\begin{array}{c}
\forall j \in J, \Delta \vdash e' : s \sigma \vdash \Delta \\
\hline
\Delta \vdash e'[\sigma] : s & \sigma \vdash s \sigma & (inter)
\end{array}
\]

By induction, for all \( j \in J \) we have \( \Delta \vdash e'[\sigma] : s \sigma \), that is \( \Delta \vdash (e'[\sigma]) : s \sigma \). Then by applying (inter) we get \( \Delta \vdash (e'[\sigma]) : s \sigma \), that is \( \Delta \vdash (e'[\sigma]) \vdash s \sigma \).

(subsum): consider the following derivation:

\[
\begin{array}{c}
\Delta \vdash e' : s \\
\hline
\Delta \vdash e' : t & (subsum)
\end{array}
\]

By induction, we have \( \Delta \vdash e' : s \). Then the rule (subsum) gives us \( \Delta \vdash e' : t \).
Definition B.7. Given two typing environments $\Gamma_1, \Gamma_2$, we define their intersection as

$$ (\Gamma_1 \cap \Gamma_2)(x) = \begin{cases} \Gamma_1(x) \land \Gamma_2(x) & \text{if } x \in \mathrm{dom}(\Gamma_1) \cap \mathrm{dom}(\Gamma_2) \\ \text{undefined} & \text{otherwise} \end{cases} $$

We define $\Gamma_2 \subseteq \Gamma_1$ if $\Gamma_2(x) \subseteq \Gamma_1(x)$ for all $x \in \mathrm{dom}(\Gamma_1)$, and $\Gamma_1 \cong \Gamma_2$ if $\Gamma_1 \subseteq \Gamma_2$ and $\Gamma_2 \subseteq \Gamma_1$.

Given an expression $e$ and a set $\Delta$ of (monomorphic) type variables, we write $e \notin \Delta$ if $\sigma_j \notin \Delta$ for all the type substitution $\sigma_j$ that occur in a subterm of $e$ of the form $e'[\sigma_j]_{j \in J}$ (in other terms, we do not consider the substitutions that occur in the decorations of $\lambda$-abstractions).

Lemma B.8 (Weakening). Let $e$ be an expression, $\Gamma, \Gamma'$ two typing environments and $\Delta'$ a set of type variables. If $\Delta \triangleright e : t$, $\Gamma' \supseteq \Gamma$ and $e \notin \Delta'$, then $\Delta \cup \Delta' \triangleright \Gamma' \triangleright e : t$.

Proof. By induction on the derivation of $\Delta \triangleright e : t$. We perform a case analysis on the last applied rule.

(const): straightforward.

(var): $\Delta \triangleright e : \Gamma(x)$. It is clear that $\Delta \cup \Delta' \triangleright \Gamma' : \Gamma(x)$ by (var). Since $\Gamma'(x) \leq \Gamma(x)$, by (subsum), we get $\Delta \cup \Delta' \triangleright \Gamma' \triangleright e : \Gamma(x)$.

(pair): consider the following derivation:

$$ \begin{array}{c} \Delta \triangleright e_1 : t_1 \\ \Delta \triangleright e_2 : t_2 \end{array} \quad \Delta \triangleright (e_1, e_2) : t_1 \times t_2 \quad \text{(pair)} $$

By applying the induction hypothesis twice, we have $\Delta \cup \Delta' \triangleright e : t_1$. Then by (pair), we get $\Delta \cup \Delta' \triangleright \Delta' \triangleright (e_1, e_2) : t_1 \times t_2$.

(proj): consider the following derivation:

$$ \Delta \triangleright e' : t_1 \times t_2 \quad \Delta \triangleright \pi_i(e') : t_i \quad \text{(proj)} $$

By the induction hypothesis, we have $\Delta \cup \Delta' \triangleright e' : t_1 \times t_2$. Then by (proj), we get $\Delta \cup \Delta' \triangleright \pi_i(e') : t_i$.

(abstr): consider the following derivation:

$$ \forall i, j \in J, \Delta'' \triangleright \Gamma, (x : t, \sigma_j) \triangleright e'[\sigma_j] : s_{i, \sigma_j} \quad \Delta'' = \Delta \cup \mathrm{var}(\bigwedge_{i \in I, j \in J} t, \sigma_j) \rightarrow s_{i, \sigma_j} \quad \text{(abstr)} $$

By induction, we have $\Delta'' \cup \Delta' \triangleright \Gamma', (x : t, \sigma_j) \triangleright e'[\sigma_j] : s_{i, \sigma_j}$ for all $i \in I$ and $j \in J$. Then by (abstr), we get $\Delta \cup \Delta' \triangleright \Delta' \triangleright (e'[\sigma_j]_{j \in J} : s_{i, \sigma_j}) \rightarrow s_{i, \sigma_j}$.

(case): consider the following derivation:

$$ \begin{array}{c} \Delta \triangleright e_0 : t' \\{ t' \notin \text{var} \Rightarrow \Delta \triangleright e_1 : s \} \\ \Delta \triangleright e_2 : s \quad \text{(case)} \end{array} \quad \quad \begin{array}{c} \Delta \triangleright e_0 : t' \\{ t' \notin \text{var} \Rightarrow \Delta \triangleright e_1 : s \} \\ \Delta \triangleright e_2 : s \quad \text{(case)} \end{array} $$

By induction, we have $\Delta \cup \Delta' \triangleright e_0 : t_0$ and $\Delta \cup \Delta' \triangleright e_i : s$ for $i$ such that $e_i$ has been type-checked in the original derivation. Then by (case), we get $\Delta \cup \Delta' \triangleright (e_0 \in t ? e_1 : e_2) : s$.

(inst): consider the following derivation:

$$ \begin{array}{c} \Delta \triangleright e' : s \quad \sigma \notin \Delta \\{ \text{inst} \} \\ \Delta \triangleright e'[\sigma] : s_{\sigma} \end{array} $$

By induction, we have $\Delta \cup \Delta' \triangleright e' : s$. Since $e \notin \Delta' (i.e., e'[\sigma] \notin \Delta')$, we have $\sigma \notin \Delta'$. Then $\sigma \notin \Delta \cup \Delta'$. Therefore, by applying (inst) we get $\Delta \cup \Delta' \triangleright e'[\sigma] : s_{\sigma}$.

(int): consider the following derivation:

$$ \forall j \in J, \Delta \triangleright e'[\sigma_j] : t_j \quad \text{(int)} $$

By induction, we have $\Delta \cup \Delta' \triangleright e'[\sigma_j] : t_j$ for all $j \in J$. Then the rule (inst) gives us $\Delta \cup \Delta' \triangleright e'[\sigma_j] : t_j$.

(subsum): there exists a type $s$ such that

$$ \begin{array}{c} \Delta \triangleright e' : s \\{ \text{subsum} \} \\ \Delta \triangleright e' : t \end{array} $$
By induction, we have \( \Delta \cup \Delta' \Gamma \vdash e' : s \). Then by applying the rule \((\text{subsum})\) we get \( \Delta \cup \Delta' \Gamma \vdash e' : t \).

The next two lemmas are used to simplify sets of type-substitutions applied to expressions when they are redundant or they work on variables that are not in the expressions.

**Lemma B.9 (Useless Substitutions).** Let \( e \) be an expression and \([\sigma_k]_{k \in K}, [\sigma'_k]_{k \in K}\) two sets of substitutions such that \( \text{dom}(\sigma'_k) \cap \text{dom}(\sigma_k) = \emptyset \) and \( \text{dom}(\sigma'_k) \cap \text{tv}(e) = \emptyset \) for all \( k \in K \). Then
\[
\Delta \Gamma \vdash e \circ [\sigma_k]_{k \in K} : t \iff \Delta \Gamma \vdash e \circ [\sigma_k \cup \sigma'_k]_{k \in K} : t
\]

**Proof.** Straightforward.

Henceforth we use "\(\text{\textcircled{\text{\textcopyright}}}\)" to denote the union of multi-sets (e.g., \(\{1, 2\} \cup \{1, 3\} = \{1, 2, 1, 3\}\)).

**Lemma B.10 (Redundant Substitutions).** Let \([\sigma_j]_{j \in J}\) and \([\sigma_j]_{j \in J'}\) be two sets of substitutions such that \( J' \subseteq J \). Then
\[
\Delta \Gamma \vdash e \circ [\sigma_j]_{j \in J} : t \iff \Delta \Gamma \vdash e \circ [\sigma_j]_{j \in J'} : t
\]

**Proof.** Similar to Lemma B.9.

Lemma B.9 states that if a type variable \( \alpha \) in the domain of a type substitution \( \sigma \) does not occur in the applied expression \( e \), namely, \( \alpha \in \text{dom}(\sigma) \setminus \text{tv}(e) \), then that part of the substitution is useless and can be safely eliminated. Lemma B.10 states that although our \([\sigma_j]_{j \in J}\) are formally multisets of type-substitutions, in practice they behave as sets, since repeated entries of type substitutions can be safely removed. Therefore, to simplify an expression without altering its type (and semantics), we first eliminate the useless type variables, yielding concise type substitutions, and then remove the redundant type substitutions. It explains why we do not apply relabeling when the domains of the type substitutions do not contain type variables in expressions in Definition A.11.

Moreover, Lemma B.10 also indicates that it is safe to keep only the type substitutions which are different from each other when we merge two sets of substitutions (e.g., Lemmas B.13 and B.14). In what follows, without explicit mention, we assume that there are no useless type variables in the domain of any type substitution and no redundant type substitutions in any set of type substitutions.

**Lemma B.11 (Relabeling).** Let \( e \) be an expression, \([\sigma_j]_{j \in J}\) a set of type substitutions and \( \Delta \) a set of type variables such that \( \sigma_j \not\subseteq \Delta \) for all \( j \in J \). If \( \Delta \Gamma \vdash e : t \), then
\[
\Delta \Gamma \vdash e \circ [\sigma_j]_{j \in J} : \bigwedge_{j \in J} t \sigma_j
\]

**Proof.** The proof proceeds by induction and case analysis on the structure of \( e \). For each case we use an auxiliary internal induction on the typing derivation. We label \( \mathbf{E} \) the main (external) induction and \( \mathbf{I} \) the internal induction in what follows.

- \( e = \mathbf{c}: e \); the typing derivation \( \Delta \Gamma \vdash e : t \) should end with either \((\text{const})\) or \((\text{subsum})\). Assume that the typing derivation ends with \((\text{const})\). Trivially, we have \( \Delta \Gamma \vdash e : b_c \). Since \( e \circ [\sigma_j]_{j \in J} = c \) and \( b_c \simeq \bigwedge_{j \in J} b_j \sigma_j \) by subsumption, we have \( \Delta \Gamma \vdash e \circ [\sigma_j]_{j \in J} : \bigwedge_{j \in J} b_j \sigma_j \).
  
- Otherwise, the typing derivation ends with an instance of \((\text{subsum})\);

\[
\Delta \Gamma \vdash e : s \iff \Delta \Gamma \vdash e : t\quad (\text{subsum})
\]

Then by \( \mathbf{I} \)-induction, we have \( \Delta \Gamma \vdash e \circ [\sigma_j]_{j \in J} : \bigwedge_{j \in J} s \sigma_j \). Since \( s \leq t \), we get \( \bigwedge_{j \in J} s \sigma_j \leq \bigwedge_{j \in J} t \sigma_j \). Then by applying the rule \((\text{subsum})\), we have \( \Delta \Gamma \vdash e \circ [\sigma_j]_{j \in J} : \bigwedge_{j \in J} t \sigma_j \).

- \( e = \mathbf{x}: e \); the typing derivation \( \Delta \Gamma \vdash e : t \) should end with either \((\text{var})\) or \((\text{subsum})\). Assume that the typing derivation ends with \((\text{var})\). Trivially, by \( \text{(var)} \), we have \( \Delta \Gamma \vdash x : \Gamma(x) \). Moreover, we have \( x \circ [\sigma_j]_{j \in J} \simeq x \) and \( \Gamma(x) = \bigwedge_{j \in J} \Gamma(x) \sigma_j \) (as \( \text{var}(\Delta) \subseteq \Delta \)). Therefore, we deduce that \( \Delta \Gamma \vdash x \circ [\sigma_j]_{j \in J} : \bigwedge_{j \in J} \Gamma(x) \sigma_j \).
  
- Otherwise, the typing derivation ends with an instance of \((\text{subsum})\), similar to the case of \( e = c \), the result follows by \( \mathbf{I} \)-induction.

- \( e = (e_1, e_2): e \); the typing derivation \( \Delta \Gamma \vdash e : t \) should end with either \((\text{pair})\) or \((\text{subsum})\). Assume that the typing derivation ends with \((\text{pair})\);

\[
\begin{align*}
\Delta \Gamma \vdash e_1 : t_1 \quad &\quad \Delta \Gamma \vdash e_2 : t_2 \\
\Delta \Gamma \vdash (e_1, e_2) : t_1 \times t_2 &\quad (\text{pair})
\end{align*}
\]

By \( \mathbf{I} \)-induction, we have \( \Delta \Gamma \vdash e_1 \circ [\sigma_j]_{j \in J} : \bigwedge_{j \in J} t_1 \sigma_j \). Then by \( \text{(pair)} \), we get \( \Delta \Gamma \vdash (e_1 \circ [\sigma_j]_{j \in J}, e_2 \circ [\sigma_j]_{j \in J}) : (\bigwedge_{j \in J} t_1 \sigma_j \times \bigwedge_{j \in J} t_2 \sigma_j) \), that is, \( \Delta \Gamma \vdash (e_1, e_2) \circ [\sigma_j]_{j \in J} : \bigwedge_{j \in J} t_1 \sigma_j \wedge \bigwedge_{j \in J} t_2 \sigma_j \).
\[ \forall \{i \in I, k \in K \} \Delta I, \Gamma \vdash e' : s \] of \( (\text{subsum}) \), similar to the case of \( e = c \), the result follows by \( \text{I}\)-induction.

\[ e = \pi_i(e') : \text{the typing derivation } \Delta I, \Gamma \vdash e : t \text{ should end with either } (\text{proj}) \text{ or } (\text{subsum}) \text{. Assume that the typing derivation ends with } (\text{proj}) \text{:} \]

\[ \Delta I, \Gamma \vdash e' : t_1 \times t_2 \] of \( \Delta I, \Gamma \vdash \pi_i(e') : t_i \] of \( (\text{proj}) \) \text{.}

By \( \text{E}\)-induction, we have \( \Delta I, \Gamma \vdash e' \circ [\sigma_j] \in J : \bigwedge_{j \in J} (t_1 \times t_2) \sigma_j \), that is, \( \Delta I, \Gamma \vdash e' \circ [\sigma_j] \in J : \bigwedge_{j \in J} (t_1 \times t_2) \sigma_j \). Then the rule \( (\text{proj}) \) gives us that \( \Delta I, \Gamma \vdash \pi_i(e' \circ [\sigma_j] \in J : \bigwedge_{j \in J} t_i \sigma_j) \), that is, \( \Delta I, \Gamma \vdash \pi_i(e' \circ [\sigma_j] \in J : \bigwedge_{j \in J} t_i \sigma_j) \) \text{. Otherwise, the typing derivation ends with an instance of } (\text{subsum}) \text{, similar to the case of } e = c \text{, the result follows by } \text{I}\)-induction.

\[ e = e_1 e_2 : \text{the typing derivation } \Delta I, \Gamma \vdash e : t \text{ should end with either } (\text{appl}) \text{ or } (\text{subsum}) \text{. Assume that the typing derivation ends with } (\text{appl}) \text{:} \]

\[ \Delta I, \Gamma \vdash e_1 : s \quad \Delta I, \Gamma \vdash e_2 : t \] of \( (\text{pair}) \) \text{.}

By \( \text{E}\)-induction, we have \( \Delta I, \Gamma \vdash e_1 \circ [\sigma_j] \in J : \bigwedge_{j \in J} (t \rightarrow s) \sigma_j \) and \( \Delta I, \Gamma \vdash e_2 \circ [\sigma_j] : \bigwedge_{j \in J} t \sigma_j \). Since \( \bigwedge_{j \in J} (t \rightarrow s) \sigma_j \leq \bigwedge_{j \in J} (t \sigma_j) \), by \( (\text{subsum}) \), we have \( \Delta I, \Gamma \vdash e \circ [\sigma_j] \in J : \bigwedge_{j \in J} t \sigma_j \). Then by \( (\text{appl}) \), we get

\[ \Delta I, \Gamma \vdash (e_1 \circ [\sigma_j] \in J : e_2 \circ [\sigma_j] \in J : \bigwedge_{j \in J} s \sigma_j) \text{ that is, } \Delta I, \Gamma \vdash (e_1 e_2) \circ [\sigma_j] \in J : \bigwedge_{j \in J} s \sigma_j \text{. Otherwise, the typing derivation ends with an instance of } (\text{subsum}) \text{, similar to the case of } e = c \text{, the result follows by } \text{I}\)-induction. \]
The proof proceeds by induction and case analysis on the structure of \( e \). Internal induction in what follows.

As an intermediary result, we first prove that the derivation can be rewritten as

\[
\Delta_1 \Gamma \vdash e : t \\
\Delta_1 \Gamma \vdash e[\sigma] : t\sigma
\]

result follows by (inst) or (subsum). Assume that the typing derivation ends with (inst):

\[
\frac{\Delta_1 \Gamma \vdash e : t}{\Delta_1 \Gamma \vdash e[\sigma] : t\sigma}
\]

Consider the set of substitutions \([\sigma_j \circ \sigma]_{j \in J}\). It is clear that \( \sigma_j \circ \sigma \not\in \Delta \) for all \( j \in J \). By E-induction, we have

\[
\Delta_1 \Gamma \vdash e[\sigma_j \circ \sigma]_{j \in J} : \bigwedge_{j \in J} t(\sigma_j \circ \sigma)
\]

that is, \( \Delta_1 \Gamma \vdash e[\sigma_j]_{j \in J} : \bigwedge_{j \in J} t\sigma_j \). Otherwise, the typing derivation ends with an instance of (subsum), similar to the case of \( e = c \), the result follows by I-induction.

\( e = e'[\sigma] \): the typing derivation \( \Delta_1 \Gamma \vdash e : t \) should end with either (inst) or (subsum). Assume that the typing derivation ends with (inter):

\[
\frac{\forall k \in K. \Delta_1 \Gamma \vdash e'[\sigma_k] : t_k}{\Delta_1 \Gamma \vdash e'[\sigma_k]_{k \in K} : \bigwedge_{k \in K} t_k}
\]

As an intermediary result, we first prove that the derivation can be rewritten as

\[
\frac{\Delta_1 \Gamma \vdash e : s \quad \sigma_k \not\in \Delta}{\Delta_1 \Gamma \vdash e'[\sigma_k] : s\sigma_k}
\]

result follows by (inst) or (subsum). Assume that the typing derivation ends with (inter):

\[
\frac{\forall k \in K. \Delta_1 \Gamma \vdash e'[\sigma_k] : s\sigma_k}{\Delta_1 \Gamma \vdash e'[\sigma_k]_{k \in K} : \bigwedge_{k \in K} s\sigma_k}
\]

We proceed by induction on the original derivation. It is clear that each sub-derivation \( \Delta_1 \Gamma \vdash e'[\sigma_k] : t_k \) ends with either (inst) or (subsum). If all the sub-derivations end with an instance of (inst), then for all \( k \in K \), we have

\[
\Delta_1 \Gamma \vdash e' : s_k \quad \sigma_k \not\in \Delta
\]

result follows by (inst) or (subsum). Assume that the typing derivation ends with (inter):

\[
\frac{\Delta_1 \Gamma \vdash e' : s_k \quad \sigma_k \not\in \Delta}{\Delta_1 \Gamma \vdash e'[\sigma_k] : s_k\sigma_k}
\]

By Lemma B.2, we have \( \Delta_1 \Gamma \vdash e'[\sigma_k] : s\sigma_k \). Let \( s = \bigwedge_{k \in K} s_k \). Then by (inst), we get \( \Delta_1 \Gamma \vdash e'[\sigma_k] : s\sigma_k \). Finally, by (inter) and (subsum), the intermediary result holds. Otherwise, at least one of the sub-derivations ends with an instance of (subsum); the intermediary result also holds by induction.

Now that the intermediary result is proved, we go back to the proof of the lemma. Consider the set of substitutions \([\sigma_j \circ \sigma_k]_{j,k \in K}\). It is clear that \( \sigma_j \circ \sigma_k \not\in \Delta \) for all \( j \in J, k \in K \). By E-induction on \( e \) (i.e., \( \Delta_1 \Gamma \vdash e' : s \)), we have

\[
\Delta_1 \Gamma \vdash e'[\sigma_j \circ \sigma_k]_{j \in J, k \in K} : \bigwedge_{j,k \in K} s(\sigma_j \circ \sigma_k)
\]

that is, \( \Delta_1 \Gamma \vdash (e'[\sigma_k]_{k \in K})[\sigma_j]_{j \in J} : \bigwedge_{j \in J} (\bigwedge_{k \in K} s\sigma_k)\sigma_j \). As \( \bigwedge_{k \in K} s\sigma_k \leq \bigwedge_{k \in K} t_k \), we get \( \bigwedge_{j \in J} (\bigwedge_{k \in K} s\sigma_k)\sigma_j \leq \bigwedge_{j \in J} (\bigwedge_{k \in K} t_k)\sigma_j \). Then by (subsum), the result follows.

Otherwise, the typing derivation ends with an instance of (subsum), similar to the case of \( e = c \), the result follows by I-induction.

\[\square\]

**Corollary B.12.** If \( \Delta_1 \Gamma \vdash e[\sigma]_{j \in J} : t \), then \( \Delta_1 \Gamma \vdash e\sigma[\sigma]_{j \in J} : t \).

**Proof.** Immediate consequence of Lemma B.11.\[\square\]

**Lemma B.13.** If \( \Delta_1 \Gamma \vdash e\sigma[\sigma]_{j \in J} : t \) and \( \Delta' \Gamma' \vdash e\sigma[\sigma]_{j \in J'} : t' \), then \( \Delta \cup \Delta' \Gamma \land \Gamma' \vdash e\sigma[\sigma]_{j \in J \cup J'} : t \land t' \).

**Proof.** The proof proceeds by induction and case analysis on the structure of \( e \). For each case we use an auxiliary internal induction on both typing derivations. We label E the main (external) induction and I the internal induction in what follows.
\( e = e \cdot \sigma \mid \Pi \vdash \sigma \mid \) Clearly, both typing derivations should end with either \((\text{const})\) or \((\text{subsum})\). Assume that both derivations end with \((\text{const})\):

\[
\begin{align*}
\Delta \vdash \Gamma \vdash e : \sigma & \quad (\text{const}) \\
\Delta' \vdash \Gamma' \vdash e : \sigma & \quad (\text{const})
\end{align*}
\]

Trivially, by \((\text{const})\) we have \(\Delta \cup \Delta' : \Gamma \vdash e : \sigma\), that is \(\Delta \cup \Delta' : \Gamma \vdash e : \sigma\), As \(b_a \subseteq b_b \cup b_c\), by \((\text{subsum})\), the result follows.

Otherwise, there exists at least one typing derivation which ends with an instance of \((\text{subsum})\), for instance,

\[
\begin{align*}
\Delta \vdash \Gamma \vdash e \mid \sigma & : \Delta \vdash \Gamma \vdash e \mid \sigma \mid s \leq t \quad (\text{subsum}) \\
\Delta \vdash \Gamma & \vdash e \mid \sigma \\
\end{align*}
\]

Then by \(I\)-induction on \(\Delta \vdash \Gamma \vdash e \mid \sigma\) and \(\Delta' \vdash \Gamma' \vdash e \mid \sigma\), we have

\[
\begin{align*}
\Delta \cup \Delta' : \Gamma \vdash e \mid \sigma & : \Delta \vdash \Gamma \vdash e \mid \sigma \\
\end{align*}
\]

Since \(s \leq t\), we have \(s \wedge t' \leq t \wedge t'\). By \((\text{subsum})\), the result follows as well.

\( e = e \cdot \sigma \mid \Pi \vdash \sigma \mid \) Clearly, both typing derivations should end with either \((\text{var})\) or \((\text{subsum})\). Assume that both derivations end with an instance of \((\text{var})\):

\[
\begin{align*}
\Delta \vdash \Gamma \vdash x : \Gamma(x) & \quad (\text{var}) \\
\Delta' \vdash \Gamma' \vdash x : \Gamma'(x) & \quad (\text{var})
\end{align*}
\]

Since \(x \in \text{dom}(\Pi)\) and \(x \in \text{dom}(\Pi')\), \(x \in \text{dom}(\Pi \wedge \Pi')\). By \((\text{var})\), we have \(\Delta \cup \Delta' : \Gamma \vdash x : (\Gamma \wedge \Gamma')(x)\), that is, \(\Delta \cup \Delta' \vdash e \mid \sigma \mid \Gamma \vdash e \mid \sigma \mid \Gamma'(x)\).

Otherwise, there exists at least one typing derivation which ends with an instance of \((\text{subsum})\), similar to the case of \(e = e\), the result follows by \(I\)-induction.

\( e = e \cdot \sigma \mid \Pi \vdash \sigma \mid \) Clearly, both typing derivations should end with either \((\text{pair})\) or \((\text{subsum})\). Assume that both derivations end with an instance of \((\text{pair})\):

\[
\begin{align*}
\Delta \vdash \Gamma \vdash e \mid \sigma \mid e \mid \sigma : \Delta \vdash \Gamma \vdash e \mid \sigma \mid e \mid \sigma \\
\Delta' \vdash \Gamma' \vdash e \mid \sigma \mid e \mid \sigma & : \Delta' \vdash \Gamma' \vdash e \mid \sigma \mid e \mid \sigma
\end{align*}
\]

By \(E\)-induction, we have \(\Delta \cup \Delta' : \Gamma \vdash e \mid \sigma \mid e \mid \sigma : \Delta \vdash \Gamma \vdash e \mid \sigma \mid e \mid \sigma\). Then by \((\text{pair})\), we get \(\Delta \cup \Delta' : \Gamma \vdash e \mid \sigma \mid e \mid \sigma : \Delta \vdash \Gamma \vdash e \mid \sigma \mid e \mid \sigma\). Since \((s_1 \wedge s_2) \subseteq (s_1' \wedge s_2')\), that is \(\Delta \cup \Delta' : \Gamma \vdash e \mid \sigma \mid e \mid \sigma\).

Otherwise, there exists at least one typing derivation which ends with an instance of \((\text{subsum})\), similar to the case of \(e = e\), the result follows by \(I\)-induction.

\( e = e \cdot \sigma \mid \Pi \vdash \sigma \mid \) Clearly, both typing derivations should end with either \((\text{app})\) or \((\text{subsum})\). Assume that both derivations end with an instance of \((\text{app})\):

\[
\begin{align*}
\Delta \vdash \Gamma \vdash e \mid \sigma \mid e \mid \sigma : \Delta \vdash \Gamma \vdash e \mid \sigma \mid e \mid \sigma \\
\Delta' \vdash \Gamma' \vdash e \mid \sigma \mid e \mid \sigma & : \Delta' \vdash \Gamma' \vdash e \mid \sigma \mid e \mid \sigma
\end{align*}
\]

By \(E\)-induction, we have \(\Delta \cup \Delta' : \Gamma \vdash e \mid \sigma \mid e \mid \sigma : \Delta \vdash \Gamma \vdash e \mid \sigma \mid e \mid \sigma\). Since \((s_1 \wedge s_2) \subseteq (s_1' \wedge s_2')\), that is \(\Delta \cup \Delta' : \Gamma \vdash e \mid \sigma \mid e \mid \sigma\).

Otherwise, there exists at least one typing derivation which ends with an instance of \((\text{subsum})\), similar to the case of \(e = e\), the result follows by \(I\)-induction.
have $(s_1 \rightarrow s_2) \land (s_1' \rightarrow s_2') \leq (s_1 \land s_1') \rightarrow (s_2 \land s_2')$. Then by the rule (subsum), we get $\Delta \cup \Delta' \land \Gamma \vdash e : \sigma_{j \in J} \Gamma_{j,J} \rightarrow (s_1 \land s_1') \rightarrow (s_2 \land s_2')$. Finally by (appi), we have $\Delta \cup \Delta' \land \Gamma \vdash (e_1 : \sigma_{j \in J} \Gamma_{j,J} \rightarrow (s_1 \land s_1')) \rightarrow (s_2 \land s_2'$. Otherwise, there exists at least one typing derivation which ends with an instance of (subsum), similar to the case of $e = e$, the result follows by I-induction.

$$e = \lambda_{[\sigma_{j \in E}]} \rightarrow x.e', e \circ \sigma_{j \in J} x.e'$$ and $e \circ \sigma_{j \in J} x.e = \lambda_{[\sigma_{j \in E}]} \rightarrow x.e'$. Clearly, both typing derivations should end with either (abstr) or (subsum). Assume that both derivations end with an instance of (abstr):

$$\forall i \in I, j \in K, \Delta \vdash e_1 : \sigma_{j \in J} \Gamma_{j,J} \rightarrow (s_1 \land s_1') \rightarrow (s_2 \land s_2')$$

By (abstr), we have

$$\Delta \cup \Delta' \land \Gamma \vdash e_1 : \sigma_{j \in J} \Gamma_{j,J} \rightarrow \sigma_{j \in K} \rightarrow x.e'$$

that is, $\Delta \cup \Delta' \land \Gamma \vdash e_1 : \sigma_{j \in J} \Gamma_{j,J} \rightarrow t : t', \Gamma_{j,J} \rightarrow (s_1 \land s_1') \rightarrow (s_2 \land s_2')$ and $t' = \lambda_{i \in I, j \in J, k \in K} t_i (s_1 \land s_1') \rightarrow s_i (s_1 \land s_1')$. Otherwise, there exists at least one typing derivation which ends with an instance of (subsum), similar to the case of $e = e$, the result follows by I-induction.

$$e = (e_0 : \sigma_{j \in J} \Gamma_{j,J}) \rightarrow \sigma_{j \in K} \rightarrow x.e'$$ and $e \circ \sigma_{j \in J} x.e = (e_0 : \sigma_{j \in J} \Gamma_{j,J} \rightarrow (s_1 \land s_1')) \rightarrow (s_2 \land s_2')$. Clearly, both typing derivations should end with either (case) or (subsum). Assume that both derivations end with an instance of (case):

$$\Delta \vdash e_0 : \sigma_{j \in J} \Gamma_{j,J} : t_0$$

By E-induction, we have $\Delta \cup \Delta' \land \Gamma \vdash e_0 : \sigma_{j \in J} \Gamma_{j,J} : t_0 \land t_0'$. Suppose $t_0 \land t_0' \leq t$, then we must have $t_0 \leq t$ and $t_0' \leq t$, and the first branch has been checked in both derivations. Therefore we have $\Delta \cup \Delta' \land \Gamma \vdash e_0 : \sigma_{j \in J} \Gamma_{j,J} : s \land s'$ by the induction hypothesis. Similarly, if $t_0 \land t_0' \leq t$, we have $\Delta \cup \Delta' \land \Gamma \vdash e_0 : \sigma_{j \in J} \Gamma_{j,J} : s \land s'$ by the induction hypothesis. And the result follows by I-induction.

$$e = e' : \sigma_{j \in J} \Gamma_{j,J} \rightarrow \sigma_{j \in K} \rightarrow x.e'$$ and $e \circ \sigma_{j \in J} x.e = e' \circ (\sigma_{j \circ \sigma_{j \in J} : (J \times J)} (J \times J))$. By E-induction on $e'$, we have

$$\Delta \cup \Delta' \land \Gamma \vdash e' : \sigma_{j \in J} \Gamma_{j,J} : t \land t'$$
that is, \( \Delta \cup \Delta' \Gamma \vdash (e'[\sigma_j]_{j \in J'}) @ \langle \sigma_j \rangle_{j \in J \cup J'} : t \wedge t' \).

\[ \]

Corollary B.14. If \( \Delta, \Gamma \vdash e \circ \emptyset[\sigma_j]_{j \in J_1} : t_1 \) and \( \Delta, \Gamma \vdash e \circ \emptyset[\sigma_j]_{j \in J_2} : t_2 \), then \( \Delta, \Gamma \vdash e \circ \emptyset[\sigma_j]_{j \in J_1 \cup J_2} : t_1 \wedge t_2 \).

\[ \]


\[ \]

B.2 Type soundness

In this section, we prove the soundness of the type system: well-typed expressions do not "go wrong". We proceed in two steps, commonly known as the subject reduction and progress theorems:

- Subject reduction: a well-typed expression keeps being well-typed during reduction.
- Progress: a well-typed expression can not be "stuck" (i.e., a well-typed expression which is not value can be reduced).

Theorem B.15 (Subject reduction). Let \( e \) be an expression and \( t \) a type. If \( \Delta, \Gamma \vdash e : t \) and \( e \rightarrow e' \), then \( \Delta, \Gamma \vdash e' : t \).

\[ \]

Proof. By induction on the derivation of \( \Delta, \Gamma \vdash e : t \). We proceed by a case analysis on the last rule used in the derivation of \( \Delta, \Gamma \vdash e : t \).

\[ \]

(const): the expression \( e \) is a constant. It cannot be reduced. Thus the result follows.

\[ \]

(var): similar to the (const) case.

\[ \]

(pair): \( e = (e_1, e_2), t = t_1 \times t_2 \). We have \( \Delta, \Gamma \vdash e_i : t_i \) for \( i = 1..2 \). There are two ways to reduce \( e \), that is

1. \( (e_1, e_2) \rightsquigarrow (e_1', e_2) \): by induction, we have \( \Delta, \Gamma \vdash e_1' : t_1 \). Then the rule (pair) gives us \( \Delta, \Gamma \vdash e : t_1 \times t_2 \).

2. The case \( e_1 \rightarrow e_1', e_2 \rightarrow e_2' \) is treated similarly.

\[ \]

(proj): \( e = \pi_\sigma(e_0), t = t_1 \). \( \Delta, \Gamma \vdash e_0 : t_1 \times t_2 \). By induction, we have \( \Delta, \Gamma \vdash e_0 : t_1 \times t_2 \). Then the rule (proj) gives us \( \Delta, \Gamma \vdash e : t_1 \).

\[ \]

(app): \( e = e_1 \cdot e_2, \Delta, \Gamma \vdash e_1 : t \rightarrow s \) and \( \Delta, \Gamma \vdash e_2 : t \).

\[ \]

1. \( e_1 \cdot e_2 \rightsquigarrow e_1' e_2' \) or \( e_1 \cdot e_2 \rightsquigarrow e_1' e_2' \): similar to the case of (pair).

2. \( e_1 = \lambda \langle \sigma_j \rangle_{j \in J}. x : e_0 , e_2 = e_2' \): by Lemma B.3, we have \( \Delta, \Gamma \vdash \lambda \langle \sigma_j \rangle_{j \in J}. x \rightarrow s, \sigma_j \leq t \rightarrow s \). From the subtyping for arrow types we deduce that \( t \leq \bigvee_{i \in I} t_i \sigma_j \) and that for any non-empty set \( P \subseteq I \times J \) if \( t \not\leq \bigvee_{(i,j) \in P} t_i \sigma_j \), then \( \bigvee_{(i,j) \in P} s \sigma_j \leq s \). Let \( P_0 = \{ (i, j) \} \mid \Delta, \Gamma \vdash v_2 : t_i \sigma_j \). Since \( \Delta, \Gamma \vdash v_2 : t_2 \) and \( t \leq \bigvee_{i \in I} t_i \sigma_j \), \( P_0 \) is non-empty. Also notice that \( t \not\leq \bigvee_{(i,j) \in P} t_i \sigma_j \), since otherwise there would exist some \( (i, j) \notin P_0 \) such that \( \Delta, \Gamma \vdash v_2 : t_i \sigma_j \).

\[ \]

\( \Delta \) and \( \Gamma \) whose premise is \( \Delta', \Gamma, (x : t_i \sigma_j) \vdash e_0 \circ \emptyset[\sigma_j] : s_i \sigma_j \) for all \( i \in I \) and \( j \in J \), where \( \Delta' = \Delta \cup \text{var}(\bigvee_{i \in I, j \in J} t_i \sigma_j \rightarrow s_i \sigma_j) \). By Lemma B.13, we get \( \Delta', \Gamma, (x : t_i \sigma_j) \vdash e_0 \circ \emptyset[\sigma_j] : P_0 \rightarrow s_i \sigma_j \). Since \( \Gamma, (x : \lambda \langle \sigma_j \rangle_{j \in J} t_i \sigma_j) \simeq \bigvee_{(i,j) \in P_0} \Gamma,(x : t_i \sigma_j) \), then from Lemma B.8 we have \( \Delta', \Gamma, (x : \lambda \langle \sigma_j \rangle_{j \in J} t_i \sigma_j) \vdash e_0 \circ \emptyset[\sigma_j] : P_0 \rightarrow s_i \sigma_j \) and a fortiori \( \Delta, \Gamma, (x : \lambda \langle \sigma_j \rangle_{j \in J} t_i \sigma_j) \vdash e_0 \circ \emptyset[\sigma_j] : P_0 \rightarrow s_i \sigma_j \). Furthermore, by definition of \( P_0 \) and the admissibility of the intersection introduction (Lemma B.2) we have that \( \Delta, \Gamma \vdash v_2 : \lambda \langle \sigma_j \rangle_{j \in J} t_i \sigma_j \).

Thus by Lemma B.6, we get \( \Delta', \Gamma \vdash e' : \lambda \langle \sigma_j \rangle_{j \in J} s_i \sigma_j \). Finally, by (subsum), we obtain \( \Delta, \Gamma \vdash e' : s \) as expected.

\[ \]

(abstr): \( e = \lambda \langle \sigma_j \rangle_{j \in J}. x : e_0 \). It cannot be reduced. Thus the result follows.

\[ \]

(case): \( e = e_0 \circ \emptyset[\sigma_j] : e_1 \).

\[ \]

1. \( e_0 \rightarrow e_0' \) or \( e_1 \rightarrow e_1' \) or \( e_2 \rightarrow e_2' \): similar to the case of (pair).

2. \( e_0 = v_0 \) and \( e_1 : s \): we have \( e' = e_1 \). The typing rule gives us \( \Delta, \Gamma \vdash e_1 : t \), thus the result follows.

3. Otherwise \( (e_0 = v_0) \): we have \( e' = e_2 \). Similar to the above case.

\[ \]

(inst): \( e = e_1[\sigma], \Delta, \Gamma \vdash e_1 : s, \sigma \not\subseteq \Delta \) and \( e \rightarrow e_1 \circ \emptyset[\sigma] \). By applying Lemma B.11, we get \( \Delta, \Gamma \vdash e_1 \circ \emptyset[\sigma] : s \sigma \).

\[ \]

(int): \( e = e_1[\sigma_j]_{j \in J}, \Delta, \Gamma \vdash e_1[\sigma_j]_{j \in J} : \bigcup_{j \in J} t_j \) and \( e \rightarrow e_1 \circ \emptyset[\sigma_j]_{j \in J} \). By applying Corollary B.12, we get \( \Delta, \Gamma \vdash e_1 \circ \emptyset[\sigma_j]_{j \in J} : \bigcup_{j \in J} t_j \).
(subsum): there exists a type \( s \) such that \( \Delta \vdash e : s \leq t \) and \( e \leadsto e' \). By induction, we have \( \Delta \vdash e' : s \), then by subsumption we get \( \Delta \vdash e' : t \).

\[
\square
\]

**Theorem B.16 (Progress).** Let \( e \) be a well-typed closed expression, that is, \( \vdash e : t \) for some \( t \). If \( e \) is not a value, then there exists an expression \( e' \) such that \( e \leadsto e' \).

**Proof.** By induction on the derivation of \( \vdash e : t \). We proceed by a case analysis of the last rule used in the derivation of \( \vdash e : t \).

(const): immediate since a constant is a value.
(var): impossible since a variable cannot be well-typed in an empty environment.
(pair): \( e = (e_1, e_2), t = t_1 \times t_2 \), and \( \vdash e_i : t_i \) for \( i = 1..2 \). If one of the \( e_i \) can be reduced, then \( e \) can also be reduced. Otherwise, by induction, both \( e_1 \) and \( e_2 \) are values, and so is \( e \).
(proj): \( e = \pi_i(e_0), t = t_i \), and \( \vdash e_0 : t_1 \times t_2 \). If \( e_0 \) can be reduced to \( e'_0 \), then \( e \leadsto \pi_i(e'_0) \). Otherwise, \( e_0 \) is a value. By Lemma B.3, we get \( e_0 = (v_1, v_2) \), and thus \( e \leadsto v_i \).
(appl): \( e = e_1 \, e_2, \vdash e_1 : t \rightarrow s \) and \( \vdash e_2 : t \). If one of the \( e_i \) can be reduced, then \( e \) can also be reduced. Otherwise, by induction, both \( e_1 \) and \( e_2 \) are values. By Lemma B.3, we get \( e_1 = \lambda^{\sigma_i \in I_i \vdash s_{i}}_{x} \cdot e_{0} \) such that \( \bigwedge_{i \in I} \bigvee_{j \in J} t_{i} \sigma_{j} \rightarrow s_{i} \sigma_{j} \leq t \rightarrow s \). By the definition of subtyping for arrow types, we have \( t \leq \bigvee_{i \in I} \bigwedge_{j \in J} t_{i} \sigma_{j} \). Moreover, as \( \vdash e_2 : t \), the set \( P = \{ j \in J \mid \exists \sigma \in I, \vdash e_2 : t_{i} \sigma_{j} \} \) is non-empty. Then \( e \leadsto (e_0 \, \pi_j | \exists j \in P \{ e_2 \}) \).
(abstr): the expression \( e \) is an abstraction which is well-typed under the empty environment. It is thus a value.
(case): \( e = e_0 \cdot s ? e_1 : e_2 \). If \( e_0 \) can be reduced, then \( e \) can also be reduced. Otherwise, by induction, \( e_0 \) is a value \( v \). If \( \vdash v : s \), then we have \( e \leadsto e_1 \). Otherwise, \( e \leadsto e_2 \).
(inst): \( e = e_1 [\sigma], t = s \sigma \) and \( \vdash e_1 : s \). Then \( e \leadsto e_1 \emptyset[\sigma] \).
(inter): \( e = e_1[\sigma_j]_{j \in J}, t = \bigwedge_{j \in J} t_j \) and \( \vdash e_1[t_j] : t_j \) for all \( j \in J \). It is clear that \( e \leadsto e_1[\emptyset]_{j \in J} \).
(subsum): straightforward application of the induction hypothesis.

\[
\square
\]

We now conclude that the type system is type sound.

**Corollary B.17 (Type soundness).** Let \( e \) be a well-typed closed expression, that is, \( \vdash e : t \) for some \( t \). Then either \( e \) diverges or it returns a value of type \( t \).

**Proof.** Consequence of Theorems B.16 and B.15.

\[
\square
\]

**B.3 Expressing intersection types**

We now prove that the calculus with explicit substitutions is able to derive the same typings as the Barendregt, Coppo, Dezani (BCD) intersection type system [11] without the universal type \( \omega \). We remind the BCD types (a strict subset of \( \mathcal{T} \)), the BCD typing rules (without \( \omega \)) and subtyping relation in Figure 5, where we use \( m \) to range over pure \( \lambda \)-calculus expressions. To make the correspondence between the systems easier, we adopt a n-ary version of the intersection typing rule. Henceforth, we use \( D \) to range over BCD typing derivations. We first remark that the BCD subtyping relation is included in the one of this work.

**Lemma B.18.** If \( t_1 \leq_{BCD} t_2 \) then \( t_1 \leq t_2 \).

**Proof.** All the BCD subtyping rules are admissible in [12] and, a fortiori, in our type system.

\[
\square
\]

In this subsection, we restrict the grammar of expressions with explicit substitutions to

\[
e ::= x \mid e \, e \mid \lambda^{\sigma} \in I \vdash S \cdot x \, e \quad (24)
\]

and we write \([e]\) for the pure \( \lambda \)-calculus expression obtained by removing all types references (i.e., interfaces and decorations) from \( e \). Given a pure \( \lambda \)-calculus expression \( m \) and a derivation of a judgement \( \Delta \vdash_{BCD} m : t \), we build an expression \( e \) such that \([e] = m \) and \( \Delta \vdash e : t \) for some set \( \Delta \) of type variables. With the restricted grammar of (24), the intersection typing rule is used only in conjunction with the abstraction typing rule. We prove that it is possible to put a similar restriction on derivations of BCD typing judgements.
Definition B.19. Let $D$ be a BCD typing derivation. We say $D$ is in intersection-abstraction normal form if (BCD inter) is used only immediately after (BCD abstr) in $D$, that is to say, all uses of (BCD inter) in $D$ are of the form

$$D_i \quad (BCD \text{ inter})$$

We prove that any BCD typing judgement can be proved with a derivation in intersection-abstraction normal form.

Lemma B.21. If $\Gamma \vdash_{BCD} m : t$, then there exists a derivation in intersection-abstraction normal form proving this judgement.

Proof. Let $D$ be the derivation proving $\Gamma \vdash_{BCD} m : t$. We proceed by induction on the size of $D$. If $|D| = 1$ then the rule (BCD var) has been used, and $D$ is in intersection-abstraction normal form. Otherwise, assume that $|D| > 1$. We proceed by case analysis on the last rule used in $D$.

1. **(BCD sub)**: $D$ ends with (BCD sub):

   $$D = D' \frac{t' \leq t}{\Gamma \vdash_{BCD} m : t}$$

   where $D'$ proves a judgement $\Gamma \vdash_{BCD} m : t'$. By the induction hypothesis, there exists a derivation $D''$ in intersection-abstraction normal form which proves the same judgement as $D'$. Then

   $$D'' \frac{t' \leq t}{\Gamma \vdash_{BCD} m : t}$$

   is in intersection-abstraction normal form and proves the same judgement as $D$.

2. **(BCD abstr)**: similar to the case of (BCD sub).

3. **(BCD app)**: similar to the case of (BCD sub).

4. **(BCD inter)**: $D$ ends with (BCD inter):

   $$D = D_i \frac{t' \leq t}{\Gamma \vdash_{BCD} m : t}$$

   where each $D_i$ proves a judgement $\Gamma \vdash_{BCD} m : t_i$ and $t = \bigwedge_{i \in I} t_i$. We distinguish several cases.
If one of the derivations ends with \((\text{BCD sub})\), there exists \(i_0 \in I\) such that
\[
D_{i_0} = \frac{D'_{i_0}}{\Gamma \vdash_{\text{BCD}} m : t_{i_0}}
\]
The derivation
\[
D' = \frac{D_i \quad D'_{i_0}}{\Gamma \vdash_{\text{BCD}} m : \bigwedge_{i \notin \{i_0\}} t_i \land t'_{i_0} \quad i \in I \setminus \{i_0\}}
\]
is smaller than \(D\), so by the induction hypothesis, there exists \(D''\) in intersection-abstraction normal form which proves the same judgement as \(D'\). Then the derivation
\[
D'' = \frac{\bigwedge_{i \notin \{i_0\}} t_i \land t'_{i_0} \leq_{\text{BCD}} \bigwedge_{i \in I} t_i}{\Gamma \vdash_{\text{BCD}} m : t}
\]
is in intersection-abstraction normal form, and proves the same judgement as \(D\).

If one of the derivations ends with \((\text{BCD inter})\), there exists \(i_0 \in I\) such that
\[
D_{i_0} = \frac{D_{j,i_0} \quad j \in J}{\Gamma \vdash_{\text{BCD}} m : \bigwedge_{j \in J} t_{j,i_0}}
\]
with \(t_{i_0} = \bigwedge_{j \in J} t_{j,i_0}\). The derivation
\[
D' = \frac{D_i \quad D_{j,i_0}}{\Gamma \vdash_{\text{BCD}} m : t \quad j \in J}
\]
is smaller than \(D\), so by the induction hypothesis, there exists \(D''\) in intersection-abstraction normal form which proves the same judgement as \(D'\), which is the same as the judgement of \(D\).

If all the derivations are uses of \((\text{BCD var})\), then for all \(i \in I\), we have
\[
D_i = \frac{\Gamma \vdash_{\text{BCD}} x : \Gamma(x)}{\Gamma \vdash_{\text{BCD}} m : t}
\]
which implies \(t = \bigwedge_{i \in I} \Gamma(x)\) and \(m = x\). Then the derivation
\[
\frac{\Gamma \vdash_{\text{BCD}} x : \Gamma(x)}{\Gamma \vdash_{\text{BCD}} m : t}
\]
is in intersection-abstraction normal form and proves the same judgement as \(D\).

If all the derivations end with \((\text{BCD app})\), then for all \(i \in I\), we have
\[
D_i = \frac{D^1_i \quad D^2_i}{\Gamma \vdash_{\text{BCD}} m_1 : t_i \quad m_2 : s_i}
\]
where \(m = m_1 m_2\), \(D^1_i\) proves \(\Gamma \vdash_{\text{BCD}} m_1 : s_i \rightarrow t_i\), and \(D^2_i\) proves \(\Gamma \vdash_{\text{BCD}} m_2 : s_i\) for some \(s_i\). Let
\[
D_1 = \frac{D^1_i}{\Gamma \vdash_{\text{BCD}} m_1 : \bigwedge_{i \in I} s_i \rightarrow t_i \quad i \in I}
\]
\[
D_2 = \frac{D^2_i}{\Gamma \vdash_{\text{BCD}} m_2 : \bigwedge_{i \in I} s_i \quad i \in I}
\]
Both \(D_1\) and \(D_2\) are smaller than \(D\), so by the induction hypothesis, there exist \(D'_1, D'_2\) in intersection-abstraction normal form which prove the same judgements as \(D_1\) and \(D_2\) respectively. Then the derivation
\[
\frac{D'_1 \quad \bigwedge_{i \in I} s_i \rightarrow t_i \leq_{\text{BCD}} \left(\bigwedge_{i \in I} s_i\right) \rightarrow \left(\bigwedge_{i \in I} t_i\right)}{\Gamma \vdash_{\text{BCD}} m_1 : \left(\bigwedge_{i \in I} s_i\right) \rightarrow \left(\bigwedge_{i \in I} t_i\right) \quad \bigwedge_{i \in I} s_i \rightarrow t_i}
\]
is in intersection-abstraction normal form and proves the same derivation as \(D\).

If all the derivations end with \((\text{BCD abstr})\), then for all \(i \in I\), we have
\[
D_i = \frac{D'_i}{\Gamma \vdash_{\text{BCD}} \lambda x.m' : t_i}
\]
with \(m = \lambda x.m'\). For all \(i \in I\), \(D'_i\) is smaller than \(D\), so by induction there exists \(D''_i\) in intersection-abstraction normal form which proves the same judgement as \(D'_i\). Then the derivation
\[
\frac{\Gamma \vdash_{\text{BCD}} \lambda x.m' : t_i \quad i \in I}{\Gamma \vdash_{\text{BCD}} \lambda x.m' : \bigwedge_{i \in I} t_i}
\]
We now sketch the principles behind the construction of e from D in intersection-abstraction normal form. If D proves a judgement $\Gamma \vdash_{BCD} \lambda x. m : t \rightarrow s$, without any top-level abstraction, then we simply put $t \rightarrow s$ in the interface of the corresponding expression $\lambda^x \cdot x.e$.

For a judgement $\Gamma \vdash_{BCD} \lambda x. m : \Lambda x \in I t_i \rightarrow s_i$, we build an expression $\lambda^{\alpha \rightarrow \beta}_{[\sigma_i]} f. x.e$ where each $\sigma_i$ corresponds to the derivation which types $\lambda x.m$ with $t_i \rightarrow s_i$. For example, let $m = \lambda f. \lambda x. f \cdot x$, and consider the judgement $\Gamma \vdash_{BCD} m : ((t_1 \rightarrow t_2) \rightarrow t_1 \rightarrow t_2) \wedge ((s_1 \rightarrow s_2) \rightarrow s_1 \rightarrow s_2)$. We first annotate each abstraction in $m$ with types $\alpha_j \rightarrow \beta_j$, where $\alpha_j, \beta_j$ are fresh, distinct variables, giving us $e = \lambda^{\alpha_1 \rightarrow \beta_1} f. \lambda^{\alpha_2 \rightarrow \beta_2} x. f \cdot x$. Comparing $\lambda^{\alpha_1 \rightarrow \beta_1} f. \lambda^{\alpha_2 \rightarrow \beta_2} x. f \cdot x$ to the judgement $f : t_1 \rightarrow t_2 \vdash_{BCD} \lambda x. f \cdot x : t_1 \rightarrow t_2$ and $e$ to $\Gamma \vdash_{BCD} m : (t_1 \rightarrow t_2) \rightarrow t_1 \rightarrow t_2$, we compute $\sigma_1 = \{ t_1 \rightarrow t_2, t_2 \rightarrow t_3, t_1 \rightarrow t_4, t_2 \rightarrow t_3, t_3 \rightarrow t_4, t_4 \rightarrow t_2 \}$, and thus $\lambda^{\alpha_1 \rightarrow \beta_1} f. \lambda^{\alpha_2 \rightarrow \beta_2} x. f \cdot x \vdash_{BCD} m : (s_1 \rightarrow s_2) \rightarrow s_1 \rightarrow s_2$, and we obtain finally

$$\vdash_{[\sigma_1, \sigma_2]} \lambda^{\alpha_1 \rightarrow \beta_1} f. \lambda^{\alpha_2 \rightarrow \beta_2} x. f \cdot x : ((t_1 \rightarrow t_2) \rightarrow t_1 \rightarrow t_2) \wedge ((s_1 \rightarrow s_2) \rightarrow s_1 \rightarrow s_2)$$

as wished.

The problem becomes more complex when we have nested uses of the intersection typing rule. For example, let $m = \lambda f. \lambda g. (\lambda x. f \cdot (\lambda y. x \cdot y))$ and consider the judgement $\Gamma \vdash_{BCD} m : (t_f \rightarrow t_g \rightarrow t_4) \wedge (s_f \rightarrow s_g \rightarrow s_7)$ with

$$t_f = (t_1 \rightarrow t_2 \rightarrow t_3)$$
$$t_g = (t_4 \rightarrow t_5 \rightarrow t_6)$$
$$s_f = (s_1 \rightarrow s_2 \rightarrow s_3) \wedge (s_4 \rightarrow s_5 \rightarrow s_6)$$
$$s_g = (s_7 \rightarrow s_8 \rightarrow s_9)$$

Notice that, to derive $\Gamma \vdash_{BCD} m : s_f \rightarrow s_g \rightarrow s_7$, we have to prove $f : s_f, g : s_g \vdash_{BCD} \lambda x. f \cdot (\lambda y. x \cdot y) : s_f$, which requires the (B(BCD) inter) rule. As before, we annotate $m$ with fresh interfaces, obtaining $\lambda^{\alpha_1 \rightarrow \beta_1} f. \lambda^{\alpha_2 \rightarrow \beta_2} g. (\lambda^{\alpha_3 \rightarrow \beta_3} x. f \cdot (\lambda^{\alpha_4 \rightarrow \beta_4} x. y \cdot x))$. Because the intersection typing rule is used twice (once to type $m$, and once to type $m' = \lambda x. f \cdot (\lambda y. x \cdot y)$), we want to compute four substitutions $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ to obtain a decorated expression $\lambda^{\alpha_1 \rightarrow \beta_1} f. \lambda^{\alpha_2 \rightarrow \beta_2} g. (\lambda^{\alpha_3 \rightarrow \beta_3} x. f \cdot (\lambda^{\alpha_4 \rightarrow \beta_4} x. y \cdot x))$. The difficult part is in computing $\sigma_3$ and $\sigma_4$: in one case (corresponding to the branch $\Gamma \vdash_{BCD} m : t_f \rightarrow t_g \rightarrow t_4$), we want $\sigma_3 = \sigma_4 = \{ t_1 \rightarrow t_2, t_3 \rightarrow t_4, t_5 \rightarrow t_6, t_7 \rightarrow t_8 \}$ to obtain $f : t_f, g : t_g \vdash_{BCD} \lambda^{\alpha_3 \rightarrow \beta_3} x. f \cdot (\lambda^{\alpha_4 \rightarrow \beta_4} x. y \cdot x) : t_f$, and in the other case (corresponding to the derivation $\Gamma \vdash_{BCD} m : s_f \rightarrow s_g \rightarrow s_7$), we want $\sigma_3 = \{ s_1 \rightarrow s_2, s_3 \rightarrow s_4, s_5 \rightarrow s_6 \}$ and $\sigma_4 = \{ s_7 \rightarrow s_8, s_9 \rightarrow s_{10}, s_1 \rightarrow s_2, s_3 \rightarrow s_4 \}$ to obtain $f : s_f, g : s_g \vdash_{BCD} \lambda^{\alpha_3 \rightarrow \beta_3} x. f \cdot (\lambda^{\alpha_4 \rightarrow \beta_4} x. y \cdot x) : s_f$. To resolve this issue, we use intermediate fresh variables $\alpha_3', \beta_3', \alpha_4', \beta_4'$ in the definition of $\sigma_3$ and $\sigma_4$. We define

$$\sigma_1 = \{ t_1 \rightarrow t_2, t_3 \rightarrow t_4, t_5 \rightarrow t_6, t_7 \rightarrow t_8, t_9 \rightarrow t_10, t_1 \rightarrow t_2, t_3 \rightarrow t_4, t_5 \rightarrow t_6, t_7 \rightarrow t_8, t_9 \rightarrow t_10 \}$$
$$\sigma_2 = \{ s_1 \rightarrow s_2, s_3 \rightarrow s_4, s_5 \rightarrow s_6, s_7 \rightarrow s_8, s_9 \rightarrow s_{10}, s_1 \rightarrow s_2, s_3 \rightarrow s_4, s_5 \rightarrow s_6, s_7 \rightarrow s_8, s_9 \rightarrow s_{10} \}$$
$$\sigma_3 = \{ \alpha_3', \beta_3', \alpha_4', \beta_4' \}$$
$$\sigma_4 = \{ \alpha_3', \beta_3', \alpha_4', \beta_4' \}$$

Because the substitutions compose themselves, we obtain

$$\vdash_{[\sigma_1, \sigma_2]} \lambda^{\alpha_1 \rightarrow \beta_1} f. \lambda^{\alpha_2 \rightarrow \beta_2} g. (\lambda^{\alpha_3 \rightarrow \beta_3} x. f \cdot (\lambda^{\alpha_4 \rightarrow \beta_4} y. x \cdot y)) : (s_f \rightarrow s_g \rightarrow s_7)$$

as wished.

In the next lemma, given $n$ derivations $D_1, \ldots, D_n$ in intersection-abstraction normal form for a same expression $m$, we construct an expression $e$ containing fresh interfaces and decorations with fresh variables if needed (as explained in the above example) and $n$ substitutions $\sigma_1, \ldots, \sigma_n$ corresponding to $D_1, \ldots, D_n$. Let $\vartheta(D_1, \ldots, D_n)$ denote the set of type variables occurring in the types in $D_1, \ldots, D_n$.

**Lemma B.22.** Let $m$ be a pure $\lambda$-calculus expression, $\Delta$ be a set of type variables, and $D_1, \ldots, D_n$ be derivations in intersection-abstraction normal form such that $D_i$ proves $\Gamma_i \vdash_{BCD} m : t_i$ for all $i$. Let $\Delta'$ be a set of type variables such that $\vartheta(D_1, \ldots, D_n) \subseteq \Delta'$. There exist $e, \sigma_1, \ldots, \sigma_n$ such that $[e] = m$, $\text{dom}(\sigma_1) = \cdots = \text{dom}(\sigma_n) \subseteq \text{tv}(e)$, $\text{tv}(e) \cap (\Delta \cup \Delta') = \emptyset$, and $\Delta' \vdash_{\Gamma_i} \vdash_{[\sigma_i]} : t_i$ for all $i$.

**Proof.** We proceed by induction on the sum of the size of $D_1, \ldots, D_n$. If this sum is equal to $n$, then each $D_i$ is a use of the (BCD $\text{var}$) rule, and we have $m = x$ for some $x$. Let $e = x$ and $\sigma_i$ be the identity; we can then easily check that the result holds. Otherwise, assume this sum is strictly greater than $n$. We proceed by case analysis on $D_1, \ldots, D_n$. 

Case 1: If one of the derivations ends with (BCD sub), there exists $i_0$ such that

$$D'_{i_0} = D'_{i_0}^1 \cup \{ e, \alpha, \beta \} \mid \Gamma \vdash \text{BCD} \ m : t_{i_0}$$

Clearly, the sum of the size of $D_1, \ldots, D_{i_0}, \ldots, D_n$ is smaller than that of $D_1, \ldots, D_n$, and $\text{var}(D_1, \ldots, D_{i_0}, \ldots, D_n) \subseteq \Delta'$. So by the induction hypothesis, we have

$$\exists \epsilon, \sigma_1, \ldots, \sigma_n \quad [\epsilon] = m$$

and

$$\text{dom}(\sigma_1) = \ldots = \text{dom}(\sigma_n) \subseteq t_v(e)$$

and

$$t_v(e) \cap (\Delta \cup \Delta') = \emptyset$$

and

$$\forall i \in \{1, \ldots, n\} \setminus \{i_0\}. \Delta' \vdash \Gamma_i \vdash e_0[\sigma_i] : t_i$$

and

$$\Delta' \vdash \Gamma_{i_0} \vdash e_0[\sigma_{i_0}] : t_{i_0}$$

Since $t'_{i_0} \leq_{BCD} t_{i_0}$, by Lemma B.18, we have $t'_{i_0} \leq t_{i_0}$. Therefore $\Delta' \vdash \Gamma_{i_0} \vdash e_0[\sigma_{i_0}] : t_{i_0}$ holds, and for all $i$, we have $\Delta' \vdash \Gamma_i \vdash e_0[\sigma_i] : t_i$ as wished.

Case 2: If all the derivations end with (BCD app), then we have $m = m_1 m_2$, and for all $i$

$$D_i = \frac{D_i^1 \cup D_i^2}{\Gamma_i \vdash \text{BCD} \ m_1 \ m_2 : t_i}$$

where $D_i^1$ proves $\Gamma_i \vdash \text{BCD} \ m_1 : s_i \rightarrow t_i$ and $D_i^2$ proves $\Gamma_i \vdash \text{BCD} \ m_2 : s_i$ for some $s_i$. Applying the induction hypothesis on $D_i^1, \ldots, D_i^2$ (with $\Delta$ and $\Delta'$), we have

$$\exists \epsilon_1, \sigma_1^1, \ldots, \sigma_n^1 \quad [\epsilon_1] = m_1$$

and

$$\text{dom}(\sigma_1^1) = \ldots = \text{dom}(\sigma_n^1) \subseteq t_v(e_1)$$

and

$$t_v(e_1) \cap (\Delta \cup \Delta') = \emptyset$$

and

$$\forall i \in \{1, \ldots, n\}. \Delta' \vdash \Gamma_i \vdash e_1[\sigma_i^1] : s_i \rightarrow t_i$$

Similarly, by induction on $D_i^2, \ldots, D_i^n$ (with $\Delta \cup t_v(e_1)$ and $\Delta'$),

$$\exists \epsilon_2, \sigma_1^2, \ldots, \sigma_n^2 \quad [\epsilon_2] = m_2$$

and

$$\text{dom}(\sigma_1^2) = \ldots = \text{dom}(\sigma_n^2) \subseteq t_v(e_2)$$

and

$$t_v(e_2) \cap (\Delta \cup t_v(e_1) \cup \Delta') = \emptyset$$

and

$$\forall i \in \{1, \ldots, n\}. \Delta' \vdash \Gamma_i \vdash e_2[\sigma_i^2] : s_i$$

From $t_v(e_2) \cap (\Delta \cup t_v(e_1) \cup \Delta') = \emptyset$, we deduce $t_v(e_1) \cap t_v(e_2) = \emptyset$. Let $i \in \{1, \ldots, n\}$. Because $\text{dom}(\sigma_1^1) \subseteq t_v(e_1)$ and $\text{dom}(\sigma_1^2) \subseteq t_v(e_2)$, we have $\text{dom}(\sigma_1^1) \cap t_v(e_2) = \emptyset$, $\text{dom}(\sigma_1^2) \cap t_v(e_1) = \emptyset$, and $\text{dom}(\sigma_1^1) \cap t_v(e_1) = \emptyset$. Consequently, by Lemma B.9, we have $\Delta' \vdash \Delta' \vdash e_1[\sigma_1^1 \cup \sigma_1^2] : s_i \rightarrow t_i$ and $\Delta' \vdash \Delta' \vdash e_2[\sigma_1^1 \cup \sigma_1^2] : s_i$. Therefore, we have $\Delta' \vdash \Gamma_i \vdash (e_1 e_2)[\sigma_1^1 \cup \sigma_1^2] : t_i$. So we have the required result with $e = e_1 e_2$ and $\sigma_i = \sigma_1^1 \cup \sigma_1^2$.

Case 3: If all the derivations end with (BCD abstr), then $m = \lambda x. m_1$, and for all $i$,

$$D_i = \frac{D_i'}{\Gamma_i \vdash \text{BCD} \ m : t_i}$$

where $D_i'$ proves $\Gamma_i, x : t_i^1 \vdash \text{BCD} \ m_1 : t_i^2$ and $t_i = t_i^1 \rightarrow t_i^2$. By the induction hypothesis,

$$\exists \epsilon_1, \sigma_1^1, \ldots, \sigma_n^1 \quad [\epsilon_1] = m_1$$

and

$$\text{dom}(\sigma_1^1) = \ldots = \text{dom}(\sigma_n^1) \subseteq t_v(e_1)$$

and

$$t_v(e_1) \cap (\Delta \cup \Delta') = \emptyset$$

and

$$\forall i \in \{1, \ldots, n\}. \Delta' \vdash \Gamma_i, x : t_i^1 \vdash e_1[\sigma_i^1] : t_i^2$$

Let $\alpha, \beta$ be two fresh type variables. So $\{\alpha, \beta\} \cap (\Delta \cup \Delta') = \emptyset$ and $\{\alpha, \beta\} \cap t_v(e_1) = \emptyset$. Take $i \in \{1, \ldots, n\}$. Let $\sigma_i = \{t_i^1, t_i^2/\alpha, t_i^2/\beta\} \cup \sigma_i^1$, and $e = \lambda^{\alpha \rightarrow \beta} x. e_1$. We have $\text{dom}(\sigma_i) = \{\alpha, \beta\} \cup \text{dom}(\sigma_i^1) \subseteq \{\alpha, \beta\} \cup t_v(e_1) = t_v(e)$. Besides, we have $t_v(e) \cap (\Delta \cup \Delta') = \emptyset$. Because $t_v(e_1) \cap \{\alpha, \beta\} = \emptyset$, we have $\text{dom}(\sigma_i) \cap t_v(e_1) = \emptyset$, and $\Delta' \vdash \Gamma_i, x : t_i^1 \vdash e_1[\sigma_i] : t_i^2$ by Lemma B.9, which is equivalent to $\Delta' \vdash \Gamma_i, x : \alpha \beta \vdash \Gamma_i \vdash e_1[\sigma_i] : \beta \sigma_i$. Because $\Delta \cup \text{var}(t_i^1 \rightarrow t_i^2) = \Delta'$, by the abstraction rule, we have $\Delta' \vdash \Gamma_i \vdash \lambda^{\alpha \rightarrow \beta} \beta x. e_1 : t_i$, i.e., $\Delta' \vdash \Gamma_i \vdash (\lambda^{\alpha \rightarrow \beta} \beta e_1)[\sigma_i] : t_i$. Therefore, we have the required result.

Case 4: If one of the derivations ends with (BCD inter), then $m = \lambda x. m_1$. The derivations end with either (BCD inter) or (BCD abstr) (we omit the already treated case of (BCD sub)). For simplicity, we suppose they all end with (BCD inter), as it is the same if some of them end with (BCD abstr) (note that Case 3 is a special case of Case 4). For all $i$, we have

$$D_i = \frac{D_i'}{\Gamma_i \vdash \text{BCD} \ m : \bigwedge_{j \in J_i} s_j^i \rightarrow t_j^i} \quad i \in J_i$$
where $D_i^j$ proves $\Gamma_i, x : s_i \vdash_{BCD} m_1 : t_i^j$ for all $j \in J_i$ and $t_i = \bigwedge_{j \in J_i} s_i \rightarrow t_i^j$ for all $i$. By the induction hypothesis on the sequence of $D_i^j$,
\[
\exists \epsilon_1, (\sigma_i^j)_{i \in J_1}, \ldots, (\sigma_i^j)_{i \in J_n}, \quad \epsilon_1 = m_1
\]
and $\forall i, i', j, j', \text{dom}(\sigma_i^j) = \text{dom}(\sigma_i^{j'})$ and $\text{dom}(\sigma_i^j) \subseteq \text{tv}(\epsilon_1)$ and $\text{tv}(\epsilon_1) \cap (\Delta \cup \Delta') = \emptyset$ and $\forall i, j, \Delta' \vdash_{\Gamma_i}, x : s_i \vdash_{\epsilon_1} \sigma_i^j : t_i^j$.

Let $p = \max_{i \in \{1, \ldots, n\}} |\{J_i\}|$. For all $i$, we complete the sequence of substitutions $(\sigma_i^j)$ so that it contains exactly $p$ elements, by repeating the last one $p - |J_i|$ times, and we number them from 1 to $p$. All the $\sigma_i^j$ have the same domain (included in $\text{tv}(\epsilon_1)$), that we number from 1 to $q$. Then $\sigma_i^j = \bigcup_{k \in \{1, \ldots, q\}} \{t_k^j/\alpha_k\}$ for some types $(t_k^j)$. Let $\alpha, \beta, (\alpha_j, k) \in (1, \ldots, p), k \in (1, \ldots, q), (\alpha_j, \beta, 0) \in (1, \ldots, p)$ be fresh pairwise distinct variables (which do not occur in $\Delta \cup \Delta' \vdash_{\epsilon_1}(\epsilon_1)$). For all $j \in \{1, \ldots, p\}$, $i \in \{1, \ldots, n\}$, we define:
\[
\sigma_j = \bigcup_{k \in \{1, \ldots, q\}} \{t_k^j/\alpha_k\} \cup \{t_k^j/\beta_k\}
\]
\[
e = \lambda^{\alpha \rightarrow \beta} e
\]
\[
\sigma_i = \bigcup_{j \in (1, \ldots, p)} \bigcup_{k \in (1, \ldots, q)} \{t_k^j/\alpha_k, k\} \cup \bigcup_{j \in (1, \ldots, p)} \{t_k^j/\beta_k, k\}
\]
For all $i, j, k$, we have by construction $(\alpha \sigma_j) \sigma_i = \alpha_k \sigma_i^j, (\alpha \sigma_j) \sigma_i = s_i^j$, and $(\beta \sigma_j) \sigma_i = t_i^j$. Moreover, since
\[
\text{tv}(\epsilon) = \text{tv}(e) \cup \bigcup_{j \in (1, \ldots, p)} \text{var}(\sigma_j) \cup \bigcup_{j \in (1, \ldots, p)} \{t_k^j/\alpha_k, k\} \cup \bigcup_{j \in (1, \ldots, p)} \{t_k^j/\beta_k, k\}
\]
The domain of $\sigma_j$ is $\text{tv}(\epsilon)$ and $\text{tv}(\epsilon) \cap (\Delta \cup \Delta') = \emptyset$. Because $\Delta' \vdash \Gamma_i, x : s_i \vdash_{\epsilon_1} \sigma_i^j : t_i^j$, by Lemma B.9, we have $\Delta' \vdash \Gamma_i, x : s_i \vdash_{\epsilon_1} \sigma_i^j : t_i^j$, which is equivalent to $\Delta' \vdash \Gamma_i, x : (\alpha_0 \sigma_j) \vdash \epsilon_1 @ [\sigma_i^j] : t_i^j$. Since $\Delta \cup \bigcup_{j \in (1, \ldots, p)} \text{var}(\sigma_i^j) \vdash_{\epsilon_1} \sigma_i^j : \Delta'$, for a given $i$, by the abstraction typing rule we have $\Delta' \vdash \Gamma_i, x : \sigma_i^j : t_i^j \vdash_{\epsilon_1} \lambda^{\alpha \rightarrow \beta} e : \epsilon_1 \bigcup_{j \in (1, \ldots, p)} s_i^j \rightarrow t_i^j = t_i$.

This is equivalent to $\Delta' \vdash \Gamma_i, x : \sigma_i^j : t_i^j \vdash_{\epsilon_1} \lambda^{\alpha \rightarrow \beta} e : |\sigma_i| : t$, hence $\Delta' \vdash \Gamma_i, e : |\sigma_i| : t$, holds for all $i$, as wished.

We are now ready to prove the main result of this subsection.

**Theorem B.23.** If $\Gamma \vdash_{BCD} m : t$, then there exist $\epsilon, \Delta$ such that $\Delta' \vdash \epsilon : t$ and $|\epsilon| = m$.

**Proof.** By Lemma B.21, there exists $D$ in intersection-abstraction normal form such that $D$ proves $\Gamma \vdash_{BCD} m : t$. Let $\Delta$ be a set of type variables such that $\text{var}(D) \subseteq \Delta$. We prove by induction on $|D|$ that there exists $\epsilon$ such that $\Delta' \vdash \epsilon : t$ and $|\epsilon| = m$.

**Case (BCD var):** The expression $m$ is a variable and the result holds with $e = m$.

**Case (BCD sub):** We have
\[
D = \frac{D' \quad t'_i \vdash_{BCD} t_i}{\Gamma \vdash_{BCD} m : t}
\]
where $D'$ in intersection-abstraction normal form and proves $\Gamma \vdash_{BCD} m : t'$. Clearly we have $|D'| < |D|$ and $\text{var}(D') \subseteq \Delta$, so by the induction hypothesis, there exists $\epsilon$ such that $|\epsilon| = m$ and $\Delta' \vdash \epsilon : t'$. By Lemma B.18, we have $t' \leq t$, therefore we have $\Delta' \vdash \epsilon : t$, as wished.

**Case (BCD app):** We have
\[
D = \frac{D_1}{\Gamma \vdash_{BCD} m : t}
\]
where $D_1$ proves $\Gamma \vdash_{BCD} m_1 : s \rightarrow t$, $D_2$ proves $\Gamma \vdash_{BCD} m_2 : s, m = m_1 m_2$, and both $D_1$ and $D_2$ are in intersection-abstraction normal form. Since $|D_1| < |D|$ and $\text{var}(D_1) \subseteq \Delta$, by the induction hypothesis, there exist $e_1$ and $e_2$ such that $|e_1| = m_1, |e_2| = m_2$, $\Delta' \vdash e_1 : s \rightarrow t$, and $\Delta' \vdash e_2 : s$. Consequently we have $\Delta' \vdash e_1 e_2 : t$, with $|e_1 e_2| = m_1 m_2$, as wished.
Proof. Given an expression and its embedding, they reduce to the same value. We proceed in several steps, using auxiliary definitions.

**Lemma B.27.** Let \( \sigma = \lambda x.m \) be an expression respecting the following grammar:

\[
\begin{align*}
\text{e} &::= e \mid x \mid (e_1 \cdot e_2) \mid \pi_i(e) \mid e \cdot e \mid \lambda_{\sigma_1 \in I}^{\alpha \rightarrow \beta} x.e \mid e \in \text{v} : e \in E
\end{align*}
\]

The set of all normalized expressions is denoted as \( \mathcal{E}_N \).

We then define an embedding of the full calculus into this subcalculus as follows:

**Definition B.24.** A normalized expression \( e \) is an expression without any subterm of the form \( e[\sigma]_{i \in I} \), i.e., an expression respecting the following grammar:

\[
\begin{align*}
\text{e} &::= e \mid x \mid (e_1 \cdot e_2) \mid \pi_i(e) \mid e \cdot e \mid \lambda_{\sigma_1 \in I}^{\alpha \rightarrow \beta} x.e \mid e \in \text{v} : e \in E
\end{align*}
\]

The embedding \( \text{emd}(.) \) is mapping from \( \mathcal{E} \) to \( \mathcal{E}_N \) defined as

\[
\begin{align*}
\text{emd}(c) &= c \\
\text{emd}(x) &= x \\
\text{emd}(e_1 \cdot e_2) &= (\text{emd}(e_1), \text{emd}(e_2)) \\
\text{emd}(\pi_i(e)) &= \pi_i(\text{emd}(e)) \\
\text{emd}(\lambda_{\sigma_1 \in I}^{\alpha \rightarrow \beta} x.e) &= \lambda_{\sigma_1 \in I}^{\alpha \rightarrow \beta} \text{emd}(e) \\
\text{emd}(e_1 \cdot e_2) &= \text{emd}(e_1) \cdot \text{emd}(e_2) \\
\text{emd}(e \in \text{v} : e_1 = e_2) &= \text{emd}(e) \in \text{v} : \text{emd}(e_1) = \text{emd}(e_2) \\
\text{emd}(e[\sigma]_{i \in I}) &= \text{emd}(e[\sigma]_{i \in I})
\end{align*}
\]

We want to prove that the subcalculus has the same expressive power as the full calculus, namely, given an expression and its embedding, they reduce to the same value. We proceed in several steps, using auxiliary lemmas.

First we show that the embedding preserves values.

**Lemma B.26.** Let \( v \in \text{v} \) be a value. Then \( \text{emd}(v) \in \text{v} \).

Proof. Straightforward.

Then we prove that values and their embeddings have the same types.

**Lemma B.27.** Let \( v \in \text{v} \) be a value. Then \( \vdash v : t \iff \vdash \text{emd}(v) : t \).

Proof. By induction and case analysis on \( v \) (note that \( \text{emd}(.) \) does not change the types in the interfaces).

We now want to prove the embedding preserves reduction, that is if an expression \( e \) reduces to \( e' \) in the full calculus, then its embedding \( \text{emd}(e) \) reduces to \( \text{emd}(e') \) in the subcalculus. Before that we show a substitution lemma.

**Lemma B.28.** Let \( e \) be an expression, \( x \) an expression variable and \( v \) a value. Then \( \text{emd}(e[\{v/x\}]) = \text{emd}(e)[\text{emd}(v)/x] \).

Proof. By induction and case analysis on \( e \).

\[
\begin{align*}
\text{emd}(e[\{v/x\}]) &= \text{emd}(e) \\
&= c[\text{emd}(v)/x] \\
&= \text{emd}(e)[\text{emd}(v)/x]
\end{align*}
\]
Lemma B.29. Let $e \in \mathcal{E}$ be an expression. If $e \leadsto e'$, then $\text{emd}(e) \leadsto^{*} \text{emd}(e')$.

Proof. By induction and case analysis on $e$.

c, x: irreducible.

(1) $e_1 \leadsto e_1'$. By induction, $\text{emd}(e_1) \leadsto^{*} \text{emd}(e_1')$. Then we have $\text{emd}(e_1), \text{emd}(e_2) \leadsto^{*} (\text{emd}(e_1), \text{emd}(e_2))$, that is, $\text{emd}(e_1, e_2) \leadsto^{*} \text{emd}(e_1', e_2)$.

(2) $e_2 \leadsto e_2'$. Similar to the subcase above.
Lemma B.30. However, the embedding of the value returned by an expression is the value returned by the embedding of $e \in E$. Similar to the case of $(e_1, e_2)$.

2) $e_0 \in \mathbf{e}$ and $e_1 \in \mathbf{e}$. Similar to the case of $(e_1, e_2)$.

3) $e_1 = \lambda^\alpha \varepsilon I_{\alpha \rightarrow \alpha} x. e_0$, $e_1, e_2 \in P$ and $e_1 e_2 \in \mathbf{e}$, according to Lemma B.27, we have $\vdash e_1 \in \mathbf{e}$; $\vdash e_2 \in \mathbf{e}$, thus we have $\{j \in J \mid \exists i \in I \vdash e \in \mathbf{e}\} = \{j \in J \mid \exists i \in I \vdash e \in \mathbf{e}\}$.

Therefore, $e \in e_0 \subseteq \mathbf{e}$ and $e_1 \in \mathbf{e}$; $e_2 \in \mathbf{e}$, according to Lemma B.27, we have $\vdash e_1 \in \mathbf{e}$; $\vdash e_2 \in \mathbf{e}$, thus we have $\{j \in J \mid \exists i \in I \vdash e \in \mathbf{e}\} = \{j \in J \mid \exists i \in I \vdash e \in \mathbf{e}\}$.

which proves this case.

$\lambda^\alpha \varepsilon I_{\alpha \rightarrow \alpha} x. e_0$; It cannot be reduced. Thus the result follows.

$e_0 \in \mathbf{e}$ and $e_1 \in \mathbf{e}$. Similar to the case of $(e_1, e_2)$.

2) $e_2 = v_0$ and $e_1 \in \mathbf{e}$, according to Lemma B.26 and B.27, $\vdash e_1 \in \mathbf{e}$ and $\vdash v_0 \in \mathbf{e}$.

3) $e_0 = v_0$ and $e_1 \in \mathbf{e}$, according to Lemma B.26 and B.27, $\vdash e_1 \in \mathbf{e}$ and $\vdash v_0 \in \mathbf{e}$.

Therefore, $\vdash e_1 \in \mathbf{e}$; $\vdash e_2 \in \mathbf{e}$, according to Lemma B.28, we have $\vdash e_0 \subseteq \mathbf{e}$; $\vdash e_1 \in \mathbf{e}$; $\vdash e_2 \in \mathbf{e}$, thus we have $\{j \in J \mid \exists i \in I \vdash e \in \mathbf{e}\} = \{j \in J \mid \exists i \in I \vdash e \in \mathbf{e}\}$.

Therefore, $\vdash e_1 \subseteq \mathbf{e}$ and $\vdash e_2 \in \mathbf{e}$, by Definition B.25, $\vdash e_1 \subseteq \mathbf{e}$; $\vdash e_2 \in \mathbf{e}$, thus we have $\{j \in J \mid \exists i \in I \vdash e \in \mathbf{e}\} = \{j \in J \mid \exists i \in I \vdash e \in \mathbf{e}\}$.

The result follows.

Although the embedding preserves the reduction, it does not indicate that an expression and its embedding reduce to the same value. This is because there may be some subterms of the form $e[\sigma_{j}]_{j \in J}$ in the body expression of an abstraction value. For example, the expression

$$(\lambda^I_{\alpha \rightarrow \alpha} \varepsilon x. e)[(\lambda^I_{\alpha \rightarrow \alpha} 0.x.x)[(\lambda^I_{\alpha \rightarrow \alpha} 0.x.x)[42]]]$$

reduces to

$$\lambda^I_{\alpha \rightarrow \alpha} y.((\lambda^I_{\alpha \rightarrow \alpha} 0.x.x)[(\lambda^I_{\alpha \rightarrow \alpha} 0.x.x)[42]])$$

while its embedding reduces to

$$\lambda^I_{\alpha \rightarrow \alpha} y.((\lambda^I_{\alpha \rightarrow \alpha} 0.x.x)[(\lambda^I_{\alpha \rightarrow \alpha} 0.x.x)[42]])$$

However, the embedding of the value returned by an expression is the value returned by the embedding of the expression. For instance, consider the example above again:

$\vdash e_1 \subseteq \mathbf{e}$; $\vdash e_2 \in \mathbf{e}$, $\vdash e_1 \subseteq \mathbf{e}$; $\vdash e_2 \in \mathbf{e}$, thus we have $\{j \in J \mid \exists i \in I \vdash e \in \mathbf{e}\} = \{j \in J \mid \exists i \in I \vdash e \in \mathbf{e}\}$.

Next, we want to prove an inversion of Lemma B.29, that is, if the embedding $\text{emd}(e)$ of an expression $e$ reduces to $e'$, then there exists $e''$ such that its embedding is $e'$ and $e$ reduces to $e''$. Prior to that we prove two auxiliary lemmas: the inversions for values and for relabeled expressions.

Lemma B.30. Let $e \in \mathcal{E}$ be an expression. If $\text{emd}(e) \in \mathcal{V}$, then there exists a value $v \in \mathcal{V}$ such that $e \approx_{\text{Rval}} v$ and $\text{emd}(e) = \text{emd}(v)$. More specifically,

1) if $e = c$, then $e \approx_{\text{Rval}} c$ and $\text{emd}(e) = c$.

2) if $e = e_0 \in \mathbf{e}$, then there exists $e_0$ such that $e \approx_{\text{Rval}} c$ and $\text{emd}(e_0) = c$.

3) if $e = (e_1, v_2)$, then there exist $v_1, v_2$ such that $e \approx_{\text{Rval}} (v_1, v_2)$ and $\text{emd}(v_i) = v_i$.

Proof. By induction and case analysis on $\text{emd}(e)$.

c$^0$ according to Definition B.25, $c$ should be of the form $c[\sigma_{j}]_{j \in J_1} \cdots [\sigma_{j_n}]_{n \in J_n}$, where $n \geq 0$. Clearly, we have $e \approx_{\text{Rval}} c$ and $\text{emd}(e) = c$.

$\lambda^\alpha \varepsilon I_{\alpha \rightarrow \alpha} x. e_0$; according to Definition B.25, $e$ should be of the form

$$\lambda^\alpha \varepsilon I_{\alpha \rightarrow \alpha} x. e_0[\sigma_{j}]_{j \in J_1} \cdots [\sigma_{j_n}]_{n \in J_n}$$

where $\text{emd}(e_0) = e_0[\sigma_{j}]_{j \in J_0} \circ \cdots \circ [\sigma_{j_1}]_{j \in J_1} \circ [\sigma_{j_0}]_{0 \in J_0} = [\sigma_{j}]_{j \in J}$, and $n \geq 0$. Moreover, it is clear that $e \approx_{\text{Rval}} \lambda^\alpha \varepsilon I_{\alpha \rightarrow \alpha} x. e_0$. Let $v = \lambda^\alpha \varepsilon I_{\alpha \rightarrow \alpha} x. e_0$ and the result follows.
(e₁, v₂): according to Definition B.25, e should be the form of (e₁, e₂)[σ₁]₀ ∈ J₁ × ... × [σₙ]₀ ∈ Jₙ, where

emed(e₁) ∘ (e₂) ∘ [σ₁]₀ ∈ J₁ × ... × [σₙ]₀ ∈ Jₙ = e₁, and n ≥ 0. Moreover, it is easy to get that

e ∈ (σᵣ)₂ (e₂ ∘ [σ₁]₀ ∈ J₁ × ... × [σₙ]₀ ∈ Jₙ)

By induction on vᵢ, there exists vᵢ’ such that e₁₀ ∘ [σ₁]₀ ∈ J₁ × ... × [σₙ]₀ ∈ Jₙ = (emed(e₁₀) ∘ [σ₁]₀ ∈ J₁ × ... × [σₙ]₀ ∈ Jₙ)

and emed(e₂) = (emed(e₄), emed(e₅)) = (e₁, v₂) = e₁. Therefore, the result follows.

Lemma B.31. Let e ∈ E be an expression and [σⱼ] ∈ J a set of substitutions. If emed(e ∘ [σⱼ] ∈ J) ∼ eᵣ, then there exists eᵣ’ such that emed(e ∘ [σⱼ] ∈ J) ∼ eᵣ’ and emed(eᵣ’) = eᵣ.’

Proof. By induction and case analysis on e.

c₁: straightforward.

(e₁, e₂): emed(e ∘ [σⱼ] ∈ J) = (emed(e₁ ∘ [σⱼ] ∈ J), emed(e₂ ∘ [σⱼ] ∈ J)). There are two ways to reduce emed(e ∘ [σⱼ] ∈ J):

(1) emed(e₁ ∘ [σⱼ] ∈ J) ∼ e₁’, Similar to the case of (e₁, e₂).

(2) emed(e₂ ∘ [σⱼ] ∈ J) ∼ e₂’, Similar to the case of (e₁, e₂).

Lemma B.32. Let e ∈ E be an expression. If emed(e) ∼ eᵣ’, then there exists eᵣ’ such that e ∼ eᵣ’ and emed(eᵣ’) = eᵣ.’

Proof. By induction and case analysis on e.

c₁: straightforward.

(e₁, e₂): emed(e) = (emed(e₁), emed(e₂)). There are two ways to reduce emed(e):

(1) emed(e₁) ∼ e₁’, Similar to the case of (e₁, e₂),

(2) emed(e₂) ∼ e₂’, Similar to the case of (e₁, e₂).
\[ \pi_i(e_0) : \text{emd}(e) = \pi_i(\text{emd}(e_0)). \] There are two ways to reduce \( \text{emd}(e) \):

1. \( \text{emd}(e_0) \leadsto e'_0 \). By induction, there exists \( e'_0 \) such that \( e_0 \leadsto^* e'_0 \) and \( \text{emd}(e'_0) = e'_0 \). Then we have \( \pi_i(e_0) \leadsto^* \pi_i(e'_0) \) and \( \text{emd}(\pi_i(e')) = \pi_i(e'_0) \).

2. \( \text{emd}(e_0) = (v_1, v_2) \) and \( \text{emd}(e) \leadsto v_i \). According to Lemma B.30, there exist \( v'_1 \) and \( v'_2 \) such that \( e_0 \leadsto^* (v'_1, v'_2) \) and \( \text{emd}(v'_i) = v_i \). Then \( \pi_i(e_0) \leadsto^* v'_i \). The result follows.

\[ e_1 \leadsto e_2 : \text{emd}(e) = \text{emd}(e_1) \text{emd}(e_2). \] There are three ways to reduce \( \text{emd}(e) \):

1. \( \text{emd}(e_1) \leadsto e'_1 \). Similar to the case of \( e_1, e_2 \).
2. \( \text{emd}(e_2) \leadsto e'_2 \). Similar to the case of \( e_1, e_2 \).
3. \( \text{emd}(e_1) = \lambda_{(i \in I, t : x, e_0, \text{emd}(e_2) = v_2 \land \text{emd}(e) \leadsto (e_0 \exists [\sigma_j] \in P \{\nu_{i,2}, \nu_{i,2} \}) \text{, where } P = \{ j \in J \mid \exists \sigma \in I. \vdash v_2 \vdash t, \sigma_j \} \). According to Lemma B.30, we have (i) there exists \( e'_0 \) such that \( e_1 \leadsto (\text{emd}(e_0) = e_0 \land \text{emd}(e_0) = e_0 \land \text{emd}(e_0) = e_0 \) and (ii) there exists \( v'_i \) such that \( e_2 \leadsto (\text{emd}(v'_i) = v_2 \land \text{emd}(v'_i) = v_2 \). Moreover, by Lemma B.27, we get \( \vdash v_2 \vdash t, \sigma_j \) \( \iff \vdash v'_i \vdash t, \sigma_j \) \( \iff \{ j \in J \mid \exists \sigma \in I. \vdash v_2 \vdash t, \sigma_j \} \). Therefore, \( e \leadsto^* (e_0 \exists [\sigma_j] \in P \{\nu_{i,2}, \nu_{i,2} \}). Finally, by Lemma B.28, \( \text{emd}(e_0) \exists [\sigma_j] \in P \{\nu_{i,2}, \nu_{i,2} \}) = \text{emd}(e_0) \exists [\sigma_j] \in P \{\nu_{i,2}, \nu_{i,2} \). Similar to the subcase above.

\( e_0 \in t \iff e_2 : \text{emd}(e) = \text{emd}(e_1) : \text{emd}(e_2) \). There are three ways to reduce \( \text{emd}(e) \):

1. \( \text{emd}(e_1) \leadsto e'_1 \). Similar to the case of \( e_1, e_2 \).
2. \( \text{emd}(e_0) = e_0 \vdash v_0 : t \land \text{emd}(e_0) \leadsto (\text{emd}(e_1)). According to Lemma B.30, there exists \( v'_0 \) such that \( e_0 \leadsto^* v_0 \land \text{emd}(v'_0) = v_0 \). Moreover, by Lemma B.27, \( v'_0 \vdash t, \sigma_j \). So \( e \leadsto^* e_1 \).
3. \( \text{emd}(e_0) = e_0, \vdash v_0 : t \land \text{emd}(e_0) \leadsto (\text{emd}(e_2)). Similar to the subcase above.

\[ e_0[\sigma_j] \in J : \text{emd}(e_0[\sigma_j] \in J) = \text{emd}(e_0 \exists [\sigma_j] \in J) \land e_0 \exists [\sigma_j] \in J \leadsto e_0 \exists [\sigma_j] \in J. By Lemma B.31, the result follows.

\[
\frac{\Delta \vdash (e) : t \land \forall j \in J. \sigma_j \notin \Delta}{\Delta \vdash (e) : t \land \forall j \in J. \sigma_j \notin \Delta} \quad \text{(instnter)}
\]

Let \( \Delta \vdash (e) : t \) denote the typing judgments derivable in the type system with the typing rule \( \text{(instnter)} \) but not \( \text{(instnter)} \) and \( \text{(inter)} \). The following theorem proves that the type system \( \vdash_m \) (\( m \) stands for “merged”) is equivalent to the original one \( \vdash \).

**Theorem C.1.** Let \( e \) be an expression. Then \( \Delta \vdash (e) : t \iff \Delta \vdash (e) : t.\)
Proof. \( \Rightarrow \): It is clear that \( \text{(inst)} \) is a special case of \( \text{(instinter)} \) where \(|J| = 1\). We simulate each instance of \( \text{(instinter)} \) where \(|J| > 1\) by using several instances of \( \text{(inst)} \) followed by one instance of \( \text{(inter)} \). In detail, consider the following derivation

\[
\frac{\Delta_1 \Gamma \vdash e' : t'}{\Delta_1 \Gamma \vdash e'[\sigma_j]_{j \in J} : \bigwedge_{j \in J} t' \sigma_j} \quad (\text{instinter})
\]

We can rewrite this derivation as follows:

\[
\frac{\Delta_1 \Gamma \vdash e' : t'}{\Delta_1 \Gamma \vdash e'[\sigma_1] : t' \sigma_1} \quad \frac{\Delta_1 \Gamma \vdash e' : t'}{\Delta_1 \Gamma \vdash e'[\sigma_{|J|}] : t' \sigma_{|J|}} \quad (\text{inst}) \quad (\text{inst})
\]

\[
\frac{\Delta_1 \Gamma \vdash e'[\sigma_j]_{j \in J} : \bigwedge_{j \in J} t' \sigma_j}{\Delta_1 \Gamma \vdash e' : t} \quad (\text{inter})
\]

We can rewrite this derivation as follows:

\[
\frac{\Delta_1 \Gamma \vdash e' : t'}{\Delta_1 \Gamma \vdash e'[\sigma_1] : t' \sigma_1} \quad \frac{\Delta_1 \Gamma \vdash e' : t'}{\Delta_1 \Gamma \vdash e'[\sigma_{|J|}] : t' \sigma_{|J|}} \quad (\text{inst}) \quad (\text{inst})
\]

\[
\frac{\Delta_1 \Gamma \vdash e'[\sigma_j]_{j \in J} : \bigwedge_{j \in J} t' \sigma_j}{\Delta_1 \Gamma \vdash e' : t} \quad (\text{inter})
\]

\[
\Delta_1 \Gamma \vdash e : t
\]

\( \Leftarrow \): The proof proceeds by induction and case analysis on the structure of \( e \). For each case we use an auxiliary internal induction on the typing derivation. We label \( E \) the main (external) induction and \( I \) the internal induction in what follows.

\( e \equiv c_1 \): the typing derivation \( \Delta_1 \Gamma \vdash e : t \) should end with either \( \text{(const)} \) or \( \text{(subsum)} \). If the typing derivation ends with \( \text{(const)} \), the result follows straightforward.

Otherwise, the typing derivation ends with an instance of \( \text{(subsum)} \):

\[
\frac{\Delta_1 \Gamma \vdash e : s \quad s \leq t}{\Delta_1 \Gamma \vdash e : t} \quad (\text{subsum})
\]

Then by \( I \)-induction, we have \( \Delta_1 \Gamma \vdash m : e \). Since \( s \leq t \), by subsumption, we get \( \Delta_1 \Gamma \vdash m : e \). \( e \equiv c_2 \): similar to the case of \( e = c_1 \).

\( e = (e_1, e_2) \): the typing derivation \( \Delta_1 \Gamma \vdash e : t \) should end with either \( \text{(pair)} \) or \( \text{(subsum)} \). Assume that the typing derivation ends with \( \text{(pair)} \):

\[
\frac{\Delta_1 \Gamma \vdash e : t_1 \quad \Delta_1 \Gamma \vdash e : t_2}{\Delta_1 \Gamma \vdash (e_1, e_2) : t_1 \times t_2} \quad (\text{pair})
\]

By \( E \)-induction, we have \( \Delta_1 \Gamma \vdash m_1 : e_1 \). Then the rule \( \text{(pair)} \) gives us \( \Delta_1 \Gamma \vdash m_1 : (e_1, e_2) : t_1 \times t_2 \).

Otherwise, the typing derivation ends with an instance of \( \text{(subsum)} \), similar to the case of \( e = c_1 \), the result follows by \( I \)-induction.

\( e = \pi_i(e') \): the typing derivation \( \Delta_1 \Gamma \vdash e : t \) should end with either \( \text{(proj)} \) or \( \text{(subsum)} \). Assume that the typing derivation ends with \( \text{(proj)} \):

\[
\frac{\Delta_1 \Gamma \vdash e' : (t_1 \times t_2) \quad \Delta_1 \Gamma \vdash \pi_i(e') : t_i}{\Delta_1 \Gamma \vdash e : t_i} \quad (\text{proj})
\]

By \( E \)-induction, we have \( \Delta_1 \Gamma \vdash m : e' : (t_1 \times t_2) \). Then the rule \( \text{(proj)} \) gives us \( \Delta_1 \Gamma \vdash m : \pi_i(e') : t_i \).

Otherwise, the typing derivation ends with an instance of \( \text{(subsum)} \), similar to the case of \( e = c_1 \), the result follows by \( I \)-induction.

\( e = e_1 e_2 \): the typing derivation \( \Delta_1 \Gamma \vdash e : t \) should end with either \( \text{(appl)} \) or \( \text{(subsum)} \). Assume that the typing derivation ends with \( \text{(appl)} \):

\[
\frac{\Delta_1 \Gamma \vdash e_1 : t_1 \rightarrow t_2 \quad \Delta_1 \Gamma \vdash e_2 : t_1}{\Delta_1 \Gamma \vdash e_1 e_2 : t_2} \quad (\text{appl})
\]

By \( E \)-induction, we have \( \Delta_1 \Gamma \vdash m : e_1 : t_1 \rightarrow t_2 \). Then the rule \( \text{(appl)} \) gives us \( \Delta_1 \Gamma \vdash m : e_1 e_2 : t_2 \).

Otherwise, the typing derivation ends with an instance of \( \text{(subsum)} \), similar to the case of \( e = c_1 \), the result follows by \( I \)-induction.

\( e = \lambda_{(\sigma_j)_{j \in J}} x.e' \): the typing derivation \( \Delta_1 \Gamma \vdash e : t \) should end with either \( \text{(abstr)} \) or \( \text{(subsum)} \).

Assume that the typing derivation ends with \( \text{(abstr)} \):

\[
\forall i \in I, j \in J. \Delta_1 \Gamma; (x : t_i \sigma_j) \vdash e'[\sigma_j] : s_i \sigma_j
\]

\[
\Delta' = \Delta \cup \text{var}(\bigwedge_{i \in I, j \in J} (t_i, s_i) : s_i \sigma_j)
\]

\[
\Delta_1 \Gamma \vdash \lambda_{(\sigma_j)_{j \in J}} x.e' : \bigwedge_{i \in I, j \in J} (t_i, s_i) \rightarrow s_i \sigma_j)
\]

(\text{abstr})
By E-induction, for all \( i \in I \) and \( j \in J \), we have \( \Delta_1; \Gamma, (x : t_i \sigma_j) \vdash_m e' \sigma_j \). Then the rule (abstr) gives us \( \Delta_1; \Gamma \vdash_m \lambda[x \leftarrow t_i]^{\Gamma}; x : \bigwedge_{i \in I, j \in J} t_i \sigma_j \). Otherwise, the typing derivation ends with an instance of \((subsum)\), similar to the case of \( e = c \), the result follows by I-induction. 

\[ e = e' \in \ell \ (e_1 : e_2) \]: the typing derivation \( \Delta_1; \Gamma \vdash e : t \) should end with either \((case)\) or \((subsum)\). Assume that the typing derivation ends with \((case)\):

\[
\begin{align*}
\Delta_1; \Gamma \vdash e' ; t'  \\
\{ t' \not\in t \Rightarrow \Delta_1; \Gamma \vdash e_1 : s \\
\} \\
\{ t' \in t \Rightarrow \Delta_1; \Gamma \vdash e_2 : s \\
\end{align*}
\]

By E-induction, we have \( \Delta_1; \Gamma \vdash_m e' ; t' \) and \( \Delta_1; \Gamma \vdash_m e_1 : s \) (for \( i \) such that \( e_i \) has been effectively type-checked). Then the rule \((case)\) gives us \( \Delta_1; \Gamma \vdash (e' \ell \ e_1 : e_2) : s \). Otherwise, the typing derivation ends with an instance of \((subsum)\), similar to the case of \( e = c \), the result follows by I-induction.

\[ e = e'[\sigma_j] ; t \]: the typing derivation \( \Delta_1; \Gamma \vdash e : t \) should end with either \((inst)\) or \((subsum)\). Assume that the typing derivation ends with \((inst)\):

\[
\Delta_1; \Gamma \vdash e' : t \quad \sigma_j \not\in \Delta 
\]

By E-induction, we have \( \Delta_1; \Gamma \vdash_m e' : t \). Since \( \sigma_j \not\in \Delta \), applying \((inst)\) where \(|J| = 1\), we get \( \Delta_1; \Gamma \vdash_m e'[\sigma_j] : t \sigma_j \). Otherwise, the typing derivation ends with an instance of \((subsum)\), similar to the case of \( e = c \), the result follows by I-induction.

\[ e = e'_{\sigma_j \in J} : t \]: the typing derivation \( \Delta_1; \Gamma \vdash e : t \) should end with either \((inter)\) or \((subsum)\). Assume that the typing derivation ends with \((inter)\):

\[
\forall j \in J, \Delta_1; \Gamma \vdash e'_{\sigma_j} : t_j \\
\Delta_1; \Gamma \vdash e'[\sigma_j \in J] : \bigwedge_{i \in J} t_j \\
\]

As an intermediary result, we first prove that the derivation can be rewritten as

\[
\forall j \in J, \Delta_1; \Gamma \vdash e'_{\sigma_j} : s_j \\
\Delta_1; \Gamma \vdash e'[\sigma_j \in J] : \bigwedge_{i \in J} s_j \\
\Delta_1; \Gamma \vdash e'[\sigma_j \in J] : \bigwedge_{j \in J} s_j \\
\]

We proceed by induction on the original derivation. It is clear that each sub-derivation \( \Delta_1; \Gamma \vdash e'[\sigma_j] : t_j \) ends with either \((inst)\) or \((subsum)\). If all the sub-derivations end with an instance of \((inst)\), then for all \( j \in J \), we have

\[
\Delta_1; \Gamma \vdash e'_{\sigma_j} : s_j \\
\Delta_1; \Gamma \vdash e'[\sigma_j] : s_j \\
\]

By Lemma B.2, we have \( \Delta_1; \Gamma \vdash e' : \bigwedge_{i \in J} s_j \). Let \( s = \bigwedge_{i \in J} s_j \). Then by \((inst)\), we get \( \Delta_1; \Gamma \vdash e'[\sigma_j] : s \sigma_j \). Finally, by \((inter)\) and \((subsum)\), the intermediary result holds. Otherwise, there is at least one of the sub-derivations ends with an instance of \((subsum)\), the intermediary result also hold by induction.

Now that the intermediary result is proved, we go back to the proof of the lemma. By E-induction on \( e' \) (i.e., \( \Delta_1; \Gamma \vdash e' : s \)), we have \( \Delta_1; \Gamma \vdash_m e' : s \). Since \( \sigma_j \not\in \Delta \), applying \((inst)\), we get \( \Delta_1; \Gamma \vdash_m e'[\sigma_j \in J] : \bigwedge_{j \in J} s_j \sigma_j \). Finally, by subsumption, we get \( \Delta_1; \Gamma \vdash_m e'[\sigma_j \in J] : \bigwedge_{j \in J} t_j \). Otherwise, the typing derivation ends with an instance of \((subsum)\), similar to the case of \( e = c \), the result follows by I-induction.

From now on we will use \( \vdash \) to denote \( \vdash_m \), that is the system with the merged rule.

### C.2 Algorithmic Typing Rules

In this section, we analyze the typing derivations produced by the rules of Section A.3 to see where subsumption is needed and where it can be pushed down the derivation tree. We need first some preliminary definitions and decomposition results about pair and function types to deal with the projection and application rules.
C.2.1 Pair types

A type \( s \) is a pair type if \( s \leq \mathbb{1} \times \mathbb{1} \). If an expression \( e \) is typeable with a pair type \( s \), we want to compute from \( s \) a valid type for \( \pi_i(e) \). In CDuce, a pair type \( s \) is a finite union of product types, which can be decomposed into a finite set of pairs of types, denoted as \( \pi(s) \). For example, we decompose \( s = (t_1 \times t_2) \lor (s_1 \times s_2) \) as \( \pi(s) = \{(t_1, t_2), (s_1, s_2)\} \). We can then compute easily a type \( \pi_i(s) \) for \( \pi_i(e) \) as \( \pi_i(s) = t_i \lor s_i \) (we used boldface symbols to distinguish these type operators from the projections used in expressions). In the calculus considered here, the situation becomes more complex because of type variables, especially top-level ones. Let \( s \) be a pair type that contains a top-level variable \( \alpha \). Since \( \alpha \not\leq \mathbb{1} \times \mathbb{1} \) and \( s \leq \mathbb{1} \times \mathbb{1} \), then it is not possible that \( s \simeq s' \lor \alpha \). In other terms the top-level variable cannot appear alone in a union: it must occur intersected with some product type so that it does not "overtake" the \( \mathbb{1} \times \mathbb{1} \) bound. Consequently, we have \( s \simeq s' \land \alpha \) for some \( s' \leq \mathbb{1} \times \mathbb{1} \). However, in a typing derivation starting from \( \Gamma \vdash e : s \) and ending with \( \Delta; \Gamma \vdash \pi_i(e) : t \), there exists an intermediary step where \( e \) is assigned a type of the form \( t_1 \times t_2 \) (and that verifies \( s \leq (t_1 \times t_2) \)) before applying the projection rule. So it is necessary to get rid of the top-level variables of \( s \) (using subsumption) before computing the projection. The example above shows that \( \alpha \) does not play any role since it is the \( s' \) component that will be used to subsume \( s \) to a product type. To say it otherwise, since \( e \) has type \( s \) for all possible assignment of \( \alpha \), then the typing derivation must hold also for \( \alpha = 1 \). In whatever way we look at it, the top-level type variables are useless and can be safely discarded when decomposing \( s \).

Given a type \( t \), we write \( \text{dnf}(t) \) for a disjunctive normal form of \( t \), which is defined in [6]. Formally, we define the decomposition of a pair type as follows:

**Definition C.2.** Let \( \tau \) be a disjunctive normal form such that \( \tau \leq \mathbb{1} \times \mathbb{1} \). We define the decomposition of \( \tau \) as follows:

\[
\pi \left( \bigvee_{i \in I} \tau_i \right) = \bigcup_{i \in I} \pi(\tau_i)
\]

\[
\pi \left( \bigwedge_{j \in J} (t_1^j \times t_2^j) \land \bigwedge_{k \in K} \neg (t_1^k \times t_2^k) \land \bigwedge_{i \in I} \alpha \land \bigwedge_{n \in N} \neg \alpha' \right) = \pi \left( \bigvee_{N \subseteq N} \left( \bigwedge_{j \in J} t_1^j \land \bigwedge_{k \in K} \neg t_2^k \right) \times \left( \bigwedge_{i \in I} t_1^i \land \bigwedge_{n \in N} \neg \alpha' \right) \right)
\]

\[
\pi \left( (t_1 \times t_2) \right) = \begin{cases} \{(t_1, t_2)\} & t_1 \neq 0 \land t_2 \neq 0 \\ \emptyset & \text{otherwise} \end{cases}
\]

and the \( i \)-th projection as \( \pi_i(\tau) = \bigvee_{\{s_1, s_2\} \subseteq \pi(\tau)} s_i \).

For all type \( t \) such that \( t \leq \mathbb{1} \times \mathbb{1} \), the decomposition of \( t \) is defined as

\[
\pi(t) = \pi \left( \text{dnf}(\mathbb{1} \times \mathbb{1} \land t) \right)
\]

and the \( i \)-th projection as \( \pi_i(t) = \bigvee_{\{s_1, s_2\} \subseteq \pi(\text{dnf}(\mathbb{1} \times \mathbb{1} \land t))} s_i \).

The decomposition of a union of pair types is the union of each decomposition. When computing the decomposition of an intersection of product types and top-level type variables, we compute all the possible distributions of the intersections over the products, and we discard the top-level variables, as discussed above. Finally, the decomposition of a product is the pair of two components, provided that both components are not empty.

We now prove that the top-level type variables can be safely eliminated in a well-founded (convex) model with infinite support (see [6] for the definitions of model, convexity and infinite support).

**Lemma C.3.** Let \( \leq \) be the subtyping relation induced by a well-founded (convex) model with infinite support. Then

\[
\bigwedge_{p \in P} (t_p \times s_p) \land \alpha \leq \bigvee_{n \in N} (t_n \times s_n) \iff \bigwedge_{p \in P} (t_p \times s_p) \leq \bigvee_{n \in N} (t_n \times s_n)
\]

**Proof.** The result trivially holds if \( \bigwedge_{p \in P} (t_p \times s_p) = 0 \) or \( |P| = 0 \) (i.e., \( \bigwedge_{p \in P} (t_p \times s_p) = \mathbb{1} \)). Let us examine the remaining cases:

\( \Leftarrow \): straightforward.

\( \Rightarrow \): Assume that \( \bigwedge_{p \in P} (t_p \times s_p) \not\leq \bigvee_{n \in N} (t_n \times s_n) \). Let \( \tau \) be the type \( \bigwedge_{p \in P} (t_p \times s_p) \land \alpha \land \bigvee_{n \in N} \neg (t_n \times s_n) \).

Then there exists an assignment \( \eta \) such that \( [\tau] \eta \neq \emptyset \) (see the subtyping relation defined in [6]). Using the procedure \( \text{explore}_\text{pos} \) defined in the proof of Lemma 3.23 in [6], we can generate an element \( d \) belonging to \( [\tau] \eta \). The procedure \( \text{explore}_\text{pos} \) also generates an assignment \( \eta_0 \) for the type variables in \( \text{var}(\tau) \). We define \( \eta' \) such that \( \eta'(\alpha) = \eta_0(\alpha) \lor \{d\} \), \( \eta'(-\alpha) = \eta_0(-\alpha) \setminus \{d\} \), and \( \eta' = \eta_0 \) otherwise. Then we have \([\tau \land \alpha] \eta' \neq \emptyset\), which implies \( \bigwedge_{p \in P} (t_p \times s_p) \land \alpha \not\leq \bigvee_{n \in N} (t_n \times s_n) \). The result follows by the contrapositive.

The decomposition of pair types defined above has the following properties:

\footnote{Strictly speaking, the procedure \( \text{explore}_\text{pos} \) of Lemma 3.23 in [6] supposes \( \tau \) contains only finite product types, but it can be extended to infinite product types by Lemma 3.24 in [6].}
Lemma C.4. Let \( \leq \) be the subtyping relation induced by a well-founded (convex) model with infinite support and \( t \) a type such that \( t \leq 1 \times 1 \). Then

(1) For all \((t_1, t_2) \in \mathfrak{t}(t)\), we have \( t_1 \neq 0 \) and \( t_2 \neq 0 \).

(2) For all \( s_1, s_2 \), we have \( t \leq (s_1 \times s_2) \iff \bigvee_{(t_1, t_2) \in \mathfrak{t}(t)} (t_1 \times t_2) \leq (s_1 \times s_2) \).

Proof. 

(1): straightforward.

(2): Since \( t \leq 1 \times 1 \), we have

\[ t \simeq \bigvee_{(P,N) \in \mathfrak{d}(t)} ((1 \times 1) \land \bigwedge_{j \in P \setminus V} (t_1^j \land t_2^j) \land \bigwedge_{k \in N \setminus V} \neg(t_1^k \land t_2^k) \land \bigwedge_{\alpha \in P \setminus V} \alpha \land \bigwedge_{\alpha' \in N \setminus V} \neg\alpha') \]

If \( t \simeq 0 \), then \( \mathfrak{t}(t) = \emptyset \), and the result holds. Assume that \( t \neq 0 \), \( |P| > 0 \) and each summand of \( \text{dnf}(t) \) is not equivalent to \( \neg t \) as well. Let \( ^t t ^m \) denote the type \( \bigvee_{(P,N) \in \mathfrak{d}(t)} (\bigwedge_{j \in P \setminus V} (t_1^j \land t_2^j) \land \bigwedge_{k \in N \setminus V} \neg(t_1^k \land t_2^k)) \). Using the set-theoretic interpretation of types we have that \( ^t t ^m \) is equivalent to

\[ \bigvee_{(P,N) \in \mathfrak{d}(t)} \left( \bigvee_{N' \subseteq N \setminus V} \left( \bigwedge_{j \in P \setminus V} t_1^j \land \bigwedge_{k \in N'} \neg t_1^k \right) \times \left( \bigwedge_{j \in P \setminus V} t_2^j \land \bigwedge_{k \in (N \setminus V) \setminus N'} \neg t_2^k \right) \right) \]

This means that, \( ^t t ^m \) is an equivalent to a union of product types. Let us rewrite this union more explicitly, that is,

\[ ^t t ^m \simeq \bigvee_{i \in I} (t_1^i \times t_2^i) \]

obtained as follows

\[ \bigvee_{(P,N) \in \mathfrak{d}(t)} \left( \bigvee_{N' \subseteq N \setminus V} \left( \bigwedge_{j \in P \setminus V} t_1^j \land \bigwedge_{k \in N'} \neg t_1^k \right) \times \left( \bigwedge_{j \in P \setminus V} t_2^j \land \bigwedge_{k \in (N \setminus V) \setminus N'} \neg t_2^k \right) \right) \]

We have

\[ \mathfrak{t}(t) = \{ (t_1^i, t_2^i) \mid i \in I \text{ and } t_1^i \neq 0 \text{ and } t_2^i \neq 0 \} \]

Finally, for all pair of types \( s_1 \) and \( s_2 \), we have

\[ t \leq (s_1 \times s_2) \iff \bigvee_{(P,N) \in \mathfrak{d}(t)} \left( \bigwedge_{j \in P \setminus V} (t_1^j \times t_2^j) \land \bigwedge_{k \in N \setminus V} \neg(t_1^k \times t_2^k) \land \bigwedge_{\alpha \in P \setminus V} \alpha \land \bigwedge_{\alpha' \in N \setminus V} \neg\alpha' \right) \leq (s_1 \times s_2) \]

\[ \iff \bigwedge_{(P,N) \in \mathfrak{d}(t)} \left( \bigwedge_{j \in P \setminus V} (t_1^j \times t_2^j) \land \bigwedge_{k \in N \setminus V} \neg(t_1^k \times t_2^k) \land (s_1 \times s_2) \right) \leq 0 \quad \text{(Lemma C.3)} \]

\[ \iff \bigwedge_{(P,N) \in \mathfrak{d}(t)} \left( \bigwedge_{j \in P \setminus V} (t_1^j \times t_2^j) \land \bigwedge_{k \in N \setminus V} \neg(t_1^k \times t_2^k) \land \neg(s_1 \times s_2) \right) \leq 0 \]

\[ \iff \bigwedge_{(P,N) \in \mathfrak{d}(t)} \left( \bigwedge_{j \in P \setminus V} (t_1^j \times t_2^j) \land \bigwedge_{k \in N \setminus V} \neg t_1^k \land \neg t_2^k \right) \leq (s_1 \times s_2) \]

\[ \iff \bigwedge_{(t_1, t_2) \in \mathfrak{t}(t)} (t_1 \times t_2) \leq (s_1 \times s_2) \]

Lemma C.5. Let \( s \) be a type such that \( s \leq (t_1 \times t_2) \). Then

(1) \( s \leq (\mathfrak{t}_1(s) \times \mathfrak{t}_2(s)) \)

(2) \( \mathfrak{t}_1(s) \leq t_1 \)

Proof. (1): according to the proof of Lemma C.4, \( \bigvee_{(s_1,s_2) \in \mathfrak{t}(s)} (s_1 \times s_2) \) is equivalent to the type obtained from \( s \) by ignoring all the top-level type variables. Then it is trivial that \( s \leq \bigvee_{(s_1,s_2) \in \mathfrak{t}(s)} (s_1 \times s_2) \) and then \( s \leq (\mathfrak{t}_1(s) \times \mathfrak{t}_2(s)) \).

(2): since \( s \leq (t_1 \times t_2) \), according to Lemma C.4, we have \( \bigvee_{(s_1,s_2) \in \mathfrak{t}(s)} (s_1 \times s_2) \leq (t_1 \times t_2) \). So for all \( (s_1, s_2) \in \mathfrak{t}(s) \), we have \( (s_1 \times s_2) \leq (t_1 \times t_2) \). Moreover, as \( s_1 \) is not empty, we have \( s_1 \leq t_1 \). Therefore, \( \mathfrak{t}_1(s) \leq t_1 \).

Lemma C.6. Let \( t \) and \( s \) be two types such that \( t \leq 1 \times 1 \) and \( s \leq 1 \times 1 \). Then \( \mathfrak{t}_1(t \land s) \leq \mathfrak{t}_1(t) \land \mathfrak{t}_1(s) \).

Proof. Let \( t = \bigvee_{j \in J_1} \tau_j \) and \( s = \bigvee_{j \in J_2} \tau_j \) such that

\[ \tau_j = (t_1^j \times t_2^j) \land \bigwedge_{\alpha \in P_j} \alpha \land \bigwedge_{\alpha' \in N_j} \neg\alpha' \]

\[ \mathfrak{t}_1(t \land s) \leq \mathfrak{t}_1(t) \land \mathfrak{t}_1(s) \]
and \( \tau_j \not\in 0 \) for all \( j \in J_1 \cup J_2 \). Then we have \( t \land s = \bigvee_{j \in J_1} t_{j_1} \land t_{j_2} \). Let \( j_1 \in J_1 \) and \( j_2 \in J_2 \).

If \( t_{j_1} \land t_{j_2} \models 0 \), we have \( \pi(t_{j_1} \land t_{j_2}) = 0 \). Otherwise, \( \pi(t_{j_1} \land t_{j_2}) = t_{j_1} \land t_{j_2} = \pi(t_{j_1}) \land \pi(t_{j_2}) \).

For both cases, we have \( \pi(t_{j_1} \land t_{j_2}) \leq \pi(t_{j_1}) \land \pi(t_{j_2}) \).

Therefore:

\[
\pi_i(t \land s) \simeq \bigvee_{j \in J_1, j_2 \in J_2} \pi_i(t_{j_1} \land t_{j_2}) \\
\leq \bigvee_{j \in J_1, j_2 \in J_2} (\pi_i(t_{j_1}) \land \pi_i(t_{j_2})) \\
\simeq (\bigvee_{j \in J_1} \pi_i(t_{j_1})) \land (\bigvee_{j \in J_2} \pi_i(t_{j_2})) \\
\simeq \pi_i(t) \land \pi_i(s).
\]

\[\square\]

For example, \( \pi_i((\text{Int} \times \text{Int}) \land (\text{Int} \times \text{Bool})) = \pi_i(0) = 0 \), while \( \pi_i((\text{Int} \times \text{Int}) \land \pi_i((\text{Int} \times \text{Bool})) = \text{Int} \land \text{Int} = \text{Int} \).

**Lemma C.7.** Let \( t \) be a type and \( \sigma \) be a type substitution such that \( t \leq 1 \times 1 \). Then \( \pi_i(t \sigma) \leq \pi_i(t) \sigma \)

**Proof.** We put \( t \) into its disjunctive normal form \( \bigvee_{j \in J} \tau_j \) such that

\[ \tau_j = (t^1_j \times t^2_j) \land \bigwedge_{\alpha \in P_j} \alpha \land \bigwedge_{\alpha' \in N_j} -\alpha' \]

and \( \tau_j \not\in 0 \) for all \( j \in J \). Then we have \( t \sigma = \bigvee_{j \in J} \tau_j \sigma \). So \( \pi_i(t \sigma) = \bigvee_{j \in J} \pi_i(\tau_j \sigma) \).

Let \( j \in J \). If \( \tau_j \sigma \models 0 \), then \( \pi_i(\tau_j \sigma) = 0 \) and trivially \( \pi_i(\tau_j \sigma) \leq \pi_i(\tau_j) \sigma \). Otherwise, we have \( \tau_j \sigma = (t^1_j \sigma \times t^2_j \sigma) \land (\bigwedge_{\alpha \in P_j} \alpha \land \bigwedge_{\alpha' \in N_j} -\alpha') \sigma \).

By Lemma C.6, we get \( \pi_i(\tau_j \sigma) \leq t^1_j \sigma \land \pi_i((\bigwedge_{\alpha \in P_j} \alpha \land \bigwedge_{\alpha' \in N_j} -\alpha') \sigma) \leq t^1_j \sigma \leq \pi_i(\tau_j) \sigma \).

Therefore, \( \bigvee_{j \in J} \pi_i(\tau_j \sigma) \leq \bigvee_{j \in J} \pi_i(\tau_j) \sigma \), that is, \( \pi_i(t \sigma) \leq \pi_i(\tau) \sigma \).

\[\square\]

For example, \( \pi_i(((\text{Int} \times \text{Int}) \land \alpha) \{((\text{Int} \times \text{Bool})/\alpha) \}) = \pi_i(((\text{Int} \times \text{Int}) \land \text{Int} \times \text{Bool})) = 0 \), while \( \pi_i(((\text{Int} \times \text{Int})) \{((\text{Int} \times \text{Bool})/\alpha) \}) = \text{Int} \).

**Lemma C.8.** Let \( t \) be a type such that \( t \leq 1 \times 1 \) and \([\sigma_k]_{k \in K} \) be a set of type substitutions. Then \( \pi_i(\bigwedge_{k \in K} \tau_k \sigma_k) \leq \bigwedge_{k \in K} \pi_i(t \sigma_k) \)

**Proof.** Consequence of Lemmas C.6 and C.7. \[\square\]

### C.2.2 Function types

A type \( t \) is a function type if \( t \leq 0 \to 1 \). In order to type the application of a function having a function type \( t \), we need to determine the domain of \( t \), that is, the set of values the function can be safely applied to. This problem has been solved for ground function types in [12]. Again, the problem becomes more complex if \( t \) contains top-level type variables. Another issue is to determine what is the result type of an application of a function type \( t \) to an argument of type \( s \) (where \( s \) belongs to the domain of \( t \)), knowing that both \( t \) and \( s \) may contain type variables.

Following the same reasoning as with pair types, if a function type \( t \) contains a top-level variable \( \alpha \), then \( t \models t' \land \alpha \) for some function type \( t' \). In a typing derivation for a judgment \( \Delta \Gamma \vdash e_1 e_2 : t \) that contains \( \Delta \Gamma \vdash e_1 : t \), there exists an intermediary step where we assign a type \( t_1 \to t_2 \to e_1 \) (with \( t_1 \leq t_1 \to t_2 \) before using the application rule. It is therefore necessary to eliminate the top-level variables from the function type \( t \) before we can type an application. Once more, the top-level variables are useless when computing the domain of \( t \) and can be safely discarded.

Formally, we define the domain of a function type as follows:

**Definition C.9 (Domain).** Let \( \tau \) be a disjunctive normal form such that \( \tau \leq 0 \to 1 \). We define \( \text{dom}(\tau) \), the domain of \( \tau \), as:

\[
\text{dom}(\bigvee_{i \in I} \tau_i) = \bigwedge_{i \in I} \text{dom}(\tau_i)
\]

\[
\text{dom}(\bigwedge_{j \in P}(t^1_j \to t^2_j) \land (\bigwedge_{k \in N} -\tau_1^j \to -\tau_2^j) \land (\bigwedge_{\alpha \in P_j} \alpha \land (\bigwedge_{\alpha' \in N_j} -\alpha')) =
\begin{cases}
1 & \text{if } \bigwedge_{j \in P} (t^1_j \to t^2_j) \land (\bigwedge_{k \in N} -\tau_1^j \to -\tau_2^j) \land (\bigwedge_{\alpha \in P_j} \alpha \land (\bigwedge_{\alpha' \in N_j} -\alpha')) \models 0 \\
\bigvee_{j \in P} t^1_j & \text{otherwise}
\end{cases}
\]

For any type \( t \) such that \( t \leq 0 \to 1 \), the domain of \( t \) is defined as:

\( \text{dom}(t) = \text{dom}(\text{dnf}((0 \to 1) \land t)) \)

We also define a decomposition operator \( \phi \) that —akin to the decomposition operator \( \pi \) for product types— decomposes a function type into a finite set of pairs:

**Definition C.10.** Let \( \tau \) be a disjunctive normal form such that \( \tau \leq 0 \to 1 \). We define the decomposition of \( \tau \) as:

\[
\phi(\bigvee_{i \in I} \tau_i) = \bigcup_{i \in I} \phi(\tau_i)
\]

\[
\phi(\bigwedge_{j \in P}(t^1_j \to t^2_j) \land (\bigwedge_{k \in N} -\tau_1^j \to -\tau_2^j) \land (\bigwedge_{\alpha \in P_j} \alpha \land (\bigwedge_{\alpha' \in N_j} -\alpha')) =
\begin{cases}
0 & \text{if } \bigwedge_{j \in P} (t^1_j \to t^2_j) \land (\bigwedge_{k \in N} -\tau_1^j \to -\tau_2^j) \land (\bigwedge_{\alpha \in P_j} \alpha \land (\bigwedge_{\alpha' \in N_j} -\alpha')) \models 0 \\
\{(\bigvee_{j \in P} t^1_j : \bigwedge_{j \in P \setminus P'} P') \mid P' \subseteq P\} & \text{otherwise}
\end{cases}
\]
For any type $t$ such that $t \leq \emptyset \rightarrow 1$, the decomposition of $t$ is defined as

$$\phi(t) = \phi(\text{dnf}((\emptyset \rightarrow 1) \land t)).$$

The set $\phi(t)$ satisfies the following fundamental property: for every arrow type $s \rightarrow s'$, the constraint $t \leq s \rightarrow s'$ holds if and only if $s \leq \text{dom}(t)$ holds and for all $(t_1, t_2) \in \phi(t)$, either $s \leq t_1$ or $t_2 \leq s'$ hold (see Lemma C.12). As a result, the minimum type

$$t \cdot s = \min\{s' \mid t \leq s \rightarrow s'\}$$

exists, and it is defined as the union of all $t_2$ such that $s \leq t_1$ and $(t_1, t_2) \in \phi(t)$ (see Lemma C.13). The type $t \cdot s$ is used to type the application of an expression of type $t$ to an expression of type $s$.

As with pair types, in a well-founded (convex) model with infinite support, we can safely eliminate the top-level type variables.

**Lemma C.11.** Let $\leq$ be the subtyping relation induced by a well-founded (convex) model with infinite support. Then

$$\forall n \in \mathbb{N} \quad (t_n \rightarrow s_n) \iff \bigwedge_{n \in \mathbb{N}} (t_n \rightarrow s_n)$$

**Proof.** Similar to the proof of Lemma C.3. \qed

**Lemma C.12.** Let $\leq$ be the subtyping relation induced by a well-founded (convex) model with infinite support and $t$ a type such that $t \leq \emptyset \rightarrow \emptyset$. Then

$$\forall s_1, s_2 \quad (t_1 \leq s_1 \rightarrow s_2) \iff \left\{ \begin{array}{l}
  s_1 \leq \text{dom}(t) \\
  \forall (t_1, t_2) \in \phi(t), (s_1 \leq t_1) \text{ or } (t_2 \leq s_2)
\end{array} \right.$$  

**Proof.** Since $t \leq \emptyset \rightarrow \emptyset$, we have

$$t \equiv \bigcup_{(P,N) \in \text{dom}(t)} ((0 \rightarrow 1) \land \bigwedge_{j \in P \setminus Y} (t_1^j \rightarrow t_2^j) \land \bigwedge_{k \in N \setminus Y} (t_1^k \rightarrow t_2^k) \land \bigwedge_{\alpha \in P \setminus Y} \alpha \land \bigwedge_{\alpha' \in N \setminus Y} \neg \alpha')$$

If $t \equiv \emptyset$, then $\text{dom}(t) = \emptyset$, $\phi(t) = \emptyset$, and the result holds. If $t \equiv t_1 \lor t_2$, then $t_1 \leq \emptyset \rightarrow 1$, $t_2 \leq \emptyset \rightarrow \emptyset$, $\text{dom}(t) = \text{dom}(t_1) \lor \text{dom}(t_2)$ and $\phi(t) = \phi(t_1) \cup \phi(t_2)$. So the result follows if it also holds for $t_1$ and $t_2$. Thus, without loss of generality, we can assume that $t$ has the following form:

$$t \equiv \bigcup_{j \in P} (t_1^j \rightarrow t_2^j) \land \bigwedge_{k \in N} \neg (t_1^k \rightarrow t_2^k) \land \bigwedge_{\alpha \in P \setminus Y} \alpha \land \bigwedge_{\alpha' \in N \setminus Y} \neg \alpha'$$

where $P \neq \emptyset$ and $t \neq \emptyset$. Then $\text{dom}(t) = \bigcup_{j \in P} t_1^j$ and $\phi(t) = \{\{\bigcup_{j \in P \setminus P'} t_1^j, \land_{j \in P \setminus P'} t_2^j) \mid P' \subseteq P\}$. For every pair of types $s_1$ and $s_2$, we have

$$t \leq (s_1 \rightarrow s_2)$$

$$\equiv \bigcup_{j \in P} (t_1^j \rightarrow t_2^j) \land \bigwedge_{k \in N} \neg (t_1^k \rightarrow t_2^k) \land \bigwedge_{\alpha \in P \setminus Y} \alpha \land \bigwedge_{\alpha' \in N \setminus Y} \neg \alpha' \leq (s_1 \rightarrow s_2)$$

$$\equiv \bigcup_{j \in P} (t_1^j \rightarrow t_2^j) \land \bigwedge_{k \in N} \neg (t_1^k \rightarrow t_2^k) \leq (s_1 \rightarrow s_2)$$

where $P' \subseteq P$. (Lemma C.11)

$$\equiv \forall P' \subseteq P, \left( s_1 \leq \bigcup_{j \in P'} t_1^j \lor P' \neq P \land \bigwedge_{j \in P \setminus P'} t_2^j \leq s_2 \right)$$

where $t' = \bigcup_{j \in P} t_1^j \land \bigwedge_{k \in N} \neg t_1^k$.

**Lemma C.13.** Let $t$ and $s$ be two types. If $t \leq s \rightarrow \emptyset$, then $t \leq s \rightarrow s'$ has a smallest solution $s'$, which is denoted as $t \cdot s$.

**Proof.** Since $t \leq s \rightarrow \emptyset$, by Lemma C.12, we have $s \leq \text{dom}(t)$. Then the assertion $t \leq s \rightarrow s'$ is equivalent to:

$$\forall (t_1, t_2) \in \phi(t). (s \leq t_1) \lor (t_2 \leq s')$$

that is:

$$\bigcup_{(t_1, t_2) \in \phi(t). (s \leq t_1)} t_2 \leq s'$$

Thus the type $s'$ is a lower bound for all the solutions.

By the subtyping relation on arrows it is also a solution, so it is the smallest one. To conclude, it suffices to take it as the definition for $t \cdot s$. \qed
We now prove some properties of the operators \( \text{dom}(\_\_\&) \) and \("\_\_\&\_\_"\).

**Lemma C.14.** Let \( t \) be a type such that \( t \leq 0 \rightarrow 1 \) and \( t',\, s,\, s' \) be types. Then

1. if \( s' \leq s \leq \text{dom}(t) \), then \( t \cdot s' \leq t \cdot s \).
2. if \( t' \leq t \), and \( s \leq \text{dom}(t') \), then \( t' \cdot s \leq t \cdot s \).

**Proof.** (1) Since \( s' \leq s \), we have \( s \rightarrow t \cdot s' \leq t \cdot s \). By definition of \( t \cdot s \), we have \( t \leq s \rightarrow t \cdot s \), therefore \( t \leq s' \rightarrow t \cdot s' \) holds. Consequently, we have \( t \cdot s' \leq t \cdot s \) by definition of \( t \cdot s' \).

(2) By definition, we have \( t \leq s \rightarrow t \cdot s \), which implies \( t' \leq s \rightarrow t \cdot s \). Therefore, \( t' \cdot s \) is a solution to \( t' \leq s \rightarrow s' \), hence we have \( t' \cdot s \leq t \cdot s \).

**Lemma C.15.** Let \( t \) and \( s \) be two types such that \( t \leq 0 \rightarrow 1 \) and \( s \leq 0 \rightarrow 1 \). Then \( \text{dom}(t) \lor \text{dom}(s) \leq \text{dom}(t \land s) \).

**Proof.** Let \( t = \bigvee_{i \in I_1} \tau_{i_1} \) and \( s = \bigvee_{i \in I_2} \tau_{i_2} \) such that \( \tau_i \neq 0 \) for all \( i \in I_1 \cup I_2 \). Then we have \( t \land s = \bigvee_{i \in I_1, j \in I_2} \tau_{i_1} \land \tau_{i_2} \). Let \( i \in I_1 \) and \( j \in I_2 \). If \( \tau_i \land \tau_j \neq 0 \), then \( \text{dom}(\tau_i \land \tau_j) = 1 \). Otherwise, \( \text{dom}(\tau_i \land \tau_j) = \text{dom}(\tau_i) \lor \text{dom}(\tau_j) \). In both cases, we have \( \text{dom}(\tau_i \land \tau_j) \geq \text{dom}(\tau_i) \lor \text{dom}(\tau_j) \). Therefore
\[
\text{dom}(t \land s) \simeq \bigwedge_{i \in I_1, j \in I_2} \text{dom}(\tau_{i_1} \land \tau_{i_2}) \\
\geq \bigwedge_{i \in I_1} \text{dom}(\tau_{i_1} \lor \text{dom}(\tau_{i_1})) \\
\geq \bigwedge_{i \in I_1} \text{dom}(\tau_{i_1}) \\
= \text{dom}(t) \\
= \text{dom}(t \land s).
\]

Similarly, \( \text{dom}(t \land s) \geq \text{dom}(s) \). Therefore \( \text{dom}(t) \lor \text{dom}(s) \leq \text{dom}(t \land s) \).

**Lemma C.16.** Let \( t \) be a type and \( \sigma \) be a type substitution such that \( t \leq 0 \rightarrow 1 \). Then \( \text{dom}(t) \sigma \leq \text{dom}(t \sigma) \).

**Proof.** We put \( t \) into its disjunctive normal form \( \bigvee_{i \in I} \tau_i \) such that \( \tau_i \neq 0 \) for all \( i \in I \). Then we have \( t \sigma = \bigvee_{i \in I} \tau_i \sigma \). So \( \text{dom}(t \sigma) = \bigwedge_{i \in I} \text{dom}(\tau_i \sigma) \). Let \( i \in I \). If \( \tau_i \sigma \neq 0 \), then \( \text{dom}(\tau_i \sigma) = 1 \). Otherwise, let \( \tau_i = \bigwedge_{p \in P} (t_i \rightarrow t_i') \land \bigwedge_{k \in K} (t_i \to t_i) \land \bigwedge_{\alpha \in \mathcal{P}} \alpha \land \bigwedge_{\alpha' \in \mathcal{P}'} \alpha' \). Then \( \text{dom}(\tau_i) = \bigvee_{k \in K} t_i \land \text{dom}(\tau_i \sigma) \). In both cases, we have \( \text{dom}(t \sigma) \leq \text{dom}(t \sigma) \). Therefore, \( \bigwedge_{i \in I} \text{dom}(\tau_i \sigma) \leq \bigwedge_{i \in I} \text{dom}(\tau_i \sigma) \) that is, \( \text{dom}(t \sigma) \leq \text{dom}(t \sigma) \).

**Lemma C.17.** Let \( t \) be a type such that \( t \leq 0 \rightarrow 1 \) and \( [\sigma_k]_{k \in K} \) be a set of type substitutions. Then
\[
\bigwedge_{k \in K} \text{dom}(t \sigma_k) \leq \bigwedge_{k \in K} \text{dom}(t \sigma_k) \quad \text{by Lemma C.16}
\]
\[
\leq \bigwedge_{k \in K} \text{dom}(t \sigma_k) \quad \text{by Lemma C.15}
\]

**Lemma C.18.** Let \( t_1, s_1, t_2, \) and \( s_2 \) be types such that \( t_1 \cdot s_1 \) and \( t_2 \cdot s_2 \) exists. Then \( (t_1 \land t_2) \cdot (s_1 \land s_2) \) exists and \( (t_1 \land t_2) \cdot (s_1 \land s_2) \leq (t_1 \cdot s_1) \cdot (t_2 \cdot s_2) \).

**Proof.** According to Lemma C.13, we have \( s_i \leq \text{dom}(t_i) \) and \( t_i \leq s_i \rightarrow (t_i \cdot s_i) \). Then by Lemma C.15, we get \( s_i \land s_2 \leq \text{dom}(t_1) \land (t_2) \leq \text{dom}(t_1 \land t_2) \). Moreover, \( t_1 \land t_2 \leq (s_1 \land s_2) \rightarrow ((t_1 \cdot s_1) \land (t_2 \cdot s_2)) \). Therefore, \( (t_1 \land t_2) \cdot (s_1 \land s_2) \) exists and \( (t_1 \land t_2) \cdot (s_1 \land s_2) \leq (t_1 \cdot s_1) \cdot (t_2 \cdot s_2) \).

**Lemma C.19.** Let \( t \) and \( s \) be types such that \( t \cdot s \) exists and \( \sigma \) be a type substitution. Then \( (t \sigma) \cdot (s \sigma) \) exists and \( (t \sigma) \cdot (s \sigma) \leq (t \cdot s) \).

**Proof.** Because \( t \cdot s \) exists, we have \( s \leq \text{dom}(t) \) and \( t \leq s \rightarrow (t \cdot s) \). Then \( s \sigma \leq \text{dom}(t \sigma) \) and \( t \sigma \leq (t \cdot s) \). By Lemma C.16, we get \( \text{dom}(t \sigma) \leq \text{dom}(t \sigma) \). So \( s \sigma \leq \text{dom}(t \sigma) \). Therefore, \( (t \sigma) \cdot (s \sigma) \) exists, since \( (t \cdot s) \) is a solution to \( t \sigma \leq (t \cdot s) \sigma \), by definition, we have \( (t \sigma) \cdot (s \sigma) \leq (t \cdot s) \sigma \).
For example, \(((\text{Int} \rightarrow \text{Int}) \land \lnot \sigma) \cdot (\text{Int} \sigma) = 0 \cdot \text{Int} = 0\), while \(((\text{Int} \rightarrow \text{Int}) \land \lnot \sigma) \cdot \text{Int} \sigma = \text{Int} \sigma = \text{Int} \cdot \text{Int} \sigma / \sigma\).

**Lemma C.20.** Let \(t\) and \(s\) be two types and \([\sigma_k]_{k \in K}\) be a set of type substitutions such that \(t \cdot s\) exists. Then 
\[
(\bigwedge_{k \in K} (\sigma_k)) \cdot (\bigwedge_{k \in K} s\sigma_k) \text{ exists and } (\bigwedge_{k \in K} t\sigma_k) \cdot (\bigwedge_{k \in K} s\sigma_k) \leq \bigwedge_{k \in K} (t \cdot s)\sigma_k.
\]

**Proof.** According to Lemmas C.19 and C.18, 
\[
(\bigwedge_{k \in K} (\sigma_k)) \cdot (\bigwedge_{k \in K} s\sigma_k) \text{ exists. Moreover,}
\]
\[
\bigwedge_{k \in K} (t \cdot s)\sigma_k \geq (\bigwedge_{k \in K} (t\sigma_k \cdot s\sigma_k)) \cdot (\bigwedge_{k \in K} s\sigma_k) \quad \text{(Lemma C.19)}
\]
\[
\geq (\bigwedge_{k \in K} t\sigma_k) \cdot (\bigwedge_{k \in K} s\sigma_k) \quad \text{(Lemma C.18)}
\]

\[\square\]

### 2.2.3 Syntax-Directed Rules

Because of subsumption, the typing rules provided in Section A.3 are not syntax-directed and so they do not yield a type-checking algorithm directly. In simply type \(\lambda\)-calculus, subsumption is used to bridge gaps between the types expected by functions and the actual types of their arguments in applications [19]. In our calculus, we identify four situations where the subsumption is needed, namely, the rules for projections, abstractions, applications, and type cases. To see why, we consider a typing derivation ending with each typing rule whose immediate sub-derivation ends with \((\text{subsum})\). For each case, we explain how the use of subsumption can be pushed through the typing rule under consideration, or how the rule should be modified to take subtyping into account.

First, we consider the case where a typing derivation ends with \((\text{subsum})\) whose immediate sub-derivation also ends with \((\text{subsum})\). The two consequent uses of \((\text{subsum})\) can be merged into one, because the subtyping relation is transitive.

**Lemma C.21.** If \(\Delta \mid \Gamma \vdash e : t\), then there exists a derivation for \(\Delta \mid \Gamma \vdash e : t\) where there are no consecutive instances of \((\text{subsum})\).

**Proof.** Assume that there exist two consecutive instances of \((\text{subsum})\) occurring in a derivation of \(\Delta \mid \Gamma \vdash e : t\), that is,
\[
\Delta' \Gamma \vdash e : s_2' \quad s_2' \leq s_1' \quad (\text{subsum})
\]
\[
\Delta' \Gamma \vdash e : s_1' \quad s_1' \leq t' \quad (\text{subsum})
\]
\[
\Delta' \Gamma \vdash e : t
\]
Since \(s_2' \leq s_1'\) and \(s_1' \leq t'\), we have \(s_2' \leq t'\). So we can rewrite this derivation as follows:
\[
\Delta' \Gamma \vdash e : s_2' \quad s_2' \leq t' \quad (\text{subsum})
\]
\[
\Delta' \Gamma \vdash e : t
\]
Therefore, the result follows.

\[\square\]

Next, consider an instance of \((\text{pair})\) such that one of its sub-derivations ends with an instance of \((\text{subsum})\), for example, the left sub-derivation:
\[
\Delta \mid \Gamma \vdash e_1 : s_1 \quad s_1 \leq t_1 \quad (\text{subsum})
\]
\[
\Delta \mid \Gamma \vdash e_2 : t_2 \quad (\text{pair})
\]
As \(s_1 \leq t_1\), we have \(s_1 \times t_2 \leq t_1 \times t_2\). Then we can move subsumption down through the rule \((\text{pair})\), giving the following derivation:
\[
\Delta \mid \Gamma \vdash e_1 : s_1 \quad \Delta \mid \Gamma \vdash e_2 : t_2 \quad (\text{pair})
\]
\[
\Delta \mid \Gamma \vdash e_1 \times e_2 \ : (s_1 \times t_2) \quad (\text{subsum})
\]
\[
\Delta \mid \Gamma \vdash e_1 \times e_2 \ : (t_1 \times t_2)
\]
The rule \((\text{proj})\) is a little trickier than \((\text{pair})\). Consider the following derivation:
\[
\Delta \mid \Gamma \vdash e : s \quad s \leq t_1 \times t_2 \quad (\text{subsum})
\]
\[
\Delta \mid \Gamma \vdash e : (t_1 \times t_2)
\]
\[
\Delta \mid \Gamma \vdash \pi_1(e) : t_1 \quad (\text{proj})
\]
\[
\Delta \mid \Gamma \vdash \pi_2(e) : t_2
\]
As \( s \leq t_1 \times t_2 \), \( s \) is a pair type. According to the decomposition of \( s \) and Lemma C.5, we can rewrite the previous derivation into the following one:

\[
\frac{\Delta \vdash e : s \quad s \leq t_1 \times t_2}{\Delta \vdash \pi_i(e) : \pi_i(s) \quad \pi_i(s) \leq t_i}
\]

Note that the subtyping check \( s \leq 1 \times 1 \) ensures that \( s \) is a pair type.

Next consider an instance of (\textbf{subsum}) (where \( \Delta' = \Delta \cup \text{var}(\bigwedge_{i \in I, j \in J} (t_i \sigma_j \rightarrow s_i \sigma_j)) \)). All the subderivations may end with (\textbf{subsum}):

\[
\frac{\forall i \in I, j \in J. \quad \Delta' \vdash (x : t_i \sigma_j) \vdash e \circ [\sigma_j] : s'_{ij} \quad s'_{ij} \leq s_i \sigma_j}{\Delta \vdash \lambda^{\bigwedge_{i \in I, j \in J} x : t_i \sigma_j} e : \bigwedge_{i \in I, j \in J} (t_i \sigma_j \rightarrow s_i \sigma_j)}
\]

Without subsumption, we would assign the type \( \bigwedge_{i \in I, j \in J} (t_i \sigma_j \rightarrow s_i \sigma_j) \) to the abstraction, while we want to assign the type \( \bigwedge_{i \in I, j \in J} (t_i \sigma_j \rightarrow s_i \sigma_j) \) to it because of the type annotations. Consequently, we have to keep the subtyping checks \( s'_{ij} \leq s_i \sigma_j \) as side-conditions of an algorithmic typing rule for abstractions.

\[
\frac{\forall i \in I, j \in J. \quad \Delta' \vdash (x : t_i \sigma_j) \vdash e \circ [\sigma_j] : s'_{ij} \quad s'_{ij} \leq s_i \sigma_j}{\Delta \vdash \lambda^{\bigwedge_{i \in I, j \in J} x : t_i \sigma_j} e : \bigwedge_{i \in I, j \in J} (t_i \sigma_j \rightarrow s_i \sigma_j)}
\]

In (\textbf{appl}) case, suppose that both subderivations end with (\textbf{subsum}):

\[
\frac{\Delta' \vdash e_1 : t \quad t \leq t' \rightarrow s' \quad \Delta' \vdash e_1 : s \quad s \leq t'}{\Delta \vdash e_1 : t' \rightarrow s' \quad \Delta \vdash e_2 : t' \quad \Delta \vdash e_1 e_2 : s'}
\]

Since \( s \leq t' \), then by the contravariance of arrow types we have \( t' \rightarrow s' \leq s \rightarrow s' \). Hence, such a derivation can be rewritten as

\[
\frac{\Delta' \vdash e_1 : t \quad t \leq t' \rightarrow s' \quad \Delta' \vdash e_1 : t' \rightarrow s' \quad e' \rightarrow s' \leq s \rightarrow s'}{\Delta \vdash e_1 e_2 : s' \quad \Delta \vdash e_1 : s}
\]

Applying Lemma C.21, we can merge the two adjacent instances of (\textbf{subsum}) into one:

\[
\frac{\Delta' \vdash e_1 : t \quad t \leq s \rightarrow s' \quad \Delta' \vdash e_1 : s \quad s \rightarrow s'}{\Delta \vdash e_1 e_2 : s'}
\]

A syntax-directed typing rule for applications can then be written as follows

\[
\frac{\Delta' \vdash e_1 : t \quad \Delta' \vdash e_2 : s \quad t \leq s \rightarrow s'}{\Delta \vdash e_1 e_2 : s'}
\]

where subsumption is used as a side condition to bridge the gap between the function type and the argument type.

This typing rule is not algorithmic yet, because the result type \( s' \) can be any type verifying the side condition. Using Lemma C.12, we can equivalently rewrite the side condition as \( t \leq 0 \rightarrow 1 \) and \( s \leq \text{dom}(t) \) without involving the result type \( s' \). The first condition ensures that \( t \) is a function type and the second one that the argument type \( s \) can be safely applied by \( t \). Moreover, we assign the type \( t \rightarrow s \) to the application, which is by definition the smallest possible type for it. We obtain then the following algorithmic typing rule.

\[
\Delta' \vdash e_1 : t \quad \Delta' \vdash e_2 : s \quad t \leq 0 \rightarrow 1 \quad s \leq \text{dom}(t)
\]

Next, let us discuss the rule (\textbf{case})::

\[
\frac{\Delta' \vdash e : t \quad \begin{cases} t' \leq \neg t & \Rightarrow \Delta' \vdash e_1 : s \\ t' \leq t & \Rightarrow \Delta' \vdash e_2 : s \end{cases}}{\Delta \vdash (e \in t ? e_1 : e_2) : s}
\]

The rule covers four different situations, depending on which branches of the type-cases are checked: (i) no branch is type-checked, (ii) the first branch \( e_1 \) is type-checked, (iii) the second branch \( e_2 \) is type-checked, and (iv) both branches are type-checked. Each case produces a corresponding algorithmic rule.

In case (i), we have simultaneously \( t' \leq t \) and \( t' \leq \neg t \), which means that \( t' = 0 \). Consequently, \( e \) does not reduce to a value (otherwise, subject reduction would be violated), and neither does the whole type case.
Consequently, we can assign type 0 to the whole type.

\[
\begin{align*}
\Delta_3 \Gamma \vdash e : 0 \\
\Delta_3 \Gamma \vdash (e \in \ell ? e_1 : e_2) : 0
\end{align*}
\]

Suppose we are in case (ii) and the sub-derivation for the first branch \( e_1 \) ends with \((\text{subsum})\):

\[
\begin{align*}
\Delta_3 \Gamma \vdash e : t' & \quad t' \leq t \\
\Delta_3 \Gamma \vdash (e \in \ell ? e_1 : e_2) : s_1 & \quad s_1 \leq s \\
\Delta_3 \Gamma \vdash (e \in \ell ? e_1 : e_2) : s \\
\end{align*}
\]

\[(\text{case})\]

Such a derivation can be rearranged as:

\[
\begin{align*}
\Delta_3 \Gamma \vdash e : t' & \quad t' \leq t \\
\Delta_3 \Gamma \vdash (e \in \ell ? e_1 : e_2) : s_1 & \quad s_1 \leq s \\
\Delta_3 \Gamma \vdash (e \in \ell ? e_1 : e_2) : s \\
\end{align*}
\]

Moreover, \((\text{subsum})\) might also be used at the end of the sub-derivation for \( e \):

\[
\begin{align*}
\Delta_3 \Gamma \vdash e : t'' & \quad t'' \leq t' \\
\Delta_3 \Gamma \vdash e : t' & \quad t' \leq t \\
\Delta_3 \Gamma \vdash (e \in \ell ? e_1 : e_2) : s \\
\end{align*}
\]

\[(\text{case})\]

From \( t'' \leq t' \) and \( t' \leq t \), we deduce \( t'' \leq t \) by transitivity. Therefore this use of subtyping can be merged with the subtyping check of the type case rule. We then obtain the following algorithmic rule.

\[
\begin{align*}
\Delta_3 \Gamma \vdash e : t'' & \quad t'' \leq t \\
\Delta_3 \Gamma \vdash (e \in \ell ? e_1 : e_2) : s \\
\end{align*}
\]

We obtain a similar rule for case (iii), except that \( e_2 \) is type-checked instead of \( e_1 \), and \( t'' \) is tested against \( \neg e \).

Finally, consider case (iv). We have to type-check both branches and each typing derivation may end with \((\text{subsum})\):

\[
\begin{align*}
\Delta_3 \Gamma \vdash e : t' \quad & \quad \{ t' \not\leq t \quad \text{and} \quad \Delta_3 \Gamma \vdash e_1 : s_1 \\
& \quad \{ t' \not\leq t \quad \text{and} \quad \Delta_3 \Gamma \vdash e_2 : s_2 \\
& \quad \Delta_3 \Gamma \vdash (e \in \ell ? e_1 : e_2) : s \\
& \quad \Delta_3 \Gamma \vdash (e \in \ell ? e_1 : e_2) : s_1 \lor s_2 \\
& \quad \Delta_3 \Gamma \vdash (e \in \ell ? e_1 : e_2) : (s_1 \lor s_2) \leq s \\
\end{align*}
\]

\[(\text{case})\]

Subsumption is used there just to unify \( s_1 \) and \( s_2 \) into a common type \( s \), which is used to type the whole type case. Such a common type can also be obtained by taking the least upper-bound of \( s_1 \) and \( s_2 \), i.e., \( s_1 \lor s_2 \). Because \( s_1 \leq s \) and \( s_2 \leq s \), we have \( s_1 \lor s_2 \leq s \), and we can rewrite the derivation as follows:

\[
\begin{align*}
\Delta_3 \Gamma \vdash e : t' \quad & \quad \{ t' \not\leq t \quad \text{and} \quad \Delta_3 \Gamma \vdash e_1 : s_1 \\
& \quad \{ t' \not\leq t \quad \text{and} \quad \Delta_3 \Gamma \vdash e_2 : s_2 \\
& \quad \Delta_3 \Gamma \vdash (e \in \ell ? e_1 : e_2) : s_1 \lor s_2 \\
& \quad \Delta_3 \Gamma \vdash (e \in \ell ? e_1 : e_2) : (s_1 \lor s_2) \leq s \\
\end{align*}
\]

\[(\text{case})\]

Suppose now that the sub-derivation for \( e \) ends with \((\text{subsum})\):

\[
\begin{align*}
\Delta_3 \Gamma \vdash e : t'' & \quad t'' \leq t' \\
\Delta_3 \Gamma \vdash e : t' \\
\end{align*}
\]

\[(\text{case})\]

The relations \( t'' \leq t', t' \not\leq t \) do not necessarily imply \( t'' \not\leq t \), and \( t'' \leq t', t' \not\leq t \) do not necessarily imply \( t'' \not\leq t \). Therefore, by using the type \( t'' \) instead of \( t' \) for \( e \), we may type-check less branches. If so, then we would be in one of the cases (i) – (iii), and the result type \((\text{i.e., a type among 0, s_1 or s_2})\) for the whole type case would be smaller than \( s_1 \lor s_2 \). It would then be possible to type the type case with \( s_1 \lor s_2 \) by subsumption. Otherwise, we type-check as many branches with \( t'' \) as with \( t' \), and we can modify the rule into

\[
\begin{align*}
\Delta_3 \Gamma \vdash e : t'' \\
\Delta_3 \Gamma \vdash e : t' \\
\end{align*}
\]

\[(\text{case})\]

\[
\begin{align*}
\Delta_3 \Gamma \vdash (e \in \ell ? e_1 : e_2) : s_1 \lor s_2 \\
\Delta_3 \Gamma \vdash (e \in \ell ? e_1 : e_2) : (s_1 \lor s_2) \leq s \\
\end{align*}
\]
Finally, consider the case where the last rule in a derivation is \((\text{instinter})\) and all its sub-derivations end with \((\text{subsum})\):

\[
\frac{\Delta_1 \vdash e : s \quad s \leq t}{\Delta_1 \vdash e : t} \quad \text{(subsum)}
\]

\[
\frac{\forall j \in J, \sigma_j \not\in \Delta}{\Delta_1 \vdash e[\sigma_j] j \in J : \bigwedge_{j \in J} t \sigma_j} \quad \text{(\text{instinter})}
\]

Since \(s \leq t\), we have \(\bigwedge_{j \in J} \sigma_j \leq \bigwedge_{j \in J} t \sigma_j\). So such a derivation can be rewritten into

\[
\frac{\Delta_1 \vdash e : s \quad \forall j \in J, \sigma_j \not\in \Delta}{\Delta_1 \vdash e[\sigma_j] j \in J : \bigwedge_{j \in J} \sigma_j} \quad \text{(\text{instinter})}
\]

\[
\frac{\Delta_1 \vdash e[\sigma_j] j \in J : \bigwedge_{j \in J} t \sigma_j}{\Delta_1 \vdash e : t} \quad \text{(subsum)}
\]

In conclusion, by applying the aforementioned transformations repeatedly, we can rewrite an arbitrary typing derivation into a special form where subsumption are used at the end of sub-derivations of projections, abstractions or applications, in the conditions of type cases and at the very end of the whole derivation. Thus, this transformations yields a set of syntax-directed typing rules, given in Figure 6. Let \(\Delta_1 \vdash_{st} e : t\) denote the typing judgments derivable by the set of syntax-directed typing rules.

\[
\begin{align*}
\Delta_1 \vdash_{st} e : b_i & \quad \text{(ALG-CONST)} & \Delta_1 \vdash_{st} x : \Gamma(x) & \quad \text{(ALG-VAR)} \\
\Delta_1 \vdash_{st} e_1 : t_1 & \quad \Delta_1 \vdash_{st} e_2 : t_2 & \quad \text{(ALG-PAIR)} & \Delta_1 \vdash_{st} e : t \quad t \leq 1 \times 1 & \quad \text{(ALG-PROJ)} \\
\Delta_1 \vdash_{st} e_1 : t & \quad \Delta_1 \vdash_{st} e_2 : s & \quad t \leq 0 \rightarrow 1 & \quad s \leq \text{dom}(t) & \quad \text{(ALG-APPL)} \\
\Delta' = \Delta \cup \text{var}(\bigwedge_{i \in J} (t_i \sigma_j \rightarrow s_i \sigma_j)) & \quad \forall i \in J, \sigma_j \not\in \Delta_1 \vdash_{st} \chi : \sigma_j \vdash_{st} e \cdot \bigwedge_{i \in J} (t_i \sigma_j \rightarrow s_i \sigma_j) & \quad \text{(ALG-ABSTR)} \\
\Delta_1 \vdash_{st} \lambda^{t \sigma_j \rightarrow s \sigma_j} x : \chi & \quad \text{dom}(t) & \quad \text{(ALG-ABSTR)} \\
\Delta_1 \vdash_{st} e : 0 & \quad \text{(ALG-CASE-NONE)} \\
\Delta_1 \vdash_{st} e : t' & \quad t' \leq t & \quad t' \not\leq t & \quad \Delta_1 \vdash_{st} e_1 : s_1 & \quad \text{(ALG-CASE-FST)} \\
\Delta_1 \vdash_{st} e : t' & \quad t' \leq -t & \quad t' \not\leq t & \quad \Delta_1 \vdash_{st} e_2 : s_2 & \quad \text{(ALG-CASE-SND)} \\
\Delta_1 \vdash_{st} e : t' & \quad \text{dom}(t) & \quad \text{(ALG-CASE-BOTH)} \\
\Delta_1 \vdash_{st} e : t & \quad \forall j \in J, \sigma_j \not\in \Delta & |J| > 0 & \quad \text{(ALG-INST)} \\
\end{align*}
\]

\textbf{Figure 6. Syntax-directed typing rules}

**Theorem C.22 (Soundness).** Let \(e\) be an expression. If \(\Gamma \vdash_{st} e : t\), then \(\Gamma \vdash e : t\).

**Proof.** By induction on the typing derivation of \(\Delta_1 \vdash_{st} e : t\). We proceed by a case analysis on the last rule used in the derivation.

\(\text{(ALG-CONST)}:\) straightforward.
(ALG-VAR): straightforward.
(ALG-PAIR): consider the derivation
\[
\Delta_1 \Gamma \vdash e_1 : t_1 \\
\Delta_2 \Gamma \vdash e_2 : t_2 \\
\Delta_3 \Gamma \vdash e : t_1 \times t_2
\]
Applying the induction hypothesis twice, we get \(\Delta_3 \Gamma \vdash e_1 : t_1\). Then by applying the rule (pair), we have \(\Delta_3 \Gamma \vdash (e_1, e_2) : t_1 \times t_2\).
(ALG-PROJ): consider the derivation
\[
\Delta_3 \Gamma \vdash e : t \\
\Delta_3 \Gamma \vdash \pi_i(e) : \pi_i(t)
\]
By induction, we have \(\Delta_3 \Gamma \vdash e : t\). According to Lemma C.5, we have \(t \leq (\pi_1(t) \times \pi_2(t))\). Then by (subsum), we get \(\Delta_3 \Gamma \vdash e : (\pi_1(t) \times \pi_2(t))\). Finally, the rule (proj) gives us \(\Delta_3 \Gamma \vdash \pi_i(e) : \pi_i(t)\).
(ALG-APPL): consider the derivation
\[
\Delta_3 \Gamma \vdash e_1 : t \\
\Delta_3 \Gamma \vdash e_2 : s \\
\Delta_3 \Gamma \vdash e_1 \cdot e_2 : t \cdot s
\]
By induction, we have \(\Delta_3 \Gamma \vdash e_1 : t\) and \(\Delta_3 \Gamma \vdash e_2 : s\). According to Lemma C.13, we have \(t \cdot s = \min\{t' \mid t \leq t' \rightarrow s\}\).
Note that the conditions \(t \leq 0 \rightarrow 1\) and \(s \leq \text{dom}(t)\) ensure that such a type exists. It is clear \(t \leq t' \rightarrow (t \cdot s)\). Then by (subsum), we get \(\Delta_3 \Gamma \vdash e_1 : (t \cdot s)\). Finally, the rule (app) gives us \(\Delta_3 \Gamma \vdash e_1 \cdot e_2 : t \cdot s\).
(ALG-ABSTR): consider the derivation
\[
\forall i \in I, j \in J. \Delta_3 \Gamma, (x : t_i \sigma) \vdash e_\emptyset[\sigma] : s_{ij} \\
\Delta_3 \Gamma \vdash \lambda_{[\sigma]}_{i \in I, j \in J}(x : e_i) : \Lambda_{i \in I, j \in J}(t_i \sigma_ j \rightarrow s_i \sigma_j)
\]
with \(\Delta' = \Delta \cup \text{var}(\Lambda_{i \in I, j \in J}(t_i \sigma_ j \rightarrow s_i \sigma_j))\). By induction, for all \(i \in I\) and \(j \in J\), we have \(\Delta'_3 \Gamma, (x : t_i \sigma_j) \vdash e_\emptyset[\sigma] : s_{ij}\). Since \(s_{ij} \leq s_i \sigma_j\), by (subsum), we get \(\Delta'_3 \Gamma, (x : t_i \sigma_j) \vdash e_\emptyset[\sigma] : s_i \sigma_j\). Finally, the rule (abstr) gives us \(\Delta_3 \Gamma \vdash \lambda_{[\sigma]}_{i \in I, j \in J}(x : e_i \rightarrow s_i \sigma_j)\).
(ALG-CASE-NONE): consider the derivation
\[
\Delta_3 \Gamma \vdash e : \emptyset \\
\Delta_3 \Gamma \vdash (e \in \emptyset ? e_1 : e_2) : \emptyset
\]
By induction, we have \(\Delta_3 \Gamma \vdash e : \emptyset\). No branch is type-checked by the rule (case), so any type can be assigned to the type case expression, and in particular we have \(\Delta_3 \Gamma \vdash (e \in \emptyset ? e_1 : e_2) : \emptyset\).
(ALG-CASE-FST): consider the derivation
\[
\Delta_3 \Gamma \vdash e : t' \\
\Delta_3 \Gamma \vdash (e \in \emptyset ? e_1 : e_2) : s_1
\]
By induction, we have \(\Delta_3 \Gamma \vdash e : t'\) and \(\Delta_3 \Gamma \vdash e : s_1\). As \(t' \leq t\), then we only need to type-check the first branch. Therefore, by the rule (case), we have \(\Delta_3 \Gamma \vdash (e \in \emptyset ? e_1 : e_2) : s_1\).
(ALG-CASE-SND): similar case of (ALG-CASE-FST).
(ALG-CASE-BOTH): consider the derivation
\[
\Delta_3 \Gamma \vdash e : t' \\
\left\{\begin{array}{ll}
\Delta_3 \Gamma \vdash e : t \\
\Delta_3 \Gamma \vdash e : s_1 \\
\end{array}\right\}
\]
By induction, we have \(\Delta_3 \Gamma \vdash e : t'\), \(\Delta_3 \Gamma \vdash e : s_1\) and \(\Delta_3 \Gamma \vdash e : s_2\). It is clear that \(s_1 \leq s_1 \lor s_2\) and \(s_2 \leq s_1 \lor s_2\). Then by (subsum), we get \(\Delta_3 \Gamma \vdash e : s_1 \lor s_2\) and \(\Delta_3 \Gamma \vdash e : s_1 \lor s_2\). Moreover, as \(t' \not\leq t\) and \(t' \not\leq t\), we have to type-check both branches. Finally, by the rule (case), we get \(\Delta_3 \Gamma \vdash (e \in \emptyset ? e_1 : e_2) : s_1 \lor s_2\).
(ALG-INST): consider the derivation
\[
\Delta_3 \Gamma \vdash e : t \\
\forall j \in J, \sigma_j \notin \Delta \\
\Delta_3 \Gamma \vdash e[\sigma]_{j \in J} : \Lambda_{j \in J} t \sigma_j
\]
By induction, we have \(\Delta_3 \Gamma \vdash e : t\). As \(\forall j \in J, \sigma_j \notin \Delta\), by (instinter), we get \(\Delta_3 \Gamma \vdash e[\sigma]_{j \in J} : \Lambda_{j \in J} t \sigma_j\).
Theorem C.23 (Completeness). Let $\leq$ be a subtyping relation induced by a well-founded (convex) model with infinite support and $e$ an expression. If $\Delta; \Gamma \vdash e : t$, then there exists a type $s$ such that $\Delta; \Gamma \vdash s : e$ and $s \leq t$.

Proof. By induction on the typing derivation of $\Delta; \Gamma \vdash e : t$. We proceed by case analysis on the last rule used in the derivation.

(const): straightforward (take $s$ as $b_0$).

(var): straightforward (take $s$ as $\Gamma(x)$).

(pair): consider the derivation

\[
\Delta; \Gamma \vdash e_1 : t_1 \quad \Delta; \Gamma \vdash e_2 : t_2 \quad (\text{pair})
\]

Applying the induction hypothesis twice, we have $\Delta; \Gamma \vdash e_1 : s_1$ where $s_1 \leq t_1$. Then the rule (ALG-PAIR) gives us $\Delta; \Gamma \vdash (e_1, e_2) : s_1 \times s_2$. Since $s_1 \leq t_1$, we deduce $(s_1 \times s_2) \leq (t_1 \times t_2)$.

(proj): consider the derivation

\[
\Delta; \Gamma \vdash e : (t_1 \times t_2) \quad (\text{proj})
\]

By induction, there exists $s$ such that $\Delta; \Gamma \vdash e : s$ and $s \leq (t_1 \times t_2)$. Clearly we have $s \leq 1 \times 1$. Applying (ALG-PROJ), we have $\Delta; \Gamma \vdash \pi_i(e) : \pi_i(s)$. Moreover, as $s \leq (t_1 \times t_2)$, according to Lemma C.5, we have $\pi_i(s) \leq t_i$. Therefore, the result follows.

(app): consider the derivation

\[
\Delta; \Gamma \vdash e_1 : t_1 \rightarrow t_2 \quad \Delta; \Gamma \vdash e_2 : t_2 \quad (\text{app})
\]

Applying the induction hypothesis twice, we have $\Delta; \Gamma \vdash e_1 : t$ and $\Delta; \Gamma \vdash e_2 : s$ where $t \leq t_1 \rightarrow t_2$ and $s \leq t_1$. Clearly we have $t \leq 0 \rightarrow 1$ and $t \leq s \rightarrow t_2$ (by contravariance of arrows). From Lemma C.12, we get $s \leq \text{dom}(t)$. So, by applying the rule (ALG-APPL), we have $\Delta; \Gamma \vdash e_1 e_2 : t \cdot s$. Moreover, it is clear that $t_2$ is a solution for $t \leq s \rightarrow s'$. Consequently, it is a super type of $t \cdot s$, that is $t \cdot s \leq t_2$.

(abst): consider the derivation

\[
\forall i, j \in J. \Delta'; \Gamma, (x : t_i \sigma_j) \vdash e \theta[\sigma_j] : s_i \sigma_j \quad (\text{abst})
\]

where $\Delta' = \Delta \cup \text{var}(\lambda_{i \in I, j \in J}(t_i \sigma_j \rightarrow s_i \sigma_j))$. By induction, for all $i \in I$ and $j \in J$, there exists $s_{ij}$ such that $\Delta' \vdash t_i \sigma_j \vdash e \theta[\sigma_j] : s_{ij}$ and $s_{ij} \leq s_i \sigma_j$. Then the rule (ALG-ABSTR) gives us $\Delta; \Gamma \vdash e \text{abstr}_{i \in I, j \in J}(t_i \sigma_j \rightarrow s_i \sigma_j)$

(case): consider the derivation

\[
\Delta; \Gamma \vdash e : t' \left\{ \begin{array}{ll}
\text{if } t' \nleq t & \Rightarrow \Delta; \Gamma \vdash e_1 : s \\
\text{if } t' \nleq t & \Rightarrow \Delta; \Gamma \vdash e_2 : s
\end{array} \right. \quad (\text{case})
\]

By induction hypothesis on $\Delta; \Gamma \vdash e : t'$, there exists a type $t''$ such that $\Delta; \Gamma \vdash e : t''$ and $t'' \leq t'$. If $t'' \approx \emptyset$, by (ALG-CASE-NONE), we have $\Delta; \Gamma \vdash (e \in \emptyset ? e_1 : e_2) : \emptyset$. The result follows straightforwardly. In what follows, we assume that $t'' \neq \emptyset$.

Assume that $t'' \leq t$. Because $t'' \leq t'$, we have $t' \nleq t$ (otherwise, $t'' \approx \emptyset$). Therefore the first branch is type-checked, and by induction, there exists a type $s_1$ such that $\Delta; \Gamma \vdash e : e_1 : s_1$ and $s_1 \leq s$. Then the rule (ALG-CASE-FST) gives us $\Delta; \Gamma \vdash (e \in t ? e_1 : e_2) : s_1$. Otherwise, $t'' \nleq t$. In this case, we have $t' \nleq t$ (otherwise, $t'' \leq t$). Then the second branch is type-checked. By induction, there exists a type $s_2$ such that $\Delta; \Gamma \vdash e_2 : s_2$ and $s_2 \leq s$. If $t'' \nleq t$, then by the rule (ALG-CASE- SND), we have $\Delta; \Gamma \vdash (e \in t ? e_1 : e_2) : s_2$. The result follows. Otherwise, we also have $t'' \nleq t$. Then we also have $t' \nleq t$ (otherwise, $t'' \leq t$). So the first branch should be type-checked as well. By induction, we have $\Delta; \Gamma \vdash e_1 : s_1$ where $s_1 \leq s$. By applying (ALG-CASE-BOTH), we get $\Delta; \Gamma \vdash (e \in t ? e_1 : e_2) : s_1 \lor s_2$. Since $s_1 \leq s$ and $s_2 \leq s$, we deduce that $s_1 \lor s_2 \leq s$. The result follows as well.
Consider the derivation

\[
\delta_1 \Gamma \vdash e : t \\
\delta_1 \Gamma \vdash e[\sigma_j] \in J : \bigwedge_{j \in J} \tau_{\sigma_j} \quad (\text{instinter})
\]

By induction, there exists a type \( s \) such that \( \delta_1 \Gamma \vdash e : s \) and \( s \leq t \). Then the rule (ALG-INST) gives us that \( \delta_1 \Gamma \vdash e[\sigma_j] : \bigwedge_{j \in J} s_{\sigma_j} \). Since \( s \leq t \), we have \( \bigwedge_{j \in J} s_{\sigma_j} \leq \bigwedge_{j \in J} \tau_{\sigma_j} \). Therefore, the result follows.

\[\square\]

**Corollary C.24 (Minimum typing).** Let \( e \) be an expression. If \( \delta_1 \Gamma \vdash e : t \), then \( t = \min \{s \mid \delta_1 \Gamma \vdash e : s\} \).

**Proof.** Consequence of Theorems C.22 and C.23.

\[\square\]

To prove the termination of the type-checking algorithm, we define the size of an expression \( e \) as follows.

**Definition C.25.** Let \( e \) be an expression. We define the size of \( e \) as:

\[
\begin{align*}
\text{size}(e) &= 1 \\
\text{size}(x) &= 1 \\
\text{size}((e_1, e_2)) &= \text{size}(e_1) + \text{size}(e_2) + 1 \\
\text{size}(\pi_i(e)) &= \text{size}(e) + 1 \\
\text{size}(e_1 \cdot e_2) &= \text{size}(e_1) + \text{size}(e_2) + 1 \\
\text{size}(\lambda_{e_1} t, x. e) &= \text{size}(e) + 1 \\
\text{size}(e[\tau'r_1 \Rightarrow r_2]) &= \text{size}(e) + \text{size}(e_1) + \text{size}(e_2) + 1 \\
\text{size}(e[\sigma_j]) &= \text{size}(e) + 1
\end{align*}
\]

The relabeling does not enlarge the size of the expression.

**Lemma C.26.** Let \( e \) be an expression and \([\sigma_j]_{j \in J}\) a set of type substitutions. Then

\[
\text{size}(e[\sigma_j]) \leq \text{size}(e).
\]

**Proof.** By induction on the structure of \( e \).

\[\square\]

**Theorem C.27 (Termination).** Let \( e \) be an expression. Then the type-inference algorithm for \( e \) terminates.

**Proof.** By induction on the sizes of the expressions to be checked.

- **(ALG-CONST):** it terminates immediately.
- **(ALG-VAR):** it terminates immediately.
- **(ALG-PAIR):** size\((e_1) + \text{size}(e_2) < \text{size}((e_1, e_2))\).
- **(ALG-PROJ):** size\((e') < \text{size}(\pi_i(e'))\).
- **(ALG-APPL):** size\((e_1) + \text{size}(e_2) < \text{size}(e_1e_2)\).
- **(ALG-ABSTR):** by Lemma C.26, we have size\((e'@[\sigma_j]) \leq \text{size}(e')\). Then we get
  \[
  \text{size}(e'[\sigma_j]) < \text{size}(\lambda_{e_1}[r_1 \Rightarrow r_2] x. e').
  \]
- **(ALG-CASE):** size\((e') + \text{size}(e_1) + \text{size}(e_2) < \text{size}(e'[\tau e_1 : e_2])\).
- **(ALG-INST):** size\((e') < \text{size}(e'@[\sigma_j])\).

\[\square\]

### D. Evaluation

As described in Section 3, below we assume that polymorphic variables are pairwise distinct and distinct from any monomorphic variables in the expressions under consideration. In particular, when substituting an expression \( e \) for a variable \( x \) in an expression \( e' \), we assume the polymorphic type variables of \( e' \) to be distinct from the monomorphic and polymorphic type variables of \( e \). Furthermore, we assume that there are no useless type variables in the domain of any type-substitution and no redundant type-substitutions in any set of type-substitutions (Lemmas B.9 and B.10).
D.1 Equivalence between small-step semantics and big-step semantics

In this section, we prove the equivalence between the small-step semantics in Section A.4 and the big-step semantics for the polymorphic language in Section 5.2 (extended with let-polymorphism in Section 5.4). For clarity, we use $v$ for the values for the small-step semantics and $v_p$ for those of the big-step semantics.

Definition D.1 (Polymorphic language).

\[
\begin{align*}
e & \ ::= \ c \mid x \mid \pi \mid \lambda y.x.e \mid ee \mid e \in t \ ? \ e \mid e \\
v & \ ::= \ c \mid \lambda y.x.e \\
v_p & \ ::= \ c \mid \langle \lambda y.x.e, \delta, \sigma \rangle (\text{for the small-step semantics})
\end{align*}
\]

Definition D.2 (Membership). The membership relation $v_p \in_p t$ for polymorphic values is inductively defined as follows:

\[
\begin{align*}
c & \in_p t \quad \triangleq \quad \exists b \leq t \quad \langle \lambda y.x.e, \delta, \sigma \rangle \in_p t \quad \triangleq \quad s(\sigma_j \circ \sigma_I) \leq t
\end{align*}
\]

Definition D.3. We define a transformation function $\{ \} : v_p \rightarrow v$ as follows:

\[
\begin{align*}
\{ c \} & \triangleq \ c \\
\{ \langle \lambda y.x.e, \delta, \sigma \rangle \} & \triangleq \ \lambda y_j \circ \lambda y.x.e[\delta]
\end{align*}
\]

where $\{ \delta \}$ is defined as follows:

\[
\begin{align*}
\{ \emptyset \} & \triangleq \ \emptyset \\
\{ \delta, x \mapsto v_p \} & \triangleq \ \{ \delta \} \cup \{ t \mapsto v \} \\
\{ \delta, \exists \mapsto v_p \} & \triangleq \ \{ \delta \} \cup \{ t \mapsto v \}
\end{align*}
\]

In the definition of $\{ \langle \lambda y_j.x.e, \delta, \sigma \rangle \}$, to avoid unwanted captures, we assume that the polymorphic type variables of $\lambda y_j.x.e$ are distinct from the monomorphic and polymorphic type variables in $\delta$, ie, $tv(\delta)$ as precisely defined in Definition D.4. This is guaranteed by the assumption we impose on variables.

Definition D.4. The sets $tv(v_p)$ and $tv(\delta)$ of type variables respectively occurring in $v_p$ and $\delta$ are defined as follows:

\[
\begin{align*}
tv(v_p) & \triangleq \ tv\{v_p\} \\
tv(\delta) & \triangleq \ \bigcup_{x \in \text{dom}(\delta)} tv(\delta(x)) \cup \bigcup_{x \in \text{dom}(\delta)} tv(\delta(x))
\end{align*}
\]

Definition D.5. Let $e$ be an expression and $\delta$ an environment. We write $e \notin \delta$ to mean that the polymorphic type variables in $e$ are distinct from $tv(\delta)$. Moreover, we define $\sharp v_p$ and $\sharp \delta$ recursively as follows:

\[
\begin{align*}
\sharp \langle \lambda y_j.x.e, \delta, \sigma \rangle & \triangleq \ \sharp \delta \hspace{1em} \text{and} \hspace{1em} \exists \delta', \vdash \langle \lambda y_j.x.e, \delta, \sigma \rangle : \delta' \hspace{1em} \text{and} \hspace{1em} \langle \lambda y_j.x.e, \delta, \sigma \rangle \notin \delta' \\
\sharp c & \triangleq \ \text{true} \\
\sharp \delta & \triangleq \ \forall x \in \text{dom}(\delta), \sharp(\delta(x)) \hspace{1em} \text{and} \hspace{1em} \forall x \in \text{dom}(\delta), \sharp(\delta(x))
\end{align*}
\]

Definition D.6. Let $\sharp$ denote a singleton set made of the empty type-substitution $\{ \}$. We also use $\sharp$ to mean the neutral element of the composition of sets of type-substitutions. We define a transformation function $\text{clos}(\cdot)$ from $(v, \delta, \sigma_i)$ to $v_p$ as follows:

\[
\begin{align*}
\text{clos}(c, \delta, \sigma_i) & \triangleq \ c \\
\text{clos}(\langle \lambda y_j.x.e, \delta, \sigma_i \rangle) & \triangleq \ \langle \lambda y_j.x.e, \delta, \sigma_i \rangle
\end{align*}
\]

For simplicity, we write $\text{clos}(v)$ for $\text{clos}(v, \emptyset, \emptyset)$.

Definition D.7. We define the reflexive and transitive closure $\leadsto^*$ of a single-step reduction $\leadsto$ by the following two rules:

\[
\begin{align*}
e & \leadsto (\text{refl}) \quad e_1 \leadsto e_2 \quad e_2 \leadsto^* e_3 \quad (\text{trans})
\end{align*}
\]

Lemma D.8. Suppose $e_1 \leadsto^* e'_1$. Then:

1. $e_1 e_2 \leadsto^* e'_1 e_2$.
2. $e_2 e_1 \leadsto^* e_2 e'_1$.
3. $e_1 \in t ? e_2 : e_3 \leadsto^* e'_1 \in t ? e_2 : e_3$.
4. let $x = e_1$ in $e_2 \leadsto^* \text{let } x = e'_1 \text{ in } e_2$.

Proof. By induction on a derivation of $e_1 \leadsto^* e'_1$.

Lemma D.9. If $e_1 \leadsto^* e_2$ and $e_2 \leadsto^* e_3$, then $e_1 \leadsto^* e_3$.

Proof. By induction on a derivation of $e_1 \leadsto^* e_2$. 

\[
\begin{align*}
e & \leadsto (\text{refl}) \quad e_1 \leadsto e_2 \quad e_2 \leadsto^* e_3 \quad (\text{trans})
\end{align*}
\]
Lemma D.10. Let \( v_p \) be a polymorphic value. The following properties hold:

1. Suppose \( v_p \in \ell \). If there exists some type \( t' \) such that \( \vdash \{v_p\} : t' \), then \( \vdash \{v_p\} : t \).
2. \( \vdash \{v_p\} : t \) implies \( v_p \in \ell \).

Proof. (1) By induction on a derivation of \( \vdash \{v_p\} : t' \). There are three cases to consider.

- **(const):** \( v_p = c \) and \( t' = \text{type of } c \). By definition D.2, \( \text{type of } c \leq t \) and by the rule (\text{subsum}) we have \( \vdash c : t \).
- **(abstr):** \( v_p = \lambda_{x.e} \). By Definition D.2, \( \text{type of } e \leq t \) and by the rule (\text{subsum}), we have \( \vdash \{v_p\} : t \).

By Definition D.2, \( \text{type of } \sigma_1 \leq t \) and by the rule (\text{subsum}), we have \( \vdash \{v_p\} : t \).

**By induction hypothesis, we have \( \vdash \{v_p\} : t \).**

(2) By induction on a derivation of \( \vdash \{v_p\} : t \). 

\[ \square \]

Lemma D.11. Let \( v_p \) be a polymorphic value.

1. Suppose \( v_p \not\in \ell \). If there exists some type \( t' \) such that \( \vdash \{v_p\} : t' \), then \( \vdash \{v_p\} : \neg t \).
2. \( \vdash \{v_p\} : \neg t \) implies \( v_p \not\in \ell \).

Proof. (1) By induction on a derivation of \( \vdash \{v_p\} : t' \). The proof is by contradiction. Suppose \( v_p \not\in \ell \). Then, by Lemma D.10, we have \( \vdash \{v_p\} : t \), which is absurd.

**Lemma D.12.** Suppose \( (I_1) e \notin \ell' \) and \( \ell' ; (2) (e @ \sigma_1)[\ell'] : t \) and (3) \( \sigma ; e \in \ell \). Then \( (4) v_p \) and (5) \( (e @ \sigma_1)[\ell'] \sim^* \{v_p\} \).

Proof. By induction on a derivation of \( \vdash \{v_p\} : t \).

**PE-CONST:** \( \sigma_1 ; e \vdash_p c \subseteq \ell \) where \( e = c \).

**PE-VAR:** \( \sigma_1 ; e \vdash_p x \subseteq \ell(x) \) where \( e = x \).

**PE-CLOSURE:** \( \sigma_1 ; e \vdash_p \rho x \subseteq \ell x \cup \ell \) where \( e = \rho x \).

**PE-APPLY:** \( e \vdash e_1 e_2 \subseteq \ell \).

Furthermore, by Theorem B.15 with (2), \( (e_1 \ell') \cap (e_2 \ell') \subseteq \{v_{p0}\} \) is also well-typed.

**By \( \ell' \) and \( \ell' \cap p \), we have \( \ell' \times p \) and by assuming \( \alpha \)-conversion, we have \( \ell' \times p \).**

By Lemma D.10 with (c) and (e), \( P = \sigma \). Moreover, by Lemma B.5 with \( (\sigma_1) \cap \ell = \emptyset \), we have \( (e_1 \ell') \cap (e_2 \ell') \subseteq \{v_{p0}\} \).

By IH on (d), \( (e_1 \ell') \cap (e_2 \ell') \subseteq \{v_{p0}\} \).

Finally, by Lemma D.9, \( (e_1 e_2 \ell') \subseteq \{v_p\} \).

**PE-TYPE CASE T:** \( e \vdash e \subseteq \ell \).

**By definition, \( (e_1 \subseteq s ? e_2 : \ell) \subseteq \{v_p\} \).** and each sub-expression \( (e_1 \ell') \) is well-typed from the assumption (2).

**By IH on (a), \( (e_1 \ell') \subseteq \{v_p\} \) and \( \neg \ell' \).**

Then, by Lemma D.10 with (b), we have \( \vdash \{v_p\} : s \).

**By IH on (c), \( v_0 \subseteq \ell' \) and \( \neg v_0 \).**

Finally, by Lemmas D.8 and D.9, we have \( \{v_0 \} \subseteq \{v_{p} \} \).

\[ \square \]
(PE-TYPE CASE F): Similar to the case for the rule (PE-TYPE CASE T), except that we exploit Lemma D.11 instead of Lemma D.10.

(PE-INSTANCE): \( e = c_0\sigma_J\).

\[
\begin{align*}
(a) \quad & \sigma_J \circ \sigma_J; \sigma \vdash p \quad c_0 \Downarrow v \\
(b) \quad & \sigma; \sigma \vdash p \quad c_0 \Downarrow v
\end{align*}
\]

- By the assumption (1), we have \( c_0 \not\sim \sigma \).
- By IH on (a), \((e_0 \circ \sigma_J)\{\sigma\} \sim^* \{v_p\}\) and \(z_{v_p}\).
- Finally, by definition, we have \(((e_0 \circ \sigma_J)\{\sigma\} = (e_0(\sigma_J \circ \sigma_J))\{\sigma\} \sim^* \{v_p\}\), thus proving (5).

(PE-PVAR₁): \( e = x \).

\[
e(\langle e \rangle) = c
\]

- \( \mathcal{L}_x \) and \((x \circ \sigma_J)\{\sigma\} \sim^* c \).

(PE-PVAR₂): \( e = x \).

\[
e(\langle e \rangle) = \langle \lambda_{x \circ \sigma_J} y. e_0 \circ \sigma_J, \sigma \circ \sigma_J \rangle
\]

- \((x \circ \sigma_J)\{\sigma\} = (x[\sigma_1 : e_1]) \equiv (\sigma_1)\{\sigma_1 \in \lambda_{x \circ \sigma_J} y. e_0 \circ \sigma_J, \sigma \circ \sigma_J \} = ((\lambda_{x \circ \sigma_J} y. e_0 \circ \sigma_J, \sigma \circ \sigma_J))\{\sigma_1 \in \lambda_{x \circ \sigma_J} y. e_0 \circ \sigma_J, \sigma \circ \sigma_J \} =

Definition D.13. \( v_p \equiv z_{v_p}' \) if and only if \( \{v_p\} = \{v_p'\} \).

Lemma D.14. Suppose \( v_p \equiv z_{v_p} \). Then \( v_p \in_p t \) if and only if \( z_{v_p} \in_p t \).

Proof. Suppose \( v_p \equiv c \). Then \( v_p' = c \), which completes the proof. Now suppose \( v_p = (\lambda_{x \circ \sigma_J} y. e_0 \circ \sigma_J) \).

Then \( v_p' = (\lambda_{x \circ \sigma_J} y. e_0 \circ \sigma_J, \sigma_\|) \) where \( \sigma_\| \circ \sigma_\| = \sigma_\| \circ \sigma_\| \circ \sigma_\| \) and \( e_0 \circ \sigma_J = e' \). By the definition of \( \in_p \), we complete the proof.

Lemma D.15. Suppose \((1) (e_1 \circ \sigma_J)\{\delta_1\} \equiv \{v_p\}\) and \((2) \sigma_1 \in_p e_1 \Downarrow v_p\). Then, there exists \( v_p' \) such that \( \sigma_1; \sigma_2 \in_p e_2 \Downarrow v_p' \) and \( \sigma_1 \equiv_p v_p \).

Proof. By induction on a derivation of \( \sigma_1; \sigma_2 \in_p e_2 \Downarrow v_p \).

(PE-CONST): \( e_1 = v_p \).

- From (1), \( e_1 \) is either \( c \), \( x \), or \( z \). If \( e_2 = c \), then the proof is easy.
- If \( e_2 = x \), then \( v_p' = \delta_2(x) \). From (1) we have \((e_2 \circ \sigma_J)\{\delta_2\} \equiv (x \circ \sigma_J)\{\delta_2\} = \delta_2(x) = c \), thus completing the proof. The proof for \( e_2 = z \) is similar.

(PE-VAR): \( e_1 = x \) and \( v_p = \delta_1(x) \).

- From (1), \( e_2 \) is either \( y \), \( \lambda_{x \circ \sigma_J} z.e \), or \( z \).
- If \( e_2 = c \), then \( v_p' = \delta_2(x) \). Moreover, from (1) we have \((e_2 \circ \sigma_J)\{\delta_2\} \equiv (x \circ \sigma_J)\{\delta_2\} = \delta_2(x) = c \), thus completing the proof.
- If \( e_2 = y \), then \( v_p' = \delta_2(y) \). From (1) we have \( \delta_1(x) = \delta_2(y) \), thus completing the proof.
- If \( e_2 = \lambda_{x \circ \sigma_J} z.e \), then \( v_p' = \delta_2(x) \). From (1) we have \( \delta_1(x) = \delta_2(x) = \{v_p'\} \), thus completing the proof.
- Now assume \( e_2 = z \). If \( \delta_2(x) = c \), then from (1) we also have \( \delta_1(x) = c \), thus completing the proof. Otherwise, \( \delta_2(x) = \lambda_{x \circ \sigma_J} z.e \) and \( v_p' = \lambda_{x \circ \sigma_J} z.e \). From (1), we have \( \delta_1(x) = \{v_p\} \), thus completing the proof.

(PE-CLOSURE): \( e_1 = \lambda_{x \circ \sigma_J} z.e \) and \( v_p = \langle e_1, \delta_1, \sigma_1 \rangle \).

- From (1), \( e_2 \) is either \( x \), \( \lambda_{x \circ \sigma_J} z.e \), or \( z \).
- If \( e_2 = x \), then the proof is similar to the third case for (PE-VAR).
- If \( e_2 = \lambda_{x \circ \sigma_J} z.e \), then \( v_p' = \lambda_{x \circ \sigma_J} z.e, \delta_2, \sigma_1 \). From (1), we have \( \{v_p\} = \{v_p'\} \), thus completing the proof.
- If \( e_2 = z \), then \( v_p' = \lambda_{x \circ \sigma_J} z.e, \delta_2, \sigma_1 \). From (1), we have \( \{v_p\} = \{v_p'\} \), thus completing the proof. 

(PE-APPLY): \( e_1 = e_1 e_2 \).

- If \( e_2 = x \), then the proof is similar to the case for (PE-APPLY).
- If \( e_2 = \lambda_{x \circ \sigma_J} z.e \), then \( v_p' = \lambda_{x \circ \sigma_J} z.e, \delta_2, \sigma_1 \). From (1), we have \( \{v_p\} = \{v_p'\} \), thus completing the proof.
- If \( e_2 = z \), then \( v_p' = \lambda_{x \circ \sigma_J} z.e, \delta_2, \sigma_1 \). From (1), we have \( \{v_p\} = \{v_p'\} \), thus completing the proof.
• From (1), we have $e_2 = e_{21} e_{22}$ and $(e_{11} \circ \sigma_I)[\delta_1] = (e_{21} \circ \sigma_I)[\delta_2]$ where $l \in \{1, 2\}$.

• From (2), we have the following assumptions:
  (a) $\sigma_I; \delta_1 \vdash_p e_{21} \upharpoonright v_0 \equiv (\lambda_{s^i_{xy}} x.e, \delta')$ where $s = \wedge_{i \in I} s_i \rightarrow s'_i$ and $tv(\delta) \cap \text{dom}(\sigma_H \circ \sigma_K) = \emptyset$;
  (b) $\sigma_I; \delta_1 \vdash_p e_{12} \downarrow v_0$;
  (c) $\sigma_I \vdash_p \sigma_K$ and $P = \{j \in J \mid \exists i \in I : v_{90} \in_p s_i \sigma_j\}$; and
  (d) $\sigma_I; \delta \vdash_p \sigma_P; x. x \mapsto v_0 \vdash_p \sigma''_1$.

• By IH on (a), $\sigma_I; \delta_2 \vdash_p e_{21} \upharpoonright (\lambda_{s^i_{xy}} x.e, \delta', \sigma_H')$ where $\sigma_H \circ \sigma_K = \sigma_H' \circ \sigma_K'$ and $e[\delta] = e'[\delta']$.

Moreover, $tv(\delta') \cap \text{dom}(\sigma_H' \circ \sigma_K') = \emptyset$.

• By IH on (b), $\sigma_i; \delta_2 \vdash_p e_{22} \downarrow v_0$ and $v_90 \equiv v_0'$.

• By Lemma D.14 with $v_{90} \equiv v_0$ and $\sigma_I \vdash_p \sigma_K \circ \sigma_K'$, we have $P = \{j \in J \mid \exists i \in I : v_{90} \in_p s_i \sigma_j\}$.

• From $e[\delta] = e'[\delta']$ and $v_{90} \equiv v_0$ and $tv(\delta') \cap \sigma_P = \emptyset$ and $tv(\delta') \cap \sigma_P = \emptyset$, we have $(e \circ \sigma_P)[\delta, x \mapsto v_{90}] = (e' \circ \sigma_P)[\delta', x \mapsto v_0']$.

• Now, by IH on (d), $\sigma_P; x. x \mapsto v_0' \vdash_p e' \downarrow v'$ and $v_90 \equiv v_0'$.

• Finally, we have $\sigma_I; \delta_2 \vdash_p e_2 \downarrow v'$ by the rule (PE-APPLY), thus completing the proof.

(PE-TYPE CASE T): $e_1 = e_{11} \in s ? e_{12} : e_{13}$.

• From (1), we have $e_2 = e_{21} \in s ? e_{22} : e_{23}$ and $(e_{11} \circ \sigma_I)[\delta_1] = (e_{21} \circ \sigma_I)[\delta_2]$ where $l \in \{1, 2, 3\}$.

• From (2), we have the following assumptions:
  (a) $\sigma_I; \delta_1 \vdash_p e_{21} \downarrow v_0$ (b) $v_{90} \in_p s$ (c) $\sigma_I; \delta_1 \vdash_p e_{12} \downarrow v_0$.

• By IH on (a), $\sigma_I; \delta_2 \vdash_p e_{21} \downarrow v_{90}$ and $v_{90} \equiv v_0'$.

• By Lemma D.14 with (b), we have $v_{90} \in_p s$.

• By IH on (c), $\sigma_I; \delta_2 \vdash_p e_{22} \downarrow v'$ and $v_90 \equiv v_0'$.

• Finally, we have $\sigma_I; \delta_2 \vdash_p e_2 \downarrow v'$ by the rule (PE-TYPE CASE T), thus completing the proof.

(PE-TYPE CASE F): $e_1 = e_{11} \in s ? e_{12} : e_{13}$.

• Similar to the case for the rule (PE-TYPE CASE F).

(PE-INSTANCE): $e_1 = e \sigma_J$.

• From (1), we have $e_2 = e' \sigma_J$ and $(e \circ (\sigma_I \circ \sigma_J))[\delta_1] = (e' \circ (\sigma_I \circ \sigma_J))[\delta_2]$.

• From (2), we have the following assumption:
  (a) $\sigma_I \circ \sigma_J; \delta_2 \vdash_p e' \downarrow v'$.

• By IH on (a), $\sigma_I \circ \sigma_J; \delta_2 \vdash_p e_2 \downarrow v'$ and $v_90 \equiv v_0'$.

• Finally, we have $\sigma_I; \delta_2 \vdash_p e_2 \downarrow v'$ by the rule (PE-INSTANCE), thus completing the proof.

(PE-PVAR): $e_1 = x$ and $v_90 = c$.

• Similar to the case for (PE-CONST).

(PE-PVAR): $e_1 = x$, $\delta_1(x) = (\lambda_{x^r_{xy}} x.e, \delta')$ and $v_90 = (\lambda_{x^r_{xy}} x.e, \delta', \sigma_I \circ \sigma_H)$.

• Similar to the case for (PE-VAR).

(PE-LET): $e_1 = \text{let } x = e_{11} \text{ in } e_{12}$.

• From (1), we have $e_2 = \text{let } x = e_{21} \text{ in } e_{22}$. Moreover, $(e_{11} \circ \sigma_I)[\delta_1] = (e_{21} \circ \sigma_I)[\delta_2]$ and $(e_{12} \circ \sigma_I)[\delta_1] = (e_{22} \circ \sigma_I)[\delta_2]$.

• From (2), we have the following assumptions:
  (a) $\sigma_I; \delta_1 \vdash_p e_{11} \downarrow v_0$ (b) $\sigma_I; \delta_1; x. x \mapsto v_0 \vdash_p e_{12} \downarrow v_0$.

• By IH on (a), $\sigma_I; \delta_2 \vdash_p e_{21} \downarrow v_{90}$ and $v_{90} \equiv v_0'$.

• By IH on (b), $\sigma_I; \delta_2; x. x \mapsto v_0 \vdash_p e_{22} \downarrow v_0$ and $v_90 \equiv v_0'$.

• We conclude $\sigma_I; \delta_2 \vdash_p e_2 \downarrow v_0$ by the rule (PE-LET).

Lemma D.16. Let $v$ be a well-typed closed value such that $\text{dom}(\sigma_I) \cap tv(v) = \emptyset$. Suppose (1) $\sigma_I; \delta \vdash_p (e \circ \sigma_L)[\{v\}] \downarrow v_0$. Then $\sigma_I \circ \sigma_H; \delta, x \mapsto \text{close}(v) \vdash_p e \downarrow v'$ and $v_90 \equiv v_0'$.

Proof. By induction on a derivation of $\sigma_I; \delta \vdash_p (e \circ \sigma_L)[\{v\}] \downarrow v_0$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.

(PE-CONSTR): $e = c$ and $v_90 = v' = c$.
Proof.
Lemma D.17. Let $v$ be a well-typed closed value such that $\text{dom}(\sigma_t) \cap \text{tv}(v) = \emptyset$. Suppose (1) $\sigma_t; e \vdash_p e_2 \downarrow v_p$. Then $\sigma_t; e; \langle x \rangle \vdash v_p$ and $v_p \equiv v'_p$.

Proof. Similar to the proof for Lemma D.16.

Lemma D.18. If (1) $\vdash e; t$, (2) $e \rightsquigarrow e'$, and (3) $i; \vdash e_2 \downarrow v_p$, then $i; \vdash e \downarrow v'_p$ and $v_p \equiv v'_p$.

Proof. By induction on a derivation of $e \rightsquigarrow e'$.

\[
\frac{e = e_1 e_2}{e_1 \rightsquigarrow e_1' e_2} \quad \text{where } e' = e_1' e_2:
\]

- From (1), we have the following assumptions:
  \[ (a) \frac{i; \not\vdash e_2 \downarrow v_p}{s \equiv \lambda_{x_{\sigma_0}} e_0; e_2, \sigma_H} \text{ where } s = \lambda_{i_{\sigma_0}} \sigma_i \to s_1' \text{ and } \text{tv}(\sigma') \cap \text{dom}(\sigma_H \circ \sigma_K) = \emptyset; \]
  \[ (b) \frac{s \equiv \lambda_{x_{\sigma_0}} e_0; e_2, \sigma_H}{\not\vdash e_2 \downarrow v_p}; \]
  \[ (c) \frac{\sigma_j = \sigma_H \circ \sigma_K \text{ and } P = \{ j \in J \mid \exists i \in I : v_p \not\equiv s_{\sigma_{i}} \}}{e_1 \rightsquigarrow e_1'} \text{ with (a) and (b)}; \]
  \[ (d) \frac{\sigma_j; e_1 \to v_p \not\equiv e_0 \not\equiv v_p}{e_1 \rightsquigarrow e_1'} \text{ with (a) and (b)}; \]
- By IH on $e_1 \rightsquigarrow e_1'$ with (a) and (b) and (c):
  \[ \lambda_{x_{\sigma_0}} e_0; e_2, \sigma_H' \text{ where } \sigma_H = \sigma_H \circ \sigma_K, \text{ and } \text{tv}(\sigma') \cap \text{dom}(\sigma_H \circ \sigma_K) = \emptyset; \]
  \[ \text{and } e_{0}(\sigma_H') = e_0(\sigma_H'). \]
- By Lemma D.15 with (d), we have $\sigma_j; e_1 \to v_p \not\equiv e_0 \not\equiv v_p$ and $v_p \equiv v'_p$.
- Finally, by the rule (PE-APPLY), we conclude $\sigma_j; e_1 \rightsquigarrow e_1 e_2 \downarrow v'_p$.

\[
\frac{e_1 = \lambda_{x_{\sigma_0}} e_{2} \rightsquigarrow e_1 e_2}{e_1 e_2 \rightsquigarrow e_2} \quad \text{where } e' = e_1 e_2:
\]

- Similar to the case for (I).

\[
\frac{e_1 = \lambda_{x_{\sigma_0}} e_{2} \rightsquigarrow e_1 e_2}{e_1 e_2 \rightsquigarrow e_1 e_2} \quad \text{where } e' = e_1 e_2:
\]
• We have $\emptyset \vdash_p e_1 \downarrow \text{clo}(e_1)$ and $\emptyset \vdash_p v \downarrow \text{clo}(v)$.
• By Lemma D.10, $P = \{ j \in J \mid \exists i \in I, \vdash e : s_i \sigma_j \} = \{ j \in J \mid \exists i \in I, \text{clo}(v) \in_p s_i \sigma_j \}$.
• By Lemma D.16 with (3), we have $\sigma_j\vdash e_1 \downarrow \text{clo}(v) \vdash_p e_0 \downarrow v_0'$ and $v_0 \equiv e v_0'$.
• Finally, by the rule (PE-APPLY), we conclude $\emptyset \vdash_p e_1 e_2 \downarrow v_p'$.

$(e = e_1 \in s ? e_2 : e_3)$:

(I) \[ e_1 \sim e_1 \] where $e' = e_1' \in s ? e_2 : e_3$:
• From (3), there are two cases to consider. Here we consider only the case for (PE-TYPE CASE T).
  • The case for (PE-TYPE CASE F) is similar.
  • We have the following assumptions:
    
    (a) $\emptyset \vdash_p e_1 \downarrow v_0$ 
    (b) $v_0 \in_p s$ 
    (c) $\emptyset \vdash_p e_2 \downarrow v_p$
  • By IH on $e_1 \sim e_1'$ with (a), $\emptyset \vdash_p e_1 \downarrow v_0$ and $v_0 \equiv e v_0'$.
  • By Lemma D.14 with (b), we have $v_0' \in_p s$.
  • Finally, by the rule (PE-TYPE CASE T), we conclude $\emptyset \vdash_p e \downarrow v_p$.

(II) \[ e_2 \sim e_2' \] where $e' = e_1 \in s ? e_2' : e_3$:
• From (3), there are two cases to consider.
  • First, consider the case for the rule (PE-TYPE CASE T). Then, we have the following assumptions:
    
    (a) $\emptyset \vdash_p e_1 \downarrow v_0$ 
    (b) $v_0 \in_p s$ 
    (c) $\emptyset \vdash_p e_2 \downarrow v_p$ 
  • By IH on $e_2 \sim e_2'$ with (c), we have $\emptyset \vdash_p e_2 \downarrow v_p'$ and $v_p \equiv e v_p'$. Then, by the rule (PE-TYPE CASE T), we complete the proof.
  • Now consider the case for the rule (PE-TYPE CASE F). Then, we have the following assumptions:
    
    (a) $\emptyset \vdash_p e_1 \downarrow v_0$ 
    (b) $v_0 \not\in_p s$ 
    (c) $\emptyset \vdash_p e_2 \downarrow v_p$ 
  • Then, we just complete the proof by the rule (PE-TYPE CASE F).

(III) \[ e_3 \sim e_3 \] where $e' = e_1 \in s ? e_2 : e_3$:
• Similar to the case for (II).

(IV) \[ e_1 = v, \vdash v : s, \text{and } e' = e_2 \] :
• We have $\emptyset \vdash_p e_1 \downarrow \text{clo}(v)$.
• By Lemma D.10 with $\vdash v : s$, we have $\text{clo}(v) \in_p s$.
• Then, by the rule (PE-TYPE CASE T) with (3), we conclude $\emptyset \vdash_p e \downarrow v_p$.

(V) \[ e_1 = v, \not\vdash v : s, \text{and } e' = e_3 \] :
• We have $\emptyset \vdash_p e_1 \downarrow \text{clo}(v)$.
• By Lemma D.11 with $\not\vdash v : s$, we have $\text{clo}(v) \not\in_p s$.
• Then, by the rule (PE-TYPE CASE F) with (3), we conclude $\emptyset \vdash_p e \downarrow v_p$.

$(e = e_0 \sigma_j)$: \[ e' = e_0 \sigma_j \] :
• By Lemma D.16 with (3), we have $\sigma_j ; \emptyset \vdash_p e_1 \downarrow v_0'$ and $v_0 \equiv e v_0'$.
• By the rule (PE-INSTANCE), we conclude $\emptyset \vdash_p e_{0 \sigma_j} \downarrow v_0'$.

$(e = \text{let } x = e_1 \in e_2)$:

(I) \[ e \sim \text{let } x = e_1' \text{ in } e_2 \] where $e' = \text{let } x = e_1'$ in $e_2$:
• From (3), we have the following assumptions:
  
  (a) $\emptyset \vdash_p e_1' \downarrow v_0$ 
  (b) $\text{let } x : v_0 \vdash_p e_2 \downarrow v_p$
  • By IH on $e_1 \sim e_1'$ with (a), we have $\emptyset \vdash_p e_1 \downarrow v_0$ and $v_0 \equiv e v_0'$.
  • By Lemma D.15 with (b), we have $\emptyset \vdash_p e \downarrow v_p'$ and $v_p \equiv e v_p'$.
  • Then, by the rule (PE-LET), we conclude $\emptyset \vdash_p e \downarrow v_p'$.

(II) \[ e_2 \sim e_2 \] where $e' = \text{let } x = e_1 \text{ in } e_2$:
• From (3), we have the following assumptions:
  
  (a) $\emptyset \vdash_p e_1 \downarrow v_0$ 
  (b) $\text{let } x : v_0 \vdash_p e_2 \downarrow v_p$
  • By IH on $e_2 \sim e_2'$ with (b), we have $\emptyset \vdash_p e \downarrow v_p'$ and $v_p \equiv e v_p'$.
  • Then, by the rule (PE-LET), we conclude $\emptyset \vdash_p e \downarrow v_p'$.

(III) \[ e_1 = v \text{ and } e' = e_2 \{v/v\} \] :
• We have $\emptyset \vdash_p e_1 \downarrow \text{clo}(v)$.
• By Lemma D.17 with (3), we have $\emptyset \vdash_p e_2 \downarrow v_p'$ and $v_p \equiv e v_p'$.
• Finally, by the rule (PE-LET), we conclude $\emptyset \vdash_p e \downarrow v_p'$.

\[\blacksquare\]

Lemma D.19. If $\vdash e : t$ and $e \sim^* v$, then $\emptyset \vdash_p e \downarrow v_p$ and $v = \{v_p\}$. 

Lemma D.25. The main reason is that $\Sigma$.

Proof. By induction on $e \leadsto^* v$. Suppose $e = v$. Then $v_\emptyset = \text{clos}(v)$ and $\{v_\emptyset\} = v$.

Now suppose $e \leadsto e_1 \leadsto^* v$. By induction hypothesis on $e_1 \leadsto^* v$, we have $\emptyset; \triangleright p \vdash e_1 \dashv v_\emptyset'$ and $v_\emptyset = \{v_\emptyset'\}$. Then, by Lemma D.18, we have $\emptyset; \triangleright p \vdash e \dashv v$ and $v_\emptyset \equiv_c v_\emptyset'$, and therefore $v = \{v_\emptyset\}$. \hfill \Box

Theorem D.20. Let $e$ be a well-typed closed explicitly-typed expression ($\vdash e : t$). Then $\emptyset; \triangleright p \vdash e \dashv v \iff e \leadsto^* v$.

Proof. Follows from Lemmas D.12 and D.19. \hfill \Box

D.2 Compilation of the polymorphic language into the intermediate language

In this section, we prove the adequacy of the compilation of the polymorphic language into the intermediate language defined in Section 5.3 (extended with let-polymorphism in Section 5.4). For clarity, in this section, we write respectively $e_o$, $v_o$, and $\delta_o$ for the expressions, values, and environments for the intermediate language.

Definition D.21 (Intermediate language).

- $e_o := c | x | x_\Sigma | \lambda y.x.e_o | e_o e_o | e_o \in t \Rightarrow e_o | \text{let } \underline{x} = e_o \text{ in } e_o$
- $v_o := c | \langle \lambda y.x.e_o, \delta_o \rangle$
- $\Sigma := \sigma_f \cup \text{comp}(\Sigma, \Sigma') \cup \text{sel}(x, t, \Sigma) \cup \{e_o, \Sigma\}$

In this definition, we added a new symbolic substitution of the form $\langle e_o, \Sigma \rangle$, which is absent from Section 5.3. Furthermore, we use the following new evaluation rule for polymorphic let-variables:

\[
\begin{align*}
\text{(OE-PVARR)} & \\
\delta_o(\underline{x}) = \langle \lambda y.x.e_o, \delta_o' \rangle & \\
\emptyset; \triangleright o \vdash x_\Sigma \Downarrow \langle \lambda \text{comp}(\Sigma, \Sigma') y.e_o, \delta_o' \rangle
\end{align*}
\]

The difference from the previous rule is that in this rule we use in the decoration $\langle e_o, \Sigma \rangle$ instead of $\Sigma$. The main reason is that $\Sigma$ in $x_\Sigma$ may contain a symbolic substitution of the form $\text{sel}(y, t, \Sigma_0)$ such that $y \notin \text{dom}(\delta_o')$. Such a symbolic substitution as $\text{sel}(y, t, \Sigma_0)$ may be generated if the let-variable $\underline{x}$ is used inside $\lambda$-abstraction in the body of the let-expression. However, in practice, only type-substitutions for the type variables introduced before or in the let-binding may be applied to the polymorphic let-variable, which are already recorded in the closure for the let-variable. Therefore, for the implementation, it is safe to ignore such substitutions as $\langle e_o, \Sigma \rangle$ without evaluating them. Still, introducing this new extra symbolic substitution makes the intermediate language and its evaluation semantics clearer.

Definition D.22 (Compilation).

\[
\begin{align*}
[e]_{\Sigma} & = c \\
[x]_{\Sigma} & = x \\
[x_\Sigma] & = x_\Sigma \\
[\lambda y.x.e]_{\Sigma} & = \lambda \text{comp}(\Sigma, \sigma_f) x. [e]_{\text{sel}(x, t, \text{comp}(\Sigma, \sigma_f))} \\
[e_1 e_2]_{\Sigma} & = [e_1]_{\Sigma} [e_2]_{\Sigma} \\
[e \sigma_f]_{\Sigma} & = [e]_{\text{comp}(\Sigma, \sigma_f)} \\
[e_1 \in t ? e_2 : e_3]_{\Sigma} & = [e_1]_{\Sigma} \in t ? [e_2]_{\Sigma} : [e_3]_{\Sigma} \\
[\text{let } \underline{x} = e_1 \text{ in } e_2]_{\Sigma} & = \text{let } \underline{x} = [e_1]_{\Sigma} \text{ in } [e_2]_{\Sigma}
\end{align*}
\]

Definition D.23 (Membership).

\[
\begin{align*}
& c \in e_o \quad t \quad \text{def} \quad b_c \leq t \\
& (\lambda y.x.e_o, \delta_o) \in e_o \quad \text{def} \quad s(\text{eval}(e_o, \Sigma)) \leq t
\end{align*}
\]

where the evaluation of the symbolic set of type-substitutions is inductively defined as

\[
\begin{align*}
\text{eval}(e_o, \sigma_f) & = \sigma_f \\
\text{eval}(e_o, \text{comp}(\Sigma, \Sigma')) & = \text{eval}(e_o, \Sigma) \circ \text{eval}(e_o, \Sigma') \\
\text{eval}(e_o, \text{sel}(x, \bigwedge_{i \in I} t_i \rightarrow s_i, \Sigma)) & = [\sigma_f \in \text{eval}(e_o, \Sigma) \mid \exists i \in I : e_o(x) \in t_i, \sigma_f] \\
\text{eval}(\delta_o, \langle e_o', \Sigma \rangle) & = \text{eval}(\delta_o, \Sigma)
\end{align*}
\]

Definition D.24. Let $\delta$ and $v_\emptyset$ be an environment and a value for the polymorphic language and $\delta_o$ and $v_o$ for the intermediate language. We define an equivalence relation $\equiv$ between them as follows:

\[
\begin{align*}
\delta \equiv e_o & \quad \iff \text{dom}(\delta) = \text{dom}(e_o) \text{ and } \forall x \in \text{dom}(\delta), \delta(x) \equiv e_o(x) \\
& \quad \text{and } \forall y \in \text{dom}(\delta), \delta'(x) \equiv e_o(x) \\
\langle \lambda y.x.e, \delta, \sigma_f \rangle \equiv (\lambda y.x.e_o, \delta_o) & \quad \iff \sigma_f \circ \sigma_K = \text{eval}(e_o, \Sigma) \text{ and } e_o = [e]_{\text{sel}(x, t, \Sigma)} \text{ and } \delta \equiv e_o
\end{align*}
\]

Moreover, for any constant $c$, by definition $c \equiv e_o c$.

Note that if $v_\emptyset \equiv e_o v_\emptyset$, then $\{v_\emptyset\} = \{v_\emptyset\}$, but not vice versa.

Lemma D.25. Let $v_\emptyset \equiv e_o v_\emptyset$. Then:

1. $v_\emptyset \in p \text{ if and only if } v_\emptyset \in c \text{ t}$.
2. $v_\emptyset \not\in p \text{ if and only if } v_\emptyset \not\in c \text{ t}$.
Proof. If \( v_p = v_0 = c \), the proof is trivial. Suppose \( v_p = (\lambda x. x e, \delta', \sigma) = (\lambda x. x e, \delta_0) \) By definition, \( v_0 \in t \) if and only if \( s(\sigma) = \delta_0 \) if and only if \( s(e) \in \delta_0 \) if and only if \( s(e) \in \delta_0 \) if and only if \( s(e) \in \delta_0 \). By IH on \( (b) \), we have \( \sigma \delta \circ \sigma_K = \delta_0 \sigma_K \), thus completing the proof.

**Lemma D.26.** Suppose \( \delta \equiv_0 \delta_0 \) and \( \sigma, \sigma_I = \delta_0, \Sigma) \vdash (e \circ \sigma) : t \). Then, \( \sigma_I ; \delta \vdash e \downarrow v_p \) if and only if \( e \circ \delta \vdash \circ \downarrow v_0 \) where \( v_p \equiv_0 v_0 \).

Proof. (Only-if part) By induction on a derivation of \( \sigma_I ; \delta \vdash e \downarrow v_p \).

**PE-CONST:** \( \sigma \equiv [e]_\Sigma = v_0 = v_0 = c \).

**PE-VAR:** \( \sigma \equiv [e]_\Sigma = x \) and \( v_0 = \delta(x) \) and \( v_0 = \delta_0(x) \). From \( \delta \equiv_0 \delta_0 \), we have \( v_0 \equiv_0 v_0 \).

**PE-CLOSE:** \( \sigma \equiv \lambda x. x e' \) and \( \Sigma \equiv [e']_\Sigma = \lambda x. x e \). From \( \delta \equiv_0 \delta_0 \), we have \( e' \equiv_0 e \) and \( e \equiv_0 e_0 \).

**PE-APPLY:** \( \sigma = \lambda x. x e \) and \( e \equiv_0 e_0 \).

**PE-INST:** \( \sigma = \lambda x. x e \) and \( \Sigma \equiv [e]_\Sigma = \lambda x. x e \). From \( \delta \equiv_0 \delta_0 \), we have \( e \equiv_0 e_0 \).

**PE-BIND:** \( \sigma = \lambda x. x e \) and \( \Sigma \equiv [e]_\Sigma = \lambda x. x e \). From \( \delta \equiv_0 \delta_0 \), we have \( e \equiv_0 e_0 \).

**PE-LET:** \( \sigma = \lambda x. x e \) and \( \Sigma \equiv [e]_\Sigma = \lambda x. x e \). From \( \delta \equiv_0 \delta_0 \), we have \( e \equiv_0 e_0 \).

(If part) By induction on a derivation of \( \delta \circ \delta \vdash \circ \downarrow v_0 \). The proof is similar to the proof for the (only-if part).
**Theorem D.27.** Let $e$ be a well-typed closed explicitly-typed expression ($\vdash e : t$). Then

$$\text{if } [v] \vdash e \Downarrow v \quad \text{then} \quad \vdash o \downarrow e \Downarrow v'$$

with $\{v\} = \{v'\}$.

**Proof.** Follows from Lemma D.26. $\square$

**Corollary D.28 (Adequacy of the compilation).** Let $e$ be a well-typed closed explicitly-typed expression ($\vdash e : t$). Then

$$\vdash o \downarrow e \Downarrow v \quad \iff \quad e \rightsquigarrow t(v)$$

**Proof.** Follows from Theorems D.20 and D.27. $\square$

**E. Syntactic sugar for type-case**

In order to make type-case closer to pattern matching, it is handy to define an extension of the type-case expression that features binding, which we write

$$(x : s = e) \in t ? e_1 : e_2,$$

where $s$ is the type of $e$, $t_1$ the type of $e_1$, and $t_2$ the type of $e_2$. We add another twist to this construct and define a particular (purely) syntactic sugar: $x \in t ? e_1 : e_2$ (notice the boldface “belongs to” symbol) which stands for $(x = x) \in t ? e_1 : e_2$. The reader may wonder what is the interest of binding a variable to itself. Actually, the two occurrences of $x$ in $(x = x) \in t$ denote two distinct variables: the one on the right is recorded in the environment with some type $s$; this variable does not occur either in $e_1$ or $e_2$ because it is hidden by the $x$ on the left; this binds the occurrences of $x$ in $e_1$ and $e_2$ but with different types, $s \land t$ in $e_1$ and $s \land \neg t$ in $e_2$. This allows the system to use different type assumptions for $x$ in each branch, as stated by the (algorithmic) typing rule directly derived from the encoding:

$$\frac{\text{(case-var)}}{\Delta, \Gamma \vdash x \in t ? e_1 : e_2} \quad \Delta \vdash x : t_1 \vdash e_1 : s_1 \quad \Delta, \Gamma \vdash x : t_2 \vdash e_2 : s_2 \quad t_1 \neq t_2 \neq 0$$

Note that $x$ is defined in $\Gamma$ but its type $\Gamma(x)$ is overridden in the premises to type the branches. We already silently used this construction in the body of the definition of `map` in Section 2 which should have been written with the boldface “belongs to” symbol:

$$\lambda^{(\alpha \to \beta)} \ell. \ell \in \text{nil } ? \text{nil} : (\ell(f(\pi_1 \ell), mf(\pi_2 \ell))).$$

since the presence of the $\pi_i$’s in the second branch needs the hypothesis $\ell : [\alpha] \setminus \text{nil}$ (the expression $\pi_i \ell$ is well-typed only if $\ell$ is a product, that is, in this case it is a non-empty list). If we do not use the boldface belong symbol, then each branch (in particular, the second one) is typed under the hypothesis $\ell : [\alpha]$ and, thus, type-check fails. In practice, any real programming language will implement just $\ell$ and not $\ell \in \text{ whenever the tested expression is a variable.}$