Generic Large Cardinals and Absoluteness

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Generic Large Cardinals 
and Absoluteness

Giorgio Audrito
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INTRODUCTION

In the wake of early Gödel’s Incompleteness Theorems, a large part of set theory has revolved around the study of the independence phenomena. The most significant boost in this field was due to the introduction of the forcing technique. Many independence results were found in different areas of mathematics since then, including among many the continuum hypothesis [32], Borel’s conjecture [34], Kaplansky’s conjecture [14], Whitehead problem [35], strong Fubini theorem [18], and the existence of outer automorphisms of the Calkin algebra [15].

A dual approach in set theory is to search for new axioms of mathematics in order to control the independence phenomena. These new axioms need to be justified, so that their mathematical properties interact with philosophical guidelines. An axiom candidate can be judged both from its premises, that is, whether the statement can be argued to be a “natural” extension of the accepted axioms of ZFC, and from its consequences, that is, whether the statement is able to entail a large number of desired properties in mathematics.

In this thesis we follow this approach, considering axioms in two of the most important families: forcing axioms (Chapters 3, 4) and large cardinals (Chapter 5), considering their effectiveness both on the premises and consequences side. While these chapters follow mostly independent paths, all of them fit in the same quest for natural and powerful axioms for mathematics, and are strongly dependent on the iterated forcing technique covered in Chapter 2. Basic notation and background topics are dealt with in Chapter 1.

In all the chapters of this thesis we shall focus on the boolean valued models approach to forcing. A partial order and its boolean completion can produce exactly the same consistency results, however:

- in a specific consistency proof the forcing notion we have in mind in order to obtain the desired result is given by a partial order and passing to its boolean completion may obscure our intuition on the nature of the problem and the combinatorial properties we wish our partial order to have;

- when the problem aims to find general properties of forcings which are shared by a wide class of partial orders, we believe that focusing on complete boolean algebras gives a more efficient way to handle the problem. In fact, the rich algebraic theory of complete boolean algebras can greatly simplify our calculations.
Since the focus of the thesis is on the general correlations between forcing, large cardinals, and forcing axioms, rather than on specific applications of the forcing method to prove the consistency of a given mathematical statement, we are naturally led to analyze forcing through an approach based on boolean valued models.

**Iterated forcing** is a procedure that describes transfinite applications of the forcing method, with a special attention to limit stages, and is one of the main tools for proving the consistency of forcing axioms. In Chapter 2 we give a detailed account of iterated forcing through boolean algebras, inspired by the algebraic approach of Donder and Fuchs [19], thus providing a solid background on which the rest of the thesis is built. All the material in this chapter is joint work with Matteo Viale and Silvia Steila.

Even though all the results in this chapter come from a well established part of the current development of set theory, the proofs are novel and (according to us) simpler and more elegant, due to a systematic presentation of the whole theory of iterated forcing in terms of algebraic constructions. In Chapter 3 we introduce the notion of weakly iterable forcing class and prove the preservation theorems for semiproper and stationary set preserving iterations (for the latter result, we assume the existence of class many supercompact cardinals). We believe there will be no problem in rearranging these techniques in order to cover also the cases of proper or ccc iterations.

**Forcing axioms** are set-theoretic principles that arise directly from the technique of forcing. It is a matter of fact that forcing is one of the most powerful tools to produce consistency results in set theory: forcing axioms turn it into a powerful instrument to prove theorems. This is done by showing that a statement $\phi$ follows from an extension $T$ of ZFC if and only if $T$ proves that $\phi$ is consistent by means of a certain type of forcing. These types of results are known in the literature as generic absoluteness results and have the general form of a completeness theorem for some $T \supseteq \text{ZFC}$ with respect to the semantic given by boolean valued models and first order calculus. More precisely generic absoluteness theorems fit within the following general framework:

Assume $T$ is an extension of ZFC, $\Theta$ is a family of first order formulas in the language of set theory and $\Gamma$ is a certain class of forcing notions definable in $T$. Then the following are equivalent for a $\phi \in \Theta$ and $S \supseteq T$:

1. $S$ proves $\phi$.
2. $S$ proves that there exists a forcing $B \in \Gamma$ such that $B$ forces $\phi$ and $T$ jointly.
3. $S$ proves that $B$ forces $\phi$ for all forcings $B \in \Gamma$ such that $B$ forces $T$.

We say that a structure $M$ definable in a theory $T$ is generically invariant with respect to forcings in $\Gamma$ and parameters in $X \subset M$ when the above situation occurs with $\Theta$ being the first order theory of $M$ with parameters in $X$. A brief overview of the main known generic absoluteness results is the following:
Shoenfield’s absoluteness theorem is a generic absoluteness result for \( \Theta \) the family of \( \Sigma_1^1 \)-properties with real parameters, \( \Gamma \) the class of all forcings, \( T = \text{ZFC} \).

The pioneering “modern” generic absoluteness results are Woodin’s proofs of the invariance under set forcings of the first order theory of \( L(\text{ON}^{\omega}) \) with real parameters in \( \text{ZFC} + \text{class many Woodin cardinals which are limit of Woodin cardinals} \) \[33\] Thm. 3.1.2] and of the invariance under set forcings of the family of \( \Sigma_1^2 \)-properties with real parameters in the theory \( \text{ZFC} + \text{CH} + \text{class many measurable Woodin cardinals} \) \[33\] Thm. 3.2.1].

Further results pin down the exact large cardinal strength of the assertion that \( L(\mathbb{R}) \) is generically invariant with respect to certain classes of forcings (among others see \[37\]).

The bounded forcing axiom \( \text{BFA}(\Gamma) \) is equivalent to the statement that generic absoluteness holds for \( T = \text{ZFC} \) and \( \Theta \) the class of \( \Sigma_1 \)-formulas with parameters in \( P(\omega_1) \), as shown in \[6\].

Recently, Hamkins and Johnstone \[23\] introduced the resurrection axioms \( \text{RA}(\Gamma) \) and Viale \[42\] showed that these axioms produce generic absoluteness for \( \Theta \) the \( \Sigma_2 \)-theory with parameters of \( H_{\omega_1}, T = \text{ZFC} + \text{RA}(\Gamma) \), \( \Gamma \) any of the standard classes of forcings closed under two step iterations.

Viale introduced the forcing axiom \( \text{MM}^{+++} \) (a natural strengthening of \( \text{MM} \)) and proved that \( L(\text{ON}^{\omega_1}) \) with parameters in \( P(\omega_1) \) is generically invariant with respect to SSP forcings for

\[
T = \text{ZFC} + \text{MM}^{+++} + \text{there are class many superhuge cardinals}.
\]

Motivated by the latter results as well as by the work of Tsaprounis \[40\], in Chapter 4 we introduce over the theory MK (i.e. the Morse-Kelley set theory with sets and classes) a new natural class of forcing axioms: the iterated resurrection axioms \( \text{RA}_\alpha(\Gamma) \) of increasing strength as \( \alpha \) runs through the ordinals, with \( \Gamma \) a definable class of forcing notions. All the material in this chapter is joint work with Matteo Viale. We remark that for most classes \( \Gamma \), \( \text{RA}_1(\Gamma) \) is substantially equal to \( \neg \text{CH} \) plus the resurrection axiom \( \text{RA}(\Gamma) \) recently introduced by Hamkins and Johnstone in \[23\].

We are able to prove over MK the consistency relative to large cardinal axioms of the axioms \( \text{RA}_\alpha(\Gamma) \) for any class of forcing notions \( \Gamma \) which is weakly iterable. The latter is a property of classes of forcing notions which we introduce in Definition \[5.3.3\] and which holds for most standard classes such as locally ccc, Axiom-A, proper, semiproper, stationary set preserving (this latter class is weakly iterable only in the presence of sufficiently strong large cardinal axioms).

From the axioms \( \text{RA}_\alpha(\Gamma) \) with \( \alpha \geq \omega \) we are able to prove the following generic absoluteness result over the theory MK.

**Theorem.** Let \( V \) be a model of MK, \( \Gamma \) be a definable class of forcing notions, \( \gamma \) be the largest cardinal preserved by forcings in \( \Gamma \). Assume \( \text{RA}_\omega(\Gamma) \) holds and \( \mathcal{B} \in \Gamma \) forces \( \text{RA}_\omega(\Gamma) \). Then \( H^V_{2^\gamma} \prec H^V_{2^\gamma}. \)
This is a generic absoluteness result for $T = \text{MK} + \text{RA}_\omega(\Gamma)$ and $\Theta$ the first order theory of $H_{2^\gamma}$ with parameters. We also prove that $\text{RA}_\alpha(\Gamma)$ fits naturally within the hierarchy of the previously known results, that is, $\text{MM}^{++} \Rightarrow \text{RA}_\alpha(\text{SSP})$ and $\text{RA}_\alpha(\Gamma) \Rightarrow \text{RA}(\Gamma)$ for any $\Gamma$ and for all $\alpha > 0$. Furthermore, the consistency strength of the axioms $\text{RA}_\alpha(\Gamma)$ is below that of a Mahlo cardinal for all relevant $\Gamma$ and for all $\alpha$ except for $\Gamma = \text{SSP}$, in which case our upper bound is below a stationary limit of supercompact cardinals.

We remark that the present result cannot be formulated in ZFC alone since the iterated resurrection axioms $\text{RA}_\alpha(\Gamma)$ are second-order statements. However, it is possible that some theory strictly weaker than MK (e.g., NBG together with a truth predicate) would suffice to carry out the arguments at hand. Notice that $\text{RA}_n(\Gamma)$ can also be formulated by an equivalent first-order sentence for all $n < \omega$, and $\text{RA}_\omega(\Gamma)$ can be formulated as the corresponding first-order axiom schema $\{\text{RA}_n(\Gamma) : n < \omega\}$ if needed.

Altogether these results show the effectiveness of the axioms $\text{RA}_\omega(\Gamma)$ both on the premises side (low consistency strength, natural generalization of well-known axioms) and on the consequences side (generic invariance of $H_{2^\gamma}$). However, the axioms $\text{RA}_\omega(\Gamma)$ are pairwise incompatible for different choices of $\Gamma$ thus making it difficult to support the adoption of a specific $\text{RA}_\omega(\Gamma)$ as a natural axiom for set theory. Towards this aim, we remark the following two special cases.

- If we focus on forcing classes preserving $\omega_1$, there is a unique largest class (the class of stationary set preserving posets SSP) which contains all the possible classes $\Gamma$ for which the axiom $\text{RA}_\omega(\Gamma)$ is consistent. Thus $\text{RA}_\omega(\text{SSP})$ gives the strongest form of generic absoluteness which can be instantiated by means of the iterated resurrection axioms for forcing classes preserving $\omega_1$.

- If we consider the forcing classes $\langle \omega_\alpha \rangle$-closed for $\alpha \in \text{ON}$, the corresponding resurrection axioms are all pairwise compatible. In fact, from a Mahlo cardinal is possible to obtain the consistency of $\text{MK} + \text{GCH} + \text{RA}_\omega(\langle \kappa \rangle \text{-closed})$ for all cardinals $\kappa$ simultaneously. This gives a very strong and uniform generic absoluteness result, that is, given any forcing $\mathbb{B}$ we have that $H_{2^\kappa} = H^\mathbb{B}_{2^\kappa}$ where $\kappa$ is such that $\mathbb{B}$ is $\langle \kappa \rangle$-closed.

We remark that the case $\Gamma = \langle \omega_\alpha \rangle$-closed pushes the limits on which generic absoluteness can be obtained above the usual threshold $2^\omega$ whenever $\alpha > 0$. Even though the class of $\langle \omega_\alpha \rangle$-closed forcing is narrow and (some notion of) $\omega_\alpha$-proper forcing would be preferable, our result highlights a strong connection of the theory of iterations with generic absoluteness. In fact, it shows that a preservation theorem for a forcing class $\Gamma$ translates in a corresponding axiom yielding a generic absoluteness result for the same class $\Gamma$.

Compared to the generic absoluteness result obtained in [43], the present results for $\Gamma = \text{SSP}$ are weaker since they regard the structure $H_{2^{\omega_1}}$ instead of $L(\text{ON}^{\omega_1})$. On the other hand, the consistency of $\text{RA}_\alpha(\Gamma)$ is obtained from (in most cases much) weaker large cardinal hypothesis and the results are more general since they also apply to interesting choices of $\Gamma \neq \text{SP, SSP}$. Moreover the arguments we employ to prove the consistency of $\text{RA}_\alpha(\Gamma)$ are considerably simpler than the arguments developed in [43].
Finally, a byproduct of our results is that the theory $T = \text{MK} + \text{RA}_{\omega}(\text{all})$ is consistent relative to the existence of a Mahlo cardinal and makes the theory of projective sets of reals generically invariant with respect to any forcing which preserves $T$. Notice that $T$ is consistent with the failure of projective determinacy. This shows that the request of generic absoluteness for the projective sets of reals (with respect to forcings preserving $T$) is much weaker than Woodin’s generic absoluteness for all sets of reals in $L(\mathbb{R})$ with respect to $T = \text{ZFC} + \text{large cardinals}$. 

**Large cardinals** can be regarded as natural strenghtenings of the accepted principle of generation of new sets such as the powerset axiom and the replacement axiom. Furthermore, they are extremely useful in set theory: on the one hand they provide a fine scale to measure the consistency strength of a rich variety of combinatorial principles, on the other hand they also solve important questions within set theory. However, such cardinals are rarely used in common mathematics outside of set theory: for example, large parts of number theory and analysis can be formalized within $H_\kappa$, and even if new subjects can push this limit beyond that point, it is uncommon for structures of inaccessible or larger size to be employed outside of set theory.

Generic large cardinal axioms try to address this point, and postulate the existence of elementary embeddings $j : V \rightarrow M$ with $M \subseteq V[G]$ a transitive class definable in a generic extension $V[G]$ of $V$. Contrary to the classical case one can consistently have generic large cardinal properties at cardinals as small as $\omega_1$. Thus, generic large cardinal axioms are fit to produce consequences on small objects, and might be able to settle questions arising in domains of mathematics other than set theory, scoring both on the premises and consequences side. A detailed presentation of this approach can be found in [16].

Due to the class nature of the elementary embeddings involved in the definitions of large cardinals (both classical and generic), a key issue concerns the possibility to define (or derive) such embeddings from set-sized objects. The first natural candidates are ideals, although it turns out that they are not able to represent various relevant large cardinal properties. For this reason many extensions of the concept have been proposed, the most important of which are extenders (see among many [9, 29, 31]) and normal towers (see for example [11, 33, 43, 44]).

In Chapter 5 we introduce the notion of $\mathcal{C}$-system of filters (see Section 5.1). This concept is inspired by the well-known definitions of extenders and towers of normal ideals, generalizes both of them, and provides a common framework in which the standard properties of extenders and towers used to define classical or generic large cardinals can be expressed in an elegant and concise way. Using the new framework given by $\mathcal{C}$-system of filters we easily generalize to the setting of generic large cardinals well-known results about extenders and towers, providing shorter and modular proofs of several well-known facts regarding classical and generic large cardinals. Furthermore, we are able to examine closely the relationship between extenders and towers, and investigate when they are equivalent or not, both in the standard case and in the generic one (see Section 5.1.5). All the material in this chapter is joint work with Silvia Steila.

The second part of this chapter investigates some natural questions regarding generic large cardinals. In particular, we first examine the difference between having
a generic large cardinal property *ideally* or *generically*, and study when a generic \( C \)-system of ultrafilters is able to reproduce a given large cardinal property. Then we focus on *ideally* large cardinals, and study how the large cardinal properties are captured by the combinatorial structure of the \( C \)-system of filters used to induce the embedding. In particular, we are able to characterize strongness-like properties via the notion of *antichain splitting*, and closure-like properties via the notion of *antichain guessing* (a generalization of the well-known concept of *presaturation* for normal towers). Finally, we investigate to what extent it is possible to collapse a generic large cardinal while preserving its properties.

We remark that the main original contribution of this chapter is to streamline the essential features common to a variety of arguments involving a notion of normality. Thus we shall assume that all results in this chapter without an explicit attribution are adaptations of well-known facts to the setting of \( C \)-systems of filters.

**Acknowledgments**

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**Notation**

As in common set-theoretic use, \( \text{trcl}(x) \), \( \text{rank}(x) \) denote respectively the transitive closure and the rank of a given set \( x \). We denote by \( V_\alpha \) the sets \( x \) such that \( \text{rank}(x) < \alpha \) and by \( H_\kappa \) the sets \( x \) such that \( |\text{trcl}(x)| < \kappa \). We use \( P(x), [x]^\kappa, [x]^{<\kappa} \) to denote the powerset, the set of subsets of size \( \kappa \) and the ones of size less than \( \kappa \). The notation \( f : A \to B \) is improperly used to denote partial functions in \( A \times B \), \( A^B \) to denote the collection of all such (partial) functions, and \( f[A] \) to denote the pointwise image of \( A \) through \( f \). We denote by \( \text{id} : V \to V \) the identity map on \( V \). We use \( s \cdot t \) for sequence concatenation and \( s \rhd x \) where \( x \) is not a sequence as a shorthand to \( s \cdot \langle x \rangle \). We use \( s \lhd t \) to denote that \( t = s \cdot (|s| - 1) \). \( \text{CH} \) denote the continuum hypothesis and \( c \) the continuum itself. We prefer the notation \( \omega_\alpha \) instead of \( R_\alpha \) for cardinals.

Let \( \mathcal{L}^2 \) be the language of set theory with two sorts of variables, one for sets and
one for classes. We will work with models of set theory with sets and classes and with theories in the language $\mathcal{L}^2$ extending the Gödel-Bernays system NBG and in most cases also the Morse-Kelley system MK (see [32, Sec. II.10] for the axioms of NBG and [1] for MK). We will identify a model $N = \langle \text{set}(N), \text{class}(N) \rangle$ of NBG with its underlying collection of classes $\text{class}(N)$: e.g. if $\kappa$ is inaccessible and $N = \langle V_\kappa, V_{\kappa+1} \rangle$, we shall just denote $N$ by $V_{\kappa+1}$. We recall that from the underlying collection $\text{class}(N)$ we can reconstruct whether $x \in \text{set}(N)$ via the formula $\exists y \in \text{class}(N) \ x \in y$.

We will use $M \prec_n N$ to denote that $(M, \in)$ is a $\Sigma_n$-elementary substructure of $(N, \in)$. Given an elementary embedding $j : V \to M$, we use $\text{crit}(j)$ to denote the critical point of $j$. We denote by $\text{SkH}^M(X)$ the Skolem Hull of the set $X$ in the structure $M$, and by $M[X]$ be the smallest model of ZFC containing $M$ and $X$ (even if $X$ is not generic for any forcing). We say that $I \subseteq \mathcal{P}(X)$ is an ideal on $X$ whenever it is closed under unions and subsets, and feel free to confuse an ideal with its dual filter when clear from the context. We denote the collection of $I$-positive sets by $I^+ = \mathcal{P}(X) \setminus I$.

We follow Jech's approach [27] to forcing via boolean valued models. The letters $\mathbb{B}, \mathbb{C}, \mathbb{D}, \ldots$ are used for set sized complete boolean algebras, and $0, 1$ denote their minimal and maximal element. We use $V^\mathbb{B}$ for the boolean valued model obtained from $V$ and $\mathbb{B}$, $\dot{x}$ for the elements (names) of $V^\mathbb{B}$, $\dot{x}$ for the canonical name for a set $x \in V$ in the boolean valued model $V^\mathbb{B}$, $[[\phi]]_\mathbb{B}$ for the truth value of the formula $\phi$. We shall sometimes confuse $\mathbb{B}$-names with their defining properties. For example, given a collection of $\mathbb{B}$-names $\{\dot{x}_\alpha : \alpha < \gamma\}$, we confuse $\{\dot{x}_\alpha : \alpha < \gamma\}$ with a $\mathbb{B}$-name $\dot{x}$ such that for all $\dot{y} \in V^\mathbb{B}$, $[[\dot{y} \in \dot{x}]]_\mathbb{B} = [[\exists \alpha < \gamma \ y = \dot{x}_\alpha]]_\mathbb{B}$. When we believe this convention may be ambiguous we shall be explicitly more careful.

We use $\text{Coll}(\kappa, <\lambda)$ for the Lévy collapse that generically adds a surjective function from $\kappa$ to any $\gamma < \lambda$. $\text{Add}(\kappa, \lambda)$ for the $<\kappa$-closed poset that generically adds $\lambda$ many subsets to $\kappa$. We prefer the notation “$X$ has the $<\kappa$-property” for all properties that are defined in terms of $\forall \gamma < \kappa \ \phi(\gamma, X)$ for some formula $\phi$. In all such cases we shall explicitly avoid the notation “$\kappa$-property” and use $<\kappa^+\text{-property}$ instead. In general we shall feel free to confuse a partial order $\mathbb{P}$ with its boolean completion $\text{RO}(\mathbb{P})$ and a boolean algebra $\mathbb{B}$ with the partial order $\mathbb{B}$ given by its positive elements. When we believe that this convention may generate misunderstandings we shall be explicitly more careful.

When convenient we also use the generic filters approach to forcing. The letters $G$, $H$ will be used for generic filters over $V$, $\dot{G}_{\mathbb{B}}$ denotes the canonical name for the generic filter for $\mathbb{B}$, $\text{val}_{\mathbb{G}}(\dot{x})$ the valuation map on names by the generic filter $G$, $V[G]$ the generic extension of $V$ by $G$. Let $\phi$ be a formula in $\mathcal{L}^2$. We shall write $V^\mathbb{B} \models \phi$ to denote that $\phi$ holds in all generic extensions $V[G]$ with $G$ generic for $\mathbb{B}$. We shall also write $H_{2\gamma}^\mathbb{B} \prec H_{2\gamma}^\mathbb{C}$ (and similarly $H_{2\gamma}^\mathbb{P_1} \prec H_{2\gamma}^\mathbb{P_2}$) to denote that for all $G$ $V$-generic for $\mathbb{B}$ and $H$ $V[G]$-generic for $\mathbb{C}$, $H_{2\gamma}^V[G] \prec H_{2\gamma}^V[G][H]$. 

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## 0 Introduction

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<tr>
<td>Formulae</td>
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<td>Strategies</td>
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<tr>
<td>Forcing classes</td>
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<tr>
<td>Domains in $C$</td>
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<tr>
<td>Other functions</td>
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<tr>
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<tr>
<td>Sequences, threads in iteration systems</td>
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<td>Ultrapower functions</td>
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<td>Systems of ultrafilters</td>
<td>$S, E, T$</td>
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<tr>
<td>Boolean algebras</td>
<td>$\mathbb{B}, \mathbb{C}, \mathbb{D}$</td>
</tr>
<tr>
<td>Forcing posets</td>
<td>$\mathbb{P}, \mathbb{Q}, \mathbb{R}$</td>
</tr>
<tr>
<td>Systems of filters</td>
<td>$S, E, T$</td>
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Table 1: Summary of the naming conventions used throughout this thesis.
CHAPTER 1

BACKGROUNDS

In this chapter we present some unrelated topics that constitute the basis on which the remainder of the thesis is built. A reader acquainted with the following topics can safely skip this chapter or parts of it, although we recommend to browse through the content in order to fix the notation and conventions used.

Section 1.1 introduces the topic of second-order elementarity and determinacy of clopen class games. Section 1.2 reviews the main theory of generalized stationary sets. Section 1.3 gives a compact presentation of forcing via boolean valued models.

1.1 Second-order elementarity and class games

Let $T$ be a theory in the language $\mathcal{L}_2$ extending NBG that provably holds in $V_{\delta+1}$ with $\delta$ inaccessible. Let $\Delta_1^1(T)$ denote the formulae in $\mathcal{L}_2$ with set parameters that are provably equivalent modulo $T$ both to a $\Sigma_1^1$ formula (i.e. a formula with one existential class quantifier and set parameters) and to a $\Pi_1^1$ formula (with one universal class quantifier and set parameters). Let $N, M$ with $\text{set}(N) \subseteq \text{set}(M)$ be models of $T$. We will write $N \models_{\Delta_1^1(T)} M$ to denote that all $\Delta_1^1(T)$ formulae with set parameters in $N$ holding in $N$ also hold in $M$ (and viceversa). We will write $N \equiv_{\Delta_1^1(T)} M$ for $N \models_{\Delta_1^1(T)} M$ when $\text{set}(N) = \text{set}(M)$.

The $\Delta_1^1(T)$ formulae are interesting for their absolute behavior with respect to models of $T$ with the same sets.

**Proposition 1.1.1.** Let $N \subseteq M$ be models of a theory $T$ extending NBG with the same sets. Then $N \equiv_{\Delta_1^1(T)} M$.

**Proof.** Let $X = \text{set}(N) = \text{set}(M)$, and $\phi = \exists C \psi_1(C, \vec{p}) = \forall C \psi_2(C, \vec{p})$ be a $\Delta_1^1(T)$ formula, with $\vec{p} \in X$. Then,

$$
N \models \phi \Rightarrow \exists C \in N \langle X, C \rangle \models \psi_1(C, \vec{p})
\Rightarrow \exists C \in M \langle X, C \rangle \models \psi_1(C, \vec{p})
\Rightarrow M \models \phi
\Rightarrow \forall C \in M \langle X, C \rangle \models \psi_2(C, \vec{p})
\Rightarrow \forall C \in N \langle X, C \rangle \models \psi_2(C, \vec{p})
\Rightarrow N \models \phi
$$
Corollary 1.1.2. Let $T$ be a theory extending NBG that provably holds in $V_{\delta+1}$ with $\delta$ inaccessible. Let $N, M$ be models of $T$ with set $(N) = \text{set } (M) = V_\delta$, $\delta$ inaccessible. Then $N \equiv_{\Delta^1_1(T)} M$.

Proof. Since $\delta$ is inaccessible, $V_{\delta+1}$ is a model of $T$ containing both $N, M$ hence by Lemma 1.1.1 $N \equiv_{\Delta^1_1(T)} V_{\delta+1} \equiv_{\Delta^1_1(T)} M$.

Corollary 1.1.2 tells us that for any inaccessible $\delta$ the truth value of a $\Delta^1_1$-formula does not depend on the choice of the particular $T$-model whose family of sets is $V_\delta$. Thus we will focus only on NBG-models of the kind $V_{\delta+1}$.

In the remainder of this thesis we will need to prove that certain statements about class games are $\Delta^1_1$ for a suitable theory $T$. In order to provide a definition of such a theory we need to introduce clopen games on class trees. Our reference text for the basic notions and properties of games is [30, Sec. 20.A].

We shall consider well-founded trees $T$ as collections of finite sequences ordered by inclusion and closed under initial segments, such that there exists no infinite chain (totally ordered subset) in $T$. Given $s, t \in T$, let $s < t$ denote that $t = s[|s| - 1]$ ($s$ is obtained extending $t$ with one more element).

The clopen game on the well-founded tree $T$ is a two-player game $G^T$ defined as follows: Player I starts with some $s_0 \in T$ of length 1, then each player has to play a $s_{n+1} < s_n$. The last player who can move wins the game. A winning strategy $\sigma$ for Player I in $G^T$ is a subtree $\sigma \subseteq T$ such that for all $s \in \sigma$ of even length $|s|$ there is exactly one $t \in \sigma$ with $t < s$, and for every $s \in \sigma$ of odd length, every $t < s$ is in $\sigma$. A winning strategy for Player II is defined interchanging odd with even in the above statement. A game $G^T$ is determined if either one of the two players has a winning strategy.

We recall that there is correspondence between games $G^T$ on a well-founded tree $T$ and games on a pruned tree (as defined in [30]) whose winning condition is a clopen set. This justifies our terminology.

In the following we will be interested in the theory $T = \text{NBG} + \text{AD}(\Delta^0_1)$, where $\text{AD}(\Delta^0_1)$ is the following axiom of determinacy for clopen class games:

Definition 1.1.3 (AD(\Delta^0_1)). $G^T$ is determined for any well-founded class tree $T$.

Games $G^T$ on well-founded set trees $T \subseteq V_\delta$ are determined in ZFC (see [30, Thm. 20.1]) and the corresponding strategies $\sigma \subseteq T$ are elements of $V_{\delta+1}$. Thus the theory $\text{NBG} + \text{AD}(\Delta^0_1)$ holds in any $V_{\delta+1}$ with $\delta$ inaccessible, and we can apply the results of this section to this theory. Moreover, a finer upper bound for $\text{NBG} + \text{AD}(\Delta^0_1)$ is given by the next proposition (we thank Alessandro Andretta for pointing this fact to us).

Proposition 1.1.4. The Morse-Kelley theory MK (with the axiom of global choice) implies $\text{AD}(\Delta^0_1)$.

Proof. Since the recursion theorem on well-founded class trees holds in MK (see [11, Prop. 2]), we can follow the classical ZFC proof of determinacy for clopen games $G^T$ on well-founded set trees $T$. 

concluding the proof.
1.2 Generalized stationarity

For any \( s \in T \) the next moving player is I if \(|s| \mod 2 = 0 \) and II otherwise. Define recursively a (class) map \( w : T \to 2 \) so that \( w(s) = |s| \mod 2 \) iff there exists a \( t \prec s \) such that \( w(t) = |s| \mod 2 \). Intuitively, we can think of the map \( w \) as assigning (coherently) to every position \( s \) in \( G^T \) a “supposedly winning” player \( w(s) \) (I if \(|s| \mod 2 = 0 \) and II otherwise).

Then we can use the map \( w \) and the axiom of global choice to define a winning strategy \( \sigma \) for Player I if \( w(\emptyset) = 0 \), and for Player II if \( w(\emptyset) = 1 \). Precisely, define \( T_w = \{ s \in T : w(s) = w(\emptyset) \} \) and \( s^+ = \min_{<} \{ t \in T_w : t \prec s \} \) where \( < \) is a well-order on \( V \). Then,

\[
\sigma = \{ s \in T_w : \forall m < |s| \ (m \mod 2 = w(\emptyset)) \to (s|m + 1) = (s|m)^+ \}
\]

is the desired strategy.

In the remainder of this thesis we shall focus on extensions of the theory \( T = NBG + AD(\Delta_0^1) \) and use \( \Delta_1^1 \) as a shorthand for \( \Delta_1^1(NBG + AD(\Delta_0^1)) \). We remark that this theory is a natural strengthening of ZFC, since it asserts second order properties that are true for natural models of ZFC (models of the kind \( V_\delta \) with \( \delta \) inaccessible). Moreover, the theory has a reasonable consistency strength since it holds in all NBG models of the form \( V^0_{\delta+1} \) with \( \delta \) inaccessible and follows from MK.

It is not clear to us at the moment (and not relevant for the purpose of this thesis) whether the theory \( NBG + AD(\Delta_0^1) \) is preserved by set forcing. Since we know that this is true for MK (see [II, Thm. 23]), the most convenient theory to present and apply the results of this section is MK. In the remainder of this thesis we shall feel free to implicitly assume MK when needed, while pointing out some passages where \( AD(\Delta_0^1) \) is essentially used.

1.2 Generalized stationarity

We now recall the main definitions and properties of generalized stationary sets. Full references on this subject can be found in [27, 33 Chp. 2], [44].

**Definition 1.2.1.** Let \( X \) be an uncountable set. A set \( C \) is a club on \( \mathcal{P}(X) \) iff there is a function \( f_C : [X]^{<\omega} \to X \) such that \( C \) is the set of elements of \( \mathcal{P}(X) \) closed under \( f_C \), i.e.

\[
C = \{ Y \in \mathcal{P}(X) : [f_C][Y]^{<\omega} \subseteq Y \}
\]

A set \( S \) is stationary on \( \mathcal{P}(X) \) iff it intersects every club on \( \mathcal{P}(X) \).

The reference to the support set \( X \) for clubs or stationary sets may be omitted, since every set \( S \) can be club or stationary only on \( \bigcup S \). Examples of stationary sets are \( \{ X \} \), \( \mathcal{P}(X) \setminus \{ X \} \) and \( [X]^\kappa \) for any \( \kappa \leq |X| \) (the latter following the proof of the well-known downwards Löwenheim-Skolem Theorem). Notice that every element of a club \( C \) must contain \( f_C(\emptyset) \), a fixed element of \( X \).

Given any first-order structure \( M \), from the set \( M \) we can define a Skolem function \( f_M : [M]^{<\omega} \to M \) (i.e., a function coding solutions for all existential first-order formulas over \( M \)). Then the set \( C \) of all elementary submodels of \( M \) contains a club (the one corresponding to \( f_M \)). Henceforth, every set \( S \) stationary on \( X \) must contain an elementary submodel of any first-order structure on \( X \).
Definition 1.2.2. The club filter on $X$ is

$$\text{CF}_X = \{ C \subseteq \mathcal{P}(X) : C \text{ contains a club} \},$$

and its dual ideal is

$$\text{NS}_X = \{ A \subseteq \mathcal{P}(X) : A \text{ not stationary} \},$$

the non-stationary ideal on $X$.

If $|X| = |Y|$, then $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are isomorphic and so are $\text{CF}_X$ and $\text{CF}_Y$ (or $\text{NS}_X$ and $\text{NS}_Y$). Thus we can suppose $X \in \text{ON}$ or $X \supseteq \alpha$ for some $\alpha \in \text{ON}$ if needed.

Lemma 1.2.3. $\text{CF}_X$ is a $\sigma$-complete filter on $\mathcal{P}(X)$, and the stationary sets are exactly the $\text{CF}_X$-positive sets.

Proof. $\text{CF}_X$ is closed under superset by definition. Given a family of clubs $C_i$, $i < \omega$, let $f_i$ be the function corresponding to the club $C_i$. Let $\pi : \omega \to \omega^2$ be a surjection, with components $\pi_1$ and $\pi_2$, such that $\pi_2(n) \leq n$. Define $g : X^{\omega} \to X$ to be $g(s) = f_{\pi_1(|s|)}(s[\pi_2(|s|)])$. It is easy to verify that $C_g = \bigcap_{i<\omega} C_i$.

The above result is optimal, that is, $\text{CF}_X$ is never $<\omega_2$-complete unlike its well-known counterpart on cardinals. Let $A \subseteq X$ be such that $|A| = \omega_1$, and $C_x$ be the club corresponding to $f_x : [X]^{<\omega} \to \{x\}$. Then $C = \bigcap_{x \in A} C_x = \{B \subseteq X : B \supseteq A\}$ is disjoint from the stationary set $[X]^{\omega}$, hence is not a club.

Definition 1.2.4. Given a family $\{ S_x \subseteq \mathcal{P}(X) : x \in X \}$, the diagonal union of the family is $\nabla_{x \in X} S_x = \{ A \in \mathcal{P}(X) : \exists x \in A \ A \in S_x \}$, and the diagonal intersection of the family is $\Delta_{x \in X} S_x = \{ A \in \mathcal{P}(X) : \forall x \in A \ A \in S_x \}$.

Lemma 1.2.5 (Fodor). $\text{CF}_X$ is closed under diagonal intersection and $\text{NS}_X$ is closed under diagonal union.

Proof. Given a family $\{ C_x : x \in X \}$ of clubs, with corresponding functions $f_x$, let $g(x \cdot s) = f_x(s)$. It is easy to verify that $C_g = \Delta_{x \in X} C_x$.

Notice that $(\Delta_{x \in X} S_x)^c = \nabla_{x \in X} S_x^c$, where $A^c$ denotes $\mathcal{P}(X) \setminus A$. It follows that $\text{NS}_X$ is closed under diagonal union.

The above property is often applied by means of the equivalence in Lemma 1.2.7.

Definition 1.2.6. Let $I \subseteq \mathcal{P}(X)$ be an ideal. We say that $I$ (or equivalently its dual filter) is normal if for any $S \subseteq I^+$ and for any choice function $f : S \to X$ (i.e. such that $f(A) \in A$ for any $A \in S$) there exists $x \in X$ such that $\{A \in S : f(A) = x\} \subseteq I^+$. We say that $I$ (or equivalently its dual filter) is fine if for any $x \in X$ the set $\{A \subseteq X : x \notin A\}$ is in $I$.

Notice that the set $C_x = \{A \subseteq X : x \in A\}$ is the club given by the constant function $f_x : \mathcal{P}(X) \to \{x\}$, thus $\{A \subseteq X : x \notin A\}$ is non stationary.

Lemma 1.2.7. Let $I \subseteq \mathcal{P}(X)$ be an ideal. Then the following are equivalent:
1. \( I \) is closed under diagonal unions;

2. \( I \) is normal.

**Proof.** \( 1 \Rightarrow 2. \) Assume by contradiction that \( f : S \to X \) is a choice function on \( S \in I^+ \) such that \( f^{-1}([x]) = S_x \in I \) for every \( x \in X \). Let \( T = \nabla_{x \in X} S_x \) which is also in \( I \) since \( I \) is closed under diagonal unions. Notice that for every \( A \in S \), \( A \in S_{f(A)} \) and \( f(A) \in A \) hence \( A \in T \). Thus, \( S \subseteq T \in I^+ \), a contradiction.

\( 2 \Rightarrow 1. \) Assume by contradiction that \( \{S_x : x \in X\} \subseteq I \) is such that \( T = \nabla_{x \in X} S_x \) is in \( I^+ \). Define

\[
\begin{align*}
  f &: T \to X \\
  A &\mapsto x_A
\end{align*}
\]

where \( x_A \in A \) is such that \( A \in S_{x_A} \). Since \( I \) is normal, there exists an \( x \) such that \( f^{-1}([x]) \in I^+ \). Since \( f^{-1}([x]) \subseteq S_x \in I \), we have a contradiction. \( \Box \)

We are now able to prove that NS

**Lemma 1.2.8.** Let \( I \subseteq \mathcal{P}(X) \) be a normal and fine ideal, \( F \) be its dual filter. Then:

1. \( I \) is \( \omega \)-fine, that is, for every \( A \in [X]^\omega \), \( \{B \in \mathcal{P}(X) : B \supseteq A\} \in F \);

2. \( I \) is \( \sigma \)-closed;

3. \( I \) is \( \omega \)-normal, that is, for any function \( f : S \to [X]^{<\omega} \) such that \( S \in I^+ \) and \( f(A) \in [A]^{<\omega} \) for all \( A \in S \), there exists an \( s \in [X]^{<\omega} \) such that \( \{A \in S : f(A) = s\} \in I^+ \);

4. \( I \supseteq \text{NS}_X \).

**Proof.**

1. Let \( A = \{x_n : n \in \omega\} \), \( S = \{B \in \mathcal{P}(X) : B \not\supseteq A\} \) and assume by contradiction that \( S \in I^+ \). Let \( T = S \cap C_{x_0} \) be in \( I^+ \). Define

\[
\begin{align*}
  f &: T \to X \\
  B &\mapsto x_n
\end{align*}
\]

where \( n \) is such that \( \{x_0, \ldots, x_n\} \subseteq B \) and \( x_{n+1} \not\in B \). Since \( f \) is a choice function and \( I \) is normal, let \( x_m \) be such that \( f^{-1}([x_m]) \in I^+ \). Since \( I \) is fine, \( f^{-1}([x_m]) \cap C_{x_{m+1}} = \emptyset \in I^+ \), a contradiction.

2. Let \( A = \{x_n : n \in \omega\} \) be a subset of \( X \), \( \{S_n : n \in \omega\} \subseteq F \). Define

\[
T_x = \begin{cases} 
  S_n \cap C_x & \text{if } x = x_n; \\
  C_x & \text{otherwise.}
\end{cases}
\]

Since \( I \) is normal, \( T = \Delta_{x \in X} T_x \) is in \( F \). Since \( I \) is \( \omega \)-fine, \( T' = T \cap \{B \subseteq X : A \subseteq B\} \) is in \( F \) and \( T' \subseteq \bigcap_{n<\omega} S_n \).
3. Let \( f : S \to [X]^{<\omega} \) be such that \( \forall A \in S, f(A) \in [A]^{<\omega} \). Define \( S_\alpha = \{ A \in S : |f(A)| = \alpha \} \). Since \( S \in I^+ \) and \( I \) is \( \sigma \)-closed there exists \( n \) such that \( S_n \in I^+ \). Let \( T_m \) be defined by induction on \( m \leq n \) as in the following. Put 
\[
T_0 = S_n. 
\]
Given \( T_m \) for \( m < n \), define \( h_m \) as 
\[
f_m : T_m \to X \\
A \mapsto f(A)(m) 
\]
Since \( I \) is normal, let \( x_m \) be such that \( T_{m+1} = f_m^{-1} \{ [x_m] \} \in I^+ \). Then \( T_n \) is in \( I^+ \) and \( f(A) = (x_0, \ldots, x_{n-1}) = t \) for all \( A \in T_n \), witnessing \( \omega \)-normality.

4. Assume by contradiction that \( S \in I^+ \) and \( S \cap C_f = \emptyset \) for some \( f : X^{<\omega} \to X \), that is, for any \( A \in S \) there exists \( t_A \in A^{<\omega} \) such that \( f(t_A) \notin A \). Define 
\[
g : S \to [X]^{<\omega} \\
A \mapsto t_A 
\]
By \( \omega \)-normality there exists \( t \in X^{<\omega} \) such that \( T = g^{-1} \{ \{ t \} \} \in I^+ \). Since \( T \subseteq \{ A \in S : f(t) \notin A \} \), \( T \cap C_{f(t)} = \emptyset \in I^+ \), a contradiction. \( \square \)

The generalized notion of club and stationary set is closely related to the well-known one defined for subsets of cardinals.

**Lemma 1.2.9.** \( C \subseteq \omega_1 \) is a club in the classical sense if and only if \( C \cup \{ \omega_1 \} \) is a club in the generalized sense. Thus, \( S \subseteq \omega_1 \) is stationary in the classical sense if and only if it is stationary in the generalized sense. 

**Proof.** Let \( C \subseteq \omega_1 + 1 \) be a club in the generalized sense. Then \( C \) is closed: given any \( \alpha = \sup \alpha_i \) with \( f[\alpha_i]^{<\omega} \subseteq \alpha_i \), \( f[\alpha]^{<\omega} = \bigcup_i f[\alpha_i]^{<\omega} \subseteq \bigcup_i \alpha_i = \alpha \). Furthermore, \( C \) is unbounded: given any \( \beta_0 < \omega_1 \), define a sequence \( \beta_i \) by taking \( \beta_{i+1} = \sup f[\beta_i]^{<\omega} \). Then \( \beta_\omega = \sup \beta_i \in C \).

Let now \( C \subseteq \omega_1 \) be a club in the classical sense. Let \( C = \{ x_\alpha : \alpha < \omega_1 \} \) be an enumeration of the club. For every \( \alpha < \omega_1 \), let \( \{ y_\alpha^i : i < \omega \} \subseteq x_\alpha \) be a cofinal sequence in \( x_\alpha \) (possibly constant), and \( \{ z_\alpha^i : i < \omega \} \subseteq \alpha \) be an enumeration of \( \alpha \). Define \( f_C : [X]^{<\omega} \to X \) as 
\[
f_C(s) = \begin{cases} 
 y_\alpha^n & \text{if } s = (x_\alpha)^n \text{ with } \alpha > 0; \\
 z_\alpha^n & \text{if } s = x_0^n \text{ with } \alpha > 0; \\
x_0 & \text{otherwise.} 
\end{cases} 
\]
The sequence \( z_\alpha^i \) forces all closure points of \( f_C \) to be ordinals, while the sequence \( y_\alpha^i \) forces the ordinal closure points of \( f_C \) being in \( C \). \( \square \)

**Lemma 1.2.10.** If \( \kappa \) is a cardinal with cofinality at least \( \omega_1 \), \( C \subseteq \kappa \) contains a club in the classical sense if and only if \( C \cup \{ \kappa \} \) contains the ordinals of a club in the generalized sense. Thus \( S \subseteq \kappa \) is stationary in the classical sense if and only if it is stationary in the generalized sense.
Proof. If \( C \) is a club in the generalized sense, then \( C \cap \kappa \) is closed and unbounded by the same reasoning of Lemma 1.2.9. Let now \( C \) be a club in the classical sense, and define \( f : \kappa^\omega \to \kappa \) to be \( f(s) = \min \{ \alpha \in C : \sup s < \alpha \} \). Then \( C_f \cap \kappa \) is exactly the set of ordinals in \( C \cup \{ \kappa \} \) that are limits within \( C \).

Notice that if \( S \) is stationary in the generalized sense on \( \omega_1 \), then \( S \cap \omega_1 \) is stationary (since \( \omega_1 + 1 \) is a club by Lemma 1.2.9), while this is not true for \( \kappa > \omega_1 \).

In this case, \( P(\kappa) \setminus (\kappa + 1) \) is a stationary set: given any function \( f \), the closure under \( f \) of \( \{ \omega_1 \} \) is countable, hence not an ordinal.

The non-stationary ideals \( NS_X \) forms a coherent system varying \( X \in V \), as shown in the following.

**Lemma 1.2.11** (Lifting and Projection). Let \( X \subseteq Y \) be uncountable sets. If \( S \) is stationary on \( P(X) \), then \( S \uparrow Y = \{ B \subseteq Y : B \cap X \in S \} \) is stationary. If \( S \) is stationary on \( P(Y) \), then \( S \downarrow X = \{ B \cap X : B \in S \} \) is stationary.

Proof. For the first part, given any function \( f : [X]^\omega \to X \), extend it in any way to a function \( g : [Y]^\omega \to Y \). Since \( S \) is stationary, there exists a \( B \subseteq S \) closed under \( g \), hence \( B \cap X \subseteq S \downarrow X \) is closed under \( f \).

For the second part, fix an element \( x \in X \). Given any function \( f : [Y]^\omega \to Y \), replace it with a function \( g : [Y]^\omega \to Y \) such that for any \( A \subseteq Y \), \( g([A]^\omega) \) contains \( A \cup \{ x \} \) and is closed under \( f \). To achieve this, fix a surjection \( \pi : (\omega \to \omega)^2 \) with projections \( \pi_1 \) and \( \pi_2 \) such that \( \pi_2(n) \leq n \) for all \( n \), and an enumeration \( \{ t^n_i : i < \omega \} \) of all first-order terms with \( n \) variables, function symbols \( f_i \) for \( i < n \) (that represent an i-ary application of \( f \)) and a constant \( x \). The function \( g \) can now be defined as \( g(s) = t^{\pi_2(|s|)}_{\pi_1(|s|)}(s|\pi_2(|s|)) \). Finally, let \( h : [X]^\omega \to X \) be defined by \( h(s) = g(s) \) if \( s \in X \), and \( h(s) = x \) otherwise. Since \( S \) is stationary, there exists a \( B \subseteq S \) with \( h([B]^\omega) \subseteq B \), but \( h([B]^\omega) = g([B]^\omega) \cap X \) (since \( x \) is always in \( g([B]^\omega) \)) and \( g([B]^\omega) \cap X = B \). Then, \( g([B]^\omega) \subseteq S \uparrow Y \) and \( g([B]^\omega) \) is closed under \( f \) (by definition of \( g \)).

Following the same proof, a similar result holds for clubs. If \( C_f \) is club on \( P(X) \), then \( C_f \uparrow Y = C_f | Y \) where \( g = f \cup \text{id} \upharpoonright (Y \setminus X) \). If \( C_f \) is club on \( P(Y) \) such that \( \bigcap C_f \) intersects \( X \) in \( x \), and \( g, h \) are defined as in the second part of Theorem 1.2.11, then \( C_f \downarrow X = C_f \bigcap X \) is club. If \( \bigcap C_f \) is disjoint from \( X \), \( C_f \downarrow X \) is not a club, but is still true that it contains a club (namely, \( (C_f \cap C_i(x)) \downarrow X \) for any \( x \in X \)).

**Theorem 1.2.12** (Ulam). Let \( \kappa \) be an infinite cardinal. Then for every stationary set \( S \subseteq \kappa^+ \), there exists a partition of \( S \) into \( \kappa^+ \) many disjoint stationary sets.

Proof. For every \( \beta \in [\kappa, \kappa^+] \), fix a bijection \( \pi_\beta : \kappa \to \beta \). For \( \xi < \kappa \), \( \alpha < \kappa^+ \), define \( A^\kappa_\xi = \{ \beta < \kappa^+ : \pi_\beta(\xi) = \alpha \} \) (notice that \( \beta > \alpha \) when \( \alpha \in \text{ran}(\pi_\beta) \)). These sets can be fit in a \( (\kappa \times \kappa^+) \)-matrix, called Ulam Matrix, where two sets in the same row or column are always disjoint. Moreover, every row is a partition of \( \bigcup_{\alpha < \kappa} A^\kappa_\alpha = \kappa^+ \), and every column is a partition of \( \bigcup_{\xi < \kappa} A^\kappa_\xi = \kappa^+ \setminus (\alpha + 1) \).

Let \( S \) be a stationary subset of \( \kappa^+ \). For every \( \alpha < \kappa^+ \), define \( f_\alpha : S \setminus (\alpha + 1) \to \kappa \) by \( f_\alpha(\beta) = \xi \) if \( \beta \in A^\kappa_\xi \). Since \( \kappa^+ \setminus (\alpha + 1) \) is a club, every \( f_\alpha \) is regressive on a stationary set, then by Fodor’s Lemma 1.2.5 there exists a \( \xi_\alpha < \kappa \) such that
stationary on $X$. Define $g : \kappa^+ \to \kappa$ by $g(\alpha) = \xi_\alpha$. If $\xi_\alpha$ is regressive on the stationary set $\kappa^+ \setminus \kappa$, then by Fodor’s Lemma \ref{lem:stationary_on_kappa}, let $\xi^* \prec \kappa$ be such that $g^{-1}[\{\xi^*\}] = T$ is stationary. Then, the row $\xi^*$ of the Ulam Matrix intersects $S$ in a stationary set for stationary many columns $T$. So $S$ can be partitioned into $S \cap A_\alpha^\kappa$ for $\alpha \in T \setminus \{\min(T)\}$, and $S \cup \bigcup_{\alpha \in T \setminus \{\min(T)\}} A_\alpha^\kappa$. \hfill \Box

In the proof of Theorem \ref{thm:stationary_on_kappa} we actually proved something more: the existence of a Ulam Matrix, i.e. a $\kappa \times \kappa^+$-matrix such that every stationary set $S \subseteq \kappa^+$ is compatible (i.e., has stationary intersection) with stationary many elements of a certain row.

Stationary sets are to be intended as large sets. Moreover, they cannot be too small even in literal sense.

**Lemma 1.2.13.** Let $S \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ be such that $|S| < |X|$. Then $S$ is non-stationary.

Proof. Let $S = S_1 \cup S_2$, $S_1 = \{Y \in S : |Y| < |S|\}$, $S_2 = \{Y \in S : |Y| \geq |S|\}$. Since $|\bigcup S_1| \leq |S| \cdot |S| = |S| < |X|$, $S_1$ is non-stationary. We now prove that $S_2$ is non-stationary as well.

Fix an enumeration $S_2 = \{Y_\alpha : \alpha < \gamma\}$ with $\gamma = |S_2| < |X|$. For all $\alpha < \gamma$, define recursively $x_\alpha \in X \setminus Y_\alpha$, $y_\alpha \in Y_\alpha \setminus \{y_\beta : \beta < \alpha\}$. Such $x_\alpha$ exists since $\{X\} \notin S$, and such $y_\alpha$ exists since $|Y_\alpha| \geq |S| = \gamma > \alpha$. Let $f : [X]^{<\omega} \to X$ be such that $f(\{y_\alpha\}) = x_\alpha$, $f(s) = x_0$ otherwise. Thus $C_f \cap S_2 = \emptyset$, hence $S_2$ is non-stationary. \hfill \Box

As previously mentioned, $[X]^\kappa$ and $[X]^{<\kappa}$ for any $\kappa \leq |X|$ are the prototypical examples of stationary sets. This encourages to consider the notion of club and stationary set relative to them.

**Definition 1.2.14.** Let $X$ be an uncountable set, $\kappa \leq |X|$ be a cardinal. A set $C$ is a club on $[X]^\kappa$ (resp. $[X]^{<\kappa}$) iff there is a function $f_C : [X]^{<\omega} \to X$ such that $C$ is the set of elements of $[X]^\kappa$ (resp. $[X]^{<\kappa}$) closed under $f_C$, i.e.

$$C = \{Y \in [X]^\kappa : f_C[Y]^{<\omega} \subseteq Y\}$$

A set $S$ is stationary on $[X]^\kappa$ (respectively $[X]^{<\kappa}$) iff it intersects every club on $[X]^\kappa$ (respectively $[X]^{<\kappa}$).

As in the general case, the club sets on $[X]^\kappa$ (resp. $[X]^{<\kappa}$) form a normal $\sigma$-complete filter on $[X]^\kappa$ (resp. $[X]^{<\kappa}$). We can also state a restricted version of Lemma \ref{lem:stationary_on_kappa} to this setting.

**Lemma 1.2.15 (Lifting and Projection II).** Let $X \subseteq Y$ be uncountable sets, $\kappa \leq |X|$ be a cardinal. If $C$ contains a club on $[Y]^\kappa$ (resp. $[Y]^{<\kappa}$), then $C \downarrow [X]^\kappa = (C \downarrow X) \cap [X]^\kappa$ (resp. $C \downarrow [X]^{<\kappa}$) contains a club on $[X]^\kappa$ (resp. $[X]^{<\kappa}$). If $C$ contains a club on $[X]^{<\kappa}$, then $C \uparrow [Y]^{<\kappa} = (C \uparrow Y) \cap [Y]^{<\kappa}$ contains a club on $[Y]^{<\kappa}$.

If $S$ is stationary on $[Y]^{<\kappa}$, then $S \downarrow [X]^{<\kappa}$ is stationary on $[X]^{<\kappa}$. If $S$ is stationary on $[X]^\kappa$ (resp. $[X]^{<\kappa}$), then $S \uparrow [Y]^\kappa$ is stationary on $[Y]^\kappa$ (resp. with $[Y]^{<\kappa}$).
The latter lemma is optimal, that is, the lifting \([X]^\kappa \uparrow [Y]^\kappa\) may not be a club on \([Y]^\kappa\) if \(|X| < |Y|\). For example, such a set is not a club if \(|Y|\) is a Completely Jónsson cardinal (see [33]) since in this case its complement \([Y]^\kappa \setminus ([X]^\kappa \uparrow [Y]^\kappa) = [X]^{<\kappa} \uparrow [Y]^\kappa\) is stationary.

1.3 Forcing and boolean valued models

We now give a brief account of forcing via boolean valued models. Our presentation is self-contained but extremely compact, more details on the material of this section can be found in [2, 8, 26, 27, 32].

Recall that \(\mathbb{P}\) is a poset iff it is a set equipped with a partial order \(\leq_{\mathbb{P}}\). Given \(p, q \in \mathbb{P}\), we say that \(p \parallel q\) (\(p\) is compatible with \(q\)) iff there exists an \(r \leq p, r \leq q;\) and \(p \perp q\) (\(p\) is incompatible with \(q\)) otherwise.

A subset \(D \subseteq \mathbb{P}\) is dense iff for every \(p \in \mathbb{P}\) there is a \(q \in D, q \leq p\) and is open iff for every \(p \in D\), any \(q \in \mathbb{P}\) such that \(q \leq p\) is in \(D\). We say that a subset \(A \subseteq \mathbb{P}\) is an antichain iff \(p \perp q\) for all \(p, q \in A\) and that it is a chain iff it is totally ordered by \(\leq_{\mathbb{P}}\). An antichain \(A \subseteq \mathbb{P}\) is maximal iff every \(p \in \mathbb{P}\) is compatible with some \(q \in A\).

We say that \(\mathbb{P}\) is a suborder of \(\mathbb{Q}\) if \(\mathbb{P} \subseteq \mathbb{Q}\) and the inclusion map of \(\mathbb{P}\) into \(\mathbb{Q}\) is preserving the order and the incompatibility relation. Furthermore, \(\mathbb{P}\) is a complete suborder of \(\mathbb{Q}\) if any maximal antichain in \((\mathbb{P}, \leq_{\mathbb{P}})\) remains such in \((\mathbb{Q}, \leq_{\mathbb{Q}})\).

Recall that a poset \(\mathbb{P}\) is a lattice iff every \(p, q \in \mathbb{P}\) have a least upper bound \(p \lor q\) and a greater lower bound \(p \land q\), and that \(\mathbb{B}\) is a boolean algebra iff it is a complemented distributive lattice. A boolean algebra \(\mathbb{B}\) is complete iff every subset \(A \subseteq \mathbb{B}\) has a unique supremum \(\lor A\) in \(\mathbb{B}\).

**Definition 1.3.1.** Let \(\mathbb{P}\) be a poset, \(p\) be in \(\mathbb{P}\). Then

\[ \mathbb{P}\downarrow p = \{ q \in \mathbb{P} : q \leq p \}, \]

is the restriction of \(\mathbb{P}\) below \(p\). If \(\mathbb{P} = \mathbb{B}\) is a boolean algebra, we define

\[ i_p : \mathbb{B} \rightarrow \mathbb{B}\downarrow p \]

\[ q \mapsto q \land p \]

the restriction map from \(\mathbb{B}\) to \(\mathbb{B}\downarrow p\).

We call forcing notion a poset (or complete boolean algebra) used for forcing, and conditions the elements of a forcing notion. Throughout the remainder of this section, let \(M\) be a transitive model of ZFC and \(\mathbb{B}\) be a complete boolean algebra in \(M\).

**Definition 1.3.2.** The class \(M\mathbb{B}\) of \(\mathbb{B}\)-names in \(M\) is

\[ M\mathbb{B} = \{ \dot{x} \in M : \dot{x} : M\mathbb{B} \rightarrow \mathbb{B} \text{ is a partial function}\}. \]

We let for the atomic formulas \(x \in y, x \subseteq y, x = y\):

\[ ^1\text{This definition is a shorthand for a recursive definition by rank. We remark that in certain cases (for example in the definition of the name } \beta \text{ in the proof of Lemma 3.4.8)} \text{ it will be convenient to allow a name } \dot{y} \text{ to be a relation (as in Kunen’s Def. 2.5)); given a } \beta\text{-name } \dot{y} \text{ according to Kunen’s definition, the corresponding intended name } f_\beta \text{ according to the above definition is given by } f_\beta(f_c) = \bigvee \{ b : \langle c, b \rangle \in \dot{y} \}. \]
Lemma 1.3.5

1 Backgrounds

| \( \hat{x}_0 \in \hat{x}_1 \)| = \( \lor \{ [\hat{y} = \hat{x}_0] \land \hat{x}_0(\hat{y}) : \hat{y} \in \text{dom}(\hat{x}_1) \} \),
| \( \hat{x}_0 \subseteq \hat{x}_1 \)| = \( \land \{ \neg \hat{x}_0(\hat{y}) \lor [\hat{y} \in \hat{x}_0] : \hat{y} \in \text{dom}(\hat{x}_0) \} \),
| \( \hat{x}_0 = \hat{x}_1 \)| = \([\hat{x}_0 \subseteq \hat{x}_1] \land [\hat{x}_1 \subseteq \hat{x}_0] \).

and for general formulas \( \phi(x_0, \ldots, x_n) \):

- \([\neg \phi]_B = \neg [\phi]_B \),
- \([[\phi \land \psi]]_B = [[\phi]]_B \land [[\psi]]_B \),
- \([[\phi \lor \psi]]_B = [[\phi]]_B \lor [[\psi]]_B \),
- \([[\exists x \phi(x, \hat{x}_1, \ldots, \hat{x}_n)]_B = \lor \{ [[\phi(\hat{y}, \hat{x}_1, \ldots, \hat{x}_n)]_B : \hat{y} \in M^B \} \). \[1\]

\( M^B \) together with the interpretation of formulas \([\ldots]_B \) given above forms a boolean valued model of set theory. When clear from the context, we shall omit the index \( B \) from \([\ldots]_B \). We also use \( p \models_B \phi \) to denote that \( p \leq [\phi]_B \).

**Definition 1.3.3.** Given \( x \in M \), we define the canonical \( B \)-name \( \hat{x} \) for \( x \) recursively by:

\( \hat{x} = \{ \langle \hat{y}, 1 \rangle : y \in x \} \),

and expand the language with an additional symbol \( \hat{M} \) for a boolean-valued subclass of \( M^B \) defined by:

\( [\hat{x} \in \hat{M}]_B = \lor \{ [\hat{y} = \hat{y}]_B : y \in M \} \).

Finally, we denote as \( \hat{G}_B \in M^B \) the canonical name for a \( B \)-generic filter for \( B \), i.e. \( \hat{G}_B = \{ \langle \hat{p}, p \rangle : p \in B \} \).

**Lemma 1.3.4** (Mixing). Let \( A \subseteq B \) be an antichain in \( M \) and \( \{ \hat{x}_p : p \in A \} \) be a family of \( B \)-names indexed by \( A \) in \( M \). Then there exists \( \hat{y} \in M^B \) such that \( [\hat{y} = \hat{x}_p]_B \geq p \) for all \( p \in A \).

**Proof.** The property is witnessed by \( \hat{y} = \bigcup_{p \in A} \{ \langle \hat{z}, q \land p \rangle : z \in \hat{x}_p \} \). \( \square \)

**Lemma 1.3.5** (Fullness). For all formula \( \phi(x, x_1, \ldots, x_n) \) and \( \hat{x}_1, \ldots, \hat{x}_n \in M^B \), there exists a \( \hat{y} \in M^B \) such that

\( [[\exists x \phi(x, \hat{x}_1, \ldots, \hat{x}_n)]_B = [[\phi(\hat{y}, \hat{x}_1, \ldots, \hat{x}_n)]_B \).

**Proof.** First notice that for any \( \hat{y} \in M^B \),

\( [[\exists x \phi(x, \hat{x}_1, \ldots, \hat{x}_n)]_B \geq [[\phi(\hat{y}, \hat{x}_1, \ldots, \hat{x}_n)]_B \).

Set \( B = \{ [[\phi(\hat{y}, \hat{x}_1, \ldots, \hat{x}_n)]_B : \hat{y} \in M^B \} \), and let \( B^\prime \) be its downward closure. Since \( q = [[\exists x \phi(x, \hat{x}_1, \ldots, \hat{x}_n)]_B = \lor B \), we can find an antichain \( A \subseteq B^\prime \) in \( M \) which is maximal below \( q \). For all \( p \in A \), let \( \hat{x}_p \) be such that \( [[\phi(\hat{x}_p, \hat{x}_1, \ldots, \hat{x}_n)]_B \geq p \), \( \hat{y} \) be obtained from \( \{ \hat{x}_p : p \in A \} \) by mixing as in Lemma 1.3.4. Then

\( [[\phi(\hat{y}, \hat{x}_1, \ldots, \hat{x}_n)]_B \geq [[\hat{y} = \hat{x}_p] \land [[\phi(\hat{x}_p, \hat{x}_1, \ldots, \hat{x}_n)]_B \geq p \land p = p \)

for all \( p \in A \), hence \( [[\phi(\hat{y}, \hat{x}_1, \ldots, \hat{x}_n)]_B \geq \lor A = q = [[\exists x \phi(x, \hat{x}_1, \ldots, \hat{x}_n)]_B \). \( \square \)

\(^2\)Contrary to the convention used throughout this thesis, here we use the notation \( \{ \hat{x}_p : p \in A \} \) to denote a family of \( B \)-names rather than a single \( B \)-name for the whole collection.
**Definition 1.3.6.** Let $G$ be any ultrafilter on $\mathbb{B}$. For $\dot{x}, \dot{y} \in M^G$ we let

- $[\dot{x}]_G = [\dot{y}]_G$ iff $\langle \dot{x} = \dot{y} \rangle \in G$,
- $[\dot{x}]_G \in [\dot{y}]_G$ iff $\langle \dot{x} \in \dot{y} \rangle \in G$,
- $M^G/G = \{ [\dot{x}]_G : \dot{x} \in M^G \}$.

**Theorem 1.3.7** (Łoś, [27] Lemma 14.14, Thm. 14.24). Let $G$ be any ultrafilter on $\mathbb{B}$. Then

1. $\langle M^G/G, \in_G \rangle$ is a model of ZFC,
2. $\langle M^G/G, \in_G \rangle$ models $\phi((\dot{x}_1)_G,\ldots,(\dot{x}_n)_G)$ iff $\langle \phi(\dot{x}_1,\ldots,\dot{x}_n) \rangle \in G$.

Even though the above theorem holds for any ultrafilter $G$, $M^G/G$ is commonly considered for $M$-generic filters $G$.

**Definition 1.3.8.** We say that $G$ is an $M$-generic filter for $\mathbb{B}$ if it is an ultrafilter and for every $D \in M$ dense subset of $\mathbb{B}$, $G \cap D \cap M$ is not empty.

We remark that the above definition can be applied as stated even to non-transitive models $M \subseteq V$ of ZFC (with the natural $\in$-relation). However, for the remainder of this section we shall still assume that $M$ is transitive.

**Definition 1.3.9.** Let $G$ be an $M$-generic filter for $\mathbb{B}$. For any $\dot{x} \in M^G$ we let

$$\text{val}_G(\dot{x}) = \{ \text{val}_G(\dot{y}) : \exists p \in G \langle \dot{y}, p \rangle \in \dot{x} \}$$

and $M[G] = \{ \text{val}_G(\dot{x}) : \dot{x} \in M^G \}$.

**Theorem 1.3.10** (Cohen’s Forcing, [27] Thm. 14.6). Let $G$ be an $M$-generic filter for $\mathbb{B}$. Then:

1. $\text{val}_G[M] = M$, that is, $\{ \text{val}_G(\dot{x}) : \langle \dot{x} \in M \rangle \in G \} = M$.
2. $M[G]$ is isomorphic to $M^G/G$ via the map which sends $\text{val}_G(\dot{x})$ to $[\dot{x}]_G$.
3. $M[G] \models \phi(\text{val}_G(\dot{x}_1),\ldots,\text{val}_G(\dot{x}_n))$ iff $\langle \phi(\dot{x}_1,\ldots,\dot{x}_n) \rangle \in G$.
4. $p \Vdash_{\mathbb{B}} \phi(\dot{x}_1,\ldots,\dot{x}_n)$ iff $M[G] \models \phi(\text{val}_G(\dot{x}_1),\ldots,\text{val}_G(\dot{x}_n))$ for all $M$-generic filters $G$ for $\mathbb{B}$ such that $p \in G$.
5. $M[G]$ is the smallest transitive model of ZFC including $M$ and containing $G$.

**Proposition 1.3.11.** Let $G$ be an $M$-generic ultrafilter for $\mathbb{B}$. Then $\bigwedge A \in G$ for any $A \subseteq G$ which belongs to $M$.

**Proof.** Suppose instead that $\bigwedge A \notin G$ and w.l.o.g. $\bigwedge A = \emptyset$, and define

$$D = \{ p \in \mathbb{B} : \exists q \in A \langle p \land q = 0 \rangle \}.$$

Since $D$ is a dense subset of $\mathbb{B}$ in $M$, there exists a $p \in G \cap D$. Let $q \in A$ be such that $p \land q = 0$. Then $p, q \in G$ hence $G$ is not a filter, a contradiction. \qed
CHAPTER 2

ITERATION SYSTEMS

This chapter presents a compact and self-contained development of the theory of iterated forcing. We shall pursue the approach to iterated forcing devised by Donder and Fuchs in [19], thus we shall present iterated forcing by means of directed systems of complete and injective homomorphisms of complete boolean algebras. The results of this chapter will constitute the basis on which the remainder of this thesis is built. All the material in this chapter is joint work with Matteo Viale and Silvia Steila (see [45]).

In Section 2.1 we introduce regular homomorphisms between complete boolean algebras (i.e., injective complete homomorphisms) and their associated retractions. In Section 2.2 we present iterated forcing in the setting of complete boolean algebras, together with the definition of revised countable support iterations. In the final part of the section we study basic properties of complete iteration systems, in particular we set up sufficient conditions to establish when the direct limit of an iteration system of length \( \lambda \) is \(<\lambda\)-cc (this corresponds to the well known result of Baumgartner on direct limits of \(<\lambda\)-cc forcings), and when \( \lim \mathcal{F} \) and \( \lim \mathcal{F} \) do overlap for a given iteration system \( \mathcal{F} \). In Section 2.3 we introduce the definition of two-step iteration following Jech’s approach [27], and present the main properties of generic quotients for iteration systems.

2.1 Embeddings and retractions

We now introduce the notions of complete homomorphism and regular embedding and their basic properties.

**Definition 2.1.1.** Let \( \mathbb{B}, \mathbb{C} \) be complete boolean algebras, \( i : \mathbb{B} \to \mathbb{C} \) is a complete homomorphism iff it is an homomorphism that preserves arbitrary suprema. We say that \( i \) is a regular embedding iff it is an injective complete homomorphism of boolean algebras.

**Definition 2.1.2.** Let \( i : \mathbb{B} \to \mathbb{C} \) be a complete homomorphism. We define

\[
\ker(i) = \bigvee \{ p \in \mathbb{B} : i(p) = 0 \}
\]
Proposition 2.1.4. Let $i: B \to C$ be a regular embedding, the retraction associated to $i$ is the map
\[
\pi_i : C \to B \\
p \mapsto \bigwedge \{q \in B : i(q) \geq p\}.
\]

Definition 2.1.3. Let $i: B \to C$ be a regular embedding, the retraction associated to $i$ is the map
\[
\pi_i : C \to B \\
p \mapsto \bigwedge \{q \in B : i(q) \geq p\}.
\]

Proposition 2.1.4. Let $i: B \to C$ be a regular embedding, $p \in B$, $q, r \in C$ be arbitrary. Then,

1. $\pi_i \circ i(p) = p$ hence $\pi_i$ is surjective;
2. $i \circ \pi_i(q) \geq q$ hence $\pi_i$ maps $C^+$ to $B^+$;
3. $\pi_i$ preserves joins, i.e. $\pi_i(\bigvee A) = \bigvee \pi_i[A]$ for all $A \subseteq C$;
4. $i(p) = \bigvee \{q : \pi_i(q) \leq p\}$.
5. $\pi_i(q \land i(p)) = \pi_i(q) \land p = \bigvee \{\pi_i(r) : r \leq q, \pi_i(r) \leq p\}$;
6. $\pi_i$ does not preserve neither meets nor complements whenever $i$ is not surjective, but $\pi_i(r \land q) \leq \pi_i(r) \land \pi_i(q)$ and $\pi_i(\neg q) \geq \neg \pi_i(q)$;

Proof. 1. Since $i$ is injective,
\[
\pi_i \circ i(p) = \bigwedge \{q \in B : i(q) \geq i(p)\} = \bigwedge \{q \in B : q \geq p\} = p,
\]
thus $\pi$ is surjective.

2. Suppose by contradiction that $q > 0$, $\pi_i(q) = 0$. Since
\[
i \circ \pi_i(q) = i \left( \bigwedge \{p \in B : i(p) \geq q\} \right) = \bigwedge \{i(p) : p \in B, i(p) \geq q\}
\geq \bigwedge \{r \in C : r \geq q\} = q,
\]
we have that $0 = i \circ \pi_i(q) \geq q > 0$, a contradiction.

3. Let $A = \{q_\alpha : \alpha < \gamma\} \subseteq C$. Thus for all $\beta < \gamma$,
\[
\pi_i \left( \bigvee \{q_\alpha : \alpha < \gamma\} \right) = \bigwedge \{p \in B : i(p) \geq \bigvee \{q_\alpha : \alpha < \gamma\}\}
\geq \bigwedge \{p \in B : i(p) \geq q_\beta\} = \pi_i(q_\beta).
\]
and we obtain the first inequality: $\pi_i(\bigvee A) \geq \bigvee \pi_i[A]$.

Let $p = \bigvee \pi_i[A]$, so that $p \geq \pi_i(q_\alpha)$ for all $\alpha < \gamma$. Thus for all $\beta < \gamma$:
\[
i(p) \geq i \circ \pi_i(q_\beta) \geq q_\beta.
\]
In particular \( i(p) \geq \bigvee \{ q_\alpha : \alpha < \gamma \} \). By definition, \( \pi_i \) is increasing then
\[
p = \pi_i(i(p)) \geq \pi \left( \bigvee \{ q_\alpha : \alpha < \gamma \} \right),
\]
that is, the second inequality \( \bigvee \pi_i[A] \geq \pi_i(\bigvee A) \) holds.

4. Let \( q \in C \) be such that \( \pi_i(q) \leq p \). Since \( i \) is order preserving, \( q \leq i(\pi_i(q)) \leq i(p) \). Thus,
\[
\bigvee \{ q : \forall \alpha \in i^{-1}(q) \leq i(p) \} \leq i(p).
\]
In order to prove the other inequality; recall that \( p = \pi_i(i(p)) \). So,
\[
i(p) \leq \bigvee \{ q : \pi_i(q) \leq \pi_i(i(p)) \} = \bigvee \{ q : \pi_i(q) \leq p \}.
\]

5. For \( p \in B, q \in C \), the following three equations hold:
\[
\begin{align*}
\pi_i(q \land i(p)) \lor \pi_i(q \land \lnot i(p)) &= \pi_i(q); \\
(\pi_i(q) \land p) \lor (\pi_i(q) \land \lnot p) &= \pi_i(q); \\
(\pi_i(q) \land p) \land (\pi_i(q) \land \lnot p) &= 0.
\end{align*}
\]
Furthermore, by definition of \( \pi_i \) we have:
\[
\pi_i(q \land \lnot i(p)) = \pi_i(q \land \lnot p) \leq \pi_i(q) \land \lnot p. \tag{2.4}
\]
By (2.4), (2.5), and (2.3) we get
\[
\pi_i(q \land i(p)) \land \pi_i(q \land \lnot i(p)) = (\pi_i(q) \land p) \land (\pi_i(q) \land \lnot p) = 0.
\]
Moreover, by (2.1) and (2.2),
\[
\pi_i(q \land i(p)) \lor \pi_i(q \land \lnot i(p)) = (\pi_i(q) \land p) \lor (\pi_i(q) \land \lnot p).
\]
We conclude that
\[
\pi_i(q \land i(p)) = \pi_i(q) \land p \text{ and } \pi_i(q \land \lnot i(p)) = \pi_i(q) \land \lnot p.
\]

6. Suppose that \( i : B \to C \) is not surjective, and pick \( q \in C \setminus i[B] \). Then \( i(\pi_i(q)) \neq q \) hence \( i(\pi_i(q)) > q \). Put \( r = i(\pi_i(q)) \land \lnot q > 0 \). Then \( \pi_i(r) > 0 \) and
\[
\pi_i(q) \lor \pi_i(r) = \pi_i(q \lor r) = \pi_i(i(\pi_i(q))) = \pi_i(q).
\]
thus \( \pi_i(r) \land \pi_i(q) = \pi_i(r) > 0 \). Since \( \pi_i(r \land q) = \pi_i(0) = 0 \), \( \pi_i \) does not preserve meets. Furthermore, \( \pi_i \) preserves joins thus it must be the case that it does not preserve complements. In addition, for all \( r, q \in C \),
\[
\pi_i(r \land q) \leq \pi_i(r \land i(\pi_i(q))) = \pi_i(r) \land \pi_i(q)
\]
and \( \lnot \pi_i(r) \leq \pi_i(\lnot r) \), since
\[
\lnot \pi_i(r) \land \lnot \pi_i(\lnot r) = \lnot(\pi_i(r) \lor \pi_i(\lnot r)) = \lnot(\pi_i(r \lor \lnot r)) = \lnot(\pi_i(1)) = 0. \tag{\square}
\]
Complete homomorphisms and regular embeddings are the boolean algebraic counterpart of two-step iterations. This will be spelled out in detail in Section 2.3.

We now outline the relation existing between generic extensions by \( \mathbb{B} \) and \( \mathbb{C} \) in case there is a complete homomorphism \( i : \mathbb{B} \to \mathbb{C} \).

**Lemma 2.1.5.** Let \( i : \mathbb{B} \to \mathbb{C} \) be a regular embedding, \( D \subset \mathbb{B} \), \( E \subset \mathbb{C} \) be predense sets. Then \( i[D] \) and \( \pi_i[E] \) are predense (i.e. predense subsets are mapped into predense subsets). Moreover \( \pi_i \) maps \( \mathbb{V} \)-generic filters to \( \mathbb{V} \)-generic filters.

**Proof.** First, let \( p \in \mathbb{C} \) be arbitrary. Since \( D \) is predense, there exists \( q \in D \) such that \( q \land \pi(p) > 0 \). Then by Property 2.1.4(5) also \( i(q) \land p > 0 \) hence \( i[D] \) is predense. Furthermore, let \( p \in \mathbb{B} \) be arbitrary. Since \( E \) is predense, there exists \( q \in E \) such that \( q \land i(p) > 0 \). Then by Property 2.1.4(6) also \( \pi_i(q) \land p > 0 \) hence \( \pi_i[E] \) is predense.

For the last point in the lemma, we first prove that \( \pi_i[G] \) is a filter whenever \( G \) is a filter. Let \( p \in G \), and suppose \( q > \pi_i(p) \). Then by Property 2.1.4(2) also \( i(q) > i(\pi_i(p)) \geq p \), hence \( i(q) \in G \) and \( q \in \pi_i[G] \), proving that \( \pi_i[G] \) is upward closed. Now suppose \( p,q \in G \), then by Property 2.1.4(4) we have that \( \pi_i(p) \land \pi_i(q) \geq \pi_i(p \land q) \in \pi_i[G] \) since \( p \land q \in G \). Combined with the fact that \( \pi_i[G] \) is upward closed this concludes the proof that \( \pi_i[G] \) is a filter.

Finally, let \( D \) be a predense subset of \( \mathbb{B} \) and assume \( G \) is \( \mathbb{V} \)-generic for \( \mathbb{C} \). We have that \( i[D] \) is predense hence \( i[D] \cap G \not= \emptyset \) by \( \mathbb{V} \)-genericity of \( G \). Fix \( p \in i[D] \cap G \), then \( \pi_i(p) \in D \cap \pi_i[G] \) concluding the proof. \( \square \)

**Lemma 2.1.6.** Let \( i : \mathbb{B} \to \mathbb{C} \) be an homomorphism of boolean algebras. Then \( i \) is a complete homomorphism iff for every \( \mathbb{V} \)-generic \( G \) for \( \mathbb{C} \), \( i^{-1}[G] \) is a \( \mathbb{V} \)-generic filter for \( \mathbb{B} \).

**Proof.** If \( i \) is a complete homomorphism and \( G \) is a \( \mathbb{V} \)-generic filter, then \( i^{-1}[G] \) is trivially a filter. Furthermore, given \( D \) dense subset of \( \mathbb{B} \), \( i[D] \) is predense so there exists a \( p \in G \cap i[D] \), hence \( i^{-1}(p) \in i^{-1}[G] \cap D \).

Conversely, suppose by contradiction that there exists an \( A \subset \mathbb{B} \) such that \( i(\lor A) \not= \lor i[A] \) (in particular, necessarily \( i(\lor A) > \lor i[A] \)). Let \( p = i(\lor A) \setminus \lor i[A] \), \( G \) be a \( \mathbb{V} \)-generic filter with \( p \in G \). Then \( i^{-1}[G] \cap A = \emptyset \) hence is not \( \mathbb{V} \)-generic below \( \lor A \in i^{-1}[G] \), a contradiction. \( \square \)

Later in this chapter we shall use the following lemma to produce local versions of various results.

**Lemma 2.1.7 (Restriction).** Let \( i : \mathbb{B} \to \mathbb{C} \) be a regular embedding, \( q \in \mathbb{C} \), then

\[
i_q : \mathbb{B} \cap i(q) \to \mathbb{C} \ni q
\]

is a regular embedding and its associated retraction is \( \pi_{i_q} = \pi_i \cap (\mathbb{C} \ni q) \).

**Proof.** First suppose that \( i_q(p) = 0 \), then by Proposition 2.1.4(5),

\[
0 = \pi_i(i_q(p)) = \pi_i(i(p) \land q) = p \land \pi_i(q) = p
\]

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that ensures the regularity of \( i_q \). Furthermore for any \( r \leq q \),
\[
\pi_{i_q}(r) = \bigwedge \{ p \leq \pi_i(q) : i(p) \land q \geq r \} \\
= \bigwedge \{ p \leq \pi_i(q) : i(p) \geq r \} \}
\]
concluding the proof.

### 2.1.1 Embeddings and boolean valued models

Complete homomorphisms of complete boolean algebras induce natural \( \Delta_1 \)-elementary maps between the corresponding boolean valued models.

**Proposition 2.1.8.** Let \( i : B \rightarrow C \) be a complete homomorphism, and define by recursion \( i : V^B \rightarrow V^C \) as
\[
i(\hat{x}) = \{ (i(\hat{y}), i(\hat{z}(\hat{y}))) : \hat{y} \in \operatorname{dom}(\hat{x}) \}
\]
Then the map \( i \) is \( \Delta_1 \)-elementary, i.e. for every \( \Delta_1 \)-formula \( \phi \),
\[
i(\|\phi(\hat{x}, \ldots, \hat{z}_n)\|_B) = \|i(\hat{x}_1), \ldots, i(\hat{x}_n)\|_C
\]

**Proof.** We prove the result by induction on the complexity of \( \phi \). For atomic formulas \( \psi \) (either \( x = y \) or \( x \in y \)), we proceed by further induction on the rank of \( \hat{x}_1, \hat{x}_2 \).

\[
i(\|\hat{x}_1 \in \hat{x}_2\|_B) = i(\bigvee \{ i(\hat{x}_2(\hat{y})) = \|\hat{x}_1 = \hat{y}\|_B : \hat{y} \in \operatorname{dom}(\hat{x}_2) \})
\]
\[
= \bigvee \{ i(\hat{x}_2(\hat{y})) \land i(\|\hat{x}_1 = \hat{y}\|_B) : \hat{y} \in \operatorname{dom}(\hat{x}_2) \}
\]
\[
= \bigvee \{ i(\hat{x}_1) \land i(\|\hat{y}\|_C : \hat{y} \in \operatorname{dom}(\hat{x}_2)) \}
\]
\[
i(\|\hat{x}_1 \subseteq \hat{x}_2\|_B) = i(\bigwedge \{ i(\hat{x}_1(\hat{y})) \rightarrow \|\hat{y}\|_B \land i(\|\hat{x}_1\|_B) : \hat{y} \in \operatorname{dom}(\hat{x}_1) \})
\]
\[
= \bigwedge \{ i(\hat{x}_1(\hat{y})) \rightarrow i(\|\hat{y}\|_C : \hat{y} \in \operatorname{dom}(\hat{x}_1)) \}
\]
\[
= \bigwedge \{ i(\hat{x}_1) \land i(\|\hat{y}\|_C : \hat{y} \in \operatorname{dom}(\hat{x}_1)) \}
\]

We used the inductive hypothesis in the last row of each case. Since \( \|\hat{x}_1 = \hat{x}_2\| = \|\hat{x}_1 \subseteq \hat{x}_2\| \land \|\hat{x}_2 \subseteq \hat{x}_1\| \), the proof for \( \psi \) atomic is complete.

For \( \psi \) quantifier-free formula the proof is immediate since \( i \) is an embedding hence preserves \( \lor, \land \). Suppose now that \( \psi = \exists x \in y \phi \) is a \( \Delta_0 \)-formula.

\[
i(\|\exists x \in \hat{x}_1 \phi(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)\|_B)
\]
\[
= \bigvee \{ i(\hat{x}_1(\hat{y})) \land i(\|\phi(\hat{y}, \hat{x}_1, \ldots, \hat{x}_n)\|_B) : \hat{y} \in \operatorname{dom}(\hat{x}_1) \}
\]
\[
= \bigvee \{ i(\hat{x}_1(\hat{y})) \land i(\|\phi(\hat{y}, \hat{x}_1, \ldots, \hat{x}_n)\|_C) : \hat{y} \in \operatorname{dom}(\hat{x}_1) \}
\]
\[
= \|\exists x \in i(\hat{x}_1) \phi(x, i(\hat{x}_1), \ldots, i(\hat{x}_n))\|_C
\]
Furthermore, if $\psi = \exists x \phi$ is a $\Sigma_1$-formula, by the fullness lemma there exists a
$\hat{y} \in V^B$ such that $\mathcal{B}[\exists x \phi(x, \hat{x}_1, \ldots, \hat{x}_n)] = \mathcal{B}[\phi(\hat{y}, \hat{x}_1, \ldots, \hat{x}_n)]$ hence

$$i(\mathcal{B}[\exists x \phi(x, \hat{x}_1, \ldots, \hat{x}_n)]) = i(\mathcal{B}[\phi(\hat{y}, \hat{x}_1, \ldots, \hat{x}_n)])$$

$$= \mathcal{C}[\phi(i(\hat{y}), i(\hat{x}_1), \ldots, i(\hat{x}_n))]$$

$$\leq \mathcal{C}[\exists x \phi(x, i(\hat{x}_1), \ldots, i(\hat{x}_n))].$$

Thus, if $\phi$ is a $\Delta_1$-formula, either $\phi$ and $\neg \phi$ are $\Sigma_1$ hence the above inequality holds
and also

$$i(\mathcal{B}[\exists x \phi(x, \hat{x}_1, \ldots, \hat{x}_n)]) = \neg i(\mathcal{B}[\neg \phi(\hat{x}_1, \ldots, \hat{x}_n)])$$

$$\geq \neg \mathcal{C}[\phi(i(\hat{x}_1), \ldots, i(\hat{x}_n))]$$

$$= \mathcal{C}[\phi(i(\hat{y}), i(\hat{x}_1), \ldots, i(\hat{x}_n))],$$

concluding the proof. \hfill \qed

## 2.2 Iteration systems

In this section we present iteration systems and some of their algebraic properties. We defer to later sections an analysis of their forcing properties. In order to develop the theory of iterations, from now on we shall consider only regular embeddings.

**Definition 2.2.1.** $\mathcal{F} = \{i_{\alpha \beta} : B_\alpha \to B_\beta : \alpha \leq \beta < \lambda\}$ is a complete iteration system
of complete boolean algebras iff for all $\alpha \leq \beta \leq \xi < \lambda$:

1. $B_\alpha$ is a complete boolean algebra and $i_{\alpha \alpha}$ is the identity on it;

2. $i_{\alpha \beta}$ is a regular embedding with associated retraction $\pi_{\alpha \beta}$;

3. $i_{\beta \xi} \circ i_{\alpha \beta} = i_{\alpha \xi}$.

If $\xi < \lambda$, we define $\mathcal{F}|\xi = \{i_{\alpha \beta} : \alpha \leq \beta < \xi\}$.

**Definition 2.2.2.** Let $\mathcal{F}$ be a complete iteration system of length $\lambda$. Then:

- The inverse limit of the iteration is
  \[
  \lim_{\leftarrow} \mathcal{F} = \left\{ s \in \prod_{\alpha < \lambda} B_\alpha : \forall \alpha \forall \beta > \alpha \pi_{\alpha \beta}(s(\beta)) = s(\alpha) \right\}
  \]
  and its elements are called *threads*.

- The direct limit is
  \[
  \lim_{\rightarrow} \mathcal{F} = \{ s \in \lim_{\rightarrow} \mathcal{F} : \exists \alpha \forall \beta > \alpha s(\beta) = i_{\alpha \beta}(s(\alpha)) \}
  \]
  and its elements are called *constant threads*. The support of a constant thread $\text{supp}(s)$ is the least $\alpha$ such that $i_{\alpha \beta} \circ s(\alpha) = s(\beta)$ for all $\beta \geq \alpha$.

- The revised countable support limit is
  \[
  \lim_{\text{rcs}} \mathcal{F} = \{ s \in \lim_{\rightarrow} \mathcal{F} : s \in \lim_{\rightarrow} \mathcal{F} \lor \exists \alpha s(\alpha) \models B_\alpha \text{ cof}(\lambda) = \omega \}.
  \]
Definition 2.2.3. Let $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ be an iteration system. We say that $\mathcal{F}$ is a $<\kappa$-support iteration iff $\lim\downarrow_{\mathcal{F}|\alpha}$ whenever $\text{cof}(\alpha) \geq \kappa$, and $\lim\uparrow_{\mathcal{F}|\alpha}$ otherwise. We say that $\mathcal{F}$ is a revised countable support (RCS) iteration iff $\lim_{\text{rcs}}\downarrow_{\mathcal{F}|\alpha}$ for all $\alpha < \lambda$.

Every thread in $\lim\downarrow_{\mathcal{F}}$ is completely determined by its tail, while every thread in $\lim\uparrow_{\mathcal{F}}$ is entirely determined by the restriction to its support (and in particular by $s(\text{supp}(s))$). Notice that $\lim\downarrow_{\mathcal{F}} \subseteq \lim_{\text{rcs}}\downarrow_{\mathcal{F}} \subseteq \lim\uparrow_{\mathcal{F}}$ are partial orders with the order relation given by pointwise comparison of threads. Furthermore, if $\lambda$ is singular and $f : \text{cof}(\lambda) \to \lambda$ is an increasing cofinal map, we have the following isomorphisms of partial orders:

$$\lim\downarrow_{\mathcal{F}} \cong \lim\uparrow_{\text{rcs}}\downarrow_{\mathcal{F}} \cong \lim\downarrow_{\text{rcs}}\downarrow_{\mathcal{F}} \cong \lim\uparrow_{\mathcal{F}}.$$

Thus we can always assume without loss of generality that $\lambda$ is a regular cardinal.

The limits of an iteration system previously defined are closely related with the elements of the iteration system, as shown by the following definition and remarks.

Definition 2.2.4. Let $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ be an iteration system. For all $\alpha < \lambda$, we define $i_{\alpha\lambda}$ as

$$i_{\alpha\lambda} : \mathbb{B}_\alpha \to \lim\downarrow_{\mathcal{F}} \quad p \mapsto (\pi_{\beta\alpha}(p) : \beta < \alpha)^\uparrow (i_{\alpha\beta}(p) : \alpha \leq \beta < \lambda)$$

and $\pi_{\alpha\lambda}$ as

$$\pi_{\alpha\lambda} : \lim\uparrow_{\mathcal{F}} \to \mathbb{B}_\alpha \quad s \mapsto s(\alpha)$$

When it is clear from the context, we will denote $i_{\alpha\lambda}$ by $i_{\alpha}$ and $\pi_{\alpha\lambda}$ by $\pi_{\alpha}$.

The maps $i_{\alpha\lambda}$ naturally extend the iteration system to $\lambda$, as the following diagram commutes:

Furthermore, these maps can naturally be seen as regular embeddings of $\mathbb{B}_\alpha$ in any of $\text{RO}(\lim\downarrow_{\mathcal{F}})$, $\text{RO}(\lim\uparrow_{\mathcal{F}})$, $\text{RO}(\lim_{\text{rcs}}\downarrow_{\mathcal{F}})$. Moreover by Property 2.1.4.(3) in all three cases $\pi_{\alpha\lambda} = \pi_{i_{\alpha\lambda}|P}$ where $P = \lim_{\text{rcs}}\downarrow_{\mathcal{F}}$, $\lim\downarrow_{\mathcal{F}}$, $\lim\uparrow_{\mathcal{F}}$.

2.2.1 Boolean algebra operations on iteration system limits

We can now equip the different limits of iteration systems with boolean algebras operations.

Definition 2.2.5. $\lim\downarrow_{\mathcal{F}}$ inherits the structure of a boolean algebra with boolean operations defined as follows:
• \( s_1 \land s_2 \) is the unique thread \( s \) whose support \( \beta \) is the maximum of the supports of \( s_1 \) and \( s_2 \) and such that \( s(\beta) = s_1(\beta) \land s_2(\beta) \),

• \( \neg s \) is the unique thread \( t \) whose support \( \beta \) is the support of \( s \) and such that \( t(\beta) = \neg s(\beta) \).

**Definition 2.2.6.** Let \( A \) be any subset of \( \lim \mathcal{F} \). We define the **pointwise supremum** of \( A \) as

\[
\bigvee A = \{ s(\alpha) : s \in A \mid \alpha < \lambda \}.
\]

The previous definition makes sense since by Proposition 2.1.4.3, \( \bigvee A \) is a thread. It must be noted that if \( A \) is an infinite subset of \( \lim \mathcal{F} \), \( \bigvee A \) might not be the least upper bound of \( A \) in \( \text{RO}(\lim \mathcal{F}) \). A sufficient condition on \( A \) for this to happen is given by Lemma 2.2.8 below.

If \( s \in \lim \mathcal{F} \) and \( t \in \lim \mathcal{F} \) we can check that \( s \land t \), defined as the thread where eventually all coordinates \( \alpha \) are the pointwise meet of \( s(\alpha) \) and \( t(\alpha) \), is the infimum of \( s \) and \( t \) in \( \lim \mathcal{F} \). There can be nonetheless two distinct incompatible threads \( s, t \in \lim \mathcal{F} \) such that \( s(\alpha) \land t(\alpha) > 0 \) for all \( \alpha < \lambda \). Thus in general the pointwise meet of two threads could not even be a thread.

In general \( \lim \mathcal{F} \) is not complete and \( \text{RO}(\lim \mathcal{F}) \) cannot be identified with a complete subalgebra of \( \text{RO}(\lim \mathcal{F}) \) (i.e. \( \lim \mathcal{F} \) and \( \lim \mathcal{F} \) as forcing notions in general share little in common). However, \( \text{RO}(\lim \mathcal{F}) \) can be identified with a subalgebra of \( \lim \mathcal{F} \) that is complete (even though it is not a complete subalgebra), as shown in the following proposition.

**Proposition 2.2.7.** Let \( \mathcal{F} = \{ i_{\alpha \beta} : \alpha \leq \beta < \lambda \} \) be an iteration system. Then \( \text{RO}(\lim \mathcal{F}) \simeq D = \{ s \in \lim \mathcal{F} : s(\alpha) = \bigwedge \{ t \in \lim \mathcal{F} : t(\alpha) \leq s(\alpha) \} \} \).

*Proof.* The isomorphism associates to a regular open \( U \in \text{RO}(\lim \mathcal{F}) \) the thread \( k(U) = \bigvee U \), with inverse \( k^{-1}(s) = \{ t \in \lim \mathcal{F} : t(\alpha) \leq s(\alpha) \} \).

First, we prove that \( k^{-1} \circ k(U) = \{ t \in \lim \mathcal{F} : t \leq U \} = U \). Since \( \bigvee U > \bigwedge U \), it follows that \( U \subseteq k^{-1} \circ k(U) \). Furthermore, since \( U \) is a regular open set, if \( t \notin U \), there exists a \( t' \leq t \) that is in the interior of the complement of \( U \) (i.e., \( \forall t'' \leq t' \ t'' \notin U \)). So suppose towards a contradiction that there exist a \( t \leq \bigwedge U \) as above (i.e., \( \forall t' \leq t \ t' \notin U \)). Let \( \alpha \) be the support of \( t \), so that \( t(\alpha) \leq \bigwedge \{ s(\alpha) : s \in U \} \). Then, there exists an \( s \in U \) such that \( s(\alpha) \) is compatible with \( t(\alpha) \), hence \( s \land t > 0 \) and is in \( U \) (since \( U \) is open). Since \( s \land t \leq t \), this is a contradiction.

It follows that \( k(U) \in D \) for every \( U \in \text{RO}(\lim \mathcal{F}) \). Moreover, \( k^{-1}(s) \) is in \( \text{RO}(\lim \mathcal{F}) \) (i.e., is regular open). In fact, it is open and if \( t \notin k^{-1}(s) \) then \( t \notin s \) and this is witnessed by some \( \alpha > \text{supp}(t) \), so that \( t(\alpha) \notin s(\alpha) \). Let \( t' = i_{\alpha}(t(\alpha) \setminus s(\alpha)) \), then for all \( t'' \leq t' \), \( t''(\alpha) \perp s(\alpha) \) hence \( t'' \notin s \), thus \( k^{-1}(s) \) is regular.

Furthermore, \( k^{-1} \) is the inverse map of \( k \) since we already verified that \( k^{-1} \circ k(U) = U \) and for all \( s \in D \), \( k \circ k^{-1}(s) = s \) by definition of \( D \). Finally, \( k \) and \( k^{-1} \) are order-preserving maps since \( U_1 \subseteq U_2 \) iff \( \bigvee U_1 \leq \bigvee U_2 \).

As noted before, the notion of supremum in \( \lim \mathcal{F} \) may not coincide with the notion of pointwise supremum. However, it is possible to give a sufficient condition for this to happen, as in the following proposition.
Proposition 2.2.8. Let \( \mathcal{F} = \{ i_{\alpha \beta} : \alpha \leq \beta < \lambda \} \) be an iteration system and \( A \subseteq \varprojlim \mathcal{F} \) be an antichain such that \( \pi_{\alpha \lambda}[A] \) is an antichain for some \( \alpha < \lambda \). Then \( \bigvee A \) is the supremum of the elements of \( A \) in \( RO(\varprojlim \mathcal{F}) \).

Proof. Suppose by contradiction that \( \bigvee A < \bigvee A \) in \( RO(\varprojlim \mathcal{F}) \). Then there exists \( t \in \varprojlim \mathcal{F} \) such that \( 0 < t \leq \neg \bigvee A \wedge \bigvee A \). Let \( \alpha < \lambda \) be such that \( \pi_{\alpha \lambda}[A] \) is an antichain and let \( s \in A \) be such that \( s(\alpha) \) is compatible with \( t(\alpha) \). Such an \( s \) exists because \( t(\alpha) \leq \bigvee \{ s(\alpha) : s \in A \} \) and \( t(\alpha) > 0 \). We now prove that \( s \) and \( t \) are compatible. Consider

\[
t' = (t(\beta) \land i_{\alpha \beta}(s(\alpha))) : \alpha \leq \beta < \lambda.
\]

Then \( t' \leq t \) and it is a thread in \( \varprojlim \mathcal{F} \). In fact, since \( i_{\alpha \beta} = i_{\xi, \beta} \circ i_{\alpha, \xi} \) for each \( \alpha \leq \xi \leq \beta < \lambda \),

\[
\pi_{\xi, \beta}(t'(\beta)) = \pi_{\xi, \beta}(t(\beta) \land i_{\alpha, \beta} \circ s(\alpha)) = \pi_{\xi, \beta}(t(\beta)) \land i_{\alpha, \xi}(s(\alpha)) = t'(\xi).
\]

It only remains to prove that \( t'(\beta) \leq s(\beta) \) for each \( \beta \geq \alpha \). Notice that \( t'(\beta) \leq t(\beta) \leq \sup \{ s'(\beta) : s' \in A \} \), \( t'(\beta) > 0 \) and \( t'(\beta) \) is incompatible with \( s'(\beta) \) for all \( s' \neq s \) in \( A \). Thus

\[
t'(\beta) \leq \bigvee \{ s'(\beta) : s' \in A \} \land \left( \neg \bigvee \{ s'(\beta) : s' \in A, s' \neq s \} \right) = s(\beta)
\]

for all \( \beta \geq \alpha \), hence \( t \) and \( s \) are compatible. Since \( s, t \in A \), we get a contradiction. \( \square \)

2.2.2 Relation between inverse and direct limits

Even though in general \( \varprojlim \mathcal{F} \) is different from \( \varinjlim \mathcal{F} \), in certain cases they happen to coincide.

Lemma 2.2.9. Let \( \mathcal{F} = \{ i_{\alpha \beta} : \alpha \leq \beta < \lambda \} \) be an iteration system such that \( \varinjlim \mathcal{F} \) is \( < \lambda \)-cc\(^1\). Then \( \varprojlim \mathcal{F} = \varinjlim \mathcal{F} \) is a complete boolean algebra.

Proof. First, since every element of \( RO(\varprojlim \mathcal{F}) \) is the supremum of an antichain in \( \varinjlim \mathcal{F} \), and since \( \varprojlim \mathcal{F} \) is \( < \lambda \)-cc and \( \lambda \) is regular, the supremum of such an antichain can be computed in some \( B_\alpha \) for \( \alpha < \lambda \) hence \( RO(\varprojlim \mathcal{F}) = \varinjlim \mathcal{F} \). Let \( s \) be in \( \varprojlim \mathcal{F} \setminus \varinjlim \mathcal{F} \). Since \( s \) is a non-constant thread, for all \( \alpha < \beta \) we have that \( i_{\alpha \beta}(s(\alpha)) \geq s(\beta) \) and for all \( \alpha \) there is an ordinal \( \beta_\alpha \) such that \( i_{\alpha \beta_\alpha}(s(\alpha)) > s(\beta_\alpha) \). By restricting to a subset of \( \lambda \) w.l.o.g. we can suppose that \( s(\beta) < i_{\alpha \beta}(s(\alpha)) \) for all \( \beta > \alpha \). Hence \( \{ i_{\alpha \lambda}(s(\alpha)) : \alpha < \lambda \} \) is a strictly descending sequence of length \( \lambda \) of elements in \( \varprojlim \mathcal{F} \). From a descending sequence we can always define an antichain in \( \varprojlim \mathcal{F} \) setting \( p_\alpha = i_{\alpha \lambda}(s(\alpha)) \land \neg i_{\alpha + 1 \lambda}(s(\alpha + 1)) \). Since \( \varprojlim \mathcal{F} \) is \( < \lambda \)-cc, this antichain has to be of size less than \( \lambda \) hence for coboundedly many \( \alpha, p_\alpha = 0 \) hence \( s(\alpha + 1) = i_{\alpha, \alpha + 1}(s(\alpha)) \) and \( s \in \varinjlim \mathcal{F} \), a contradiction. \( \square \)

Theorem 2.2.10 (Baumgartner). Let \( \mathcal{F} = \{ i_{\alpha \beta} : \alpha \leq \beta < \lambda \} \) be an iteration system such that \( B_\alpha \) is \( < \lambda \)-cc for all \( \alpha \) and \( S = \{ \alpha : B_\alpha \cong RO(\varprojlim \mathcal{F}[\alpha]) \} \) is stationary. Then \( \varprojlim \mathcal{F} \) is \( < \lambda \)-cc.

\(^1\)We refer to Definition 3.1.1 for a definition of \( < \lambda \)-cc boolean algebra.
Proof. Suppose by contradiction that there exists an antichain \( \langle s_\alpha : \alpha < \lambda \rangle \). Let \( f : \lambda \to \lambda \) be such that \( f(\alpha) > \alpha, \supp(s_\alpha) \). Let \( C \) be the club of closure points of \( f \) (i.e. such that for all \( \alpha \in C, f[\alpha] \subseteq f(\alpha) \)). Then we can define a regressive function

\[
g : S \to \lambda \\
\alpha \mapsto \min \{ \supp(s) : s \in (\lim \downarrow F \upharpoonright \alpha)^+, s < s_\alpha(\alpha) \}
\]

and a corresponding function \( h : S \to \lim \downarrow F \) such that \( \supp(h(\alpha)) = g(\alpha) \), \( h(\alpha)(\alpha) < s_\alpha(\alpha) \). By Fodor’s Lemma let \( \xi \in \lambda, T \subset S \) be stationary such that \( g[T] = \{ \xi \} \).

Since \( h[T \cap C] \) has size \( \lambda \) and \( B_\xi \) is \( <\lambda \)-cc, there are \( \alpha, \beta \in T \cap C \) such that \( h(\alpha) \wedge h(\beta) \geq p > 0 \) for some \( p \) with \( \supp(p) = \xi \). Moreover \( s_\alpha(\alpha) > p(\alpha) \) holds and the support of \( p \) is below \( \alpha \), so that \( s_\alpha \wedge p > 0 \). Furthermore, \( s_\beta(\beta) > p(\beta) \geq (s_\alpha \wedge p)(\beta) \) and the support of \( s_\alpha \wedge p \) is below \( \beta \), thus \( s_\beta \wedge s_\alpha \wedge p > 0 \) contradicting the hypothesis that \( \langle s_\alpha : \alpha < \lambda \rangle \) is an antichain.

Notice that for an iteration system \( F \) as above of regular length \( \delta > \lambda \), \( \lim \downarrow F \) is also \( <\lambda \)-cc by an easier regularity argument.

### 2.3 Two-step iterations and generic quotients

In the first part of this section we define the two-step iteration \( B \ast \dot{C} \) following [27, Chp. 16] and study the basic properties of the natural regular embedding of \( B \) into \( B \ast \dot{C} \) where \( \dot{C} \) is a \( B \)-name for a complete boolean algebra.

In the second part of this section we study the properties of generic quotients given by \( B \)-names \( C/i[\dot{G}] \) where \( i : B \to C \) is a complete homomorphism and show that if we have a commutative diagram of complete homomorphisms:

\[
\begin{array}{ccc}
B & \xrightarrow{i_0} & C_0 \\
\downarrow{i_1} & & \downarrow{k} \\
C_1
\end{array}
\]

and \( G \) is a \( V \)-generic filter for \( B \), then the map defined by \( k/G([p]_{i_0[\dot{C}]}) = [k(p)]_{i_1[\dot{C}]} \) is a complete homomorphism in \( V[G] \). We also show a converse of this property.

In the third part of this section we show that the two approaches are equivalent, that is, \( i : B \to C \) is a complete homomorphism iff \( C \) is isomorphic to \( B \ast C/i[\dot{G}_0] \) and prove a converse of the above factorization property when we start from \( B \)-names for regular embeddings \( k : \dot{C} \to \dot{D} \).

Finally, in the last part we apply the above results to analyze generic quotients of iteration systems.

All the results in this section can be generalized to complete (not necessarily injective) homomorphisms \( i \), by considering \( i|\operatorname{coker}(i) \), which is a regular embedding (see Definition 2.1.2).
2.3 Two-step iterations and generic quotients

2.3.1 Two-step iterations

We present two-step iterations following [27].

**Definition 2.3.1.** Let \( B \) be a complete boolean algebra, and \( \hat{C} \) be a \( B \)-name for a complete boolean algebra. We denote by \( B \ast \hat{C} \) the boolean algebra defined in \( V \) whose elements are the equivalence classes of \( B \)-names for elements of \( \hat{C} \) (i.e. \( \dot{p} \in V^B \) such that \( \dot{p} \in \hat{C} \) modulo the equivalence relation:

\[
\dot{p} \equiv \dot{q} \iff \|\dot{p} = \dot{q}\|_B = 1,
\]

with the following operations:

\[
\|\dot{p} \lor \dot{q}\|_B = \|\dot{r}\|_B \iff \|\dot{r} = \dot{p} \lor \dot{q}\|_B = 1;
\]

\[
\|\neg \dot{p}\|_B = \|\dot{r}\|_B \iff \|\dot{r} = \neg \dot{q}\|_B = 1.
\]

Literally speaking our definition of \( B \ast \hat{C} \) yields an object whose domain is a family of proper classes of \( B \)-names. By means of Scott’s trick we can arrange so that \( B \ast \hat{C} \) is indeed a set. We leave the details to the reader.

**Lemma 2.3.2.** Let \( B \) be a complete boolean algebra, and \( \hat{C} \) be a \( B \)-name for a complete boolean algebra. Then \( B \ast \hat{C} \) is a complete boolean algebra and the maps \( i_{B \ast \hat{C}}, \pi_{B \ast \hat{C}} \) defined as

\[
\begin{align*}
    i_{B \ast \hat{C}} &: B \to B \ast \hat{C} \\
    \pi_{B \ast \hat{C}} &: B \ast \hat{C} \to B
\end{align*}
\]

where \( \pi_p \in V^B \) is a \( B \)-name for an element of \( \hat{C} \) such that \( \|\pi_p = 1\|_B = p \) and \( \|\pi_p = 0\|_B = \neg p \), are a regular embedding with its associated retraction.

**Proof.** We leave to the reader to verify that \( B \ast \hat{C} \) is a boolean algebra. We can also check that

\[
\|\dot{p} \leq \dot{q}\|_B \iff \|\dot{p} \lor \dot{q}\|_B = 1 \iff \|\dot{p} \leq \dot{q}\|_B = 1.
\]

Observe that \( B \ast \hat{C} \) is complete: if \( \{\dot{p}_\alpha : \xi < \delta\} \subseteq B \ast \hat{C} \), let \( \dot{q} \) be such that \( \|\dot{q} = \bigvee \{\dot{p}_\xi : \xi < \delta\}\|_B = 1 \). Then \( \|\dot{q} \geq \bigvee \{\|\dot{p}_\xi\|_B : \xi < \delta\}\|_B = 1 \). Since for all \( \alpha < \delta \)

\[
\|\bigvee \{\dot{p}_\xi : \xi < \delta\}\|_B = 1.
\]

Moreover if \( \|\dot{r} \geq \dot{p}_\alpha\|_B = 1 \) for all \( \alpha < \delta \), then

\[
\bigwedge \{\|\dot{r} \geq \dot{p}_\alpha\|_B : \alpha < \delta\} = 1
\]

thus \( \|\dot{r} \geq \dot{q}\|_B = 1 \), hence \( \|\dot{r}\| \geq \|\dot{q}\| \), which gives that \( \|\dot{q}\| = \bigvee \{\|\dot{p}_\alpha\| : \alpha < \delta\} \).

We now prove that \( i_{B \ast \hat{C}} \) is a regular embedding and that \( \pi_{B \ast \hat{C}} \) is its associated retraction.

- First of all, a standard application of the mixing lemma to the maximal antichain \( \{p, \neg p\} \) and the family of \( B \)-names \( \{1, 0\} \) shows that for each \( p \in B \) there exists a unique \( \tau_p \in B \ast \hat{C} \) such that \( \|\tau_p = 1\|_B = p \) and \( \|\tau_p = 0\|_B = \neg p \).
• $i_{\mathcal{B}^{+}\mathcal{C}}$ preserves negation since $\lceil \neg\tau_p = \tau_{\neg_p}\rceil_B = 1$.

• $i_{\mathcal{B}^{+}\mathcal{C}}$ preserves joins. Consider $\{p_\alpha \in \mathcal{B} : \alpha < \delta\}$. We have that

$$\bigvee \tau_{p_\alpha} = 0 \equiv \bigwedge \tau_{p_\alpha} = 0 \equiv \bigwedge (\neg p_\alpha) = \tau(\bigvee p_\alpha).$$

Furthermore,

$$\bigvee \tau_{p_\alpha} = 1 \leq \bigvee \tau_{p_\alpha} > 0 \equiv \bigvee \tau_{p_\alpha} > 0 \equiv \bigvee \tau_{p_\alpha} = 1.$$  

then $\bigvee \tau_{p_\alpha} = 1 = \bigvee \tau_{p_\alpha} = 1 = \bigvee p_\alpha$. Thus,

$$i_{\mathcal{B}^{+}\mathcal{C}}(\bigvee p_\alpha) = \bigvee \tau_{p_\alpha} = \bigvee [i_{\mathcal{B}^{+}\mathcal{C}}(p_\alpha)].$$

• $i_{\mathcal{B}^{+}\mathcal{C}}$ is regular since $i_{\mathcal{B}^{+}\mathcal{C}}(p) = i_{\mathcal{B}^{+}\mathcal{C}}(p') \Rightarrow p = \tau_p = 1_B = \tau_{p'} = 1_B = p'$.

• Finally, we show that $\pi_{i_{\mathcal{B}^{+}\mathcal{C}}}(\hat{p}) = \lceil \hat{p} > 0 \rceil_B$. By definition of retraction associated to $i_{\mathcal{B}^{+}\mathcal{C}}$,

$$\pi_{i_{\mathcal{B}^{+}\mathcal{C}}}(\hat{p}) = \bigwedge \{q \in \mathcal{B} : i_{\mathcal{B}^{+}\mathcal{C}}(q) \geq \hat{p}\}.$$  

If $q$ is such that $i_{\mathcal{B}^{+}\mathcal{C}}(q) \geq \hat{p}$, then $\tau_q \geq \hat{p}$ hence

$$q = \lceil \tau_q = 1 \rceil_B = \lceil \tau_q > 0 \rceil_B \geq \lceil \hat{p} > 0 \rceil_B \wedge \lceil \tau_q \geq \hat{p} \rceil_B = \lceil \hat{p} > 0 \rceil_B,$$

so we have the first inequality $\pi_{i_{\mathcal{B}^{+}\mathcal{C}}}(\hat{p}) \geq \lceil \hat{p} > 0 \rceil_B$.

Let now $i_{\mathcal{B}^{+}\mathcal{C}}(\lceil \hat{p} > 0 \rceil_B) = \lceil \tau \rceil$. Then,

$$\neg \lceil \hat{p} = 0 \rceil_B = \lceil \hat{p} > 0 \rceil_B = \lceil \tau = 1 \rceil_B \leq \lceil \hat{p} \leq \tau \rceil_B$$

and $\lceil \hat{p} = 0 \rceil_B \leq \lceil \hat{p} \leq \tau \rceil_B$. If follows that $\lceil \hat{p} \leq \tau \rceil_B \geq \neg \lceil \hat{p} = 0 \rceil_B \lor \lceil \hat{p} = 0 \rceil_B = 1$, hence $\lceil \hat{p} \rceil \leq \lceil \tau \rceil = i_{\mathcal{B}^{+}\mathcal{C}}(\lceil \hat{p} > 0 \rceil_B)$. Thus, $\pi_{i_{\mathcal{B}^{+}\mathcal{C}}}(\lceil \hat{p} \rceil) \leq \lceil \hat{p} > 0 \rceil_B$ as was to be shown. $\square$

When clear from the context, we shall feel free to omit the subscripts in $i_{\mathcal{B}^{+}\mathcal{C}}$, $\pi_{\mathcal{B}^{+}\mathcal{C}}$. This definition is provably equivalent to Kunen’s two-step iteration of posets $\mathcal{Q}$, i.e. $\text{RO}(\mathcal{P} \ast \hat{\mathcal{Q}})$ is isomorphic to $\text{RO}(\mathcal{P}) \ast \text{RO}(\hat{\mathcal{Q}})$.

**Proposition 2.3.3.** $A = \{\hat{p}_\alpha : \alpha \in \lambda\}$ is a maximal antichain in $\mathcal{D} = \mathcal{B} \ast \mathcal{C}$ iff

$$\lceil \{\hat{p}_\alpha : \alpha \in \lambda\} \rceil_B = \lceil \hat{1} \rceil_B = 1.$$

**Proof.** It is sufficient to observe the following:

$$\lceil \hat{p}_\alpha \wedge \hat{p}_\beta \rceil_B = \hat{0} = 1 \iff \lceil \hat{p}_\alpha \rceil_B \wedge \lceil \hat{p}_\beta \rceil_B = \lceil \hat{0} \rceil_B,$$

$$\lceil \bigvee \hat{p}_\alpha \rceil_B = \hat{1} \iff \bigvee \lceil \hat{p}_\alpha \rceil_B = \bigvee \lceil \hat{p}_\alpha \rceil_B = \lceil \hat{1} \rceil_B.$$

$\square$
2.3 Two-step iterations and generic quotients

2.3.2 Generic quotients

We now outline the main definition and properties of generic quotients.

**Proposition 2.3.4.** Let \( i : \mathbb{B} \rightarrow \mathbb{C} \) be a regular embedding of complete boolean algebras and \( G \) be a \( V \)-generic filter for \( \mathbb{B} \). Then \( \mathbb{C}/G \), defined with abuse of notation as the quotient of \( \mathbb{C} \) with the filter generated by \( i[G] \), is a boolean algebra in \( V[G] \).

**Proof.** We have that

\[ V[G] \models \mathbb{C} \text{ is a boolean algebra and } i[G] \text{ generates a filter on } \mathbb{C}. \]

Thus \( \mathbb{C}/G \) is a boolean algebra in \( V[G] \) such that

- \( [p] = [q] \) if and only if \( p \triangle q = (p \setminus q) \lor (q \setminus p) \in i[G]^* \);
- \( [p] \lor [q] = [p \lor q] \);
- \( \neg [p] = [\neg p] \);

where \( i[G]^* \) is the dual ideal of the filter \( i[G] \). \( \square \)

**Lemma 2.3.5.** Let \( i : \mathbb{B} \rightarrow \mathbb{C} \) be a regular embedding, \( \dot{G} \) be the canonical name for a generic filter for \( \mathbb{B} \) and \( \dot{p} \) be a \( \mathbb{B} \)-name for an element of \( \mathbb{C}/\dot{G} \). Then there exists a unique \( q \in \mathbb{C} \) such that \( \left[ \dot{p} = [q]_{i[G]} \right]_\mathbb{B} = 1 \).

**Proof.** First, notice that the \( \mathbb{B} \)-name for the dual of the filter generated by \( i[\dot{G}] \) is \( \dot{I} = \{ \langle q, -\pi_i(q) \rangle : q \in \mathbb{C} \} \).

**Uniqueness.** Suppose that \( q_0, q_1 \) are such that \( \left[ \dot{p} = [q_k]_I \right]_\mathbb{B} = 1 \) for \( k < 2 \). Then \( \left[ [q_0]_I = [q_1]_I \right] = 1 \) hence \( \left[ q_0 \triangle q_1 \in \dot{I} \right]_\mathbb{B} = -\pi_i(q_0 \triangle q_1) = 1 \). This implies that \( \pi_i(q_0 \triangle q_1) = 0 \Rightarrow q_0 \triangle q_1 = 0 \Rightarrow q_0 = q_1 \).

**Existence.** Let \( A \subset \mathbb{B} \) be a maximal antichain deciding the value of \( \dot{p} \), and for every \( r \in A \) let \( q_r \) be such that \( r \Vdash \mathbb{B} \dot{p} = [q_r]_I \). Let \( q \in \mathbb{C} \) be such that \( q = \bigvee \{ i(r) \land q_r : r \in A \} \), so that

\[ \left[ [q]_I = [q_r]_I \right]_\mathbb{B} = \left[ q \triangle q_r \in \dot{I} \right]_\mathbb{B} = -\pi_i(q \triangle q_r) \geq -\pi_i(i(\neg r)) = r \]

since \( q \triangle q_r \leq \neg i(r) = i(\neg r) \). Thus,

\[ \left[ \dot{p} = [q]_I \right]_\mathbb{B} \geq \left[ \dot{p} = [q_r]_I \right]_\mathbb{B} \land \left[ [q]_I = [q_r]_I \right]_\mathbb{B} \geq r \land r = r. \]

The above inequality holds for any \( r \in A \), so \( \left[ \dot{p} = [q]_I \right]_\mathbb{B} \geq \bigvee A = 1 \) concluding the proof. \( \square \)

**Proposition 2.3.6.** Let \( i : \mathbb{B} \rightarrow \mathbb{C} \) be a regular embedding of complete boolean algebras and \( G \) be a \( V \)-generic filter for \( \mathbb{B} \). Then \( \mathbb{C}/G \) is a complete boolean algebra in \( V[G] \).
By Proposition 2.3.6 we only need to prove that $\mathbb{C}/G$ is complete. Let $\{\dot{p}_\alpha : \alpha < \delta\} \in V$ be a set of $\mathbb{B}$ names for elements of $\mathbb{C}/G$. Then, by Lemma 2.3.5 for each $\alpha < \delta$ there exists $q_\alpha \in \mathbb{C}$ such that

$$\left[ p_\alpha = [q_\alpha]_{i(G)} \right]_\mathbb{B} = 1.$$ 

We have that $\bigvee q_\alpha \in \mathbb{C}$, since $\mathbb{C}$ is complete. Let $r \in \mathbb{C}$ be such that $[r] \geq [q_\alpha]$ in $V[G]$ for all $\alpha < \delta$. Then,

$$\neg \pi(q_\alpha \land \neg r) = \left[ q_\alpha \land \neg r \in i(\dot{G})^* \right]_\mathbb{B} = \left[ [r] \geq [q_\alpha] \right]_\mathbb{B} \in G$$

So $\pi(q_\alpha \land \neg r) \notin G$ for all $\alpha < \delta$. In particular since $\{\pi(q_\alpha \land \neg r) : \alpha < \delta\} \in V$ is disjoint from $G$,

$$q = \bigvee \{\pi(q_\alpha \land \neg r) : \alpha < \delta\} = \pi(\neg r \land \bigvee \{q_\alpha : \alpha < \delta\}) \notin G.$$ 

Thus if $\pi(r) \in G$, $V[G] \models [r] \geq [\bigvee q_\alpha]$ while if $\pi(r) \notin G$, $\pi(\neg r) \in G$ and

$$\bigvee \{\pi(q_\alpha) : \alpha < \delta\} = \bigvee \{\pi(q_\alpha \land \neg r) : \alpha < \delta\} \lor \bigvee \{\pi(q_\alpha \land r) : \alpha < \delta\} \leq q \lor \pi(r) \notin G,$$

hence $[q_\alpha]$ and $[\bigvee \{q_\alpha : \alpha < \delta\}]$ are all equal to $0$. In either cases $[\bigvee \{q_\alpha : \alpha < \delta\}]$ is the least upper bound of the family $\{[q_\alpha] : \alpha < \delta\}$ in $V[G]$. This shows that $V[G] \models \mathbb{C}/G$ is complete for all $V$-generic filters $G$. \qed

The construction of generic quotients can be defined also for regular embeddings:

**Proposition 2.3.7.** Let $\mathbb{B}, \mathbb{C}_0, \mathbb{C}_1$ be complete boolean algebras, and let $G$ be a $V$-generic filter for $\mathbb{B}$. Let $i_0, i_1, k$ form a commutative diagram of regular embeddings as in the following picture:

$$\begin{array}{ccc}
\mathbb{B} & \xrightarrow{i_0} & \mathbb{C}_0 \\
\downarrow i_1 & & \downarrow k \\
\mathbb{C}_1
\end{array}$$

Then $k/G : \mathbb{C}_0/G \to \mathbb{C}_1/G$ defined by $k/G([p]_{i_0(G)}) = [k(p)]_{i_1(G)}$ is a well-defined regular embedding of complete boolean algebras in $V[G]$ with associated retraction $\pi$ such that $\pi([q]_{i_1(G)}) = [\pi_k(q)]_{i_0(G)}$.

**Proof.** By Proposition 2.3.6, $k/G$ is a map between complete boolean algebras. We can also verify that $k/G$ is:

**well-defined.** Suppose $[p]_{i_0(G)} = [q]_{i_0(G)}$. Then $p \Delta q \in i_0[G]^*$ and $k(p) \Delta k(q) = k(p \Delta q) \in i_1[G]^*$, so that $[k(p)]_{i_1[G]} = [k(q)]_{i_1[G]}$.

**complete homomorphism of boolean algebras.** By Proposition 2.3.6

$$k/G(\neg [p_\alpha]_{i_0(G)}) = k/G([\neg p_\alpha]_{i_0(G)}) = [k(\neg p_\alpha)]_{i_1(G)} = \neg [k(p_\alpha)]_{i_1(G)}$$
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and
\[ k/G \left( \bigvee [p_\alpha]_{i_0[G]} \right) = k/G \left( \bigvee p_\alpha \right)_{i_0[G]} = \bigvee k(p_\alpha)_{i_1[G]} = \bigvee [k(p_\alpha)]_{i_1[G]} \]

injective. Let \( p, q \in C_0 \) be such that \( k/G([p]_{i_0[G]}) = k/G([q]_{i_0[G]}) \). Then \( k(p \Delta q) \in i_1[G]^* \) and there exists \( r \not\in G \) such that \( k(p \Delta q) \leq i_1(r) = k(i_0(r)) \). Since \( k \) is injective, \( p \Delta q \in i_0[G]^* \).

projection is \( \pi([q]_{i_1[G]}) = [\pi_k(q)]_{i_0[G]} \). For any \( q \in C_1 \), by definition of projection
\[ \pi([q]_{i_1[G]}) = \bigwedge \{ [p]_{i_0[G]} : k/G([p]_{i_0[G]}) \geq [q]_{i_1[G]} \} \]

Given any \( p \in C_0 \),
\[ \left[ k/G([p]_{i_0[G]} \geq [q]_{i_1[G]} \right] = \left[ k(p) \right]_{i_1[G]} \geq [q]_{i_1[G]} \right] \]
\[ = [q \wedge \neg k(p) \in i_1[G]^*] \]
\[ = \neg \pi_{i_1}(q \wedge \neg k(p)) \]
\[ = \neg \pi_{i_0}(\pi_k(q \wedge k(-p))) = \neg \pi_{i_0}(\pi_k(q) \wedge \neg p) \]

thus \( k/G([p]_{i_0[G]} \geq [q]_{i_1[G]} \iff \pi_{i_0}(\pi_k(q) \wedge \neg p) \in G \) if \( \pi_k(q) \wedge \neg p \in i_0[G]^* \iff [p]_{i_0[G]} \geq [\pi_k(q)]_{i_0[G]} \). It follows that
\[ \pi([q]_{i_1[G]}) = \bigwedge \{ [p]_{i_0[G]} : [p]_{i_0[G]} \geq [\pi_k(q)]_{i_0[G]} \} = [\pi_k(q)]_{i_0[G]} \].

2.3.3 Equivalence of two-step iterations and regular embeddings

We are now ready to prove that two-step iterations and regular embeddings capture the same concept.

Theorem 2.3.8. Let \( i : B \rightarrow C \) be a regular embedding of complete boolean algebras, \( \hat{G} = \hat{G}_B \) be the canonical name for the \( V \)-generic filter for \( B \). Then \( B * C / i[G] \cong C \).

Proof. Define
\[ i^* : C \rightarrow B * C / i[G] \]
\[ p \mapsto [p]_{i[G]} \approx \]

By Proposition 2.3.6 and definition of two-step iteration,
\[ i^*(-p) = [\neg p]_{i[G]} \approx = \neg [p]_{i[G]} \approx = \neg i^*(p) \]
\[ i^*(\bigvee p_\alpha) = [\bigvee p_\alpha]_{i[G]} \approx = [\bigvee p_\alpha]_{i[G]} \approx = \bigvee i^*(p_\alpha) \]

Furthermore, given \( \hat{q} \) a \( B \)-name for an element of \( C/G \), by Lemma 2.3.5 there exists a unique \( p \in C \) such that \( [p]_{i[G]} = \hat{q} \approx \), that is, \( [p]_{i[G]} = [\hat{q}]_B \). Thus \( i^* \) is a bijection.
Proposition 2.3.9. Let \( \hat{C}_0, \hat{C}_1 \) be \( B \)-names for complete boolean algebras, and let \( \hat{k} \) be a \( B \)-name for a regular embedding from \( \hat{C}_0 \) to \( \hat{C}_1 \). Then there is a regular embedding \( i : B * \hat{C}_0 \to B * \hat{C}_1 \) such that

\[
\left[ k = i/G \right]_B = 1.
\]

Proof. Let \( \hat{G} = \hat{G}_B \). Define

\[
i : B * \hat{C}_0 \to B * \hat{C}_1
\]

\[
[p]_{\approx} \mapsto [k(p)]_{\approx}.
\]

Since \( k \) is a \( B \)-name for a regular embedding with boolean value 1,

\[
[p]_{\approx} = [q]_{\approx} \iff [k(p)]_{\approx} = [k(q)]_{\approx}
\]

hence \( i \) is well defined and injective. Furthermore, \( i \) is a complete homomorphism:

\[
i (\neg [p]_{\approx}) = i ([\neg p]_{\approx}) = [k (\neg p)]_{\approx}
\]

\[
= [\neg k (\neg p)]_{\approx} = [\neg k (p)]_{\approx} = \neg [k(p)]_{\approx},
\]

\[
i (\bigvee [p_\alpha]_{\approx}) = i \left( \left[ \bigvee p_\alpha \right]_{\approx} \right) = [k \left( \bigvee p_\alpha \right)]_{\approx}
\]

\[
= \bigvee [k(p_\alpha)]_{\approx} = \bigvee \left[ k(p_\alpha) \right]_{\approx} = \bigvee i ([p_\alpha]_{\approx}).
\]

Observe that the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{i_{\hat{C}_0}} & B * \hat{C}_0 \\
\downarrow{i_{\hat{C}_1}} & & \downarrow{i} \\
B & \xrightarrow{i} & B * \hat{C}_1
\end{array}
\]

commutes. Thus by Proposition 2.3.7

\[
\left[ i/G ([p]_{\approx}) \right]_{i_{\hat{C}_0} \{G\}} = \left[ i ([p]_{\approx}) \right]_{i_{\hat{C}_1} \{G\}} = \left[ \left[ k (\hat{p}) \right]_{\approx} \right]_{i_{\hat{C}_1} \{G\}} = 1
\]

hence \( \hat{k} = i/G \). \( \square \)

2.3.4 Generic quotients of iteration systems

The results on generic quotients of the previous sections generalize without much effort to iteration systems. In the following we outline how this occurs.

Lemma 2.3.10. Let \( F = \{ i_{\alpha \beta} : B_\alpha \to B_\beta : \alpha \leq \beta < \lambda \} \) be a complete iteration system of complete boolean algebras, \( G_\xi \) be a \( V \)-generic filter for \( B_\xi \). Then \( F/G_\xi = \{ i_{\alpha \beta}/G_\xi : \xi < \alpha \leq \beta < \lambda \} \) is a complete iteration system in \( V[G_\xi] \).
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Proof. Follows from Proposition 2.3.7.

**Lemma 2.3.11.** Let \( F = \{ i_{\alpha\beta} : \mathbb{B}_\alpha \to \mathbb{B}_\beta : \alpha \leq \beta < \lambda \} \) be a complete iteration system of complete boolean algebras, \( \dot{G}_\alpha \) be the canonical name for a generic filter for \( \mathbb{B}_\alpha \) and \( \dot{s} \) be a \( \mathbb{B}_\alpha \)-name for an element of \( \varprojlim F / \dot{G}_\alpha \). Then there exists a unique \( t \in \varprojlim F \) such that \( \dot{s} = [\dot{t}] \dot{G}_\alpha \) \( \mathbb{B}_\alpha = 1 \).

Proof. We proceed applying Lemma 2.3.5 at every stage \( \beta > \alpha \).

**Existence.** For every \( \beta > \alpha \), by hypothesis \( \dot{s}(\beta) \) is a name for an element of the quotient \( \mathbb{B}_\beta / i_{\alpha\beta}[\dot{G}_\alpha] \). Let \( t(\beta) \) be the unique element of \( \mathbb{B}_\beta \) such that
\[
\left[\dot{s}(\beta) = [\dot{t}(\beta)] i_{\alpha\beta}[\dot{G}_\alpha] \right]_{\mathbb{B}_\alpha} = 1.
\]
Then,
\[
\left[\dot{s} = [\dot{t}] \dot{G}_\alpha \right]_{\mathbb{B}_\alpha} = \left[ \forall \beta \in \lambda \dot{s}(\beta) = [\dot{t}(\beta)] \dot{G}_\alpha \right]_{\mathbb{B}_\alpha} = \left[ \dot{s}(\beta) = [\dot{t}(\beta)] i_{\alpha\beta}[\dot{G}_\alpha] : \beta \in \lambda \right] = \bigwedge 1 = 1.
\]
Furthermore, by Proposition 2.3.7 \( \pi_{\alpha\beta}(t(\beta)) = t(\alpha) \) for all \( \beta > \alpha \) hence \( t \) is a thread in \( \varprojlim F \).

**Uniqueness.** If \( t' \) is such that \( \left[\dot{s} = [\dot{t}] \dot{G}_\alpha \right]_{\mathbb{B}_\alpha} = 1 \) then for every \( \beta > \alpha \),
\[
\left[\dot{s}(\beta) = [\dot{t}(\beta)] i_{\alpha\beta}[\dot{G}_\alpha] \right]_{\mathbb{B}_\alpha} = 1.
\]
Such an element is unique by Lemma 2.3.5 hence \( t'(\beta) = t(\beta) \) defined above completing the proof.
2 Iteration systems
CHAPTER 3

FORCING AXIOMS

In the first part of this section we introduce the most common forcing classes, with a special focus on properness and semiproperness, and their corresponding forcing axioms. In the second part we introduce the definition of weakly iterable forcing class and outline its main properties. In this context we shall be able to explore how the forcing classes previously introduced behave with respect to the limit operations on iteration systems as introduced in Section 2.2. This will require to analyze them with respect to the following order.

Definition 3.0.1. Let \( \Gamma \) be a definable class of complete boolean algebras and let \( \mathbb{B}, \mathbb{C} \) be complete boolean algebras.

We say that \( \mathbb{B} \leq^*_\Gamma \mathbb{C} \) iff there is a complete homomorphism \( i : \mathbb{C} \to \mathbb{B} \) such that the quotient algebra \( \mathbb{B}/i[G_C] \) is in \( \Gamma \) with boolean value \( 1 \).

We say that \( \mathbb{B} \leq_{\Gamma} \mathbb{C} \) iff there is a regular embedding with the same properties as above.

Notice that we do not require neither \( \mathbb{B} \) nor \( \mathbb{C} \) to be in \( \Gamma \) when \( \mathbb{B} \leq^*_\Gamma \mathbb{C} \). In the final part of this section, we shall prove that SP and SSP forcings are weakly iterable. The other relevant cases (ccc, Axiom-A, proper) are left to the reader, since the corresponding proofs are either simpler or given by straightforward modifications of the arguments presented.

Section 3.1 introduces the main examples of forcing classes and their basic properties. Section 3.2 introduces the corresponding forcing axioms, using the terminology developed in Chapter 2. In Section 3.3 we introduce the definition of weakly iterable forcing class, outline its main consequences and argue that most relevant forcing classes fall into the scope of this definition. In Section 3.4 we apply the machinery developed so far to prove the preservation of semiproperness through two step iterations and revised countable support iterations. The proof of the latter splits in three cases according to the cofinality of the length of the iteration system \( (\omega, \omega_1, \text{bigger than} \omega_1) \) and mimics in this new setting the original proof of Shelah of these results. We then prove that stationary set preserving forcings are weakly iterable (provided there are class many supercompact cardinals), building on the results obtained for semiproper iterations.
3 Forcing axioms

3.1 Forcing classes

We list below the main forcing classes $\Gamma$ we shall analyze throughout this thesis.

**Definition 3.1.1.** A boolean algebra $B$ is $\kappa$-cc (satisfies the $\kappa$-chain condition) iff every antichain has size less than $\kappa$. $B$ is locally $\kappa$-cc iff there exists a maximal antichain $A \subseteq B$ such that $B \upharpoonright p$ is $\kappa$-cc for all $p \in A$. $B$ is ccc (locally ccc) iff it is $\omega_1$-cc (locally $\omega_1$-cc).

**Definition 3.1.2.** A poset $P$ is $\kappa$-closed iff every descending chain $\langle p_\alpha : \alpha < \gamma \rangle$ with $\gamma < \kappa$ has a lower bound in $P$.

**Definition 3.1.3.** A boolean algebra $B$ is $\kappa$-distributive iff every family $D$ of $\kappa$-open dense sets in $B$ has dense intersection $\bigcap D$.

For a fixed $\kappa$, any $\kappa$-cc non trivial boolean algebra $B$ is not $\kappa^+$-distributive and conversely. On the other hand, $\kappa$-distributivity is strongly related to $\kappa$-closure for posets: every $\kappa$-closed poset has a $\kappa$-distributive boolean completion, even though the converse does not hold. In addition, $\kappa$-closed posets and $\kappa$-distributive boolean algebras share their main forcing property (not adding subsets of $V$ of size less than $\kappa$).

**Definition 3.1.4.** A boolean algebra $B$ is axiom-A iff there exists a dense subset $D$ of $P$ and sequence $\langle \leq_n : n < \omega \rangle$ of partial orderings on $D$ such that the following hold:

i. $p \leq_0 q \implies p \leq q$ and $p \leq_{n+1} q \implies p \leq_n q$ for all $p, q \in D$, $n < \omega$;

ii. if $\langle p_n : n < \omega \rangle$ is such that $p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 \ldots$, then there exists a $q \in D$ such that $q \leq_n p_n$ for all $n < \omega$;

iii. given any $p \in D$, $n < \omega$ and $\dot{\alpha}$ name for an ordinal there exist a $q \leq_n p$ and a countable set $A$ such that $q \Vdash_B \dot{\alpha} \in \dot{A}$.

**Definition 3.1.5.** Let $B$ be a boolean algebra, $\theta > 2^{|B|}$ be a cardinal, $M \prec H_\theta$ such that $B \in M$. We say that a condition $p \in B$ is $M$-generic iff for every $\dot{\alpha} \in M \cap V^B$ name for an ordinal, $p \Vdash_B \dot{\alpha} \in \dot{M}$. We say that a condition $p \in B$ is $M$-semigeneric iff for every $\dot{\alpha} \in M \cap V^B$ name for a countable ordinal, $p \Vdash_B \dot{\alpha} \in \dot{M}$.

**Definition 3.1.6.** A boolean algebra $B$ is proper iff for every $M$ as in the previous definition and $q \in M \cap B$ there is an $M$-generic condition below $q$. Similarly, we say that $B$ is semiproper (SP) iff for every $M$ as in the previous definition and $q \in M \cap B$ there is an $M$-semigeneric condition below $q$.

Equivalently, a boolean algebra $B$ is proper iff it preserves stationary sets on $[\lambda]^\omega$ for any $\lambda$ uncountable cardinal (see [27, Thm. 31.7]). If we require the latter property to hold only for $\lambda = \omega_1$, we obtain the following larger forcing class.

**Definition 3.1.7.** A boolean algebra $B$ is stationary set preserving (in short, SSP) iff for every stationary set $S \subseteq \omega_1$, $1 \Vdash_B \dot{S}$ is stationary.
The following diagram summarizes the relations between the various forcing classes just introduced.

![Diagram showing relations between forcing classes]

### 3.1.1 Algebraic formulation of properness and semiproperness

The notions of properness and semiproperness can also be stated in a more algebraic fashion. This will later be used throughout this chapter in order to simplify the treatment of semiproper iterations.

**Definition 3.1.8.** Let $\mathcal{B}$ be a complete boolean algebra, $M \prec H_\theta$ for some $\theta > 2^{\mathcal{B}}$, $\text{PD}(\mathcal{B})$ be the collection of predense subsets of $\mathcal{B}$, and $\text{PD}_{\omega_1}(\mathcal{B})$ be $\text{PD}(\mathcal{B}) \cap [\mathcal{B}]^{\leq \omega_1}$.

The boolean value

$$
g(\mathcal{B}, M) = \bigwedge \left\{ \bigvee (D \cap M) : D \in \text{PD}(\mathcal{B}) \cap M \right\}
$$

is the *degree of genericity* of $M$ with respect to $\mathcal{B}$. The boolean value

$$
sg(\mathcal{B}, M) = \bigwedge \left\{ \bigvee (D \cap M) : D \in \text{PD}_{\omega_1}(\mathcal{B}) \cap M \right\}
$$

is the *degree of semigenericity* of $M$ with respect to $\mathcal{B}$.

The degree of semigenericity can be also calculated from maximal antichains, and behaves well with respect to the restriction operation.

**Proposition 3.1.9.** Let $\mathcal{B}$, $M$ be as in the previous definition, and let $\text{A}(\mathcal{B}) \subseteq \text{PD}(\mathcal{B})$, $\text{A}_{\omega_1}(\mathcal{B}) \subseteq \text{PD}_{\omega_1}(\mathcal{B})$ be the subcollections given by maximal antichains of $\mathcal{B}$. Then

$$
g(\mathcal{B}, M) = \bigwedge \left\{ \bigvee (A \cap M) : A \in \text{A}(\mathcal{B}) \cap M \right\}
$$

and

$$
sg(\mathcal{B}, M) = \bigwedge \left\{ \bigvee (A \cap M) : A \in \text{A}_{\omega_1}(\mathcal{B}) \cap M \right\}.
$$

**Proof.** Since $\text{A}(\mathcal{B}) \subseteq \text{PD}(\mathcal{B})$, the inequality

$$
g(\mathcal{B}, M) \leq \bigwedge \left\{ \bigvee (A \cap M) : A \in \text{A}(\mathcal{B}) \cap M \right\}
$$

is trivial. Conversely, if $D = \{b_\alpha : \alpha < \lambda\} \in \text{PD}(\mathcal{B}) \cap M$, define

$$
A_D = \left\{ a_\alpha = b_\alpha \land \neg \bigvee \{ b_\beta : \beta < \alpha \} : \alpha < \lambda \right\}.
$$

By elementarity, since $D \in M$ also $A_D$ is in $M$. It is straightforward to verify that $A_D$ is an antichain, and since $\bigvee A_D = \bigvee D = 1$ it is also maximal. Moreover,
Forcing axioms

\[ a_\alpha \leq b_\alpha \text{ we have that } \bigvee A \cap M \leq \bigvee D \cap M. \text{ Thus, for any } D \in \text{PD}(B) \cap M, \text{ we have that } \bigwedge \{ \bigvee (A \cap M) : A \in A(B) \cap M \} \leq \bigvee D \cap M \text{ hence} \]

\[ \bigwedge \{ \bigvee (A \cap M) : A \in A(B) \cap M \} \leq g(B, M) \]

and the same can be proved also for sg(B, M) following the same procedure. \qed

**Proposition 3.1.10.** Let \( B \) be a complete boolean algebra and \( M \prec H_\theta \) for some \( \theta > 2^{\aleph_1} \). Then for all \( p \in M \cap B \),

\[ g(B \upharpoonright p, M) = g(B, M) \wedge p \]

and

\[ \text{sg}(B \upharpoonright p, M) = \text{sg}(B, M) \wedge p. \]

**Proof.** Observe that if \( A \) is a maximal antichain in \( B \), then \( A \upharpoonright p = \{ q \wedge p : q \in A \} \) is a maximal antichain in \( B \upharpoonright p \). Moreover for each maximal antichain \( A_p \) in \( B \upharpoonright p \cap M \), \( A = A_p \cup \{ \neg p \} \) is a maximal antichain in \( B \cap M \). Therefore

\[ g(B, M) \wedge p = \bigwedge \bigvee (A \cap M) \wedge p = \bigwedge \bigvee (A \upharpoonright p) \cap M = g(B \upharpoonright p, M) \]

and the same holds for \( \text{sg}(B, M) \).

We are now ready to introduce the algebraic definition of semiproperness and properness for complete boolean algebras.

**Theorem 3.1.11.** Let \( B \) be a complete boolean algebra, \( \theta > 2^{\aleph_1} \) be a cardinal. Then \( B \) is proper iff for club many \( M \prec H_\theta \), \( g(B, M) \) is compatible with every \( p \in B \cap M \), and \( B \) is semiproper iff for club many \( M \prec H_\theta \), \( \text{sg}(B, M) \) is compatible with every \( p \in B \cap M \).

**Lemma 3.1.12.** Let \( B \) be a complete boolean algebra and fix \( M \prec H_\theta \). Then

\[ g(B, M) = \bigvee \{ p \in B : p \text{ is } M\text{-generic} \} \]

and

\[ \text{sg}(B, M) = \bigvee \{ p \in B : p \text{ is } M\text{-semigeneric} \}. \]

**Proof.** Given \( A = \{ p_\beta : \beta < \omega_1 \} \in A_{\omega_1}(B) \), define \( \dot{A}_A = \{ (\gamma, p_\beta) : \gamma < \beta < \omega_1 \} \). Then we can easily check that

\[ \| \dot{A}_A \prec \omega_1 \|_B = \bigvee \{ \| \dot{A}_A = \beta \|_B : \beta < \omega_1 \} = \bigvee \{ p_\beta : \beta < \omega_1 \} = 1. \]

Conversely, given \( \dot{A} \in V^B \cap M \) such that \( \| \dot{A} \prec \omega_1 \|_B = 1 \), define

\[ A_\dot{A} = \{ p_\beta = \| \dot{A} = \beta \|_B : \beta < \omega_1 \}. \]

Then we can easily check that \( A_\dot{A} \in A_{\omega_1}(B) \).
3.2 Forcing axioms

Suppose now that $p$ is an $M$-semigeneric condition, and fix an arbitrary $A \in A_{\omega_1}(\mathbb{B}) \cap M$. Then $\dot{\alpha}_A \in M$ and $\langle \dot{\alpha}_A, \omega_1 \rangle_B = 1$, hence

$$p \leq \left[ \dot{\alpha}_A < (M \cap \omega_1) \right]_B = \bigvee \{ \left[ \dot{\alpha}_A = \dot{\beta} \right]_B : \beta \in M \cap \omega_1 \}$$

$$= \bigvee \{ p_\beta : \beta \in M \cap \omega_1 \} = A \cap M.$$

It follows that $p \leq \bigwedge \{ \bigvee \{ A \cap M : A \in A_{\omega_1}(\mathbb{B}) \cap M \} = sg(\mathbb{B}, M)$, hence

$$sg(\mathbb{B}, M) \geq \bigvee \{ p \in \mathbb{B} : p \text{ is } M\text{-semigeneric} \}. $$

Finally, we show that $sg(\mathbb{B}, M)$ is $M$-semigeneric itself. Fix an arbitrary $\dot{\alpha} \in V^\mathbb{B} \cap M$ such that $1 \Vdash_\mathbb{B} \dot{\alpha} < \omega_1$, and let $A_{\dot{\alpha}} \in A_{\omega_1}(\mathbb{B})$ be as above. Since $\dot{\alpha} \in M$, also $A_{\dot{\alpha}} \in M$. Moreover,

$$\left[ \dot{\alpha} < (M \cap \omega_1) \right]_B = \bigvee \{ \left[ \dot{\alpha} = \dot{\beta} \right]_B : \beta \in M \cap \omega_1 \}$$

$$= \bigvee \{ a_\beta : \beta \in M \cap \omega_1 \} = A_{\dot{\alpha}} \cap M \geq sg(\mathbb{B}, M)$$

and the same holds for $g(\mathbb{B}, M)$. □

Proof of Theorem 3.1.11. First, suppose that $\mathbb{B}$ is semiproper and given $M \prec H_\theta$, $p \in \mathbb{B} \cap M$ let $q \leq p$ be an $M$-semigeneric condition. Then $q > 0$ and by Proposition 3.1.12, $q \leq sg(\mathbb{B}, M)$, hence $sg(\mathbb{B}, M) \cap p \geq q > 0$.

Conversely, suppose that $\mathbb{B}$ is as in the hypothesis of the theorem, $M \prec H_\theta$, $p \in \mathbb{B} \cap M$. Then $sg(\mathbb{B}, M) \cap p = q > 0$. Since the set of $M$-semigeneric conditions is dense below $sg(\mathbb{B}, M)$ (hence below $q$), we can find a $q' < q$ that is $M$-semigeneric.

The same reasoning holds also for $g(\mathbb{B}, M)$ with trivial modifications. □

3.2 Forcing axioms

Forcing is well-known as a versatile tool for proving consistency results. The purpose of forcing axioms is to turn it into a powerful tool for proving (conditional) theorems. Let $\Gamma$ denote a definable class of complete boolean algebras, $\kappa$ be a cardinal.

Definition 3.2.1. The bounded forcing axiom $BFA_\kappa(\Gamma)$ holds if for all $B \in \Gamma$ and all families $\{ D_\alpha : \alpha < \kappa \}$ of predense subsets of $B$ of size at most $\kappa$, there is a filter $G \subset B$ meeting all these sets.

Definition 3.2.2. The forcing axiom $FA_\kappa(\Gamma)$ holds if for all $B \in \Gamma$ and all families $\{ D_\alpha : \alpha < \kappa \}$ of predense subsets of $B$, there is a filter $G \subset B$ meeting all these sets.

Definition 3.2.3. The forcing axiom $FA_\kappa^{++}(\Gamma)$ holds if for all $B \in \Gamma$ and all families $\{ D_\alpha : \alpha < \kappa \}$ of predense subsets of $B$ and all families $\{ \dot{S}_\alpha : \alpha < \kappa \}$ of $B$-names for stationary subsets of $\kappa$, there is a filter $G \subset B$ meeting all these dense sets and evaluating each $\dot{S}_\alpha$ as a stationary subset of $\kappa$. 35
For \( \kappa = \omega_1 \), the forcing axioms are widely studied for many different classes \( \Gamma \) of complete boolean algebras. Thus, in the following we shall feel free to omit the index \( \kappa \) whenever \( \kappa = \omega_1 \). In particular, for the classes of posets:

\[
\text{ccc} \subset \text{proper} \subset \text{SP} \subset \text{SSP}
\]

the forcing axiom \( \text{FA}_{\omega_1} \) is called respectively \( \text{MA} \) (Martin’s Axiom), \( \text{PFA} \) (Proper Forcing Axiom), \( \text{SPFA} \) (Semiproper Forcing Axiom), \( \text{MM} \) (Martin’s Maximum). The corresponding bounded versions are \( \text{BMA}, \text{BPFA}, \text{BSPFA}, \text{BMM} \), while the strengthened versions are \( \text{MA}^{++}, \text{PFA}^{++}, \text{SPFA}^{++}, \text{MM}^{++} \). \( \text{BMA} \) is provably equivalent to \( \text{MA} \) [32], while \( \text{MM}^{++} \) is a strengthening of \( \text{MM} \) [41]. These axioms also have distinct consistency strengths: for example, \( \text{MA} \) is consistent relative to \( \text{ZFC} \) [27, Thm. 16.13], \( \text{BPFA} \) and \( \text{BSPFA}^{++} \) are consistent relative to a reflecting cardinal [21], while \( \text{BMM} \) is consistent relative to \( \omega \)-many Woodin cardinals [46], and \( \text{MM}^{++} \) is consistent relative to a supercompact cardinal (Theorem 3.4.15).

The class of SSP posets play a special role in the development of forcing axioms. In fact, \( \text{MM} \) is the strongest possible form of forcing axiom for \( \omega_1 \).

**Definition 3.2.4.** \( \mathbb{B} \) is **locally SSP** iff there exists a \( p \in \mathbb{B} \) such that \( \mathbb{B} \upharpoonright p \) is SSP.

**Proposition 3.2.5** (Shelah). If \( \mathbb{B} \) is not locally SSP then \( \text{FA}_{\omega_1}(\mathbb{B}) \) is false.

**Proof.** Assume that \( \mathbb{B} \) is not locally SSP and let \( S \) be a stationary set on \( \omega_1 \), \( \dot{C} \in V^{\mathbb{B}} \) be such that \( 1 \Vdash \dot{C} \subseteq \omega_1 \text{ club} \land \dot{S} \cap \dot{C} = \emptyset \). Define:

\[
\begin{align*}
D_\alpha &= \{ p \in \mathbb{B} : p \Vdash \dot{\alpha} \in \dot{C} \lor p \Vdash \dot{\alpha} \notin \dot{C} \} \\
E_\beta &= \{ p \in \mathbb{B} : p \Vdash \dot{\beta} \notin \dot{C} \Rightarrow \exists \gamma < \beta \ p \Vdash \dot{C} \cap \dot{\beta} \subseteq \dot{\gamma} \} \\
F_\gamma &= \{ p \in \mathbb{B} : \exists \alpha > \gamma \ p \Vdash \dot{\alpha} \in \dot{C} \}
\end{align*}
\]

Those sets are dense in \( \mathbb{B} \) by Loš theorem since \( \dot{C} \) is forced to be a club. Suppose by contradiction that \( \text{FA}_{\omega_1}(\mathbb{B}) \) holds, and let \( G \) be a filter that intersects all the sets \( D_\alpha, E_\beta, F_\gamma \). Then the set \( C = \{ \alpha < \omega_1 : \exists p \in G \ p \Vdash \dot{\alpha} \in \dot{C} \} \) is a club in \( V \), so there is a \( \beta \in S \cap C \). By definition of \( C \), there exists a condition \( q \in G \) such that \( q \Vdash \beta \in \dot{C} \), and \( \beta \in S \Rightarrow q \Vdash \beta \in \dot{S} \cap \dot{C} \neq \emptyset \), a contradiction. \( \square \)

Since \( \text{FA}(\text{locally SSP}) \) is easily seen to be equivalent to \( \text{MM} \), Proposition 3.2.5 implies that \( \text{MM} \) is the strongest possible form of \( \text{FA}_{\omega_1}(\Gamma) \) for some \( \Gamma \).

The forcing axioms have many equivalent formulations. In Section 3.2.1 we shall see how many of them can be expressed as density properties of the corresponding forcing classes. Furthermore, they can also be formulated in terms of the existence of generic filters \( G \in V \) for small set models \( M \in V \) of \( \text{ZFC} \).

**Definition 3.2.6.** We say that \( M \) is a \( \kappa \)-model iff \( |M| = \kappa, \kappa \subseteq M \) and \( M \) models \( \text{ZFC}^- \) (i.e. \( \text{ZFC} \) without powerset and with collection and separation, as in [20]).

We remark that this definition of \( \kappa \)-model is weaker than other definitions for the same term present in literature (which often require also that \( \kappa \in M \) and \( <^\kappa M \subseteq M \)).
Definition 3.2.7. Let $\mathbb{B}$ be a complete boolean algebra, $M$ be a $\kappa$-model such that $\mathbb{B} \in M$. We say that $G$ is an $M$-generic filter for $\mathbb{B}$ if $G \cap D \cap M \neq \emptyset$ for all $D \in M$ dense subsets of $\mathbb{B}$. We say that $G$ is an $M$-correct filter for $\mathbb{B}$ if it is $M$-generic and for every $\dot{S} \in M$ $\mathbb{B}$-name for a stationary subset of $P(\kappa)$, $\text{val}_G(\dot{S})$ is stationary in $V$.

Proposition 3.2.8 ([Ho] Thm. 2.53). Let $\mathbb{B}$ be a complete boolean algebra with $\mathcal{P}(\mathbb{B}) \in H_\theta$. Then the following are equivalent:

1. $\text{FA}_\kappa(\mathbb{B})$,

2. there exists a $\kappa$-model $M \prec H_\theta$ and a filter $G$ such that $\mathbb{B} \in M$ and $G$ is $M$-generic for $\mathbb{B}$,

3. there are stationary many $\kappa$-models $M \prec H_\theta$ and a filter $G$ such that $\mathbb{B} \in M$ and $G$ is $M$-generic for $\mathbb{B}$.

Proof. 1 $\Rightarrow$ 2. First, suppose that $\text{FA}_\kappa(\mathbb{B})$ holds and let $M \prec H_\theta$ be a $\kappa$-model such that $\mathbb{B} \in M$. There are at most $\kappa$ dense subsets of $\mathbb{B}$ in $M$, hence by $\text{FA}_\kappa(\mathbb{B})$ there is a filter $G$ meeting all those sets. However, $G$ might not be $M$-generic since for some $D \in M$, the intersection $G \cap D$ might be disjoint from $M$. Define:

$$N = \left\{ x \in H_\theta : \exists \tau \in M \cap V^\mathbb{B} \exists q \in G (q \Vdash_\mathbb{B} \tau = \dot{x}) \right\}$$

Clearly, $N$ contains $M$ (hence contains $\kappa$), and the cardinality $|N| \leq |M \cap V^\mathbb{B}| = \kappa$ since every $\tau$ can be evaluated in an unique way by the elements of the filter $G$. To prove that $N \prec H_\theta$, let $\exists x \phi(x,a_1,\ldots,a_n)$ be any formula with parameters $a_1,\ldots,a_n \in N$ which holds in $V$. Let $\tau_1 \in M^\mathbb{B}$, $q_i \in G$ be such that $q_i \Vdash_\mathbb{B} \tau_1 = \dot{a}_i$ for all $i < n$. Define $A_\phi = \{ p \in \mathbb{B} : p \Vdash_\mathbb{B} \exists x \in V \phi(x,\tau_1,\ldots,\tau_n) \}$, this set is definable in $M$ hence $A_\phi \in M$. Furthermore, $A_\phi \cap G$ is not empty since it contains any $q \in G$ below all $q_i$. By fullness in $H_\theta$, we have that:

$$H_\theta \models \forall p \in A_\phi \exists x \in V^\mathbb{B} \exists q \in G (q \Vdash_\mathbb{B} \tau = \dot{x}) \Rightarrow$$

$$H_\theta \models \exists \tau \forall p \in A_\phi \exists x \in V \phi(x,\tau,\tau_1,\ldots,\tau_n) \Rightarrow$$

$$M \models \exists \tau \forall p \in A_\phi \exists x \in V \phi(x,\tau,\tau_1,\ldots,\tau_n)$$

Fix such a $\tau$, by elementarity the last formula holds also in $H_\theta$ and in particular for $q \in A_\phi$. Since the set $\{ p \in \mathbb{B} : \exists x \in H_\theta p \Vdash_\mathbb{B} \tau = \dot{x} \}$ is an open dense set definable in $M$, there is a $q' \in G$ below $q$ belonging to this dense set, and an $a \in H_\theta$ such that $q' \Vdash_\mathbb{B} \tau = \dot{a}$. Then $q'$, $\tau$ witness that $a \in N$ hence the original formula $\exists x \phi(x,a_1,\ldots,a_n)$ holds in $N$.

Finally, we need to check that $G$ is $N$-generic for $\mathbb{B}$. Let $D \in N$ be a dense subset of $\mathbb{B}$, and $\dot{D} \in M$ be such that $1 \Vdash_\mathbb{B} \dot{D}$ is dense $\wedge \dot{D} \in V$ and for some $q \in G$, $q \Vdash_\mathbb{B} \dot{D} = D$. Since $1 \Vdash_\mathbb{B} \dot{D} \cap \dot{G} \neq \emptyset$, by fullness lemma there exists a $\tau \in H_\theta$ such that $1 \Vdash_\mathbb{B} \tau \in \dot{D} \cap \dot{G}$, and by elementarity there is such a $\tau$ also in $M$. Let $q' \in G$ below $q$ be deciding the value of $\tau$, $q' \Vdash_\mathbb{B} \tau = \dot{p}$. Since $q'$ forces that $\dot{p} \in \dot{G}$, it must be $q' \leq q$ so that $p \in G$ hence $p \in G \cap D \cap N$ is not empty.

2 $\Rightarrow$ 3. Let $M \prec H_{\theta+}$, $G$ be as in the hypothesis of the proposition. Define:

$$S = \{ N \prec H_\theta : \kappa \subset N \wedge |N| = \kappa \wedge \exists G \text{ filter } N\text{-generic} \}$$

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Note that $S$ is definable in $M$ then $S \in M$. Furthermore, $M \cap H_\theta \prec H_\theta$ and $M \cap H_\theta$ is in $S$. Given any $C_f \in M$ club on $H_\theta$, since $f \in M$ we have that $M \cap H_\theta \in C_f$.

Then $V \models S \cap C_f \neq \emptyset$ and by elementarity the same holds for $M$. Thus, $S$ is stationary in $M$ and again by elementarity $S$ is stationary also in $V$.

$3 \Rightarrow 1$. Fix a collection $D = \langle D_\alpha : \alpha < \kappa \rangle$ of dense subsets of $\mathbb{B}$. Since the set of $\kappa$-models with a generic filter is stationary, we can find a $\kappa$-model $M$ such that $D \in M$ and there is an $M$-generic filter $G$ for $\mathbb{B}$. Since $\kappa \subset M$ and $D$ has size $\kappa$, $D_\alpha \in M$ for every $\alpha < \kappa$. Thus, the $M$-generic filter $G$ will meet all dense sets in $D$, verifying FA$_\kappa(\mathbb{B})$ for this collection.

Under some mild restrictions a similar result holds also for FA$^{++}(\mathbb{B})$, as shown in the following proposition.

**Proposition 3.2.9.** Let $\mathbb{B}$ be a complete boolean algebra with $|\mathcal{P}(\mathbb{B})|^+ \in H_\theta$. Then the following are equivalent:

1. FA$^{++}(\mathbb{B} \ast \text{Coll}(\omega_1, (2^{\mathbb{B}})^+))$,

2. there exists a $\omega_1$-model $M \prec H_\theta$ and a filter $G$ such that $\mathbb{B} \in M$ and $G$ is $M$-correct for $\mathbb{B} \ast \text{Coll}(\omega_1, (2^{\mathbb{B}})^+)$,

3. there are stationary many $\omega_1$-models $M \prec H_\theta$ with a filter $G$ such that $\mathbb{B} \in M$ and $G$ is $M$-correct for $\mathbb{B} \ast \text{Coll}(\omega_1, (2^{\mathbb{B}})^+)$. 

**Proof.** Along the same lines of the proof of Proposition 3.2.8 (see [46, Thm. 2.53]).

---

### 3.2.1 Forcing axioms as density properties

Many of the common forcing axioms can be formulated in terms of density properties of their corresponding forcing classes $\Gamma$. This kind of formulation will be natural for the resurrection axioms that we shall consider in the rest of the chapter.

**Theorem 3.2.10** (Bagaria, [6]). BFA($\Gamma$) is equivalent to the assertion that the class

$$\left\{ \mathbb{B} \in \Gamma : H_{\omega_2} \prec_1 V^\mathbb{B} \right\}$$

is dense in $(\Gamma, \leq_{all})$.

We remark that the latter assertion is actually equivalent to requiring this class to coincide with the whole $\Gamma$ (since $\Sigma_1$-formulas are always upwards absolute).

Under suitable large cardinal assumptions the unbounded versions of the forcing axioms can also be reformulated as density properties, but only for $\Gamma =$ SSP (at least to our knowledge).

**Theorem 3.2.11** (Woodin, [33]). Assume there are class many Woodin cardinals. Then MM is equivalent to the assertion that the class

$$\left\{ \mathbb{B} \in \text{SSP} : \mathbb{B} \text{ is a presaturated tower forcing}^1 \right\}$$

is dense in $(\text{SSP}, \leq_{all})$.

---

1We refer the reader to [43, Def. 4.14] for a definition of presaturated tower and to [43, Def. 5.8] for a definition of strongly presaturated towers (see below).
3.3 Weak iterability

**Theorem 3.2.12** (Viale, [43]). Assume there are class many Woodin cardinals, then MM$^+$ is equivalent to the assertion that the above class is dense in $(\text{SSP}, \leq_{\text{SSP}})$.

In this chapter we shall also refer to the following strengthening of MM$^+$, which is defined by a density property of the class SSP as follows:

**Definition 3.2.13** (Viale, [43]). MM$^{+++}$ is the assertion that the class

$$\{ B \in \text{SSP} : B \text{ is a strongly presaturated tower}\}$$

is dense in $(\text{SSP}, \leq_{\text{SSP}})$.

### 3.2.2 Resurrection axioms

The resurrection axiom, introduced by Hamkins and Johnstone in [23], can be concisely stated as a density property for the class partial order $\leq_\Gamma$ defined above.

**Definition 3.2.14** (Hamkins, Johnstone [23]). The *resurrection axiom* $\text{RA}(\Gamma)$ is the assertion that the class

$$\{ B \in \Gamma : H_c \prec H^B_\Gamma \}$$

is dense in $(\Gamma, \leq_\Gamma)$.

The *weak resurrection axiom* $\text{wRA}(\Gamma)$ is the assertion that for all $B \in \Gamma$, there exists a $C \subseteq \text{all } B$ such that $H_c \prec H^C_\Gamma$.

Several variations of this axiom have been proposed.

**Definition 3.2.15** (Hamkins, Johnstone [24]). The *boldface resurrection axiom* $\text{RA} \simeq (\Gamma)$ asserts that for every $A \subseteq c$ the following class:

$$\{ B \in \Gamma : \exists A^* \subseteq c^B \langle H_c, \in, A \rangle \prec \langle H^B_\Gamma, \in, A^* \rangle \}$$

is dense in $(\Gamma, \leq_\Gamma)$.

**Definition 3.2.16** (Tsaprounis [40]). The *unbounded resurrection axiom* $\text{UR}(\Gamma)$ asserts that for every cardinal $\theta > c$ the following class:

$$\{ B \in \Gamma : \exists j : H_\theta \rightarrow H(j(\theta))^B \text{ elementary } \land \text{crit}(j) = c \land j(c) > \theta \}$$

is dense in $(\Gamma, \leq_\Gamma)$ below $\Gamma \cap H_\theta$.

### 3.3 Weak iterability

We are now ready to introduce the definition of *weakly iterable* class of forcing notions. Given a definable class $\Gamma$ of forcing notions, let $\Gamma_{\text{lim}}$ denote the (definable) class of complete iteration systems $\mathcal{F} = \{ i_{\alpha\beta} : B_\alpha \rightarrow B_\beta : \alpha \leq \beta < \lambda \}$ such that $i_{\alpha\beta}$ witnesses that $B_\beta \leq_\Gamma B_\alpha$ for all $\alpha \leq \beta < \lambda$.
3 Forcing axioms

Definition 3.3.1. Let $T$ be a theory extending ZFC, $\Gamma$ a definable class of complete boolean algebras, $\Sigma : \Gamma^{\text{lim}} \to \Gamma^{\text{lim}}$ a definable class function, $\kappa$ a definable cardinal.

We say that an iteration system $F \in \Gamma^{\text{lim}}$ of length $\lambda$ follows $\Sigma$ if and only if for all $\beta$ even,

$$F|((\beta + 1)\cdot \lambda) = \Sigma(F|\beta).$$

We say that $\Sigma$ is a weak iteration strategy for $\Gamma$ if and only if we can prove in $T$ that for every $F$ of length $\lambda$ which follows $\Sigma$, $\Sigma(F)$ has length $\lambda + 1$ and $F = \Sigma(F)|\lambda$.

We say that $\Sigma$ is a $\kappa$-weak iteration strategy for $\Gamma$ if in addition $\Sigma(F) = \lim F$ whenever $\text{cof}(\lambda) = \kappa$ or $\text{cof}(\lambda) > \kappa$, $\text{cof}(\lambda) > |B|$ for all $B$ in $F$.

Definition 3.3.2. Let $\mathcal{B}$ be a collection of complete boolean algebras. We denote as $\prod \mathcal{B}$ the lottery sum of the algebras in $\mathcal{B}$, defined as the boolean algebra obtained by the cartesian product of the respective boolean algebras with pointwise operations.

The name lottery sum is justified by the intuition that forcing with $\prod \mathcal{B}$ corresponds with forcing with a “random” algebra in $\mathcal{B}$. In fact, since the set of $p \in \prod \mathcal{B}$ that are 1 in one component and 0 in all the others form a maximal antichain, every $V$-generic filter $G$ for $\prod \mathcal{B}$ concentrates only on a specific $B \in \mathcal{B}$ (determined by the generic).

Definition 3.3.3. Let $T$ be a theory extending ZFC by a finite number of axioms, $\Gamma$ a definable class of complete boolean algebras, $\Sigma : \Gamma^{\text{lim}} \to \Gamma^{\text{lim}}$ a definable class function, $\kappa$ a definable cardinal.

We say that $\Gamma$ is $\kappa$-weakly iterable through $\Sigma$ iff we can prove in $T$ that:

• $\Gamma$ is closed under two-step iterations and set-sized lottery sums;
• $\Sigma$ is a $\kappa$-weak iteration strategy for $\Gamma$;
• $\langle \Gamma, \Sigma \rangle$ as computed in $V_\alpha$ is equal to $\langle \Gamma \cap V_\alpha, \Sigma \cap V_\alpha \rangle$ whenever $\alpha$ is inaccessible and $V_\alpha \models T$.

We say that $\Gamma$ is weakly iterable iff it is $\kappa$-weakly iterable for some $\kappa$.

We highlight that the latter definition (for a $T \supseteq \text{ZFC}$) is not related to a specific model $V$ of $T$, and requires that the above properties are provable in $T$ hence hold for every $T$-model $M$: for example, if $T = \text{ZFC}$ they must hold in every $V_\kappa$ where $\kappa$ is inaccessible. We shall feel free to omit the reference to $T$ when clear from the context, and in particular when $T = \text{ZFC}$.

Many notable cases for $\Gamma$ are $\omega_i$-weakly iterable for some $i = 0, 1$: i.e., $\Gamma = \text{all, locally ccc (using finite supports iteration strategy), } \sigma\text{-closed, axiom-A, proper (using countable supports iteration strategy). Furthermore, locally } <\kappa^+\text{-cc and } <\kappa\text{-closed are } \kappa\text{-weakly iterable (using } <\kappa\text{-sized supports iteration strategy).}^3$

In Section 3.4 we shall prove that semiproper forcings are $\omega_1$-weakly iterable in $\text{ZFC}$ and stationary set preserving forcings are $\omega_1$-weakly iterable in the theory $T = \text{ZFC + “there exists a proper class of supercompact cardinals”}$, thus concluding that most notable cases for $\Gamma$ are weakly iterable. We leave all the other cases mentioned above to the interested reader (see [7] for further details), since the corresponding

3We remark that every limit ordinal is even.

4Note that since $\alpha$ is inaccessible, this statement is equivalent to $V_\alpha \models T \setminus \text{ZFC}$ which is finite.
3.3 Weak iterability

proofs are either simpler or given by straightforward modifications of the arguments presented in Section 3.4.1. Contrary to the cases mentioned above, the strategies $\Sigma$ involved in the iteration of SP and SSP forcings will have to make careful choices for $\Sigma(F)$ also when $\mathcal{F}$ has length $\alpha$ even successor ordinal.

The definition of weak iterability for a definable class of forcing notions $\Gamma$ seem rather technical, and gets its motivation in providing the right conditions to carry out the lottery iteration $\mathcal{P}_\kappa^\Gamma$ with respect to a partial function $f : \kappa \to \kappa$ where $\kappa$ is an inaccessible cardinal. The lottery iteration has been studied extensively by Hamkins [22] and is one of the main tools for proving the consistency of forcing axioms. We will employ these type of iterations in Section 3.4.3 and in Chapter 4.

**Definition 3.3.4.** Let $\Gamma$ be weakly iterable through $\Sigma$ and $f : \kappa \to \kappa$ be a partial function. Define $\mathcal{F}_\xi = \{ i_{\alpha\beta} : \mathcal{P}_\alpha^{\Gamma,f} \to \mathcal{P}_\beta^{\Gamma,f} : \alpha \leq \beta < \xi \}$ by recursion on $\xi \leq \kappa + 1$ as:

1. $\mathcal{F}_0 = \emptyset$ is the empty iteration system;
2. $\mathcal{F}_{\xi+1} = \Sigma(\mathcal{F}_\xi)$ if $\xi$ is even;
3. $\mathcal{F}_{\xi+2}^\Gamma = \mathcal{P}_\xi^{\Gamma,f}$ if $\xi + 1$ is odd and $f(\xi)$ is undefined;
4. $\mathcal{F}_{\xi+2}^\Gamma$ has $\mathcal{P}_\xi^{\Gamma,f} = \mathcal{P}_\xi^{\Gamma,f} * \check{\mathcal{C}}$ otherwise, where $\check{\mathcal{C}}$ is a $\mathcal{P}_\xi^{\Gamma,f}$-name for the lottery sum (as computed in $V^{\mathcal{P}_\xi^{\Gamma,f}}$) of all complete boolean algebras in $\Gamma$ of rank less than $f(\xi)$, i.e. a $\mathcal{P}_\xi^{\Gamma,f}$-name for $\prod (\Gamma \cap V_{\gamma(\xi)})$.

We say that $\mathcal{P}_\kappa^{\Gamma,f}$ is the lottery iteration of $\Gamma$ relative to $f$.

**Proposition 3.3.5.** Let $T \supseteq \text{ZFC}$ be a theory, $\Gamma$ be $\gamma$-weakly iterable through $\Sigma$, $f : \kappa \to \kappa$ be a partial function with $\kappa$ inaccessible cardinal such that $V_\kappa \models T$. Then:

1. $\mathcal{P}_\kappa^{\Gamma,f}$ exists and is in $\Gamma$;
2. $\mathcal{P}_\kappa^{\Gamma,f}$ is $<\kappa$-cc and for all $\alpha < \kappa$, $1 \Vdash_{\mathcal{P}_\kappa^{\Gamma,f}} 2^\alpha \leq \check{\kappa}$;
3. $\mathcal{P}_\kappa^{\Gamma,f}$ is definable within $V_\kappa$ using the class parameter $f$;
4. Let $g : \lambda \to \lambda$ with $\lambda$ inaccessible be such that $f = g|\kappa$, $V_\lambda \models T$. Then $\mathcal{P}_\lambda^{\Gamma,g}$ absorbs every forcing in $\Gamma \cap V_{g(\kappa)}$ as computed in $V^{\mathcal{P}_\kappa^{\Gamma,f}}$. That is, for every $\check{\mathcal{B}}$ in $(\Gamma \cap V_{g(\kappa)})^{V^{\mathcal{P}_\kappa^{\Gamma,f}}}$, there is a condition $p \in \mathcal{P}_\lambda^{\Gamma,g}$ such that $\mathcal{P}_\lambda^{\Gamma,g}[p \leq \check{\mathcal{B}}] = \mathcal{P}_\kappa^{\Gamma,f} * \check{\mathcal{B}}$.

**Proof.** 1. Follows from $\Sigma$ being a weak iteration strategy for $\alpha < \kappa$ even, and from closed under two-step iterations and lottery sums for $\alpha$ odd.

2. Since $\Sigma \cap V_\kappa$ is $\Sigma$ as computed in $V_\kappa$, we can prove by induction on $\alpha < \kappa$ that $|\mathbb{B}_\alpha| < \kappa$. Furthermore, $\mathcal{P}_\kappa^{\Gamma,f} = \Sigma(\mathcal{F}_\kappa) = \lim \mathcal{F}_\alpha$ since $\kappa = \text{cof}(\kappa) > \gamma, |\mathbb{B}_\alpha|$ for all $\alpha < \kappa$ and $\Sigma$ is a $\gamma$-weak iteration strategy, and the set

$$S = \{ \alpha < \kappa : \mathcal{P}_\alpha^{\Gamma,f} = \lim \mathcal{F}[\alpha] \supseteq \{ \alpha < \kappa : \text{cof}(\alpha) = \gamma \}$$

is stationary. Thus by Theorem 2.2.10, $\mathcal{P}_\kappa^{\Gamma,f}$ is $<\kappa$-cc.
3 Forcing axioms

For the second part, given \( \alpha < \kappa \) let \( \dot{x} \) be a \( \mathbb{P}^\Gamma_f \)-name for a subset of \( \alpha \). Then \( \dot{x} \) is decided by \( \alpha < \kappa \) antichains of size \( \kappa \), hence \( \dot{x} = i_\beta(\dot{y}) \) for some \( \dot{y} \in V^\mathbb{P}_\beta, \beta < \kappa \). Since \( |\mathbb{P}^\Gamma_f| < \kappa \) and \( \kappa \) is inaccessible, there are less than \( \kappa \)-many names for subsets of \( \alpha \) in \( V^\mathbb{P}_\beta, \beta < \kappa \). Thus there are at most \( \kappa \)-many names for subsets of \( \alpha \) in \( V^\mathbb{P}_\beta, \beta < \kappa \).

3. Follows since \( (\Gamma \cap V_\kappa, \Sigma \cap V_\kappa) \) is \( (\Gamma, \Sigma) \) as computed in \( V_\kappa \).

4. Let \( g : \lambda \rightarrow \lambda \) with \( \lambda \) inaccessible be such that \( f = g|\kappa, V_\lambda \models T \). Since \( (\Gamma \cap V_\kappa, \Sigma \cap V_\kappa) \) is \( (\Gamma, \Sigma) \) as computed in \( V_\kappa \) and the same holds for \( V_\lambda \), \( \mathbb{P}^\Gamma_f \) is an initial part of \( \mathbb{P}^\Gamma_\lambda \). Furthermore given any \( \dot{\mathbb{B}} \) in \( (\Gamma \cap V_\theta(\kappa))V^\mathbb{P}_\kappa, \dot{\mathbb{B}} \) is a factor of \( \dot{\mathbb{C}} = \prod (\Gamma \cap V_\theta(\alpha))V^\mathbb{P}_\kappa, \) as defined in Definition 3.3.1.(4). Let \( \dot{q} \in \dot{\mathbb{C}} \) be the condition choosing \( \dot{\mathbb{B}} \) in the lottery sum (so that \( \dot{q} \) is \( \Gamma \) in the component corresponding to \( \dot{\mathbb{B}} \) and \( \mathbf{0} \) in all the others), and let \( p \in \mathbb{P}^\Gamma_\lambda \) be \( p = i_\lambda([\dot{q}]) \).

Thus,
\[
\mathbb{P}^\Gamma_f \ast \dot{\mathbb{B}} = \mathbb{P}^\Gamma_f \ast \dot{\mathbb{C}} \dot{q} = \mathbb{P}^\Gamma_\lambda \ast [\dot{q}] \geq \mathbb{P}^\Gamma_\lambda \ast [p].
\]

\[\Box\]

3.4 Semiproper and SSP iterations

We now prove that semiproper forcings are \( \omega_1 \)-weakly iterable and SSP forcings are \( \omega_1 \)-weakly iterable in the theory \( T = \text{ZFC} + \text{“there exists a proper class of supercompact cardinals”} \). The treatment of semiproper forcings will be split in the two-step and limit cases, then we shall explore the SSP case building on the results previously obtained.

3.4.1 Semiproper two-step iterations

We recall here the algebraic definition of semiproperness, stated in a way that is more convenient to us.

**Definition 3.4.1.** Let \( \mathbb{B} \) be a complete boolean algebra, \( S \) be a stationary set on \( H_\theta \) with \( \theta > 2|\mathbb{B}| \). Then \( \mathbb{B} \) is \( S \)-SP iff for club many \( M \in S, \text{sg}(\mathbb{B}, M) \land p > \mathbf{0} \) for all \( p \in \mathbb{B} \cap M \).

Similarly, \( i : \mathbb{B} \rightarrow \mathbb{C} \) is \( S \)-SP iff \( \mathbb{B} \) is \( S \)-SP and for club many \( M \in S, \)

\[
\pi(q \land \text{sg}(\mathbb{C}, M)) = \pi(q) \land \text{sg}(\mathbb{B}, M).
\]

for all \( q \) in \( \mathbb{C} \cap M \).

Finally, an iteration system \( \mathcal{F} = \{i_\alpha : \alpha \leq \beta < \lambda \} \) is \( S \)-SP iff \( i_\alpha \beta \) is \( S \)-SP for all \( \alpha \leq \beta < \lambda \).

The previous definitions can be reformulated with a well-known trick in the following form.

**Proposition 3.4.2.** \( \mathbb{B} \) is \( S \)-SP iff for every \( \delta \gg \theta \) regular, \( M \prec H_\delta \) with \( \mathbb{B}, S \in M \) and \( M \cap H_\theta \in S \) we have that \( \text{sg}(\mathbb{B}, M) \land p > \mathbf{0} \) for every \( p \in \mathbb{B} \cap M \).
Similarly, \( i : B \rightarrow C \) is \( S \text{-SP} \) iff \( B \) is \( S \text{-SP} \) and for every \( \delta \gg \theta \) regular, \( M \prec H_\delta \) with \( i, S \in M \) and \( M \cap H_\theta \in S \) we have that
\[
\pi(q \wedge \text{sg}(C, M)) = \pi(q) \wedge \text{sg}(B, M).
\]
for every \( q \in C \cap M \).

**Proof.** First, suppose that \( B, i : B \rightarrow C \) satisfy the above conditions. Then \( C = \{ M \cap H_\theta : M \prec H_\delta, B, S \in M \} \) contains a club (since it is the projection of a club), and witnesses that \( B, i : B \rightarrow C \) are \( S \text{-SP} \).

Conversely, suppose that \( B, i : B \rightarrow C \) are \( S \text{-SP} \) and fix \( \delta \gg \theta \) regular and \( M \prec H_\delta \) with \( B, S \in M, M \cap H_\theta \in S \). Since the sentence that \( B, i : B \rightarrow C \) are \( S \text{-SP} \) is entirely computable in \( H_\delta \) and \( M \prec H_\delta \), there exists a club \( C \in M \) witnessing that \( B, i : B \rightarrow C \) are \( S \text{-SP} \). Furthermore, \( M \) models that \( C \) is a club hence \( M \cap H_\theta \in C \) and \( \text{sg}(B, M) \wedge p > 0, \pi(q \wedge \text{sg}(C, M)) = \pi(q) \wedge \text{sg}(B, M) \) hold for any \( p \in B \cap M, q \in C \cap M \) since \( C \) witnesses that \( B, i : B \rightarrow C \) are \( S \text{-SP} \) and \( M \cap H_\theta \in S \cap C \).

Observe that if \( i : B \rightarrow C \) is \( S \text{-SP} \), then \( C \) is \( S \text{-SP} \). As a matter of fact \( q \in C \cap M \) is such that \( \text{sg}(C, M) \wedge q = 0 \) if and only if
\[
0 = \pi(q \wedge \text{sg}(C, M)) = \pi(q) \wedge \text{sg}(B, M),
\]
which contradicts the assumption that \( B \) is \( S \text{-SP} \) since \( \pi(q) \in B \cap M \).

The notion of being \( S \text{-SP} \) can change when we move to a generic extension: for example, \( S \) can be no longer stationary. In order to recover the “stationarity” in \( V[G] \) of an \( S \) which is stationary in \( V \), we are led to the following definition:

**Definition 3.4.3.** Let \( S \) be a subset of \( \mathcal{P}(H_\theta) \), \( B \in H_\theta \) be a complete boolean algebra, \( G \) be a \( V \)-generic filter for \( B \). We define
\[
S(G) = \{ M[G] : B \in M \in S \}.
\]

**Proposition 3.4.4.** Let \( S \) be a stationary set on \( H_\theta \), \( B \in H_\theta \) be a complete boolean algebra, \( G \) be a \( V \)-generic filter for \( B \). Then \( S(G) \) is stationary in \( V[G] \).

**Proof.** Let \( \dot{C} \in V^B \) be a name for a club \( C \) on \( \mathcal{P}(H_\theta) \), and let \( M \prec H_\theta^+ \) be such that \( M \cap H_\theta \in S, B, \dot{C} \in M \). Then \( C \in M[G] \) and \( M[G] \) models that \( C \) is a club, hence \( M[G] \cap H_\theta \in C \). Furthermore, \( M[G] \cap H_\theta = (M \cap H_\theta)[G] \) since \( B \) is \( <\theta\text{-cc} \) and \( M[G] \cap H_\theta \in S(G) \cap C \).

**Proposition 3.4.5.** Let \( B \) be a \( S \text{-SP} \) complete boolean algebra, \( \dot{C} \) be such that
\[
\left[ \dot{C} \text{ is } \langle \dot{G} \rangle \text{-SP} \right]_B = 1,
\]
then \( D = B \ast \dot{C} \) and \( i_{B \ast \dot{C}} \) are \( S \text{-SP} \).

**Proof.** First, we verify that \( i = i_{B \ast \dot{C}} \) is \( S \text{-SP} \). Let \( \dot{C}_1 \) be the club that witnesses \( \left[ \dot{C} \text{ is } \langle \dot{G} \rangle \text{-SP} \right]_B = 1 \), and let \( M \) be such that \( \dot{C}_1 \in M \). This guarantees that \( V[G] \models M[G] \cap H_\theta \in \text{val}(C_1, G) \).
First, we prove that $\pi(\text{sg}(\mathbb{D}, M)) = \text{sg}(\mathbb{B}, M)$. By Lemma 2.1.5, $\text{sg}(\mathbb{D}, M) \leq \pi(\text{sg}(\mathbb{B}, M))$ hence $\pi(\text{sg}(\mathbb{D}, M)) \leq (\text{sg}(\mathbb{B}, M))$. In order to prove the opposite inequality, suppose that $G$ is a $V$-generic filter for $\mathbb{B}$ such that $\text{sg}(\mathbb{B}, M) \in G$. By semiproperness of $\mathbb{B}$, $\forall G \in V[G] \ni M \cap \omega_1 = M[G] \cap \omega_1$ and by Lemma 2.3.3, $V[G] \models [\text{sg}(\mathbb{D}, M)]_{i[G]} = \text{sg}(\mathcal{C}, M[G])$. It follows that

$$\left[\text{sg}(\mathbb{D}, M)]_{i[G]} = \text{sg}(\mathcal{C}, M[G])\right]_\mathbb{B} \cong \text{sg}(\mathbb{B}, M)$$

and

$$\text{sg}(\mathbb{B}, M) \cap \left[\text{sg}(\mathbb{D}, M)]_{i[G]} > 0\right]_\mathbb{B} = \text{sg}(\mathbb{B}, M) \cap \left[\text{sg}(\mathcal{C}, M[G]) > 0\right]_\mathbb{B} = \text{sg}(\mathbb{B}, M)$$

by semiproperness of $\mathcal{C}$. Thus,

$$\pi(\text{sg}(\mathbb{D}, M)) = \left[\text{sg}(\mathbb{D}, M)]_{i[G]} > 0\right]_\mathbb{B} \cong \text{sg}(\mathbb{B}, M).$$

Finally, by Lemma 3.1.10 and 2.1.7 we can repeat the proof for $\mathbb{D}[\dot{\mathcal{C}}] = \mathbb{B}[\pi(\dot{\mathcal{C}})] \ast \mathcal{C}[\dot{\mathcal{D}}]$ (that is a two-step iteration of $S$-SP boolean algebras) and obtain that

$$\pi(\text{sg}(\mathbb{D}, M) \cap [\dot{\mathcal{C}}] = \text{sg}(\mathbb{B}, M) \cap \pi(\dot{\mathcal{C}}))$$

hence $i$ is $S$-SP. Since $\mathbb{B}$ is $S$-SP, it follows that $\mathbb{D}$ is also $S$-SP.

**Lemma 3.4.6.** Let $\mathbb{B}$, $\mathcal{C}_0$, $\mathcal{C}_1$ be $S$-SP complete boolean algebras, and let $G$ be any $V$-generic filter for $\mathbb{B}$. Let $i_0$, $i_1$, $k$ form the following commutative diagram of regular embeddings:

$$\begin{array}{ccc}
\mathbb{B} & \xrightarrow{i_0} & \mathcal{C}_0 \\
\downarrow{i_1} & & \downarrow{k} \\
\mathcal{C}_1 & & \\
\end{array}$$

Moreover assume that $\mathcal{C}_0/i_0[G]$ is $S(G)$-SP and

$$\left[\mathcal{C}_1/k[i_0]\mathcal{C}_0\right]_{\mathcal{C}_0} \cong S(G)\mathcal{C}_0 \cdot \text{SP}. $$

Then in $V[G]$, $k/G : \mathcal{C}_0/G \rightarrow \mathcal{C}_1/G$ is an $S(G)$-SP embedding.

**Proof.** Let $K$ be any $V[G]$-generic filter for $\mathcal{C}_0/i_0[G]$, and let

$$H = \{p \in \mathcal{C}_0 : [p]_{i_0[G]} \in K\}$$

so that

$$K = H/i_0[G] = \{[p]_{i_0[G]} : p \in H\}. $$

Then $H$ is $V$-generic for $\mathcal{C}_0$, $V[H] = V[G][K]$, $S(H) = S(G)(K)$. Furthermore, the latter equalities hold for any choice of $K$ thus in $V[G]$ the map $k/G : \mathcal{C}_0/G \rightarrow \mathcal{C}_1/G$ is such that

$$\left[\mathcal{C}_1/G/k[G]\mathcal{C}_0\right]_{\mathcal{C}_0} \cong S(G)\mathcal{C}_0 \cdot \text{SP}. $$

By Proposition 3.4.5, $k/G$ is $S(G)$-SP in $V[G]$. 

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3.4.2 Semiproper iteration systems

The limit case needs a slightly different approach depending on the length of the iteration. We shall start with some general lemmas, then we will proceed to examine the different cases one by one.

**Proposition 3.4.7.** Let $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ be an S-SP iteration system, $s$ be in $\operatorname{lim} \mathcal{F}$. Then

$$\mathcal{F}|s = \{(i_{\alpha\beta})_s : \mathbb{B}_\alpha|s(\alpha) \rightarrow \mathbb{B}_\beta|s(\beta) : \alpha \leq \beta < \lambda\}$$

is an S-SP iteration system and its associated retraction are the restriction of the original retractions.

**Proof.** By Lemma 2.1.7, $\mathcal{F}|s$ is indeed an iteration system and its associated retractions are the restriction of the original retractions. By Proposition 3.1.10, $\pi_{\alpha\beta}(\sigma(\mathbb{B}_\alpha|s(\alpha))) = \sigma(\mathbb{B}_\alpha|s(\alpha))$ for all $\alpha < \lambda$ thus for any $M \in S$ with $\mathcal{F}, s \in M$,

$$\pi_{\alpha\beta}(\sigma(\mathbb{B}_\alpha|s(\beta))) = \pi_{\alpha\beta}(\sigma(\mathbb{B}_\beta \wedge s(\beta))) = \sigma(\mathbb{B}_\alpha \wedge s(\alpha) = \sigma(\mathbb{B}_\alpha|s(\alpha)))$$

for all $\alpha, \beta \in M$, since in this case also $s(\alpha), s(\beta)$ are in $M$ and $i_{\alpha\beta}$ is S-SP. \(\Box\)

**Lemma 3.4.8.** Let $\mathcal{F} = \{i_{\alpha\beta} : \mathbb{B}_\alpha \rightarrow \mathbb{B}_\beta : \alpha \leq \beta < \lambda\}$ be an RCS and S-SP iteration system with $S$ stationary on $[H_\theta]^{\omega}$. Let $M$ be in $S$, $t \in M$ be any condition in $\operatorname{lim} \mathcal{F}$, $\dot{\alpha} \in M$ be a $\operatorname{lim} \mathcal{F}$-name for a countable ordinal, $\zeta \in M$ be an ordinal smaller than $\lambda$. Then:

- if $\operatorname{lim} \mathcal{F} = \operatorname{lim} \mathcal{F}$, there exists a condition $t' \in \operatorname{lim} \mathcal{F} \cap M$ below $t$ with $t'(\zeta) = t(\zeta)$ such that $t' \land i_\zeta(\sigma(\mathbb{B}_\zeta))$ forces that $\dot{\alpha} < M \cap \omega_1$;
- if $\lambda = \omega_1$, there exists a condition $t' \in \operatorname{lim} \mathcal{F} \cap M$ as above and such that $t' \land i_\zeta(\sigma(\mathbb{B}_\zeta))$ is in $\operatorname{lim} \mathcal{F}$ and has support contained in $M \cap \omega_1$.

**Proof.** Let $D \in M$ be the set of conditions in $\operatorname{lim} \mathcal{F}$ deciding the value of $\dot{\alpha}$ ($D$ is open dense by the forcing theorem):

$$D = \{s \in \operatorname{lim} \mathcal{F} : \exists \beta < \omega_1 s \forces_{\operatorname{lim} \mathcal{F}} \dot{\alpha} = \dot{\beta}\}.$$

Consider the set $\pi_\zeta[D|t]$ (which is open dense below $t(\zeta)$ by Lemma 2.1.5) and fix $A \subseteq \mathbb{B}_\zeta$ a maximal antichain in $M$ contained in it, so that $\bigvee A = t(\zeta)$. Let $f : A \rightarrow D$ be a map in $M$ such that $\pi_\zeta(f(p)) = p$ for every $p \in A$, and define $t' \in \operatorname{lim} \mathcal{F} \cap M$ by $t' = \bigvee f[A]$. Observe that $t'(\zeta) = t(\zeta)$ by definition of pointwise supremum and $t' \leq t$ since $\bigvee f[A]$ is really the supremum of $f[A]$ in $\operatorname{lim} \mathcal{F}$ by Lemma 2.2.8.

Furthermore, if $\operatorname{lim} \mathcal{F} = \operatorname{lim} \mathcal{F}$, $t'$ is in $\operatorname{lim} \mathcal{F}$ as required. If instead $\lambda = \omega_1$, $\operatorname{lim} \mathcal{F} = \operatorname{lim} \mathcal{F}$ and $t' \in \operatorname{lim} \mathcal{F}$ as characterized in Proposition 2.2.7, in both cases, $t' \in \operatorname{lim} \mathcal{F}$ is $\bigvee f[A]$. Then we can define a name $\mathcal{F}|\dot{\beta} \in \bigwedge \mathcal{F} \cap M$ as:

$$\dot{\beta} = \{\dot{\xi}, p : p \in A, f(p) \forces_{\operatorname{lim} \mathcal{F}} \dot{\alpha} < \dot{\xi}\}.$$

\footnote{Literally speaking this is not a $\mathbb{B}_\zeta$-name according to our definition. See the footnote below 1.3.2 to resolve this ambiguity.}
so that for any \( p \in A \), \( p \models \bar{\Delta} = \xi \) iff \( f(p) \models \Delta = \bar{\xi} \). It follows that
\[
\left[ i_\xi(\bar{\Delta}) = \Delta \right]_{\lim F} \geq \bigvee f[A] = t'.
\]

Moreover, \( \text{sg}(\mathbb{B}_\xi, M) \leq \left[ \bar{\Delta} < M \cap \omega_1 \right]_{\mathbb{B}_\xi} \) and is compatible with \( t'(\xi) \in M \) (since \( \mathbb{B}_\xi \) is \( S-\text{SP} \)), so that \( \left[ \bar{\Delta} < M \cap \omega_1 \right]_{\lim F} \geq t' \wedge i_\xi(\text{sg}(\mathbb{B}_\xi, M)) \). If \( \lim F = \lim F_{\text{rcs}} \), this concludes the proof.

If \( \lambda = \omega_1 \), we still need to prove that the support of \( t' \wedge i_\xi(\text{sg}(\mathbb{B}_\xi, M)) \) is contained in \( M \cap \omega_1 \). Define a name \( \check{\eta} \in V^{\mathbb{B}_\xi} \cap M \) for a countable ordinal setting:
\[
\check{\eta} = \{ (\bar{\xi}, p) : p \in A, \xi < \text{supp}(f(p)) \}.
\]
Notice that \( \check{\eta} \) is defined in such a way that for all \( \beta < \omega_1 \),
\[
\left[ \check{\eta} = \bar{\Delta} \right]_{\mathbb{B}_\xi} = \bigvee \{ p \in A : \text{supp}(f(p)) = \bar{\beta} \}.
\]

hence
\[
i_\xi(\left[ \check{\eta} = \bar{\Delta} \right]_{\mathbb{B}_\xi}) \wedge t' = i_\xi(\bigvee \{ p \in A : \text{supp}(f(p)) < \beta \}) \wedge \bigvee \{ f(p) : p \in A \} = \bigvee \{ f(p) : p \in A \wedge \text{supp}(f(p)) < \beta \}.
\]

Since \( t' \wedge i_\xi(\text{sg}(\mathbb{B}_\xi, M)) = \bigvee \{ (f(p), i_\xi(\text{sg}(\mathbb{B}_\xi, M)) : p \in A \} \) and \( \text{sg}(\mathbb{B}_\xi, M) \leq \left[ \check{\eta} < M \cap \omega_1 \right]_{\mathbb{B}_\xi} \), we get that
\[
t' \wedge i_\xi(\text{sg}(\mathbb{B}_\xi, M)) = t' \wedge i_\xi(\text{sg}(\mathbb{B}_\xi, M)) \wedge i_\xi(\left[ \check{\eta} < M \cap \omega_1 \right]_{\mathbb{B}_\xi}) = \bigvee \{ f(p) : p \in A \wedge \text{supp}(f(p)) < M \cap \omega_1 \} \wedge i_\xi(\text{sg}(\mathbb{B}_\xi, M)).
\]

It is now immediate to check that \( t'' = \bigvee \{ f(p) : p \in A \wedge \text{supp}(f(p)) < M \cap \omega_1 \} \) has support contained in \( M \cap \omega_1 \) as required. \( \Box \)

**Lemma 3.4.9.** Let \( F = \{ i_{nm} : n \leq m < \omega \} \) be an \( S-\text{SP} \) iteration system with \( S \) stationary on \( |H_0|^\omega \). Then \( \lim F \) and the corresponding \( i_{nm} \) are \( S-\text{SP} \).

**Proof.** By Proposition 3.4.2 any countable \( M < H_\delta \) with \( \delta > \theta \), \( M \cap H_\theta \in S \) and \( F, S \in M \) witnesses the semiproperness of every \( i_{nm} \). We need to show that for every \( s \in \lim F \cap M \) and \( n < \omega \),
\[
\pi_{n \omega}(\text{sg}(\text{RO}(\lim F)) \wedge s) = \text{sg}(\mathbb{B}_n, M) \wedge s(n)
\]
this would also imply that \( \text{RO}(\lim F) \) is \( S-\text{SP} \) by the same reasoning of the proof of Lemma 3.4.5. Without loss of generality, we can assume that \( n = 0 \) and by Lemma 3.1.10 and 3.4.7 we can also assume that \( s = 1 \). Thus is sufficient to prove that
\[
\pi_{0 \omega}(\text{sg}(\text{RO}(\lim F)), M)) = \text{sg}(\mathbb{B}_0, M)
\]
Let \( \{ \dot{\alpha}_n : n \in \omega \} \) be an enumeration of the \( \lim_{\leftarrow} F \)-names in \( M \) for countable ordinals. Let \( t_0 = 1 \), \( t_{n+1} \) be obtained from \( t_n, \dot{\alpha}_n, n \) as in Lemma 3.4.8 so that
\[
\left\lfloor \dot{\alpha}_n < M \cap \omega_1 \right\rfloor_{\lim_{\leftarrow} F} \geq t_{n+1} \land i_n(\text{sg}(\mathbb{B}_n, M))
\]

Consider now the sequence \( \bar{\ell}(n) = t_n(n) \land \text{sg}(\mathbb{B}_n, M) \). This sequence is a thread since \( i_{n,n+1} \) is \( S \)-SP and \( t_n(n) \in M \) for every \( n \), hence
\[
\pi_{n,n+1}(\text{sg}(\mathbb{B}_{n+1}, M) \land t_{n+1}(n+1)) = \text{sg}(\mathbb{B}_n, M) \land \pi_{n,n+1}(t_{n+1}(n+1))
\]
and \( \pi_{n,n+1}(t_{n+1}(n+1)) = t_{n+1}(n) = t_n(n) \) by Lemma 3.4.8. Furthermore, for every \( n \in \omega \), \( \bar{\ell} \leq t_n \) since the sequence \( t_n \) is decreasing, and \( \bar{\ell} \leq i_n(\text{sg}(\mathbb{B}_n, M)) \) since \( \bar{\ell}(n) \leq \text{sg}(\mathbb{B}_n, M) \). It follows that \( \bar{\ell} \) forces that \( \dot{\alpha}_n < M \cap \omega_1 \) for every \( n \), thus \( \bar{\ell} \leq \text{sg}(\text{RO}(\lim_{\leftarrow} F), M) \) by Lemma 3.1.12 Then,
\[
\pi_0(\text{sg}(\text{RO}(\lim_{\leftarrow} F), M)) \geq \bar{\ell}(0) = t_0(0) \land \text{sg}(\mathbb{B}_0, M) = \text{sg}(\mathbb{B}_0, M)
\]
and the opposite inequality is trivial, completing the proof. \( \square \)

**Lemma 3.4.10.** Let \( F = \{ \iota_{\alpha\beta} : \mathbb{B}_{\alpha} \to \mathbb{B}_{\beta} : \alpha \leq \beta < \omega_1 \} \) be an RCS and \( S \)-SP iteration system with \( S \) stationary on \( [H_0]^{\omega} \). Then \( \lim_{\leftarrow} F \) and the corresponding \( i_{\omega_1} \) are \( S \)-SP.

**Proof.** The proof follows the same pattern of the previous Lemma 3.4.9 By Proposition 3.4.2 any countable \( M < H_3 \) with \( \delta > \theta, F, S \in M, M \cap H_0 \in S \), witnesses the semiproperness of every \( i_{\alpha\beta} \) with \( \alpha, \beta \in M \cap \omega_1 \). As before, by Lemma 3.1.10 and 3.4.7 we only need to show that
\[
\pi_0(\text{sg}(\text{RO}(\lim_{\leftarrow} F), M)) \geq \text{sg}(\mathbb{B}_0, M),
\]
the other inequality being trivial. Let \( \langle \zeta_n : n \in \omega \rangle \) be an increasing sequence of ordinals such that \( \zeta_0 = 0 \) and \( \sup \zeta_n = \zeta = M \cap \omega_1 \), and \( \{ \dot{\alpha}_n : n \in \omega \} \) be an enumeration of the \( \lim_{\leftarrow} F \)-names in \( M \) for countable ordinals. Let \( t_0 = 1 \), \( t_{n+1} \) be obtained from \( t_n, \dot{\alpha}_n, \zeta_n \) as in Lemma 3.4.8 so that
\[
\left\lfloor \dot{\alpha}_n < M \cap \omega_1 \right\rfloor_{\lim_{\leftarrow} F} \geq t_{n+1} \land i_{\zeta}(\text{sg}(\mathbb{B}_{\zeta_n}, M)).
\]

Consider now the sequence \( \bar{\ell}(\zeta_n) = t_n(\zeta_n) \land \text{sg}(\mathbb{B}_{\zeta_n}, M) \). This sequence induces a thread on \( F \cap \zeta \) so that \( \bar{\ell} \in \mathbb{B}_{\zeta} \) since \( F \) is an RCS iteration, \( \zeta \) has countable cofinality and thus we can naturally identify \( \lim_{\leftarrow} F \cap \zeta \) as a dense subset of \( \mathbb{B}_{\zeta} \). Moreover we can also check that \( i_{\zeta}(\bar{\ell}) \) is a thread in \( \lim_{\leftarrow} F \) with support \( \zeta \) such that \( i_{\zeta}(\bar{\ell})(\alpha) = \bar{\ell}(\alpha) \) for all \( \alpha < \zeta \). Since by Lemma 3.4.8
\[
\text{supp}(t_{n+1} \land i_{\zeta}(\text{sg}(\mathbb{B}_{\zeta_n}, M))) \leq \zeta,
\]
the relation \( i_{\zeta}(\bar{\ell}) \leq t_{n+1} \land i_{\zeta}(\text{sg}(\mathbb{B}_{\zeta_n}, M)) \) holds pointwise hence \( i_{\zeta}(\bar{\ell}) \) forces that \( \dot{\alpha}_n < M \cap \omega_1 \) for every \( n \). Thus, \( i_{\zeta}(\bar{\ell}) \leq \text{sg}(\text{RO}(\lim_{\leftarrow} F), M) \) by Lemma 3.1.12 and \( \pi_0(\text{sg}(\text{RO}(\lim_{\leftarrow} F), M)) \geq \bar{\ell}(0) = t_0(0) \land \text{sg}(\mathbb{B}_0, M) = \text{sg}(\mathbb{B}_0, M) \) as required. \( \square \)
Lemma 3.4.11. Let \( F = \{ i_{\alpha \beta} : B_\alpha \to B_\beta : \alpha \leq \beta < \lambda \} \) be an RCS and S-SP iteration system with S stationary on \([H_\theta]^{\omega} \) such that \( \lim_\rightarrow F \) is \(<\lambda, \text{cc}) \). Then \( \lim_\rightarrow F \) and the corresponding \( i_{\alpha \lambda} \) are S-SP.

**Proof.** The proof follows the same pattern of the previous Lemmas 3.4.9 and 3.4.10. By Proposition 3.4.2, any countable \( M < H_\delta \) with \( \delta > \theta \), \( F, S \in M \), \( M \cap H_\delta \subseteq S \), witnesses the semiproperness of every \( i_{\alpha \beta} \) with \( \alpha, \beta \in M \cap \lambda \). By Lemma 3.4.10 and 3.4.7, we only need to show that

\[
\pi_0(\sg(\RO(\lim_\rightarrow F), M)) \geq \sg(\emptyset, M).
\]

Let \( \langle \zeta_n : n \in \omega \rangle \) be an increasing sequence of ordinals such that \( \zeta_0 = 0 \) and \( \sup_n \zeta_n = \zeta = \sup(M \cap \lambda) \), \( \langle \alpha_n : n \in \omega \rangle \) be an enumeration of the \( \lim_\rightarrow F \)-names in \( M \) for countable ordinals, and \( t_0 = 1 \) be in \( \lim F \).

Since \( \lim F \) is \(<\lambda, \text{cc} \), by Theorem 2.2.9, we have that \( \lim_\rightarrow F = \lim \res F = \lim F \).

Thus, we can define \( t_{n+1} \) from \( t_n, \alpha_n, \zeta_n \) as in Lemma 3.4.8, so that \( t_n \in \lim_\rightarrow F \cap M \)

\[
\left[ \alpha_n < \lambda \cap \omega_1 \right] \lim_\rightarrow F \geq t_{n+1} \wedge i_{\alpha_n}(\sg(\emptyset, M)).
\]

Since \( t_n \in \lim_\rightarrow F \cap M \), \( M \) has to model \( t_n \) to be eventually constant hence \( \sup(t_n) < \zeta \). Then the sequence \( \bar{t}(\zeta_n) = t_n(\zeta_n) \wedge \sg(\emptyset, \zeta_n, M) \) induces a thread on \( F(\zeta) = \RO(\lim_\rightarrow F(\zeta)) \) by the countable cofinality of \( \zeta \) and \( i_{\bar{t}}(\bar{t}) \leq t_{n+1} \wedge i_{\alpha_n}(\sg(\emptyset, \zeta_n, M)) \) for every \( n \), so that \( i_{\bar{t}}(\bar{t}) \leq \sg(\RO(\lim F), M) \) by Lemma 3.1.12 and \( \pi_0(\sg(\RO(\lim_\rightarrow F), M)) \geq \bar{t}(0) = t_0(0) \wedge \sg(\emptyset, M) = \sg(\emptyset, M) \) as required.

**Theorem 3.4.12.** Let \( F = \{ i_{\alpha \beta} : B_\alpha \to B_\beta : \alpha \leq \beta < \lambda \} \) be an RCS and S-SP iteration system with S stationary on \([H_\theta]^{\omega} \), such that for all \( \alpha < \beta < \lambda \),

\[
\left[ B_\beta / i_{\alpha \beta}(G_\alpha) \right]_{B_\alpha} \quad S(G_\alpha) - \text{SP}
\]

and for all \( \alpha \) there is a \( \beta > \alpha \) such that \( 1 \models_{B_\beta} |B_\alpha| \leq \omega_1 \). Then \( \lim F \) and the corresponding \( i_{\alpha \lambda} \) are S-SP.

**Proof.** First, suppose that for all \( \alpha \) we have that \( |B_\alpha| < \lambda \). Then, by Theorem 2.2.10, \( \lim_\rightarrow F \) is \(<\lambda, \text{cc} \) and \( \lim \res F = \lim_\rightarrow F \) hence by Lemma 3.4.11, we have the thesis.

Now suppose that there is a \( \alpha \) such that \( |B_\alpha| \geq \lambda \). Then by hypothesis there is a \( \beta > \alpha \) such that \( 1 \models_{B_\beta} |B_\alpha| \leq \omega_1 \), thus \( 1 \models_{B_\beta} \text{cof } \lambda \leq \omega_1 \). So by Lemma 3.4.6, \( F/G_\beta \) is a \( B_\beta \)-name for an \( S(G_\beta) \)-SP iteration system that is equivalent to a system of length \( \omega \) or \( \omega_1 \) hence its limit is \( S(G_\beta) \)-SP by Lemma 3.4.9 or Lemma 3.4.10 applied in \( V_{B_\beta} \). Finally, \( \lim F \) can always be factored as a two-step iteration of \( B_\beta \) and \( \lim \res F/G_\beta \), hence by Proposition 3.4.5, we have the thesis.

**Corollary 3.4.13.** Semiproper forcings are \( \omega_1 \)-weakly iterable.

**Proof.** It is easy to check that S-SP forcings are closed under lottery sums and in Section 3.4.1, we proved that they are closed under two-step iterations. Consider the strategy \( \Sigma : \Gamma \mapsto \Gamma \) which extends an iteration system \( F \) with \( \lim F \) if \( F \)
has limit length, and with $\mathcal{B} \ast \text{Coll}(\omega_1, \mathcal{B})$ if $\mathcal{F}$ has successor length and terminates with $\mathcal{B}$. It is easy to check that $(\Gamma, \Sigma)$ as computed in $V_\alpha$, where $V_\alpha \models \text{ZFC}$ is equal to $(\Gamma \cap V_\alpha, \Sigma \cap V_\alpha)$. Finally, if $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ follows $\Sigma$ it also satisfies the hypothesis of Theorem 3.4.12 hence $\Sigma(\mathcal{F}) = \lim_{\text{res}} \mathcal{F}$ extends $\mathcal{F}$ to an $S$-SP iteration system of length $\lambda + 1$ whenever $\lambda$ is limit. If instead $\lambda$ is successor, $\Sigma(\mathcal{F})$ is $S$-SP by Proposition 3.4.5.

**3.4.3 Stationary set preserving iterations**

In order to prove that SSP forcings are weakly iterable, we first prove the consistency of the semiproper forcing axiom SPFA from a supercompact cardinal. This axiom will play a crucial role in the iteration theorem for SSP forcings since it implies that $\text{SSP} = \text{SP}$ (Shelah, [27] Thm. 37.10). We will then be able to carry out the proof of an iteration theorem for SSP forcings in the theory $T = \text{ZFC} + \text{“there exists a proper class of supercompact cardinals”}$ along the lines of the iteration theorem for SP forcings.

**Definition 3.4.14.** A cardinal $\delta$ is supercompact and $f : \delta \to V_\delta$ is its Menas function iff for every $\delta$ cardinal there exists an elementary embedding $j : V \to M$ such that $\text{crit}(j) = \kappa$, $j(f)(\kappa) > \delta$ and $\langle \delta, M \rangle \subseteq M$.

**Theorem 3.4.15** (Magidor, Foreman, Shelah). Let $\kappa$ be supercompact with Menas function $f$. Then $\mathbb{P}_\kappa^{\text{SP}, f}$, the lottery iteration of SP forcings guided by $f$, forces SPFA and collapses $\kappa$ to $\omega_2$.

**Proof.** Let $\mathbb{B}$ be a $\mathbb{P}_\kappa^{\text{SP}, f}$-name for a semiproper forcing, $\mathcal{D}$ be a $\mathbb{P}_\kappa^{\text{SP}, f}$-name for a collection of $\omega_1$ many open dense sets of $\mathbb{B}$. Let $j : V \to M$ be elementary such that $\text{crit}(j) = \kappa$, $j(f)(\kappa) > \delta$ and $\langle \delta, M \rangle \subseteq M$ where $\delta = \text{rank}(\mathbb{B})$, $\mathbb{P}_\kappa^{\text{SP}, f} \ast \mathbb{B}$.

Let $G$ be $V$-generic for $\mathbb{P}_\kappa^{\text{SP}, f}$, $H$ be $V[G]$-generic for $\text{val}_G(\mathbb{B})$. Since $j(\mathbb{P}_\kappa^{\text{SP}, f}) = \left(\mathbb{P}_j(f)\right)^M$ and $\mathbb{B}$ is in $(V_j(f)(\kappa))^M$, by Proposition 3.3.5 there is a $p \in j(\mathbb{P}_\kappa^{\text{SP}, f})$ such that $j(\mathbb{P}_\kappa^{\text{SP}, f}) \models \mathbb{P}_\kappa^{\text{SP}, f} \ast \mathbb{B}$ hence we can find a further $G'$ such that $G \ast H \ast G'$ is $V$-generic for $j(\mathbb{P}_\kappa^{\text{SP}, f})$.

Since $\text{crit}(j) = \kappa$ and $\mathbb{P}_\kappa^{\text{SP}, f}$ is a direct limit of smaller forcings, $j[G] = G \subseteq G \ast H \ast G'$ thus we can extend $j$ to an elementary map $\bar{j}$:

$$\begin{align*}
\bar{j} : V[G] &\to M[G \ast H \ast G'] \\
\text{val}_G(\bar{x}) &\mapsto \text{val}_{G \ast H \ast G'}(j(\bar{x})).
\end{align*}$$

Let $\mathbb{B} = \text{val}_G(\mathbb{B})$, $\mathcal{D} = \text{val}_G(\mathcal{D})$. Since $H$ is a filter on $\mathbb{B}$ which meets every element of $\mathcal{D}$ and $\mathcal{D}$ has size $\omega_1 < \kappa$, $\bar{j}(\mathcal{D}) = j(\mathcal{D})$ and for any $\bar{j}(A) \in j(\mathcal{D})$ there exists a $p \in H \cap A$ so that $\bar{j}(p) \in j[H] \cap \bar{j}(A)$. It follows that $\bar{j}(H) \in V[G \ast H \ast G']$ is a filter on $j(\mathbb{B})$ which meets every element of $j(\mathcal{D})$.

Since $H = \text{val}_{G \ast H \ast G'}(j[G])$ and $M$ is closed under $\langle \delta \rangle$-sequences with $\delta > \mathbb{P}_\kappa^{\text{SP}, f} \ast \mathbb{B}$, $j[G] \in M$ hence $\bar{j}[H] = \text{val}_{G \ast H \ast G'}(j[G]) \in M[G \ast H \ast G']$. Thus,

$$M[G \ast H \ast G'] \models \exists F \subseteq \bar{j}(\mathbb{B}) \text{ filter } \forall A \in j(\mathcal{D}) \ F \cap A \neq \emptyset \Rightarrow \ V[G] \models \exists F \subseteq \mathbb{B} \text{ filter } \forall A \in D \ F \cap A \neq \emptyset$$

concluding the proof.
We are now able to prove that SSP forcings are weakly iterable.

**Lemma 3.4.16.** The class of SSP forcings is closed under two-step iterations and lottery sums.

**Proof.** Let \( \mathbb{B} \ast \hat{\mathbb{C}} \) be a two-step iteration of SSP forcings and \( S \subseteq \omega_1 \) a stationary set in \( V \). Since \( \mathbb{B} \) is SSP, \( \hat{S} \) is a stationary subset of \( \hat{\omega}_1 \) in \( V^\mathbb{B} \). Since \( \hat{\mathbb{C}} \) is SSP in \( V^\mathbb{B} \), it follows that \( \hat{S} \) is a stationary subset of \( \hat{\omega}_1 \) in \( V^\mathbb{B} \ast \hat{\mathbb{C}} \) as well.

Let now \( \mathbb{B} = \prod \{ \mathbb{B}_\alpha : \alpha < \lambda \} \) be a lottery sum of SSP forcings, \( S \subseteq \omega_1 \) be a stationary set in \( V \). Then the set of \( p \in \prod \mathbb{B} \) that are 1 in one component and 0 in all the others form a maximal antichain in \( \mathbb{B} \) which forces \( \hat{S} \) to be a stationary subset of \( \hat{\omega}_1 \) in \( V^\mathbb{B} \).

**Theorem 3.4.17.** Stationary set preserving forcings are \( \omega_1 \)-weakly iterable in \( T = \text{ZFC} + \text{“there exists a proper class of supercompact cardinals”} \).

**Proof.** By Lemma 3.4.16 we only need to show a definable iteration strategy for SSP. Let \( \hat{\mathbb{P}}^\mathbb{B} \) denote the lottery sum of all the semiproper boolean algebras of minimal rank which force SPFA and collapse \( |\mathbb{B}| \) to \( \omega_1 \) as computed in \( V^\mathbb{B} \). That is,

\[
\hat{\mathbb{P}}^\mathbb{B} = \left( \prod \{ C \in \text{SP} : \text{rank}(C) = \hat{\alpha} \land 1 \Vdash C \text{ SPFA} \land |\mathbb{B}| \leq \omega_1 \} \right)^{V^\mathbb{B}}
\]

where \( \alpha \) is minimal such that the above set is not empty. Such an \( \alpha \) exists by Theorem 3.4.15, together with the fact that any supercompact cardinal \( \kappa > |\mathbb{B}| \) is supercompact in \( V^\mathbb{B} \). Consider the following definable class \( \Sigma : \text{SSP}^\lim \rightarrow \text{SSP}^\lim \).

- \( \Sigma(\emptyset) = \hat{\mathbb{P}}^2 \), where \( 2 \) is the two-valued boolean algebra;
- \( \Sigma(\{ \mathbb{B}_\alpha : \alpha < \lambda + 1 \}) = \mathbb{B}_\lambda \ast \hat{\mathbb{P}}^\mathbb{B}_\lambda \);
- \( \Sigma(\{ \mathbb{B}_\alpha : \alpha < \lambda \}) = \lim_{\text{rcs}} \{ \mathbb{B}_\alpha : \alpha < \lambda \} \) if \( \lambda \) is a limit ordinal.

It is easy to check that the above definition grants that for all \( \kappa \) with \( V_\kappa \models T \), \( \langle \text{SSP} \cap V_\kappa, \Sigma \cap V_\kappa \rangle \) is \( \langle \text{SSP}, \Sigma \rangle \) as computed in \( V_\kappa \). Thus we only need prove that \( \Sigma \) is indeed an iteration strategy.

Let \( \mathcal{F} = \{ \mathbb{B}_\alpha : \alpha < \lambda \} \) be an iteration system following \( \Sigma \). We prove by induction on \( \lambda \) that \( \mathcal{F} \) is an SP iteration system. If \( \lambda = 0 \), it follows since \( \hat{\mathbb{P}}^2 \) is semiproper. If \( \lambda = \alpha + 1 \) is odd, it follows since \( \mathbb{B}_\alpha \) forces SPFA hence SSP = SP thus \( \mathbb{B}_{\alpha + 1} \leq_{\text{SSP}} \mathbb{B}_\alpha \) has to be semiproper. If \( \lambda = \alpha + 2 \) is even successor, it follows since \( \hat{\mathbb{P}}^\mathbb{B}_{\alpha + 1} \) is semiproper. If \( \lambda \) is limit, it follows by Theorem 3.4.12 since every \( \mathbb{B}_\alpha \ast \hat{\mathbb{P}}^\mathbb{B}_\alpha \) collapses the size of \( \mathbb{B}_\alpha \) to \( \omega_1 \). \( \square \)
CHAPTER 4

GENERIC ABSOLUTENESS

In this chapter we introduce the iterated resurrection axioms, which are forcing axioms with a relatively low consistency strength, and show that strong generic absoluteness results can be obtained from them. These axioms are variations of the resurrection axioms \[23\]. We shall assume from now on that the reader is acquainted with the content of Section 1.1 (second order elementarity and clopen class games), Chapter 2 (iterated forcing), Sections 3.1 - 3.3 (forcing axioms and weakly iterable forcing classes). All the material in this chapter is joint work with Matteo Viale \[5\].

Section 4.1 introduces the definition of the iterated resurrection axioms together with their basic properties, and proves the main Theorem 4.1.12. Section 4.2 develops the necessary technical devices for the consistency proofs of the axioms RA$_\alpha$(Γ). These proofs are carried out in Section 4.3 adapting the consistency proofs for the resurrection axioms introduced in \[23\] to this new setting of iterated resurrection. Section 4.4 outlines the main possible directions of further research on the topic.

4.1 Iterated resurrection and absoluteness

In this section we introduce the iterated resurrection axioms RA$_\alpha$(Γ) with Γ a suitable definable class of complete boolean algebras, and prove that the iterated resurrection axiom RA$_\omega$(Γ) gives generic absoluteness for the first order theory (with parameters) of $H_{2^\gamma}$ for a certain cardinal $\gamma$ which is computed in terms of the combinatorial properties of Γ. In particular, we aim to choose $\gamma$ as high as possible while still being able to consistently prove the generic absoluteness of the theory of $H_{2^\gamma}$ with respect to Γ. We are inspired by the resurrection axioms introduced by Hamkins and Johnstone in \[23\], which are formulated in similar terms when $\gamma = \omega$.

**Definition 4.1.1.** Let Γ be a definable class of complete boolean algebras closed under two step iterations. The **cardinal preservation degree** cpd(Γ) of Γ is the maximum cardinal $\kappa$ such that every $\mathcal{B} \in \Gamma$ forces that every cardinal $\nu \leq \kappa$ is still a cardinal in $V^\mathcal{B}$. If all cardinals are preserved by Γ, we say that cpd(Γ) = $\infty$.

The **distributivity degree** dd(Γ) of Γ is the maximum cardinal $\kappa$ such that every $\mathcal{B} \in \Gamma$ is $<\kappa$-distributive.
4 Generic absoluteness

We remark that the supremum of the cardinals preserved by $\Gamma$ is preserved by $\Gamma$, and similarly for the property of being $\lt \kappa$ distributive. Furthermore, $\text{dd}(\Gamma) \leq \text{cpd}(\Gamma)$ and $\text{dd}(\Gamma) \neq \infty$ whenever $\Gamma$ is non-trivial (i.e. it contains a boolean algebra that is not forcing equivalent to the trivial boolean algebra).

**Definition 4.1.2.** Let $\Gamma$ be a definable class of complete boolean algebras. Then we let $\gamma = \gamma_\Gamma$ be:

- $\text{cpd}(\Gamma)$ if $\text{cpd}(\Gamma) < \infty$;
- $\text{dd}(\Gamma)$ otherwise.

For example, $\gamma = \omega$ if $\Gamma$ is among all, ccc, while for Axiom-A, proper, semiproper, SSP we have that $\gamma = \omega_1$ and for $\lt \kappa$–closed we have that $\gamma = \kappa$.

We aim to isolate an axiom $\text{AX}(\Gamma)$ with the following properties:

1. assuming certain large cardinal axioms, the family of $B \in \Gamma$ which force $\text{AX}(\Gamma)$ is dense in $(\Gamma, \leq)_{\Gamma}$,
2. $\text{AX}(\Gamma)$ gives generic absoluteness for the theory with parameters of $H_{2^\gamma}$ with respect to all forcings in $\Gamma$ which preserve $\text{AX}(\Gamma)$.

Towards this aim remark the following:

- $\text{cpd}(\Gamma)$ is the maximum possible cardinal $\gamma$ for which an axiom $\text{AX}(\Gamma)$ as above can grant generic absoluteness with respect to $\Gamma$ of the theory of $H_{2^\gamma}$ with parameters. To see this, let $\Gamma$ be such that $\text{cpd}(\Gamma) = \gamma < \infty$, and assume towards a contradiction that there is an axiom $\text{AX}(\Gamma)$ yielding generic absoluteness with respect to $\Gamma$ for the theory with parameters of $H_{2^{\gamma^+}}$.

Assume that $\text{AX}(\Gamma)$ holds in $V$. Since $\text{cpd}(\Gamma) = \gamma$, there exists a $B \in \Gamma$ which collapses $\gamma^+$. Let $C \leq_{\Gamma} B$ be obtained by (1) so that $\text{AX}(\Gamma)$ holds in $V^C$, and remark that $\gamma^+$ cannot be a cardinal in $V^C$ as well. Then $\gamma^+$ is a cardinal in $H_{2^{\gamma^+}}$ and not in $H_{2^{\gamma^+}}^{\leq_{\Gamma}}$, witnessing failure of generic absoluteness and contradicting (2).

- $\text{dd}(\Gamma)$ is the least possible cardinal $\gamma$ such that $\text{AX}(\Gamma)$ is a non-trivial axiom asserting generic absoluteness for the theory of $H_{\gamma^+}$ with parameters. In fact, $H_{\text{dd}(\Gamma)}$ is never changed by forcings in $\Gamma$.

In the remainder of this chapter we shall see that the axiom $\text{RA}_\omega(\Gamma)$ satisfies both of the above requirements and is consistent for a variety of forcing classes $\Gamma$. In particular, we shall prove the consistency of $\text{RA}_\omega(\Gamma)$ for forcing classes which are weakly iterable (see Definition 3.3.3) and satisfy the following property.

**Definition 4.1.3.** Let $\Gamma$ be a definable class of complete boolean algebras. We say that $\Gamma$ is *well behaved* iff it is closed under two-step iterations and either:

- if $\text{cpd}(\Gamma) < \infty$, for all $\kappa > \gamma$ there are densely many $B \in \Gamma$ collapsing $\kappa$ to $\gamma$;
- if $\text{cpd}(\Gamma) = \infty$, for all $\kappa > \gamma$ there are densely many $B \in \Gamma$ forcing that $2^\gamma > \kappa$.  


The first requirement ensures that the cardinal preservation degree of $\Gamma$ gives a uniform bound for the cardinals which are preserved by the forcings in $\Gamma$. The second requirement ensures that above the distributivity degree of $\Gamma$ there are no further bounds on the possible values of $2^\gamma$ in extensions by forcings in $\Gamma$. We remark that this property is not significantly restrictive, as it holds for any forcing class $\Gamma$ we defined in Section 3.1. Thus, throughout all of this chapter we shall always assume that $\Gamma$ is a well behaved definable class of complete boolean algebras and $\gamma = \gamma_\Gamma$ is given by Definition 4.1.2.

In order to prove the consistency of $\text{RA}_\alpha(\Gamma)$, we shall also need to assume that $\Gamma$ is weakly iterable (Definition 3.3.3). Even though this hypothesis will not be needed for most of the results presented in this section, we shall sometimes assume it when convenient to carry out the proofs.

### 4.1.1 Resurrection game

Motivated by Hamkins and Johnstone’s [23], as well as by Tsaprounis’ [40], we introduce the following new class games and corresponding forcing axioms.

**Definition 4.1.4.** The $\Gamma$-weak resurrection game $G^{w\text{RA}}$ is as follows. Player I (Kill) plays couples $(\alpha_n, B_n)$ where $\alpha_n$ is an ordinal such that $\alpha_{n+1} < \alpha_n$ and $B_n$ is such that $B_{n+1} \leq_\Gamma C_n$. Player II (Resurrect) plays boolean algebras $C_n$ such that $H^C_n < H^C_{n+1}$ and $C_n \leq \gamma B_n$. The last player who can move wins.

The $\Gamma$-resurrection game $G^{\text{RA}}$ is the same game as $G^{w\text{RA}}$ with the additional requirement for Player II (Resurrect) to play so that $C_n \leq \Gamma B_n$ for all $n$.

The $\Gamma$-strong resurrection game $G^{\text{RFA}^{++}}$ is the same game as $G^{\text{RA}}$ with the further requirement for Player II (Resurrect) to play so that $V^C_n \models \text{FA}^{++}_\gamma(\Gamma)$ for all $n$.

**Definition 4.1.5.** The $\alpha$-weak resurrection axiom $w\text{RA}_\alpha(\Gamma)$ is the assertion that Player II (Resurrect) wins the $\Gamma$-weak resurrection game after $\langle (\alpha, 2), 2^\gamma \rangle$.

The $\alpha$-resurrection axiom $\text{RA}_\alpha(\Gamma)$ is the assertion that Player II (Resurrect) wins the $\Gamma$-resurrection game after $\langle (\alpha, 2), 2^\gamma \rangle$.

The $\alpha$-strong resurrection axiom $\text{RFA}^{++}_\alpha(\Gamma)$ is the assertion that Player II (Resurrect) wins the $\Gamma$-strong resurrection game after $\langle (\alpha, 2), 2^\gamma \rangle$.

We say that $w\text{RA}_\text{ON}(\Gamma)$ (respectively $\text{RA}_\text{ON}(\Gamma)$, $\text{RFA}^{++}_\text{ON}(\Gamma)$) holds iff the corresponding axioms hold for all $\alpha \in \text{ON}$.

Note that all these games are clopen class games, since Player I plays descending sequences of ordinals. Since we require $\Gamma$ to be (first-order) definable, the corresponding axioms asserting the existence of a class winning strategy for Player II in the relevant game (equivalently, that all strategies in the relevant game are

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1With $2$ we denote the two-valued boolean algebra $\{0, 1\}$. 

---
not winning for Player I are $\Delta^1_1$-statements (in set parameters) over the theory
$\text{NBG} + \text{AD}(\Delta^1_0)$.

A posteriori these axioms can also be formulated in recursive terms, as we will see in the
next proposition. However, this type of formulation cannot be directly
used as a definition in $L^2$. We will come back in more details to this
delicate point after the proof of the next proposition.

**Proposition 4.1.6** $(\text{AD}(\Delta^0_0))$, $\text{wRA}_\alpha(\Gamma)$ holds iff for all $\beta < \alpha$ and $B \in \Gamma$ there is
a $C \subseteq_{\text{all}} B$ such that $H_{2^\gamma} < H^C_{2^\gamma}$ and $V^C \models \text{RA}_\beta(\Gamma)$.

$\text{RA}_\alpha(\Gamma)$ holds iff the same holds with $C \leq B$; or equivalently for all $\beta < \alpha$ the class
$$\left\{ B \in \Gamma : H_{2^\gamma} < H^B_{2^\gamma} \land V^B \models \text{RA}_\beta(\Gamma) \right\}$$
is dense in $(\Gamma, \leq_{\Gamma})$.

Similarly, $\text{RFA}_\alpha^{++}(\Gamma)$ holds iff for all $\beta < \alpha$ the class
$$\left\{ B \in \Gamma : H_{2^\gamma} < H^B_{2^\gamma} \land V^B \models \text{FA}^{++}_\beta(\Gamma), \text{RFA}^{++}_\beta(\Gamma) \right\}$$
is dense in $(\Gamma, \leq_{\Gamma})$.

**Proof.** First we prove that for any $C$, $\text{wRA}_\beta(\Gamma)$ holds in $V^C$ iff Player II ($\text{Resurrect}$)
wins $G^{w\text{RA}}$ in $V$ after $(\langle \beta, C \rangle, \text{C})$. Let $G$ be $V$-generic for $C$ and $D \subseteq_{\text{all}} C$ with
witnessing map $i : C \to D$ ($i$ being the identity for $D = C$), and define $D_G := D/i[G]$ in $V[G]$. Let $s$ be in $G^{w\text{RA}}$ extending
$(\langle \beta, C \rangle, \text{C})$ and define $s_G$ as the sequence
obtained by substituting every boolean algebra $D$ appearing in $s$ with $D_G$. By
$\text{AD}(\Delta^0_0)$, let $\sigma$ be a winning strategy for player I or II in $G^{w\text{RA}}$.

Similarly, $\text{RFA}_\alpha^{++}(\Gamma)$ holds for all $\beta < \alpha$ the class
$$\left\{ B \in \Gamma : H_{2^\gamma} < H^B_{2^\gamma} \land V^B \models \text{FA}^{++}_\beta(\Gamma), \text{RFA}^{++}_\beta(\Gamma) \right\}$$
is dense in $(\Gamma, \leq_{\Gamma})$.

Furthermore we observe that whenever there is a winning strategy $\sigma$ for player I or II in $G^{w\text{RA}}$
and $s = (\langle \alpha, 2 \rangle, \langle \beta, B \rangle, C)$ in $\sigma$, we can define a winning strategy
for the same player $\sigma_s = \{ (\langle \beta, C \rangle, \text{C}) \cup u : s' \in \sigma \}$ in the game $G^{w\text{RA}}\{\langle \beta, C \rangle, \text{C}\}$.

Now we can prove the proposition: suppose that $\text{wRA}_\alpha(\Gamma)$ holds, and fix $\beta < \alpha$, $B \in \Gamma$. Let $\sigma$ be a winning strategy for Player II ($\text{Resurrect}$) in $G^{w\text{RA}}$
after $s_0 = (\langle \alpha, 2 \rangle, 2)$. Then $s_1 = s_0(\beta, B) \in \sigma$ and there is exactly one $s_2 < s_1$ in $\sigma$
with $C \subseteq B$ and $H^{C}_{2^\gamma} = H_{2^\gamma} \wedge H^{C}_{2^\gamma}$. Moreover, $\sigma_s$ is a winning strategy for Player II ($\text{Resurrect}$) in $G^{w\text{RA}}$
after $\langle \beta, C \rangle, C)$, thus $\text{wRA}_\beta(\Gamma)$ holds in $V^C$.

Conversely, suppose that for all $\beta < \alpha$, $B \in \Gamma$ there is a $C$ such that $\text{wRA}_\beta(\Gamma)$
holds in $V^C$ and $H_{2^\gamma} < H^{C}_{2^\gamma}$. Assume towards a contradiction that $\text{wRA}_\alpha(\Gamma)$ fails.
Then by $\text{AD}(\Delta^0_0)$ Player I ($\text{Kill}$) has a winning strategy $\sigma$ in $G^{w\text{RA}}$ after $s_0 = (\langle \alpha, 2 \rangle, 2)$, and there is exactly one $s_1 = s_0(\beta, B) \in \sigma$. Let $C$ be such that $\text{wRA}_\beta(\Gamma)$
holds in $V^C$, $H_{2^\gamma} < H^{C}_{2^\gamma}$.

Moreover, $\sigma_{s_2}$ is a valid move (hence is in $\sigma$) and by the
first part of this proof, since $V^C \models \text{wRA}_\beta(\Gamma)$, Player II ($\text{Resurrect}$) wins the game
$G^{w\text{RA}}\{\langle \beta, C \rangle, \text{C}\}$. Since $\sigma_{s_2}$ is a winning strategy for Player I ($\text{Kill}$) in the same
game, we get a contradiction.

Similar arguments yield the thesis also for $\text{RA}_\alpha(\Gamma)$, $\text{RFA}_\alpha^{++}(\Gamma)$. \qed

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2We remark that $s_2 \ll s_1$ iff $s_1 = s_2 \setminus \{s_2 \setminus 1\}$.
Remark 4.1.7. Assume $\langle V, C, \in \rangle$ is a model of MK. Let $\phi_0(x, y)$ be the formula

$$\phi_0(x, y) \equiv x, y \text{ are complete boolean algebras and } H^x_{\gamma} \prec H^y_{\gamma},$$

and for all $n < \omega$, let $\phi_{n+1}(x, y)$ be the formula

$$\phi_{n+1}(x, y) \equiv \phi_0(x, y) \land (\forall z \leq \Gamma y \exists w \leq \Gamma z \phi_n(y, w)).$$

Then for all $n < \omega$ the assertion $\phi_n(2, 2)$ is equivalent to $RA_n(\Gamma)$ in $\langle V, C, \in \rangle$ and it is a formula with no class quantifier. In particular we get that

$$\langle V, C, \in \rangle \models RA_n(\Gamma) \iff \langle V, \in \rangle \models \phi_n(2, 2).$$

hence the formulae $\phi_n(2, 2)$ can be used as a first order formulation of the axioms $RA_n(\Gamma)$ expressible in ZFC with no sort for class variables. However, if

$$\langle V, \in \rangle \models RA_\omega(\Gamma)$$

we can infer that for all $n < \omega$, $\langle V, \in \rangle \models \phi_n(2, 2)$ but it is not at all clear whether we can express in the structure $\langle V, \in \rangle$ that $RA_\omega(\Gamma)$ holds. In fact, the simplest strategy to express this property of $V$ would require us to perform an infinite conjunction of the formulae $\phi_n(2, 2)$ for all $n < \omega$, thus getting out of first order syntax.

This problem can be overcome in models of MK appealing to the class-game formulation of these axioms.

From now on, we will stick to the recursive formulation of the $\alpha$-resurrection axioms given by the latter proposition in order to prove the main results by induction on $\alpha$. The same will be done with the subsequent Definitions 4.2.1, 4.2.6, 4.2.9 of other class games. Note that by Proposition 4.1.6, $wRA_0(\Gamma)$, $RA_0(\Gamma)$ hold vacuously true for any $\Gamma$. Thus $wRA_1(\Gamma)$, $RA_1(\Gamma)$ imply the non-iterated formulations of resurrection axioms given in [23].

The different forcing axioms we have just introduced are connected by the following implications:

- if $\beta < \alpha$, $wRA_\alpha(\Gamma) \Rightarrow wRA_\beta(\Gamma)$ (same with RA, RFA++),
- if $\Gamma_1 \subseteq \Gamma_2$ and their associated $\gamma_1 \leq \gamma_2$, $wRA(\Gamma_2) \Rightarrow wRA(\Gamma_1)$,
- $RFA_\alpha^+(\Gamma) \Rightarrow RA_\alpha(\Gamma) \Rightarrow wRA_\alpha(\Gamma)$ whenever $\Gamma$ is closed under two step iterations: for the latter implication notice that the winning strategy $\sigma$ for II in $G^{RA}$ starting from $((\alpha, 2), 2)$ can also be used in $G^{wRA}$ and will force I to play always a $B_n$ in $\Gamma$. In particular $\sigma$ will remain a winning strategy also in $G^{wRA}$.

We shall be mainly interested in $RA_\alpha(\Gamma)$, even though $wRA_\alpha(\Gamma)$ will be convenient to state theorems in a modular form (thanks to its monotonic behavior with respect to $\Gamma$), and $RFA_\alpha^{++}(\Gamma)$ will be convenient to handle the $\Gamma = SSP$ case.

Some implications can be drawn between iterated resurrection axiom and the usual forcing axioms, as shown in the following theorems.

Theorem 4.1.8. $wRA_1(\Gamma)$ implies $H_{2^\gamma} \prec_1 V^B$ for all $B \in \Gamma$ and $BFA_\kappa(\Gamma)$ for all $\kappa < 2^\gamma$. 

Proof. Let $\mathcal{B}$ be any boolean algebra in $\Gamma$, and $\mathcal{C} \subseteq \mathcal{B}$ be such that $H_{2^\gamma} \prec H^{\mathcal{C}}_\omega$, hence $H_{2^\gamma} \prec H^\mathcal{C}_\omega \prec_1 V^\mathcal{C}$ by Levy’s absoluteness. Let $\phi = \exists x \psi(x)$ be a $\Sigma_1$ formula. If $\phi$ holds in $H_{2^\gamma}$, it trivially holds in $V^\mathcal{B}$ since $\Sigma_1$ formulas are upwards absolute. If $\phi$ holds in $V^\mathcal{B}$, it holds in $V^\mathcal{C}$ as well hence in $H_{2^\gamma}$, concluding the first part.

Let now $\mathcal{B}$ be in $\Gamma$ and $\mathcal{D}$ be a family of $\kappa$ many predense subsets of $\mathcal{B}$ of size at most $\kappa$. Let $\mathcal{B}' \subseteq \mathcal{B}$ be boolean algebra finitely generated by $\bigcup \mathcal{D}$ in $\mathcal{B}$, so that $|\mathcal{B}'| \leq \kappa$. Without loss of generality we can assume that both $\mathcal{B}'$ and $\mathcal{D}$ are in $H_{\kappa^+} \subseteq H_{2^\gamma}$, by replacing $\mathcal{B}$ with an isomorphic copy if necessary. Let $G$ be a $V$-generic filter for $\mathcal{B}$. Then $G$ meets every predense in $\mathcal{D}$, that is,

$$V[G] \models \exists F \subseteq \mathcal{B}' \text{ filter } \forall A \in \mathcal{D} F \cap A \neq \emptyset$$

and since $H_{2^\gamma} \prec_1 V^\mathcal{B}$, $H_{2^\gamma}$

and models the same completing the proof. \hfill \Box

Theorem 4.1.9. Assume there are class many super almost huge cardinals. Then $\text{MM}^{++}$ implies $\text{RFA}^{++}_{\text{ON}}(\text{SSP})$.

Proof. Recall that $\gamma_{\text{SSP}} = \omega_1$ and MM implies that $2^{\omega} = 2^{\omega_1} = \omega_2$. We prove that $\text{MM}^{++}$ implies $\text{RFA}^{++}_{\text{ON}}(\text{SSP})$ by induction on $\alpha$. For $\alpha = 0$ it follows since $\text{MM}^{++} \Rightarrow \text{MM}^{++}$, suppose now that $\alpha > 0$ and the thesis holds for all $\beta < \alpha$.

Let $A$ be the class of all super almost huge cardinals in $V$. Let $\mathcal{U}_\delta^{\text{SSP}}$ be the forcing whose condition are the SSP-cbas in $\text{SSP} \cap \mathcal{V}_\delta$ ordered by $\leq_{\text{SSP}}$. Since $A$ is a proper class, by [43 Thm. 3.5, Lemma 3.12] the class $\{\mathcal{U}_\delta^{\text{SSP}} : \delta \in A\}$ is predense in $(\text{SSP}, \leq_{\text{SSP}})$. Moreover $H_{2^\omega} = H_{\omega_2} \prec H_{\omega_2}^{\mathcal{U}_\delta^{\text{SSP}}}$ for all $\delta \in A$, since by [43 Lemma 5.19] $\mathcal{U}_\delta^{\text{SSP}}$ is forcing equivalent to a (strongly) presaturated tower for any such $\delta$. Finally, by [43 Cor. 5.20], every such $\mathcal{U}_\delta^{\text{SSP}}$ forces $\text{MM}^{++}$ and preserves that there are class many super almost huge cardinals (since large cardinals are indestructible by small forcings). It follows by inductive hypothesis that every such $\mathcal{U}_\delta^{\text{SSP}}$ forces $\text{RFA}_{\beta}^{++}(\text{SSP})$ for any $\beta < \alpha$ as well, hence $\text{RFA}_{\alpha}^{++}(\text{SSP})$ holds in $V$. \hfill \Box

It is also interesting to examine the consequences of the iterated resurrection axioms on the cardinal arithmetic.

Proposition 4.1.10. If $\text{cpd}(\Gamma) = \gamma < \infty$, $\text{wRA}_1(\Gamma)$ implies that $2^\gamma = \gamma^+$.\hfill \Box

Proof. Suppose by contradiction that $2^\gamma \nless \kappa^+$. By definition of cpd, there exists a $\mathcal{B} \in \Gamma$ such that $\kappa$ is collapsed in $V^\mathcal{B}$. Let $\mathcal{C} \subseteq \mathcal{B}$ be obtained from $\text{wRA}_1(\Gamma)$ so that $H_{2^\gamma} \prec H^{\mathcal{C}}_\omega$. Since $\kappa$ is a cardinal in $H^{V}_{2^\gamma}$ and it is not a cardinal in $H^{\mathcal{C}}_{2^\gamma}$, this is a contradiction. Thus $2^\gamma \leq \gamma^+$, concluding the proof. \hfill \Box

The latter proposition suggests that formulating the resurrection axioms in terms of the theory of $H_{2^\gamma}$ instead of that of $H_{\kappa^+}$ might be misleading, since in most cases they turn out to be the same. However, this is not the case whenever $\text{cpd}(\Gamma) = \infty$.

Proposition 4.1.11. If $\text{cpd}(\Gamma) = \infty$, then $\text{wRA}_1(\Gamma)$ implies that $2^\gamma$ is a limit cardinal.

Proof. Since $\text{cpd}(\Gamma)$ is infinite, $\gamma = \text{dd}(\Gamma)$. Suppose by contradiction that $2^\gamma = \kappa^+$ is a successor cardinal. Since $\Gamma$ is well behaved, there exists a $\mathcal{B} \in \Gamma$ such that $2^\gamma \nless (\kappa^+)^V$ in $V^\mathcal{B}$. Then by $\text{wRA}_1(\Gamma)$ we can find a $\mathcal{C} \subseteq \mathcal{B}$ such that $H^{\mathcal{C}}_\omega \nless H^{\mathcal{C}}_{2^\gamma}$. Since $\kappa$ is the maximum cardinal in $H_{2^\gamma}$ and not in $H^{\mathcal{C}}_{2^\gamma}$, we get a contradiction. \hfill \Box

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4.1 Iterated resurrection and absoluteness

We remark that similar results were obtained by Hamkins and Johnstone from their formulation of the resurrected axiom RA(\Gamma) (see Definition 3.2.14). In fact, RA_1(\Gamma) seem to overlap with RA(\Gamma) whenever 2^\gamma = \kappa. However, even in this case there are some subtle differences. In particular, let \Gamma be such that \gamma = \omega_1, 2^{\omega_1} = 2^\omega and Add(\omega_1, 1) is in \Gamma (e.g. \Gamma is among axiom-A, proper, semiproper, SSP). Then:

- Add(\omega_1, 1) preserves RA(\Gamma). Assume that RA(\Gamma) holds in V and let G be V-generic for Add(\omega_1, 1) so that 2^\omega = \omega_1 in V[G] and H_\kappa^{V[G]} = H_\omega^{V[\Gamma]} = H_\omega^V. Then, any B \in \Gamma in V[G] which resurrect the theory of H_\kappa^V will also resurrect the theory of H_\kappa^{V[G]} = H_\omega^{V[G]} = H_\omega^V. Thus B * Add(\omega_1, 1) will resurrect the theory of H_\kappa^{V[G]} and there are densely many such forcing since there are densely many B resurrecting the theory of H_\kappa^V in V.

- There is no reason to expect that Add(\omega_1, 1) preserves RA_1(\Gamma). In fact, in this latter case we want to resurrect the theory of H_{2^\omega_1} as computed in V[G] and H_{2^\omega_1}[G] \supseteq H_{\kappa^V}[G] \supseteq H_{\kappa^V}^V, and the theory of H_{\kappa^V} is not at all controlled by RA_1(\Gamma).

Thus, it is not transparent under which conditions on \Gamma the axiom RA_1(\Gamma) can be compatible with the GCH, unlike its non-iterated counterpart RA(\Gamma) which is compatible with CH whenever \Gamma contains an <\omega_1-distributive boolean algebra forcing CH.

4.1.2 Resurrection axioms and generic absoluteness

The main motivation for the iterated resurrection axioms can be found in the following result:

**Theorem 4.1.12.** Suppose \( n \in \omega \), RA_\( n \)(\( \Gamma \)) holds and \( B \in \Gamma \) forces RA_\( n \)(\( \Gamma \)). Then \( H_{2^n} \prec_n H_B^{V[B]} \).

**Proof.** We proceed by induction on \( n \). Since \( 2^{\gamma} \leq (2^{\gamma})^{V[B]} \), \( H_{2^n} \subseteq H_B^{V[B]} \) and the thesis holds for \( n = 0 \) by the fact that all transitive structures \( M \subseteq N \) are \Sigma_0\-elementary. Suppose now that \( n > 0 \), and fix \( G \) V-generic for \( B \). By Proposition 4.2.2 and RA_\( n \)(\( \Gamma \)), let \( C \subseteq V[G] \) be such that whenever \( H \) is V[G]\-generic for \( C \), \( V[G * H] \models RA_{n-1}(\Gamma) \) and \( H_{2^n}^V \times H_{2^n}^{V[G * H]} \). Then we have the following diagram:

\[
\begin{array}{c}
H_{2^n}^V \\
\Sigma_\omega \\
\Sigma_{n-1} \\
\Sigma_{n-1}
\end{array}
\]

\[
\begin{array}{c}
H_{2^n}^{V[G]} \\
H_{2^n}^{V[G * H]}
\end{array}
\]

obtained by inductive hypothesis applied both on \( V \), \( V[G] \) and on \( V[G], V[G * H] \) since in all those classes RA_{n-1}(\( \Gamma \)) holds.

Let \( \phi = \exists x \psi(x) \) be any \( \Sigma_\omega \) formula with parameters in \( H_{2^n}^V \). First suppose that \( \phi \) holds in \( V \), and fix \( \bar{x} \in V \) such that \( \psi(\bar{x}) \) holds. Since \( H_{2^n}^V \prec_{n-1} H_{2^n}^{V[G]} \) and \( \psi \) is \( \Pi_{n-1} \), it follows that \( \psi(\bar{x}) \) holds in \( V[G] \) hence so does \( \phi \). Now suppose that \( \phi \) holds
in $V[G]$ as witnessed by $\bar{x} \in V[G]$. Since $H_{2^\gamma}^{V[G]} \prec_n H_{2^\gamma}^{V[G+H]}$ it follows that $\psi(\bar{x})$ holds in $V[G+H]$, hence so does $\phi$. Since $H_{2^\gamma}^V \prec H_{2^\gamma}^{V[G+H]}$, the formula $\phi$ holds also in $V$ concluding the proof.

Corollary 4.1.13. Assume $\text{RA}_\omega(\Gamma)$ holds and $B \in \Gamma$ forces $\text{RA}_\omega(\Gamma)$. Then $H_{2^\gamma} \prec H_{2^\gamma}^B$.

A similar absoluteness result for $L(\text{ON}^{\omega_1})$ concerning SSP forcings in the $\text{MM}^{+++}$ setting was previously obtained in [43]. Corollary 4.1.13 for $\Gamma = \text{SSP}$ provides a weaker statement (since it concerns the smaller model $H_{\omega_2}$). However, it is a more general result since it holds also for $\Gamma \neq \text{SSP}$, and the consistency of $\text{RA}_\omega(\Gamma)$ follows from much weaker large cardinal assumptions than the principle $\text{MM}^{+++}$ needed to prove absoluteness for $L(\text{ON}^{\omega_1})$. Furthermore, the above result can be applied to classes $\Gamma$ such that the corresponding $\gamma$ is above $\omega_1$ thus providing generic absoluteness for the (partially) larger model $H_{2^\gamma}$ with $2^\gamma \geq \omega_3$.

4.2 Uplifting cardinals and definable Menas functions

In order to measure the consistency strength of the $\alpha$-resurrection axiom, we need to introduce the following large cardinal notion:

Definition 4.2.1. The uplifting game $G^{\text{UP}}$ is as follows. Player I (challenge) plays couples of ordinals $(\alpha_n, \theta_n)$ such that $\alpha_{n+1} < \alpha_n$. Player II (uplift) plays regular cardinals $\kappa_n$ such that $V_{\kappa_n+1} \prec_1 V_{\kappa_{n+1}+1}$, $\kappa_n \geq \theta_n$. The last player who can move wins.

$$
\begin{align*}
\text{I (Challenge).} & \quad (\alpha_0, \theta_0) \\
& \quad (\alpha_1, \theta_1) \\
& \quad (\alpha_2, \theta_2) \\
\text{II (Uplift).} & \quad \kappa_0 \rightarrow \kappa_1 \rightarrow \ldots
\end{align*}
$$

We say that $\kappa$ is $(\alpha)$-uplifting iff Player II (Uplift) wins the uplifting game after $\langle (\alpha, 0), \kappa \rangle$. We say that $\kappa$ is $(\text{ON})$-uplifting if it is $(\alpha)$-uplifting for all $\alpha \in \text{ON}$.

The uplifting game is a clopen class game, since Player I (Challenge) plays a descending sequence of ordinals. It follows that upliftingness is a $\Delta^1_1$-property. As for the iterated resurrection axioms, we can give a formulation in recursive terms.

Proposition 4.2.2 $(\text{AD}(\Delta^1_1))$. $\kappa$ is $(\alpha)$-uplifting iff it is regular and for all $\beta < \alpha$, $\theta > \kappa$ there is a $\lambda > \theta$ that is $(\beta)$-uplifting and $V_{\kappa+1} \prec_1 V_{\lambda+1}$.

Proof. Follows the one of Proposition 4.1.6. Let $\sigma$ be a winning strategy for Player I or II in $G^{\text{UP}}$, $s = \langle (\alpha, 0), \kappa, (\beta, \theta), \lambda \rangle$ be in $\sigma$. Then $\sigma_s = \{ (\langle \beta, 0 \rangle, \lambda)^{\uparrow} u : s^{\uparrow} u \in \sigma \}$ is a winning strategy for the same player in $G^{\text{UP}} | \langle (\beta, 0), \lambda \rangle$. Then we can follow step by step the last part of the proof of Proposition 4.1.6 using the reduction $\sigma \rightarrow \sigma_s$ defined above.

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4.2 Uplifting cardinals and definable Menas functions

Note that replacing “regular” with “inaccessible” in Definition 4.2.1 would produce no difference for $\alpha > 0$, since no successor cardinal $\kappa$ can satisfy $V_\kappa \prec V_\lambda$ with $\lambda > \kappa$. However, the formulation hereby chosen for Definition 4.2.1 is more convenient for the proofs in Section 4.3.

We remark that the definition of $(0)$-uplifting cardinal overlaps with that of a regular cardinal. Although similar, the class of $(1)$-uplifting cardinals does not coincide with the class of uplifting cardinals as defined by Johnstone and Hamkins in [23] (see Definition 4.3.11 below). In fact, being $(1)$-uplifting is a stronger property than being uplifting, since the family of $\Delta^1_1$-formulae for the MK-model $V_{\kappa+1}$ strictly includes all formulae in the language of set theory with parameters in $V_\kappa$. However the consistency strength of (ON)-uplifting cardinals is close to that of uplifting cardinals as defined by Johnstone and Hamkins. This is shown in Proposition 4.2.4 below.

The key reason which led us to introduce the notion of $(\alpha)$-uplifting cardinal as a natural strengthening of the Hamkins and Johnstone original notion of upliftingness is to be found in Lemma 4.2.5 below which states a very nice reflection property of $(\alpha)$-uplifting cardinal which we cannot predicate for the recursive strengthenings of Hamkins and Johnstone notion of upliftingness (see Def. 4.3.11). These reflection properties are a key ingredient in our proof of the consistency of the iterated resurrection axioms. We shall come back to these issues in more details in Section 4.3.

4.2.1 Consistency strength of $(\alpha)$-uplifting cardinals

Lemma 4.2.3. Assume $\delta$ is Mahlo. Then there are stationarily many inaccessible $\kappa \prec \delta$ such that $V_{\kappa+1} \prec V_{\delta+1}$.

Proof. Let $C \subseteq \delta$ be a club, $M_0$ be the Skolem hull of $\{C\}$ in $V_{\delta+1}$. Define a sequence $\langle M_\alpha, \kappa_\alpha : \alpha < \delta \rangle$ where $\kappa_\alpha = \max(\alpha, \text{rank}(M_\alpha \cap V_\delta))$, $M_{\alpha+1} \prec V_{\delta+1}$ is obtained by Lowenheim-Skolem Theorem from $M_\alpha \cup V_{\kappa_\alpha}$, and $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ for $\alpha$ limit ordinal.

Since $M_\alpha < \delta$ implies that $\kappa_\alpha < \delta$ and $|M_{\alpha+1}| = |M_\alpha \cup V_{\kappa_\alpha}| < \delta$, by induction on $\alpha$ we have that $|M_\alpha|, \kappa_\alpha < \delta$ for all $\alpha$. Furthermore, the sequence $\langle \kappa_\alpha : \alpha < \delta \rangle$ is a club on $\delta$, which is Mahlo, thus we can find an $\bar{\alpha} < \delta$ limit such that $\kappa_{\bar{\alpha}}$ is inaccessible.

Since $\bar{\alpha}$ is limit, $M_{\bar{\alpha}} \cap V_\delta = V_{\kappa_{\bar{\alpha}}}$. Since $V_{\bar{\alpha}+1} \models C$ is a club and $M_{\bar{\alpha}} \prec V_{\bar{\alpha}+1}$, $C \in M_{\bar{\alpha}}$, also $M_{\bar{\alpha}} \models C$ is a club hence $\kappa_{\bar{\alpha}}$ is a limit point of $C$. Thus, $\kappa_{\bar{\alpha}}$ is an inaccessible cardinal in $C$ and by Corollary 1.1.2 $V_{\kappa_{\bar{\alpha}}+1} \equiv \Delta^1_1(T)$ $M_{\bar{\alpha}} \prec V_{\bar{\alpha}+1}$, concluding the proof.

Proposition 4.2.4. Assume $\delta$ is Mahlo. Then $V_{\delta+1}$ models $\text{MK} + \text{there are class many (ON)-uplifting cardinals}$.  

Proof. Since $\delta$ is inaccessible, $V_{\delta+1}$ models $\text{MK}$. Furthermore,

\[ S = \left\{ \kappa < \delta : \kappa \text{ inaccessible } \land V_{\kappa+1} \prec V_{\delta+1} \right\} \]

is stationary by Lemma 4.2.3. We prove that every element of $S$ is $(\alpha)$-uplifting in $V_{\delta+1}$ by induction on $\alpha < \delta$.  

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First, every element of \( S \) is \((0)\)-uplifting by definition. Suppose now that every element of \( S \) is \((\beta)\)-uplifting for every \( \beta < \alpha \), and let \( \kappa \) be in \( S \). Since \( S \) is unbounded, for every \( \beta < \alpha, \theta > \kappa \) in \( V_\delta \) there is a \( \lambda \in S, \lambda > \theta \). Such \( \lambda \) is \((\beta)\)-uplifting by inductive hypothesis and \( V_{\kappa+1}, V_{\lambda+1} \prec_{\Delta^1_1} V_{\delta+1} \) implies that \( V_{\kappa+1} \prec_{\Delta^1_1} V_{\lambda+1} \). We can now use Property 4.2.2 to conclude.

As shown in [23, Thm. 11], if there is an uplifting cardinal then there is a transitive model of ZFC + ON is Mahlo. So the existence of an \((ON)\)-uplifting cardinal is in consistency strength strictly between the existence of a Mahlo cardinal and the scheme “ON is Mahlo”. We take these bounds to be rather close together and low in the large cardinal hierarchy.

4.2.2 Reflection properties of \((\alpha)\)-uplifting cardinals

The following proposition outlines a key reflection property of \((\alpha)\)-uplifting cardinals:

**Lemma 4.2.5.** Let \( \kappa \) be an \((\alpha)\)-uplifting cardinal with \( \alpha < \kappa \), and \( \delta < \kappa \) be an ordinal. Then \((\delta \text{ is } (\alpha)\text{-uplifting}) V_\kappa \iff \delta \text{ is } (\alpha)\text{-uplifting} \).

**Proof.** Let \( \phi(\alpha) \) be the statement of this theorem, i.e:

\[
\forall \kappa > \alpha \ (\alpha\text{-uplifting}) \ orall \delta < \kappa \ (\delta \text{ is } (\alpha)\text{-uplifting}) V_\kappa \iff \delta \text{ is } (\alpha)\text{-uplifting}
\]

We shall prove \( \phi(\alpha) \) by induction on \( \alpha \) using the recursive formulation given by Proposition 4.2.2. For \( \alpha = 0 \) it is easily verified, suppose now that \( \alpha > 0 \).

For the forward direction, suppose that \((\delta \text{ is } (\alpha)\text{-uplifting}) V_\kappa \), and let \( \beta < \alpha, \theta > \delta \) be ordinals. Let \( \lambda > \theta \) be a \((\beta)\)-uplifting cardinal with \( V_{\kappa+1} \prec_{\Delta^1_1} V_{\lambda+1} \), so that \((\delta \text{ is } (\alpha)\text{-uplifting}) V_\lambda \) since upliftingness is a \( \Delta^1_1 \)-property under AD(\( \Delta^0_1 \)) (and AD(\( \Delta^0_1 \)) holds at inaccessible cardinals). Then there is a \( \nu > \theta \) in \( V_\lambda \) with \( V_{\delta+1} \prec_{\Delta^1_1} V_{\nu+1} \) and \((\nu \text{ is } (\beta)\text{-uplifting}) V_\lambda \). By inductive hypothesis, since \( \beta < \alpha \) and \( \lambda \) is \((\beta)\)-uplifting in \( V \), also \( \nu \) is \((\beta)\)-uplifting in \( V \) concluding this part.

Conversely, suppose that \( \delta \) is \((\alpha)\)-uplifting in \( V \) and let \( \beta < \alpha, \theta > \delta \) be ordinals in \( V_\kappa \). Let \( \nu > \theta \) be a \((\beta)\)-uplifting cardinal such that \( V_{\delta+1} \prec_{\Delta^1_1} V_{\nu+1} \), and let \( \lambda > \nu \) be a \((\beta)\)-uplifting cardinal such that \( V_{\kappa+1} \prec_{\Delta^1_1} V_{\lambda+1} \). By inductive hypothesis, since \( \beta < \alpha \) and \( \nu \), \( \lambda \) are \((\beta)\)-uplifting in \( V \), \((\nu \text{ is } (\beta)\text{-uplifting}) V_\lambda \), thus

\[
V_{\lambda+1} \models \exists \nu > \theta \ V_{\delta+1} \prec_{\Delta^1_1} V_{\nu+1} \wedge \nu \text{ is } (\beta)\text{-uplifting}.
\]

By \( \Delta^1_1 \)-elementarity, \( V_{\kappa+1} \) models the same concluding the proof.

4.2.3 Variations and strengthenings of \((\alpha)\)-upliftingness

To achieve a consistency result also for RFA\(+\), we need to introduce the following definition:

**Definition 4.2.6.** The \textit{uplifting for supercompacts game} \( \mathcal{G}^{\text{UP++}} \) is the same game as \( \mathcal{G}^{\text{UP}} \) (see Definition 4.2.1) with the additional requirement that Player II (Uplift) has to play supercompact cardinals.
We say that $\kappa$ is ($\alpha$)-uplifting for supercompacts iff it is supercompact and Player II ($\text{Uplift}$) wins the uplifting for supercompacts game after $\langle (\alpha, 0), \kappa \rangle$. We say that $\kappa$ is (ON)-uplifting for supercompacts iff it is ($\alpha$)-uplifting for supercompacts for all $\alpha \in \text{ON}$.

**Proposition 4.2.7** ($\text{AD}(\Delta^0_1)$). $\kappa$ is ($\alpha$)-uplifting for supercompacts iff it is supercompact and for all $\beta < \alpha$, $\theta > \kappa$ there is a $\lambda > \theta$ that is ($\beta$)-uplifting for supercompacts and $V_{\kappa+1} \prec_{\Delta^1_1} V_{\lambda+1}$.

**Proof.** Follows the one of Proposition 4.2.2. \hfill \Box

A bound for this large cardinal notion can be obtained in a completely similar way to Proposition 4.2.4:

**Proposition 4.2.8.** There are class many (ON)-uplifting for supercompacts cardinals consistently relative to an inaccessible cardinal which is a stationary limit of supercompact cardinals.

### 4.2.4 Menas functions for uplifting cardinals

As previously mentioned, to obtain the consistency results at hand we shall use a lottery iteration relative to a fast-growing function $f : \kappa \to \kappa$ for a sufficiently large cardinal $\kappa$. The exact notion of fast-growth we will need is given by the Menas property schema introduced in [36] and developed by Hamkins for several different cardinal notions in [23, 22].

We remark that it is always possible to define Menas functions for cardinals that have a Laver function, while it is also possible to define such functions for some cardinals that don’t have a Laver function. Moreover, from Menas functions we can obtain many of the interesting consequences given by Laver functions.

**Definition 4.2.9.** The Menas uplifting game $G^{\text{M-UP}}$ is as follows. Player I ($\text{Challenge}$) plays couples of ordinals $\langle \alpha_n, \theta_n \rangle$ such that $\alpha_{n+1} < \alpha_n$. Player II ($\text{Uplift}$) plays partial functions $f_n : \kappa_n \to \kappa_n$ with $\kappa_n$ regular cardinal such that $\langle V_{\kappa_{n+1}}, f_n \rangle \prec_{\Delta^1_1} \langle V_{\kappa_{n+1}+1}, f_{n+1} \rangle$ and $f_{n+1}(\kappa_n) \geq \theta_{n+1}$. The last player who can move wins.

\[
\begin{array}{c}
\text{I (Challenge).} & (\alpha_0, \theta_0) & (\alpha_1, \theta_1) \quad (\alpha_2, \theta_2) \\
\text{II (Uplift).} & f_0 \quad f_1 & \ldots
\end{array}
\]

We say that a partial function $f : \kappa \to \kappa$ is Menas for ($\alpha$)-uplifting iff Player II ($\text{Uplift}$) wins the Menas uplifting game after $\langle (\alpha, 0), f \rangle$. We say that $f$ is Menas for (ON)-uplifting iff it is Menas for ($\alpha$)-uplifting for all $\alpha \in \text{ON}$.

**Proposition 4.2.10** ($\text{AD}(\Delta^0_1)$). A partial function $f : \kappa \to \kappa$ is Menas for ($\alpha$)-uplifting iff $\kappa$ is regular and for all $\beta < \alpha$, $\theta > \kappa$ there is a Menas for ($\beta$)-uplifting function $g : \lambda \to \lambda$ with $g(\kappa) > \theta$ and $\langle V_{\kappa+1}, f \rangle \prec_{\Delta^1_1} \langle V_{\lambda+1}, g \rangle$.

---

3. $\langle M, C \rangle \models (\forall \alpha < \beta, \lambda < \kappa, \theta)(\forall \alpha < \beta, \lambda < \kappa, \theta)(\exists \gamma \prec \kappa)(\vec{x}, Y)(\exists \gamma \prec \kappa)(\vec{x}, Y)$ with $\vec{x}$ a tuple of set variables and $Y$ a class variable, $\langle M, C \rangle \models \phi(\vec{a}, C)$ iff $\langle N, D \rangle \models \phi(\vec{a}, D)$. 

---
Proof. Follows the one of Proposition 4.2.2.

We can now prove the existence of definable Menas functions for \((\alpha\)-uplifting cardinals):

**Proposition 4.2.11.** If \(\kappa\) is \((\alpha\)-uplifting, then there is a definable Menas function for \((\alpha\)-uplifting on \(\kappa\).

**Proof.** We shall prove by induction on \(\alpha\) that whenever \(\kappa\) is \((\alpha\)-uplifting, such a function is given by the following definition (failure of upliftingness function) relativized to \(V_{\kappa+1}\):

\[
f(\xi) = \sup \left\{ \nu : V_{\xi+1} \prec_{\Delta_1} V_{\nu+1} \wedge \exists \beta \, \nu \text{ is } (\beta\)-uplifting } \wedge \xi \text{ is not } (\beta+1\)-uplifting} \right\}
\]

Note that \(f(\xi)\) is undefined only if \(\xi\) is (ON)-uplifting (in the domain of \(f\)), since otherwise \(\xi\) would be an element considered in the supremum. Thus \(f(\xi) \geq \xi\) when it is defined.

If \(\kappa\) is (0)-uplifting any function \(g : \kappa \to \kappa\) is Menas, in particular our \(f\). Now suppose that \(\kappa\) is \((\alpha\)-uplifting with \(\alpha > 0\) and let \(\beta < \alpha, \theta > \kappa\) be ordinals. Let \(\nu > \theta\) be a \((\beta\)-uplifting cardinal such that \(V_{\kappa+1} \prec_{\Delta_1} V_{\nu+1}\), and let \(\lambda\) be the least \((\beta\)-uplifting cardinal bigger than \(\nu\) such that \(V_{\kappa+1} \prec_{\Delta_1} V_{\lambda+1}\).

Thus no \(\nu' \in (\nu, \lambda)\) with \(V_{\kappa+1} \prec_{\Delta_1} V_{\nu'+1}\) can be \((\beta\)-uplifting in \(V\), hence by Lemma 4.2.5 neither in \(V_{\lambda}\). It follows that \(\kappa\) cannot be \((\beta+1\)-uplifting in \(V_{\lambda}\), while again by Lemma 4.2.5 it is \((\beta\)-uplifting in \(V_{\lambda}\). Then any \(\nu'\) considered in calculating \(f^{V_{\lambda+1}}(\kappa)\) must be witnessed by a \(\beta' \geq \beta\) (since \(\kappa\) is \((\beta'+1\)-uplifting in \(V_{\lambda+1}\) for any \(\beta' < \beta\)) hence be \((\beta\)-uplifting.

It follows that \(f^{V_{\lambda+1}}(\kappa) = \nu\) and \(\langle V_{\kappa+1}, f^{V_{\kappa+1}} \rangle \prec_{\Delta_1} \langle V_{\lambda+1}, f^{V_{\lambda+1}} \rangle\) since \(f\) is \(\Delta_1\)-definable, concluding the proof.

Note that in the proof of this Lemma we used in key steps the reflection properties of \((\alpha\)-uplifting cardinals given by Lemma 4.2.5.

### 4.3 Consistency strength

The results of this section expand on the ones already present in [23] and [43]. In the previous section we outlined the large cardinal properties we shall need for our consistency proofs. Now we shall apply the machinery developed by Hamkins and Johnstone in their proof of the consistency of RA(\(\Gamma\)) for various classes of \(\Gamma\), and show that with minor adjustments their techniques will yield the desired consistency results for RA\(_{\alpha}(\Gamma)\) for weakly iterable \(\Gamma\) when applied to lottery preparation forcings guided by suitable Menas functions. To prove the consistency of RFA\(_{++}^{\alpha}(SSP)\) we will instead employ the technology introduced by the Viale in [43]. For this reason we shall feel free to sketch some of the proofs leaving to the reader to check the details, that can be developed in analogy to what is done in [23] and [43].
4.3 Consistency strength

4.3.1 Upper bounds

Let \( \Gamma \) be a weakly iterable class of forcing notions. In this section we shall prove that \( \mathbb{P}_\Gamma \), the lottery iteration of \( \Gamma \) relative to a function \( f : \kappa \to \kappa \), forces \( RA_\alpha(\Gamma) \) whenever \( f \) is Menas for \((\alpha)\)-uplifting. In order to prove this result, we will need to ensure that \( \mathbb{P}_\Gamma \) "behaves well" as a class forcing with respect to \( V_\kappa \).

There are two possible approaches. In the first one, we can consider \( \langle V_\kappa, f \rangle \) as a ZFC model extended with an additional unary predicate for \( f \), so that \( \mathbb{P}_\Gamma \) can be handled as a definable class forcing. Thus we can proceed following step by step the analogous argument carried out in [23]. In the second one, we can consider the MK model \( V_{\kappa+1} \) and expand on the results in [1] to prove that \( \mathbb{P}_\Gamma \) preserve enough of MK and behaves well with respect to elementarity. Even though the second approach is more general and natural in some sense, the first approach is considerably simpler. Thus we will follow the argument in [23] and give the following definition.

**Definition 4.3.1.** Let \( \langle M, C \rangle \) be a model of ZFC expanded with an additional class predicate \( C \), and \( \mathbb{P} \) be a class partial order definable in \( \langle M, C \rangle \).

An \( \langle M, C \rangle \)-generic filter \( G \) for \( \mathbb{P} \) is a filter meeting all dense subclasses of \( \mathbb{P} \) which are definable in \( \langle M, C \rangle \) with parameters.

\( \mathbb{P} \) is nice for forcing in \( \langle M, C \rangle \) if the forcing relation \( \Vdash_\mathbb{P} \) is definable in \( \langle M, C \rangle \) and the forcing theorem holds, i.e. for every first-order formula \( \phi \) with parameters in \( \langle M, C \rangle \) and every \( \langle M, C \rangle \)-generic filter \( G \) for \( \mathbb{P} \),

\[ (M[G], C) \models \phi \iff \exists p \in G \langle M, C \rangle \models (p \Vdash_\mathbb{P} \phi). \]

Note that the definition is interesting only when \( \mathbb{P} \notin M \) is a class forcing with respect to \( M \). The next lemmas provide a sufficient condition for being nice for forcing in \( H_\lambda \).

**Lemma 4.3.2.** Let \( \lambda \) be a regular cardinal in \( V \) and \( \mathbb{P} \subseteq H_\lambda \) be a partial order preserving the regularity of \( \lambda \). Assume \( G \) is \( V \)-generic for \( \mathbb{P} \). Then

\[ H_\lambda[G] = \{ \text{val}_G(\dot{x}) : \dot{x} \in V^\mathbb{P} \cap H_\lambda \} = H_\lambda^{V[G]}. \]

The above Lemma is rather standard but we sketch a proof since we cannot find a precise reference for it.

**Proof.** Since every element of \( H_\lambda \) with \( \lambda \) regular is coded with a bounded subset of \( \lambda \), and \( \mathbb{P} \) preserves the regularity of \( \lambda \), we can assume that every \( \mathbb{P} \)-name for an element of \( H_\lambda^{V[G]} \) is coded by a \( \mathbb{P} \)-name for a function \( \dot{f} : \lambda \to 2 \) such that \( \dot{f} \) is allowed to assume the value 1 only on a bounded subset of \( \lambda \). In particular we let for any such \( \dot{f} \),

\[ D_{\dot{f}} = \{ p \in \mathbb{P} : \exists \alpha \ p \Vdash \dot{f}^{-1}([1]) \subseteq \alpha \} \]

and for all \( \xi < \lambda \),

\[ E_{\xi, \dot{f}} = \{ p \in \mathbb{P} : \exists i < 2 \ \dot{f}(\xi) = i \} \]

Notice that \( p \in D_{\dot{f}} \) as witnessed by \( \alpha \) implies that \( p \in E_{\xi, \dot{f}} \) for all \( \xi \geq \alpha \). In particular to decide the values of \( \dot{f} \) below such conditions \( p \) we just need to consider the dense sets \( E_{\xi, \dot{f}} \) for \( \xi < \alpha \).
Let \( p \in \mathbb{P} \) be arbitrary, \( A_\xi \subseteq E_\xi \cap D_\xi \) be maximal antichains. Since \( \mathbb{P} \) preserves the regularity of \( \lambda \), it is \( <\lambda \)-presaturated (see Definition 5.2.5) hence we can find \( q \leq p \) such that \( q \in D_\xi \) as witnessed by \( \alpha \) and

\[
B_\xi = \{ r \in A_\xi : r \text{ is compatible with } q \}
\]

has size less than \( \lambda \) for all \( \xi < \alpha \). We can now use these antichains \( B_\xi \) to cook up a name \( \dot{g}_q \in H_\lambda \cap V^\mathbb{P} \) such that \( q \) forces that \( \dot{f} = \dot{g}_q \). By standard density arguments, the thesis follows.

**Lemma 4.3.3.** Let \( \mathbb{P} \subseteq H_\kappa \) be a partial order preserving the regularity of \( \kappa \) that is definable in \( \langle H_\kappa, C \rangle \). Then \( \mathbb{P} \) is nice for forcing in \( \langle H_\kappa, C \rangle \).

**Proof.** Since \( \mathbb{P} \) is a definable class in \( \langle H_\kappa, C \rangle \), the corresponding forcing relation \( \Vdash_\kappa \) given by formulas with parameters in \( H_\kappa \cap V^\mathbb{P} \) and whose quantifiers range only over the \( \mathbb{P} \)-names in \( H_\kappa \) is clearly definable in \( \langle H_\kappa, C \rangle \). Moreover, we can prove by induction on \( \phi \) that this relation coincides with the forcing relation as calculated in \( V \), i.e. \( p \Vdash_\kappa \phi^H_\kappa \) iff \( (p \Vdash_\kappa \phi)^H_\kappa \). The case \( \phi \) atomic follows from absoluteness of \( \Delta_1 \) formulas and the case of propositional connectives is easily handled, so we focus on the case \( \phi = \exists x \psi(x) \). Using the previous Lemma 4.3.2

\[
p \Vdash_\kappa \phi^H_\kappa \iff \left\{ q \leq p : \exists \tau \in V^\mathbb{P} q \Vdash (\tau)^H_\kappa \wedge \tau \in H_\kappa \right\} \text{ open dense}
\]

\[
\iff \left\{ q \leq p : \exists \sigma_q \in H_\kappa \cap V^\mathbb{P} q \Vdash (\sigma_q)^H_\kappa \right\} \text{ open dense}
\]

\[
\iff (p \Vdash_\kappa \phi)^H_\kappa
\]

since the intersection of two open dense sets is open dense. It follows that

\[
H_\kappa[G] \models \phi \iff H_\kappa^V[G] \models \phi \iff V[G] \models \phi^H_\kappa \iff \exists p \in G p \Vdash_\kappa \phi^H_\kappa \iff \exists p \in G H_\kappa \models p \Vdash_\kappa \phi
\]

concluding the proof.

**Lemma 4.3.4 (Lifting Lemma, [23 Lemma 17]).** Let \( \langle M, C \rangle \prec \langle M', C' \rangle \) be models of ZFC expanded with additional class predicates \( C \) and \( C' \). Let \( \mathbb{P} \) be a definable class poset in \( \langle M, C \rangle \) that is nice for forcing. Let \( \mathbb{P}' \) be defined by the same formula in \( \langle M', C' \rangle \) (obtained replacing \( C \) with \( C' \)) and suppose that \( \mathbb{P}' \) is also nice for forcing. Then for any \( G \langle M, C \rangle \)-generic for \( \mathbb{P} \) and \( G' \langle M', C' \rangle \)-generic for \( \mathbb{P}' \) such that \( G' \cap M = G \), we have that \( \langle M[G], C, G \rangle \prec \langle M'[G'], C', G' \rangle \).

We remark that since \( \mathbb{P}^\kappa_{\Gamma^f} \) is definable in \( \langle V_\kappa, f \rangle \), the above results are applicable to this kind of iteration (even though the \( \mathbb{P}^\kappa_{\Gamma^f} \) we will be interested with are non-definable classes in \( V_\kappa \) alone).

**Theorem 4.3.5.** Let \( \Gamma \) be weakly iterable in ZFC. Then \( RA_\alpha(\Gamma) \) is consistent relative to MK + the existence of an \( (\alpha) \)-uplifting cardinal.

**Proof.** The proof follows the one of [23 Thm. 18]. Let \( V \) be the standard model of MK. We prove by induction on \( \alpha \) that \( \mathbb{P}_\alpha = \mathbb{P}^{\Gamma^f}_\alpha \), the lottery iteration of \( \Gamma \) relative to a function \( f : \kappa \to \kappa \), forces \( RA_\alpha(\Gamma) \) whenever \( f \) is Menas for \( (\alpha) \)-uplifting.
By Lemma 4.2.11 the existence of such an $f$ follows from the existence of an $(\alpha)$-uplifting cardinal, giving the desired result.

Since RA$_\omega(\Gamma)$ holds vacuously true, the thesis holds for $\alpha = 0$. Suppose now that $\alpha > 0$. Let $\mathcal{Q} \in V_{\kappa^+}$ be a name for a forcing in $\Gamma$, $\beta < \alpha$ be an ordinal. Using the Menas property for $f$, let $g : \lambda \to \lambda$ be such that $(V_{\kappa+1}, f) \triangleleft (V_{\lambda+1}, g)$, $g(\kappa) \geq \text{rank}(\mathcal{Q})$ and $g$ is a Menas for $(\beta)$-uplifting function on $\lambda$. Let $F_\lambda = P_\lambda[g]$ be the lottery iteration of $\Gamma$ relative to $g$.

Since $g|\kappa = f$, by Proposition 4.3.7 we have that:

- $F_\kappa$ is $<\kappa$-cc and is definable in $(V_\kappa, f)$ thus by Lemma 4.3.3 is nice for forcing.
- Similarly, $F_\lambda$ is $<\lambda$-cc and definable in $(V_\lambda, g)$ thus nice for forcing.
- $F_\kappa$ forces $2^\gamma \leq \kappa$ and $F_\lambda$ forces $2^\gamma \leq \lambda$.
- Since $g(\kappa) > \text{rank}(\mathcal{Q})$ and $\hat{\mathcal{Q}}$ is in $\Gamma^{V_\kappa}$, $F_\lambda | p \leq F_\kappa \ast \hat{\mathcal{Q}}$ for a certain $p \in F_\lambda$.

Furthermore, by inductive hypothesis $F_\lambda$ forces RA$_\beta(\Gamma)$. Thus, we only need to prove that $(H_{2^\gamma})^{V_\kappa} \prec (H_{2^\gamma})^{V_\lambda}$. The thesis will then follow by Proposition 4.1.6 since $F_\lambda | p$ would be a legal (and winning) move in $G^{RA}$ after $F_\kappa \ast \hat{\mathcal{Q}}$.

Let $G$ be any $V$-generic filter for $F_\kappa$, $H$ be a $V[G]$-generic filter for val$_G(\mathcal{Q})$. Since $g(\kappa) \geq \text{rank}(\mathcal{Q})$, $\hat{\mathcal{Q}}$ is one of the elements of the lottery sum considered at stage $\kappa + 1$ so that $G \ast H$ is $V$-generic for $F_\lambda | (\kappa + 1)$. Let $G'$ be $V[G \ast H]$-generic for $F_\lambda$. Since $F_\kappa, F_\lambda$ are nice for forcing in the respective models and $(H_{\kappa}, f) \prec (H_{\lambda}, f)$, we can apply Lemma 4.3.2 to obtain that $H_{\kappa}[G] \prec H_{\lambda}[G \ast H \ast G']$. Furthermore by Lemma 4.3.2

$$H_{\kappa}[V[G]] = H_{\kappa}[G] \prec H_{\lambda}[G \ast H \ast G'] = H_{\lambda}[V[G \ast H \ast G']]$$

Since $2^\gamma \leq \kappa$ in $V[G]$ and $2^\gamma \leq \lambda$ in $V[G \ast H \ast G']$, we can restrict the above elementarity obtaining that $H_{2^\gamma}[V[G]] \prec H_{2^\gamma}[V[G \ast H \ast G']]$ and concluding the proof. \hfill $\square$

We also mention the following interesting cases, where generic absoluteness can be obtained at the level of any cardinal $\kappa$.

**Corollary 4.3.6.** Generic absoluteness for the theory of $H_{2^\kappa}$ and $<\kappa$-closed forcing follows from RA$_\omega(<\kappa$-closed) and is consistent relative to an $(\omega)$-uplifting cardinal.

In general, the axioms RA$_\omega(\Gamma)$ for different choices of $\Gamma$ are pairwise incompatible. However, the axioms RA$_\omega(<\kappa$-closed) for different values of $\kappa$ are all compatible together, as shown in the following result.

**Lemma 4.3.7.** RA$_\alpha(<\kappa$-closed) is preserved under $<\kappa^+$-closed forcing.

**Proof.** Assume RA$_\alpha(<\kappa$-closed) holds in $V$, and $B$ is a $<\kappa^+$-closed forcing. We prove that RA$_\alpha(<\kappa$-closed) holds in $V^B$ by induction on $\alpha$ and appealing to the formulation given by Proposition 4.1.6.

Given $\beta < \alpha$ and $C \leq \Gamma_\alpha$ $B$ where $\Gamma_\alpha = <\kappa$-closed, notice that $B$ is $<\kappa^+$-closed thus $<\kappa$-closed hence so is $C$. Then by RA$_\alpha(<\kappa$-closed) in $V$ we can find a $D \leq \Gamma_\alpha \subseteq C$ such that RA$_\beta(<\kappa$-closed) holds in $V^B$ and $H_{2^\kappa} < H_{2^\beta}$. Since $B$ is $<\kappa^+$-closed and $2^\alpha = \kappa^+$ by Proposition 4.1.10, $H_{2^\beta} = H_{2^\alpha}$ hence $H_{2^\beta} < H_{2^\alpha}$, witnessing that RA$_\alpha(<\kappa$-closed) holds in $V^B$. \hfill $\square$
The last lemma directly implies that the axioms \( RA_\alpha(<\omega_\alpha\text{-closed}) \) are pairwise compatible for any two ordinals \( \alpha, \beta \). Moreover, we can also show that these axioms are all simultaneously compatible together.

**Definition 4.3.8.** Let \( \mathbb{B} \) be a complete boolean algebra. Then its closure degree \( \text{cd}(\mathbb{B}) \) is the largest cardinal \( \kappa \) such that there exists a dense subset \( D \subseteq \mathbb{B} \) which is a \( <\kappa \)-closed poset.

**Theorem 4.3.9.** \( RA_\omega(<\kappa\text{-closed}) \) for all cardinals \( \kappa \) simultaneously is consistent relative to a Mahlo cardinal.

**Proof.** Let \( \delta \) be a Mahlo cardinal, and let

\[
S = \{ \kappa < \delta : \kappa \text{ inaccessible } \land \, V_{\kappa+1} \subsetneq V_{\delta+1} \}
\]

be stationary by Lemma 4.2.3. Fix an increasing enumeration \( \langle \kappa_\alpha : \alpha < \delta \rangle \) of a subset of \( S \) which is discontinuous at the limits, and let \( \langle f_\alpha : \kappa_\alpha \rightarrow \kappa_\alpha : \alpha < \delta \rangle \) be a sequence of corresponding Menas for \( (\omega)\)-uplifting functions.

Define the following forcing iteration \( \mathcal{F} = \langle \mathbb{B}_\alpha : \alpha < \delta \rangle 

1. \( \mathbb{B}_0 = 2 \) the trivial boolean algebra;
2. \( \mathbb{B}_{\alpha+1} = \mathbb{B}_\alpha \ast \mathbb{P}_{\Gamma_{\kappa_\alpha}} f_\alpha \) where \( \mathbb{P}_{\Gamma_{\kappa_\alpha}} f_\alpha \) is a \( \mathbb{B}_\alpha \)-name for the lottery iteration of length \( \kappa_\alpha \) guided by \( f_\alpha \) of \( <\omega_\alpha \)-closed forcings;
3. \( \mathbb{B}_\alpha = \lim_{\rightarrow} \{ \mathbb{B}_\beta : \beta < \alpha \} \) whenever \( \alpha \) is inaccessible;
4. \( \mathbb{B}_\alpha = \lim_{\leftarrow} \{ \mathbb{B}_\beta : \beta < \alpha \} \) otherwise.

By Theorem 4.3.5 for any \( \alpha < \delta \) \( \mathbb{B}_{\alpha+1} \) forces that \( RA_\omega(<\omega_\alpha\text{-closed}) \) holds in \( V_{\mathbb{B}_{\alpha+1}} \) thus \( 2^{2^{\omega_\alpha}} = \omega_{\alpha+1} \) by Proposition 4.1.10. We now prove by induction on \( \alpha \leq \delta \) that for every \( \beta < \alpha \), \( \mathbb{B}_\alpha / i_{\beta \alpha}(\dot{G}_\beta) \) is a \( <\omega_\beta \)-closed forcing.

Remark that given any \( <\gamma\text{-closed iteration} \) \( \mathcal{F} = \langle \mathbb{C}_\alpha : \alpha < \lambda \rangle \) (i.e. an \( \mathcal{F} \) such that \( \text{cd}(\mathbb{C}_\beta / i_{\alpha \beta}(\dot{G}_\alpha)) \geq \gamma \) \( \mathbb{C}_\alpha \) = 1 for all \( \alpha \leq \beta < \lambda \) ), we have that \( \lim_{\rightarrow} \mathcal{F} \) is always \( <\gamma\)-closed and \( \lim_{\leftarrow} \mathcal{F} \) is \( <\gamma\)-closed if \( \text{cof}(\lambda) > \gamma \). The proof splits in three cases:

- If \( \alpha = \xi + 1 \) is successor, the thesis holds since \( \dot{C} = \mathbb{B}_\xi / i_{\beta \xi}(\dot{G}_\beta) \) is \( <\omega_\beta \)-closed by inductive hypothesis and \( \mathbb{B}_\alpha / i_{\beta \alpha}(\dot{G}_\beta) \) is the two-step iteration \( \dot{C} * \mathbb{P}_{\Gamma_{\kappa_\xi}} f_\xi \) of \( <\omega_\beta \)-closed forcings.
- If \( \alpha \) is inaccessible, by inductive hypothesis \( (\mathcal{F}|\alpha)/\dot{G}_\beta \) is a \( <\omega_\beta \)-closed iteration system of regular length \( \alpha > \omega_\beta \). It follows that its direct limit \( \lim_{\rightarrow} (\mathcal{F}|\alpha)/\dot{G}_\beta = \mathbb{B}_\alpha / \dot{G}_\beta \) is \( <\omega_\beta \)-closed as well.

---

4 We do not specify here the embeddings corresponding to the iteration system, since they can be easily derived from the standard embeddings defined in Sections 2.2 and 2.3.
• If \( \alpha \) is a limit ordinal which is not inaccessibile, there exists a \( \xi < \alpha, \xi > \beta \) such that \( \omega_\xi > \text{cof}(\alpha) \). By inductive hypothesis, \( \mathcal{C} = \mathcal{B}_{\xi}/_{i_{\xi}\{\check{G}_\beta}\}} \) is \( \omega_\beta \)-closed. Furthermore, by inductive hypothesis \( (\mathcal{F}|\alpha)/_{\check{G}_\xi} \) is a \( \omega_\beta \)-closed iteration system of length \( \text{cof}(\alpha) < \omega_\beta \). It follows that its inverse limit \( \lim\downarrow (\mathcal{F}|\alpha)/_{\check{G}_\xi} = B_\alpha/_{\check{G}_\xi} = \check{D} \) is \( \omega_\beta \)-closed as well. Finally, \( B_\alpha/_{\check{G}_\beta} \) can be split as the two step iteration \( \check{C} * \check{D} \) thus it is \( \omega_\beta \)-closed, concluding the induction.

Since \( B_{\alpha+1} \) forces that \( 2^{2^\alpha} = \omega_{\alpha+1} \) hold in \( V^B_{\alpha+1} \), and \( B_{\delta}/_{\alpha+1,\delta}[\check{G}_{\alpha+1}] \) is \( <\omega_{\alpha+1} = 2^{2^\alpha} \)-closed, in \( V^B_\delta \) the GCH holds up to \( \delta \) which is a limit cardinal. Furthermore, the inaccessible cardinals below \( \delta \) are stationary in \( \delta \) hence in \( \mathcal{F} \) the direct limit is taken stationarily often and by Theorem 2.2.10 \( B_\delta \) is \( \delta \)-cc. Thus, \( \delta \) is inaccessible in \( V^B_\delta \) and \( V^B_\delta[G] = (V^B_\delta)^{V[G]} \) is a model of MK for any \( G \) a \( V \)-generic filter for \( B_\delta \).

We are now able to prove that \( RA_\omega(<>\kappa\text{-closed}) \) holds in \( V^B_\delta \) for all \( \kappa < \delta \). Since \( B_{\delta}/_{\alpha,\delta}[\check{G}_{\alpha}] \) is a class forcing for any \( \alpha < \delta \), we cannot directly apply Lemma 4.3.7. However, its proof can be adapted to work also in this setting.

We prove that \( RA_\omega(<>\omega_\alpha\text{-closed}) \) holds in \( V^B_\delta \) by induction on \( n < \omega \) and appealing to the formulation given by Proposition 4.1.6. Let \( \check{C} \) be a \( B_\delta \)-name for a \( <>\omega_\alpha \)-closed boolean algebra in \( V^B_\delta \). Since \( B_\delta \) is \( <>\delta \)-cc and \( \delta \) is inaccessible in \( V^B_\xi \), there exists a \( \xi < \delta, \xi > \omega_{\alpha+1} \) such that \( \check{C} \) is indeed a \( B_\xi \)-name. Furthermore, \( \check{C} \) has to be \( <>\omega_\alpha \)-closed in \( V^B_\xi \) as well. Since by Lemma 4.3.7 \( RA_\omega(<>\omega_\alpha\text{-closed}) \) holds in \( V^B_\xi \), there exists a further \( \check{D} \leq_{g_{\omega_\alpha}} \check{C} \) such that \( H^B_{2^\omega_{\alpha}} \prec H^B_{2^\omega_{\alpha} * \check{D}} \). Since \( B_{\delta}/_{\xi,\xi}[\check{G}_{\xi}] \) is \( <2^{\omega_\alpha} \)-closed, \( \check{D} \) is \( <>\omega_\alpha \)-closed in \( V^B_\delta \) as well and

\[
H^B_{2^\omega_{\alpha}} = H^B_{2^\omega_{\alpha}} \prec H^B_{2^\omega_{\alpha} * \check{D}} = H^B_{2^\omega_{\alpha} * \check{D}}
\]

concluding the proof. \( \square \)

In Theorem 4.3.5 one notable case was excluded, i.e. \( \Gamma = SSP \) since SSP forcings are not weakly iterable in ZFC. The best known upper bound for the consistency strength of RA(SSP) is given in [40, Thm. 3.1] where it is shown that in the presence of class many Woodin cardinals MM++ implies RA(SSP) (according to Hamkins and Johnstone’s terminology), although it is not clear how this result can be generalized to the axioms RA_\alpha(SSP) we introduced.

Assuming the existence of an \( (\alpha) \)-uplifting cardinal \( \kappa \) which is a limit of supercompact cardinals, we can obtain the consistency of RA_\alpha(SSP) since in this case \( V_\kappa \) models that there are class many supercompact cardinals and under such assumption the class SSP is weakly iterable. We remark that an \( (\alpha) \)-uplifting cardinal which is a limit of supercompact cardinals has lower consistency strength than a Mahlo cardinal which is a stationary limit of supercompact cardinals.

We know that MM++ can be forced by a semiproper iteration of length a supercompact cardinal to get SPFA and \( SP = SSP \). For RA_\alpha(SSP) a similar idea can be applied iteratively, in order to ensure that after each resurrection the equality \( SP = SSP \) still holds. This led to the definition of \( RA_\alpha^{++}(\Gamma) \) and \( (\alpha) \)-uplifting for supercompacts cardinal. In fact, we can now prove that the stronger axiom \( RFA_\alpha^{++}(SSP) \) can be obtained from an \( (\alpha) \)-uplifting for supercompacts cardinal.
Notice that we cannot directly generalize the previous result in order to obtain $\text{RFA}^{++}_\alpha(\text{SSP})$. In fact, a function $f$ that is Menas for $(\alpha)$-uplifting for supercompacts defined in analogy to the one in Lemma 4.2.11 might fail to produce a lottery iteration $\pi^{\text{SSP},f}_\kappa$ that forces $\text{MM}^{++}$. Thus, in order to prove a consistency upper bound for $\text{RFA}^{++}_\alpha(\text{SSP})$ we will change slightly the approach, replacing the lottery iteration with the category forcing $\mathbb{U}_\kappa^{\text{SSP}}$ introduced in [43].

**Theorem 4.3.10.** $\text{RFA}^{++}_\alpha(\text{SSP})$ is consistent relative to an $(\alpha)$-uplifting for supercompacts cardinal.

*Proof.* We prove by induction on $\alpha$ that $\mathbb{U}_\kappa^{\text{SSP}}$, the category forcing of height $\kappa$, forces $\text{RFA}^{++}_\alpha(\text{SSP})$ whenever $\kappa$ is $(\alpha)$-uplifting for supercompacts. Since $\text{RFA}^{++}_0(\text{SSP})$ holds vacuously true, the thesis holds for $\alpha = 0$.

Suppose now that $\alpha > 0$. Let $\mathcal{Q} \in V^{U^{\text{SSP}}}_\kappa$ be a name for a forcing in SSP, $\beta < \alpha$ be an ordinal. Since $\kappa$ is $(\alpha)$-uplifting for supercompacts, let $\lambda > \text{rank}(\mathcal{Q})$ be a $(\beta)$-uplifting for supercompacts cardinal such that $V_{\lambda+1} \prec \Delta_{\lambda}^1 V_{\lambda+1}$. By inductive hypothesis, $U^{\text{SSP}}_\lambda$ forces $\text{RFA}^{++}_\beta(\text{SSP})$, and by [43] Thm. 3.5.2 it forces also $\text{MM}^{++}$.

Let $G$ be any $V$-generic filter for $\mathbb{U}_\kappa^{\text{SSP}}$, $H$ be a $V[G]$-generic filter for $\text{val}_G(\mathcal{Q})$. Since $\lambda \geq \text{rank}(\mathcal{Q})$, $\mathbb{U}_\kappa^{\text{SSP}} \ast \mathcal{Q}$ is in $\mathbb{U}_\kappa^{\text{SSP}}$ hence by [43] Lemma 3.12, Lemma 3.22 there is a $G'$ such that $G \ast H \ast G'$ is generic for $\mathbb{U}_\lambda^{\text{SSP}}$ with $\mathbb{U}_\kappa^{\text{SSP}} \ast \mathcal{Q} \in G \ast H \ast G'$.

By [43] Lemma 3.19, $\mathbb{U}_\kappa^{\text{SSP}}$ preserves the regularity of $\kappa$, hence by Lemma 4.3.3 $\mathbb{U}_\kappa^{\text{SSP}}$ is nice for forcing in $H_\kappa$. Then we can apply Lemma 4.3.4 to obtain $H_\kappa[G] \prec H_\lambda[G \ast H \ast G']$.

Furthermore, since $\mathbb{U}_\kappa^{\text{SSP}}$ preserves the regularity of $\kappa$ and $\mathbb{U}_\lambda^{\text{SSP}}$ preserves the regularity of $\lambda$, we have that $H_\kappa[G] = H_\kappa^{V[G]}$ and $H_\lambda[G \ast H \ast G'] = H_\lambda^{V[G \ast H \ast G']}$. Finally, by [43] Thm. 3.23 $\kappa = \omega_2 = 2^\omega$ in $V[G]$ and $\lambda = \omega_2 = 2^\omega$ in $V[G \ast H \ast G']$, so that $H_\omega^{V[G]} < H_\omega^{V[G \ast H \ast G']}$ concluding the proof.

### 4.3.2 Lower bounds

A lower bound for the axioms $\text{RA}_\alpha(\Gamma)$ can be quickly found noticing that $\text{RA}_\alpha(\Gamma)$ implies the resurrection axiom $\text{RA}(\Gamma)$ introduced by Hamkins and Johnstone for any $\alpha \geq 1$, and lower bounds for the latter with several choices of $\Gamma$ were already given in [23]. For a more detailed analysis, we need the following definition:

**Definition 4.3.11.** The HJ-uplifting game $G^{\text{HJ}}$ is the weakening of game $G^{\text{UP}}$ in Definition 4.2.1 where Player II (uplift) is allowed to play regular cardinals $\kappa_\alpha$ such that $V_{\kappa_\alpha} \prec V_{\kappa_{\alpha+1}}$ (instead of requiring $\Delta^1_\alpha$-elementarity).

We say that $\kappa$ is HJ$(\alpha)$-uplifting iff Player II $(\text{Uplift})$ wins the HJ-uplifting game after $(\langle (\alpha, 0), \kappa \rangle)$.

**Proposition 4.3.12** (AD$(\Delta^0_\theta)$), $\kappa$ is HJ$(\alpha)$-uplifting iff it is regular and for all $\beta < \alpha$, $\theta > \kappa$ there is a $\lambda > \theta$ that is HJ$(\beta)$-uplifting and such that $V_\kappa \prec V_\lambda$.

*Proof.* Follows the one of Proposition 4.2.2. \qed

The HJ$(1)$-uplifting cardinals are exactly the uplifting cardinals introduced by Hamkins and Johnstone in [23], by a reasoning similar to the one shown in Remark
4.3 Consistency strength

4.1.7 Notice that if \( \kappa \) is \((\alpha)\)-uplifting according to our definition it is also \( \text{HJ}(\alpha)\)-uplifting. Since Definition 4.3.11 and Proposition 4.3.12 are based on a second order formalization of set theory such as the one provided by MK, we need to translate all the notions involved in the proof of [23, Thm. 16] to the MK setting.

In particular, we need to define a natural expansion of \( L \) into a model of MK. If the starting model of MK is of the kind \( V_{\kappa+1} \) with \( \kappa \) inaccessible, such expansion is naturally given by considering \( C_L = (V_{\kappa+1})^L = \mathcal{P}(L_\kappa) \cap L \). By [32, Thm. II.6.23], \( C_L = \mathcal{P}(L_\kappa) \cap L_{\kappa^+} \) and every \( x \in L_{\kappa^+} \) has \( |\text{trcl}(x)| \leq \kappa \). This suggests a natural strategy to define the canonical constructible model of the theory MK.

**Definition 4.3.13.** Let \( \langle V, C \rangle \) be a (transitive) model of MK.

Given a well-founded extensional class relation \( E \subseteq X^2 \) in \( C \) and \( x \in X \), we define \( \text{trcl}_E(x) \) as the subclass of \( X \) that represents the transitive closure of \( x \) in the well-founded extensional structure \( \langle X, E \rangle \), identifying \( E \) with the membership relation on \( X \) (it can be easily shown under the above assumptions that \( \text{trcl}_E(x) \in C \)). Let \( E|_x \) denote \( E \cap \text{trcl}_E(x)^2 \).

We say that \( \langle Y, F \rangle \) is a definable powerset of \( \langle X, E \rangle \) if \( F \subseteq Y^2 \) is a well-founded extensional relation, \( X \subseteq Y \), \( F \cap X^2 = E \) and for all \( y \in Y \) there is some formula \( \phi \) in the language of set theory and some \( \bar{a} \in X^{<\omega} \) such that for all \( z \in Y \):

\[
\begin{align*}
z \in F \iff z \in X & \land \langle X, E \rangle \models \phi(z, \bar{a}) \tag{4.1}
\end{align*}
\]

and conversely for all formulae \( \phi \) in the language of set theory and some \( \bar{a} \in X^{<\omega} \) there is some \( y \in Y \) such that (4.1) holds.

**Definition 4.3.14.** Let \( \langle V, C \rangle \) be a (transitive) model of MK. Given a well-founded extensional class relation \( E \subseteq X^2 \) and a class \( A \subseteq X \) all in \( C \), we say that \( \langle X, E \rangle \) is a **constructible initial segment** (c.i.s.) of length \( A \) iff:

- \( L \subseteq X \) and \( E \cap L^2 = \in \),
- \( E' = E|_A \) is a well-order on \( A \) with a maximum and such that the sequence \( \langle \{L_\alpha : \alpha \in \text{ON}\}, \in \rangle \) is an initial segment of \( \langle A, E' \rangle \),
- if \( y \in A \) is the successor of \( x \) in \( E' \), then \( \langle y, E|_y \rangle \) is a definable powerset of \( \langle x, E|x \rangle \),
- if \( y \in A \) is limit in \( E' \), then \( E|_y \) is the union of \( E|x \) for \( x \in \text{E'} \) with \( x \in A \),
- if \( y \in A \) is the maximum of \( E' \), then \( x \in E \) for every \( x \in X \).

Define \( L^+ = \langle L, C_L \rangle \), where \( C_L \) is the collection of classes in \( C \) defined by:

\[
C_L = \{ C \subseteq L : \exists X, E \in C \text{ with } \langle X, E \rangle \text{ c.i.s. } \exists x \in X \text{ such that } \forall y \in L \quad y \in C \iff y \in E \}
\]

Note that the structure \( \langle X, E \rangle \) in the previous definition can be thought as representing the hyper-class \( L_\alpha \) for some \( \alpha \) a meta-ordinal of size \( \text{ON} \), while the class \( A \) which is the length of the c.i.s \( \langle X, E \rangle \) can be thought as the collection of representatives in \( E \) of hyper-classes \( L_\beta \) for \( \beta < \alpha \). Such c.i.s. can be proved
to exist for arbitrary meta-ordinal length, and can be proved to be unique and absolute (modulo isomorphisms) for a fixed length by an adaptation of the usual ZFC arguments for $L_\alpha$ with $\alpha$ ordinal.

Moreover, by adapting the proof of [32, Thm. II.6.23] we can prove that MK holds in $L^+ = \langle L, C_L \rangle$, that $L^+$ is absolute between models of MK with the same ordinals, and hence that $L^+$ is the canonical extension of $L$ to a model $\langle V, C \rangle$ of MK. We are now able to prove the following theorems.

**Theorem 4.3.15.** Assume that $\Gamma$ is weakly iterable and $\text{RA}_1(\Gamma)$ implies that $2^\gamma$ is regular. Then $\text{RA}_\alpha(\Gamma)$ implies that $(2^\gamma)^V$ is HJ($\alpha$)-uplifting in $L^+$.

**Proof.** Since $L^+$ is a model of MK, AD($\Delta_0^1$) holds in $L^+$ and set($L^+)$ = $L$, hence we can resort to the recursive formulation given by Proposition 4.3.12 relativized to $L$. We proceed by induction on $\alpha$. For $\alpha = 0$, $\kappa = 2^\gamma$ is regular in $V$, hence in $L$, concluding the proof.

Assume now that $\alpha > 0$ and let $\beta < \alpha$, $\theta > \kappa$ be arbitrary. If cpd($\Gamma) = \gamma < \infty$, let $\mathbb{B} \in \Gamma$ be such that $\theta^+$ is collapsed to $\gamma$. If cpd($\Gamma) = \infty$ and dd($\Gamma) = \gamma$, let $\mathbb{B} \in \Gamma$ be adding $\theta^+$ subsets of $\gamma$. Thus in both cases $\mathbb{B}$ forces that $2^\gamma$ is bigger than $\theta$. Let $C \leq_\Gamma \mathbb{B}$ be such that $H_{2^\gamma} < H_\gamma^C$ and $\text{RA}_\beta(\Gamma)$ holds in $V^C$. Then $\lambda = (2^\gamma)^C$ is bigger than $\theta$, $H_\kappa < H_\lambda^C$, and by inductive hypothesis $\lambda$ is HJ($\beta$)-uplifting in $L^+$.

Restrict the latter elementarity to $L$ obtaining that $(H_\kappa)^L \prec (H_\lambda)^L$. It follows that $\kappa$, $\lambda$ cannot be successor cardinals in $L$, hence they need to be inaccessible obtaining that $(H_\kappa)^L = (V_\kappa)^L$ and the same for $\lambda$, concluding the proof. \qed

We remark that the hypothesis that wRA$_1(\Gamma)$ implies that $2^\gamma$ is regular is true whenever $\gamma = \text{cpd}(\Gamma) < \infty$ by Proposition 4.1.10.

There might be a gap between the lower and upper bounds showed in this section, due to the fact that the $\Delta^1_1$-elementarity between $V_\kappa$ and $V_\lambda$ is not exploited to give additional properties between the corresponding $H_{2^\gamma}$ in the forcing extensions. Nonetheless these bounds are very close together (since both are between the axioms scheme “ON is Mahlo” and “there exists a Mahlo cardinal”), so we claim these results to be quite satisfactory.

### 4.4 Conclusions and open problems

In this chapter we showed how strong generic absoluteness results can be obtained from forcing axioms of relatively low consistency strength. This was achieved via an iterated version of a base axiom. Thus it is natural to ask whether this same procedure can be carried out for different base axioms and to inquire on the properties that such base axioms might have. This gives rise to the following.

**Question 4.4.1.** What is the consistency strength of $\text{RA}_\omega(\Gamma)$, and what degree of generic absoluteness can entail?

This is currently a work in progress [3]. Whereas the consistency strength in this case seems to be quite mild, it is not clear whether it is possible to infer a stronger generic absoluteness result from $\text{RA}_\omega(\Gamma)$ than the one we can obtain from $\text{RA}_\omega(\Gamma)$.  

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4.4 Conclusions and open problems

**Question 4.4.2.** What is the consistency strength of UR_α(Γ), and what degree of generic absoluteness can entail?

We suspect an I_3 cardinal to be needed for the consistency of UR_α(Γ). On the other hand, it is likely that UR_α(Γ) would be able to extend the generic absoluteness we have for H_{2^ω} to all initial segments of L(ON^{<ω}). This would give an intermediate axiom in between MM^{++} (see [43]) and RA_ω(SSP).

In this chapter we focused on the ω-iterated axiom RA_ω(Γ). Another possible direction of research is considering whether a larger amount of iteration might affect the axioms significantly. This not only regards RA_α(Γ) for α > ω, but also a stronger RA_∞(Γ), which can be defined as follows.

**Definition 4.4.3.** The long Γ-resurrection game G^{RA_∞} is the game of length ω defined as follows. Player I (Kill) plays boolean algebras B_{n+1} such that B_{n+1} ≤_Γ C_n. Player II (Resurrect) plays boolean algebras C_n such that H_{2^γ}^{C_n} ⊊ H_{2^γ}^{C_{n+1}} and C_n ≤_Γ B_n. Player II looses if it cannot move, and wins if the game reaches full length.

\[ \text{I (Kill).} \]
\[ \begin{array}{ccc} B_0 & \cdots & B_1 \\ \vdots & \ddots & \vdots \\ C_0 & \cdots & C_1 \end{array} \]

II (Resurrect).

RA_∞(Γ) is the assertion that Player II (Resurrect) wins the long Γ-resurrection game after (2, 2).

Since the game G^{RA_∞} is open, it is still determined in MK and a theory similar to that of G^{RA} can be carried out. Furthermore, a similar definition of (∞)-uplifting cardinal can be outlined and proved to follow from a Mahlo cardinal, while an easy adaptation of the results in this chapter can show that RA_∞(Γ) is consistent relative to a Menas for (∞)-uplifting cardinal. However, it is not clear how to obtain a Menas for (∞)-uplifting cardinal from an (∞)-uplifting cardinal.

**Question 4.4.4.** What is the consistency strength of RA_∞(Γ)? Does RA_∞(Γ) or RA_{ON}(Γ) entail stronger generic absoluteness properties than RA_ω(Γ)?

Furthermore, we mentioned after Proposition 4.1.11 that even though RA(Γ) as defined by Hamkins and Johnstone is compatible with CH for most classes Γ, their proof fails for RA_1(Γ) and it is not at all clear whether it can be recovered by other means. This leads to the following question.

**Question 4.4.5.** For which classes Γ can RA_ω(Γ) be compatible with the GCH?

In Theorem 4.3.15 we were not able to prove that RA_ω(Γ) implies that 2^γ is regular when γ = dd(Γ) (hence when any forcing in Γ preserves all cardinals). This implies also that a precise lower bound for RA_ω(Γ) is missing in this case. However, we know that RA_ω(Γ) implies BFA_κ(Γ) for all κ ∈ [γ, 2^γ), which in the case Γ = ccc (the only case for cpd(Γ) = ∞ we considered) gives that 2^{ω} = 2^{κ} for all such κ and hence that 2^γ is regular. It is therefore natural to investigate on the following question.
**Question 4.4.6.** Does $\text{RA}_\omega(\Gamma)$ implies that $2^\gamma$ is regular?

Finally, in this chapter we developed the axiom $\text{RA}_\omega(\Gamma)$ and we showed that it gives generic absoluteness for the theory of $H_{2^\gamma}$ and forcings in $\Gamma$. However, the unique example with $2^\gamma > 2^\omega$ is given by choosing $\Gamma = <\gamma$-closed for some $\gamma > \omega$. This is a very narrow forcing class among the ones which preserve $\gamma$.

This situation is due to the lack of fully satisfactory results regarding forcing iterations preserving cardinals $\gamma > \omega_1$. However, the arguments presented in this chapter show that the two topics are strongly connected, so that an iteration theorem for (a certain notion of) $\gamma$-proper forcings immediately entails a generic absoluteness theorem for the same class of forcings and the theory of $H_{2^\gamma}$. Conversely, a counterexample to generic absoluteness for $H_{2^\gamma}$ with respect to (a certain notion of) $\gamma$-proper forcings prevents the possibility of having an iteration theorem for the same class of forcings. This naturally leads us to the last open problem.

**Question 4.4.7.** Can $\text{RA}_\alpha(\Gamma)$ be consistent for $\text{cpd}(\Gamma) = \gamma > \omega_1$, $\Gamma \supseteq <\gamma$-closed? That is, are there weakly iterable $\Gamma \supseteq <\gamma$-closed which preserve $\gamma$?
CHAPTER 5

SYSTEMS OF FILTERS AND GENERIC LARGE CARDINALS

In this chapter we introduce the notion of $C$-system of filters, a concept generalizing the well-known definitions of both extenders and towers. In this framework we investigate when extenders and towers happen to be equivalent, together with some natural questions regarding definability of generic large cardinals. In particular, we consider the difference between having a generic large cardinal property ideally or generically and study how the large cardinal properties of an embedding reflect into the structure of the derived $C$-system of filters. All the material in this chapter is joint work with Silvia Steila [4]. We remark that the main contribution of this chapter is given by the definition of $C$-system of filters, and that almost all mentioned results are adaptations of well-known arguments to this new setting.

Section 5.1 introduces the concept of $C$-system of filters and develops their general theory. Section 5.2 addresses some issues regarding generic large cardinals, using the machinery previously developed. Section 5.3 gives an example showing that the towers in $V$ induced from a generic embedding could have very little in common with the forcing initially used to originate them.

5.1 Systems of filters

In this section we present the definition and main properties of $C$-systems of filters. This notion has both classical extenders [29, 31], ideal extenders (recently introduced by Claverie in [9]) and towers [11, 33, 43] as special cases, and it is able to generalize and subsume most of the standard results about extenders and towers. We recall here the standard definitions of $(\kappa, \lambda)$-extender (see e.g. [31]) and tower of height $\lambda$ (see e.g. [43]) in the form that is more convenient to us.

Given $a, b \in [\lambda]^{<\omega}$ such that $b = \{\alpha_0, \ldots, \alpha_n\} \supseteq a = \{\alpha_{i_0}, \ldots, \alpha_{i_m}\}$ and $s = \{s_0, \ldots, s_n\}$, let $\pi_{ba}(s) = \{s_{i_0}, \ldots, s_{i_m}\}$.

\(^1\)Here and in the following we assume that finite sets of ordinals are always implicitly ordered by the natural ordering on the ordinals.
5 Systems of filters and generic large cardinals

**Definition 5.1.1.** $E = \{ F_a : a \in [\lambda]^{<\omega} \}$ is a standard $\langle \kappa, \lambda \rangle$-extender with supports $\langle \kappa_a : a \in [\lambda]^{<\omega} \rangle$ iff the following holds.

1. (Filter property) For all $a \in [\lambda]^{<\omega}$, $F_a$ is a $<\kappa$-complete filter on $[\kappa_a]^{[a]}$ and $\kappa_a$ is the least $\xi$ such that $[\xi]^{[a]} \in F_a$;

2. (Compatibility) if $a \subseteq b \in [\lambda]^{<\omega}$ then
   
   (a) $\kappa_a \leq \kappa_b$;
   
   (b) if $\max(a) = \max(b)$, then $\kappa_a = \kappa_b$;
   
   (c) $A \in F_a$ iff $\pi_{ba}^{-1} [A] \in F_b$;

3. (Uniformity) $\kappa_{\{\kappa\}} = \kappa$;

4. (Normality) Assume that $a \in [\lambda]^{<\omega}$, $A \in I_a^+$ where $I_a$ is the dual of $F_a$, $u : A \rightarrow \kappa_a$, $i \in |a|$ are such that $u(s) \in s_i$ for all $s \in A$. Then there exist $\beta \in a_i$, $b \supseteq a \cup \{ \beta \}$ and $B \leq_{\mathbb{E}} A$ (i.e. such that $\pi_{pa}^{-1} [A] \supseteq B$) with $B \in I_b^+$ such that for all $s \in B$, $u(\pi_{ba}(s)) = s_j$, where $b_j = \beta$.

**Definition 5.1.2.** $T = \{ F_a : a \in V_\lambda \}$ is a standard tower of height $\lambda$ iff the following holds.

1. (Filter property) For all $a \in V_\lambda$, $F_a$ is a non trivial filter on $\mathcal{P}(a)$;

2. (Compatibility) For all $a \subseteq b$, $A \in F_a$ iff $A \uparrow b = \{ X \subseteq b : X \cap a \in A \} \in F_b$;

3. (Fineness) For all $a \in V_\lambda$ and $x \in a$ we have $\{ X \subseteq a : x \in X \} \in F_a$;

4. (Normality) Given $A \in I_a^+$, $u : A \rightarrow V$ such that $u(X) \in X$ for any $X \in A$, there exist $b \supseteq a$, $B \in I_b^+$ with $B \leq_{\mathbb{E}} A$ (i.e. such that $A \uparrow b \supseteq B$), and a fixed $y$ such that $u(X \cap a) = y$ for all $X \in B$.

### 5.1.1 Main definitions

Throughout this section let $V$ denote a transitive model of ZFC.

**Definition 5.1.3.** We say that a set $C \in V$ is a directed set of domains iff the following holds:

1. (Ideal property) $C$ is closed under subsets and unions;

2. (Transitivity) $\bigcup C$ is transitive, i.e. for every $y \in x \in a \in C$ we have $y \in \bigcup C$ (or, equivalently in presence of the ideal property, $\{y\} \in C$).

We say that $C$ has length $\lambda$ iff $\text{rank}(C) = \lambda$, and that $C$ is $<\gamma$-directed iff it is closed under unions of size $<\gamma$ in $V$.

**Example 5.1.4.** In the case of extenders, $C$ will be $[\lambda]^{<\omega}$, while for towers it will be $V_\lambda$. The first is absolute between transitive models of ZFC, while the latter is $<\lambda$-directed whenever $\lambda$ is regular. These two different properties entail most of the differences in behaviour of these two objects.
5.1 Systems of filters

Definition 5.1.5. Let $\mathcal{C} \in V$ be a directed set of domains. Given a domain $a \in \mathcal{C}$, we define $O_a$ as the set of functions

$$O_a = \{ \pi_M \mid (a \cap M) : M \subseteq \text{trcl}(a), M \in V \text{ extensional} \}$$

where $\pi_M$ is the Mostowski collapse map of $M$. If $a \subseteq b$, we define the standard projection $\pi_{ba} : O_b \rightarrow O_a$ by $\pi_{ba}(f) = f\mid a$.

We shall sometimes denote $\pi_{ba}$ by $\pi_a$ and $\pi^{-1}_{ba}$ by $\pi^{-1}_b$ when convenient. Notice that every $f \in O_b$ is $\in$-preserving, and that $\pi_{ba}(f) = f\mid a \in O_a$ for any $a \subseteq b$, so that $\pi_{ba}$ is everywhere defined. From now on we shall focus on filters on the boolean algebra $\mathcal{P}^V(O_a)$ for $a \in \mathcal{C}$ and $\mathcal{C} \in V$ a directed set of domains.

Example 5.1.6. In the case of extenders, any $f \in O_a$ will be an increasing function from the sequence $a \in [\lambda]^{<\omega}$ to smaller ordinals. $O_a$ can be put in correspondence with the domain $\kappa_a[a]$ of a standard extender via the mapping $f \mapsto \text{ran}(f)$, $\pi_{ba}$ will correspond in the new setting to the usual notion of projection for extenders.

In the case of towers, any $f \in O_a$ with a transitive will be the collapsing map of a $M \subseteq a$. In this case $O_a$ can be put in correspondence with the classical domain $\mathcal{P}^V(a)$ via the mapping $f \mapsto \text{dom}(f)$, and $\pi_{ba}$ will correspond to the usual notion of projection for towers.

A complete proof of the above mentioned equivalences can be found in Section 5.1.2

Definition 5.1.7. Define $x \preceq y$ as $x \in y \lor x = y$. We say that $u : O_a \rightarrow V$ is regressive on $A \subseteq O_a$ iff for all $f \in A$, $u(f) \preceq f(x_f)$ for some $x_f \in \text{dom}(f)$. We say that $u$ is guessed on $B \subseteq O_b$, $b \supseteq a$ iff there is a fixed $y \in b$ such that for all $f \in B$, $u(\pi_{ba}(f)) = f(y)$.

Definition 5.1.8. Let $V \subseteq W$ be transitive models of ZFC and $\mathcal{C} \in V$ be a directed set of domains. We say that $S = \{ F_a : a \in \mathcal{C} \} \in W$ is a $\mathcal{C}$-system of $V$-filters, and we equivalently denote $S$ also by $\{ I_a : a \in \mathcal{C} \}$ where $I_a$ is the dual ideal of $F_a$, iff the following holds:

1. (Filter property) for all $a \in \mathcal{C}$, $F_a$ is a non-trivial filter on the boolean algebra $\mathcal{P}^V(O_a)$;
2. (Fineness) for all $a \in \mathcal{C}$ and $x \in a$, $\{ f \in O_a : x \in \text{dom}(f) \} \in F_a$;
3. (Compatibility) for all $a \subseteq b$ in $\mathcal{C}$ and $A \subseteq O_a$, $A \in F_a \iff \pi^{-1}_{ba}[A] \in F_b$;
4. (Normality) every function $u : A \rightarrow V$ in $V$ that is regressive on a set $A \in I^+_a$ for some $a \in \mathcal{C}$ is guessed on a set $B \in I^+_b$ for some $b \in \mathcal{C}$ such that $B \subseteq \pi^{-1}_{ba}[A]$;

We say that $S$ is a $\mathcal{C}$-system of $V$-ultrafilters if in addition:

5. (Ultrafilter) for all $a \in \mathcal{C}$, $F_a$ is an ultrafilter on $\mathcal{P}^V(O_a)$.

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We shall feel free to drop the reference to $V$ when clear from the context, hence denote the $C$-systems of $V$-filters as $C$-systems of filters. When we believe that this convention may generate misunderstandings we shall be explicitly more careful. To clearly distinguish $C$-systems of filters from $C$-systems of ultrafilters, in the following we shall use $S$, $E$, $T$ for the first and $S$, $E$, $T$ for the latter.

**Definition 5.1.9.** Let $S$ be a $C$-system of filters, $a$ be in $C$. We say that $\kappa_a$ is the *support* of $a$ iff it is the minimum $\alpha$ such that $O_a \cap {}^a V_\alpha \in F_a$. We say that $S$ is a $\langle \kappa, \lambda \rangle$-system of filters if and only if:

- it has length $\lambda$ and $\kappa \subseteq \bigcup C$,
- $F(\gamma)$ is principal generated by $id \mid \{ \gamma \}$ whenever $\gamma < \kappa$,
- $\kappa_a \leq \kappa$ whenever $a \in V_{\kappa+2}$.

Notice that $\kappa_a \leq \text{rank}(a)$, and $\kappa_a = \text{rank}(a)$ when $F_a$ is principal as in the above definition. In particular, $\kappa_{\langle \gamma \rangle} = \gamma + 1$ in this case. The definition of $C$-system of filters entails several other properties commonly required for coherent systems of filters.

**Proposition 5.1.10.** Let $S$ be a $C$-system of filters. Then $dF_a = \{ \text{dom}[A] : A \in F_a \}$ is a normal and fine filter on $P(a)$, for any $a$ in $C$ infinite. In particular if $a$ is uncountable, $\text{dom}[A]$ is stationary for all $A \in F_a$.

**Proof.** Filter property and fineness follow directly from restricting the corresponding points in Definition [5.1.8](#) to $\text{dom}[O_a]$. We now focus on normality. Let $u : D \rightarrow a$ where $D = \text{dom}[A]$ be such that $u(X) \in X$ for all $X \in D$ (i.e. $X = \text{dom}(f)$ for some $f$ in $A$). Then we can define $v : A \rightarrow V$ as $v(f) = f(u(\text{dom}(f)))$. Let $B \in I^+_b$, $y \in b$ be such that $v(\pi_{ba}(f)) = f(u(\text{dom}(\pi_{ba}(f)))) = f(y)$ for all $f \in B$ by normality. Since every $f \in B$ is injective, $u(\text{dom}(\pi_{ba}(f))) = y$ for all $f \in B$ hence $u$ is constant on $\text{dom}[B] \in dI^+_a$. By Lemma [1.2.8](#) if $a$ is uncountable we conclude that $\text{dom}[A]$ is stationary for any $A \in F_a$. □

**Proposition 5.1.11.** Let $S$ be a $C$-system of filters, $a, b$ be in $C$. Then $\text{rank}(a) \leq \text{rank}(b)$ implies that $\kappa_a \leq \kappa_b$, i.e. the supports depend (monotonically) only on the ranks of the domains.

**Proposition 5.1.12.** Let $S$ be a $\langle \kappa, \lambda \rangle$-system of filters, $a$ be in $C$. Then $F_a$ is $\langle \kappa, \lambda \rangle$-complete.

We defer the proof of the last two propositions to Section [5.1.4](#) (just before Proposition [5.1.30](#)) for our convenience. We are now ready to introduce the main practical examples of $C$-system of filters.

**Definition 5.1.13.** Let $V \subseteq W$ be transitive models of ZFC.

- $E \in W$ is an *ideal extender* on $V$ iff it is a $\langle \lambda \rangle^{<\omega}$-system of filters on $V$ for some $\lambda$. $E \in W$ is an *extender* on $V$ iff it is a $\langle \lambda \rangle^{<\omega}$-system of ultrafilters on $V$.
- $E \in W$ is an *ideal $\gamma$-extender* on $V$ iff it is a $\langle \lambda \rangle^{<\gamma}$-$V$-system of filters for some $\lambda$. $E$ is a $\gamma$-extender on $V$ iff it is a $\langle \lambda \rangle^{<\gamma}$-$V$-system of ultrafilters on $V$.
- $T \in W$ is an *ideal tower* iff it is a $V_\lambda$-system of filters for some $\lambda$. $T \in W$ is a *tower* iff it is a $V_\lambda$-system of ultrafilters.
The above definitions of extender and tower can be proven equivalent to the classical ones presented at the beginning of this section (see also [31, 33]) via the mappings \( \text{ran}_a : O_a \to \mathcal{P}(a) \), \( f \mapsto \text{ran}(f) \) (for extenders) and \( \text{dom}_a : O_a \to \mathcal{P}(a) \), \( f \mapsto \text{dom}(f) \) (for towers). Furthermore, \( \langle \kappa_a : a \in \mathcal{C} \rangle \) correspond to the supports of long extenders as defined in [31]. A detailed account of this correspondence is given in Section 5.1.2.

Given a \( \mathcal{C} \)-system of \( V \)-filters \( S \), we can define a preorder \( \leq_S \) on the collection \( S^+ = \{ A : \exists a \in \mathcal{C} A \in I^+_a \} \) as in the following.

**Definition 5.1.14.** Given \( A \in I^+_a, B \in I^+_b \) we say that \( A \leq_S B \) iff \( \pi^{-1}_a[A] \leq_{c_b} \pi^{-1}_b[B] \) where \( c = a \cup b \), and \( A =_S B \) iff \( A \leq_S B \) and \( B \leq_S A \).

Consider the quotient \( S^+/=_S \). With an abuse of notation for \( p, q \in S^+/=_S \), we let \( p \leq_S q \) iff \( A \leq_S B \) for any (some) \( A \in p, B \in q \). The partial order \( \langle S^+/=_S, \leq_S \rangle \) is a boolean algebra which is the limit of a directed system of boolean algebras, and can be used as a forcing notion in order to turn \( S \) into a system of ultrafilters. This process will be described in Section 5.1.3.

**Proposition 5.1.15.** Let \( \mathcal{C} \) be a \( <\gamma \)-directed set of domains, \( S \) be a \( \mathcal{C} \)-system of filters. Then \( \langle S^+/=_S, \leq_S \rangle \) forms a \( <\gamma \)-closed boolean algebra.

**Proof.** Let \( \mathcal{A} = \langle A_\alpha : \alpha < \mu \rangle \subseteq S^+ \) be such that \( \mu < \gamma \) with \( A_\alpha \in I^+_a \). Since \( \mathcal{C} \) is \( <\gamma \)-directed, there is a domain \( a \in \mathcal{C} \) with \( |a| \geq \mu \) such that \( a_\alpha \subseteq a \) for all \( \alpha < \mu \). Fix \( \langle x_\alpha : \alpha < \mu \rangle \) a (partial) enumeration of \( a \), and define

\[
B = \{ f \in O_a : \forall \alpha < \mu \ x_\alpha \in \text{dom}(f) \Rightarrow f \in \pi^{-1}_a[A_\alpha] \}.
\]

First, \( B \leq_S A_\alpha \) for all \( \alpha < \mu \) by fineness, since \( \{ f \in B : x_\alpha \in \text{dom}(f) \} \subseteq \pi^{-1}_a[A_\alpha] \). Suppose now by contradiction that for some \( c \supset a \), \( C \subseteq I^+_c \) is such that \( C \leq_S A_\alpha \) for all \( \alpha < \mu \) and \( C \cap \pi^{-1}_c[B] = \emptyset \). Then for any \( f \in C \) we can find an \( \alpha_f < \mu \) such that \( x_{\alpha_f} \in \text{dom}(f) \) and \( f \notin \pi^{-1}_c[A_{\alpha_f}] \). Define

\[
u : C \longrightarrow V
\]

\[
u(f) = f(x_{\alpha_f})
\]

By normality we can find a single \( \bar{\alpha} \) and a \( \bar{c} \supset c \cup \{ \bar{\alpha} \} \) such that

\[D = \{ f \in \pi^{-1}_c[C] : \nu(p_d(f)) = f(x_{\bar{\alpha}}) \} \in I^+_d.
\]

Thus \( D \cap A_\alpha = \emptyset \) and \( D \leq_S C \leq_S A_{\bar{\alpha}}, \) a contradiction.

5.1.2 Standard extenders and towers as \( \mathcal{C} \)-systems of filters

**Extenders.** We now compare the definition of \( \langle \kappa, \lambda \rangle \)-extender just introduced (Definition 5.1.13) with the definition of standard \( \langle \kappa, \lambda \rangle \)-extender (Definition 5.1.1).

Let \( \mathcal{E} \) be a \( \langle \kappa, \lambda \rangle \)-extender with supports \( \langle \kappa_a : a \in [\lambda]^{<\omega} \rangle \) according to Definition 5.1.13. Notice that given any \( a \in [\lambda]^{<\omega} \), the collection

\[O'_a \equiv \{ f \in O_a : \text{dom}(f) = a \land \text{ran}(f) \subseteq \kappa_a \}\]
is in $F_a$ by fineness and definition of $\kappa_a$. Consider the injective map $\text{ran}_a : O'_a \to [\kappa_a]^{\omega}$, which maps $F_a$ into a corresponding filter $F'_a$ on $[\kappa_a]^{\omega}$ that is the closure under supersets of $\{\text{ran}_a[A \cap O'_a] : A \in F_a\}$. Notice that many sequences $s \in [\kappa_a]^{\omega}$ cannot be obtained as the range of Mostowski collapse maps, e.g. $s = \{\beta, \beta + 2\}$ whenever $a$ is of the kind $\{\alpha, \alpha + 1\}$.

Let us denote with $\pi'_{ba}$ the projection map from $F'_b$ to $F'_a$ in the standard case. Notice that for any $a \subseteq b \in [\lambda]^{<\omega}$ and $f \in O_b$, $\text{ran}_a(\pi_{ba}(f)) = \pi'_{ba}(\text{ran}_b(f))$. Define $E' = \{F'_a : a \in [\lambda]^{<\omega}\}$. We claim that $E'$ is a $\langle \kappa, \lambda \rangle$-extender with respect to the standard definition whenever $E$ is a $\langle \kappa, \lambda \rangle$-extender.

**Proposition 5.1.16.** If $E$ is a $\langle \kappa, \lambda \rangle$-extender then $E'$ is a standard $\langle \kappa, \lambda \rangle$-extender.

**Proof.** 1. (Filter property) It follows since $F'_a$ is an injective image of $F_a | O'_a$.

2. (Compatibility)

(a-b) Follow by Proposition 5.1.11 since rank($a$) depends only on max($a$).

(c) By compatibility of $E$, it follows: $A' = \text{ran}_a[A] \in F'_a$ iff $A \in F_a$ iff $\pi_{ba}^{-1}[A] \in F_b$ iff $\text{ran}_b(\pi_{ba}^{-1}[A]) = \pi'_{ba}(\text{ran}_b(f))$.

3. (Uniformity) By definition of $\langle \kappa, \lambda \rangle$-system of filters.

4. (Normality) Given $a \in [\lambda]^{<\omega}$, $A' = \text{ran}_a[A] \in F'_a$, $u : A' \to \kappa_a$, $i < |a|$ such that $u(s) \in s(i)$ for all $s \in A'$, let $\alpha = a_i$. Define

$$v : A \to V \quad f \mapsto u(\text{ran}_a(f)).$$

Since $s_i = f(\alpha)$, $v$ is regressive. By normality of $E$, there exist $\beta$ and $B \subseteq \pi_{ba}^{-1}[A]$, where $\beta \in b \supseteq a$ such that for all $f \in B$, $v(\pi_{ba}(f)) = f(\beta)$. Since $\pi_{ba}(f) \in A$, $f(\beta) = v(\pi_{ba}(f)) \in f(\alpha)$. Since $f$ is $\in$-preserving, $\beta \in \alpha$ and $B' = \text{ran}_b[B]$ witnesses normality of $E'$. \[ \square \]

On the other hand, given a standard $\langle \kappa, \lambda \rangle$-extender $E'$ we can define a collection of corresponding filters $F_a$ on $O'_a$ for any $a \in [\lambda]^{<\omega}$. This can be achieved since $\text{ran}_a[O'_a] \subseteq F'_a$ for any $a \in [\lambda]^{<\omega}$ and standard $\langle \kappa, \lambda \rangle$-extender $E[2]$. Let $E$ consists of the closure of $F_a$ under supersets in $O_a$, for any $a \in [\lambda]^{<\omega}$. Then we can show the following.

**Proposition 5.1.17.** If $E'$ is a standard $\langle \kappa, \lambda \rangle$-extender then $E$ is a $\langle \kappa, \lambda \rangle$-extender.

**Proof.** 1. (Filter property) Follows directly from the filter property of $E'$. 2. (Compatibility) By compatibility of $E'$ and unfolding definitions, $A \in F_a$ iff $A \cap O'_a \subseteq F_a$ iff $\text{ran}_a[A \cap O'_a] \in F'_a$ iff $\text{ran}_b(\pi_{ba}^{-1}[A \cap O'_a]) \in F'_b$ iff $\pi_{ba}^{-1}[A \cap O'_a] \subseteq F_b$ iff $\pi_{ba}^{-1}[A] \subseteq F_b$. 3. (Fineness) For any $a \in O_a$, $\{f \in O_a : f \in \text{dom}(f)\} \supseteq O'_a \subseteq F_a$. 2This fact can be proved directly using the corresponding version of Proposition 5.1.28 (i.e. $A \in F'_a$ iff $a \in j(A)$) and Los Theorem 5.1.26 for standard extenders.
4. (Normality) Assume that $A \in I^+_a$ and that $u : A \to V$ is regressive. By definition of regressive function we have that $A = A_0 \cup A_1$, where

$$A_0 = \{ f \in A : \exists x \in \text{dom}_a(f)(u(f) = f(x)) \};$$

$$A_1 = \{ f \in A : \exists x \in \text{dom}_a(f)(u(f) \in f(x)) \}.$$

We have two cases.

- If $A_0 \in I^+_a$ there exists a fixed $x \in a$ such that $B = \{ f \in A : u(f) = f(x) \}$ is in $I^+_a$ since $a$ is finite. Hence $B$ and $x$ witness normality for $E$.

- Otherwise, if $A_0 \in I_a$, we have that $A_1 \in I^+_a$. Since $a$ is finite, there exists $x = a_i \in a$ such that $A^* = \{ f \in A : u(f) \in f(x) \} \in I^+_a$. Let $A' = \text{ran}_a[A^* \cap O^a]$. Let $v : A' \to \kappa_a$ be such that $v(s) = u(f)$ for any $s = \text{ran}_a(f)$, so that $v(s) \in s$. By normality of $E'$ there exist $\beta = a_i$ and $B' \subset V$ with $B' = \text{ran}_a[\beta] \in I^+_b$ for $b = a \cup \{ \beta \}$ such that for all $s \in B'$, $v(\pi_{ba}(s)) = s$, where $b_j = \beta$. Hence for any $f \in B$, $u(\pi_{ba}(f)) = f(\beta)$.

**Towers.** We now compare the definition of tower just introduced (Def. 5.1.13) with the definition of standard tower (Def. 5.1.2). Let $T$ be a tower of length $\lambda$. Notice that the whole tower can be induced from the filters $F_a$ where $a$ is a transitive set. Furthermore, whenever $a$ is transitive the map $\text{dom}_a : O_a \to \mathcal{P}(a)$ is a bijection.

In fact, any $\lambda$ is isomorphic to $\mathcal{P}(a)$. Thus we can map any $F_a$ with a transitive into an isomorphic filter $F'_a = \{ \text{dom}_a[A] : A \in F_a \}$ on $\mathcal{P}(a)$. Define $T' = \{ F'_a : a \in V_\lambda \}$, then we can prove the following.

**Proposition 5.1.18.** If $T$ is a tower of length $\lambda$ then $T'$ is a standard tower of length $\lambda$.

**Proof.**

1. (Filter property) It follows since $F'_a$ is isomorphic to $F_a$.

2. (Compatibility) Due to compatibility of $T$, $A' = \text{dom}_a[A] \in F'_a$ iff $A \in F_a$ iff $\pi_{ba}^{-1}[A] \in F_b$ iff $\text{dom}_a[\pi_{ba}^{-1}[A]] = \{ X \subseteq b : X \cap a \in A' \} \in F'_b$.

3. (Fineness) Follows directly from fineness of $T$.

4. (Normality) Let $A' = \text{dom}_a[A] \in I^+_a$, $u : A' \to V$ be such that $u(X) \in X$ for all $X \in A'$. Since $u(\text{dom}_a(f)) \in \text{dom}_a(f)$, we can define

$$v : A \to V$$

$$f \mapsto f(u(\text{dom}_a(f))).$$

that is regressive on $A$. Thus by normality of $T$ there exist $a \subseteq b \in V_\lambda$, $B \subseteq \pi_{ba}^{-1}[A]$ and $y \in b$ such that $B \in I^+_b$ and for any $f \in B$, $v(\pi_{ba}(f)) = f(u(\text{dom}(f) \cap a)) = f(y)$. Since $f$ is injective, $u(\text{dom}(f) \cap a) = y$, hence $B' = \text{dom}_b[B]$ and $y$ witness normality of $T'$.

**Proposition 5.1.19.** If $T'$ is a standard tower of length $\lambda$ then $T$ is a tower of length $\lambda$.

**Proof.**

1. (Filter property) It follows since $F'_a$ is isomorphic to $F_a$.  

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2. (Compatibility) By compatibility of $\mathbb{T}'$, $A \in F_a$ iff $\text{dom}_a[A] \in F'_a$ iff

$$ \{ X \subseteq b : X \cap a \in \text{dom}_a[A] \} = \text{dom}_b[\pi^{-1}_{ba}[A]] \in F'_b $$

iff $\pi^{-1}_{ba}[A] \in F_b$.

3. (Fineness) Follows directly from fineness of $\mathbb{T}'$.

4. (Normality) Assume that $A \in I^+_a$ and that $u : A \to V$ is regressive, i.e. for all $f \in A$ there exists $x_f \in \text{dom}(f)$ such that $u(f) \subseteq f(x_f)$. Since $f = \pi_{\text{dom}(f)}$ and $a$ is transitive, $u(f) \in f(x_f)$ implies that there is an $y_f \in x_f \in \text{dom}(f)$ such that $u(f) = f(y_f)$. Thus we can assume without loss of generality that $x_f$ is such that $u(f) = f(x_f)$ and define

$$ v : \text{dom}_a[A] \to V $$

$$ \text{dom}(f) \mapsto x_f. $$

Then $v(X) \in X$ for all $X \in \text{dom}_a[A]$ and we can apply normality of $\mathbb{T}'$ to find a $\text{dom}_b[B] <_\mathbb{T} \text{dom}_a[A]$, $B \in I^+_b$ and a fixed $y$ such that for any $\text{dom}_b(f) \in \text{dom}_b[B]$ $v(\text{dom}_b(f) \cap a) = v(\text{dom}_a(\pi_{ba}(f))) = x_{\pi_{ba}(f)} = y$. Thus for any $f \in B$, $u(\pi_{ba}(f)) = f(y)$. \qed

### 5.1.3 Systems of filters in $V$ and generic systems of ultrafilters

In this section we shall focus on ideal extenders and ideal towers in $V$, and their relationship with the corresponding generic systems of ultrafilters. This relation will expand from the following bidirectional procedure, mapping a $V$-ultrafilter in a generic extension with an ideal in $V$ and vice versa. Full references on this procedure can be found in [16].

**Definition 5.1.20.** Let $\hat{F}$ be a $\mathbb{B}$-name for an ultrafilter on $\mathcal{P}^V(X)$. Let $\mathbf{I}(\hat{F}) \in V$ be the ideal on $\mathcal{P}^V(X)$ defined by:

$$ \mathbf{I}(\hat{F}) = \left\{ Y \subseteq X : \left[ \check{Y} \in \hat{F} \right]_{\mathbb{B}} = 0 \right\} $$

Conversely, let $I$ be an ideal in $V$ on $\mathcal{P}(X)$ and consider the poset $\mathbb{C} = \mathcal{P}(X)/I$. Let $\hat{F}(I)$ be the $\mathbb{C}$-name for the $V$-generic ultrafilter for $\mathcal{P}^V(X)$ defined by:

$$ \hat{F}(I) = \left\{ (\check{Y}, [Y]_I) : Y \subseteq X \right\} $$

Notice that $\mathbf{I}(\hat{F}(I)) = I$, while the $\mathbb{B}$-name $\hat{F}$ and the $\mathbb{C}$-name $\hat{F}(\mathbf{I})$ might be totally unrelated (since $\mathbb{C} = \mathcal{P}(X)/\mathbf{I}(\hat{F})$ does not necessarily embeds completely into $\mathbb{B}$). We refer to Theorem 5.3.12 and subsequent corollary for an example of this behavior.

**Definition 5.1.21.** Let $\hat{F}$ be a $\mathbb{B}$-name for an ultrafilter on $\mathcal{P}^V(X)$. Set $\mathbb{C} = \mathcal{P}(X)/\mathbf{I}(\hat{F})$. The immersion map $i_{\hat{F}}$ is defined as follows:

$$ i_{\hat{F}} : \mathbb{C} \to \mathbb{B} $$

$$ [A]_{\mathbf{I}(\hat{F})} \mapsto \left[ \check{A} \in \hat{F} \right]_{\mathbb{B}} $$

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Proposition 5.1.22. Let \( \hat{F}, \hat{i}_F \) be as in the previous definition. Then \( i_F \) is a (not necessarily complete) morphism of boolean algebras.

Proof. By definition of \( I(\hat{F}) \), the morphism is well-defined. Since \( \hat{F} \) is a \( \mathcal{B} \)-name for a filter \( i_F \) preserves the order of boolean algebras, and since \( \hat{F} \) satisfies the ultrafilter property it also preserves complementation. \( \square \)

The above can be immediately extended to systems of filters, by means of the following.

Definition 5.1.23. Let \( \hat{S} = \langle \hat{F}_a : a \in C \rangle \) be a \( \mathcal{B} \)-name for a \( C \)-system of ultrafilters. Then \( I(\hat{S}) = \langle I_a = I(\hat{F}_a) : a \in C \rangle \) is the corresponding system of filters in \( V \). Conversely, let \( S = \langle I_a : a \in C \rangle \) be a \( C \)-system of filters in \( V \). Then \( \hat{F}(S) = \langle \hat{F}_a = \hat{F}(I_a) : a \in C \rangle \) is the corresponding \( \mathcal{S} \)-name for a system of ultrafilters.

Proposition 5.1.24. Let \( \hat{S} \) be a \( \mathcal{B} \)-name for a \( C \)-system of ultrafilters. Then \( I(\hat{S}) \) is a \( C \)-system of filters in \( V \). Conversely, let \( S \) be a \( C \)-system of filters in \( V \). Then \( \hat{F}(S) \) is the canonical \( (\mathcal{S}^+, <_{\mathcal{S}}) \)-name for the \( V \)-generic filter on \( \mathcal{S} \) and defines a \( C \)-system of \( V \)-ultrafilters.

Proof. \( I(\hat{S}) \) and \( \hat{F}(S) \) satisfy the following properties.

1. (Filter and ultrafilter property) Left to the reader.

2. (Fineness) Fix \( a \in C \).
   \( I(\hat{S}) \): Since \( F_a \) is fine and \( \hat{F}(I_a) \) contains \( F_a \) with boolean value \( 1 \), the latter is also fine.
   \( \hat{F}(S) \): Let \( A = \{ f \in O_a : x \notin \text{dom}(f) \} \) be obtained from any \( x \in a \). Since \( \hat{F}_a \) is fine with boolean value \( 1 \), \( [A \in \hat{F}_a] = 0 \) hence \( A \in I(\hat{F}_a) \).

3. (Compatibility) Fix \( a \subseteq b \in C, A \subseteq O_a \).
   \( I(\hat{S}) \): \( A \in I(\hat{F}_a) \iff [\hat{A} \in \hat{F}_a] = 0_B = \pi^{-1}_{ba}[\hat{A}] \in \hat{F}_b \iff \pi^{-1}_{ba}[A] \in I(\hat{F}_b) \).
   \( \hat{F}(S) \): \( [\hat{A} \in \hat{F}(I_a)] = [A]_S = [\pi^{-1}_{ba}[A]]_S = [\pi^{-1}_{ba}[\hat{A}] \in \hat{F}(I_a)] \).

4. (Normality) Let \( u : O_a \to V \) in \( V \) be regressive on \( A \).
   \( I(\hat{S}) \): Suppose that \( A \in I(\hat{S})^+ \) (i.e. \( [\hat{A} \in \hat{F}_a] = p > 0 \)). Then \( p \models \exists X \leq S \hat{A} X \in \hat{S}^+ \exists y \in b \forall f \in X \hat{u}(\pi_{ba}(f)) = f(y) \)
   Thus by the forcing theorem there is a \( q < p, q > 0 \) and fixed \( B \subseteq \pi^{-1}_{ba}[A], y \in b \) such that \( q \) forces the above formula with the quantified \( X \) replaced by \( B \). Then \( [B \in \hat{S}^+] \geq q > 0 \iff B \in I(\hat{S})^+, \forall f \in B \hat{u}(\pi_{ba}(f)) = f(y) \) holds true in \( V \).
   \( \hat{F}(S) \): Consider the system \( \hat{F}(S) \). Suppose that \( A \in I^+_c \). Given any \( C \subseteq S A \in I^+_c \), we can find \( B \subseteq S C \in S^+ \) witnessing the normality of \( S \) for the regressive
map on $C$ defined by $h \mapsto u(h \upharpoonright a)$. We conclude that there are densely many $B$ below $A$ such that $\exists y \in b \ \forall f \in B \ u(\pi_{ba}(f)) = f(y)$, hence
\[
\exists B < F(S) \ \hat{A} B = F(S)^+ \ \exists y \in b \ \forall f \in B \ \hat{u}(\pi_{ba}(f)) = f(y) \geq [A]_S
\]
and $[A]_S = [\hat{A} \in F(S)^+]$. \hfill $\Box$

As already noticed for single filters, the maps $I$ and $\hat{F}$ are not inverse of each other and $\hat{F}(I(S))$ might differ from $S$.

### 5.1.4 Embedding derived from a system of ultrafilters

We now introduce a notion of *ultrapower* induced by a $C$-system of $V$-ultrafilters $S$. Notice that the results of the last section allows to translate any result about $C$-systems of $V$-ultrafilters to a result on $C$-systems of filters in $V$, by simply considering the $C$-system of $V$-ultrafilters $F(S)$.

**Definition 5.1.25.** Let $V \subseteq W$ be transitive models of ZFC and $S \in W$ be a $C$-system of $V$-ultrafilters. Let
\[
U_S = \{ u : O_a \to V : a \in C, \ u \in V \}.
\]

Define the relations
\[
u S v \iff \{ f \in O_c : u(\pi_{ca}(f)) = v(\pi_{cb}(f)) \} \in F_c
\]
\[
u S v \iff \{ f \in O_c : u(\pi_{ca}(f)) \in v(\pi_{cb}(f)) \} \in F_c
\]
where $O_a = \text{dom}(u)$, $O_b = \text{dom}(v)$, $c = a \cup b$. The ultrapower of $V$ by $S$ is
\[\text{Ult}(V, S) = \langle U_S / \sim_S, \in_S \rangle.\]

We leave to the reader to check that the latter definition is well-posed. From now on, we identify the well-founded part of the ultrapower with its Mostowski collapse.

**Theorem 5.1.26 (Łoś).** Let $\phi(x_1, \ldots, x_n)$ be a formula and let $u_1, \ldots, u_n \in U_S$. Then $\text{Ult}(V, S) \models \phi([u_1]_S, \ldots, [u_n]_S)$ if and only if
\[
\{ f \in O_b : \phi(u_1(\pi_{ba_1}(f)), \ldots, u_n(\pi_{ban}(f))) \} = A \in F_b
\]
where $O_{a_i} = \text{dom}(u_i)$ for $i = 1 \ldots n$, $b = \bigcup a_i$.

**Proof.** We proceed by induction on $\phi$. The case $\phi$ atomic follows directly from the definition of ultrapower, and the case for $\neg$ and $\lor$ is easily handled, so we focus on the case $\phi(\bar{x}) = \exists y \ \psi(y, \bar{x})$. First, suppose that $A \in F_b$ and define $v : O_b \to V$ so that $\psi(v(f), u_1(\pi_{ba_1}(f)), \ldots, u_n(\pi_{ban}(f)))$ holds if $f \in A$, $v(f) = \emptyset$ otherwise. Then $\text{Ult}(V, S) \models \psi([v]_S, [u_1]_S, \ldots, [u_n]_S)$ by inductive hypothesis and the thesis follows.

Conversely, suppose that $\text{Ult}(V, S) \models \exists y \ \psi([v]_S, [u_1]_S, \ldots, [u_n]_S)$, and fix $v \in U_S$ such that $\text{Ult}(V, S) \models \psi([v]_S, [u_1]_S, \ldots, [u_n]_S)$, $v : O_c \to V$ with $c \supseteq b$. Then, by inductive hypothesis,
\[
\{ f \in O_c : \psi(v(f), u_1(\pi_{ca_1}(f)), \ldots, u_n(\pi_{can}(f))) \} = B \in F_c,
\]
hence $\pi_{cb}^{-1}[A] \supseteq B$ is also in $F_c$ and by coherence $A \in F_b$. \hfill $\Box$
As in common model-theoretic use, define \( j_S : V \to \text{Ult}(V, S) \) by \( j_S(x) = [c_x]_S \) where \( c_x : O_b \to \{ x \} \). From the last theorem it follows that the map \( j_S \) is elementary. Notice that the proof of the last theorem does not use neither fineness nor normality of the system of ultrafilters. However these properties allows us to study the elements of the ultrapower by means of the following proposition.

**Proposition 5.1.27.** Let \( S \) be a \( \mathcal{C} \)-system of ultrafilters, \( j : V \to M = \text{Ult}(V, S) \) be the derived embedding. Then,

1. \([c_x]_S = j(x)\) for any \( x \in V \);
2. \([\text{proj}_x]_S = x\) for any \( x \in \bigcup \mathcal{C} \), where
   \[
   \text{proj}_x : O(x) \longrightarrow V
   \]
   \[
   f \longmapsto f(x)
   \]
3. \([\text{ran}_a]_S = a\) for any \( a \in \mathcal{C} \);
4. \([\text{dom}_a]_S = j(a)\) for any \( a \in \mathcal{C} \);
5. \([\text{id}_a]_S = (j(a))^{-1}\) for any \( a \in \mathcal{C} \).

**Proof.**
1. Follows from the definition of \( j \).
2. By induction on \( \text{rank}(x) \). Fix \( x \in \bigcup \mathcal{C} \). If \( y \in x \), then \( y = [\text{proj}_y]_S \in [\text{proj}_x]_S \) since all \( f \) in \( O_{\{x,y\}} \) are \( \epsilon \)-preserving. Conversely, assume that \( x \in a \) and \([ u : O_a \to V ]_S \in [\text{proj}_x]_S \). By Loś’s Theorem, \( u \) is regressive on \( A = \{ f \in O_a : u(f) \in f(x) \} \in F_a \) thus there exist \( y \in b \supseteq a \) and \( B \subseteq \pi^{-1}_b[A] \) in \( F_b \) such that \( u(\pi_a(f)) = f(y) \) for all \( f \in B \). Since \( f(y) = u(\pi_a(f)) \in f(x) \) and any \( f \in B \) is \( \epsilon \)-preserving, it follows that \( y \in x \). Finally, by Loś’s Theorem and inductive hypothesis \([u]_S = [\text{proj}_y]_S = y \in x \).
3. Fix \( a \in \mathcal{C} \). If \( y = [\text{proj}_y]_S \in a \), by fineness
   \[
   \{ f \in O_a : f(y) \in \text{ran}_a(f) \} = \{ f \in O_a : y \in \text{dom}(f) \} \in F_a.
   \]
   thus \( y = [\text{proj}_y]_S \in \text{ran}_a]_S \) by Loś’s Theorem. Conversely, assume that \( u : O_b \to V \) is such that \([u]_S \in [\text{ran}_a]_S \), \( b \supseteq a \). By Loś’s Theorem, \( u \) is regressive on
   \[
   A = \{ f \in O_b : u(f) \in \text{ran}_a(\pi_a(f)) = f[a] \} \in F_b
   \]
   thus by normality there exist \( y \in c \supseteq b \) and \( B \subseteq \pi^{-1}_c[A] \) such that \( u(\pi_b(f)) = f(y) \) for all \( f \in B \). Since \( B \subseteq \pi^{-1}_c[A] \), \( f(y) = u(\pi_b(f)) = f(x) \) for some \( x \in a \).
   Since \( f \) is injective, \( x = y \in a \) and \([u]_S = [\text{proj}_y]_S = y \) by Loś’s Theorem.
4. Fix \( a \in \mathcal{C} \). If \( x \in a \), by fineness \( \{ f \in O_a : x \in \text{dom}_a(f) \} \in F_a \) hence \( j(x) = [c_x]_S \in [\text{dom}_a]_S \). Conversely, assume \([u : O_b \to V]_S \in [\text{dom}_a]_S \) with \( b \supseteq a \). By Loś’s Theorem, \( A = \{ f \in O_b : u(f) \in \text{dom}_a(\pi_a(f)) \} \in F_b \) and we can define
   \[
   v : A \longrightarrow V
   \]
   \[
   f \longmapsto f(u(f)).
   \]
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that is regressive on $A$. Then by normality there exist $y \in c \supseteq b$ and $B \subseteq \pi^{-1}(A)$ such that $v(\pi_0(f)) = f(u(\pi_0(f))) = f(y)$ for all $f \in B$. Since $f$ is injective, $u(\pi_0(f)) = y$ hence $y$ is in $\text{dom}_u(\pi_0(f)) = \text{dom}(f) \cap \alpha$. Thus by Łoś’s Theorem $[u]_S = [c_y]_S = j(y)$.

5. Follows from points 3 and 4, together with the observation that $[\text{id}_a]_S$ has to be an $\in$-preserving function by Łoś’s Theorem and $(j|a)^{-1}$ is the only such function with domain $j[a]$ and range $a$.

These canonical representatives can be used in order to prove many general properties of $\mathcal{C}$-system of filters and of the induced ultrapowers. In particular, we shall use them to prove Propositions 5.1.11 and 5.1.12 and other related properties.

**Proposition 5.1.28.** Let $\mathcal{S}$ be a $\mathcal{C}$-system of ultrafilters, $A \subseteq O_a$ be such that $a \in \mathcal{C}$. Then $A \in F_a$ if and only if $(j_S|a)^{-1} \in j_S(A)$.

**Proof.** By Łoś’s Theorem, we have $A = \{f \in O_a : f = \text{id}_a(f) \in A\} \in F_a$ if and only if $(j_S|a)^{-1} = [\text{id}_a]_S \in [c_a]_S = j_S(A)$.

**Lemma 5.1.29.** Let $\mathcal{S}$ be a $\mathcal{C}$-system of filters and $a \in \mathcal{C}$. Then $\kappa_a$ is the minimum $\alpha$ such that $\left\lceil j_{F(\mathcal{S})}(\hat{a}) \geq \text{rank}(\hat{a}) \right\rceil_\mathcal{S} = 1$.

**Proof.** Let $j$ be the elementary embedding derived from $\check{F}(\mathcal{S})$ in a generic extension by $\mathcal{S}$. Notice that $O_a \cap {}^aV_a \in F_a$ is equivalent by Proposition 5.1.28 to

$$(j|a)^{-1} \in j(O_a \cap {}^aV_a) = j(O_a) \cap j(a)V_{j(a)}$$

which is in turn equivalent to $a \subseteq V_{j(\alpha)}$ i.e. $j(\alpha) \geq \text{rank}(a)$. Since this holds in all generic extensions by $\mathcal{S}$, we are done.

**Proof of Proposition 5.1.11.** By the previous proposition $\kappa_a$ is the minimum $\alpha$ such that $1 \equiv_\mathcal{S} j(\alpha) \geq \text{rank}(a)$, hence it depends (monotonically) only on $\text{rank}(a)$. The conclusion of Proposition 5.1.11 follows.

**Proposition 5.1.30.** Let $\mathcal{S}$ be a $\mathcal{C}$-system of filters. Then $\kappa$ is the critical point of $j = j_{F(\mathcal{S})}$ with boolean value $1$ iff $\mathcal{S}$ is a $\langle \kappa, \lambda \rangle$-system of filters.

**Proof.** Suppose that $\kappa$ is the critical point of $j$ with boolean value $1$. If $\gamma < \kappa$, $A \in F_{\{\gamma\}}$ iff $1 \equiv_\mathcal{S} (j|\{\gamma\})^{-1} = j(\text{id} \mid \{\gamma\}) \in j(A)$ iff $\text{id} \mid \{\gamma\} \in A$. Thus $F_{\{\gamma\}}$ is principal generated by $\text{id} \mid \{\gamma\}$. If $a \in C \cap V_{\kappa+2}$, $\text{rank}(a) \leq \kappa + 1 \leq j_{F(\mathcal{S})}(\kappa)$ with boolean value $1$, thus $\kappa_a \leq \kappa$ by Lemma 5.1.29.

Conversely, suppose that $\{\text{id} \mid \{\gamma\}\} \in F_{\{\gamma\}}$ for $\gamma < \kappa$, and $\kappa_a \leq \kappa$ for $a \in V_{\kappa+2}$. If there is an $A \in \mathcal{S}^+$ forcing that $j$ has no critical point or has critical point bigger than $\kappa$, $\kappa_a = \text{rank}(a) > \kappa$ for $a \in V_{\kappa+2} \setminus V_{\kappa+1}$, a contradiction. If there is a $B \in \mathcal{S}^+$ forcing that $j$ has critical point $\gamma$ smaller than $\kappa$, $F_{\{\gamma\}}$ cannot be principal generated by $\text{id} \mid \{\gamma\}$, again a contradiction.
5.1 Systems of filters

Proof of Proposition 5.1.12. Let $S$ be a $\langle \kappa, \lambda \rangle$-system of filters, $a$ be in $C$, $j$ be derived from $F(S)$. We need to prove that $F_a$ is $<\kappa$-complete for all $a$. Suppose that $A \subseteq F_a$ is such that $|A| < \kappa$. Hence by Proposition 5.1.30, $j(A) = j[\bigcap A]$. Then $\bigcap A \in F_a$ iff

$$1 \models_S (j|a)^{-1} \in j(\bigcap A) = \bigcap j(A) = \bigcap j[\bigcap A]$$

which is true since $A \in F_a \Rightarrow 1 \models_S (j|a)^{-1} \in j(A)$ for all $A \in A$.

The ultrapower $\text{Ult}(V, S)$ happens to be the direct limit of the directed system of ultrapowers $\langle \text{Ult}(V, F_a) : a \in C \rangle$ with the following factor maps:

$$k_{ab} : \text{Ult}(V, F_a) \rightarrow \text{Ult}(V, F_b)$$

$$[u]_{F_a} \mapsto [u \circ \pi_{ba}]_{F_b}$$

$$k_a : \text{Ult}(V, F_a) \rightarrow \text{Ult}(V, S)$$

$$[u]_{F_a} \mapsto [u]_S$$

The ultrapower $\text{Ult}(V, S)$ is also the direct limit of the ultrapowers given by the restrictions of $S$, as shown in the following.

Definition 5.1.31. Let $S$ be a $C$-system of filters of length $\lambda$, $\alpha < \lambda$ be an ordinal. The restriction of $S$ to $\alpha$ is the $\langle C \cap V_\alpha \rangle$-system of filters $S|\alpha = \{ F_a : a \in C \cap V_\alpha \}$. Moreover, if $S$ is a $C$-system of ultrafilters the corresponding factor map is

$$k_\alpha : \text{Ult}(V, S|\alpha) \rightarrow \text{Ult}(V, S)$$

$$[u]_{S|\alpha} \mapsto [u]_S$$

Proposition 5.1.32. Let $S$ be a $C$-system of ultrafilters of length $\lambda$, $\alpha < \lambda$ be an ordinal. Then

1. $k_\alpha$ is elementary;
2. $k_\alpha \circ j_{S|\alpha} = j_S$;
3. $k_\alpha \upharpoonright \bigcup (C \cap V_\alpha) = \text{id} \upharpoonright \bigcup (C \cap V_\alpha)$, hence $\text{crit}(k_\alpha) \geq \alpha$.

Proof. A particular case of Proposition 5.1.36 to follow.

5.1.5 System of ultrafilters derived from an embedding

We now present the definitions and main properties of $C$-system of ultrafilters derived from a generic elementary embedding. With abuse of notation, we denote as generic elementary embedding any map $j : V \rightarrow M$ which is elementary and such that $M \subseteq W$ for some $W \supseteq V$. In the following we shall assume that $j$ is a definable class in $W$. However, we believe that it should be possible to adapt the present results to non-definable $j$, provided we are working in a strong enough set theory with sets and classes (e.g. MK). We also provide a comparison between derived $C$-systems of ultrafilters for different choices of $C$.

Let $S$ be a $C$-system of ultrafilters, $A \subseteq O_a$ be such that $a \in C$. Then by Proposition 5.1.28

$$A \in F_a \iff (j_S|a)^{-1} \in j_S(A)$$

and this relation actually provides a definition of $S$ from $j_S$. This justifies the following definition.
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**Definition 5.1.33.** Let $V \subseteq W$ be transitive models of ZFC. Let $j : V \to M \subseteq W$ be a generic elementary embedding definable in $W$, $C \in V$ be a directed set of domains such that for any $a \in C$, $(j|a)^{-1} \in M$. The **C-system of ultrafilters derived from** $j$ is $S = \{F_a : a \in C\}$ such that:

$$F_a = \left\{ A \subseteq O_a : (j|a)^{-1} \in j(A) \right\}.$$  

Definition 5.1.33 combined with Proposition 5.1.28 guarantees that for a given $C$-system of ultrafilters $S$, the $C$-system of ultrafilters derived from $j_S$ is $S$ itself. We now show that the definition is meaningful for any embedding $j$.

**Proposition 5.1.34.** Let $j, C, S$ be as in the definition above. Then $S$ is a $C$-system of $V$-ultrafilters.

*Proof.* 1. **(Filter and ultrafilter property)** Fix $a \in C$ and assume that $A, B \in F_a$. Then $(j|a)^{-1} \in j(A) \cap j(B) = j(A \cap B)$. Moreover if $C \subseteq O_a$ and $A \subseteq C$, then $(j|a)^{-1} \subseteq j(A) \subseteq j(C)$. Finally, if $(j|a)^{-1} \notin j(A)$ we have that $(j|a)^{-1} \in j(O_a) \setminus j(A) = j(O_a \setminus A)$.

2. **(Fineness)** Fix $x \in a$ so that $j(x) \in j[a]$. Then $j(x) \in \text{dom}(j|a)^{-1}$ hence we have $\{ f \in O_a : x \in \text{dom}(f) \} \in F_a$ by definition of $F_a$.

3. **(Compatibility)** Assume that $a \subseteq b \in C$ and $A \subseteq O_a$. Then

$$(j|b)^{-1} \in j(\pi_{ba}[A]) = \{ f \in O_{j(b)} : \pi_{j(a)}(f) \in j(A) \}$$

if and only if $(j|a)^{-1} = \pi_{j(a)}((j|b)^{-1}) \in j(A)$.

4. **(Normality)** Let $u : A \to V$ be regressive on $A \in F_a$ and in $V$. By elementarity,

$$M \models \forall f \in j(A) \exists x \in \text{dom}(f) j(u)(f) \subseteq f(x)$$

Since $(j|a)^{-1} \in j(A)$, there exists $x \in j[a]$ with $j(u)((j|a)^{-1}) \subseteq (j|a)^{-1}(x)$. Define $y = j(u)((j|a)^{-1})$, and put $b = a \cup \{y\}$. Note that $y \subseteq j^{-1}(x) \in a$ hence by transitivity of $\bigcup C$, $\{y\} \in C$. Define $B = \{ f \in O_b : u(\pi_{ba}(f)) = f(y) \}$. Then $(j|b)^{-1} \in j(B)$, i.e. $B \in F_b$, since

$$(j|b)^{-1}(j(y)) = y = j(u)((j|a)^{-1}) = j(u)(\pi_{j(a)}((j|b)^{-1})).$$

Given a $C$-system of ultrafilters $S$ derived from a generic embedding $j$, we can factor out the embedding $j$ through $j_S$.

**Definition 5.1.35.** Let $j : V \to M \subseteq W$ be a generic elementary embedding, $C \in V$ be a directed set of domains of length $\lambda$, $S$ be the $C$-system of ultrafilters derived from $j$. Then

$$k : \text{Ult}(V, S) \to M$$

$$[u : O_a \to V]_S \mapsto j(u)((j|a)^{-1})$$

is the factor map associated to $S$.

**Proposition 5.1.36.** Let $j, C, \lambda, S, k$ be as in the previous definition. Then
1. \( k \) is elementary;
2. \( k \circ j_S = j \);
3. \( k \mid \bigcup C = \text{id} \mid \bigcup C \) hence \( \text{crit}(k) \geq \lambda \);
4. if \( \lambda = j(\gamma) \) for some \( \gamma \), then \( \text{crit}(k) > \lambda \).

**Proof.** 1. Let \( \phi(x_1, \ldots, x_n) \) be a formula, and for any \( i \in n \) let \( u_i : O_{a_i} \to V, a_i \in C \). Put \( b = \bigcup \{ a_i : 1 \leq i \leq n \} \). Then \( \text{Ult}(V, S) \models \phi([u_1]_S, \ldots, [u_n]_S) \) if and only if (by Loš Theorem)

\[
B = \{ f \in O_b : \phi(u_1(\pi a_1(f)), \ldots, u_n(\pi a_n(f))) \} \in F_b.
\]

if and only if \( (j|b)^{-1} \in j(B) \) (by definition of \( F_b \)) i.e.

\[
M \models \phi(j(u_1)(\pi j(a_2)(j|b)^{-1}), \ldots, j(u_n)(\pi j(a_n)(j|b)^{-1}))
\]

if and only if (by definition of \( \pi j(a_i) \))

\[
M \models \phi(j(u_1)((j|a_1)^{-1}), \ldots, j(u_n)((j|a_n)^{-1}))
\]

i.e. \( M \models \phi(k([u_1]_S), \ldots, k([u_n]_S). \)

2. For any \( x \in V \),

\[
k(j_S(x)) = k([c_x]_S) = j(c_x)() = c_j(x)() = j(x).
\]

3. Let \( x \in \bigcup C \). Then by Proposition 5.1.27 for some \( a \in C \) with \( x \in a \),

\[
k(x) = k([\text{proj}_x]_S) = j(\text{proj}_x)((j|a)^{-1}) = j^{-1}(j(x)) = x.
\]

4. If \( \lambda = j(\gamma) \), the following diagram commutes:

\[
\begin{array}{ccc}
V & \xrightarrow{j} & M \\
\downarrow{j_S} & & \downarrow{k} \\
\text{Ult}(V, S) & \subseteq & V[S]
\end{array}
\]

Thus \( \text{crit}(k) \geq j(\gamma) \) and \( k \circ j_S(\gamma) = j(\gamma) \). Therefore \( j(\gamma) \in \text{ran}(k) \) hence \( j(\gamma) \) cannot be the critical point of \( k \), showing that the above inequality is strict. \( \square \)

Observe that in Definition 5.1.33 \( (j|a)^{-1} \notin M \) would imply that the derived filter \( F_a \) is empty. Thus, depending on the choice of \( C \), there can be a limit on the maximal length attainable for a \( C \)-system of ultrafilter derived from \( j \). If \( C = [\lambda]^{<\omega} \), \( (j|a)^{-1} \) is always in \( M \) thus there is no limit on the length of the extenders derived from \( j \). If \( C = V_\lambda \), the maximal length is the minimal \( \lambda \) such that \( j[V_\lambda] \notin M \). These bounds are relevant, as shown in the following proposition.

\[\text{Remark that in the above diagram } V[S] \text{ is the smallest transitive model } N \text{ of ZFC such that } V, \{S\} \subseteq N \subseteq W.\]
Proposition 5.1.37. Let $\mathcal{T} \in W$ be a tower of length a limit ordinal $\lambda$, $j : V \to M \subseteq W$ be the derived embedding. Then the tallest tower derivable from $j$ is $\mathcal{T}$.

Proof. Since $\text{dom}_V \alpha$ represents $j[V_\alpha]$ for all $\alpha < \lambda$, $j[V_\alpha] \in M$ for all $\alpha < \lambda$ and we only need to prove that $j[V_\lambda] \notin M$. Suppose by contradiction that $u : O_a \to V$ is such that $[u]_\mathcal{T} = j[V_\lambda]$. Let $\alpha < \lambda$ be such that $a \in V_\alpha$, and let $v : O_a \to V$ be such that $v(x) = u(x) \cap V_{\alpha+1}$. Thus $[v]_\mathcal{T} = j[V_\lambda] \cap j(V_{\alpha+1}) = j[V_{\alpha+1}]$, and by Łoś Theorem

$$A = \{ f \in O_{V_{\alpha+1}} : v(\pi_a(f)) = \text{dom}(f) \} \in F_{V_{\alpha+1}}$$

Since $|\text{dom}[A]| \leq |\text{ran}(v)| \leq |O_a| < |V_{\alpha+1}|$, $\text{dom}[A]$ is a non-stationary subset of $V_{\alpha+1}$ by Lemma 1.2.13 contradicting Proposition 5.1.10.

We now consider the relationship between different $C$-systems of ultrafilters derived from a single $j$.

Proposition 5.1.38. Let $j : V \to M \subseteq W$ be a generic elementary embedding definable in $W$, $C_1 \subseteq C_2$ be directed sets of domains in $V$, $S_n$ be the $C_n$-system of $V$-ultrafilters derived from $j$ for $n = 1, 2$. Then $\text{Ult}(V, S_2)$ can be factored into $\text{Ult}(V, S_1)$, and $\text{crit}(k_1) \leq \text{crit}(k_2)$ where $k_1, k_2$ are the corresponding factor maps.

Proof. We are in the following situation:

$$\begin{align*}
  &V \xrightarrow{j} M \\
  &\downarrow j_2 \quad \downarrow j_1 \\
  &\text{Ult}(V, S_2) \quad \text{Ult}(V, S_1) \\
  &\text{Ult}(V, S_1) \quad \text{Ult}(V, S_2)
\end{align*}$$

where $k$ is defined as

$$k : \text{Ult}(V, S_1) \xrightarrow{} \text{Ult}(V, S_2)
\quad [u]_{S_1} \xrightarrow{} [u]_{S_2}$$

Observe that $j_1$, $j_2$ and $k$ commute. Moreover given $u : O_a \to V$ with $a \in C_1$,

$$k_2 \circ k([u]_{S_1}) = j(u) \left( (j|a)^{-1} \right) = k_1([u]_{S_1})$$

therefore the diagram commutes. Since $k_1$ and $k_2$ are elementary, $k$ has to be elementary as well and $\text{crit}(k_1) \leq \text{crit}(k_2)$.

Notice that the last proposition can be applied whenever $S_1$ is an extender and $S_2$ is a tower, both of the same length $\lambda$ and derived from the same generic elementary embedding $j : V \to M \subseteq W$. It is also possible for a “thinner” system of filters (i.e. an extender) to factor out a “fatter” one.

Definition 5.1.39. Let $F$ be an ultrafilter. We denote by $\text{non}(F)$ the minimum of $|A|$ for $A \in F$. Let $S$ be a $C$-system of ultrafilters. We denote by $\text{non}(S)$ the supremum of $\text{non}(F_a) + 1$ for $a \in C$.
If the length of $S$ is a limit ordinal $\lambda$, $\text{non}(S)$ is bounded by $\beth_\lambda$. If $E$ is a $\gamma$-extender of regular length $\lambda > \gamma$, $\text{non}(S)$ is also bounded by $2^{<\lambda} + 1$.

**Theorem 5.1.40.** Let $C$ be a directed set of domains. Let $j : V \rightarrow M \subseteq W$ be a generic elementary embedding definable in $W$, $S$ be the $C$-system of filters derived from $j$, $E$ be the extender of length $\lambda \supseteq j[\text{non}(S)]$ derived from $j$. Then $\text{Ult}(V, E)$ can be factored into $\text{Ult}(V, S)$, and $\text{crit}(k_S) \leq \text{crit}(k_E)$.

**Proof.** Let $\rho : [\text{non}(F_a)]^1 \rightarrow O_a$ be an enumeration of an $A \in F_a$ of minimum cardinality, so that $(j|a)^{-1} \in j(A) = \text{ran}(j(\rho_a))$. Let $k$ be defined by

$$k : \text{Ult}(V, S) \rightarrow \text{Ult}(V, E)$$

$$[u : O_a \rightarrow V]_S \mapsto [u \circ \rho_a \circ \text{ran}(\beta)]_E$$

where $\beta < j(\text{non}(F_a)) \leq \lambda$ is such that $j(\rho_a)(\{\beta\}) = (j|a)^{-1}$. We are in the following situation:

![Diagram](image.png)

Observe that $j_S$, $j_E$ and $k$ commute. Moreover given $u : O_a \rightarrow V$ with $a \in C$,

$$k_E \circ k([u]_S) = j(u \circ \rho_a \circ \text{ran}(\beta)) \left((j|\{\beta\})^{-1}\right)$$

$$= j(u \circ \rho_a)(\{\beta\})$$

$$= j(u) \left((j|a)^{-1}\right) = k_S([u]_S)$$

therefore the diagram commutes. Since $k_S$ and $k_E$ are elementary, $k$ has to be elementary as well and $\text{crit}(k_S) \leq \text{crit}(k_E)$. \hfill \Box

The last proposition with $j = j_S$ shows that from any $C$-system of filters $S$ can be derived an extender $E$ of sufficient length such that $\text{Ult}(V, S) = \text{Ult}(V, E)$. The derived extender $E$ might have the same length as $S$, e.g. when $\lambda = \beth_\lambda$ and $j[\lambda] \subseteq \lambda$. In particular, this happens in the notable case when $S$ is the full stationary tower of length $\lambda$ a Woodin cardinal.

### 5.2 Generic large cardinals

Generic large cardinal embeddings are analogous to classical large cardinal embeddings. The difference between the former and the latter is that the former is definable in some forcing extension of $V$ and not in $V$ itself as the latter. An exhaustive survey on this topic is given in [16]. Most of the large cardinal properties commonly considered can be built from the following basic blocks.

**Definition 5.2.1.** Let $V \subseteq W$ be transitive models of ZFC. Let $j : V \rightarrow M \subseteq W$ be a generic elementary embedding with critical point $\kappa$. We say that
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- \( j \) is \( \gamma \)-tall iff \( j(\kappa) \geq \gamma \);
- \( j \) is \( \gamma \)-strong iff \( V^j_\gamma \subseteq M \);
- \( j \) is \( <\gamma \)-closed iff \( <\gamma M \subseteq M \) from the point of view of \( W \).

Notice that the definition of a large cardinal property through the existence of an embedding \( j \) with (some version of) the above properties is not a first-order statement, since it quantifies over a class object. In the theory of large cardinals in \( V \), this problem is overcome by showing that an extender \( E \) of sufficient length is able to capture all the aforementioned properties of \( j \) in \( jE \). For generic large cardinals the same can be done with some additional limitations, as shown in Section 5.2.1.

In contrast with the classical case, this process requires the use of \( C \)-systems of ultrafilters in some generic extension. However, it would be a desirable property to be able to obtain a description of such generic elementary embeddings from objects living in \( V \). Natural intuition suggest the feasibility of this option (see e.g. [33]). We thus introduce the following definition schema (already suggested in [9]).

**Definition 5.2.2** (Claverie). Let \( P \) be a large cardinal property of an elementary embedding \( \kappa \) be a cardinal. We say that \( \kappa \) has property \( P \) iff there exists an elementary embedding \( j : V \rightarrow M \subseteq V \) with critical point \( \kappa \) and satisfying property \( P \).

We say that \( \kappa \) has *generically* property \( P \) iff there exists a forcing extension \( V[G] \) and an elementary embedding \( j : V \rightarrow M \subseteq V[G] \) definable in \( V[G] \) and with critical point \( \kappa \) satisfying property \( P \).

We say that \( \kappa \) has *ideally* property \( P \) iff there exist a \( C \)-system of filters \( S \) in \( V \) such that the corresponding generic ultrapower embedding \( j^F(S) \) satisfies property \( P \) in the corresponding generic extension.

Observe that for any \( \kappa, P(\kappa) \Rightarrow \text{ideally } P(\kappa) \Rightarrow \text{generically } P(\kappa) \). On the other side, it is not clear whether generically \( P(\kappa) \Rightarrow \text{ideally } P(\kappa) \) as pointed out in [9, 10]. In Section 5.3, an example is given suggesting that the natural procedure of inducing a \( C \)-system of filters in \( V \) from a generic elementary embedding might fail to preserve large cardinal properties, thus giving some hints against the equivalence of these two concepts.

Furthermore, having ideally property \( P \) can be much weaker than having property \( P \) in \( V \), e.g. the consistency of an ideally \( I_1 \) cardinal follows from the consistency of a Woodin cardinal [33]. Nonetheless, upper bounds on the consistency of generic large cardinals similar to those for classical large cardinals can be proven (see [25, 39]), e.g. the inconsistency of a set-generic Reinhardt cardinal. Since from a stationary tower of height a Woodin cardinal we can obtain a *class*-generic Reinhardt cardinal, it is clear that the strength of a generic large cardinal very much depends on the nature of the forcing allowed to obtained it. In fact, the strength of a generic

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\(^*\)We remark that the present definition of \( \gamma \)-tall for an embedding does not coincide with the classical notion of \( \gamma \)-tall for cardinals, which is witnessed (in the present terms) by a \( \gamma \)-tall and \( \kappa \)-closed embedding with critical point \( \kappa \). We believe that the present definition is more convenient to our purposes since it avoids overlapping of concepts and simplifies the corresponding combinatorial version for \( C \)-systems of filters.

\(^\dagger\)More precisely, we can assume that \( P \) is a first-order property in the class parameter \( j \).
large cardinal hypothesis depends on the interaction of three parameters, as outlined in [16]: the size of the critical point, the closure properties of the embedding, and the nature of the forcing used to define it. We shall not expand on the impact of the nature of forcing, while we shall spend some time on the size of the critical point. In this setting, the trivial observation that \( P(\kappa) \Rightarrow \text{ideally} \ P(\kappa) \Rightarrow \text{generically} \ P(\kappa) \) is not really satisfying, since we are interested in the consistence of small cardinals \( \kappa \) having ideally (or generically) property \( P \). However, it is sometimes possible to collapse a large cardinal in order to obtain a small generic large cardinal. Examples of positive results on this side can be found in [9, 10, 17, 28, 35], we present and generalize some of them in Section 5.2.2.

Notice that having ideally property \( P \) is inherently a statement on the structure of the relevant \( C \)-system of filters \( S \) in question. In Section 5.2.3 we provide a characterization of these properties as combinatorial statements on \( S \).

Since having a generic large cardinal property is possibly weaker than having the same property in \( V \), two large cardinal properties which are inequivalent for classical large cardinals may turn out to be equivalent for their generic counterparts. In Section 5.2.4 we show some examples of embeddings separating different generic large cardinal properties. These examples are an application of the techniques introduced throughout all this section.

5.2.1 Deriving large cardinal properties from generic systems of filters

All over this section \( G \) is \( V \)-generic for some forcing \( \mathbb{B} \) and \( j : V \rightarrow M \subseteq V[G] \) is a generic elementary embedding definable in \( V[G] \) with some large cardinal property \( P \) and critical point \( \kappa \). We aim to approximate \( j \) via a suitable \( C \)-system of \( V \)-ultrafilters \( S \) derived from \( j \). Then \( j_S \) is \( \gamma \)-tall.

**Proposition 5.2.3.** Let \( j \) be \( \gamma \)-tall, \( C \in V \) be a directed set of domains with \( \lambda \subseteq \bigcup C \), \( S \) be the \( C \)-system of ultrafilters of length \( \lambda \geq j(\kappa) \) derived from \( j \). Then \( j_S \) is \( \gamma \)-tall.

**Proof.** By Proposition 5.1.36 \( \text{crit}(k) > j(\kappa) \) hence \( j_S(\kappa) = k(j_S(\kappa)) = j(\kappa) > \gamma \). \( \square \)

**Proposition 5.2.4.** Let \( j \) be \( \gamma \)-strong and \( \lambda \) be such that either \( \lambda > \gamma \) or \( \lambda = j(\mu) = \gamma \) for some \( \mu \). Let \( C \in V \) be a directed set of domains with \( \lambda \subseteq \bigcup C \), and \( S \) be the \( C \)-system of ultrafilters derived from \( j \). Then \( j_S \) is \( \gamma \)-strong; i.e. \( V_{\gamma}^{\text{Ult}(V,S)} = V_{\gamma}^{V[S]} = V_{\gamma}^{\text{Ult}[G]} \).

**Proof.** By Proposition 5.1.36 we have that \( \text{crit}(k) > \gamma \). Thus
\[
V_{\gamma}^{\text{Ult}(V,S)} = k(V_{\gamma}^{\text{Ult}(V,S)}) = V_{\gamma}^M = V_{\gamma}^{V[G]}.
\]
Furthermore, since \( V_{\gamma}^{\text{Ult}(V,S)} \subseteq V_{\gamma}^{V[S]} \subseteq V_{\gamma}^{\text{Ult}[G]} \) they must all be equal. \( \square \)

While tallness and strongness are easily handled, in order to ensure preservation of closure we need some additional technical effort.

---

6Once again \( V[S] \) is the minimal transitive model \( N \) of ZFC such that \( V, \{S\} \subseteq N \subseteq V[G] \).
Definition 5.2.5. A boolean algebra \( B \) is \( \lambda \)-presaturated is for any \( \gamma < \lambda \) and family \( \mathcal{A} = \langle A_\alpha : \alpha < \gamma \rangle \) of maximal antichains of size \( \lambda \), there are densely many \( p \in B^+ \) such that
\[
\forall \alpha < \gamma \ [ \{ a \in A_\gamma : a \land p > 0 \} ] < \lambda.
\]

Proposition 5.2.6. Let \( \lambda \) be a regular cardinal. A boolean algebra \( B \) is \( \lambda \)-presaturated if and only if it preserves the regularity of \( \lambda \).

Lemma 5.2.7. Let \( S \) be a \( C \)-system of ultrafilters in \( V[G] \) and \( N = \text{Ult}(V,S) \) be such that:

- \( V[G] \) is a \( \lambda^+ \)-cc forcing extension for some \( \lambda \) regular in \( V[G] \);
- \( V^{|V[G]}_\lambda = V^N_\lambda \) and \( C \) has length at least \( \lambda \);
- \( N \) is closed for \( \lambda \)-sequences in \( V \).

Then \( N \) is closed for \( \lambda \)-sequences in \( V[G] \).

Proof. Let \( s \) be the name for a sequence of length \( \gamma < \lambda \) of elements of \( N \). Since \( \lambda \) is regular in \( V[G] \), the forcing \( C \) which defines \( V[G] \) is \( \langle \lambda, \gamma \rangle \)-presaturated. Moreover, \( C \) is \( \lambda^+ \)-cc hence for any \( \alpha < \gamma \) there are at most \( \lambda \)-many possibilities for \( \dot{s}(\alpha) \).

Therefore we can apply presaturation and find a condition \( p \in G \) such that
\[
p \models \dot{s} = \{ (\langle \alpha, [u^0_\alpha]_S \rangle, q^0_\beta) : \alpha < \gamma, \beta < \mu \},
\]
for some \( \mu < \lambda \).

Let \( \langle x_\alpha : \alpha < \lambda \rangle \) be a (partial) enumeration of \( C \cap V_\lambda \). Define \( t : \gamma \times \mu \to V \) so that \( t(\alpha, \beta) = \langle u^0_\beta, \text{ran}(x_\alpha), \text{ran}(\{ x_\beta \}) \rangle \) is a sequence in \( V \). Since \( N \) is closed for sequences in \( V \), the sequence represented by \( t \) is in \( N \); i.e.
\[X = \{ (\langle u^0_\beta \rangle_S, \langle \{ x_\alpha \}, \{ x_\beta \} \rangle) : \alpha < \gamma, \beta < \delta \} \in N.
\]
Moreover, \( Y \in V[G] \) where \( Y = \{ (\langle x_\alpha \rangle, \{ x_\beta \}) : q^0_\beta \in G \} \). Since \( Y \in V^{|V[G]}_\lambda = V^N_\lambda \), inside \( N \) we can define \( \text{val}_C(s) = \{ \langle \alpha, [u^0_\beta]_S \rangle \in N : \exists y \in Y \langle u^0_\beta, y \rangle \in X \} \).

Theorem 5.2.8. Let \( j \) be \( \lambda \)-closed with \( \lambda \) regular cardinal, \( V[G] \) be a \( \lambda^+ \)-cc forcing extension. Let \( C \in V \) be a \( \lambda \)-directed set of domains in \( V \), let \( S \) be the \( C \)-system of filters derived from \( j \). Then \( j_S \) is \( \lambda \)-closed.

Proof. Let \( u_\alpha : O_{u_\alpha} \to V \) for \( \alpha < \gamma \) be a sequence of length \( \gamma < \lambda \) in \( V \) of elements of \( \text{Ult}(V,S) \). Since \( C \) is \( \lambda \)-directed, there is a \( b \supseteq \bigcup \{ a_\alpha : \alpha < \gamma \} \in C \) such that \( |b| \geq \gamma \). Let \( \langle x_\alpha : \alpha < \gamma \rangle \) be a (partial) enumeration of \( b \). Define
\[
v : O_b \to V \quad f \mapsto \{ u_\alpha(f) : x_\alpha \in \text{dom}(f) \}
\]
so that by fineness and normality \( [v]_S = \{ [u_\alpha]_S : \alpha < \gamma \} \). Thus \( \text{Ult}(V,S) \) is closed under \( \lambda \)-sequences in \( V \) and is \( \lambda \)-strong by Proposition 5.2.4. We can apply Lemma 5.2.7 to obtain that \( j_S \) is \( \lambda \)-closed. \( \square \)
Note that since in the hypothesis of the previous theorem $j$ is $<\lambda$-closed, it is always possible to derive a system of filters with a $<\lambda$-directed set of domains. In particular, it is possible to derive towers of length $\lambda$ and $\lambda$-extenders of any length. Thus the only significant limitation is the hypothesis that $V[G]$ is $<\lambda^+$-cc where $\lambda$ is the amount of closure required for $M$. However, this hypothesis is satisfied in most classical examples of generic elementary embeddings with high degrees of closure, as e.g. the full stationary tower of length a Woodin cardinal.

It is also possible to ensure the same closure properties with non-directed system of filters, as e.g. extenders. This can be done by means of Theorem 5.1.40 and the following remarks.

### 5.2.2 Consistency of small generic large cardinals

In this section we shall prove that in most cases the assertion that a small cardinal (e.g. $\omega_1$) has generically or ideally property $P$ consistently follows from the existence of any such cardinal (Corollary 5.2.13). Similar results were proved independently in [28, 35] and echoed by [9]; we generalize them to $C$-system of filters, obtaining a simpler proof.

In the following we shall need to lift embeddings and systems of filters in forcing extensions. We refer to [13] Chp. 9 for a complete treatment of the topic. Recall that if $j : V \to M$ is an elementary embedding and $B \in V$ is a boolean algebra, $j$ is also an elementary embedding of the boolean valued model $V^B$ into $M^j(B)$. Furthermore, $j$ can be lifted to the generic extensions $V[G]$ and $M[H]$ where $G$ is $V$-generic for $B$ and $H$ is $j(B)$-generic for $M$ whenever $j[G] \subseteq H$.

For sake of simplicity, we shall focus on the boolean valued models approach and avoid explicit use of generic filters. This will be convenient to handle several different forcing notions at the same time. All the proofs will then be carried out in $V$ using names and explicitly mentioning in which boolean valued model $V^B$ every sentence is to be interpreted. We recall the following definition of two-step iteration.

**Definition (2.3.1).** Let $B$ be a complete boolean algebra, and $\dot{C}$ be a $B$-name for a complete boolean algebra. We denote by $\dot{B} * \dot{C}$ the boolean algebra defined in $V$ whose elements are the equivalence classes of $B$-names for elements of $\dot{C}$ (i.e. $\dot{a} \in V^B$ such that $[\dot{a} \in \dot{C}]_B = 1$) modulo the equivalence relation $\dot{a} \approx \dot{b} \iff [\dot{a} = \dot{b}]_B = 1$.

We refer to [44, 45] for further details on two-step iterations and iterated forcing.

**Definition 5.2.9.** Let $S$ be a $C$-system of filters, $C$ be a cBa. Then $S^C = \{F^C_a : a \in C\}$ where $F^C_a = \{A \subseteq (O_a)^{V^C} : \exists B \in F_a A \supseteq B\}$.

We remark that the following theorem is built on the previous results by Kakuda and Magidor [28, 35] for single ideals and by Claverie [9] for ideal extenders.

**Theorem 5.2.10.** Let $j : V \to M \subseteq V^B$ be elementary with critical point $\kappa$, and $C \in V$ be an $<\kappa$-cc cBa. Then $B * j(C)$ factors into $C$, and the embedding $j$ lifts to $j^C : V^C \to M^{j(C)}$. 

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\[
\begin{align*}
V \xrightarrow{j} M & \subseteq V^\mathcal{B} \\
V^\mathcal{C} \xrightarrow{j^\mathcal{C}} M^\mathcal{C}(\mathcal{C}) & \subseteq V^{\mathcal{B} \ast j(\mathcal{C})}
\end{align*}
\]

Furthermore, if \( \mathcal{B} = \mathcal{S} = (F_a : a \in \mathcal{C}) \) is a \( (\kappa, \lambda) \)-system of filters and \( j = j_{\mathcal{F}(\mathcal{S})} \), then \( \mathcal{C} \ast \mathcal{S}^\mathcal{C} \) is isomorphic to \( \mathcal{S} \ast j(\mathcal{C}) \) and \( j^\mathcal{C} \) is the embedding induced by \( \mathcal{S}^\mathcal{C} \).

\[
\begin{align*}
V \xrightarrow{j_{\mathcal{F}(\mathcal{S})}} M & \subseteq V^\mathcal{S} \\
V^\mathcal{C} \xrightarrow{j_{\mathcal{F}(\mathcal{S}^\mathcal{C})}} M^\mathcal{C}(\mathcal{C}) & \subseteq V^{\mathcal{S} \ast j(\mathcal{C})} = V^{\mathcal{C} \ast \mathcal{S}^\mathcal{C}}
\end{align*}
\]

Proof. For the first part, consider the embedding:

\[
i_1 : \mathcal{C} \rightarrow \mathcal{B} \ast j(\mathcal{C})
\]

By elementarity of \( j \), \( i_1 \) must preserve \( \subseteq, \perp \). Given any maximal antichain \( \mathcal{A}, \mathcal{C} \) is \( <\kappa \cdot \text{cc} \) hence \( j[\mathcal{A}] = j(\mathcal{A}) \) which is maximal again by elementarity of \( j \). Then \( i_1 \) is a complete embedding hence \( \mathcal{B} \ast j(\mathcal{C}) \) is a forcing extension of \( \mathcal{C} \). Thus we can lift \( j \) to a generic elementary embedding \( j^\mathcal{C} \).

For the second part, consider the embedding:

\[
i_2 : \mathcal{C} \ast \mathcal{S}^\mathcal{C} \rightarrow \mathcal{S} \ast j(\mathcal{C})
\]

\[
\dot{A} \subseteq O_a \mapsto \left\{ [\text{id}_a]_{\mathcal{F}(\mathcal{S})} \in j(\dot{A}) \right\} \in \mathcal{S} \ast j(\mathcal{C})
\]

This map is well-defined since the set of \( \dot{A} \in \mathcal{C} \ast \mathcal{S}^\mathcal{C} \) such that \( \dot{A} \subseteq O_a \) for some fixed \( a \in \mathcal{C} \) is dense in \( \mathcal{C} \ast \mathcal{S}^\mathcal{C} \). Suppose now that \( \dot{A} \subseteq \mathcal{C} \ast \mathcal{S}^\mathcal{C} \) \( \hat{B} \) with \( \hat{B} \subseteq O_b, b \in \mathcal{C}, c = a \cup b \). Then,

\[
1 \models_{\mathcal{C}} \left( \pi_c^{-1}[\hat{A}] \setminus \pi_c^{-1}[\hat{B}] \right) \in I_c^\mathcal{C} \Rightarrow 1 \models_{\mathcal{C}} \exists C \in I_c \left( \pi_c^{-1}[\hat{A}] \setminus \pi_c^{-1}[\hat{B}] \right) \subseteq C
\]

and we can find a maximal antichain \( \mathcal{A} \subseteq \mathcal{C} \) such that \( p \models_{\mathcal{C}} \left( \pi_c^{-1}[\hat{A}] \setminus \pi_c^{-1}[\hat{B}] \right) \subseteq C_p \) for every \( p \in \mathcal{A} \) and corresponding \( C_p \in I_c \Rightarrow 1 \models_{\mathcal{C}} [\text{id}_a]_{\mathcal{F}(\mathcal{S})} \notin j(C_p) \). Thus by elementarity of \( j \), for all \( p \in \mathcal{A} \) we have that

\[
\dot{j}(p) \models_{\mathcal{C} \ast j(\mathcal{C})} \left( \pi_c^{-1}[\dot{j}(\hat{A})] \setminus \pi_c^{-1}[\dot{j}(\hat{B})] \right) \subseteq \dot{j}(C_p) \notin [\text{id}_a]_{\mathcal{F}(\mathcal{S})}
\]

and since \( j[\mathcal{A}] \) is maximal in \( j(\mathcal{C}) \),

\[
1 \models_{j(\mathcal{C})} [\text{id}_a \cup b]_{\mathcal{F}(\mathcal{S})} \notin \left( \pi_c^{-1}[\dot{j}(\hat{A})] \setminus \pi_c^{-1}[\dot{j}(\hat{B})] \right) \Rightarrow 1 \models_{j(\mathcal{C})} [\text{id}_a]_{\mathcal{F}(\mathcal{S})} \notin j(\dot{A}) \Rightarrow i_2(\hat{B}) \vee \lnot i_2(\dot{A}) = 1 \Rightarrow i_2(\dot{A}) \leq i_2(\hat{B})
\]
Since \( u \) is a \( \kappa \)-system of filters in \( V \) derived from \( j_{F(\mathbb{S})} \), it is a \( \mathcal{C} \)-system of filters by Propositions 5.1.34 and 5.1.24.

**Proposition 5.2.12.** Let \( j : V \rightarrow M \subseteq V^B \) be elementary with critical point \( \kappa \), \( \gamma < \kappa \) be a cardinal, and \( j^C : V^C \rightarrow M^j(\mathbb{C}) \) be obtained from \( j \) and \( \mathbb{C} = \text{Coll}(\gamma, < \kappa) \). Suppose that \( j(\kappa) \) is regular in \( V^B \).
Corollary 5.2.13. Let $j$ be a property among $(n)$-huge, almost $(n)$-huge (for \( n > 0 \)), $\alpha$-superstrong (for $\alpha > \kappa$), $(n)$-superstrong (for $n > 1$).

If $\kappa$ is generically (resp. ideally) $P$, then it is so after $\text{Coll}(\gamma, < \kappa)$ for any $\gamma < \kappa$. Thus the existence of a generically (resp. ideally) $P$ cardinal is equiconsistent with $\omega_1$ being such a cardinal.

Note that the previous corollary applies only to generically and ideally $P$: the existence of a large cardinal with property $P$ in $V$ is usually stronger than $\omega_1$ being generically $P$. Due to the fact that a generically superstrong cardinal does not guarantee that $j(\kappa)$ is regular in $V^B$, the previous result does not apply to superstrong cardinals. We recall that the case of a strong cardinal was already treated in [9 Corollary 4.14], which showed the following.

Theorem 5.2.14. The existence of a strong cardinal is equiconsistent with $\omega_1$ being ideally strong.

As in Proposition 5.2.12, it is possible to prove that forcing with $\text{Coll}(\gamma, < \kappa)$ with $\kappa$ a strong cardinal preserves the ideally strongness of $\kappa$. However, starting with an ideally strong cardinal would not suffice in this case. In order to get a $j^C$ with strength $\gamma$ we need an embedding $j: V \to M \subseteq V^B$ with enough strength so as to contain in $M$ a name for $V^B$. Although, since the complexity of such a name depends on $B$, and $B$ depends on the amount of strength that we wish to achieve, there is no hope to sort out this circular reference. On the other hand, a generically strong cardinal is preserved under $\text{Cohen}$ forcing under some assumptions [10].

Notice that the previous corollary does not apply also to generically supercompact cardinals. However, this is not surprising since $\kappa = \gamma^+$ being generically supercompact is equivalent to being generically almost huge: in fact, if $j: V \to M \subseteq V[G]$ is a $\gamma$-closed embedding obtained by $\gamma$-supercompactness, it is also almost huge since $j(\kappa) = (\gamma^+)_{V[G]}$. Thus such a preservation theorem for supercompactness would in turn imply the equiconsistency of generically supercompactness and generically almost hugeness, which is not expected to hold. However, if we restrict the class of forcing to $\text{proper}$ forcings, is possible to obtain a similar preservation theorem [17].
5.2.3 Combinatorial equivalents of ideally large cardinal properties

The ideal properties of cardinals given in Definition 5.2.2 are inherently properties of a \( C \)-system of filters, it is therefore interesting to reformulate them in purely combinatorial terms. In this section we review the main results on this topic present in literature, adapted to the paradigm introduced in Section 5.1, and we integrate them with a characterization of strongness that, to our knowledge, is not yet present in literature.

Critical point and tallness

In order to express any large cardinal property, we need to be able to identify the critical point of an embedding \( j_{\mathcal{F}(S)} \) derived from some \( C \)-system of filters \( S \).

**Definition 5.2.15.** Let \( S \) be a \( C \)-system of filters in \( V \). The completeness of \( S \) is the minimum of the completeness of \( F_\alpha \) for \( \alpha \in C \), i.e. the unique cardinal \( \kappa \) such that every \( F_\alpha \) is \(<\kappa\)-complete and there is an \( F_\alpha \) that is not \(<\kappa^+\)-complete.

We say that \( S \) has densely completeness \( \kappa \) iff it has completeness \( \kappa \) and there are densely many \( B \in S^+ \) disproving \(<\kappa^+\)-completeness (i.e. that are the union of \( \kappa \)-many \( B_\alpha \) in the relevant ideal).

**Proposition 5.2.16.** Let \( S \) be a \( C \)-system of filters in \( V \). Then the following are equivalent:

1. the ultrapower map \( \dot{k} = j_{\mathcal{F}(S)} \) has critical point \( \kappa \) with boolean value \( 1 \);
2. \( S \) is a \( (\kappa, \lambda) \)-system of filters;
3. \( S \) has densely completeness \( \kappa \);
4. \( \{ id : \alpha < \kappa \} \in F_\kappa \).

**Proof.** (1) \( \Leftrightarrow \) (2): has already been proved in Proposition 5.1.30

(2) \( \Rightarrow \) (3): By Proposition 5.1.12 we know that \( F_\alpha \) is \(<\kappa\)-complete for all \( \alpha \in C \). Let \( \dot{u} \) be a name for a function representing \( \kappa \) in \( \text{Ult}(V, \mathcal{F}(S)) \). Then there are densely many \( B \in S^+ \) deciding that \( \dot{u} = \dot{v} \), for some \( v : B \rightarrow \kappa \). Since

\[
\left[ [u]_{\mathcal{F}(S)} \neq \alpha = [\text{dom}_\alpha]_{\mathcal{F}(S)} \right]_S \geq B
\]

for all \( \alpha < \kappa \), \( B_\alpha = B \land v^{-1}([\alpha]) \in I_b \) for any such \( B \) hence \( B = \bigcup_{\alpha < \kappa} B_\alpha \) disproves \(<\kappa^+\)-completeness.

(3) \( \Rightarrow \) (1): We prove by induction on \( \alpha < \kappa \) that \( 1 \models j(\dot{\alpha}) = \dot{\alpha} \). Let \( u : A \rightarrow \alpha \) with \( A \in I_\kappa^+ \) be representing an ordinal smaller than \( j(\alpha) \) in the ultrapower, and let \( A_\beta = u^{-1}([\beta]) \) for \( \beta < \alpha \). Since \( A = \bigcup_{\beta < \alpha} A_\beta \) and \( S \) is \(<\kappa\)-complete, the conditions \( A_\beta \) form a maximal antichain below \( A \) hence \( [u]_{\mathcal{F}(S)} \) is forced to represent some \( \beta < \alpha \). Furthermore, there are densely many \( B \in I_\kappa^+ \) that are a union of \( \kappa \)-many \( B_\alpha \). From any one of them we can build a function \( u : B \rightarrow \kappa \), \( u(f) = \alpha_f \).
where \( f \in B_{\alpha_f} \), so that \( B \) forces that \([u]_{\mathcal{F}(\mathcal{S})} j(\kappa) \) and \([u]_{\mathcal{F}(\mathcal{S})} > \alpha \) for all \( \alpha < \kappa \). Thus \( B \models j(\kappa) > \kappa \) for densely many \( B \).

Assume now that \( \kappa \in \mathcal{C} \). Then (1) \( \Leftrightarrow \) (4) follows from Proposition 5.1.27 and Loś theorem, since \( \{ \alpha : \alpha < \kappa \} \) is equal to

\[
\bigwedge_{\alpha < \kappa} \left[ \left[ \text{ran}_\alpha \right]_{\mathcal{F}(\mathcal{S})} = j(\alpha) \right] \land \left[ \left[ \text{ran}_\alpha \right]_{\mathcal{F}(\mathcal{S})} < j(\kappa) \right] \mathcal{S} = [j[\kappa] = \kappa \land j(\kappa) > \kappa] \mathcal{S} \text{.} \quad \square
\]

A similar approach can apply also to tallness-related properties.

**Proposition 5.2.17.** Let \( \mathcal{S} \) be a \((\kappa, \lambda)\)-system of filters in \( V \). The ultrapower map \( j = j_{\mathcal{F}(\mathcal{S})} \) is \( \gamma \)-tall for \( \gamma < \lambda \) iff \( \{ f \in O_{\langle x \rangle} : \text{rank}(f(x)) \leq \kappa \} \in F_{\langle x \rangle} \) for some \( x \in \bigcup \mathcal{C} \) with \( \text{rank}(x) = \gamma \).

**Proof.** By Proposition 5.1.27 and Loś theorem the above set is equal to

\[
\left[ \gamma = \text{rank}(x) = \text{rank}(\text{proj}_x)_{\mathcal{F}(\mathcal{S})} \leq j(\kappa) \right] \mathcal{S} \text{.} \quad \square
\]

### Measurability

We say that a cardinal is measurable iff there is an elementary embedding \( j : V \rightarrow M \) with critical point \( \kappa \) such that the image is well-founded. Its generic counterpart can be characterized for \( \mathcal{C} \)-systems of filters by means of the following definition.

**Definition 5.2.18.** Let \( \mathcal{S} \) be a \( \mathcal{C} \)-system of filters in \( V \). We say that \( \mathcal{S} \) is precipitous iff for every \( B \in \mathcal{S}^+ \) and sequence \( \langle A_\alpha : \alpha < \omega \rangle \in V \) of maximal antichains in \( \mathcal{S} \) below \( B \), there are \( \tilde{A}_\alpha \in A_\alpha, A_\alpha \in I_{\tilde{a}_\alpha}^+ \) and \( h : \bigcup \tilde{a}_\alpha \rightarrow V \) such that \( \pi_{\tilde{a}_\alpha}(h) \in A_\alpha \) for all \( \alpha < \omega \).

This definition is equivalent to [9, Def. 4.4.ii] for ideal extenders, and to \( \mathcal{S} \)-closure for extenders in \( V \) (see [31]), while being applicable also to other systems of filters. The results relating these definitions with well-foundedness are subsumed in the following.

**Theorem 5.2.19.** Let \( \mathcal{S} \) be a \( \mathcal{C} \)-system of filters in \( V \). The ultrapower map \( j = j_{\mathcal{F}(\mathcal{S})} \) is well-founded iff \( \mathcal{S} \) is precipitous.

**Proof.** First, suppose that \( \mathcal{S} \) is precipitous and assume by contradiction that \( B \) forces the ultrapower to be ill-founded. Let \( \langle \hat{u}_n : n < \omega \rangle \) be \( \mathcal{S} \)-names for functions \( \hat{u}_n : O_{\alpha_n} \rightarrow V \) in \( U_{\mathcal{S}} \) such that \( \left[ \hat{u}_{n+1}[\mathcal{F}(\mathcal{S})] \in [\hat{u}_n]_{\mathcal{F}(\mathcal{S})} \right] \mathcal{S} \geq B \). Define \( \hat{b}_n = \bigcup \{ \hat{a}_m : m \leq n \} \), \( B_0 = O_{\hat{b}_0} \), and

\[
\hat{B}_{n+1} = \{ x \in O_{\hat{b}_{n+1}} : \hat{u}_{n+1}(\pi_{\hat{a}_{n+1}}(x)) \in \hat{u}_n(\pi_{\hat{a}_n}(x)) \}
\]

so that \( \left[ \hat{B}_n \in \hat{\mathcal{F}(\mathcal{S})} \right] \mathcal{S} \geq B \). Fix \( n < \omega \). By the forcing theorem there is a dense set of \( A \) in \( \mathcal{S} \) below \( B \) deciding the values of \( \hat{u}_n \), \( \hat{B}_n \), and every such \( A \models \hat{B}_n = \hat{B}_n \) must force that \( \hat{B}_n \in \hat{\mathcal{F}(\mathcal{S})} \) hence satisfy \( A \leq \mathcal{S} B_n \). It follows that the set of \( A \in I^+_{\mathcal{S}} \) deciding \( \hat{u}_n \), \( \hat{B}_n \) and with the additional property that every such \( A \) satisfy \( a \geq b_n \),
we can define a rank-like function \( \pi \) such that \( \pi \subseteq V \).

Let \( \bar{A}_n, \bar{a}_n, h : \bar{a} = \bigcup_n \bar{a}_n \) be obtained from \( \langle A_n : n < \omega \rangle \) by precipitousness of \( S \). Let also \( u_n, B_n \) be such that \( \bar{A}_n \vDash u_n = \bar{a}_n \land \bar{B}_n = \bar{B}_n \). Then \( \pi_{\bar{a}_n}(h) \in \bar{A}_n \subseteq \pi_{\bar{a}_n}^{-1}[B_n] \) and \( \pi_{\bar{a}_n}(h) \in B_n \) for all \( n < \omega \). Thus, \( u_{n+1}(\pi_{a_{n+1}}(h)) \in u_n(\pi_{a_n}(h)) \) is an descending chain in \( V \), a contradiction.

Suppose now that \( S \) is not precipitous, and fix \( B, \langle A_n : n < \omega \rangle \) witnessing it. Define a tree \( T \) of height \( \omega \) consisting of couples of sequences \( \langle B, f \rangle \) such that \( B_n \in A_n \) for all \( n < |B| < \omega \) and \( f \in \bigwedge B \), ordered by member-wise inclusion. Since \( \langle A_n : n < \omega \rangle \) contradicts precipitousness, the tree \( T \) has no infinite chain and we can define a rank-like function \( r : T \to ON \) by well-founded recursion on \( T \) as \( r(x) = \bigcup \{ r(y) + 1 : y <_T x \} \). Notice that \( y <_T x \) implies \( r(y) < r(x) \).

Let \( \mathcal{B} \) be as above, and define \( u_\mathcal{B} : \bigwedge \mathcal{B} \to V \) by \( u_\mathcal{B}(f) = r(\langle \mathcal{B}, f \rangle) \). Let \( \bar{u}_n \) be the \( S \)-name defined by \( \bar{u}_n = \{ (\bar{u}_\mathcal{B}, \bigwedge \mathcal{B}) : B \in \Pi_{m \leq n} A_m \} \). Then any \( \mathcal{B} \in \Pi_{m \leq n+1} A_m \) forces \( u_{n+1} \) to be \( u_\mathcal{B} \), \( \bar{u}_n \) to be \( \bar{u}_\mathcal{B} \), and \( \bar{u}_{n+1} = \bar{u}_n \) since for all \( f \in \bigwedge \mathcal{B} \),

\[
\bar{u}_\mathcal{B}(f) = r(\langle \mathcal{B}, f \rangle) < r(\langle \mathcal{B}[n], f' \rangle) = u_{\mathcal{B}[n]}(f')
\]

where \( f' = f \upharpoonright \text{dom}(\bigwedge \mathcal{B}[n]) \). Since \( \{ \mathcal{B} : B \in \Pi_{m \leq n+1} A_m \} \) forms a maximal antichain below \( B \) for every \( i \), \( B \) forces that \( \langle \bar{u}_n : n < \omega \rangle \) is a name for an ill-founded chain. 

**Strongness**

In this section we cover large cardinal properties defined in terms of the existence of elementary embeddings \( j : V \to M \subseteq V[G] \) with certain degree of strongness (i.e. such that \( V_\gamma^{V[G]} \subseteq M \) for some appropriate \( \gamma \)). Main examples of such properties are strongness, superstrongness and variants of them. We now present a criterion to characterize \( \gamma \)-strongness for an elementary embedding \( j_{\mathcal{F}(S)} \), which can in turn be applied in order to characterize all of the aforementioned large cardinal properties.

To our knowledge, there is no equivalent version of the content of this section in the classical tower or extender setting.

**Definition 5.2.20.** Let \( \mathcal{S} \) be a \( \mathcal{C} \)-system of filters, \( A_0 \cup A_1 \) be an antichain in \( \mathcal{S}^+ \). We say that \( \langle A_0, A_1 \rangle \) is split by \( \mathcal{S} \) iff there exist a \( b \in \mathcal{C} \) and \( B_0, B_1 \) disjoint in \( \mathcal{P}(O_b) \) such that \( A \subseteq B_n \) for all \( A \in A_n, n < 2 \).

We say that a family of antichains \( \langle A_{a_0} \cup A_{a_1} : \alpha < \mu \rangle \) is simultaneously split by \( \mathcal{S} \) if for all \( \alpha < \mu \) there is a single \( b \in \mathcal{C} \) witnessing splitting for all of them.

**Definition 5.2.21.** Let \( \mathcal{S} \) be a \( \mathcal{C} \)-system of filters. We say that \( \mathcal{S} \) is \( < \gamma \)-splitting (\( \mathcal{C} \)-splitting) iff for all sequences \( \langle A_{a_0} \cup A_{a_1} : \alpha < \mu \rangle \) of maximal antichains with \( \mu < \gamma \), there are densely many \( B \in \mathcal{S}^+ \) such that the antichains \( \langle A_{a_0} | B, A_{a_1} | B \rangle \) for \( \alpha < \mu \) are simultaneously split by \( \mathcal{S} \).

**Theorem 5.2.22** (A., Viale). Let \( \mathcal{S} \) be a \( < \gamma \)-directed \( \mathcal{C} \)-system of filters. Then the ultrapower \( \text{Ult}(V, \mathcal{F}(\mathcal{S})) \) contains \( \mathcal{P}^{V[G]}(\mu) \) for all \( \mu < \gamma \) iff \( \mathcal{S} \) is \( < \gamma \)-splitting.

**Proof.** Let \( a \in \mathcal{C}, u_\alpha : O_a \to ON \) be such that \( \bigg[ [u_\alpha]_{\mathcal{F}(\mathcal{S})} = \check{\alpha} \bigg]_{\mathcal{S}} = 1 \) for all \( \alpha < \gamma \).

First, suppose that \( \mathcal{S} \) is \( < \gamma \)-splitting and let \( \check{X} \) be a name for a subset of \( \mu < \gamma \).
Let \( A_{00} \cup A_{01} \) for \( \alpha < \mu \) be a maximal antichain deciding whether \( \check{\alpha} \in \check{X} \) and \( S \) be generic for \( \mathcal{S} \). By \(<\gamma\)-splitting let \( B \in \mathcal{S} \) be such that \( a \subseteq b \in \mathcal{C} \), \( B \subseteq O_b \) and \( \langle A_{00}|B, A_{01}|B \rangle \) is split by \( \mathcal{S} \) in \( B_{a0}, B_{a1} \) partitioning \( B \) for all \( \alpha < \mu \). Then we can define

\[
v : B \to \mathcal{P}(\text{ON})
\]

\[
f \mapsto \{u_\alpha(\pi_{ba}(f)) : f \in B_{a1}, \ \alpha \in \mu\}
\]

Then \( B \) forces that \( [v]_{\mathcal{F}(\check{S})} = \check{X} \), and \( B \in \mathcal{S} \) so \( \text{val}(\check{X}, \mathcal{S}) = [v]_\mathcal{S} \) is in \( \text{Ult}(V, \mathcal{S}) \).

Suppose now that \( [v]_{\mathcal{F}(\check{S})} = \check{X} \), and \( B \in \mathcal{S} \) so \( \text{val}(\check{X}, \mathcal{S}) = [v]_\mathcal{S} \) is in \( \text{Ult}(V, \mathcal{S}) \).

**Definition 5.2.24.** Let \( \mathcal{S} \) be a \( \mathcal{C} \)-system of filters, \( \mathcal{A} = \{A_\alpha : \alpha < \delta\} \) be an antichain in \( \mathcal{S}^+ \). We say that \( \mathcal{A} \) is guessed by \( \mathcal{S} \) iff there exist a \( b \in \mathcal{C} \) and \( B = \{B_\alpha : \alpha < \delta\} \) antichain in \( \mathcal{P}(O_b) \) such that \( A_\alpha = B_\alpha \) for all \( \alpha < \delta \).

We say that a family of antichains \( \langle A_\alpha : \alpha < \mu \rangle \) is simultaneously guessed by \( \mathcal{S} \) iff there is a single \( b \in \mathcal{C} \) witnessing guessing for all of them.

**Definition 5.2.25.** Let \( \mathcal{S} \) be a \( \mathcal{C} \)-system of filters. We say that \( \mathcal{S} \) is \(<\gamma\text{-guessing} \) iff for all sequences \( \langle A_\alpha : \alpha < \mu \rangle \) of maximal antichains with \( \mu < \gamma \), there are densely many \( B \in \mathcal{S}^+ \) such that the antichains \( A_\alpha \upharpoonright B \) for \( \alpha < \mu \) are simultaneously guessed by \( \mathcal{S} \).

Notice that if an antichain is guessed by \( \mathcal{S} \), every partition of it is split by \( \mathcal{S} \). It follows that \(<\gamma\text{-guessing} \) implies \(<\gamma\text{-splitting} \). Furthermore, if \( T \) is a tower of inaccessible length \( \lambda \), \(<\lambda\text{-guessing} \) as defined above is equivalent to \(<\lambda\text{-presaturation} \) for the boolean algebra \( \langle T^+, \leq_T \rangle \).

**Theorem 5.2.26.** Let \( \lambda \) be an inaccessible cardinal, \( \mathcal{S} \) be a \(<\lambda\text{-directed} \mathcal{C} \)-system of filters of length \( \lambda \), \( \gamma < \lambda \) be a cardinal. Then the ultrapower \( \text{Ult}(V, \mathcal{F}(\check{S})) \) is \(<\gamma\text{-closed} \) iff \( \mathcal{S} \) is \(<\gamma\text{-guessing} \).
5.2 Generic large cardinals

Proof. Let $a \in C$, $u_\alpha : O_\alpha \to ON$ be such that $\left[\left[\check{u}_\alpha\right]_{\check{F}(S)} = \check{\alpha}\right]_S = 1$ for all $\alpha < \gamma$.

First, suppose that $\mathcal{S}$ is $<\gamma$-guessing and let $\check{s}$ be a name for a sequence $\check{s} : \mu \to Ult(V, F(S))$ for some $\mu < \gamma$. Let $\mathcal{A}_\alpha$ for $\alpha < \mu$ be a maximal antichain deciding the value of $\check{s}(\check{\alpha})$, so that given any $A \in \mathcal{A}_\alpha$, $A \Vdash \check{s}(\check{\alpha}) = [\check{v}_A]_{\check{F}(S)}$ for some $\check{v}_A : O_{\alpha A} \to V$.

Let $\mathcal{S}$ be generic for $\mathcal{S}$. Then by $<\gamma$-guessing there is a $B \in \mathcal{S}$ such that $\mathcal{A}_\alpha \cap B$ is guessed by $\mathcal{S}$ in $B_\alpha \subseteq P(O_b)$ for all $\alpha < \mu$. Furthermore, there can be only $|b| < \lambda$ elements $A \in \mathcal{A}_\alpha$ such that the corresponding $A' \in B_\alpha$ is not empty. Since $\mathcal{S}$ is $<\lambda$-directed, there is a single $c \in C$, $c \supseteq a, b$, such that $a_A \subseteq c$ for any $A \in \mathcal{A}_\alpha$ that is guessed in an $A' \neq \emptyset$.

Let $B'_\alpha$ denote the unique element of $B_\alpha$ such that $\pi_\alpha(f) \in B'_\alpha$, and $v'_\alpha : O_{\alpha A} \to V$ be such that $B'_\alpha \Vdash \check{s}(\check{\alpha}) = [\check{v}'_\alpha]_{\check{F}(S)}$. Then for any $\alpha < \mu$ we can define

$$
v'_\alpha : O_\alpha \to V \quad f \mapsto \{\{u_i(\pi_\alpha(f)), v'_\alpha(f)\} : \alpha < u_\mu(\pi_\alpha(f))\}
$$

so that $B \Vdash [s]_{\check{F}(S)}(\check{\alpha}) = [s'_{\alpha}]_{\check{F}(S)}$. Since all the $v'_\alpha$ have the same domain $O_\alpha$, we can glue them together forming a single function

$$
v' : O_\alpha \to V \quad f \mapsto \{\{u_i(\pi_\alpha(f)), v'_\alpha(f)\} : \alpha < u_\mu(\pi_\alpha(f))\}
$$

Then $B$ forces that $[s']_{\check{F}(S)} = \check{s}$, and $B \in \mathcal{S}$ so $\text{val}(\check{s}, \mathcal{S}) = [v']_S$ is in $Ult(V, \mathcal{S})$.

Suppose now that $\mathcal{S}$ is $<\gamma$-closed and let $\langle A_\alpha : \alpha < \mu \rangle$, $\mathcal{A}_\alpha = \langle A_{\alpha \beta} : \beta < \xi_\alpha \rangle$ be as in the definition of $<\gamma$-guessing. Let $f$ be a name for a sequence $\check{s} : j_{\check{F}(S)}[\check{\mu}] \to ON$ such that $\left[\check{s}(j_{\check{F}(S)}(\check{\alpha})) = j_{\check{F}(S)}(\check{\beta})\right]_S = [A_{\alpha \beta}]_S$. Since $j_{\check{F}(S)}$ is $<\gamma$-closed, we can find densely many $B \subseteq O_b$ for $b \in C$, $\check{v} : B \to ON^\mu$ such that $B \Vdash \check{s} = [\check{v}]_{\check{F}(S)}$. Then given any $\alpha < \mu$, $\beta < \xi_\alpha$ we can define $B_{\alpha \beta} = \{f \in B : v(f)(\alpha) = \beta\}$ witnessing guessing for $\langle A_\alpha : B \supseteq A_\alpha \rangle$.

The above result gives a good characterization of $<\gamma$-closure for ideal towers, since ideal towers $T$ of inaccessible height $\lambda$ are always $<\lambda$-directed. On the other hand, this result does not apply to ideal extenders since their associated system of domains is never $<\omega_1$-directed. Since it is not known whether there is such a characterization of $<\gamma$-closure for extenders in $V$, we cannot expect to have one in the more general case of ideal extenders.

We can also determine an upper bound to the amount of closure that a system of filters might possibly have.

Definition 5.2.27. Let $\mathcal{S}$ be a $C$-system of filters of length $\lambda$, $\alpha < \lambda$ be an ordinal. We say that $\mathcal{S} \upharpoonright \alpha$ does not express $\mathcal{S}$ iff $1 \Vdash_\mathcal{S} M \supseteq M_\alpha$ where $M = Ult(V, \check{F}(S))$, $M_\alpha = Ult(V, \check{F}(\mathcal{S} \upharpoonright \alpha))$.

Notice that $1 \Vdash_\mathcal{S} M \supseteq M_\alpha$ is equivalent to $1 \Vdash_\mathcal{S} M \supseteq k_\alpha[M_\alpha]$. In fact, $M = M_\alpha$ implies that $k_\alpha = \text{id}$ by Kunen’s inconsistency, and $M = k_\alpha[M_\alpha]$ implies that $k_\alpha$ has no critical point thus is the identity.

Theorem 5.2.28. Let $\mathcal{S}$ be a $C$-system of filters of length $\lambda$ such that $\mathcal{S} \upharpoonright \alpha$ does not express $\mathcal{S}$ for any $\alpha < \lambda$. Then $M = Ult(V, \check{F}(S))$ is not closed under $\text{cof}(\lambda)$-sequences.
5 Systems of filters and generic large cardinals

Proof. Let \( \langle \xi_{\alpha} : \alpha < \gamma = \text{cof}(\lambda) \rangle \) be a cofinal sequence in \( \lambda \). For all \( \alpha < \gamma \), let \( \dot{u}_{\alpha} \) be a name for an element of \( M \setminus k_{\xi_{\alpha}}[M_{\xi_{\alpha}}] \), \( M_{\xi_{\alpha}} = \text{Ult}(V, \dot{F}(S_{\xi_{\alpha}})) \). Let \( \dot{s} \) be a name such that

\[
\dot{s}(j_{\dot{F}(S)}(\alpha)) = [\dot{u}_{\alpha}]_{\dot{F}(S)}
\]

for all \( \alpha < \gamma \), i.e. \( \dot{s} \) is a name for a \( \gamma \)-sequence of elements of \( M \) indexed by \( j_{\dot{F}(S)}[\gamma] \).

Suppose by contradiction that \( M \) is closed under \( \gamma \)-sequences, so that \( 1 \Vdash_{S} \dot{s} \in M \), then there is an \( A \in S^{+} \), \( v : A \rightarrow V \) such that \( A \Vdash_{S} \dot{s} = [\dot{\alpha}]_{\dot{F}(S)} \). Let \( \alpha < \gamma \) be such that \( A \in (S_{\xi_{\alpha}})^{+} \). Let \( v' : A \rightarrow V \) be such that \( v'(f) = v(f)(\alpha) \) if \( \alpha \in \text{dom}(f) \).

By fineness \( A \Vdash [v']_{\dot{F}(S)} = [\dot{u}_{\alpha}]_{\dot{F}(S)} \) for all \( \alpha < \gamma \). Thus \( A \Vdash [\dot{u}_{\alpha}]_{\dot{F}(S)} \in k_{\xi_{\alpha}}[M_{\xi_{\alpha}}] \), a contradiction.

The situation described in the previous theorem occurs in several cases, as shown by the following proposition.

Proposition 5.2.29. Let \( T \) be an ideal tower of height \( \lambda \) limit ordinal, \( \alpha < \lambda \) be an ordinal. Then \( T|\alpha \) does not express \( T \).

Proof. Suppose by contradiction that there is an \( A \in T^{+} \) such that \( A \Vdash T \geq k_{\alpha}[M_{\alpha}] \). Then in particular \( A \Vdash T \vdash [\text{id}_{V_{\alpha+1}}]_{\dot{F}(T)} \in k_{\alpha}[M_{\alpha}] \) hence let \( A' \in T^{+} \), \( A' \leq A \) be such that \( A' \Vdash T \vdash [\text{id}_{V_{\alpha+1}}]_{\dot{F}(T)} = [\dot{u}]_{\dot{F}(T)} \) for some \( u : O_{\alpha} \rightarrow V \), \( a \in C|\alpha \). Thus by Löś Theorem,

\[
B = \{ f \in O_{V_{\alpha+1}} : f = u(\pi_{a}(f)) \} \in T^{+}
\]

Since \( |B| \leq |\text{ran}(u)| \leq |O_{\alpha}| \leq |\Delta_{\alpha}| < |V_{\alpha+1}| \), \( \text{dom}(B) \) is a non-stationary subset of \( V_{\alpha+1} \) by Lemma 5.2.30 contradicting Proposition 5.1.10.

Lemma 5.2.30. Let \( S \) be a \( C \)-system of filters of length \( \lambda \), \( \gamma \) be a cardinal such that \( |a| \leq \gamma \) for all \( a \in C \). Then \( j_{\dot{F}(S)}(\gamma) < (|\Delta_{\lambda} \cdot 2^{\gamma}|^{+})^{V} \).

Proof. Consider the set \( U \) of functions \( u : a \rightarrow \gamma \) for some \( a \in C \). The total number of such functions is bounded by

\[
|C| \cdot \gamma^{\text{sup} a \in |a|} \leq |\Delta_{\lambda} \cdot \gamma^{\gamma}| = |\Delta_{\lambda} \cdot 2^{\gamma}| = \delta
\]

Let \( U = \{ u_{\alpha} : \alpha < \delta \} \), \( A_{\alpha} \) be the maximal antichain in \( S^{+} \) deciding the value of \( [u_{\alpha}]_{\dot{F}(S)} \), and let \( X_{\alpha} = \{ \xi : \exists A \in A_{\alpha} \quad A \vdash [\dot{u}_{\alpha}]_{\dot{F}(S)} = \dot{\xi} \} \), \( X = \bigcup_{\alpha < \delta} X_{\alpha} \). Since \( |X_{\alpha}| \leq |A_{\alpha}| \leq |\Delta_{\lambda}| \), we have that \( |X| \leq \delta \cdot |\Delta_{\lambda}| = \delta \). Let now \( \dot{v} \) be such that \( [\dot{v}]_{\dot{F}(S)} = j_{\dot{F}(S)}(\gamma) \). Then there is a dense set \( A \in S^{+} \) such that \( A \vdash [\dot{v}]_{\dot{F}(S)} = [\dot{u}_{\alpha}]_{\dot{F}(S)} \Rightarrow A \vdash [\dot{v}]_{\dot{F}(S)} \in \dot{X} \). Thus \( j_{\dot{F}(S)}(\gamma) \subseteq X \) (actually, \( j_{\dot{F}(S)}(\gamma) = X \)) and \( |X| \leq \delta \), hence \( j_{\dot{F}(S)}(\gamma) < (\delta^{+})^{V} \).

Proposition 5.2.31. Let \( E \) be a \( \langle \kappa, \lambda \rangle \)-ideal extender such that \( \lambda = |\Delta_{\lambda}| \). Suppose that \( 1 \vdash j_{\dot{F}(E)}(\gamma) \geq \lambda \) for some \( \gamma < \lambda \). Then \( E|\alpha \) does not express \( E \) for any \( \alpha < \lambda \).

Proof. Since \( j_{\dot{F}(E)}(\gamma) \geq \lambda \), \( \kappa|\alpha \} \leq \gamma \) for any \( \alpha < \lambda \) hence we can apply Lemma 5.2.30 to obtain \( j_{\dot{F}(E|\alpha)}(\gamma) < (|\Delta_{\lambda} \cdot 2^{\gamma}|^{+})^{V} \) which is smaller than \( \lambda \) since \( \alpha, \gamma < \lambda \) and \( \lambda \) is a \( \Delta \)-fixed point. It follows that the critical point of \( k_{\alpha} : M_{\alpha} \rightarrow M \) is at most \( j_{\dot{F}(E|\alpha)}(\gamma) \) and in particular \( k_{\alpha}[M_{\alpha}] \neq M \).

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5.2 Generic large cardinals

We remark that the conditions of the previous proposition are often fulfilled. In particular they hold whenever $E$ is the $\langle \kappa, \lambda \rangle$-ideal extender derived from an embedding $j$ and the length $\lambda$ is a $\beth$-fixed point but not a $j$-fixed point.

5.2.4 Distinction between generic large cardinal properties

Let $j : V \to M \subseteq M[G]$ be a generic elementary embedding with critical point $\kappa$. In this section we provide examples separating the following generic large cardinals notions at a successor cardinal $\kappa = \gamma +$.

- $j$ is almost superstrong if $V^M_{j(\kappa)} \prec V_{\gamma^+}$;
- $j$ is superstrong if it is $j(\kappa)$-strong;
- $j$ is almost huge if it is $<j(\kappa)$-closed.

These examples will all be obtained by collapsing with $C = \operatorname{Coll}(\gamma, <\kappa)$ a suitable large cardinal embedding in $V$, so that by Theorem 5.2.10 a generic large cardinal embedding $j_C$ is obtained with the desired properties.

**Proposition 5.2.32.** Let $\kappa$ be a 2-superstrong cardinal. Then there is a generic elementary embedding on $\kappa = \gamma +$ that is almost superstrong and not superstrong.

**Proof.** Let $j$ be a 2-superstrong embedding with critical point $\kappa$, and let $E$ be the $\langle \kappa, j(\kappa) \rangle$-extender derived from $j$. Since $V$ models that $E$ is a superstrong $\langle \kappa, j(\kappa) \rangle$-extender, by elementarity $M$ models that $j(E)$ is a superstrong $\langle j(\kappa), j^2(\kappa) \rangle$-extender. Thus

$$j_{j(E)} : M_{j(\kappa)} \to N = \operatorname{Ult}(M, j(E)) \supseteq M_{j^2(\kappa)} = V_{j^2(\kappa)}$$

Since $M \subseteq V$, also $\operatorname{Ult}(V, j(E)) \supseteq \operatorname{Ult}(M, j(E)) \supseteq V_{j^2(\kappa)}$ hence $j(\kappa)$ is superstrong as witnessed by $j(E)$ also in $V$.

Consider now $C = \operatorname{Coll}(\gamma, <\kappa)$, $j_1$ induced by $C$ and $j_E$, $j_2$ induced by $j(C)$ and $j_{j(E)}$. By Proposition 5.2.12, $j_1$ and $j_2$ are still superstrong and we get the following diagram, where all the inclusions are superstrong:

$$\begin{array}{ccc}
V^C & \longrightarrow & M_{j(C)} \\
\upharpoonright & & \upharpoonright \\
\downarrow & & \downarrow \\
V^{j(C)} & \longrightarrow & V_{j^2(C)} \\
j_0 & : & M_{j(C)} \subseteq M_{j^2(C)} \\
j_1 & : & N_{j^2(C)} \subseteq M_{j^2(C)} \\
j_2 & : & V_{j^2(C)} \subseteq V_{j^2(C)}
\end{array}$$

Thus $j_0$ considered as a generic elementary embedding in $V_{j^2(C)}$ is almost superstrong:

$$M_{j(C)} = V_{j(C)}_{j(\kappa)} \prec V_{j^2(C)}_{j^2(\kappa)} = V_{j^2(C)}_{j^2(\kappa)} = V^{j^2(C)}_{\gamma^+}$$

but not superstrong, since $V_{\gamma^+}$ has cardinality $\delta \in (\gamma, j^2(\kappa))$ in $V^{j(C)}$ hence in $V_{j^2(C)}$ its cardinality is collapsed to $\gamma$ and bijection between $\gamma$ and $V_{\gamma^+}$ is added.

$\square$
Proposition 5.2.33. Let $\kappa$ be a 2-huge cardinal. Then there is a generic elementary embedding on $\kappa = \gamma^+$ that is superstrong and not $<\omega$-closed.

Proof. Let $j$ be a 2-huge embedding in $V$ with critical point $\kappa$. Then we can derive a $(\kappa, j(\kappa) + \omega)$-tower $T$ from $j$, so that $j_T$ is still $\kappa + \omega$-superstrong but by Theorem 5.2.28 is not closed under $\omega$-sequences.

Let $j^C : V^C \rightarrow M^{j^C}$ be derived from $j_T$ as in Theorem 5.2.10. Since $j_T$ is $\kappa + \omega$-superstrong, by Proposition 5.2.12 $j^C$ is still $\kappa + \omega$-superstrong (hence superstrong). Moreover $j^C$ is not $<\omega$-closed. In fact given any $A \in \omega \setminus M$, $j(C)$ cannot add $A$ since it is a set of size $\omega$ and $C$ is closed under $\omega$-sequences.

5.3 Derived ideal towers in $V$ failing preservation of large cardinal properties

In this section we shall present an example of a tower $T$ (of maximal length) derived from a generically superstrong embedding $j : V \rightarrow M \subseteq V[G]$, such that the natural embedding of $T$ into $B$ is densely incomplete (even though $T$ could still induce a superstrong embedding). The construction will be built on Foreman and Woodin’s work on self-genericity for towers (see e.g. [16, Sec. 9.4]). The superstrong embedding $j$ will be obtained as the restriction of the huge embedding induced by a full stationary tower of length a Woodin cardinal.

Definition 5.3.1. We say that $T_\delta = \langle NS_a : a \in V_\delta \rangle$ is the full stationary tower of length $\delta$. We denote by $T_\delta | A$ with $A \in T_\delta$ the tower of ideals $NS_a | A = \{ B \in NS_a : A \wedge B \text{ is stationary} \}$.

We say that $T_\delta^\gamma = \langle NS_a^\gamma : a \in V_\delta \rangle$ is the $\gamma$-stationary tower of length $\delta$ where $NS_a^\gamma = \{ A : A \cap [a]^{<\gamma} \in NS_a \}$.

By the results of Appendix 5.1.2 we can identify $T_\delta$ with a $C$-system of filters. Filter property and fineness can be directly verified, compatibility follows from Lemma 1.2.11 and normality from Lemma 1.2.5. If $\delta$ attains sufficient large cardinal properties, these towers provably induce embeddings with strong closure properties.

Theorem 5.3.2 (Woodin [33 Thm. 2.5.8]). Let $\delta$ be a Woodin cardinal, $j : V \rightarrow M \subseteq V[G]$ be induced by $T_\delta$. Then $j$ is $<\delta$-closed.

We shall follow the following procedure:

- carefully choose a $T \in NS_\lambda^+$ such that $T \Vdash_{T_\delta} j(\text{crit}(j)) \leq \lambda < \delta$;
- consider the huge embedding $j : V \rightarrow M \subseteq V[G]$ induced by $T_\delta | T$;
- consider the reduced superstrong embedding $j_\lambda$ induced by $\hat{F}(T_\delta | T) | \lambda$;
- derive a tower $T_\lambda^\gamma | S'$ from $j_\lambda$ (of maximal height $\lambda$ by Proposition 5.1.37);
- show that the natural immersion of $T_\lambda^\gamma | S'$ into $T_\delta | T$ is densely incomplete.

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5.3 Derived ideal towers in $V$ failing preservation of large cardinal properties

**Definition 5.3.3.** Let $\lambda < \delta$ be ordinals, $M \subseteq V_\delta$, and

$$G_M = \{ A \in T_\lambda \cap M : M \cap \bigcup A = A \}$$

We say that $G_M$ is $M$-generic for $T_\lambda$ iff for every $A \in M$ predense subset of $T_\lambda$, $A \cap G_M \neq \emptyset$.

We aim to prove that

$$S = \{ M \in [V_{\lambda+2}]^{<\gamma} : G_M \text{ not } M\text{-generic for } T_\lambda \}$$

is stationary and compatible with every $A \in T^*_\lambda$, for appropriately chosen $\gamma$, $\lambda$. This will be done by showing that $S$ has the end-extension property.

**Definition 5.3.4.** We say that $M$ is a rank initial segment of $N$ iff $N \cap V_{\text{rank}(M)} = M$. We say that $N < V_\delta$ is a $\lambda$-end-extension of $M < V_\delta$ iff $N \supseteq M$ and $M \cap V_\lambda$ is a rank initial segment of $N \cap V_\lambda$. We say that an end-extension is proper iff $N \cap V_\lambda \supseteq M \cap V_\lambda$, and that is sup-preserving iff sup$(N \cap \alpha) = \text{sup}(M \cap \alpha)$ for all $\alpha \in M$ with cof$(\alpha) > \lambda$.

We say that $A \subseteq [V_\delta]^{<\gamma}$ has the $\lambda$-end-extension property iff for club many $M \in [V_{\delta+2}]^{<\gamma}$ there is $N \in A \uparrow V_{\delta+2}$ that is a $\lambda$-end-extension of $M$.

**Proposition 5.3.5.** Suppose that $A \subseteq [V_\delta]^{<\gamma}$ has the $\lambda$-end-extension property. Then it is stationary and compatible with every $B \in T^*_\lambda$.

**Proof.** Fix $B \in T^*_\lambda$ and $C_f$ any club on $V_\delta$. We prove that $A \cap B \cap C_f \neq \emptyset$. Let $D$ be the club witnessing the end-extension property of $A$, and let $M \prec V_{\delta+2}$ be such that $A, B, f \in M$ and $M \in B \cap D$. Let $N$ be a $\lambda$-end extension of $M$ in $A \uparrow V_{\delta+2}$. Then $N \cap V_\lambda \in A \cap B \cap C_f$ as required. \hfill \Box

Using a measurable cardinal, it is possible to produce end-extensions satisfying many requirements. The main building block is the following.

**Theorem 5.3.6.** Let $M \prec V_\delta$ be such that $|M| < \lambda \in M$ with $\lambda$ a measurable cardinal. Then there exists an $N$ proper sup-preserving $\lambda$-end-extension of $M$ such that $|N| = |M|$.

**Proof.** Let $F \in M$ be a $<\lambda$-complete ultrafilter on $\lambda$, so that $(F \cap M) \in F$ hence it is unbounded. Let $\gamma$ be in $(F \cap M) \setminus \text{sup}(M \cap \lambda)$, and let

$$N = \{ f(\gamma) : f : \lambda \to V_\delta \land f \in M \}.$$ 

Then $N$ is a $\lambda$-end-extension of $M$ as shown in $[33$, Lemma 1.1.21$]$, and it is proper since $\gamma = \text{id}(\gamma)$ and $\text{id} |\lambda \in M$ thus $\gamma \in N$.

We now prove that it is also sup-preserving. Suppose by contradiction that there is an $\alpha \in M$, cof$(\alpha) > \lambda$ and $\xi \in N \cap \alpha$ such that $\xi \geq \text{sup}(M \cap \alpha)$. Let $f : \lambda \to \alpha$ in $M$ be such that $f(\gamma) = \xi$. Since $V_\delta$ models that $f$ is bounded below $\alpha$, so does $M$ hence let $\beta = \text{sup}(\text{ran}(f)) < \alpha$ be in $M$. Then $\beta \in (M \cap \alpha) \setminus \xi$, a contradiction. \hfill \Box

\footnote{Notice that $G_M \subseteq M$ hence $G_M = G_M \cap M$ and the present definition coincides with the usual notion of $M$-genericity.}
End-extensions are powerful in capturing the following type of stationary sets.

**Definition 5.3.7.** Let $A \subseteq \lambda$ be stationary in the classical sense. Then,

$$\tilde{A} = \{ M \prec V_\lambda : \sup(M \cap \lambda) \in A \}$$

is the corresponding generalized stationary set on $V_\lambda$.

**Lemma 5.3.8.** Let $M \prec V_\delta$ be such that $|M| < \gamma < \lambda \in M$ with $\lambda$ measurable cardinal, $A \subseteq \{ \alpha < \lambda : \text{cof}(\alpha) < \gamma \}$ be stationary in the classical sense. Then there exists an $N$ sup-preserving $\lambda$-end-extension of $M$ such that $|N| < \gamma$ and $N \in \tilde{A} \uparrow V_\delta$.

**Proof.** Define $M_\alpha$ for any $\alpha < \lambda$ as follows.

- $M_0 = M$
- Let $M_{\alpha+1} \supseteq M_\alpha$ be the witness provided by Theorem 5.3.6 for $M, \delta, \lambda$. Hence $\sup(M_{\alpha+1} \cap \lambda) > \sup(M_\alpha \cap \lambda)$, $M_{\alpha+1} \cap \eta = M_\alpha \cap \eta = M \cap \eta$ for $\eta = \sup(M \cap \lambda)$ and all sups of cofinality greater than $\lambda$ are preserved.
- If $\alpha$ is a limit ordinal, $M_\alpha = \bigcup \{ M_\beta : \beta < \alpha \}$.

Therefore $C = \{ \sup(M_\alpha \cap \lambda) : \alpha < \lambda \}$ is a club such that there exists $\tilde{\alpha} > \gamma$ with $\sup(M_{\tilde{\alpha}} \cap \lambda)$. Hence $|X| < \gamma$. Define $N = \text{SkH}^{M_\delta}(M \cup X)$. Then $N \prec M_\delta$. Hence $N$ is a sup-preserving $\lambda$-end-extension in $\tilde{A} \uparrow V_\delta$ such that $|N| < \gamma$.

In presence of a limit of measurable cardinals, this process can be iterated in order to capture simultaneously a sequence of stationary sets.

**Lemma 5.3.9.** Let $s = (\lambda_\alpha : \alpha < \xi)$ be an increasing sequence of measurable cardinals such that $s \cap \text{lim}(s) = \emptyset$ and $\alpha < \lambda_\alpha$ for all $\alpha < \xi$. Let $M \prec V_\delta$ be such that $|M| < \gamma < \lambda_0$ and $s \in M$. Let $A_\alpha \subseteq \{ \beta < \lambda_\alpha : \text{cof}(\beta) < \gamma \}$ be stationary in the classical sense for all $\alpha < \xi$. Then there exists a $\lambda_0$-end-extension $N$ of $M$ such that $|N| < \gamma$ and $N \in (\Delta_{\alpha<\xi} \tilde{A}_\alpha) \uparrow V_\delta$.

**Proof.** For any $i \in \omega$ define $M^i$ by induction over $i$. Put $M^0 = M$. Let $M^i \cap \xi = \{ \alpha^i_\beta : \beta < \nu_i \}$, where $\nu_i = \text{otp}(M^i \cap \xi) < \gamma$. Let $M^{i+1} = \bigcup \{ M^i_\beta : \beta < \nu_i \}$, where $M^i_\beta$ are defined by induction as follows.

- $M^i_0 = M^i$.
- If $M^i_\beta \notin \tilde{A}_{\alpha^i_\beta} \uparrow V_\delta$, define $M^i_{\beta+1} \supseteq M^i_\beta$ given by Lemma 5.3.8 applied to $A_{\alpha^i_\beta}$.
- Otherwise let $M^i_{\beta+1} = M^i_\beta$.
- If $\beta$ is a limit ordinal, $M^i_\beta = \bigcup \{ M^j_\beta : j < \beta \}$.

Let $M^\omega = \bigcup \{ M^i : i < \omega \}$. Hence for any $\nu \in M^\omega \cap \gamma$, there exists $i$ such that $\nu \in M^i$. Thus there exists $\beta$ such that $\nu = \alpha^i_\beta$. It follows that $\sup(M^{i+1} \cap \lambda_\nu) \in A_\nu$, and by Lemma 5.3.8 for any $j > i$, $\sup(M^j \cap \lambda_\nu) = \sup(M^{i+1} \cap \lambda_\nu)$. Hence it holds also for $j = \omega$.

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Let now \( \delta \) be a Woodin cardinal, \( \kappa < \delta \) be regular. Let \( \mu < \delta \) be such that
\[
S' = \{ X \subseteq \mu : X \cap \kappa \in \kappa \land \text{otp}(X) \geq \kappa \}
\]
is stationary. Notice that \( S' \) forces the critical point of \( j \) to be \( \kappa \) and \( j(\kappa) \leq \mu \). Let \( \lambda \in (\mu, \delta) \) be a limit of measurable cardinals, and \( s = (\lambda_\alpha : \alpha < \xi) \) be a cofinal sequence of measurable cardinals as in the hypothesis of the previous lemma and such that \( \lambda_0 > \mu \). Let \( \gamma \in (\mu, \lambda_0) \) be a cardinal, let \( A_\alpha, B_\alpha \) be disjoint stationary subsets of \( \{ \beta < \lambda_\alpha : \text{cof}(\beta) < \gamma \} \) for all \( \alpha < \xi \) by Ulam’s Theorem 1.2.12, and let \( C_\alpha = \{ M \prec V_{\lambda_\alpha} : |M| \geq \gamma \} \) for \( \alpha < \xi \).

**Lemma 5.3.10.** The set \( A_\alpha = \{ A_\beta, C_\beta : \beta \in [\alpha, \xi) \} \) is predense in \( T_\lambda \) for all \( \alpha < \xi \).

**Proof.** Let \( D \in T_\lambda \) be arbitrary, and \( \beta \in [\alpha, \xi) \) be such that \( D \in V_{\lambda_\beta} \). Partition \( D \) in \( D_1, D_2 \) such that \( D_1 = \{ M \in D : |M| < \gamma \} \), \( D_2 = D \setminus D_1 \). At least one among \( D_1, D_2 \) is stationary. If \( D_2 \) is stationary it is compatible with \( C_\beta \) (since \( D_2 < C_\beta \)), otherwise \( D_1 \) is compatible with \( A_\beta \). In fact, given any club \( C_\beta \) on \( V_{\lambda_\beta} \) and \( M \prec V_{\lambda_\beta} \) such that \( |M| < \gamma \), \( D_1, A_\beta, f \in M, M \in D_1 \uparrow V_\lambda \), by Lemma 5.3.8 we can find an \( N \in A_\beta \uparrow V_\lambda \) so that \( N \cap V_{\lambda_\beta} \in D_1 \land A_\beta \land C_\beta \).

**Theorem 5.3.11.** Let \( \gamma, s, \lambda, \delta \) be as above. Then,
\[
S = \{ M \in [V_{\lambda + 2}]^{<\gamma} : G_M \text{ not } M\text{-generic for } T_\lambda \}
\]
is stationary and compatible with any \( A \in T_\lambda^s \).

**Proof.** We show that \( S \) has the \( \lambda_\alpha \)-end-extension property for any \( \alpha < \xi \), the thesis will then follow by Proposition 5.3.5. Let \( M \prec V_{\lambda + 4} \) be such that \( |M| < \gamma \), \( A_\lambda, s \in M \). Let \( N \) be a \( \lambda_\alpha \)-end-extension of \( M \) in \( (\Delta_{\beta \in [\alpha, \xi)} B_\beta) \uparrow V_{\lambda + 4} \) by Lemma 5.3.9 (considering the sequence \( s \) after \( \alpha \)). Then \( G_N \) is not \( N \)-generic for \( T_\lambda \), since \( \beta \in N \cap [\alpha, \xi) \Rightarrow N \in B_\beta \uparrow V_{\lambda + 4} \Rightarrow N \notin A_\beta \uparrow V_{\lambda + 4} \Rightarrow \forall \bar{A}_\beta \notin G_N \) hence \( A_\alpha \cap G_N = \emptyset \).

Let \( j \) be the generic elementary embedding derived from \( T_\delta \uparrow T \), \( T = S \land S' \), and consider the reduced embedding \( j_\lambda \) derived from \( \mathbf{F}(T_\delta \uparrow T) \cap V_\lambda \). By Theorem 5.3.11 the elements of \( T_\lambda^s \) which are compatible with \( T \) are exactly the elements of \( T_\lambda^s \uparrow S' \). Furthermore, no element \( A \subseteq \mathcal{P}(X) \) of \( T_\lambda^s \) which concentrates on \( |X| \geq \gamma \) can be compatible with \( S \) (thus with \( T \)). It follows that the tallest tower derivable from \( j_\lambda \) is \( T_\lambda^s \uparrow S' \), and the corresponding embedding \( i : T_\lambda^s \uparrow S' \to T_\delta \uparrow T \) is the inclusion map.

**Theorem 5.3.12.** The embedding \( i : T_\lambda^s \uparrow S' \to T_\delta \uparrow T \) is densely incomplete.

**Proof.** Let \( B \in T_\lambda^s \uparrow S' \) be arbitrary. Since \( T \land B \) is stationary, consider the function
\[
f : T \land B \longrightarrow V_{\lambda + 2} \to M \longmapsto A \in M \text{ such that } G_M \cap A = \emptyset
\]
By Fodor’s Lemma 1.2.5 there is a stationary set \( C \subseteq T \land B \) such that \( f[C] = \{ A \} \). Then \( A \uparrow B \) is maximal below \( B \) in \( T_\lambda^s \uparrow S' \), but is not maximal in \( T_\delta \uparrow T \) as witnessed by \( C \).

**Corollary 5.3.13.** Let \( \delta \) be a Woodin cardinal. Then for any \( \kappa \in [\omega_1, \delta) \) there is a generically superstrong embedding \( j \) with critical point \( \kappa \) such that the tallest tower derivable from \( j \) embeds in the original forcing in a densely incomplete way.
5.4 Conclusions and open problems

In the last chapter we investigated some topics related to the definability of generic large cardinal properties. We gave a unified treatment of extenders and towers, and some partial results on how generic large cardinal embeddings are induced by set-sized objects. However, many questions remain open. We list them according to the ordering of sections of this chapter.

Question 5.4.1. Is \((j_S[a])^{-1}\) the unique element of \(\bigcap j_S[F_a]\)?

Question 5.4.2. Assume that \(j\) is a \(<\gamma\)-closed embedding in \(V[G]\), with \(G\) \(V\)-generic for \(B\). Can this be witnessed by a generic \(\gamma\)-extender of sufficient length independently of the chain condition satisfied by \(B\)?

In Theorem 5.1.40, we showed that a generic extender can have enough expressive power to approximate any other generic \(C\)-system of ultrafilters. This observation suggests the following question.

Question 5.4.3. Assume \(\kappa\) has ideally property \(P\), can this be witnessed by an extender in the ground model \(V\)?

Since \(\kappa\) has ideally property \(P\), there is a \(C\)-system of filters \(S\) in \(V\) which witness property \(P\) for \(\kappa\). However, in general \(S\) might not be an extender (e.g. a tower or a \(\gamma\)-extender). We already know that a generic extender \(E\) is able to fully approximate \(S\), is this possible also for an ideal extender \(E\) in \(V\)? We believe that this question could be an important cornerstone to uncover the following.

Question 5.4.4. Is having ideally property \(P\) equivalent to having generically property \(P\)?

A final question arises from inspecting the machinery used in Section 5.3.

Question 5.4.5 (Cox). What is the consistency strength of self-genericity with respect to towers?

More precisely, let \(T\) be a tower of inaccessible height \(\lambda\), \(S^T\) be the set of self-generic \(M \prec H_{\lambda^+}\) for \(T\), and consider the following two versions of self-genericity (inspired by the analogous definitions for ideals introduced in [12]):

- StatCatch\((T)\) abbreviates the statement: \(S^T\) is stationary;
- ProjectiveCatch\((T)\) abbreviates the statement: \(S^T\) is stationary and compatible with all \(A \in T\).

What is the consistency strength of these properties holding for a tower \(T\) with critical point \(\omega_1\) (or \(\omega_2\))? A similar problem was inspected in [12] for single ideals, but it is not at all clear whether a similar argument can be applied to towers.


