Martin's maximum revisited

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Abstract

We present several results relating the general theory of the stationary tower forcing developed by Woodin with forcing axioms. In particular we show that, in combination with class many Woodin cardinals, the forcing axiom MM^{++} makes the Π_2-fragment of the theory of $H_{\aleph_2}$ invariant with respect to stationary set preserving forcings that preserve BMM. We argue that this is a promising generalization to $H_{\aleph_2}$ of Woodin's absoluteness results for $L(\mathbb{R})$. In due course of proving this, we shall give a new proof of some of these results of Woodin. Finally we relate our generic absoluteness results with the resurrection axioms introduced by Hamkins and Johnstone and with their unbounded versions introduced by Tsaprounis.

This[1] paper is meant as an introductory exposition containing some preliminary results to the research I've undertaken to generalize Woodin's absoluteness results. More precisely it is a survey over a different approach to present Woodin's generic absoluteness results for $L(\mathbb{R})$ and how this approach can lead to generalize Woodin's results to larger fragments of the universe.

Woodin shows that the first order theory of $L(\mathbb{R})$ with real parameters is invariant under set forcing assuming large cardinals. In this paper we shall show that in models V of MM^{++} the Π_2-theory of $H^V_{\aleph_2}$ with parameters in $P(\omega_1)^V$ is invariant with respect to stationary set preserving forcings which preserve BMM. We shall also argue that the restriction to the class of stationary set preserving forcings is a necessary requirement if one wishes to admit as parameters of the generically invariant theory all subsets of ω_1 which are in V. A complete account on the (close to) optimal absoluteness results we can obtain for models of strenghtenings of MM are presented in [19] and in [1] (this latter with Giorgio Audrito) which are the natural continuation of this article.

The paper is organized as follows: In the introduction (Section 1) we shall take a long detour to motivate the absoluteness results we want to present and to show how they stem out of Woodin's work on Ω-logic. Section 2 presents background material on forcing (Subsection 2.1), the stationary tower forcing (Subsection 2.2), forcing axioms (Subsection 2.3), the relation between the stationary tower forcing and forcing axioms (Subsection 2.4), and a new characterization of the forcing axiom MM^{++} in terms of complete embeddings of stationary set preserving posets into stationary tower forcings (Subsection 2.5). Section 3 gives a new elementary proof of the invariance of the theory of $H_{\aleph_1}$ with respect to set forcing in the presence of class many Woodin cardinals,

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while Section 4 presents the proof of the main result i.e. the Π_2-absoluteness for the theory of H_{ω_2} in models of MM^{++} with respect to stationary set preserving forcings that preserve BMM. Section 5 extends the results of Section 4 to the setting of the resurrection axioms introduced by Hamkins and Johnstone [7] and of their unbounded version introduced by Tsaprounis [17].

While the paper is meant to be as much self-contained as possible, we presume that familiarity with forcing axioms (in particular with Martin's maximum) and with the stationary tower forcing are of valuable help for the reader. A good reference for background material on Martin's maximum is [8, Chapter 37]. For the stationary tower forcing a reference text is [11].

The reader who is interested only in the proofs of the new generic absoluteness results and is already acquainted with forcing axioms and the stationary tower may skip the introduction, have a glance at the results of Section 2 with a particular attention to the content of Subsection 2.5 and then move directly to Sections 3, 4, 5.

1 Introduction

We tried to make this introduction comprehensible to any person acquainted with the theory of forcing as presented for example in [9]. The reader may refer to subsection 1.1 for unexplained notions and to Section 2 for the background material he may need to follow our presentation.

Since its discovery in the early sixties by Paul Cohen [3], forcing has played a central role in the development of modern set theory. It was soon realized its fundamental role to establish the undecidability in ZFC of all the classical problems of set theory, among which Cantor's continuum problem. Moreover, up to date, forcing (or class forcing) is the most efficient method to obtain independence results over ZFC. This method has found applications in virtually all fields of pure mathematics: in the last forty years natural problems of group theory, functional analysis, operator algebras, general topology, and many other subjects were shown to be undecidable by means of forcing (see [5, 14] among others). Perhaps driven by these observations Woodin introduced Ω-logic, a non-constructive semantics for ZFC which rules out the independence results obtained by means of forcing.

Definition 1.1. Given a model V of ZFC and a family Γ of partial orders in V, we say that V models that ϕ is Γ-consistent if $V^{\mathbb{B}} \models \phi$ for some $\mathbb{B} \in \Gamma$.

The notions of Γ-validity and of Γ-logical consequence $\models_\Gamma$ are defined accordingly. Woodin's Ω-logic is the Γ-logic obtained by letting Γ be the class of all partial orders[2]. Prima facie Γ-logics appear to be even more intractable than β-logic (the logic given by

[2]There is a slight twist between Woodin's original definition of Ω-consistency and our definition of Γ-consistency when Γ is the class of all posets. We shall explain in this footnote why we decided to modify Woodin's original definition. On a first reading the reader may skip it over. Woodin states that ϕ is Ω-consistent in V if there is some α and some $\mathbb{B} \in V_\alpha$ such that $V_\alpha^{\mathbb{B}} \models \phi$. For our purposes the advantage of our definition (with respect to Woodin's) is that it allows for a simpler formulation of the forcing absoluteness results which are the motivation of this paper and which assert that over any model V of some theory T which extends ZFC any statement ϕ of a certain form which V models to be Γ-consistent actually holds in V. To appreciate the difference between Woodin's definition of Ω-consistency and the current definition, assume

the class of well founded models of ZFC). However this is a misleading point of view, and, as we shall see below, it is more correct to view these logics as means to radically change our point of view on forcing:

> Γ-logics transform forcing in a tool to prove theorems over certain natural theories T which extend ZFC.

The following corollary of Cohen's forcing theorem (which we dare to call Cohen's Absoluteness Lemma) is an illuminating example:

Lemma 1.2 (Cohen's Absoluteness). *Assume $T \supset$ ZFC and $\phi(x, r)$ is a Σ_0-formula in the parameter r such that $T \vdash r \subset \omega$. Then the following are equivalent:*

- $T \vdash [H_{\omega_1} \models \exists x\phi(x, r)]$.
- $T \vdash \exists x\phi(x, r)$ is Ω-consistent[3].

Observe that for any model V of ZFC, $H^V_{\omega_1} \prec_{\Sigma_1} V$ and that for any theory $T \supseteq$ ZFC there is a recursive translation of Σ^1_2-properties (provably Σ^1_2 over T) into Σ_1-properties over H_{ω_1} (provably Σ_1 over the same theory T) [8, Lemma 25.25]. Summing up we get that a Σ^1_2-statement is provable in some theory $T \supseteq$ ZFC iff the corresponding Σ_1-statement over H_{ω_1} is provably Ω-consistent over the same theory T. This shows that already in ZFC forcing is an extremely powerful tool to prove theorems. Lemma 1.2 complements Shoenfield's absoluteness theorem [8, Theorem 25.20] and gives another powerful argument to prove the validity of some Σ^1_2-property by means of an absoluteness argument.

We briefly sketch why Lemma 1.2 holds since this will outline many of the ideas we are heading for:

Proof. We shall actually prove the following slightly stronger formulation[4] of the non-trivial direction in the equivalence:

> Assume V is a model of T. Then $H_{\omega_1} \models \exists x\phi(x, r)$ if and only if $V \models \exists x\phi(x, r)$ *is* Ω*-consistent*.

that ϕ is a Π_2-formula and that ϕ is Ω-consistent in V in the sense of Woodin: this means that there exist α and $\mathbb{B}$ such that $V^{\mathbb{B}}_\alpha \models \phi$, nonetheless it is well possible that $V^{\mathbb{B}} \not\models \phi$ and thus that $\mathbb{B}$ does not witness that ϕ is Ω-consistent according to our definition. Now if V models ZFC+*there are class many Woodin cardinals* and $\phi^{L(\mathbb{R})}$ is Ω-consistent in V in the sense of Woodin, this can be reflected in the assertion that $\exists\alpha \in V$, $V_\alpha \models \phi^{L(\mathbb{R})}$, but not in the statement that $\phi^{L(\mathbb{R})}$ holds in V. On the other hand if V models ZFC+*there are class many Woodin cardinals* and $\phi^{L(\mathbb{R})}$ is Ω-consistent in V according to our definition, we can actually reflect this fact in the assertion that $V \models \phi^{L(\mathbb{R})}$. There is no real discrepancy on the two definitions because for each n we can find a formula ϕ_n such that if V is any model of ZF, $V_\alpha \models \phi_n$ if and only if $V_\alpha \prec_{\Sigma_n} V$. Thus, if we want to prove that a certain Σ_n-formula ϕ is Ω-consistent according to our definition, we just have to prove that $\phi_n \wedge \phi$ is Ω-consistent in V according to Woodin's definition. On the other hand the set of Γ-valid statements (according to Woodin's definition) is definable in V in the parameters used to define Γ, while (unless we subsume that there is some δ such that $V_\delta \prec V$ and all the parameters used to define Γ belong to V_δ) we shall encounter the same problems to define in V the class of Γ-valid statements (according to our definition) as we do have troubles to define in V the set of V-truths.

[3] I.e. $T \vdash$ *There is a partial order* $\mathbb{B}$ *such that* $\Vdash_{\mathbb{B}} \exists x\phi(x, r)$.

[4] In the statement below we do not require that the existence of a partial order witnessing the Ω-consistency of $\exists x\phi(x, r)$ in V is provable in T.

To simplify the exposition we prove it with the further assumption that V is a *transitive* model. With the obvious care in details essentially the same argument works for any first order model of T. So assume $\phi(x, \vec{y})$ is a Σ_0-formula and $\exists x\phi(x, \vec{r})$ is Ω-consistent in V with parameters $\vec{r} \in \mathbb{R}^V$. Let $\mathbb{P} \in V$ be a partial order that witnesses it. Pick a model $M \in V$ such that $M \prec (H_{|\mathbb{P}|^+})^V$, M is countable in V, and $\mathbb{P}, \vec{r} \in M$. Let $\pi_M : M \to N$ be its transitive collapse and $\mathbb{Q} = \pi_M(\mathbb{P})$. Notice also that $\pi(\vec{r}) = \vec{r}$. Since π_M is an isomorphism of M with N,

$$N \models (\Vdash_{\mathbb{Q}} \exists x\phi(x, \vec{r})).$$

Now let $G \in V$ be N-generic for $\mathbb{Q}$ (G exists since N is countable), then, by Cohen's fundamental theorem of forcing applied in V to N, we have that $N[G] \models \exists x\phi(x, \vec{r})$. So we can pick $a \in N[G]$ such that $N[G] \models \phi(a, \vec{r})$. Since $N, G \in (H_{\aleph_1})^V$, we have that V models that $N[G] \in H^V_{\omega_1}$ and thus V models that a as well belongs to $H^V_{\omega_1}$. Since $\phi(x, \vec{y})$ is a Σ_0-formula, V models that $\phi(a, \vec{r})$ is absolute between the transitive sets $N[G] \subset H_{\omega_1}$ to which $a, \vec{r}$ belong. In particular a witnesses in V that $H^V_{\omega_1} \models \exists x\phi(x, \vec{r})$.$\square$

If we analyze the proof of this Lemma, we immediately realize that a key observation is the fact that for any poset $\mathbb{P}$ there is some countable $M \prec H_{|\mathbb{P}|^+}$ such that $\mathbb{P} \in M$ and there is an M-generic filter for $\mathbb{P}$. The latter statement is an easy outcome of Baire's category theorem and is provable in ZFC. For a given regular cardinal λ and a partial order $\mathbb{P}$, let $S^{\lambda}_{\mathbb{P}}$ be the set consisting of $M \prec H_{\max(|\mathbb{P}|^+, \lambda)}$ such that there is an M-generic filter for $\mathbb{P}$ and $M \cap \lambda \in \lambda > |M|$. Then an easy outcome of Baire's category theorem is that $S^{\aleph_1}_{\mathbb{P}}$ is a club subset of $P_{\omega_1}(H_{|\mathbb{P}|^+})$ for every partial order $\mathbb{P}$. If we analyze the above proof what we actually needed was just the stationarity of $S^{\aleph_1}_{\mathbb{P}}$ to infer the existence of the desired countable model $M \prec H_{|\mathbb{P}|^+}$ such that $r \in M$ and there is an M-generic filter for $\mathbb{P}$. For any regular cardinal λ, let Γ_λ be the class of posets such that $S^{\lambda}_{\mathbb{P}}$ is stationary. In particular we can generalize Cohen's absoluteness Lemma as follows:

Lemma 1.3 (Generalized Cohen Absoluteness). *Assume V is a model of* ZFC *and λ is regular and uncountable in V. Then $H^V_\lambda \prec_{\Sigma_1} V^P$ if $P \in \Gamma_\lambda$.*

Let $\mathsf{FA}_\nu(\mathbb{P})$ assert that: *P is a partial order such that for every collection of ν-many dense subsets of P there is a filter $G \subset P$ meeting all the dense sets in this collection.* Let $\mathsf{BFA}_\nu(\mathbb{P})$ assert that $H^V_{\nu^+} \prec_{\Sigma_1} V^P$.

Given a class of posets Γ, let $\mathsf{FA}_\nu(\Gamma)$ ($\mathsf{BFA}_\nu(\Gamma)$) hold if $\mathsf{FA}_\nu(P)$ ($\mathsf{BFA}_\nu(P)$) holds for all $P \in \Gamma$. Then Baire's category theorem just says that $\mathsf{FA}_{\aleph_0}(\Omega)$ holds where Ω is the class of all posets. It is not hard to check that if S^{λ}_{P} is stationary, then $\mathsf{FA}_\gamma(P)$ holds for all $\gamma < \lambda$. Woodin [21, Proof of Theorem 2.53] proved that if $\lambda = \nu^+$ is a successor cardinal $P \in \Gamma_\lambda$ *if and only if* $\mathsf{FA}_\nu(P)$ *holds* (see for more details subsection 2.3 and Lemma 2.7). In particular for all cardinals ν we get that Γ_{ν^+} is the class of partial orders P such that $\mathsf{FA}_\nu(P)$ holds or (equivalently) such that $S^{\nu^+}_{P}$ is stationary. With this terminology Cohen's absoluteness Lemma states that $\mathsf{FA}_\nu(P)$ implies $\mathsf{BFA}_\nu(P)$ for all infinite cardinals ν.

Observe that many interesting problems of set theory can be formulated as Π_2-properties of H_{ν^+} for some cardinal ν (an example is Suslin's hypothesis, which can be formulated as a Π_2-property of $H_{\aleph_2}$). Lemma 1.3 gives a very powerful general framework to prove in any given model V of ZFC whether a Π_2-property $\forall x \exists y \phi(x, y, z)$ (where ϕ is Σ_0) holds for some $H^V_{\nu^+}$ with $p \in H^V_{\nu^+}$ replacing z: It suffices to prove that for any $a \in H^V_{\nu^+}$, V models that $\exists y \phi(a, y, p)$ is Γ_{ν^+}-consistent. This shows that if we are in a model V of ZFC where $\Gamma^V_{\nu^+}$ contains interesting and manageable families of partial orders $\Gamma^V_{\nu^+}$-logic is a powerful tool to study the Π_2-theory of $H^V_{\nu^+}$. In particular this is always the case for $\nu = \aleph_0$ in any model of ZFC, since $\Gamma_{\aleph_1}$ is the class of all posets. Moreover this is certainly one of the reasons of the success the forcing axiom Martin's Maximum MM and its bounded version BMM have had in settling many relevant problems of set theory which can be formulated as Π_2-properties of the structure $H_{\aleph_2}$ and that boosted the study of bounded versions of forcing axioms[5].

For any set theorist willing to accept large cardinal axioms, Woodin has been able to show that Ω-logic gives a natural non-constructive semantics for the full first order theory of $L(\mathbb{R})$ and not just for the Σ_1-fragment of $H_{\aleph_1} \subset L(\mathbb{R})$ which is given by Cohen's absoluteness Lemma. Woodin [11, Theorem 2.5.10] has proved that assuming large cardinals Ω-*truth is* Ω-*invariant* i.e.:

> Let V be any model of ZFC+*there are class many Woodin cardinals*. Then for any statement ϕ with parameters in $\mathbb{R}^V$,
>
> $$V \models (\phi \text{ is } \Omega\text{-consistent})$$
>
> if and only if there is $\mathbb{B} \in V$ such that
>
> $$V^{\mathbb{B}} \models (\phi \text{ is } \Omega\text{-consistent}).$$

Thus Ω-logic, the logic of forcing, has a notion of truth which forcing itself cannot change. Woodin [11, Theorem 3.1.7] also proved that the theory ZFC+*large cardinals* decides in Ω-logic the theory of $L(\mathbb{R})$ and actually, by strengthening the large cardinal assumptions, even of the larger structure $L(P_{\omega_1}\mathrm{Ord})$, i.e.:

> For any model V of ZFC+*there are class many Woodin cardinals which are a limit of Woodin cardinals* and any first order formula ϕ, $L(P_{\omega_1}\mathrm{Ord})^V \models \phi$ if and only if
>
> $$V \models [L(P_{\omega_1}\mathrm{Ord}) \models \phi] \text{ is } \Omega\text{-consistent}.$$

He pushed further these result and showed that if T extends ZFC+ *There are class many measurable Woodin cardinals*, then T decides in Ω-logic any mathematical problem expressible as a (provably in T) Δ^2_1-statement. These are optimal and sharp results: it is well known that the Continuum hypothesis CH (which is provably not a Δ^2_1-statement) and the first order theory of $L(P(\omega_1))$ cannot be decided by ZFC+ *large cardinal axioms* in Ω-logic. Martin and Steel's result that projective determinacy holds in ZFC^* complements the fully satisfactory description Ω-logic and large cardinals give of the first order theory of the structure $L(\mathbb{R})$ in models of ZFC^*. Moreover we can make these results meaningful also for a non-platonist, for example we can reformulate the statement that ZFC^* decides in Ω-logic the theory of $L(\mathbb{R})$ as follows:

[5] Bagaria [2] and Stavi, Väänänen [15] are the first who realize that bounded forcing axioms are powerful tools to describe the Π_2-theory of $H_{\mathfrak{c}}$ exactly for the reasons we are pointing out.

Assume T extends ZFC+*there are class many Woodin cardinals which are a limit of Woodin cardinals*. Let $\phi(r)$ be a formula in the parameter r such that $T \vdash r \subseteq \omega$. Then the following are equivalent:

- $T \vdash [L(P_{\omega_1}\mathrm{Ord}) \models \phi(r)]$.
- $T \vdash \phi(r)^{L(P_{\omega_1}\mathrm{Ord})}$ *is* Ω*-consistent.*

The next natural stage is to determine to what extent Woodin's results on Ω-logic and the theory of $H_{\aleph_1}$ and $L(\mathbb{R})$ can be reproduced for $H_{\aleph_2}$ and $L(P(\omega_1))$. There is also for these theories a fundamental result of Woodin: he introduced an axiom $(*)$ which is a strengthened version of BMM with the property that the theory of $H_{\aleph_2}$ *with real parameters* is invariant with respect to *all* forcings which preserve this axiom[6]. The $(*)$-axiom is usually formulated [10, Definition 7.9] as the assertion that $L(\mathbb{R})$ is a model of the axiom of determinacy and $L(P(\omega_1))$ is a generic extension of $L(\mathbb{R})$ by the homogeneous forcing $\mathbb{P}_{\max} \in L(\mathbb{R})$.

There are two distinctive features of $(*)$:

1. It asserts the "proximity" of $L(\mathbb{R})$ with $L(P(\omega_1))$: on the one hand the homogeneity of $\mathbb{P}_{\max}$ entails that the first order theory of $L(P(\omega_1))$ is essentially determined by the theory of the underlying $L(\mathbb{R})$. On the other hand $(*)$ implies that $L(P(\omega_1)) = L(\mathbb{R})[A]$ for any $A \in P(\omega_1) \setminus L(\mathbb{R})$.
2. $(*)$ entails that $(H^V_{\omega_2}, \in, \mathbb{R}^V) \prec (H^{V^P}_{\omega_2}, \in, \mathbb{R}^V)$ for any notion of forcing $P \in V$ which preserves $(*)$ even if $\mathsf{FA}_{\aleph_1}(P)$ may be false for such a P.

In this paper we propose a different approach to the analysis of the theory of $H_{\aleph_2}$ then the one given by $(*)$. We do not seek for an axiom system $T \supseteq \mathsf{ZFC}$ which makes the theory of $H_{\aleph_2}$ for formulae with *real parameters* invariant with respect to *all* forcing notions which preserve a suitable fragment of T. Our aim is to show that the strongest forcing axioms in combination with large cardinals give an axiom system T which extends ZFC and makes the theory of $H_{\aleph_2}$ for formulae with *arbitrary parameters* in the structure invariant with respect to all forcing notions P which preserve a suitable fragment of T *and for which we can predicate* $\mathsf{FA}_{\aleph_1}(P)$ (i.e. forcings P which are in the class $\Gamma_{\aleph_2}$).

This leads us to analyze the properties of the class $\Gamma_{\aleph_2}$ in models of ZFC^*. This is a delicate matter, first of all Shelah proved that $\mathsf{FA}_{\aleph_1}(P)$ fails for any P which does not preserve stationary subsets of ω_1. Nonetheless it cannot be decided in ZFC whether this is a necessary condition for a poset P in order to have the failure of $\mathsf{FA}_{\aleph_1}(P)$. For example let P be a forcing which shoots a club of ordertype ω_1 through a projectively stationary and costationary subset of $P_{\omega_1}(\omega_2)$ by selecting countable initial segments of this club: It is provable in ZFC that P preserve stationary subsets of ω_1 for all such P. However in L, $\mathsf{FA}_{\aleph_1}(P)$ fails for some such P while in a model of Martin's maximum MM, $\mathsf{FA}_{\aleph_1}(P)$ holds for all such P. This shows that we cannot hope to prove general theorems about $H_{\aleph_2}$ in ZFC^* alone using forcing, but just theorems about the properties of $H_{\aleph_2}$ for particular theories T which extend ZFC^* and for which we have a nice description of the class $\Gamma_{\aleph_2}$.

[6]We refer the reader to [10] for a thorough development of the properties of models of the $(*)$-axiom.

In this respect it is well known that the study of the properties of $H_{\aleph_2}$ in models of Martin's maximum MM, of the proper forcing axiom PFA, or of their bounded versions BMM and BPFA has been particularly successful. Moreover it is well known that the strongest such theories (MM and PFA) are able to settle many relevant questions about the whole universe V and to show that many properties of the universe reflect to $H_{\aleph_2}$[7]. The reason is at least two-fold:

- First of all there is a manageable description of the class $\Gamma_{\aleph_2}$ in models of MM (PFA,MA): this is the class of stationary set preserving posets for MM (respectively contains the class of proper forcings for PFA, and the class of CCC partial orders for MA).
- MM realizes the slogan that $\mathsf{FA}_{\aleph_1}(P)$ holds for any partial order P for which we cannot prove that $\mathsf{FA}_{\aleph_1}(P)$ fails, thus MM substantiates a natural maximality principle for the class $\Gamma_{\aleph_2}$.

We believe that the arguments we presented so far already show that for any model V of ZFC and any successor cardinal $\lambda \in V$ it is of central interest to analyze what is the class Γ_λ in V, since this gives a powerful tool to investigate the Π_2-theory of H_λ^V. Moreover in this respect ZFC + MM is particularly appealing since it asserts the maximality of the class $\Gamma_{\aleph_2}$. The main result of this paper is to show that a natural strengthening of MM (denoted by MM^{++}) which holds in the standard models of MM, in combination with Woodin cardinals, makes $\Gamma_{\aleph_2}$-logic the correct semantics to describe completely the Π_2-theory of $H_{\aleph_2}$ in models of MM^{++}. In particular we shall prove the following theorem:

Theorem 1.4. *Assume* MM^{++} *holds and there are class many Woodin cardinals. Then*

$$H_{\aleph_2}^V \prec_{\Sigma_2} H_{\aleph_2}^{V^P}$$

for all stationary set preserving posets P which preserve BMM.

Notice that we can reformulate the theorem in the same fashion of Woodin's and Cohen's results as follows:

Theorem 1.5. *Assume T extends* ZFC + MM^{++}+ There are class many Woodin cardinals. *Then for every Π_2-formula $\phi(x)$ in the free variable x and every parameter p such that $T \vdash p \in H_{\omega_2}$ the following are equivalent:*

- $T \vdash [H_{\aleph_2} \models \phi(p)]$
- $T \vdash$ There is a stationary set preserving partial order P such that $\Vdash_P \phi^{H_{\aleph_2}}(p)$ and $\Vdash_P$ BMM.

[7]The literature is vast, we mention just a sample of the most recent results with no hope of being exhaustive: [12, 16, 21] present different examples of well-ordering of the reals definable in $H_{\aleph_2}$ (with parameters in $H_{\aleph_2}$) in models of BMM (BPFA), [4, 18, 20] present several different reflection properties between the universe and $H_{\aleph_2}$ in models of MM^{++} (PFA,MM), [5, 13] present applications of PFA to the solution of problems coming from operator algebra and general topology and which can be formulated as (second order) properties of the structure $H_{\aleph_2}$.

1.1 Notation and prerequisites

We adopt standard notation which is customary in the subject, our reference text is [8].

For models (M, E) of ZFC, we say that $(M, E) \prec_{\Sigma_n} (M', E')$ if $M \subset M'$, $E = E' \cap M^2$ and for any Σ_n-formula $\phi(p)$ with $p \in M$, $(M, E) \models \phi(p)$ if and only if $(M', E') \models \phi(p)$. We usually write $M \prec_{\Sigma_n} M$ instead of $(M, E) \prec_{\Sigma_n} (M', E')$ when E, E' is clear from the context. We let $(M, E) \prec (M', E')$ if $(M, E) \prec_{\Sigma_n} (M', E')$ for all n.

We let Ord denote the class of ordinals. For any cardinal κ $P_\kappa X$ denote the subsets of X of size less than κ. Given $f : X \to Y$ and $A \subset X, B \subset Y$, $f[A]$ is the pointwise image of A under f and $f^{-1}[B]$ is the preimage of B under f. A set S is stationary if for all $f : P_\omega(\cup S) \to \cup S$ there is $X \in S$ such that $f[X] \subseteq X$ (such an X is called a closure point for f). A set C is a club subset of S if it meets all stationary subsets of S or, equivalently, if it contains all the closure points in S of some $f : P_\omega(\cup S) \to \cup S$. Notice that $P_\kappa X$ is always stationary if κ is a cardinal and X, κ are both uncountable.

If V is a transitive model of ZFC and $(P, \leq_P) \in V$ is a partial order with a top element 1_P, V^P denotes the class of P-names, and $\dot{a}$ or τ denote an arbitrary element of V^P, if $\check{a} \in V^P$ is the canonical name for a set a in V we drop the superscript and confuse $\check{a}$ with a. We also feel free to confuse the approach to forcing via boolean valued models as done by Scott and others or via the forcing relation. Thus we shall write for example $V^P \models \phi$ as an abbreviation for

$$V \models [1_P \Vdash \phi].$$

If $M \in V$ is such that $(M, \in)$ is a model of a sufficient fragment of ZFC and $(P, \leq_P)$ in M is a partial order, an M-generic filter for P is a filter $G \subset P$ such that $G \cap A \cap M$ is non-empty for all maximal antichains $A \in M$ (notice that if M is non-transitive, $A \nsubseteq M$ is well possible). If N is a *transitive* model of a large enough fragment of ZFC, $P \in N$ and G is an N-generic filter for P, let $\sigma_G : N \cap V^P \to N[G]$ denote the evaluation map induced by G of the P-names in N.

We say that $(M, E) \prec_{\Sigma_n} (\dot{N}, \dot{E})$ for some $\dot{N} \in V^P$ if

$$V^P \models \dot{E} \cap M^2 = E$$

and for any Σ_n-formula $\phi(p)$ with $p \in M$, $(M, E) \models \phi(p)$ if and only if

$$V^P \models [(\dot{N}, \dot{E}) \models \phi(p)].$$

We will write $M \prec_{\Sigma_n} \dot{N}$ if $(M, E) \prec_{\Sigma_n} (\dot{N}, \dot{E})$ and $E, \dot{E}$ are clear from the context.

For any set M we denote $\pi_M : M \to N_M$ the unique transitive collapse map which is an homomorphism of the structure $(M, \in)$ with the structure $(N_M, \in)$.

We shall also frequently refer to Woodin cardinals, however for our purposes we won't need to recall the definition of a Woodin cardinal but just its effects on the properties of the stationary tower forcing. This is done in subsection 2.2.

2 Preliminaries

We shall briefly outline some general results on the theory of forcing which we shall need for our exposition. The reader may skip Subsections 2.1, 2.2, 2.3 and eventually refer back to them.

2.1 Preliminaries I: complete embeddings

For a poset Q and $q \in Q$, let $Q \upharpoonright q$ denote the poset Q restricted to conditions $r \in Q$ which are below q and $\mathbb{B}(Q)$ denote its boolean completion, i.e. the complete boolean algebra of regular open subsets of Q, so that Q is naturally identified with a dense subset of $\mathbb{B}(Q)$. We say that:

- P completely embeds into Q if there is a map $i : P \to \mathbb{B}(Q)$ which preserves the order relation and maps maximal antichains of P into maximal antichains of $\mathbb{B}(Q)$. With abuse of notation we shall call a complete embedding of P into Q any such homomorphism $i : P \to \mathbb{B}(Q)$ (notice that our definition does not prevent that i may map large portions of P to $0_{\mathbb{B}(Q)}$).
- P regularly embeds into Q if there is an *injective* map $i : P \to Q$ which is also a complete embedding of P into Q.
- $i : P \to \mathbb{B}(Q)$ is locally complete if for some $q \in \mathbb{B}(Q)$, $i : P \to \mathbb{B}(Q \upharpoonright q)$ is a complete embedding (we shall also call any locally complete embedding a *locally regular* embedding).

We remark that what we define here as a *complete embedding* is a weaker notion than the one appearing in [9, Definition VII.7.1] with this same terminology, which instead corresponds exactly to what we defined here to be a *regular embedding*. The following facts are well known and we just state them without a proof.

Lemma 2.1. *The following are equivalent:*

1. *P completely embeds into Q,*
2. *for any V-generic filter G for Q there is in $V[G]$ a V-generic filter H for P,*
3. *There is a complete homomorphism $i : \mathbb{B}(P) \to \mathbb{B}(Q)$ of complete boolean algebras.*

Remark 2.2. Observe that if $i : P \to \mathbb{B}(Q)$ is a complete embedding then for all $q \in Q$ such that $i(p) \wedge q > 0_{\mathbb{B}}$, the map $i_q : P \to \mathbb{B}(Q \upharpoonright q)$ which maps p to $q \wedge i(p)$ is also a complete embedding. Moreover if $q \Vdash_Q \check{p} \in \dot{H}$ where $\dot{H} = i^{-1}[\dot{G}] \in V^Q$ and $\dot{G}$ is the canonical $\mathbb{B}(Q)$-name for a V-generic filter for $\mathbb{B}(Q)$, we have that $i_q(r) = 0_{\mathbb{B}(Q)}$ for all $r \in P$ incompatible with p. Thus in general a complete embedding (according to our terminology) $i : P \to \mathbb{B}(Q)$ may map a large portion of P to $0_{\mathbb{B}(Q)}$.

The quotient forcing $\mathbb{B}/i[\mathbb{Q}]$ is some object belonging to $V^{\mathbb{Q}}$ such that $\mathbb{Q} * (\mathbb{B}/i[\mathbb{Q}])$ is forcing equivalent to $\mathbb{B}$.

Remark 2.3. There might be a variety of complete embeddings of a poset P into a poset Q. These embeddings greatly affect the properties the generic extensions by Q attributes to elements of the generic extensions by P. For example the following can occur:

There is a P-name $\dot{S}$ which is forced by P to be a stationary subset of ω_1 and there are $i_0 : P \to \mathbb{B}(Q)$, $i_1 : P \to \mathbb{B}(Q)$ distinct complete embeddings of P into Q such that if G is V-generic for $\mathbb{B}(Q)$ and $H_j = i_j^{-1}[G]$, then $\sigma_{H_0}(\dot{S})$ is stationary in $V[G]$, $\sigma_{H_1}(\dot{S})$ is stationary in $V[H_1]$ but non-stationary in $V[G]$.

If $i : P \to \mathbb{B}(Q)$ is a locally complete embedding and $p \in P$, $q \in Q$ are such that i can be extended to a complete homomorphism of $\mathbb{B}(P \restriction p)$ into $\mathbb{B}(Q \restriction q)$ we shall also denote $\mathbb{B}(Q \restriction q)/i[\mathbb{B}(P \restriction p)]$ by $Q/i[P]$, if i is clear from the context we shall even denote such quotient forcing as Q/P.

2.2 Preliminaries II: stationary sets and the stationary tower forcing

S is stationary if for all $f : P_\omega(\cup S) \to (\cup S)$ there is an $X \in S$ such that $f[P_\omega(X)] \subset X$.

For a stationary set S and a set X, if $\cup S \subseteq X$ we let $S^X = \{M \in P(X) : M \cap \cup S \in S\}$, if $\cup S \supseteq X$ we let $S \restriction X = \{M \cap X : M \in S\}$.

If S and T are stationary sets we say that S and T are compatible if

$$S^{(\bigcup S)\cup(\bigcup T)} \cap T^{(\bigcup S)\cup(\bigcup T)}$$

is stationary.

We let $S \wedge T$ denote the set of $X \in P(\cup S \cup \cup T)$ such that $X \cap \cup S \in S$ and $X \cap \cup T \in T$ and for all η $\bigwedge\{S_\alpha : \alpha < \eta\}$ is the set of $M \in P(\bigcup_{\alpha<\eta} S_\alpha)$ such that $M \cap \cup S_\alpha \in S_\alpha$ for all $\alpha \in M \cap \eta$.

For a set M we let $\pi_M : M \to V$ denote the transitive collapse of the structure $(M, \in)$ onto a transitive set $\pi_M[M]$ and we let $j_M = \pi_M^{-1}$.

For any regular cardinal λ

$$R_\lambda = \{X : X \cap \lambda \in \lambda \text{ and } |X| < \lambda\}.$$

and for any Woodin cardinal $\delta > \lambda$, $\mathbb{T}_\delta^\lambda$ is the stationary tower whose elements are stationary sets $S \in V_\delta$ such that $S \subset R_\lambda$ with order given by $S \leq T$ if, letting $X = \cup(T) \cup \cup(S)$, S^X is contained in T^X modulo a club.

Notice that $\mathbb{T}_\delta^\lambda/\equiv$ where $\equiv$ is the equivalence relation induced by its order is easily seen to be a $<\delta$-complete boolean algebra whose positive elements give a forcing which is the separative quotient of $\mathbb{T}_\delta^\lambda$. We shall thus feel free to confuse $\mathbb{T}_\delta^\lambda/\equiv$ with $\mathbb{T}_\delta^\lambda$, for example in the proof of 2 implies 3 in Theorem 2.4 below. $\mathbb{T}_\delta$ will denote $\mathbb{T}_\delta^{\aleph_2}$.

We recall that if G is V-generic for $\mathbb{T}_\delta^\lambda$, then G induces in a natural way a direct limit ultrapower embedding $j_G : V \to Ult(V, G)$ where $[f]_G \in Ult(V, G)$ if $f : P(X_f) \to V$ in V and $[f]_G\ R_G\ [h]_G$ iff for some $\alpha < \delta$ such that $X_f, X_h \in V_\alpha$ we have that

$$\{M \prec V_\alpha : f(M \cap X_f)\ R\ h(M \cap X_h)\} \in G.$$

If $Ult(V, G)$ is well founded it is customary to identify $Ult(V, G)$ with its transitive collapse.

We recall the following results about the stationary tower (see [11, Chapter 2]):

Theorem 2.4 (Woodin). *Assume δ is a Woodin cardinal, $\lambda = \nu^+ < \delta$ is a successor and G is V-generic for $\mathbb{T}^{\lambda}_{\delta}$. Then*

1. *$Ult(V, G)$ is a definable class in $V[G]$ and*
$$V[G] \models (Ult(V, G))^{<\delta} \subseteq Ult(V, G).$$
2. *$V_\delta, G \subseteq Ult(V, G)$ and $j_G(\lambda) = \delta$.*
3. *$Ult(V, G) \models \phi([f_1]_G, \dots, [f_n]_G)$ if and only for some $\alpha < \delta$ such that $f_i : P(X_i) \to V$ are such that $X_i \in V_\alpha$ for all $i \leq n$:*
$$\{M \prec V_\alpha : V \models \phi(f_1(M \cap X_1), \dots, f_n(M \cap X_n))\} \in G.$$

Moreover by 1 $Ult(V, G)$ is well founded and thus can be identified with its transitive collapse. With this identifications we have that for every $\alpha < \delta$ and every set $X \in V_\alpha$,
$$X = [\{\langle M, \pi_M(X)\rangle : M \prec V_\alpha, X \in M\}]_G$$
and
$$j_G[X] = [\{\langle M, X\rangle : M \prec V_\alpha, X \in M\}]_G.$$
In particular these identifications show that:

- $(H_{j_G(\lambda)})^{M[G]} = V_\delta[G] = (H_\delta)^{V[G]}$.
- *$j_G \restriction H^V_\theta \in Ult(V, G)$ and $j_G[H^V_\theta] \prec H^{Ult(V,G)}_{j_G(\theta)}$ for all $\theta < \delta$.*
- *$j_G \restriction H^V_\lambda$ is the identity and witnesses that $H^V_\lambda \prec H^{V[G]}_{j_G(\lambda)}$.*
- *$S \in G$ iff $j_G[\cup S] \in j_G(S)$ for all $S \in \mathbb{T}^{\lambda}_{\delta}$.*

2.3 Preliminaries III: forcing axioms

Definition 2.5. Given a cardinal λ and a partial order P, $\mathsf{FA}_\lambda(P)$ holds if:

> For every collection of λ-many dense subsets of P there is a filter $G \subset P$ meeting all the dense set in this collection.

$\mathsf{FA}_{<\lambda}(P)$ holds if $\mathsf{FA}_\nu(P)$ holds for all $\nu < \lambda$.
$\mathsf{BFA}_\lambda(P)$ holds if $H_{\lambda^+} \prec_{\Sigma_1} V^P$.
If Γ is a family of partial orders, $\mathsf{FA}_\lambda(\Gamma)$ ($\mathsf{FA}_{<\lambda}(\Gamma)$, $\mathsf{BFA}_\lambda(\Gamma)$) asserts that $\mathsf{FA}_\lambda(P)$ ($\mathsf{FA}_{<\lambda}(P)$, $\mathsf{BFA}_\lambda(P)$) holds for all $P \in \Gamma$.
For any partial order P

$$S^\lambda_P = \{M \prec H_{|P|^+} : M \cap \lambda \in \lambda > |M| \text{ and there is an } M\text{-generic filter for } P\}$$

We shall abbreviate $S^{\aleph_2}_P$ by S_P.
For any regular uncountable cardinal λ, we let Γ_λ be the family of P such that S^λ_P is stationary.
In the introduction we already showed:

Lemma 2.6. *Assume λ is an infinite cardinal. Then $P \in \Gamma_{\lambda^+}$ implies $\mathsf{BFA}_\lambda(P)$.*

MM asserts that $\mathsf{FA}_{\aleph_1}(\mathsf{SSP})$ holds, where SSP is the family of posets which preserve stationary subsets of ω_1. BMM asserts that $\mathsf{BFA}_{\aleph_1}(\mathsf{SSP})$ holds. It is not hard to see that if S^λ_P is stationary, then $\mathsf{FA}_{<\lambda}(P)$ holds. It is not clear whether the converse holds if λ is inaccessible. However the converse holds if λ is a successor cardinal and Woodin's proof of (1) imlies (2) in [21, Theorem 2.53] gives a special case of the following Lemma for $\lambda = \omega_2$.

Lemma 2.7. *Let $\lambda = \nu^+$ be a successor cardinal. Then the following are equivalent:*

1. $\mathsf{FA}_\nu(P)$ *holds.*
2. S^λ_P *is stationary.*

2.4 Preliminaries IV: Woodin cardinals are forcing axioms

The following is an outcome of Woodin's work on the stationary tower and slightly generalizes [21, Theorem 2.53].

Theorem 2.8. *Woodin [21, Theorem 2.53]*

Assume V is a model of ZFC+ there are class many Woodin cardinals*, and $\lambda = \nu^+$ is a successor cardinal in V.*

Then the following are equivalent for any partial order $P \in V$:

1. S^λ_P *is stationary.*
2. $\mathsf{FA}_\nu(P)$ *holds.*
3. *There is a complete embedding of P into $\mathbb{T}^\lambda_\delta \restriction S$ for some Woodin cardinal $\delta > |P|$ and some $S \in \mathbb{T}^\lambda_\delta$.*

Proof. We just sketch it. The equivalence of the firts two items has already been stated in Lemma 2.7.

We prove that the third item implies the second item: If the third item holds, let H be V-generic for $\mathbb{T}^\lambda_\delta \restriction S$ and $G \in V[H]$ be V-generic for P.

Let $j : V \to Ult(V, H)$ be the generic ultrapower embedding, let $\theta = |P|^+$. Then $j[G]$ is a $j[H_\theta]$-generic filter for $j(P)$. Now observe that $j[G]$, $j[H_\theta]$, $j(P)$ are all elements of $Ult(V, H)$ and that

$$Ult(V, H) \models j[H_\theta] \prec H_{j(\theta)}.$$

By standard arguments we can infer that $Ult(V, H)$ models that $(S^{j(\lambda)}_{j(P)})^{Ult(V,H)}$ is stationary. Now we can conclude by elementarity that S^λ_P is stationary in V and moreover that S^λ_P belongs to H since $j[H_\theta] \in j(S^\lambda_P)$. This shows that S^λ_P is stationary and belongs to H whenever H is a V-generic filter for $\mathbb{T}^\lambda_\delta$ which adds a V-generic filter for P.

Now we prove that the second item implies the third item: Assume S^λ_P is stationary and let H_M be an M generic filter for P for any $M \in S^\lambda_P$. Consider the map

$$i : P \to \mathbb{T}^\lambda_\delta \restriction S^\lambda_P$$

which maps p to the set of $M \in S^{\lambda}_{P}$ such that $p \in H_M$. It is immediate to check that i is a complete embedding (though it may map large portions of P to non stationary subsets of S^{λ}_{P}). $\square$

SSP denote the class of posets which preserve stationary subsets of ω_1. Martin's maximum MM asserts that $\mathsf{FA}_{\aleph_1}(P)$ holds for all $P \in \mathsf{SSP}$.

The following sums up the current state of affair regarding the classes Γ_λ for $\lambda \leq \aleph_2$.

Theorem 2.9. *Assume there are class many Woodin cardinals. Then:*

1. *$\Gamma_{\aleph_1}$ is the class of all posets and for any poset P there is a regular embedding into $\mathbb{T}^{\aleph_1}_{\delta}$ for any Woodin cardinal $\delta > |P|$.*
2. *$\mathbb{T}^{\aleph_2}_{\delta} \in \mathsf{SSP}$ for any Woodin cardinal δ.*
3. *MM holds if and only if SSP is the class of all posets which regularly embeds into $\mathbb{T}^{\aleph_2}_{\delta} \upharpoonright S$ for some Woodin cardinal δ and $S \in \mathbb{T}^{\aleph_2}_{\delta}$. (Foreman, Magidor, Shelah [6]).*

Proof. We sketch a proof.

1 Trivial by Theorem 2.8.

2 Let $S \in V$ be a stationary subset of ω_1, G be V-generic for $\mathbb{T}^{\aleph_2}_{\delta}$ and $\dot{C}$ be a $\mathbb{T}^{\aleph_2}_{\delta}$-name for a club subset of ω_1. Then $\sigma_G(\dot{C}) \in (H_{\omega_2})^{V[G]} = V_\delta[G] = (H_{\omega_2})^{Ult(V,G)}$. In particular since $j_G : V \to Ult(V, G)$ is elementary, $Ult(V, G) \models \sigma_G(\dot{C}) \cap j_G(S) \neq \emptyset$. Now, since $j_G(\omega_1) = \omega_1$, we have that $j_G(S) = S$. The conclusion follows.

3 $\aleph_2$ is a a successor cardinal. For this reason, if MM holds, we can use the equivalence given by Theorem 2.8 to get that any $P \in \mathsf{SSP}$ regularly embeds into $\mathbb{T}^{\aleph_2}_{\delta} \upharpoonright S_P$ for any Woodin cardinal δ. We can then use 2 to argue that if P regularly embeds into some $\mathbb{T}^{\aleph_2}_{\delta} \upharpoonright S$ with δ a Woodin cardinal and $S \in \mathbb{T}^{\aleph_2}_{\delta}$, then $P \in \mathsf{SSP}$. $\square$

2.5 Preliminaries V: MM^{++}

The ordinary proof of the consistency of MM actually gives more information than what is captured by Theorem 2.9.3: the latter asserts that any stationary set preserving poset $\mathbb{P}$ can be completely embedded into $\mathbb{T}^{\aleph_2}_{\delta} \upharpoonright S_{\mathbb{P}}$ for any Woodin cardinal $\delta > |\mathbb{P}|$ via some complete embedding i. However MM doesn't give much information on the nature of the complete embedding i. On the other hand the standard model of MM provided by Foreman, Shelah and Magidor's consistency proof actually show that for any stationary set preserving poset $\mathbb{P}$ and any Woodin cardinal $\delta > |\mathbb{P}|$ we can get a complete embedding $i : \mathbb{P} \to \mathbb{B}(\mathbb{T}^{\aleph_2}_{\delta} \upharpoonright T)$ with a "nice" quotient forcing $(\mathbb{T}^{\aleph_2}_{\delta} \upharpoonright T)/i[\mathbb{P}]$. For this reason we introduce the following well known variation of Martin's maximum:

Definition 2.10. MM^{++} holds if $T_{\mathbb{P}}$ is stationary for all $\mathbb{P} \in \mathsf{SSP}$, where $M \in T_{\mathbb{P}}$ iff

- $M \prec H_{|\mathbb{P}|^+}$ is in $R_{\aleph_2}$,

- There is an M-generic filter H for $\mathbb{P}$ such that, if $G = \pi_M[H]$, $Q = \pi_M(\mathbb{P})$ and $N = \pi_M[M]$, then $\sigma_G : N^Q \to N[G]$ is an evaluation map such that $\sigma_G(\pi_M(\dot{S}))$ is stationary for all $\dot{S} \in M$ $\mathbb{P}$-name for a stationary subset of ω_1.

We shall call *correct M-generic filter for* $\mathbb{P}$ any M-generic filter H as above.

The following is a well-known by-product of the ordinary consistency proofs of MM which to my knowledge is seldom explicitly stated:

Theorem 2.11 (Foreman, Magidor, Shelah). MM^{++} *is relatively consistent with respect to the existence of a super compact cardinal.*

A variaton of the proof of [21, Theorem 2.53] gives the following:

Theorem 2.12. *Assume there are class many Woodin cardinals. Then the following are equivalent:*

1. MM^{++} *holds.*
2. *For every Woodin cardinal δ and every stationary set preserving poset $\mathbb{P} \in V_\delta$ there is a complete embedding $i : \mathbb{P} \to \mathbb{B}$ where $\mathbb{B} = \mathbb{B}(\mathbb{T}_\delta^{\aleph_2} \upharpoonright T)$ for some stationary set $T \in V_\delta$ such that*

$$\Vdash_{\mathbb{P}} \mathbb{B}/i[\mathbb{P}] \text{ is stationary set preserving.}$$

Proof. This is a straightforward variation of the proof of Theorem 2.8. The proof that the first item implies the second is based on the following observation:

Fact 2.13. For each $M \in T_P$ let H_M be a correct M-generic filter for P. Then the map

$$i : P \to \mathbb{T}_\delta \upharpoonright T_P$$

which maps p to the set of M such that $p \in H_M$ is a complete embedding such that P forces that $(\mathbb{T}_\delta \upharpoonright T_P)/i[P]$ is stationary set preserving.

Proof. If G is V-generic for $\mathbb{T}_\delta \upharpoonright T_P$, then for any $\dot{S} \in V^P$ P-name for a stationary subset of ω_1 the map $f_{\dot{S}} : M \mapsto \sigma_{\pi_M[H_M]}(\dot{S})$ represents a stationary subset of ω_1 in $Ult(V, G)$ as well as in $V[G]$ (where π_M is the transitive collapse mapping of M onto a transitive set). Moreover we also have that if $H = i^{-1}[G]$ is the V-generic filter for P induced by G and i,

$$[f_{\dot{S}}]_G = \sigma_H(\dot{S})$$

holds in $V[G]$. This means that $V[G]$ is a generic extension over $V[H]$ for the forcing $\sigma_H((\mathbb{T}_\delta \upharpoonright T_P)/i[P])$ which preserve the stationarity of the subsets of ω_1 in $V[H]$. Since G is an arbitrary V-generic filter for $\mathbb{T}_\delta \upharpoonright T_P$, the conclusion follows. □

The proof that the second item implies the first is a variation of the corresponding proof in Theorem 2.8 which takes into account the extra features of the complete embedding $i : P \to \mathbb{B}$. □

3 Woodin's absoluteness results for $H_{\aleph_1}$

We shall motivate the results of the next section proving the following weak version of Woodin's original absoluteness theorem:

Theorem 3.1. *Assume there are class many Woodin cardinals. Then the theory of $H_{\aleph_1}$ with parameters is invariant with respect to set forcing.*

Proof. We prove by induction on n the following Lemma, of which the Theorem is an immediate consequence:

Lemma 3.2. *Assume V is a model of* ZFC *in which there are class many Woodin cardinals. Let $P \in V$ be a forcing notion.*
Then for all n, $H_{\aleph_1}^V \prec_{\Sigma_n} H_{\aleph_1}^{V^P}$.

Proof. By Cohen's absoluteness Lemma 1.2, we already know that for all models M of ZFC and all forcing $P \in M$

$$H_{\aleph_1}^M \prec_{\Sigma_1} H_{\aleph_1}^{M^P}.$$

Now assume that for all models M of ZFC+*there are class many Woodin cardinals* and all $P \in M$ we have shown that

$$H_{\aleph_1}^M \prec_{\Sigma_n} H_{\aleph_1}^{M^P}.$$

First observe that M^P is still a model of ZFC+*there are class many Woodin cardinals*. Now pick V an arbitrary model of ZFC+*there are class many Woodin cardinals* and $P \in V$ a forcing notion.

Let $\delta \in V$ be a Woodin cardinal in V such that $P \in V_\delta$.

To simplify the argument we assume V is transitive and there is a V-generic filter G for $\mathbb{T}_\delta^{\aleph_1}$ (we leave to the reader to remove these unnecessary assumptions).

Then, since $\mathsf{FA}_{\aleph_0}(P)$ holds in V and $P \in V_\delta$, by Theorem 2.9.1 there is in V a complete embedding $i : P \to \mathbb{T}_\delta^{\aleph_1}$. Let G be V-generic for $\mathbb{T}_\delta^{\aleph_1}$ and $H = i^{-1}[G]$. Then by our inductive assumptions applied to V (with respect to $V[H]$) and to $V[H]$ (with respect to $V[G]$) we have that:

$$H_{\aleph_1}^V \prec_{\Sigma_n} H_{\aleph_1}^{V[H]} \prec_{\Sigma_n} H_{\aleph_1}^{V[G]}.$$

By Woodin's work on the stationary tower forcing we also know that

$$H_{\aleph_1}^V \prec H_{\aleph_1}^{V[G]}.$$

Now we prove that

$$H_{\aleph_1}^V \prec_{\Sigma_{n+1}} H_{\aleph_1}^{V[H]}.$$

Since this argument holds for any V, $P \in V$ and G V-generic for P, the proof will be completed.

We have to prove the following for any Σ_n-formula $\phi(x, z)$ and any Π_n-formula $\psi(x, z)$:

1. If
$$H^V_{\aleph_1} \models \forall x \phi(x, p)$$
for some $p \in \mathbb{T}^V$, then also
$$H^{V[H]}_{\aleph_1} \models \forall x \phi(x, p).$$
2. If
$$H^V_{\aleph_1} \models \exists x \psi(x, p)$$
for some $p \in \mathbb{T}^V$, then also
$$H^{V[H]}_{\aleph_1} \models \exists x \psi(x, p).$$

To prove 1 we note that, since $H^V_{\aleph_1} \prec H^{V[G]}_{\aleph_1}$, we have that
$$H^{V[G]}_{\aleph_1} \models \forall x \phi(x, p).$$
In particular we have that for any $q \in H^{V[H]}_{\aleph_1}$ we have that $H^{V[G]}_{\aleph_1}$ models that $\phi(q, p)$. Now, since by inductive assumptions
$$H^{V[H]}_{\aleph_1} \prec_{\Sigma_n} H^{V[G]}_{\aleph_1},$$
we get that
$$H^{V[H]}_{\aleph_1} \models \phi(q, p)$$
for all $q \in H^{V[H]}_{\aleph_1}$, from which the desired conclusion follows.

To prove 2 we note that for some $q \in H^V_{\aleph_1}$ we have that
$$H^V_{\aleph_1} \models \psi(q, p).$$
Then, since by inductive assumptions we have that
$$H^V_{\aleph_1} \prec_{\Sigma_n} H^{V[H]}_{\aleph_1},$$
we conclude that
$$H^{V[H]}_{\aleph_1} \models \psi(q, p).$$
The conclusion now follows.

The lemma is now completely proved. □

The Theorem is proved. □

Remark 3.3. Theorem 3.1 has a weaker conclusion than [11, Theorem 3.1.12] where from the same assumptions it is drawn the conclusion that the first order theory of $L(\mathbb{R})$ is invariant with respect to set forcing. We had to weaken the conclusion of Theorem 3.1 with respect to [11, Theorem 3.1.12] since it is not clear whether we can replace $H_{\aleph_1}$ with $L(\mathbb{R})$ in the proof of the above Lemma. The reason is given by the different range of the quantifiers, since an element of $H_{\aleph_1}$ is essentially a real while an element of $L(\mathbb{R})$ is essentially determined by a real and an arbitrary large ordinal. Now in the notation of the Lemma our inductive assumption to generalize it to $L(\mathbb{R})$ would be that

$$\langle L(\mathbb{R})^V, \in, \mathbb{R}^V\rangle \prec_{\Sigma_n} \langle L(\mathbb{R})^{V[H]}, \in, \mathbb{R}^V\rangle,$$

and

$$\langle L(\mathbb{R})^{V[H]}, \in, \mathbb{R}^{V[H]}\rangle \prec_{\Sigma_n} \langle L(\mathbb{R})^{V[G]}, \in, \mathbb{R}^{V[H]}\rangle.$$

If a Π_{n+1} formula $\forall x\phi(x, \vec{r})$ relativized to $L(\mathbb{R})$ with parameters in $\mathbb{R}^V$ holds in V we can infer that it holds in $L(\mathbb{R}^{V[G]})$ and thus that for any real $r \in V[H]$ $\phi(r, \vec{r})$ holds in in $L(\mathbb{R}^{V[H]})$. However this is not sufficient to infer that $\forall x\phi(x, \vec{r})$ holds in in $L(\mathbb{R}^{V[H]})$ since we do not have control on the elements of $V[H]$ which are not reals.

4 Π_2-absoluteness of the theory of H_{ω_2} in models of MM^{++}

In this section we prove Theorem 1.4. We leave to the reader to convert it into a proof of Theorem 1.5.

Theorem 4.1. *Assume MM^{++} holds in V and there are class many Woodin cardinals. Then the Π_2-theory of $H_{\aleph_2}$ with parameters cannot be changed by stationary set preserving forcings which preserve* BMM.

Proof. Assume V models MM^{++} and let $P \in M$ be such that V^P models BMM.

Let δ be a Woodin cardinal larger than $|P|$. By Theorem 2.12 there is a complete embedding $i : P \to Q = \mathbb{T}_\delta \restriction T_P$ for some stationary set $T_P \in V_\delta$ such that

$$\Vdash_P Q/i[P] \text{ is stationary set preserving.}$$

Now let G be V-generic for Q and $H = i^{-1}[G]$ be V generic for P. Then $V \subset V[H] \subset V[G]$ and $V[G]$ is a generic extension of $V[H]$ by a forcing which is stationary set preserving in $V[H]$. Moreover by Woodin's theorem on stationary tower forcing 2.4, we have that $H^V_{\aleph_2} \prec H^{V[G]}_{\aleph_2}$.

We show that

$$H^V_{\aleph_2} \prec_{\Sigma_2} H^{V[H]}_{\aleph_2}.$$

This will prove the Theorem, modulo standard forcing arguments.

We have to prove the following for any Σ_0-formula $\phi(x, y, z)$:

1. If
$$H^V_{\aleph_2} \models \exists y \forall x \phi(x, y, p)$$
for some $p \in H^V_{\aleph_2}$, then also
$$H^{V[H]}_{\aleph_2} \models \exists y \forall x \phi(x, y, p).$$
2. If
$$H^V_{\aleph_2} \models \forall y \exists x \phi(x, y, p)$$
for some $p \in H^V_{\aleph_2}$, then also
$$H^{V[H]}_{\aleph_2} \models \forall y \exists x \phi(x, y, p).$$

To prove 1 we note that for some $q \in H^V_{\aleph_2}$ we have that

$$H^V_{\aleph_2} \models \forall x \phi(x, q, p).$$

Then, since

$$H^V_{\aleph_2} \prec H^{V[G]}_{\aleph_2},$$

we have that

$$H^{V[G]}_{\aleph_2} \models \forall x \phi(x, q, p).$$

In particular, since $q, p \in H^{V[H]}_{\aleph_2}$ and $H^{V[H]}_{\aleph_2}$ is a transitive substructure of $H^{V[G]}_{\aleph_2}$, we get that

$$H^{V[H]}_{\aleph_2} \models \forall x \phi(x, q, p)$$

as well. The conclusion now follows.

To prove 2 we note that, since

$$H^V_{\aleph_2} \prec H^{V[G]}_{\aleph_2},$$

we have that

$$H^{V[G]}_{\aleph_2} \models \forall y \exists x \phi(x, y, p).$$

In particular we have that for any $q \in H^{V[H]}_{\aleph_2}$ we have that

$$H^{V[G]}_{\aleph_2} \models \exists x \phi(x, q, p).$$

Now, since $V[H]$ models BMM and $V[G]$ is an extension of $V[H]$ by a stationary set preserving forcing, we get that

$$H^{V[H]}_{\aleph_2} \prec_{\Sigma_1} H^{V[G]}_{\aleph_2}.$$

In particular we can conclude that

$$H^{V[H]}_{\aleph_2} \models \exists x \phi(x, q, p)$$

for all $q \in H^{V[H]}_{\aleph_2}$, from which the desired conclusion follows.

The proof of the theorem is completed. $\square$

We conclude this section with the a variation of the above result. Recall that BMM^{++} asserts that

$$\langle H_{\omega_2}, \in, \mathsf{NS}_{\omega_1} \rangle \prec_{\Sigma_1} \langle H^{V^P}_{\omega_2}, \in, \mathsf{NS}^{V^P}_{\omega_1} \rangle$$

for any stationary set preserving poset P, where NS_{ω_1} is a unary predicate for the non-stationary ideal on ω_1.

A straightforward variation of the above proof shows also that:

Theorem 4.2. *Assume MM^{++} holds in V and there are class many Woodin cardinals. Then the Π_2-theory of the structure $\langle H_{\aleph_2}, \in, \mathsf{NS}_{\omega_1} \rangle$ with parameters cannot be changed by stationary set preserving forcings which preserve BMM^{++}.*

5 Resurrection axioms vs generic absoluteness for the theory of MM^{++}

There is a close analogy between the line of research pursued in this paper and in its sequels [19] "Category forcings, MM^{+++} and generic absoluteness for strong forcing axioms" and [1] "Absoluteness via resurrection" and a line of research initiated by unpublished work of Chalons and Veličković and which has brought Hamkins and Johnstone to the formulation of the resurrection axioms [7] and Tsaprounis to the formulation of the unbounded resurrection axioms [17].

Hamkins and Johnstone state that the weak resurrection axiom $\mathsf{wRA}(\Gamma)$ holds for a class of posets Γ if for any $P \in \Gamma$ there is $\dot{Q} \in V^P$ such that $H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{P*\dot{Q}}$. $\mathsf{RA}(\Gamma)$ holds if the witness $\dot{Q} \in V^P$ of the fact that $H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{P*\dot{Q}}$ can also be found so that P forces $\dot{Q}$ to be in Γ as well.

Tsaprounis formulate $\mathsf{URA}(\Gamma)$ as the statement that for all $P \in \Gamma$ and for all regular θ there is $\dot{Q} \in V^P$ which is forced by P to be in Γ as well so that, whenever $G * H$ is V-generic for $P * \dot{Q}$, there is in $V[G * H]$ an elementary embedding $j : H_\theta^V \to H_\lambda^{V[G*H]}$ with critical point $\mathfrak{c}$.

We can define $\mathsf{wURA}(\Gamma)$ by dropping the requirement that the name $\dot{Q} \in V^P$ used to obtain the map $j : H_\theta^V \to H_\lambda^{V[G*H]}$ is forced by P to be in Γ.

A close inspection of Woodin's proof of [21, Theorem 2.53] actually shows that in the presence of class many Woodin cardinals $\mathsf{FA}_{\omega_1}(\Gamma)$ is equivalent to $\mathsf{wURA}(\Gamma)$. Tsaprounis and Asperò show that under the same large cardinal assumptions MM^{++} is equivalent to $\mathsf{URA}(\mathsf{SSP})$ essentially arguing along the same lines of Theorem 2.12 of the present paper.

We can translate our proof of the Σ_2-absoluteness result for models of MM^{++} to the resurrection axioms setting and use it to separate $\mathsf{RA}(\mathsf{SSP})$ and $\mathsf{RA}(\mathit{proper})$ as follows:

Theorem 5.1. *Assume* CH *fails and* Γ *is a definable class of posets such that* $\mathsf{RA}(\Gamma)$ *holds. Then*

$$H_{\mathfrak{c}}^V \prec_{\Sigma_2} H_{\mathfrak{c}}^{V^P}$$

for every $P \in \Gamma$ *which forces* $\mathsf{BFA}_{<\mathfrak{c}}(\Gamma)$.

Proof. First of all it is not hard to check that $\mathsf{RA}(\Gamma) + \neg\mathsf{CH}$ implies $\mathsf{BFA}_{<\mathfrak{c}}(\Gamma)$ (see the proof of [7, Observation 3] and check that the same argument would work with $\mathsf{RA}(\Gamma)$ in place of $\mathsf{wRA}(\Gamma)$). Given some $P \in \Gamma$ which forces $\mathsf{BFA}_{<\mathfrak{c}}(\Gamma)$ we can follow the same pattern of the proof of Theorem 4.1 recalling that $\mathsf{RA}(\Gamma)$ grants that there is $\dot{Q} \in V^P$ such that:

- $H_{\mathfrak{c}}^V \prec H_{\mathfrak{c}}^{V^{P*\dot{Q}}}$,
- P forces $\dot{Q} \in \Gamma$.

Now we can use that $\mathsf{BFA}_{<\mathfrak{c}}(\Gamma)$ holds in V and V^P to argue that:

$$H_{\mathfrak{c}}^V \prec_{\Sigma_1} H_{\mathfrak{c}}^{V^P} \prec_{\Sigma_1} H_{\mathfrak{c}}^{V^{P*\dot{Q}}}.$$

We can now follow the usual pattern to reach the desired conclusion. □

We remark the following:

Fact 5.2. Assume there is a reflecting cardinal. Then $\mathsf{RA}(proper) + \neg\mathsf{CH}$ implies that the canonical functions on $\omega_1^{\omega_1}$ are not a dominating family modulo the club filter.

Proof. First of all we have already pointed out that $\mathsf{RA}(proper) + \neg\mathsf{CH}$ implies BPFA (by which we abbreviate $\mathsf{BFA}_{<\mathfrak{c}}(proper)$). Now the proof relies on the following:

Fact 5.3. Assume $\dot{Q} \in V^{Coll(\omega_1,\omega_1)}$ is a name for the usual proper iteration of length a reflecting cardinal δ which gives a model of BPFA. Then the canonical functions are not dominating in $V^{P*\dot{Q}}$.

We roughly sketch why this is the case, this argument has been explained to me by Asperò.

Proof. We can assume that $Coll(\omega_1,\omega_1)$ is the poset which shoots a new function in $\omega_1^{\omega_1}$ using countable initial segments. Now let $\alpha < \delta$ be an ordinal and let $M \prec H_{\delta^+}$ be a countable model such that $\alpha \in M$. Let $g : M \cap \omega_1 \to M \cap \omega_1$ be M-generic for $Coll(\omega_1,\omega_1)$ and use the properness of $\dot{Q}$ in $V^{Coll(\omega_1,\omega_1)}$ to find $\dot{q}$ such that $\langle g, \dot{q}\rangle$ is an M-generic condition for $Coll(\omega_1,\omega_1) * \dot{Q}$. Now we extend g to

$$g' = g \cup \{\langle M \cap \omega_1, \mathrm{otp}(M \cap \alpha)\rangle\}.$$

This gives that whenever $\langle g', \dot{q}\rangle \in f * G$ for some $f * G$ V-generic for $Coll(\omega_1,\omega_1) * \dot{Q}$ we have that

$$M[f * G] \prec H_{\delta^+}[f * G]$$

is a countable substructure with $\alpha, f \in M[f * G]$, $M[f * G] \cap \delta^+ = M \cap \delta^+$ and $f(M \cap \omega_1) = \mathrm{otp}(M \cap \alpha)$. α has size ω_1 in $V[f * G]$ since the latter models that δ is ω_2. Since $\alpha \in M$, we have that in $M[f * G]$ there is a continuous chain of countable sets $\{X_i : i < \omega_1\}$ such that $\alpha = \bigcup_{i<\omega_1} X_i$. This chain can be used to define the α-th canonical function by $g_\alpha(i) = \mathrm{otp}(X_i)$. We can now check that $M \cap \alpha = X_{M\cap\omega_1}$ and that

$$f(M \cap \omega_1) = \mathrm{otp}(M \cap \alpha) = \mathrm{otp}(X_{M\cap\omega_1}) = g_\alpha(M \cap \omega_1).$$

A standard reflection argument gives that $M[f * G]$ models that the set of $i < \omega_1$ such that $f(i) \geq g_\alpha(i)$ is stationary. Since $M[f*G] \prec H_{\delta^+}[f*G]$, we get that this set is really stationary in $H_{\delta^+}[f * G]$. Since this argument can be repeated for all $\alpha < \delta$ we get that f is a function which is not dominated by any canonical function in $V[f * G]$.

Thus $V^{Coll(\omega_1,\omega_1)*\dot{Q}}$ models that the canonical functions are not a dominating family modulo the club filter. □

This is a Σ_2-statement in H_{ω_2} which holds in $V^{Coll(\omega_1,\omega_1)*\dot{Q}}$ which is a model of BPFA. The previous theorem grants that it holds also in V. □

The above shows that (under mild large cardinal assumptions) $\mathsf{RA}(\mathsf{SSP})$ and $\mathsf{RA}(proper)$ give a different Π_2-theory of $H_{\aleph_2}$, since it is well known that BMM (which follows from $\mathsf{RA}(\mathsf{SSP}) + \neg\mathsf{CH}$) gives that the canonical functions are dominating in $\omega_1^{\omega_1}$ modulo the club filter.

We finally remark that the forcing axioms $\mathsf{RA}_\alpha(\Gamma)$ formulated in [1] are the natural outcome of the further investigations of the connection between resurrection axioms and generic absoluteness results outlined in this last section. Moreover the forcing axiom MM^{+++} introduced in [19] is actually equivalent to a strengthening of $\mathsf{URA}(\mathsf{SSP})$ (in the final section of [19] this equivalent formulation is spelled out in detail).

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