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A Maple algorithm for k-statistics, polykays and their multivariate generalization

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▼ Introduction

Abstract: Through the classical umbral calculus, we provide a unifying syntax for single and multivariate k-statistics, polykays and multivariate polykays. Classical umbral calculus is a symbolic method for handling ordinary and exponential formal power series. A feature of the classical umbral calculus is that an umbra has the structure of a random variable but with non reference to a probability space. This brings the umbral syntax closer to statistical methods.

Application Areas/Subject: Combinatorics & algebraic methods in statistics

Keyword: umbral calculus, symmetric polynomials, set partitions, multiset, cumulants, k-statistics, polykeys.

See Also: Maple algorithm [3]

▼ Initialization

> restart

> with(combinat, partition, multinomial)

[partition, multinomial]

(2.1)

▼ Subdivision of Multiset and Augmented symmetric function to Power Sum

The following algorithm function is used for listing all subdivision of a multiset. These algorithms is fully discussed in [3]

▼ Subdivision of Multiset

See [3] for complete discussion about this algorithm.

> nRep := proc(u) mul($x_2!$, $x = \text{convert}(u, \text{multiset})$) end proc:

```
URv := proc(u, v)
  local U, ou, i, ptr, vI;
  ou := NULL; U := [ ]; vI := indets(v);
  for ptr from nops(u) by -1 to 2 do
    if has(uptr, v) then break end if
  end do;
  for i from ptr to nops(u) do
    if not (ui = ou or has(ui, vI)) then
      ou := ui;
      U := [op(U), [op(u1..i-1), ui·v, op(ui+1..-1)] ]
    end if
  end do;
  op(U), [op(u), v]
end proc:
```

```
URV := proc( )
  local U, V, i;
  U := [args1,1]; V := args2,1;
  for i to nops(V) do U := [seq(URv(u, Vi), u = U) ] end do;
  seq([x, args1,2·args2,2], x = U)
```

end proc:

```
URmV := proc ( )
  local U, i;
  if nargs = 1 then args else
    U := URV( args_1, args_2 );
    for i from 3 to nargs do
      U := seq( URV( u, args_i ), u = [U] )
    end do;
    seq( [ x_1,  $\frac{x_2}{nRep(x_1)}$  ], x = U )
  end if
end proc:
```

```
comb := proc( V, ptr, Y )
  if ptr = nops( V ) + 1 then return Y end if;
  seq( comb( V, ptr + 1, [ op( Y ), L ] ), L = V_ptr )
end proc:
```

The output of the following new function is a disjoint union of vectors.

```
> unionVects := proc( U::list, V::list )
  if nops( U ) = 0 then V
  elif nops( V ) = 0 then U
  else [ seq( seq( [ sort( [ op( v_1 ), op( u_1 ) ] ), v_2 · u_2 ], u = U ), v = V ) ] end if
end proc:
```

Example

```
> A := [ [ [ a, a, a ], 1 ], [ [ a, a^2 ], 3 ], [ [ a^3 ], 1 ] ];
B := [ [ [ b, b ], 1 ], [ [ b^2 ], 1 ] ];
DisjointUnion = unionVects( A, B )
```

$$A := [[[a, a, a], 1], [[a, a^2], 3], [[a^3], 1]]$$

$$B := [[[b, b], 1], [[b^2], 1]]$$

$$DisjointUnion = [[[a, a, a, b, b], 1], [[a, b, b, a^2], 3], [[b, b, a^3], 1], [[a, a, a, b^2], 1], \quad (3.1.1) \\ [[a, b^2, a^2], 3], [[b^2, a^3], 1]]$$

The output of the following function is a multiset subdivision. This algorithm is described in [3].

Remark: here the function *maketab* is slightly different from the function *maketab* in [3]. The input parameters are changed. In [3] we just give the multiplicities of elements. In this function we also name the elements. For example if all partitions of the multiset [a,a,b] are needed we can recall *makeTab*([a,a,b]) while in [3] the function calls *makeTab*(2,1).

Note: at the end of the function we have inserted a code block in order to calculate additional values of the coefficients for every term of the final expression. This is useful either in the

construction of the k-statistics either in the conversion between augmented symmetric functions and power sums.

```

> makeTab := proc ( )
  local vParts, vEvalTo, vEvalBack, vIndets, U, V, v;
  vParts := sort(convert(args_1, multiset), proc(x, y) evalb(y_2 < x_2) end proc);
  vEvalBack := [seq(P || i = vParts_i_1, i = 1 .. nops(vParts) ) ];
  vEvalTo := [seq(vParts_i_1 = P || i, i = 1 .. nops(vParts) ) ];
  vIndets := { op( sort( eval( args_1, vEvalTo ) ) ) };
  vParts := eval(vParts, vEvalTo);
  U := [seq( [ seq( [ seq( p_1^z, z = y ) ], multinomial( p_2, seq(r, r = y) ) ], y
    = partition( p_2 ) ) ], p = vParts );
  if nops(U) = 1 then
    U := [ seq( [ x_1, x_2 / nRep(x_1) ], x = op(U) ) ]
  else
    U := [ seq( URmV( op(x) ), x = [ comb(U, 1, [ ]) ] ) ]
  end if;
  if nargs = 1 then
    elif args_2 = 'augToPs( )' then
      U := [ seq( [ y_1, mul( ( - 1 ) ^ degree(x) - 1 . ( degree(x) - 1 ) !, x = y_1 ) . y_2 ], y = U ) ]
    elif args_2 = 'polykays( )' then
      U := [ seq( [ y_1, ( - 1 ) ^ nops(y_1) - 1 . ( nops(y[1]) - 1 ) ! . y_2 ], y = U ) ]
    end if;
    eval(U, vEvalBack)
  end proc;

```

> makeTab([a, a, a])

$$[[[a, a, a], 1], [[a, a^2], 3], [[a^3], 1]] \quad (3.1.2)$$

> makeTab([a, a, b])

$$[[[a b, a], 2], [[a, a, b], 1], [[a^2 b], 1], [[a^2, b], 1]] \quad (3.1.3)$$

> makeTab([a, a, a^3])

$$[[[a^4, a], 2], [[a, a, a^3], 1], [[a^5], 1], [[a^2, a^3], 1]] \quad (3.1.4)$$

▼ Augmented symmetric function to Power Sum

The output of this function is a symmetric function expressed in terms of power sum symmetric function, e.g.

$$S_r = \sum_{i=1}^n X_i^r$$

Note:

the symbol $[a, a, a]$ represents the subdivision $\{\{a\}\{a\}\{a\}\}$ and the symbol $[a^2, a]$ represents the

subdivision $\{\{a, a\}, \{a\}\}$.

The number associated to each symbol represents a product of $(-1)^{k-1} \cdot (k-1)!$, where k is the degree of the monomial in each block, multiplied by the integer associated to the same symbol in the output of *makeTab*.

For example: for the symbol $[a^2, a] = \{\{a, a\}, \{a\}\}$ we calculate $[(-1)^{2-1} \cdot (2-1)!] \cdot [(-1)^{1-1} \cdot (1-1)!] = [-1 \cdot 1] = -1$ and we multiply -1 with 3 that is the multiplicity of the subdivision:

Example: how we calculate *augToPs*([a,a,a])

1) we recall *makeTab*([a,a,a]) having following result:

$$[[[a, a, a], 1], [[a, a^2], 3], [[a^3], 1]]$$

2) we calculate:

2.1) block $[a, a, a]$:

$$[(-1)^{1-1} (1-1)!] [(-1)^{1-1} (1-1)!] [(-1)^{1-1} (1-1)!] = 1 \cdot 1 \cdot 1 = 1$$

2.2) block $[a, a^2]$:

$$[(-1)^{1-1} (1-1)!] [(-1)^{2-1} (2-1)!] = -1 \cdot 1 = -1$$

2.3) block $[a^3]$:

$$[(-1)^{3-1} (3-1)!] = 2$$

3) Multiplying the multiplicities in the step 1) with the results found in step 2), we have:

$$[[[a, a, a], 1], [[a, a^2], -3], [[a^3], 2]]$$

4) from which we can express the following result:

$$S_1^3 - 3 S_1 S_2 + 2 S_3$$

```

> augToPs := proc ( )
  local vIndets;
  if nargs = 0 then
    return NULL
  elif nargs = 1 then vIndets := sort([op(indets(args_1))])
    else vIndets := args_2 end if;
  add(mul(S_seq(degree(x, vIndets[i]), i = 1..nops(vIndets)), x = y_1) · y_2, y = makeTab(args_1,
  'procname(') ) )
end proc:

```

Examples

> *augToPs*([a, a, a])

$$S_1^3 - 3 S_1 S_2 + 2 S_3 \quad (3.2.1)$$

> *augToPs*([a, b])

$$-S_{1,1} + S_{0,1} S_{1,0} \quad (3.2.2)$$

> *augToPs*([a, a²]);

$$-S_3 + S_1 S_2 \quad (3.2.3)$$

the step 3) in the previous note.

> *makeTab*([a, a, a], 'augToPs()')

$$[[[a, a, a], 1], [[a, a^2], -3], [[a^3], 2]]$$

(3.2.4)

▼ A unifying framework for k-statistics, polykays and their multivariate generalizations

The nth k-statistic is the unique symmetric unbiased estimator of the cumulant κ_n of a given statistical distribution.

$$k_n \text{ is defined so that } E[k_n] = \kappa_n$$

The symmetric statistic $k_{r, s, \dots}$ is defined as

$$E[k_{r, s, \dots}] = \kappa_r \kappa_s \dots$$

where κ_r is a cumulant. These statistics called polykays, generalize the k-statistics. K-statistics, polykays and their multivariate generalization are commonly defined in terms of power sums, that are sums of the rth powers of the data points:

$$S_r = \sum_{i=1}^n X_i^r$$

Note: in polyk we calculate an additional value: the cardinality of subdivision. For every subdivision we calculate $(-1)^{k-1} (k-1)!$ where k is its cardinality.

For example $[a^2, a, b] = \{\{a, a\}, \{a\}, \{b\}\}$ has 3 blocks and we calculate $(-1)^{3-1} (3-1)! = 2$

Example: how we calculate *Polykays*(3)

1) we call *makeTab*([a,a,a]) and this is the output:

$$[[[a, a, a], 1], [[a, a^2], 3], [[a^3], 1]]$$

2) we calculate:

$$2.1) \text{ block } [a, a, a]: [(-1)^{3-1} (3-1)!] = 2$$

$$2.2) \text{ block } [a, a^2]: [(-1)^{2-1} (2-1)!] = -1$$

$$2.3) \text{ block } [a^3]: [(-1)^{1-1} (1-1)!] = 1$$

3) Multiplying the multiplicities in step 1) with the results found in step 2) we have:

$$[[[a, a, a], 2], [[a, a^2], -3], [[a^3], 1]]$$

4) from which we calculate:

$$\frac{2 \text{ augToPs}([a, a, a])}{n(n-1)(n-2)} - \frac{3 \text{ augToPs}([a, a^2])}{n(n-1)} + \frac{1 \text{ augToPs}([a^3])}{n}$$

$$= \frac{2 S_1^3 - 6 S_1 S_2 + 4 S_3}{n (n - 1) (n - 2)} + \frac{3 S_3 - 3 S_1 S_2}{n (n - 1)} + \frac{S_3}{n}$$

$$= \frac{S_3 n^2 - 3 S_1 S_2 n + 2 S_1^3}{n (n - 1) (n - 2)}$$

The following function gives the expression for k-statistics and polykays and their multivariate generalizations depending on the input parameters:

- for generate k-statistics k_r , the parameter is: [r]
- for generate polykays $k_{r,s}$, the parameter is: [r], [s]
- for generate multivariate k-statistics $k_{r,s}$, the parameter is: [r , s]
- for generate multivariate polykays $k_{r,s;u,v}$, the parameter is: [r , s], [u , v]

```

> polykays := proc ( )
  local nRes, nNum, vPar, vParP, vEval, vIndets, vTab, k, i, x, T, TotN, px;
  if args_1 = [ 0 ] and nargs = 1 then return 1 end if;
  vPar := [ ]; vEval := [ ]; vParP := [ ]; vIndets := [ ];
  TotN := add(add(x, x = y), y = args) - 1;
  vIndets := [ seq( X || i, i = 1 .. max(seq(nops(x), x = args)) ) ];
  for k to nargs do
    if args_k = [ 0 ] then next end if;
    for i to nops(args_k) do
      vParP := [ op(vParP), X || i || Y || k $ args_k i ];
      vEval := [ op(vEval), X || i || Y || k = X || i ];
    end do;
    vPar := [ op(vPar), vParP ];
    vParP := [ ];
  end do;
  vTab := makeTab(vPar_1, 'procname( )');
  for x from 2 to nops(vPar) do
    vTab := unionVects(vTab, makeTab(vPar_x, 'procname( )'));
  end do;
  vTab := eval(vTab, vEval);
  if nops(vTab) > 1 then
    vTab := [ seq( [ op(i)_1, op(i)_2 ], i = mul( u_1^{2·u_2}, u = vTab ) ) ];
  end if;
  for x from 0 to TotN do
    T_{x+1} := 0;
  end do;
  for x in vTab do
    px := nops(x_1);
  end do;
end proc;

```

```

    Tpx := Tpx + x2 · augToPs(x1, vIndets)
  end do;
  nRes := add(
    (
      Tx+1
      /
      mul(n - t, t = 0 .. x)
    ), x = 0 .. TotN);
  nNum := numer(nRes);
  nNum := collect(nNum, `minus`(indets(nNum), {n}), distributed);
  nRes := nNum / denom(nRes);
  return nRes
end proc:

```

examples: k-statistics

> polykays([3])

$$\frac{S_3 n^2 - 3 S_1 S_2 n + 2 S_1^3}{n (n - 1) (n - 2)} \quad (4.1)$$

example: polykays

> polykays([2], [1])

$$\frac{-S_1^3 - S_3 n + (n + 1) S_1 S_2}{n (n - 1) (n - 2)} \quad (4.2)$$

example: multivariate k-statistics

> polykays([2, 1])

$$\frac{S_{2,1} n^2 - 2 S_{1,0} S_{1,1} n - S_{2,0} S_{0,1} n + 2 S_{1,0}^2 S_{0,1}}{n (n - 1) (n - 2)} \quad (4.3)$$

example: multivariate polykays

> polykays([1, 1], [1])

$$\frac{-S_{2,1} n + S_{1,0} S_{1,1} n - S_{1,0}^2 S_{0,1} + S_{2,0} S_{0,1}}{n (n - 1) (n - 2)} \quad (4.4)$$

▼ Example: steps for k-statistics and polykays construction

k-Statistics k_3

> vTab := makeTab([a, a, a], 'polykays()')

$$vTab := [[[a, a, a], 2], [[a, a^2], -3], [[a^3], 1]] \quad (4.1.1)$$

> kstat := $\frac{2 \text{augToPs}([a, a, a])}{n (n - 1) (n - 2)} - \frac{3 \text{augToPs}([a, a^2])}{n (n - 1)} + \frac{1 \text{augToPs}([a^3])}{n}$

$$kstat := \frac{2 (S_1^3 - 3 S_1 S_2 + 2 S_3)}{n (n - 1) (n - 2)} - \frac{3 (-S_3 + S_1 S_2)}{n (n - 1)} + \frac{S_3}{n} \quad (4.1.2)$$

> $k_3 := \text{simplify}(kstat)$

$$k_3 := \frac{2 S_1^3 - 3 S_1 S_2 n + S_3 n^2}{n (n - 1) (n - 2)} \quad (4.1.3)$$

Test previous result

> *polykays*([3])

$$\frac{2 S_1^3 - 3 S_1 S_2 n + S_3 n^2}{n (n - 1) (n - 2)} \quad (4.1.4)$$

> *evalb*(%=*k*[3])

true (4.1.5)

polykays $k_{2;1}$

> *A* := *makeTab*([*a*, *a*], '*polykays*()'

$$A := [[[*a*, *a*], -1], [[*a*²], 1]] \quad (4.1.6)$$

> *B* := *makeTab*([*b*], '*polykays*()'

$$B := [[[*b*], 1]] \quad (4.1.7)$$

> *vTab* := *unionVects*(*A*, *B*)

$$vTab := [[[*a*, *a*, *b*], -1], [[*b*, *a*²], 1]] \quad (4.1.8)$$

> *vTab* := *eval*(*vTab*, [*b* = *a*])

$$vTab := [[[*a*, *a*, *a*], -1], [[*a*, *a*²], 1]] \quad (4.1.9)$$

> *kstat* := $-\frac{1 \text{ augToPs}([a, a, a])}{n (n - 1) (n - 2)} + \frac{1 \text{ augToPs}([a, a^2])}{n (n - 1)}$

$$kstat := -\frac{S_1^3 - 3 S_1 S_2 + 2 S_3}{n (n - 1) (n - 2)} + \frac{-S_3 + S_1 S_2}{n (n - 1)} \quad (4.1.10)$$

> *kstat* := *simplify*(*kstat*)

$$kstat := \frac{-S_1^3 + S_1 S_2 - S_3 n + S_1 S_2 n}{n (n - 1) (n - 2)} \quad (4.1.11)$$

> *nNum* := *numer*(*kstat*); *nDenom* := *denom*(*kstat*)

$$\begin{aligned} nNum &:= -S_1^3 + S_1 S_2 - S_3 n + S_1 S_2 n \\ nDenom &:= n (n - 1) (n - 2) \end{aligned} \quad (4.1.12)$$

> *nNum* := *collect*(*nNum*, *indets*(*nNum*) \setminus {*n*}, *distributed*)

$$nNum := -S_1^3 - S_3 n + (1 + n) S_1 S_2 \quad (4.1.13)$$

> *k*['2;1'] := $\frac{nNum}{nDenom}$

$$k_{2;1} := \frac{-S_1^3 - S_3 n + (1 + n) S_1 S_2}{n (n - 1) (n - 2)} \quad (4.1.14)$$

test previous result

> *polykays*([2], [1])

$$\frac{-S_1^3 - S_3 n + (1+n) S_1 S_2}{n(n-1)(n-2)} \quad (4.1.15)$$

> evalb(%=k[2;1])

true (4.1.16)

multivariate k-Statistics $k_{2,1}$

> vTab := makeTab([a, a, b], 'polykays()')

$$vTab := [[a b, a], -2], [[a, a, b], 2], [[a^2 b], 1], [[a^2, b], -1]] \quad (4.1.17)$$

$$kstat := -\frac{2 \text{augToPs}([a b, a])}{n(n-1)} + \frac{2 \text{augToPs}([a, a, b])}{n(n-1)(n-2)} + \frac{1 \text{augToPs}([a^2 b])}{n} - \frac{1 \text{augToPs}([a^2, b])}{n(n-1)}$$

$$kstat := -\frac{2(-S_{2,1} + S_{1,0}S_{1,1})}{n(n-1)} + \frac{2(-2S_{1,0}S_{1,1} + S_{1,0}^2S_{0,1} + 2S_{2,1} - S_{2,0}S_{0,1})}{n(n-1)(n-2)} + \frac{S_{2,1}}{n} - \frac{-S_{2,1} + S_{2,0}S_{0,1}}{n(n-1)} \quad (4.1.18)$$

> k[2, 1] := simplify(kstat)

$$k_{2,1} := \frac{-2S_{1,0}S_{1,1}n + 2S_{1,0}^2S_{0,1} + S_{2,1}n^2 - S_{2,0}S_{0,1}n}{n(n-1)(n-2)} \quad (4.1.19)$$

test previous result

> polykays([2, 1])

$$\frac{-2S_{1,0}S_{1,1}n + 2S_{1,0}^2S_{0,1} + S_{2,1}n^2 - S_{2,0}S_{0,1}n}{n(n-1)(n-2)} \quad (4.1.20)$$

> evalb(%=k[2, 1])

true (4.1.21)

multivariate polykays $k_{1,1;1,0}$

> A := makeTab([a, b], 'polykays()')

$$A := [[a b], 1], [[b, a], -1]] \quad (4.1.22)$$

> B := makeTab([a], 'polykays()')

$$B := [[a], 1]] \quad (4.1.23)$$

> vTab := unionVects(A, B)

$$vTab := [[a, a b], 1], [[a, a, b], -1]] \quad (4.1.24)$$

$$kstat := \frac{1 \text{augToPs}([a, a b])}{n(n-1)} - \frac{1 \text{augToPs}([a, a, b])}{n(n-1)(n-2)}$$

$$kstat := \frac{-S_{2,1} + S_{1,0}S_{1,1}}{n(n-1)} - \frac{-2S_{1,0}S_{1,1} + S_{1,0}^2S_{0,1} + 2S_{2,1} - S_{2,0}S_{0,1}}{n(n-1)(n-2)} \quad (4.1.25)$$

> *kstat* := *simplify*(*kstat*)

$$kstat := \frac{-S_{2,1}n + S_{1,0}S_{1,1}n - S_{1,0}^2S_{0,1} + S_{2,0}S_{0,1}}{n(n-1)(n-2)} \quad (4.1.26)$$

> *nNum* := *numer*(*kstat*); *nDenom* := *denom*(*kstat*)

$$\begin{aligned} nNum &:= -S_{2,1}n + S_{1,0}S_{1,1}n - S_{1,0}^2S_{0,1} + S_{2,0}S_{0,1} \\ nDenom &:= n(n-1)(n-2) \end{aligned} \quad (4.1.27)$$

> *nNum* := *collect*(*nNum*, *indets*(*nNum*) \ {*n*}, *distributed*)

$$nNum := -S_{2,1}n + S_{1,0}S_{1,1}n - S_{1,0}^2S_{0,1} + S_{2,0}S_{0,1} \quad (4.1.28)$$

> *k*[*'1,1; 1'*] := $\frac{nNum}{nDenom}$

$$k_{1,1;1} := \frac{-S_{2,1}n + S_{1,0}S_{1,1}n - S_{1,0}^2S_{0,1} + S_{2,0}S_{0,1}}{n(n-1)(n-2)} \quad (4.1.29)$$

test previous result

> *polykays*([1, 1], [1])

$$\frac{-S_{2,1}n + S_{1,0}S_{1,1}n - S_{1,0}^2S_{0,1} + S_{2,0}S_{0,1}}{n(n-1)(n-2)} \quad (4.1.30)$$

> *evalb*(% = *k*[*'1,1; 1'*])

$$true \quad (4.1.31)$$

▼ Replacing symbols with numerical data

Sums of the *r*th powers of the data points:

> *powS* := **proc**()

if *nargs* = 1 **then**

$$Sum\left('X_i^{args_1}, i = 1..n'\right)$$

else

$$Sum\left(mul\left('X_{i,j}^{args_j}, j = 1..nargs'\right), i = 1..n'\right)$$

end if

end proc:

Example

> *powS*(5, 3, 1)

$$\sum_{i=1}^n X_{i,1}^5 X_{i,2}^3 X_{i,3} \quad (4.2.1)$$

> *powS*(9);

$$\sum_{i=1}^n X_i^9 \quad (4.2.2)$$

This function allows us to process a k-statistic or polykay replacing the symbols with numerical data. The parameter is the following:

- for generate k-statistics k_r , the parameter is: [r], [[n1, n2, ...]]
- for generate polykays $k_{r,s}$ the parameter is: [[r], [s]], [[n1, n2, ...]]
- for generate multivariate k-statistics $k_{r,s}$ the parameter is: [[r , s]], [[n1a, n2a], [n1b, n2b] , ...]
- for generate multivariate polykays $k_{r,s; u,v}$ the parameter is: [[r , s], [u , v]], [[n1a, n2a], [n1b, n2b] , ...]

```

> npolyk:=proc(V, data)
  local res, ind, N, vE;
  if nops(V) = 1 then
    if nops(op(V)) = 1 then
      N := nops(op(data))
    else
      N := nops(data) end if;
  elif add( if(nops(x) = 1, 0, 1), x = V) = 0 then
    N := nops(op(data))
  else
    N := nops(data)
  end if;
  res := polykays(op(V));
  ind := `minus`(indets(res), {n});
  vE := seq(y1 = powS(op(y2)), y = seq( [x, [op(x)]] , x = ind));
  eval(res, [eval(evalf(eval(vE, [n = N])), [X = op(data)]), n = N])
end proc:

```

Examples: k-statistics and polykays

```

> data := [[16.34, 10.76, 11.84, 13.55, 15.85, 18.20, 7.51, 10.22, 12.52, 14.68, 16.08, 19.43,
            8.12, 11.20, 12.95, 14.77, 16.83, 19.80, 8.55, 11.58, 12.10, 15.02, 16.83, 16.98, 19.92,
            9.47, 11.68, 13.41, 15.35, 19.11]]
data := [[16.34, 10.76, 11.84, 13.55, 15.85, 18.20, 7.51, 10.22, 12.52, 14.68, 16.08,          (4.2.3)
          19.43, 8.12, 11.20, 12.95, 14.77, 16.83, 19.80, 8.55, 11.58, 12.10, 15.02, 16.83, 16.98,
          19.92, 9.47, 11.68, 13.41, 15.35, 19.11]]

```

The estimator for the mean is given by

```

> npolyk([[1]], data)
14.02166667 (4.2.4)

```

The estimator for the variance is given by

```

> npolyk([[2]], data)
12.65006954 (4.2.5)

```

The estimator for the skewness is given by $k_3 / \sqrt{k_2^3}$

```

> 
$$\frac{\text{npolyk}([[3]], \text{data})}{\sqrt{\text{npolyk}([[2]], \text{data})^3}}$$


```

$$-0.03216238591 \quad (4.2.6)$$

The estimator for the kurtosis is given by k_4 / k_2^2

$$\begin{aligned} > \frac{\text{npolyk}([4], \text{data})}{\text{npolyk}([2], \text{data})^2} \\ & -0.8852921296 \quad (4.2.7) \end{aligned}$$

The estimator for the $\kappa_3 \kappa_2$ is given by

$$\begin{aligned} > \text{npolyk}([3], [2], \text{data}) \\ & -15.56133309 \quad (4.2.8) \end{aligned}$$

Examples: multivariate k-tstatistics and multivariate polykays

$$\begin{aligned} > \text{data} := [[5.31, 11.16], [3.26, 3.26], [2.35, 2.35], [8.32, 14.34], [13.48, 49.45], [6.25, \\ & 15.05], [7.01, 7.01], [8.52, 8.52], [0.45, 0.45], [12.08, 12.08], [19.39, 10.42]] \\ > \text{data} := [[5.31, 11.16], [3.26, 3.26], [2.35, 2.35], [8.32, 14.34], [13.48, 49.45], [6.25, \\ & 15.05], [7.01, 7.01], [8.52, 8.52], [0.45, 0.45], [12.08, 12.08], [19.39, 10.42]] \quad (4.2.9) \end{aligned}$$

The estimator for the $\kappa_{2,1}$ is given by

$$\begin{aligned} > \text{npolyk}([2, 1], \text{data}) \\ & -23.73790606 \quad (4.2.10) \end{aligned}$$

The estimator for the $\kappa_{2,1} \kappa_{1,0}$ is given by

$$\begin{aligned} > \text{npolyk}([1, 1], [1], \text{data}) \\ & 294.2657622 \quad (4.2.11) \end{aligned}$$

The estimator for the $\kappa_{2,1} \kappa_{2,0} \kappa_{1,0}$ is given by

$$\begin{aligned} > \text{npolyk}([2, 1], [2], [1], \text{data}) \\ & 12369.47450 \quad (4.2.12) \end{aligned}$$

▼ Conclusions

Umbral formulae for k-statistics and polykays, either in single or multivariate cases, share a common algorithm to construct multiset subdivisions. When the multiset has the form $\{a^{(i)}\}$, an efficient way is to resort integer partitions.

In general we may construct multiset subdivisions by using suitable set partitions, but this procedure result non efficient from computational point of view [4].

Indeed, subdivision may occur more than one time in the same formula so that it is necessary to built a procedure generating only different subdivision with their multiplicity i.e. the number of corresponding set partitions.

To accomplish this task, the algorithm makeTab takes into account the connection between multisets and integer partitions, reducing the overall computational complexity. To accomplish this task, the algorithm makeTab takes into account the connection between multisets and integer partitions, reducing the overall computational complexity. It is possible to build very fast algorithms for k-statistics, polikays and their generalizations by forfeiting the elegant idea of producing only one algorithm for the whole subject see [5],[6].

▼ References

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