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An approximate solution to rent-seeking contests with private information

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Abstract

We propose an approximate solution to rent-seeking contests in which participants have heterogeneous and private valuations. The procedure only requires common knowledge about the mean of the distribution of valuations. We obtain a closed-form expression for an agent's level of investment and subject it to comparative statics analysis. We then assess the performance of the model and find that the proposed solution provides a remarkably effective approximation of the optimal solution for a wide range of parameter specifications.

Keywords: rent-seeking contests; private information; approximate solution; behavioural OR.

1 Introduction

A rent-seeking contest is a probabilistic contest in which players invest resources in order to influence the probability that they will win a prize. Rent-seeking contests were first investigated by Tullock (1980). Tullock's seminal model has since been generalized in many directions (see Congleton et al., 2008, for a comprehensive literature review) and applied in different domains, such as the analysis of political lobbying (Hillman &

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Riley, 1989), conflicts (Garfinkel & Skaperdas, 2006), R&D races (Baye & Hoppe, 2003), and sporting competitions (Szymanski, 2003). Rent-seeking games are also increasingly used in OR, such as in modelling interactions between attackers and defenders (Hausken, 2008; Hausken & Bier, 2011; Zhuang et al., 2011; Rinott et al., 2012; Deck et al., 2015; Mo et al., 2015) or selecting suppliers in outsourcing decisions (Benjaafar et al., 2007; Fu et al., 2016).

One interesting line of research involves optimal player behaviour (i.e., the level of investment that maximizes expected payoff) when the standard hypothesis of all participants sharing a common prize valuation does not hold. Indeed, in many typical applications of rent-seeking contests, the alternative assumption of asymmetric and private valuations seems more realistic. For instance, in the case of R&D expenditures, different competitors may assess the potential of a patent according to different information or in light of different scenarios.

Early research allowed for heterogeneity in player valuations but maintained the assumption of their common knowledge (Hillman & Riley, 1989; Nti, 1999; Stein, 2002). In addition to asymmetry, other papers investigate the consequences of the privacy of player valuations in various contexts:[1] two-player games with one-sided private information and continuous types (Hurley & Shogren, 1998a), n-player games with one-sided private information and discrete types (Schoonbeek & Winkel, 2006), and two-player games with two-sided private information and discrete (Hurley & Shogren, 1998b; Malueg & Yates, 2004) or continuous types (Ewerhart, 2010). However, a full analysis remains elusive for the general case in which $n \geq 2$ players have asymmetric and private valuations that are distributed over a continuous support. Indeed, it has been shown that the problem rapidly becomes intractable (Fey, 2008; Wasser, 2013; Ewerhart, 2014). As such, a proper characterization of a participant's optimal level of investment is either unfeasible

[1] This line of research (including this paper) assumes that agents know their own type (i.e., their own valuation of the prize) but are uncertain about the other participants' valuations. A different strand of the literature (Wärneryd, 2003, Wärneryd, 2013) investigates rent-seeking games where players are uncertain about their own valuation. In a more general sense, participants in a contest may be heterogeneous across other dimensions as well, such as the effectiveness of their lobbying efforts, their cost functions, or their financial constraints (see Yamazaki, 2008).

or extremely cumbersome (and thus of little practical use).

For these reasons, the aim of this paper is to propose an approximate solution to rent-seeking games where participants have private information. Such a solution can be useful for all those subjects (say, the organizer of the contest, other stakeholders, researchers and external observers) that may want to have a feeling about how participants will behave and how the contest will evolve. For instance, the promoter of a sport event may be interested in having an estimate of the total amount of effort that the players will exert (so called rent-dissipation, we will investigate this issue in the Appendix) and how this amount varies as a function of agents' valuations of the prize. Moreover, an important feature of the method that we propose is its simplicity. In particular, the method only requires knowledge of the mean of the distribution from which players draw their valuations. The information structure that we adopt is thus minimal.[2] As such, the approach is also well-tailored for capturing the behaviour of boundedly rational agents, i.e., individuals that suffer from behavioural biases or face computational limits and thus base their decisions on some heuristics.[3] For instance, a participant in the contest may use our procedure as a rule of thumb for computing his level of investment and thus evaluate his chances of winning and his expected payoff. In this respect, the paper joins a number of recent articles that investigate the role and implications of bounded rationality and psychological heuristics in the theory and practice of OR (e.g., Hämäläinen et al., 2013; Becker, 2016; Brocklesby, 2016; Keller & Katsikopoulos, 2016; White, 2016).[4]

[2]Notice in this respect that the method that we propose does not require common knowledge of the distribution of agents' valuations. As such, our approach is viable under weaker assumptions with respect to the standard private values setting. In fact, the private values assumption requires agents to know the entire distribution of types and not just the mean. The private values assumption is commonly used in auction theory (see Krishna, 2002) and in many OR applications that postulate the existence of private information (for instance, see Li & Balachandran, 1997, about the design of optimal pricing transfer schemes within a firm, Lee & Ferguson, 2010, about the strategic disclosure of private information in negotiations, or Wang & Zhuang, 2011, about the trade-off between the depth of screening and congestion in selecting applicants).

[3]The notion of bounded rationality was originally proposed by Simon (1955). However, there are currently multiple views of bounded rationality (see Rubinstein, 1998; Gigerenzer & Selten, 2001; Lee, 2011). Katsikopoulos (2014) provides a detailed assessment of the differences in terms of both premises and implications between the two main approaches to bounded rationality. We discuss how our method can be interpreted within such a framework in Section 2.2.

[4]The last four papers are part of a special issue of the European Journal of Operational Research

We show that our approach leads to a closed-form solution for an agent's level of investment. We then subject this solution to comparative statics analysis and investigate the issue of entry in the contest. We find that an agent invests a strictly positive amount if and only if his private valuation is above a certain threshold. In particular, we show that a "strong" player (one whose valuation is above average) always participates, while a "weak" player's decision of whether to participate or not depends on the number of competitors. The literature already highlights how asymmetric valuations may act as a barrier to entry (Hillman & Riley, 1989; Stein, 2002). However, our analysis shows how the combination of heterogeneity and private information can sometimes exacerbate or contrast with this effect.

Most importantly, we examine how the proposed solution performs in approximating optimal behaviour. To this goal, we use two different benchmarks. First, we compare our solution with the optimal solution that characterizes the private information case. Since our model aims at delivering a closed-form solution that could be used to approximate optimal behaviour in rent-seeking games in which participants have private valuations, the optimal solution of the private information case is indeed the most natural and appropriate benchmark to use. However, and as already mentioned, as of today a general solution to the problem of identifying the optimal level of investment when individuals have private valuations does not exist. What the literature has uncovered are only a few ad-hoc solutions, i.e., optimal solutions that are valid under specific distributional assumptions. We thus study how our approximate solution performs in comparison with two of these optimal solutions as identified by Malueg and Yates (2004) and Ewerhart (2010).[5] Despite of substantial differences between the two settings (Malueg and Yates, 2004, consider a framework in which agents' valuations follow a discrete bivariate uniform distribution, whereas Ewerhart, 2010, considers a framework in which valuations stem from a continuous distribution), we find that our solution provides a

(Vol. 249, Issue 3, March 2016) devoted to behavioural operational research, for which additional references and more details are available in the editorial of the issue (Franco & Hämäläinen, 2016).

[5] To the best of our knowledge, these are actually the only two existing explicit solutions for rent-seeking contests that match the assumptions of our model, i.e., agents' private valuations that are independently and identically distributed random variables.

remarkably good approximation of optimal behaviour. Indeed, the approximate solution differs from the optimal solution by less than 5% in terms of an agent's level of investment and by less than 2% in terms of expected payoffs. To further validate our method we then also use a second benchmark, namely the optimal solution that characterizes a setting with perfect information. In this case, we ran a series of simulations that assume different functional forms for the underlying distribution of agent valuations and different numbers of participants. The results show that the model is also effective in approximating the optimal solution that an agent would implement if all individual valuations were common knowledge. Here as well, we find that the approximate solution differs from the optimal solution by less than 5% (and often by less than 2%) in terms of both the level of investment and the expected payoff across a wide range of parameter specifications. This last set of results corroborates the possible role of the proposed solution as a heuristic for approximating optimal behaviour at a lower computational cost. We show in fact that the functional form of the approximate solution is much simpler than the optimal solution that characterizes a framework with perfect information.

Elaborating on this last point, we then discuss the trade-off between the performance of the method and the simplicity of its functional form. Admittedly, the functional form of the approximate solution remains non-trivial as it still requires some calculations on the part of the agent. However, we argue that this is an acceptable shortcoming for two reasons. First, the functional form of the approximate solution is far simpler than that of the optimal solution, regardless of whether it is computed in a context of imperfect or perfect information. Second, we show that the performance of the model is improved by an order of magnitude compared to an alternative approach that stems from different premises and leads to a simpler functional form.

The remainder of the paper is organized as follows. Section 2 introduces the model, discusses the approximate solution, and presents exercises of comparative statics. Section 3 studies the performance of the model in approximating optimal behaviour. Section 4 concludes the paper, and the appendix explores additional properties of the approximate solution.

2 The model

Consider a rent-seeking contest in which $n \geq 2$ risk-neutral players compete to win a prize. Let $v_i \in [v_{\min}, v_{\max}]$ with $v_{\max} > v_{\min} > 0$ indicate the valuation of player $i \in N$, where $N = \{1, ..., n\}$. The actual realization of v_i is agent i's private information.[6] It is common knowledge that all valuations are identically and independently distributed according to a possibly unknown probability distribution F with mean v_m.

Players can invest resources in order to influence their chances of winning the prize. Let $x_i \in [0, v_i]$ be the level of investment chosen by player i (we measure the investment in units commensurate with the rent), and let the vector $x = (x_1, ..., x_n)$ collect the choices of all the players. The probability $P_i(x)$ of generic player i winning the prize follows the well-known logit specification originally proposed by Tullock (1980). In particular, we adopt the formulation that features constant returns to scale such that $P_i(x) = \frac{x_i}{x_i + \sum_{j \neq i} x_j}$.[7] We also assume that $P_i(x) = \frac{1}{n}$ if $x = (0, ..., 0)$.

Each player must simultaneously choose how much effort to exert. The optimal level is the one that maximizes the player's expected payoff $\pi_i(x)$:

$$\max_{x_i} \pi_i(x) = \left(\frac{x_i}{x_i + \sum_{j \neq i} x_j} \right) v_i - x_i \tag{1}$$

However, player i neither knows nor can he infer the levels of effort that his opponents will choose. In fact, the optimal investment of generic agent $j \neq i$ depends on the valuation v_j, which is agent j's private information.

It has been shown that there exists an equilibrium solution to problem (1) in the form of a profile of optimal and mutually consistent levels of investment (Cornes & Hartley, 2005; Fey, 2008; Wasser, 2013; Ewerhart, 2014; Einy et al., 2015). However,

[6] Importantly, the problem of private information about asymmetric valuations is analogous to the problem of common and publicly known valuation of the prize paired with private information about the cost of investing resources (Fey, 2008; Ryvkin, 2010; Wasser, 2013; Ryvkin, 2013). As such, our method is applicable to both cases.

[7] A more general formulation of the success function is given by $P_i(x) = \frac{x_i^r}{x_i^r + \sum_{j \neq i} x_j^r}$ where the parameter $r > 0$ measures the returns to scale of a player's investment on effort (e.g., Tullock, 1980; Nti, 1999). The rent-seeking technology shows decreasing returns to scale if $r \in (0, 1)$, constant returns to scale if $r = 1$, and increasing returns to scale if $r > 1$.

the fact that agent valuations are private information generally seems to preclude the possibility of finding the solution analytically. For instance, Fey (2008) shows that the problem is already intractable when the distribution F is uniform and there are only two contestants. Malueg and Yates (2004) and Ewerhart (2010) identify an explicit solution in two different scenarios but only under very specific distributional assumptions.[8] We thus propose a simplified approach to overcome this problem and approximate the optimal solution.

2.1 An approximate solution

The method that we propose postulates that an agent assigns a mean valuation v_m to any of his $n-1$ opponents. Therefore, from agent i's point of view, $x_j = x_j(v_m)$ for any $j \neq i$. Problem (1) thus becomes:

$$\max_{x_i} \pi_i(x) = \left(\frac{x_i(v_i)}{x_i(v_i) + (n-1)x_j(v_m)} \right) v_i - x_i(v_i) \tag{2}$$

Within this simplified framework, we further postulate that agent i behaves as if every other participant in the game was adopting the same behaviour that he adopts. More precisely, player i expects any player $j \neq i$ to attach a valuation v_m to all of his opponents $k \neq j$ (notice that the set of these players includes agent i himself). As such, agent i expects generic agent $j \neq i$ to face the problem:

$$\max_{x_j} \pi_j(x) = \left(\frac{x_j(v_j)}{x_j(v_j) + (n-1)x_k(v_m)} \right) v_j - x_j(v_j) \tag{3}$$

However, agent i does not know $x_j(v_j)$ since v_j is j's private information. Coherently with the premises of the procedure, agent i then assumes $v_j = v_m$ and thus approximates $x_j(v_j)$ with $x_j(v_m)$. Therefore, agent i behaves as if all of his opponents $j \neq i$ were facing the following problem:

[8] We will more explicitly discuss the models by Malueg and Yates (2004) and Ewerhart (2010) in Section 3.1.

$$\max_{x_j} \pi_j(x) = \left(\frac{x_j(v_m)}{x_j(v_m) + (n-1)x_k(v_m)} \right) v_m - x_j(v_m) \tag{4}$$

Agent i thus ascribes to every agent $j \neq i$ the same behaviour that player j would adopt in a rent-seeking game in which all the participants have a homogeneous valuation v_m.[9] As such, agent i postulates that $x_j(v_m) = \left(\frac{n-1}{n^2}\right) v_m$ for any $j \neq i$ where the term $\left(\frac{n-1}{n^2}\right) v_m$ is the Cournot-Nash equilibrium solution of a standard rent-seeking game among n players with homogeneous valuation v_m (Tullock, 1980).

By substituting $x_j(v_m) = \left(\frac{n-1}{n^2}\right) v_m$ in (2), agent i's problem thus becomes:

$$\max_{x_i} \pi_i(x) = \left(\frac{x_i(v_i)}{x_i(v_i) + \left(\frac{n-1}{n}\right)^2 v_m} \right) v_i - x_i(v_i) \tag{5}$$

Solving for x_i, one obtains:

$$x_i(v_i) = \frac{n-1}{n}\sqrt{v_i v_m} - \left(\frac{n-1}{n}\right)^2 v_m \tag{6}$$

where it is clear that $x_i(v_i) > 0$ if and only if $v_i > \left(\frac{n-1}{n}\right)^2 v_m$.

Therefore, we can properly define the approximate solution (which we denote by the superscript AS) as follows:

$$x_i^{AS} = \begin{cases} \frac{n-1}{n}\sqrt{v_i v_m} - \left(\frac{n-1}{n}\right)^2 v_m & \text{if} \quad v_i > \left(\frac{n-1}{n}\right)^2 v_m \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

2.2 Discussion

Before proceeding with the formal analysis of the approximate solution, we discuss its foundations in more detail. The major simplification of the method that we propose is using the expected value v_m as a proxy for the opponents' valuations. This seems to be a valid approach since v_m is indeed the best predictor of a rival's valuation when no further

[9] Notice that agent i thus expects generic agent j to rely on the same approximation of the aggregate level of investment of the opponents (compare the term $(n-1)x_j(v_m)$ in (2) with the term $(n-1)x_k(v_m)$ in (3) and (4)). However, the estimate of the total level of investment of all the players (i.e., the denominator in (2), (3), and (4)) differs because of the two agents' actual or perceived investment).

information about the underlying distribution is available. Moreover, if the agent knew the underlying distribution or even the actual valuations of his opponents, he might still deliberately decide to ignore this information in order to simplify his task. In this respect, our approach is thus in line with the notion of bounded rationality postulating that decision makers may consciously abstain from computing the "optimal solution" of the general problem but instead focus on a simplified environment and rely on rules of thumb that may still lead to satisfactory outcomes (Katsikopoulos, 2011; Katsikopoulos, 2014). The use of v_m as a proxy for $v_{j\neq i}$ allows agent i to focus on Problem (2) rather than on the more general formulation displayed in (1).

The proposed solution also requires an agent to presume that all rivals adopt the same behaviour and thus use v_m to approximate the valuation of their opponents. The agent is thus implicitly assuming that all of his opponents are as rational as he is (at least along this dimension; we will show that this does not hold along other dimensions). To appreciate this point, note that agent i uses the vector of valuations $\left(v_i, \{v_m\}_{j\neq i}\right)$ as model inputs. By the same token, agent i assumes that generic opponent l uses the vector $\left(v_l, \{v_m\}_{j\neq l}\right)$. However, agent i does not know v_l. Consistent with the general approach of the model, he thus approximates v_l with v_m. Therefore, agent i behaves as if l uses the vector $\left(v_m, \{v_m\}_{j\neq l}\right)$. These logical steps lead agent i to further simplify the problem from (2) to (5).

Finally, note that the approximate solution still emerges as the solution of a maximization problem. This feature of utility function maximization is in line with the standard approach used in economic modelling, even when one departs from the neoclassical framework of unbounded rationality (which is for instance the case in behavioural economics; e.g., Camerer & Loewenstein, 2004). As such, if one wants to interpret our model in terms of heuristics, our solution does not exactly fit in the most common definition of a heuristic, i.e., a cognitive shortcut for solving complex problems. Moreover, there is no evidence that agents use the approach that we suggest in actual rent-seeking contests. Our solution can be rather interpreted as a proposal for a possible heuristic that agents may find useful to follow. In this respect, the approach would then more appropriately qualify as an "as if" heuristic, i.e., a formal description of an agent's behaviour that is not meant to fully describe the underlying psychological process but rather the resulting outcome (see Katsikopoulos, 2014, for a very detailed discussion of the differences that exist between the "as if" approach to bounded rationality and the approach that instead relies on the use of psychological heuristics).

We now discuss two potential drawbacks of the proposed solution. First, the functional form defined in (7) is not that simple. The computation of x_i^{AS} on the side of the agent is not instantaneous and still requires some calculations. However, we show that the computation of the approximate solution is far simpler than that of the optimal solution, no matter if this is computed in a framework of private or perfect infomation. We already mentioned that the computation of the optimal solution in a framework of private information is either unfeasible or extremely cumbersome and only applicable to highly specific environments. We show in Section 3 that the computation of the approximate solution that we propose is also much simpler than that of the optimal solution in a context of perfect information.

The second drawback is that agent i chooses his level of investment under the assumption that *all* of his opponents will actively participate in the contest. The agent postulates that $x_j(v_m) = \left(\frac{n-1}{n^2}\right) v_m > 0$ for any j but the investment strategy defined in (7) explicitly allows for the possibility of deciding not to participate in the contest (i.e., the agent plays $x_i^{AS} = 0$). The procedure thus appears to be somehow inconsistent as agent i builds his investment strategy without considering the possibility that some of his opponents (those with the lowest valuations) may actually decide not to participate in the contest. In other words, and consistent with the literature on "level-k reasoning" (Stahl & Wilson, 1995; Crawford et al., 2013), the agent does not internalize the fact that his opponents could be as rational as he is in terms of entry decisions. In particular, a more sophisticated agent should realize that not all of his opponents will actively participate in the game. This in turn implies that the average valuation of those who do invest a positive amount will be larger than v_m. A more sensible approach (but inevitably more complex) would thus require the player to account for this possibility and adjust his investment strategy accordingly. The solution described in (7) fails to do so and thus tends to overestimate the number of participants and underestimate their valuations.[10] Luckily, these two conflicting forces tend to cancel out, thus making the proposed solution effective in approximating the total amount of resources that agent i's rivals invest in rent seeking. In other words, the second term in the denominator of (5) is similar on average to the second term in the denominator of (1). More formally, $\left(\frac{n-1}{n}\right)^2 v_m \simeq \sum_{j \neq i} x_j(v_j)$. We investigate this relationship in more detail in Section 3.

2.3 Comparative statics analysis

Simple exercises of comparative statics highlight how the agent's level of investment x_i^{AS} is influenced by the parameters of the model. In terms of active participation in the game, expression (7) shows that the agent exerts positive effort if and only if his valuation v_i is larger than the threshold $\lambda^{AS} = \left(\frac{n-1}{n}\right)^2 v_m$. Clearly, λ^{AS} is increasing in both n and v_m. However, note that the condition $\lambda^{AS} < v_m$ always holds. This implies that a "strong" player (one with a valuation $v_i \geq v_m$) always invests a strictly positive amount. On the other hand, a "weak" player may decide to participate ($\lambda^{AS} < v_i < v_m$) or abstain ($v_i \leq \lambda^{AS} < v_m$). With all else being equal, a player of type $v_i < v_m$ may thus invest in rent-seeking activities if the game features only a few competitors or instead abstain if the competition looks tougher.[11]

We now focus on the case in which the agent invests a strictly positive amount. It can be verified immediately that x_i^{AS} is increasing and strictly concave in the agent's private valuation v_i. The effect of the average valuation v_m on x_i^{AS} is instead non-monotonic. In particular, x_i^{AS} is increasing in v_m as far as $v_m < \frac{1}{4}\left(\frac{n}{n-1}\right)^2 v_i$. In such a situation, agent i perceives himself as a strong player and thus increments his level of investment to maintain his good chances of winning the contest. In contrast, x_i^{AS}

[10] We thank an anonymous referee for mentioning this point.

[11] More precisely, a player that perceives himself as being very weak (i.e., whose private valuation is much lower than v_m) may refuse to participate even when he faces a single opponent (i.e., $n = 2$). This happens when the condition $v_i \leq \frac{1}{4} v_m$ holds. This is consistent with the analysis by Schoonbeek & Winkel (2006), although it can never happen in models of perfect information such as Nti's (1999) model, where the two players always participate, or Stein's (2002) model, where non-entry of some player can occur only when $n \geq 3$.

is decreasing in v_m for any $v_m > \frac{1}{4}\left(\frac{n}{n-1}\right)^2 v_i$, given that in such a situation the agent expects to face more aggressive opponents and thus adopts a softer strategy. Finally, x_i^{AS} is always decreasing in the number of participants n whenever $v_i \leq v_m$, whereas it might be increasing initially in n when $v_i > v_m$.

The following panel provides a graphical illustration of the effects that v_i (first row), v_m (second row), and n (third row) have on the agent's level of investment x_i^{AS} for different parametrizations of the model.

[**Insert Figure 1 here (see the file at the end of the manuscript)**]

Figure 1. Comparative statics on x_i^{AS}.

3 Benchmark analysis

The ideal benchmark for evaluating the performance of the approximate solution would be the optimal solution of the problem under scrutiny, i.e., the level of investment that maximizes an agent's expected payoff in a setting in which individual valuations are private information. As already noticed, it has been shown that such a solution exists (Cornes & Hartley, 2005; Fey, 2008; Wasser, 2013; Ewerhart, 2014; Einy et al., 2015). However, its analytical form generally cannot be pinned down and this precludes the possibility to perform a full-fledged comparison between the approximate solution that we propose and the optimal solution. To overcome this problem, Wasser (2013) notes that the private information solution can be compared with some alternative benchmarks. For instance, Hurley and Shogren (1998b) use as a benchmark the optimal solution that characterizes a setting with no information, i.e., a contest in which agents do not even know their own valuation. Instead, Malueg and Yates (2004) use as a benchmark the solution of the perfect information case.

In what follows, we adopt a similar approach and evaluate the performance of our approximate solution against two different benchmarks. In Section 3.1, we compare our proposal with the few existing closed-form solutions for rent-seeking contests where agents have private valuations. In Section 3.2, we instead use as a benchmark the optimal solution that emerges in a context of perfect information. All in all, we find that our solution performs well in both comparisons. It thus emerges as a promising candidate for approximating an agent's optimal level of investment in rent-seeking games with private information, as well as a valid shortcut for approximating the solution of the perfect information case.

3.1 Comparison with the private information case

To the best of our knowledge, Malueg and Yates (2004) and Ewerhart (2010) are the only papers that provide a closed-form solution for an agent's optimal level of investment in rent-seeking contests where individual valuations are private and identically distributed.[12] To achieve such a result, both papers rely on strong distributional assumptions. It follows that the optimal solutions that they compute are only valid in the specific contexts that they consider.

[12] In the Introduction we also mentioned the article by Hurley and Shogren (1998b) as another study that finds an explicit solution to the private information case. However, Hurley and Shogren (1998b) consider a situation in which agents' valuations are drawn from different distributions. Therefore, their framework does not match the assumptions of our model.

A comparison with Malueg and Yates (2004)

Malueg and Yates (2004) study a rent-seeking contest with two participants whose valuations are drawn from a discrete distribution. In particular, they consider a setting in which each agent's valuation can be either low or high with equal probability. More formally, $n = 2$, $v_i \in \{v_L, v_H\}$ and $F \sim U$. Malueg and Yates (2004) show that in such a context the optimal solution is given by:

$$x_i^{MY04} = \left(\frac{1-\sigma}{\left[\left(\frac{v_L}{v_H} \right)^{-\frac{r}{2}} + \left(\frac{v_L}{v_H} \right)^{\frac{r}{2}} \right]^2} + \frac{\sigma}{4} \right) r v_i \quad \text{for any } i \in \{L, H\} \tag{8}$$

where the parameter $r > 0$ reflects the returns to scale of a player's investment (see footnote 7 in this paper) and $\sigma \in [0, 1]$ measures the degree of correlation among agents' valuations with $\sigma = 0$ corresponding to perfect negative correlation and $\sigma = 1$ corresponding to perfect positive correlation. In our framework, returns to scale are constant (i.e., $r = 1$) and agents' valuations are independent (i.e., $\sigma = \frac{1}{2}$). As such, the optimal solution defined in (8) becomes:

$$x_i^{MY04} = \left(\frac{0.5}{\left[\left(\frac{v_L}{v_H} \right)^{-\frac{1}{2}} + \left(\frac{v_L}{v_H} \right)^{\frac{1}{2}} \right]^2} + 0.125 \right) v_i \quad \text{for any } i \in \{L, H\} \tag{9}$$

Given that $v_m = \frac{v_L + v_H}{2}$, our approximate solution (see (7)) takes the following form:

$$x_i^{AS} = \frac{1}{2} \sqrt{\left(\frac{v_L + v_H}{2} \right) v_i} - \frac{1}{4} \left(\frac{v_L + v_H}{2} \right) \quad \text{for any } i \in \{L, H\} \tag{10}$$

The following figures compare the two solutions when $v_L = 100$ and $v_H \in [100, 200]$. Figure 2.a depicts x_i^{MY04} and x_i^{AS} for an agent whose valuation is low (i.e., $v_i = v_L$). Figure 2.b plots x_i^{MY04} and x_i^{AS} for an agent whose valuation is high (i.e., $v_i = v_H$).[13]

[13] In both cases, the pattern of the approximate solution is obviously consistent with the comparative statics analysis discussed in Section 2.3. In particular, x_L^{AS} is decreasing in v_H since $v_m = \frac{v_L + v_H}{2}$ is increasing in v_H and the condition $v_m > \frac{1}{4} \left(\frac{n}{n-1} \right)^2 v_i$ (i.e., $v_m > v_L$) holds. Notice also that both types

The approximate solution x_i^{AS} closely follows the optimal solution x_i^{MY04} in both cases. Indeed, the difference between the two solutions is smaller than 5% along the entire interval. If instead of comparing the two levels of investments one compares the expected payoffs that stem from investing x_i^{AS} rather than x_i^{MY04}, the difference between the approximate and the exact solution often becomes negligible.[14]

[Insert Figure 2.a here] **[Insert Figure 2.b here]**

Figure 2: x_i^{MY04} and x_i^{AS} for an agent with $v_i = v_L$ (Fig. 2.a) and $v_i = v_H$ (Fig. 2.b).

A comparison with Ewerhart (2010)

Ewerhart (2010) provides the unique closed-form solution for the case in which agents' valuations are drawn from a continuous probability distribution. More precisely, Ewerhart (2010) considers a scenario in which the valuations of the two participants are distributed according to the following cumulative distribution function:

$$F(v_i \mid v_{\min}, v_{\max}) = \frac{\ln\left(\sqrt{\frac{(v_{\max}-v_{\min})^2}{4} + 2\left(v_{\max}+v_{\min}\right)v_i} - \frac{v_{\max}+v_{\min}}{2}\right) - \ln\left(v_{\min}\right)}{\ln\left(v_{\max}\right) - \ln\left(v_{\min}\right)} \tag{11}$$

of individuals invest a strictly positive amount since the condition that determines entry in (7) becomes $v_i > \frac{1}{8}\left(v_L + v_H\right)$ and thus holds for any $v_i \in \{v_L, v_H\}$. More in general, and contrary to the model of Malueg and Yates (2004), our model indicates that an agent with low valuation should not enter the contest whenever $v_L < \frac{1}{7} v_H$.

[14] For instance, if $v_H = 150$ then $\pi_L^{MY04} = 20.5$ and $\pi_L^{AS} = 20.5$ (i.e., exactly the same value), whereas $\pi_H^{MY04} = 45.75$ and $\pi_H^{AS} = 45.747$. If instead $v_H = 200$ then $\pi_L^{MY04} = 18.056$ and $\pi_L^{AS} = 17.593$, whereas $\pi_H^{MY04} = 40.278$ and $\pi_H^{AS} = 39.776$. Consistent with the formulation in (1), the expected payoff of investing x_i^{MY04} is computed as $\pi_i^{MY04} = \frac{1}{2}\left(\frac{x_i^{MY04}}{x_i^{MY04}+x_L^{MY04}} + \frac{x_i^{MY04}}{x_i^{MY04}+x_H^{MY04}}\right)v_i - x_i^{MY04}$ for $i \in \{L, H\}$. Similarly, the expected payoff of investing π_i^{AS} is computed as $\pi_i^{AS} = \frac{1}{2}\left(\frac{x_i^{AS}}{x_i^{AS}+x_L^{MY04}} + \frac{x_i^{AS}}{x_i^{AS}+x_H^{MY04}}\right)v_i - x_i^{AS}$ for $i \in \{L, H\}$.

He then shows that in the symmetric Bayesian equilibrium of the contest, each agent $i \in \{1, 2\}$ invests the amount

$$x_i^{E10} = \rho \left(\sqrt{A^2 + v_i} - B \right) \tag{12}$$

where

$$B = \left(\frac{1}{2} \sqrt{\frac{v_{\max} + v_{\min}}{2}} \right), \ A = \left(\frac{v_{\max} - v_{\min}}{v_{\max} + v_{\min}} B \right), \text{ and } \rho = \left(\frac{2A}{\ln(v_{\max}/v_{\min})} \right). \tag{13}$$

Therefore, the functional form of x_i^{E10} is rather complicated. Ewerhart (2010) then presents an explicit example in which $v_{\min} = 8$ and $v_{\max} = 24$. Substituting these values in (13) and (12), the optimal level of investment of agent $i \in \{1, 2\}$ becomes

$$x_i^{E10} = \frac{2}{\ln 3} \left(\sqrt{v_i + 1} - 2 \right) \tag{14}$$

Given that $v_m = \int_8^{24} v_i f(v_i \mid v_{\min}, v_{\max}) dv_i$, where $f(v_i \mid v_{\min}, v_{\max}) = \frac{\partial F(v_i \mid v_{\min}, v_{\max})}{\partial v_i}$ is the density, one obtains that $v_m = 13.923$. It follows that our approximate solution is given by

$$x_i^{AS} = \frac{1}{2} \sqrt{13.923 v_i} - \left(\frac{1}{2} \right)^2 13.923 \tag{15}$$

Figure 3 plots the two solutions, x_i^{E10} and x_i^{AS}. The function x_i^{AS} again provides a good approximation of the optimal solution x_i^{E10}. The difference between the approximate and the exact solution is always below 5%. The difference between the two methods becomes negligible if one considers the agent's expected payoffs.[15]

[15] For instance, if $v_i = 8$ then $\pi_i^{E10} = 1.1321$ and $\pi_i^{AS} = 1.1320$; if $v_i = 16$ then $\pi_i^{E10} = 4.9268$ and $\pi_i^{AS} = 4.9249$; if $v_i = 24$ then $\pi_i^{E10} = 9.6809$ and $\pi_i^{AS} = 9.6765$. The expected payoff of investing x_i^{E10} is computed as $\pi_i^{E10} = \left(\int_8^{24} \frac{x_i^{E10}}{x_i^{E10} + \frac{2}{\ln 3}\left(\sqrt{v_j+1} - 2\right)} f(v_j \mid v_{\min}, v_{\max}) dv_j \right) v_i - x_i^{E10}$. Similarly, the expected payoff of investing x_i^{AS} is computed as $\pi_i^{AS} = \left(\int_8^{24} \frac{x_i^{AS}}{x_i^{AS} + \frac{2}{\ln 3}\left(\sqrt{v_j+1} - 2\right)} f(v_j \mid v_{\min}, v_{\max}) dv_j \right) v_i - x_i^{AS}$.

[Insert Figure 3 here (see the file at the end of the manuscript)]

Figure 3: A comparison between the two solutions x_i^{E10} and x_i^{AS}.

3.2 Comparison with the perfect information case

As an alternative benchmark for evaluating the performance of the approximate solution, we use the case of perfect information. Stein (2002) examines a rent-seeking game among $n \geq 2$ players with heterogeneous and publicly known valuations and presents explicit solutions for the case of constant returns to scale success function. Therefore, the only difference between Stein's framework and our framework lies in the different information structure that agents can rely on. Stein finds that the optimal strategy of generic agent $i \in N$ takes the following form (the superscript PI indicates the case of perfect information):

$$x_i^{PI} = \begin{cases} \frac{(p-1)\Phi_p}{p}\left[1 - \frac{(p-1)\Phi_p}{pv_i}\right] & \text{if } \ i \leq p \\ 0 \ \ \text{otherwise} \end{cases} \tag{16}$$

where $p \in \{1, ..., n\}$ is the largest number for which the condition $v_p > \frac{(p-1)}{p}\Phi_p$ holds (players are ordered in terms of their valuations such that $v_1 \geq v_2 \geq ...v_n > 0$), and $\Phi_p = \left[\frac{1}{p}\sum_{i\leq p}\frac{1}{v_i}\right]^{-1}$ is the harmonic mean of the first p values of $\{v_i\}_{i\in N}$. The computation of x_i^{PI} is thus rather demanding and far more complex than that of x_i^{AS} (see expression (7)). Obviously, the two solutions x_i^{PI} and x_i^{AS} usually differ.[16]

[16]The two approaches lead to the same solution only when $v_i = v_m$ for all $i \in N$, in which case both solutions collapse to the standard solution of a rent-seeking game with symmetric valuation (Tullock, 1980). More formally, if $v_i = v_m$ for all $i \in N$ then $x_i^{AS} = x_i^{PI} = \left(\frac{n-1}{n^2}\right) v_m$.

The following example illustrates this point.

Example 1 *Consider a rent-seeking contest with* 4 *players and let individual valuations be drawn from a uniform distribution defined on the interval* $[90, 100]$. *Therefore,* $v_m = 100$. *Let* $v_1 = 101$, $v_2 = 106$, $v_3 = 93$, *and* $v_4 = 98$. *Now, consider the situation of agent* 1. *If information is perfect, the agent plays according to (16). He thus invests the amount* $x_1^{PI} = 19.567$ *and his expected payoff equals* $\pi_1^{PI} = 6.975$. *If valuations are instead private and the agent relies on the proposed method, he plays according to (7). He thus invests the amount* $x_1^{AS} = 19.124$, *and his expected payoff equals* $\pi_1^{AS} = 6.973$.[17]

In this example, the differences between the approximate solution and the optimal solution are minimal in terms of both of the level of investment (i.e., x_1^{AS} vs. x_1^{PI}) and expected payoff (i.e., π_1^H vs. π_1^{PI}). This similarity appears to be a robust feature of the approximate solution. Indeed, the differences between the optimal and the approximate solution turn out to be extremely small for a wide range of possible parametrizations of the model.

Table 1 reports the results of a series of simulations aimed at addressing this issue. We considered three possible functional forms for the underlying distribution of agent valuations: a uniform distribution ($F \sim U(a, b)$), normal distribution ($F \sim N(\mu, \sigma^2)$), and beta distribution ($F \sim B(\alpha, \beta)$). In the cases of the uniform and the normal distribution, we simulated six alternative scenarios that differ in terms of the number of participants ($n = 2$, $n = 4$, and $n = 10$) or the variance of agent valuations (low vs. high). In the case of the beta distribution, we simulated nine scenarios that differ in terms of the number of participants (again $n = 2$, $n = 4$, and $n = 10$) or the relation between the mean and the median of the distribution (=, >, or <). The mean of F always equals $v_m = 100$ across all 21 scenarios.

[17] The expected payoff under perfect information is computed as $\pi_i^{PI} = \left(\frac{x_i^{PI}}{x_i^{PI} + \sum_{j \neq i} x_j^{PI}}\right) v_i - x_i^{PI}$, whereas the expected payoff that stems from using the approximate solution is computed as $\pi_i^{AS} = \left(\frac{x_i^{AS}}{x_i^{AS} + \sum_{j \neq i} x_j^{PI}}\right) v_i - x_i^{AS}$.

(1) F	(2) n	(3) $\bar{x}_1^{AS}$	(4) $\bar{x}_1^{PI}$	(5) $\boldsymbol{\Delta\bar{x}_1^{AS}}$	(6) $\bar{\pi}_1^{AS}$	(7) $\bar{\pi}_1^{PI}$	(8) $\boldsymbol{\Delta\bar{\pi}_1^{AS}}$	(9) $\bar{X}_{-1}^{AS}$	(10) $\bar{X}_{-1}^{PI}$	(11) $\Delta\bar{X}_{-1}^{AS}$
$U(90,110)$	2	24.94	24.92	**0.06**	24.75	24.76	**-0.01**	25.00	25.23	-0.90
$U(90,110)$	4	18.61	18.54	**0.27**	6.35	6.36	**-0.40**	56.25	56.28	-0.05
$U(90,110)$	10	8.52	8.38	**1.67**	1.12	1.16	**-2.90**	81.00	81.11	-0.13
$U(50,150)$	2	25.01	24.41	**2.45**	28.86	28.91	**-0.17**	25.00	24.38	2.53
$U(50,150)$	4	17.66	17.93	**-1.46**	9.89	10.05	**-1.51**	56.25	54.78	2.68
$U(50,150)$	10	9.77	9.31	**4.99**	2.71	2.80	**-2.94**	81.00	87.72	-7.67
$N(100,10)$	2	24.84	24.78	**0.26**	24.88	24.89	**-0.01**	25.00	25.16	-0.64
$N(100,10)$	4	18.40	18.31	**0.49**	7.06	7.12	**-0.85**	56.25	55.66	1.07
$N(100,10)$	10	9.27	9.34	**-0.72**	1.71	1.77	**-3.40**	81.00	80.79	0.25
$N(100,50)$	2	25.10	23.65	**6.11**	33.25	33.79	**-1.60**	25.00	23.62	5.85
$N(100,50)$	4	19.62	19.83	**-1.06**	15.75	16.14	**-2.41**	56.25	54.27	3.64
$N(100,50)$	10	12.60	11.29	**11.61**	4.77	4.85	**-1.67**	81.00	94.83	-14.59
$B(5,5)$	2	23.17	22.45	**3.19**	26.78	26.90	**-0.45**	25.00	23.96	4.33
$B(5,5)$	4	18.05	17.86	**1.08**	10.45	10.60	**-1.37**	56.25	55.36	1.60
$B(5,5)$	10	11.06	10.35	**6.85**	3.30	3.39	**-2.75**	81.00	87.07	-6.97
$B(2,5)$	2	23.82	22.89	**4.06**	35.40	36.08	**-1.88**	25.00	21.83	14.50
$B(2,5)$	4	20.52	19.99	**2.62**	17.97	18.27	**-1.65**	56.25	54.85	2.55
$B(2,5)$	10	13.48	11.97	**12.60**	6.51	6.65	**-2.07**	81.00	97.44	-16.87
$B(5,3)$	2	25.32	24.95	**1.46**	26.86	26.90	**-0.16**	25.00	25.84	-3.23
$B(5,3)$	4	19.38	19.03	**1.86**	9.88	10.03	**-1.50**	56.25	55.27	1.78
$B(5,3)$	10	9.99	9.30	**7.50**	2.92	3.04	**-3.75**	81.00	86.63	-6.50

Table 1: A comparison between the approximate solution and the optimal solution.

For each scenario, we simulated 100 different realizations of the vector of agents' valuations, $v = (v_1, ..., v_n)$. For each realization of v, we computed agent 1's level of investment and expected payoff under the approximate solution (x_1^{AS} and π_1^H, respectively). Columns 3 and 6 in the table report the average values of these statistics across the 100 realizations ($\bar{x}_1^{AS}$ and $\bar{\pi}_1^{AS}$). Similarly, we computed the optimal level of investment and the expected payoff of agent 1 in a framework of perfect information (x_1^{PI} and π_1^{PI}). Table 1 reports the average values in columns 4 and 7 ($\bar{x}_1^{PI}$ and $\bar{\pi}_1^{PI}$).

We then calculated the *percentage* differences between the results of the two methods. Column 5 shows the difference between the two levels of investment, $\Delta\bar{x}_1^{AS} = \left(\frac{\bar{x}_1^{AS} - \bar{x}_1^{PI}}{\bar{x}_1^{PI}}\right) \cdot 100$, whereas column 8 shows the difference in terms of the expected payoff, $\Delta\bar{\pi}_1^{AS} = \left(\frac{\bar{\pi}_1^{AS} - \bar{\pi}_1^{PI}}{\bar{\pi}_1^{PI}}\right) \cdot 100$. The table shows that the solution that we propose performs remarkably well in approximating the optimal solution that characterizes the perfect information case. In terms of levels of investment ($\Delta\bar{x}_1^{AS}$ in column 5), the two methods differ by less than 5% in the vast majority of cases. In terms of expected payoff ($\Delta\bar{\pi}_1^{AS}$ in column 8, arguably the most relevant comparison), the two methods always differ by less than 5% and often by less than 2%. The differences appear to be particularly small when the contest features only few participants.[18]

[18] Indeed, Table 1 seems to suggest that as n increases the quality of the approximation tends to decrease. To address this issue more carefully, Table 1.1 below reports the results of two further sets of simulations that feature many participants ($n = 30$ and $n = 50$). In particular, Table 1.1 shows the differences between the approximate solution and the optimal solution in terms of the agent's expected payoff ($\Delta\bar{\pi}_1^{AS}$, as reported in column 8 in Table 1).

(1) F	(2) n	(8) $\mathbf{\Delta\bar{x}_1^{AS}}$	(1) F	(2) n	(8) $\mathbf{\Delta\bar{x}_1^{AS}}$	(1) F	(2) n	(8) $\mathbf{\Delta\bar{x}_1^{AS}}$
$U(90, 110)$	30	**-4.10**	$N(100, 10)$	30	**-4.85**	$B(5, 5)$	30	**-4.70**
$U(90, 110)$	50	**-5.04**	$N(100, 10)$	50	**-5.40**	$B(5, 5)$	50	**-8.73**
$U(50, 150)$	30	**-5.74**	$N(100, 50)$	30	**-5.51**	$B(2, 5)$	30	**-4.45**
$U(50, 150)$	50	**-15.36**	$N(100, 50)$	50	**-7.93**	$B(2, 5)$	50	**-9.81**
						$B(5, 3)$	30	**-5.90**
						$B(5, 3)$	50	**-9.34**

Table 1.1. The differences in expected payoffs when the number of participants is large.

The numerical results confirm that the performance of the approximate solution tends to decrease as n gets large. This is a limitation of the procedure to be kept in mind. On the other hand, it is true that rent-seeking games that emerge in practical situations usually involve only a handful of participants.

Table 1 also shows why the performance of the approximate solution is so precise, despite the potential drawbacks in its foundations (see Section 2.2). We already noted that the method tends to overestimate the number of active participants while simultaneously underestimating their valuations. The last three columns in the table show that these effects do little harm and tend to cancel out. Column 9 reports $\bar{X}_{-1}^{AS} = \left(\frac{n-1}{n}\right)^2 v_m$, which is the presumed sum of investments of agent 1's opponents (all $n-1$ opponents are assumed to invest a positive amount). Column 10 reports $\bar{X}_{-1}^{PI} = \sum_{j \neq 1} x_j^{PI}(v_j)$, the sum of the actual investments of agent 1's opponents in a framework of perfect information (in this case, some of the x_j^{PI} may be zero since participants with low valuations may decide not to participate). Finally, column 11 shows the percentage difference between the two statistics, $\Delta \bar{X}_{-1}^{AS} = \left(\frac{\bar{X}_{-1}^{AS} - \bar{X}_{-1}^{PI}}{\bar{X}_{-1}^{PI}}\right) \cdot 100$. Again, the difference is usually small. This indicates that the simplified problem that the agent solves (see problem (5) in Section 2.1) is similar to the original problem (see problem (1)).

3.3 The trade-off between simplicity and performance

The previous sections showed that the proposed solution performs well in approximating the optimal solution, both in a context of private information and of perfect information. Moreover, the functional form of the approximate solution (see expression (7)) is clearly simpler than the one of the optimal solution, no matter if this is computed in the private information case (see (9), (12), and (13)) or in the perfect information case (see (16)).

However, since the computation of x_i^{AS} is not trivial, one may be tempted to rely on other functional forms that can further simplify the task of finding an approximate solution. In this section, we briefly discuss an alternative proposal. This alternative method still postulates that an agent assigns the mean valuation v_m to any of his $n-1$ rivals. However, the process through which the player approximates the investment $x_j(v_m)$ of generic rival j is now different. The approach conjectures that the agent is able to understand the two main forces that shape $x_j(v_m)$. First, $x_j(v_m)$ increases in the agent's (presumed) valuation, v_m. Second, $x_j(v_m)$ decreases in the number of competitors that opponent j faces, $n-1$. The simplest formulation that captures these two forces is given by $\tilde{x}_j(v_m) = \frac{v_m}{n-1}$.

Substituting $\tilde{x}_j(v_m)$ in Problem (2) and solving for x_i, one obtains that according to this alternative approximate solution (which we indicate by superscript $AS2$), the agent should invest the amount:

$$x_i^{AS2} = \begin{cases} \sqrt{v_i v_m} - v_m & \text{if } v_i > v_m \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

The functional form of x_i^{AS2} is indeed simpler than that of the original approach x_i^{AS}. Moreover, x_i^{AS2} displays similar qualitative features to x_i^{AS} in terms of comparative statics. In particular, x_i^{AS2} is strictly increasing in v_i and strictly concave in both v_i and v_m.[19] However, the performance of this alternative functional form turns out to be extremely disappointing.

The following figure shows how the solution x_i^{AS2} performs in approximating the optimal solution identified by Ewerhart (2010) and discussed in Section 3.1. The figure can thus be directly compared with Figure 3. The fact that the alternative solution x_i^{AS2} provides a much worst (and actually totally unreliable) approximation of the optimal solution than the original proposal x_i^{AS} is evident.[20]

[Insert Figure 4 here (see the file at the end of the manuscript)]

Figure 4: A comparison between the two solutions x_i^{E10} and x_i^{AS2}.

[19] However, in contrast to x_i^{AS}, x_i^{AS2} is independent of n.

[20] The same qualitative conclusion holds if one compares how x_i^{AS} and x_i^{AS2} performs in approximating the optimal solution x_i^{MY04} identified by Malueg and Yates (2004). In particular, the solution x_i^{AS2} underestimates x_i^{MY04} by an amount that ranges between 50% and 100%.

Table 2 below instead compares the level of investment and the expected payoff of an agent that uses the alternative approximate solution (x_i^{AS2} and π_i^{AS2}, respectively) with those resulting in a context of perfect information (x_i^{PI} and π_i^{PI}). To make this comparison, we used the same simulated data that we used in Section 3.2 (note that columns 4 and 7 display the same values in both Table 1 and Table 2). The results are thus comparable between the two tables.[21]

(1) F	(2) n	(3) $\bar{x}_1^{AS2}$	(4) $\bar{x}_1^{PI}$	(5) $\boldsymbol{\Delta\bar{x}_1^{AS2}}$	(6) $\bar{\pi}_1^{AS2}$	(7) $\bar{\pi}_1^{PI}$	(8) $\boldsymbol{\Delta\bar{\pi}_1^{AS2}}$
$U(90,110)$	2	1.03	24.92	**-95.85**	2.82	24.76	**-88.61**
$U(90,110)$	4	1.17	18.54	**-93.69**	0.99	6.36	**-84.38**
$U(90,110)$	10	0.96	8.38	**-88.49**	0.32	1.16	**-72.31**
$U(50,150)$	2	6.15	24.41	**-74.82**	15.64	28.91	**-45.90**
$U(50,150)$	4	5.30	17.93	**-70.44**	5.90	10.05	**-41.23**
$U(50,150)$	10	5.16	9.31	**-44.52**	2.31	2.80	**-17.29**

Table 2: A comparison between the alternative solution and the optimal solution.

The percentage differences are reported in columns 5 and 8. The performance of the alternative method in approximating the optimal solution is extremely unsatisfactory in terms of both the level of investment and the expected payoff. The alternative solution often falls short of the optimal solution by more than 70%.

All in all, the results of this section further validate the original approximate solution proposed in (7). Even though the computation of x_i^{AS} is slightly more complex than the computation of x_i^{AS2}, this negative aspect is more than compensated by the fact that the solution leads to estimates that are more precise by more than an order of magnitude.

[21] For the sake of space, Table 2 only reports the scenarios in which F is uniform. The results remain equally disappointing in the scenarios that use the normal or beta distributions.

4 Conclusions

As of today, a generic closed-form solution for an agent's optimal level of investment in rent-seeking contests with private information does not exist. In this paper, we proposed a method to overcome this problem and compute an approximate solution. In addition to the knowledge of one's own type and the number of participants, the method only requires knowledge of the mean of the distribution of valuations. Focusing on contests with constant returns to scale, we obtained a relatively simple closed-form solution and showed that such a solution performs well in approximating the few ad hoc optimal solutions that the literature has so far identified.

We thus feel that the method can be useful in a variety of practical situations. First, it can be used by agents who occasionally find themselves involved in a rent-seeking contest and wants to find a "near-optimal" level of investment. One example is a litigant who is involved in a dispute over a contested good and must decide how much to spend on legal expenses. More in line with typical OR problems, attackers and defenders of single or multiple assets (such as in computer security) may rely on the proposed approximate solution to choose their level of commitment to various sites. The method can also be used by some external agents that want to forecast some of the features of the final outcome (say, the identity of the winner or the amount of rent dissipation). This for instance could be the case of the organizer of the contest who must decide upon the institutional details of the competition. Similarly, the method may appeal to researchers who may want to study participants' behaviour and the properties of the resulting outcome. In the course of the analysis, we showed that the solution that we propose also provides a good approximation of the optimal solution that characterizes the perfect information case. Therefore, within such a framework, the method can also be used for identifying an almost optimal solution at a lower computational cost.

More in general, our framework allows to compute an approximate solution when agents can only rely (or decide to only rely) on some very general summary statistic, such as the mean or expected valuation. Knowledge about this statistic could in turn stem from different sources. For instance, the agent may use the mean value that emerged in previous contests with similar prizes as a proxy. Alternatively, some external player (perhaps the organizer of the contest, an authoritative expert, a think-tank, or a governmental agency) may publicly provide a valuation of the prize that thus becomes a natural focal point that participants may use to attribute a valuation to their rivals.[22] This last point opens interesting paths for future research. The strategic release of selected information by some external agent (e.g., value assessment of a certain asset by a rating agency) may affect the way agents behave and thus influence some of the relevant outcomes of the contest, such as the number of active participants or their levels of investment.

[22] Concerning this last point, many countries recently set up specific agencies (both at the national and local levels) to provide so-called standard costs (i.e., mean valuation) for the supply of goods and services to the public sector that are assigned through procurement auctions.

Appendix: Rent dissipation

Here we discuss additional properties of the proposed heuristic in more detail. We focus in particular on the notion of rent dissipation. Rent dissipation provides a measure of the resources that agents waste in rent-seeking activities and has thus always played a prominent role in evaluating the efficiency of rent-seeking contests (Tullock, 1980; Nitzan, 1994; Hurley, 1998; Stein, 2002; Congleton et al., 2008; Alcalde & Dahm, 2010). At the individual level, rent dissipation is defined as the fraction of an agent's valuation that is invested in rent-seeking activities. Given the approximate solution x_i^{AS} (see expression (7) in the main text), a player with valuation v_i thus dissipates the amount:

$$RD_i^{AS} = \begin{cases} \frac{n-1}{n}\sqrt{\frac{v_m}{v_i}} - \left(\frac{n-1}{n}\right)^2 \frac{v_m}{v_i} & \text{if } \; v_i > \left(\frac{n-1}{n}\right)^2 v_m \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

This expression subsumes the results of the standard model with homogeneous valuations (see also footnote 11). When $v_i = v_m$, the expression simplifies to $RD_i^{AS} = \frac{(n-1)}{n^2}$. This well-known constant ratio does not depend on v_i (Tullock, 1980). However, expression (18) shows that RD_i^{AS} does depend on v_i in general. In particular, rent dissipation is increasing in the agent's valuation for $v_i < \tilde{v}_i$ and decreasing for $v_i > \tilde{v}_i$, where $\tilde{v}_i = \left(\frac{n-1}{n}\right)^2 4v_m$. Similarly, when holding v_i fixed, rent dissipation is a concave function of v_m that reaches its maximum at $\tilde{v}_m = \frac{1}{4}\left(\frac{n}{n-1}\right)^2 v_i$. The shape of the RD_i^{AS} function is driven by the behaviour of x_i^{AS}, which is strictly concave in v_i and v_m, as shown in the main text.

The notion of rent dissipation can also be related with the concept of an agent's relative resolve, as introduced by Hurley and Shogren (1998a, 1998b). The relative resolve of agent i with respect to a generic agent j is defined as $\rho_i = \sqrt{\frac{v_i}{v_j}}$ and thus provides a measure of the relative strength of a player. In particular, agent i is stronger than j when $\rho_i > 1$ (and weaker when $\rho_i < 1$). In the context of our heuristic approach, we can thus define $\rho_i = \sqrt{\frac{v_i}{v_m}}$ as the relative resolve of agent i with respect to his "representative rival" of type v_m. Rearranging expression (18) yields:

$$RD_i^{AS} = \begin{cases} \left(\frac{n-1}{n}\right)\frac{1}{\rho_i} - \left(\frac{n-1}{n}\right)^2 \frac{1}{\rho_i^2} & \text{if } v_i > \left(\frac{n-1}{n}\right)^2 v_m \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

Whenever positive, the amount of rent dissipation is thus first increasing and then decreasing in ρ_i. Rent dissipation reaches its maximum for $\tilde{\rho}_i = \frac{1}{n}(2n-2)$, in which case $RD_i^{AS}(\tilde{\rho}_i) = \frac{1}{4}$. Note that this maximum is a constant that does not depend on the number of participants n. Therefore, an agent that plays according to the heuristic will never dissipate more than 25% of his valuation, no matter the value of his relative resolve (and thus the values of v_i and v_m) or the number of participants in the game.

Figure 5 illustrates this peculiar result. The figure depicts RD_i^{AS} as a function of ρ_i in three different contests with $n = 2$, $n = 4$, and $n = 100$. In all cases, the participation constraint $v_i > \left(\frac{n-1}{n}\right)^2 v_m$ implies that the agent invests a positive amount if and only if $\rho_i > \left(\frac{n-1}{n}\right)$.

[Insert Figure 5 here (see the file at the end of the manuscript)]

Figure 5: Rent dissipation with x_i^{AS}.

Consistent with the approach commonly adopted in the literature (e.g., Hurley & Shogren, 1998a, 1998b; Stein, 2002), we define rent dissipation at the aggregate level as the total expenditures by all the players.[23] More formally, $RD^{AS} = \sum_i x_i^{AS}$ (note that

[23] Whenever agent valuations are heterogeneous, the sum of individual rent dissipations (i.e., $\sum_i RD_i$) makes no sense. Each RD_i is in fact computed with respect to the agent's specific valuation v_i.

we compute RD^{AS} under the assumption that all of the participants play according to the proposed approximate solution). In our framework, one must then consider that not all the players necessarily invest a positive amount. We thus define the set of active players as $\tilde{N} = \left\{ i \in N \mid v_i > \left(\frac{n-1}{n}\right)^2 v_m \right\}$; i.e., those agents that play $x_i^{AS} > 0$. Therefore, $\tilde{N} \subseteq N$, or equivalently, $\tilde{n} \leq n$. Rent dissipation at the aggregate level is then given by:

$$RD^{AS} = \sum\nolimits_{i \in \tilde{N}} \left[\left(\frac{n-1}{n} \right) \sqrt{v_i v_m} \right] - \tilde{n} \left(\frac{n-1}{n} \right)^2 v_m \tag{20}$$

Aggregate dissipation is thus weakly increasing and weakly concave in any individual valuation v_i with $i \in N$. The "weakness" of these relations stems from the fact that a small increase in the valuation of an agent that does not participate may still not be enough to convince him to actually invest a positive amount. If this is the case, then RD^{AS} obviously would not change. In contrast, the possible positive effect on total rent dissipation of an increase in an individual valuation can flow through two channels. First, a higher v_i pushes up the optimal amount $x_i^{AS} > 0$ of an agent that would have invested anyway. Second, a higher v_i may convince a player to enter the contest and thus change his investment strategy from $x_i^{AS} = 0$ to $x_i^{AS} > 0$.

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