Amplitude Analysis of the Decays $\eta' \to \pi^+\pi^-\pi^0$ and $\eta' \to \pi^0\pi^0\pi^0$
The decays $\eta' \to \pi^+\pi^-\pi^0$ are isospin-violating processes. Because the electromagnetic contribution is strongly suppressed [1,2], they are induced dominantly by the strong interaction via the explicit breaking of chiral symmetry by the $d-u$ quark mass difference. In recent years, there has been considerable interest in these decays because they allow the determination of the light quark mass difference using the ratios of decay widths, $r_{\pi\pi} = B(\eta' \to \pi^+\pi^-\pi^0)/B(\eta' \to \pi^+\pi^-\eta)$ and $r_0 = B(\eta' \to \pi^0\pi^0\pi^0)/B(\eta' \to \pi^0\pi^0\eta)$ [3,4]. Within the framework of chiral effective field theory combined with a relativistic coupled-channel approach, Ref. [5] predicts that the $\eta' \to \rho^+\pi^-$ P-wave contribution should be large for $\eta' \to \pi^+\pi^-\pi^0$. For the channel with three neutral pions, $\eta' \to \pi^0\pi^0\pi^0$, the P-wave contribution in two-body rescattering is forbidden by Bose symmetry. In general, the final-state interaction is expected to be very important because it was already found to be essential to explain the decay width of $\eta \to \pi\pi\pi$ [6,7]. In the case of $\eta'$ decays, the final-state interaction is further enhanced due to the presence of nearby resonances and is expected to strongly affect the values of the branching fractions and the Dalitz plot distributions.

So far, there is no direct experimental evidence of an intermediate $\rho^+$ contribution to the decay $\eta' \to \pi^+\pi^-\pi^0$. In 2009, the CLEO-c experiment [8] reported the first observation of $\eta' \to \pi^+\pi^-\pi^0$ with 20.2$^{+6.1}_{-4.8}$ events, corresponding to a branching fraction of $(37 \pm 11) \times 10^{-4}$, and a Dalitz plot consistent with a flat distribution. Recently the decay was also observed by the BESIII experiment [9] with a branching fraction consistent with the CLEO-c result; however, no Dalitz plot analysis was presented. Interest in the decay channel $\eta' \to \pi^0\pi^0\pi^0$ stems from the observed $4\sigma$ discrepancy between the recent branching fraction measurement by BESIII [(35.6$^{+4.0}_{-3.4}) \times 10^{-3}$] [9] and those from all previous experiments [10–12]. The BESIII result indicates a value for the ratio $r_0$ that is two times larger than previous experiments. Furthermore, the recent determination of the Dalitz plot slope parameter for $\eta' \to \pi^0\pi^0\pi^0$ decay gave $\alpha = -0.687 \pm 0.061$ [13], which deviates significantly from that for the phase-space distribution ($\alpha = 0$). This implies that final-state interactions play an essential role. In this Letter, we present an amplitude analysis combining $\eta' \to \pi^+\pi^-\pi^0$ and $\eta' \to \pi^0\pi^0\pi^0$ events originating from $J/\psi$ radiative decays using $1.31 \times 10^9 J/\psi$ events [14,15] accumulated by the BESIII detector, which is described in detail in Ref. [16].

For a $J/\psi \to \gamma\eta'$ with $\eta' \to \pi^+\pi^-\pi^0$ candidate event, two tracks with opposite charge and at least three photon candidates are required. The selection criteria for charged tracks and photon candidates are the same as those in Ref. [13]. Because the radiative photon from the $J/\psi$ is always more energetic than the photons from the $\pi^0$ decays, the photon candidate with the maximum energy in the event is taken as the radiative one. For each $\pi^+\pi^-\gamma\gamma\gamma$ combination, a six-constraint (6C) kinematic fit is performed, and the $\chi^2_{6C}$ is required to be less than 25. The fit enforces energy-momentum conservation and constrains the invariant masses of the other photon pair and $\pi^+\pi^-\pi^0$ to the nominal $\pi^0$ and $\eta'$ mass, respectively. If there are more than three photon candidates in an event, the combination with the smallest $\chi^2_{6C}$ is retained. To reject possible backgrounds with two or four photons in the final states, we further require that the probability of the 4C kinematic fit imposing energy-momentum conservation for the $J/\psi \to \pi^+\pi^-\gamma\gamma\gamma$ signal hypothesis is larger than that for the $J/\psi \to \pi^+\pi^-\gamma\gamma\gamma\gamma\gamma$ background hypotheses. Additionally, events with $|M(\gamma\pi^0) - m_{\omega}| < 0.05$ GeV/$c^2$ are rejected to suppress background from $J/\psi \to \omega\pi^+\pi^-$. With the above requirements, a sample of 8267 events is selected, and the corresponding Dalitz plot is shown in Fig. 1(a), where two clusters of events corresponding to the decays of $\eta' \to \rho^+\pi^-$ are observed. The possible background events are investigated with a Monte Carlo (MC) sample of $1.2 \times 10^9 J/\psi$ inclusive decays generated with the LUNDCHARM and EVTGEN models [17,18]. Using the same selection criteria, the surviving background events mainly originate from the decay $\eta' \to \gamma\rho$ with $\rho \to \pi\pi$ or $\rho \to \gamma\pi\pi$, which accumulate in a peak around the $\eta'$ mass.
A Dalitz plot analysis based on the formalism of the isobar model [23] is performed. The resonant $\pi\pi S$-wave ($L = 0$ for $\sigma$) and $P$-wave ($L = 1$ for $\rho^0$) amplitudes are described following the formalism from Ref. [24].
BS and likelihood fit using both the PDFs are optimized with an unbinned maximum likelihood fit, where

\[ W(s) = \frac{1}{\cot \delta_L(s) - i}, \]

where

\[ \cot \delta_0(s) = \frac{\sqrt{s}}{2k} \frac{M^2_\pi}{s - M^2_\pi/2} \left( \frac{M^2_\pi}{\sqrt{s}} + B^0_0 + B^0_1 \omega_0(s) \right), \]

\[ \cot \delta_1(s) = \frac{\sqrt{s}}{2k^2} (M^2_\rho - s) \left( \frac{2M^2_\rho}{M^2_\rho - \sqrt{s}} + B^0_0 + B^0_1 \omega_1(s) \right), \]

\[ \omega_L(s) = \frac{\sqrt{s} - \sqrt{s_L(s) - s}}{\sqrt{s + \sqrt{s_L(s) - s} - 1}}. \]

Here \( s \) is the \( \pi \pi \) invariant mass square, \( k = \sqrt{s/4 - M^2_\pi} \), \( \sqrt{s_0} = 2M_K \), the masses \( M_\rho, M_K \), and \( M_\pi \) are fixed to the world average values [20], \( \sqrt{s_1} = 1.05 \text{ GeV} \) is a constant, and \( B^0_0, B^0_1, B^0_2 \) are free parameters.

The free parameters of the probability density function (PDF) are optimized with an unbinned maximum likelihood fit using both the \( \eta' \to \pi^+ \pi^- \pi^0 \) and \( \eta' \to \pi^0 \pi^0 \pi^0 \) events, where the background contributions are included as noninterfering terms in the PDF and are fixed according to the MC simulation, the mass resolution, and the detection efficiency obtained from the MC simulation are taken into account in the signal PDF. The fit minimizes the negative log-likelihood value \( -\ln L = -\sum_{i=1}^{N_1} \ln p_i - \sum_{j=1}^{N_2} \ln p'_j \), where \( p_i \) and \( p'_j \) are the PDFs for an \( \eta' \to \pi^+ \pi^- \pi^0 \) event \( i \) and an \( \eta' \to \pi^0 \pi^0 \pi^0 \) event \( j \), respectively. The sum runs over all accepted events. From charge conjugation invariance, the magnitude and phase for \( \rho^+ \) and \( \rho^- \) are taken to be the same in the nominal fit.

Projections of the data and fit results are displayed in Fig. 3. The data are well described by three components: \( P \) wave (\( \rho^+ \pi^\mp \)), resonant \( S \) wave (\( \sigma \pi^0 \)), and phase-space \( S \) wave (\( \pi \pi \pi \)). The interference between \( \sigma \) and the nonresonant term is large and strongly depends on the parametrization of \( \sigma \). Therefore we are unable to determine the individual contributions and consider only the sum of the \( S \)-wave amplitudes in this analysis. To estimate the significance of each component, the fit is repeated with the corresponding amplitude excluded and the statistical significance is then determined by the changes of the \( -2 \ln L \) value, with the number of degrees of freedom equal to twice the number of extra parameters in the fit [25]. The statistical significances of all three components are found to be larger than 24σ. To check for an additional contribution, we add an amplitude for the scalar meson \( f_0(980) \), described by the Flatté function [26] with the parameters fixed using values from Ref. [27]. The corresponding statistical significance is only 0.3σ, and the contribution is therefore neglected.

With the fitted values of \( B^0_0 = 2.685 \pm 0.006 \), \( B^0_1 = 1.740 \pm 0.004 \), \( B^0_2 = -39.09 \pm 5.66 \), and \( B^0_3 = -39.18 \pm 4.64 \), the corresponding poles of \( \rho \) and \( \sigma \) are determined to be 775.49 (fixed) \( -i(68.5 \pm 0.2) \) MeV and (512 \( \pm 15 \)) \( -i(188 \pm 12) \) MeV, respectively, and are therefore in reasonable agreement with the \( \rho \) and \( \sigma \) values from the Particle Data Group (PDG) [20]. The signal yields defined as the integrals over the Dalitz plot of a single decay amplitude squared, the detection efficiencies obtained from the MC sample weighted with each amplitude, and the branching fractions for each component are summarized in Table I. In the calculation, the number of \( J/\psi \) is taken from Refs. [14,15], and the branching fraction for \( J/\psi \to \gamma \eta' \) and \( \pi^0 \to \gamma \gamma \) are taken from the PDG [20].

In order to compare with previous measurements, which did not consider the \( P \)-wave contribution [8,9], we also provide the branching fraction of \( \eta' \to \pi^+ \pi^- \pi^0 \) calculated with the total number of observed signal events, which is presented in Table I.
To check charge conjugation in the $P$-wave process, alternative fits were performed with different magnitudes and phases for $\rho^+$ and $\rho^-$. The result is consistent with charge symmetry, $\left|B(\eta' \rightarrow \rho^+\pi^-) - B(\eta' \rightarrow \rho^-\pi^+)\right|/\left|B(\eta' \rightarrow \rho^+\pi^-) + B(\eta' \rightarrow \rho^-\pi^+)\right| = 0.053 \pm 0.060(\text{stat}) \pm 0.010(\text{syst})$.

As an alternative model, the Gounaris-Sakurai parametrization [28] is used to describe the $\rho^\pm$ contribution with the mass and width fixed to the PDG values [20]. The $-\ln L$ value is only worse by 0.9. In another check the $\pi - \pi S$ wave for $\sigma$ is replaced with a relativistic Breit-Wigner function. This fit also provides a reasonable description of the data, and the $-\ln L$ value only changes by 3.5. The mass and width determined from this fit are $(538 \pm 12)$ MeV/$c^2$ and $(363 \pm 20)$ MeV, respectively, which are compatible with the pole position of the $\pi\pi$ elastic scattering amplitude.

Based on the symmetry imposed by Bose-Einstein statistics and isospin [29,30], the magnitude of the nonresonant $S$-wave amplitude in $\eta' \rightarrow \pi^+\pi^-\pi^0$ is three times that in $\eta' \rightarrow \pi^+\pi^-\pi^0$. If this constraint is introduced, the fitted yields are compatible with the unconstrained result, while the change in $-\ln L$ is 8.4, corresponding to a statistical significance of 3.7$\sigma$.

The differences of the branching fractions for the above tests contribute to the systematic uncertainties, denoted as model and constraint in Table II, respectively. In addition, the following sources of the systematic uncertainty are considered: The uncertainties in main drift chamber (MDC) tracking, photon selection and $\eta^0$ reconstruction efficiency (including photon detection efficiency) are studied using a high purity control sample of $J/\psi \rightarrow \rho\pi$. The differences between data and MC simulation are less than 1% per charged track, 1% for the radiative photon and 2% per $\pi^0$. The uncertainties associated with kinematic fits are studied using the control sample $J/\psi \rightarrow \gamma\eta \rightarrow \gamma\pi\pi\pi$. The preliminary selection conditions for good charged tracks, good photons, and $\eta^0$ candidates are the same as those for $J/\psi \rightarrow \gamma\eta' \rightarrow \gamma\pi\pi\pi$. The differences between data and MC simulation for the requirements of $\chi_{\text{MC}}^2(\gamma\pi^+\pi^-\pi^0) < 25$ and $\chi_{\text{MC}}^2(\eta^0\pi^0\pi^0) < 70$ are determined as 1.7% and 1.6%, respectively.

To investigate the uncertainties of the background determination, alternative fits are performed on the background components one at a time. The peaking backgrounds $\eta' \rightarrow \gamma\rho$ and $\eta' \rightarrow \pi^0\pi^0\eta'$ are varied according to the errors of the branching fraction for $J/\psi \rightarrow \gamma\eta'$ and the cascade decays in the PDG [20]. The continuum background is varied according to the uncertainties of the fits to the $\pi\pi\pi$ mass spectra. Different selection criteria for vetoing $\omega$ background are also used. The differences of the branching fractions with respect to the default values are taken as the uncertainties associated with backgrounds.

All the systematic uncertainties including the uncertainty from the number of $J/\psi$ events and the branching fraction of $J/\psi \rightarrow \gamma\eta'$ are summarized in Table II, where the total systematic uncertainty is given by the quadratic sum, assuming all sources to be independent.

In summary, using a combined amplitude analysis of $\eta' \rightarrow \pi^+\pi^-\pi^0$ and $\eta' \rightarrow \pi^0\pi^0\pi^0$ decays, the $P$-wave contribution from $\rho^\pm$ is observed for the first time with high statistical significance. The pole position of $\rho^\pm$, 775.49 (fixed) - $i(68.5 \pm 0.2)$ MeV, is consistent with previous measurements, and the branching fraction $B(\eta' \rightarrow \rho^\pm\pi^\mp)$ is determined to be $(7.44 \pm 0.60 \pm 1.26 \pm 1.84) \times 10^{-4}$.

In addition to the nonresonant $S$ wave, the resonant $\pi\pi S$ wave with a pole at $(512 \pm 15) - i(188 \pm 12)$ MeV, interpreted as the broad $\sigma$ meson, plays an essential role in the $\eta' \rightarrow \pi\pi\pi$ decays. Because of the large interference between nonresonant and resonant $S$ waves, only the

### Table I

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<th>Decay mode</th>
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<th>$\varepsilon$ (%)</th>
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### Table II

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<th>Source</th>
<th>$\rho^+\pi^+$ (%)</th>
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<th>$\pi^+\pi^-\pi^0$ (%)</th>
<th>$\pi^0\rho\pi^0$ (%)</th>
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<td>11.9</td>
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sum is used to describe the S-wave contribution, and the branching fractions are determined to be \( B(\eta' \rightarrow \pi^+\pi^-\pi^0)_S = (37.63 \pm 0.77 \pm 2.22 \pm 4.48) \times 10^{-4} \) and \( B(\eta' \rightarrow \pi^0\pi^0\pi^0) = (35.22 \pm 0.82 \pm 2.54) \times 10^{-4} \), respectively. The branching fractions of \( \eta' \rightarrow \pi^+\pi^-\pi^0 \) and \( \eta' \rightarrow \pi^0\pi^0\pi^0 \) are in agreement with and supersede the previous BESIII measurements [9]. The value for \( B(\eta' \rightarrow \pi^0\pi^0\pi^0) \) is two times larger than that from GAMS [(16.0 \pm 3.2) \times 10^{-4}] [11]. The significant resonant S-wave contribution also provides a reasonable explanation for the negative slope parameter of the \( \eta' \rightarrow \pi^0\pi^0\pi^0 \) Dalitz plot [13]. The ratio of the branching fractions between the S-wave components \( B(\eta' \rightarrow \pi^0\pi^0\pi^0) / B(\eta' \rightarrow \pi^+\pi^-\pi^0) \) is determined to be 0.94 \pm 0.03 \pm 0.13, where the common systematic uncertainties cancel out. With the branching fractions of \( \eta' \rightarrow \pi^0\pi^0\pi^0 \) taken from the PDG [20], \( r_\pm \) and \( r_0 \) are now calculated to be \((8.77 \pm 1.19) \times 10^{-3}\) and \((15.86 \pm 1.33) \times 10^{-3}\), respectively. While the previous values based on the PDG [20] are \((8.86 \pm 0.94) \times 10^{-3}\) and \((9.64 \pm 0.97) \times 10^{-3}\), respectively.

The observed substantial \( P- \) and S-wave resonant contributions have to be properly considered by theory before attempting to determine light quark masses from \( r_\pm \) and \( r_0 \). In particular, one of the previously most comprehensive analyses of hadronic decays of \( \eta \) and \( \eta' \) mesons relied on \( r_0 \), which is now two times larger, and \( r_\pm \) was not known [4]. Further progress will depend on the development of dispersive approaches such as Refs. [31–34] for \( \eta' \) hadronic decays.

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