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Cover: A 1754 painting by H.J. Detouche shows Galileo Galilei displaying his telescope to Leonardo Donato and the Venetian Senate.

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Numerical solution of surface integral equations based on spline quasi-interpolation

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Abstract

In this paper we propose a modified version of the classical collocation method and two spline collocation methods with high order of convergence, for the solution of integral equations on surfaces of $\mathbb{R}^3$. Such methods are based on optimal superconvergent quasi-interpolants defined on type-2 triangulations and based on the Zwart-Powell quadratic box spline.

Key words: surface integral equation, spline quasi-interpolation
MSC 2000: 65R20, 65D07

1 Introduction

In this paper we consider the surface integral equation

$$\rho(\mathbf{P}_1) - \int_S K(\mathbf{P}_1, \mathbf{P}_2)\rho(\mathbf{P}_2) dS_{\mathbf{P}_2} = \psi(\mathbf{P}_1), \quad \mathbf{P}_1 \in S, \tag{1}$$

where S is a connected surface in $\mathbb{R}^3$, described by a sufficiently smooth map $\mathbf{F} : \Omega \to S$, with Ω a polygonal domain in $\mathbb{R}^2$, and the kernel $K(\mathbf{P}_1, \mathbf{P}_2)$ is continuous for $\mathbf{P}_1$, $\mathbf{P}_2 \in S$.

Therefore, (1) can be written as

$$\rho(\mathbf{F}(u,v)) - \int_\Omega K(\mathbf{F}(u,v), \mathbf{F}(s,t))\rho(\mathbf{F}(s,t)) \left|(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)\right| \, ds \, dt = \psi(\mathbf{F}(u,v)), \ (u,v) \in \Omega,$$

where $|(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)|$ is the Jacobian of the map $\mathbf{F}(s,t)$.

If we denote by $\mathcal{K}: C(S) \longrightarrow C(S)$ the integral operator defined by

$$\mathcal{K}\rho(\mathbf{F}(u,v)) := \int_\Omega K(\mathbf{F}(u,v), \mathbf{F}(s,t))\rho(\mathbf{F}(s,t)) \left|(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)\right| \; ds\; dt,$$

for $(u,v) \in \Omega$, then we can write (1) in the following operator form

$$(\mathcal{I} - \mathcal{K})\rho = \psi. \tag{2}$$

We remark that (2) has a unique solution $\rho \in C(S)$ for any given $\psi \in C(S)$ [3].

In the literature, standard methods for solving (2) consist in Nyström, Galerkin and collocation methods. For instance, we recall the collocation ones based on a sequence of linear interpolatory projection operators onto finite dimensional subspaces $\mathcal{X}_{mn}$ of $C(S)$, converging to the identity operator pointwise. A classical choice of $\mathcal{X}_{mn}$ is the space of C^0 piecewise polynomials of a given degree d (usually $d = 2$) on a triangulation of Ω (see [3, 5]).

In this paper we propose three collocation methods for (2), based on a sequence of optimal superconvergent spline quasi-interpolating operators $\{Q_{mn}\}$, that are not projectors and are defined on the space $\mathcal{X}_{mn} = S_2^1(\Omega, T_{mn})$ of the C^1 quadratic splines on a uniform type-2 triangulation T_{mn} of Ω, with Ω a rectangular domain. We recall [12] that the above quasi-interpolating splines are expressed by means of the scaled/translates of the Zwart-Powell quadratic box spline (ZP-element) (see e.g. [4, Chap. 1], [14, Chap. 2]). From a computational point of view, this is more convenient than the use of other spanning sets, for instance formed by bivariate B-splines with support completely included in Ω [1, 7, 9, 13], that, having different supports, have different expressions in the domain, while the ZP-element is always the same.

Given a rectangular domain $\Omega = [a,b] \times [c,d]$, by dividing it into mn equal squares $\{\Omega_{ij}\}_{i=1,\ j=1}^{m\quad n}$ with a given edge h, $m, n \geq 4$, each of them being subdivided into 4 triangles by its diagonals, we obtain a uniform type-2 triangulation T_{mn} of Ω. We denote by $S_2^1(\Omega, T_{mn})$ the space of C^1 quadratic splines on T_{mn}, whose dimension is $(m+2)(n+2) - 1$ ([14] and the reference therein).

This space is generated by the $(m+2)(n+2)$ B-spline functions $\{B_{i,j}, (i,j) \in A_{mn}\}$, where $A_{mn} = \{(i,j),\, 0 \leq i \leq m+1,\, 0 \leq j \leq n+1\}$, obtained by dilation/translation of the ZP-element. Moreover, in order to obtain a B-spline basis for $S_2^1(\Omega, T_{mn})$ we have to neglect one B-spline from the spanning set ([14] and the reference therein).

In the space $S_2^1(\Omega, T_{mn})$ we consider special optimal quasi-interpolants (abbr. QIs) of the form

$$Q_{mn}f := \sum_{(i,j)\in A_{mn}} \lambda_{i,j}(f) B_{i,j}, \tag{3}$$

with $\{\lambda_{i,j},\ (i,j) \in A_{mn}\}$ a family of local linear functionals defined in this way

$$\lambda_{i,j}(f) := \sum_{(k,l)\in F_{i,j}} \sigma_{i,j}(k,l) f(M_{k,l}), \tag{4}$$

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where the finite set of points $\{M_{k,l},\ (k,l)\in F_{i,j}\}$, $F_{i,j}\subset A_{mn}$, lies in some neighbourhood of $\mathrm{supp}B_{i,j}\cap\Omega$ and the $\sigma_{i,j}(k,l)$'s are chosen such that $Q_{mn}f\equiv f$ for all f in $\mathbb{P}_2$ (the space of bivariate polynomials of total degree two) and superconvergence is induced at some specific points, i.e. the vertices, the centers, the midpoints of horizontal and vertical edges of each subsquare of the partition. The coefficient functional expression (4) is given in [12] and we recall that $\|Q_{mn}\|_\infty\leq 2$. The points $M_{k,l}$ in (4) are the mn centers of the squares, the $2(m+n)$ midpoints of boundary segments and the four vertices of Ω.

We remark that the QIs (3) can also be written in quasi-Lagrange form

$$Q_{mn}f:=\sum_{(i,j)\in A_{mn}} f(M_{i,j})L_{i,j},$$

by means of the fundamental functions $L_{i,j}$, obtained as linear combination of the $B_{i,j}$'s.

Standard results in approximation theory and other specific ones given in [6] allow us to deduce the following theorem, where $D^\beta=D^{\beta_1\beta_2}=\frac{\partial^{|\beta|}}{\partial x^{\beta_1}\partial y^{\beta_2}}$, with $|\beta|=\beta_1+\beta_2$, $\|D^\nu f\|_\infty=\max_{|\beta|=\nu}\left\|D^\beta f\right\|_\infty$, $\omega\left(D^\nu f,h\right)=\max\left\{\omega\left(D^\alpha f,h\right),|\alpha|=\nu\right\}$, where $\omega\left(f,h\right)=\max\left\{|f(P_1)-f(P_2)|\,;P_1,P_2\in\Omega,\|P_1-P_2\|\leq h\right\}$ is the modulus of continuity of $f\in C(\Omega)$, and $\|\cdot\|$ is the Euclidean norm.

Theorem 1 *Let $f\in C^\nu(\Omega)$, $0\leq|\alpha|\leq\nu\leq 2$, $|\alpha|=0,1$ then*

$$\|D^\alpha(f-Q_{mn}f)\|_\infty\leq K_{\alpha,\nu}h^{\nu-|\alpha|}\omega\left(D^\nu f,h\right),$$

where the error constant $K_{\alpha,\nu}$ is independent of h and depends only on α and ν.

If, in addition, $f\in C^3(\Omega)$, then

$$\|D^\alpha(f-Q_{mn}f)\|_\infty\leq K_{\alpha,3}h^{3-|\alpha|}\left\|D^3 f\right\|_\infty.$$

We underline that Q_{mn} has superconvergence properties. In particular, for $f\in C^4(\Omega)$, we have that $|(f-Q_{mn}f)(P)|=O(h^4)$ at specific points P in Ω, that are the vertices, the centers, the midpoints of horizontal and vertical edges of each subsquare of Ω partition.

Finally, the above superconvergent QIs can be applied to numerical integration, getting cubature rules that we will use in Section 2.2.

For any function $f\in C(\Omega)$, we can numerically evaluate the integral

$$I(f)=\int_\Omega f(s,t)\,ds\;dt$$

by the cubature rule defined by

$$I(Q_{mn}f)=\sum_{(i,j)\in A_{mn}} w_{i,j}f(M_{i,j}),\tag{5}$$

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where the weights

$$w_{i,j} = \int_\Omega L_{i,j}(s,t)\, ds\, dt.$$

are reported in [8].

From Theorem 1, we can easily deduce the following result.

Theorem 2 *Let $f \in C(\Omega)$ and $E(f) = I(f) - I(Q_{mn}f)$.*
Then, $|E(f)| \le \bar{C}\omega(f,h)$, where $\bar{C}$ is a positive constant independent of m and n. Moreover if $f \in C^\nu(\Omega)$, $\nu = 1,2,3$, then $E(f) = O(h^\nu)$.

We remark that the above cubature has precision degree at least 2, because Q_{mn} is exact on $\mathbb{P}_2$. However, since uniform partitions are special cases of the ones with symmetric knots with respect to the center of Ω, Corollary 1 of [10] can be generalized to our case, getting

$$I(f) = I(Q_{mn}f) \text{ for } f(s,t) = s^{r_1}t^{r_2},$$

with $0 \le r_1, r_2 \le 3$, $r_1 + r_2 = 3$ and $r_1, r_2 = 1, 3$, with $r_1 + r_2 = 4$. Therefore the precision degree of the cubature (5) is 3 and, if $f \in C^4(\Omega)$, then $E(f) = O(h^4)$.

2 Collocation methods for surface integral equations

In this section we present and analyse three collocation methods (see [8] for details) based on the sequence $\{Q_{mn}\}$ of spline QI operators defined in Section 1.

2.1 Modified collocation method

In this method, that we call *modified collocation method*, in (2) we replace the operator $\mathcal{K}$ by $Q_{mn}\mathcal{K}$ and the right hand side ψ by $Q_{mn}\psi$. We remark that the idea of defining a collocation method by operators that are not projectors has been proposed in [2] for univariate integral equations.

Therefore, we approximate the integral equation (2) by

$$(\mathcal{I} - Q_{mn}\mathcal{K})\rho_{mn} = Q_{mn}\psi. \tag{6}$$

We write the approximated solution ρ_{mn}, belonging to $S_2^1(\Omega, T_{mn})$, as

$$\rho_{mn}(\mathbf{F}(u,v)) = \sum_{\alpha \in A_{mn}} X_\alpha L_\alpha(u,v), \quad \text{with} \quad \alpha = (i,j).$$

Substituting the expressions of Q_{mn} and ρ_{mn} into (6), by identifying the coefficients of L_α, we obtain

$$X_\alpha - \sum_{\beta \in A_{mn}} X_\beta \bar{L}_\beta(M_\alpha) = \psi(\mathbf{F}(M_\alpha)), \quad \alpha \in A_{mn},$$

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with $\bar{L}_\beta = \mathcal{K} L_\beta$. This is a linear system of $(m+2)(n+2)$ equations, that can be written in the form

$$(I - A)\mathbf{X} = \mathbf{a} \tag{7}$$

where A is the matrix with entries

$$A_{\alpha\beta} := \bar{L}_\beta(M_\alpha) = \int_\Omega K(\mathbf{F}(M_\alpha), \mathbf{F}(s,t)) \left|(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)\right| L_\beta(s,t)\ ds\ dt \tag{8}$$

and $\mathbf{a}$ is the vector with elements $\mathbf{a}_\alpha := \psi(\mathbf{F}(M_\alpha))$.

Concerning the convergence, we can state the following theorem.

Theorem 3 *Let $\rho \in C^3(\Omega)$, then $\|\rho - \rho_{mn}\|_\infty = O\left(h^3\right)$.*

2.2 Collocation methods with high order of convergence

In these methods, that we call *collocation methods with high order of convergence*, in (2) we replace $\mathcal{K}$ by one of the two following finite rank operators

$$\mathcal{K}_{mn,i} := Q_{mn}\mathcal{K} + \mathcal{K}^*_{mn,i} - Q_{mn}\mathcal{K}^*_{mn,i}, \qquad i = 1, 2,$$

where

1. $\mathcal{K}^*_{mn,1}$ is the degenerate kernel operator defined by

$$\begin{aligned} &\mathcal{K}^*_{mn,1}\rho(\mathbf{F}(u,v)) \\ &:= \int_\Omega Q_{mn}\left(K(\mathbf{F}(u,v), \mathbf{F}(s,t)) \left|(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)\right|\right) \rho(\mathbf{F}(s,t))\ ds\ dt \\ &= \sum_{\alpha \in A_{mn}} K(\mathbf{F}(u,v), \mathbf{F}(M_\alpha)) \left|(D_s\mathbf{F} \times D_t\mathbf{F})(M_\alpha)\right| \cdot \int_\Omega L_\alpha(s,t)\rho(\mathbf{F}(s,t))\ ds\ dt, \end{aligned} \tag{9}$$

2. $\mathcal{K}^*_{mn,2}$ is the Nyström operator based on Q_{mn} and defined by

$$\mathcal{K}^*_{mn,2}\rho(\mathbf{F}(u,v)) := \sum_{\alpha \in A_{mn}} w_\alpha K(\mathbf{F}(u,v), \mathbf{F}(M_\alpha)) \left|(D_s\mathbf{F} \times D_t\mathbf{F})(M_\alpha)\right| \rho(\mathbf{F}(M_\alpha)), \tag{10}$$

according to (5).

We remark that such methods are defined by a logical scheme similar to that one used in [1] to construct methods for 2D integral equations, based on other quasi-interpolants.

Therefore, we approximate (2) by

$$\rho_{mn,i} - (Q_{mn}\mathcal{K} + \mathcal{K}^*_{mn,i} - Q_{mn}\mathcal{K}^*_{mn,i})\rho_{mn,i} = \psi, \qquad i = 1, 2. \tag{11}$$

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that can be reduced to two systems of $2(m+2)(n+2)$ linear equations.

After some algebra, from (9) and (11), we can write the approximate solution $\rho_{mn,1}$ as:

$$\begin{aligned}\rho_{mn,1}(\mathbf{F}(u,v)) = \; & \psi(\mathbf{F}(u,v)) + \sum_{\alpha\in A_{mn}} X_\alpha L_\alpha(u,v) \\ & + \sum_{\alpha\in A_{mn}} Y_\alpha K(\mathbf{F}(u,v),\mathbf{F}(M_\alpha))\,|(D_s\mathbf{F}\times D_t\mathbf{F})(M_\alpha)|\,,\end{aligned}$$

where the unknowns $\{X_\alpha\}$ and $\{Y_\alpha\}$, $\alpha \in A_{mn}$, are obtained by solving the linear system $(I-R)\mathbf{Z} = \mathbf{d}$, with

$$R := \begin{bmatrix} A & D-B \\ C & E \end{bmatrix}, \quad \mathbf{Z} := \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}, \quad \mathbf{d} := \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} \tag{12}$$

and A, B, C, D, $E \in \mathbb{R}^{(m+2)(n+2)\times(m+2)(n+2)}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^{(m+2)(n+2)}$, whose entries are given by

- $A_{\alpha,\beta} := \bar{L}_\beta(M_\alpha)$, see (8),
- $B_{\alpha,\beta} := K(\mathbf{F}(M_\alpha),\mathbf{F}(M_\beta))\,|(D_s\mathbf{F}\times D_t\mathbf{F})(M_\beta)|$,
- $C_{\alpha,\beta} := \int_\Omega L_\alpha(s,t)L_\beta(s,t)\;ds\;dt$,
- $D_{\alpha,\beta} := \int_\Omega K(\mathbf{F}(M_\alpha),\mathbf{F}(s,t))|(D_s\mathbf{F}\times D_t\mathbf{F})(s,t)|K(\mathbf{F}(s,t),\mathbf{F}(M_\beta))|(D_s\mathbf{F}\times D_t\mathbf{F})(M_\beta)|\;ds\;dt$,
- $E_{\alpha,\beta} := \int_\Omega K(\mathbf{F}(s,t),\mathbf{F}(M_\beta))\,|(D_s\mathbf{F}\times D_t\mathbf{F})(M_\beta)|\,L_\alpha(s,t)\;ds\;dt$,
- $\mathbf{b}_\alpha := \mathcal{K}\psi(\mathbf{F}(M_\alpha)) = \int_\Omega K(\mathbf{F}(M_\alpha),\mathbf{F}(s,t))\,|(D_s\mathbf{F}\times D_t\mathbf{F})(s,t)|\,\psi(\mathbf{F}(s,t))\;ds\;dt$,
- $\mathbf{c}_\alpha := \int_\Omega \psi(\mathbf{F}(s,t))L_\alpha(s,t)\;ds\;dt$.

Similarly, from (10) and (11), we can get that the solution $\rho_{mn,2}$ is

$$\begin{aligned}\rho_{mn,2}(\mathbf{F}(u,v)) = \; & \psi(\mathbf{F}(u,v)) + \sum_{\alpha\in A_{mn}} X_\alpha L_\alpha(u,v) \\ & + \sum_{\alpha\in A_{mn}} w_\alpha Y_\alpha K(\mathbf{F}(u,v),\mathbf{F}(M_\alpha))\,|(D_s\mathbf{F}\times D_t\mathbf{F})(M_\alpha)|\,,\end{aligned}$$

where the unknowns $\{X_\alpha\}$ and $\{Y_\alpha\}$, $\alpha \in A_{mn}$, are obtained by solving the linear system $(I-T)\mathbf{Z} = \mathbf{f}$, with

$$T := \begin{bmatrix} A & F-G \\ H & G \end{bmatrix}, \quad \mathbf{Z} := \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}, \quad \mathbf{f} := \begin{bmatrix} \mathbf{a} \\ \mathbf{e} \end{bmatrix} \tag{13}$$

and F, G, $H \in \mathbb{R}^{(m+2)(n+2)\times(m+2)(n+2)}$, $\mathbf{e} \in \mathbb{R}^{(m+2)(n+2)}$, whose entries are given by

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- $F_{\alpha,\beta} := w_\beta \int_\Omega K(\mathbf{F}(M_\alpha), \mathbf{F}(s,t)) |(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)| K(\mathbf{F}(s,t), \mathbf{F}(M_\beta)) |(D_s\mathbf{F} \times D_t\mathbf{F})(M_\beta)| \, ds \, dt$,
- $G_{\alpha,\beta} := w_\beta K(\mathbf{F}(M_\alpha), \mathbf{F}(M_\beta)) \, |(D_s\mathbf{F} \times D_t\mathbf{F})(M_\beta)|$,
- $H_{\alpha,\beta} := L_\beta(M_\alpha)$,
- $\mathbf{e}_\alpha := \psi(\mathbf{F}(M_\alpha)$.

Concerning the convergence, we can state the following theorem.

Theorem 4 *If ρ is differentiable with bounded derivatives, $K(\cdot,\cdot) \in C^4(S \times S)$ and $\mathbf{F} \in C^5(\Omega)$, then $\|\rho - \rho_{mn,1}\|_\infty = O(h^7)$.*

If $\rho \in C^4(S)$, $K(\cdot,\cdot) \in C^4(S \times S)$ and $\mathbf{F} \in C^5(\Omega)$, then $\|\rho - \rho_{mn,2}\|_\infty = O(h^7)$.

3 Numerical results

By using the collocation methods (6) and (11), we have to evaluate many integrals and usually it must be done by suitable numerical integration formulas. Therefore, we have to discretize the proposed methods by introducing convenient cubatures and we denote by ρ^D_{mn}, $\rho^D_{mn,i}$, $i = 1, 2$, the corresponding solutions.

Here, we decide to compute the entries of the matrices and vectors appearing in (7), (12), (13), by using a composite Gaussian cubature on triangular domains (see [11]), implemented by the Matlab function `triquad` (see [15]), with N^2 nodes in each triangle of T_{mn} and with precision degree $2N - 1$. The number of nodes is chosen to preserve the approximation order of the method. Therefore, we choose $N = 2$ for the modified collocation method (6) and $N = 4$ for the two collocation methods with high order of convergence (11).

We test the performances of the proposed methods in the numerical solution of the surface integral equation from [3]

$$\rho(\mathbf{P}_1) - \frac{1}{30} \int_S \rho(\mathbf{P}_2) \frac{\partial}{\partial \mathbf{n}_{\mathbf{P}_2}} \left(\|\mathbf{P}_1 - \mathbf{P}_2\|^2 \right) dS_{\mathbf{P}_2} = \frac{1}{30} \psi(\mathbf{P}_1), \ \mathbf{P}_1 \in S,$$

where S is the ellipsoidal surface given by $x^2 + \left(\frac{4y}{3}\right)^2 + (2z)^2 = 1$, $\mathbf{n}_{\mathbf{P}_2}$ is the inner normal to S at $\mathbf{P}_2$ and

$$\mathbf{F}(s,t) = \begin{bmatrix} \sin(s)\cos(t) \\ \frac{3}{4}\sin(s)\sin(t) \\ \frac{1}{2}\cos(s) \end{bmatrix}, \qquad (s,t) \in \Omega = [0,\pi] \times [0, 2\pi].$$

We choose $\rho(\mathbf{P}) = e^{\frac{1}{2}\cos(s)}$ and define ψ accordingly.

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For each method we compute the maximum absolute errors

$$E_{mn} = \max_{(u,v)\in G} |\rho(u,v) - \rho^D_{mn}(u,v)|, \quad E_{mn,i} = \max_{(u,v)\in G} |\rho(u,v) - \rho^D_{mn,i}(u,v)|, \quad i = 1,2,$$

for increasing values of m and n, where G is a uniform grid of 100×100 points in Ω. We also compute the corresponding numerical convergence orders o_{mn}, $o_{mn,i}$, $i = 1,2$.

The results are shown in Table 1 and we can notice that they agree with the theoretical ones.

Table 1: Maximum absolute errors and numerical convergence orders.

m	n	E_{mn}	o_{mn}	$E_{mn,1}$	$o_{mn,1}$	$E_{mn,2}$	$o_{mn,2}$
4	8	7.56e-03	-	2.51e-05	-	3.17e-05	-
8	16	8.11e-04	3.22	2.09e-07	6.91	1.29e-07	7.94
16	32	8.21e-05	3.30	1.48e-09	7.14	1.71e-09	6.24
32	64	8.34e-06	3.30	1.12e-11	7.04	1.46e-11	6.87

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