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ON THE \( k \)-REGULARITY OF THE \( k \)-ADIC VALUATION OF LUCAS SEQUENCES

NADIR MURRU AND CARLO SANNA

Abstract. For integers \( k \geq 2 \) and \( n \neq 0 \), let \( \nu_k(n) \) denotes the greatest nonnegative integer \( e \) such that \( k^e \) divides \( n \). Moreover, let \((u_n)_{n \geq 0} \) be a nondegenerate Lucas sequence satisfying 
\[ u_0 = 0, \quad u_1 = 1, \quad \text{and} \quad u_{n+2} = au_{n+1} + bu_n, \]
for some integers \( a \) and \( b \). Shu and Yao showed that for any prime number \( p \) the sequence \( \nu_p(u_{n+1})_{n \geq 0} \) is \( p \)-regular, while Medina and Rowland found the rank of \( \nu_p(F_{n+p})_{n \geq 0} \), where \( F_n \) is the \( n \)-th Fibonacci number. We prove that if \( k \) and \( b \) are relatively prime then \( \nu_k(u_{n+1})_{n \geq 0} \) is a \( k \)-regular sequence, and for \( k \) a prime number we also determine its rank. Furthermore, as an intermediate result, we give explicit formulas for \( \nu_k(u_n) \), generalizing a previous theorem of Sanna concerning \( p \)-adic valuations of Lucas sequences.

1. Introduction

For integers \( k \geq 2 \) and \( n \neq 0 \), let \( \nu_k(n) \) denotes the greatest nonnegative integer \( e \) such that \( k^e \) divides \( n \). In particular, if \( k = p \) is a prime number then \( \nu_p(\cdot) \) is the usual \( p \)-adic valuation. We shall refer to \( \nu_k(\cdot) \) as the \( k \)-adic valuation, although, strictly speaking, for composite \( k \) this is not a “valuation” in the algebraic sense of the term, since it is not true that \( \nu_k(mn) = \nu_k(m) + \nu_k(n) \) for all integers \( m, n \neq 0 \).

Valuations of sequences with combinatorial meanings have been studied by several authors (see, e.g., [4, 6, 7, 8, 9, 10, 12, 14, 15, 18]). To this end, an important role is played by the family of \( k \)-regular sequences, which were first introduced and studied by Allouche and Shallit [1, 2, 3] with the aim of generalizing the concept of automatic sequences.

Given a sequence of integers \( s(n)_{n \geq 0} \), its \( k \)-kernel is defined as the set of subsequences
\[ \ker_k(s(n)_{n \geq 0}) := \{ s(k^e n + i)_{n \geq 0} : e \geq 0, \ 0 \leq i < k^e \}. \]
Then \( s(n)_{n \geq 0} \) is said to be \( k \)-regular if the \( \mathbb{Z} \)-module \((\ker_k(s(n)_{n \geq 0})) \) generated by its \( k \)-kernel is finitely generated. In such a case, the rank of \( s(n)_{n \geq 0} \) is the rank of this \( \mathbb{Z} \)-module.

Allouche and Shallit provided many examples of regular sequences. In particular, they showed that the sequence of \( p \)-adic valuations of factorials \( \nu_p(n!)_{n \geq 0} \) is \( p \)-regular [1, Example 9], and that the sequence of \( 3 \)-adic valuations of sums of central binomial coefficients
\[ \nu_3 \left( \sum_{i=0}^{n} \binom{2i}{i} \right)_{n \geq 0} \]
is \( 3 \)-regular [1, Example 23]. Furthermore, for any polynomial \( f(x) \in \mathbb{Q}[x] \) with no roots in the natural numbers, Bell [5] proved that the sequence \( \nu_p(f(n))_{n \geq 0} \) is \( p \)-regular if and only if \( f(x) \) factors as a product of linear polynomials in \( \mathbb{Q}[x] \) times a polynomial with no root in the \( p \)-adic integers.

Fix two integers \( a \) and \( b \), and let \((u_n)_{n \geq 0} \) be the Lucas sequence of characteristic polynomial \( f(x) = x^2 - ax - b \), i.e., \((u_n)_{n \geq 0} \) is the integral sequence satisfying \( u_0 = 0, u_1 = 1, \) and \( u_{n+2} = au_{n+1} + bu_n \), for each integer \( n \geq 0 \). Assume also that \((u_n)_{n \geq 0} \) is nondegenerate, i.e., \( b \neq 0 \) and the ratio \( \alpha/\beta \) of the two roots \( \alpha, \beta \in \mathbb{C} \) of \( f(x) \) is not a root of unity.

Using \( p \)-adic analysis, Shu and Yao [16, Corollary 1] proved the following result.

\begin{itemize}
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  \item \textbf{Key words and phrases.} Lucas sequence; Fibonacci numbers; \( p \)-adic valuation; \( k \)-regular sequence; automatic sequence.
\end{itemize}
Theorem 1.1. For each prime number $p$, the sequence $\nu_p(u_{n+1})_{n \geq 0}$ is $p$-regular.

In the special case $a = b = 1$, i.e., when $(u_n)_{n \geq 0}$ is the sequence of Fibonacci numbers $(F_n)_{n \geq 0}$, Medina and Rowland [11] gave an algebraic proof of Theorem 1.1 and also determined the rank of $\nu_p(F_{n+1})_{n \geq 0}$. Their result is the following.

Theorem 1.2. For each prime number $p$ the sequence $\nu_p(F_{n+1})_{n \geq 0}$ is $p$-regular. Precisely, for $p \neq 2, 5$ the rank of $\nu_p(F_{n+1})_{n \geq 0}$ is $\alpha(p) + 1$, where $\alpha(p)$ is the least positive integer such that $p \mid F_{\alpha(p)}$, while for $p = 2$ the rank is 5, and for $p = 5$ the rank is 2.

In this paper, we extend both Theorem 1.1 and Theorem 1.2 to $k$-adic valuations with $k$ relatively prime to $b$. Let $\Delta := a^2 + 4b$ be the discriminant of $f(x)$. Also, for each positive integer $m$ relatively prime to $b$ let $\tau(m)$ denotes the rank of apparition of $m$ in $(u_n)_{n \geq 0}$, i.e., the least positive integer $n$ such that $m \mid u_n$ (which is well-defined, see, e.g., [13]).

Our first two results are the following.

Theorem 1.3. If $k \geq 2$ is an integer relatively prime to $b$, then the sequence $\nu_k(u_{n+1})_{n \geq 0}$ is $k$-regular.

Theorem 1.4. Let $p$ be a prime number not dividing $b$, and let $r$ be the rank of $\nu_p(u_{n+1})_{n \geq 0}$.

- If $p \mid \Delta$ then:
  - $r = 2$ if $p \in \{2, 3\}$ and $\nu_p(u_p) = 1$, or if $p \geq 5$;
  - $r = 3$ if $p \in \{2, 3\}$ and $\nu_p(u_p) \neq 1$.
- If $p \nmid \Delta$ then:
  - $r = 5$ if $p = 2$ and $\nu_2(u_6) \neq \nu_2(u_3) + 1$;
  - $r = \tau(p) + 1$ if $p > 2$, or if $p = 2$ and $\nu_2(u_6) = \nu_2(u_3) + 1$.

Note that Theorem 1.2 follows easily from our Theorem 1.4, since in the case of Fibonacci numbers $b = 1$, $\Delta = 5$, $\nu_2(F_3) = 1$, $\nu_2(F_5) = 3$, and $\tau(p) = \alpha(p)$.

As a preliminary step in the proof of Theorem 1.3, we obtain some formulas for the $k$-adic valuation $\nu_k(u_n)$, which generalize a previous result of the second author. Precisely, Sanna [15] proved the following formulas for the $p$-adic valuation of $u_n$.

Theorem 1.5. If $p$ is a prime number such that $p \nmid b$, then

$$\nu_p(u_n) = \begin{cases} \nu_p(n) + \varrho_p(n) & \text{if } \tau(p) \mid n, \\ 0 & \text{if } \tau(p) \nmid n, \end{cases}$$

for each positive integer $n$, where

$$\varrho_2(n) := \begin{cases} \nu_2(u_3) & \text{if } 2 \nmid \Delta, 2 \nmid n, \\ \nu_2(u_6) - 1 & \text{if } 2 \nmid \Delta, 2 \mid n, \\ \nu_2(u_2) - 1 & \text{if } 2 \mid \Delta, \end{cases}$$

and

$$\varrho_p(n) = \varrho_p := \begin{cases} \nu_p(u_{\tau(p)}) & \text{if } p \nmid \Delta, \\ \nu_3(u_3) - 1 & \text{if } p \mid \Delta, p = 3, \\ 0 & \text{if } p \mid \Delta, p \geq 5, \end{cases}$$

for $p \geq 3$.

Actually, Sanna’s result [15, Theorem 1.5] is slightly different but it quickly turns out to be equivalent to Theorem 1.5 using [15, Lemma 2.1(v), Lemma 3.1, and Lemma 3.2]. Furthermore, in Sanna’s paper it is assumed $\gcd(a, b) = 1$, but the proof of [15, Theorem 1.5] works exactly in the same way also for $\gcd(a, b) \neq 1$.

From now on, let $k = p_1^{a_1} \cdots p_h^{a_h}$ be the prime factorization of $k$, where $p_1 < \cdots < p_h$ are prime numbers and $a_1, \ldots, a_h$ are positive integers.

We prove the following generalization of Theorem 1.5.
Theorem 1.6. If $k \geq 2$ is an integer relatively prime to $b$, then

$$
\nu_k(u_n) = \begin{cases} 
\nu_k(c_k(n)n) & \text{if } \tau(p_1 \cdots p_h) | n, \\
0 & \text{if } \tau(p_1 \cdots p_h) \nmid n,
\end{cases}
$$

for any positive integer $n$, where

$$
c_k(n) := \prod_{i=1}^{h} p_i^{\varphi_i(n)}.
$$

Note that Theorem 1.6 is indeed a generalization of Theorem 1.5. In fact, if $k = p$ is a prime number then obviously

$$
\nu_p(c_p(n)n) = \nu_p(p^{\varphi_p(n)}n) = \nu_p(n) + \varphi_p(n),
$$

for each positive integer $n$.

2. Preliminaries

In this section we collect some preliminary facts needed to prove the results of this paper. We begin with some lemmas on $k$-regular sequences.

Lemma 2.1. If $s(n)_{n \geq 0}$ and $t(n)_{n \geq 0}$ are two $k$-regular sequences, then $(s(n) + t(n))_{n \geq 0}$ and $s(n)t(n)_{n \geq 0}$ are $k$-regular too. Precisely, if $A$ is a finite set of generators of $\langle \ker_k(s(n)_{n \geq 0}) \rangle$ and $B$ is a finite set of generators of $\langle \ker_k(t(n)_{n \geq 0}) \rangle$, then $A \cup B$ is a set of generators of $\langle \ker_k((s(n) + t(n))_{n \geq 0}) \rangle$.

Proof. See [1, Theorem 2.5].

Lemma 2.2. If $s(n)_{n \geq 0}$ is a $k$-regular sequence, then for any integers $c \geq 1$ and $d \geq 0$ the subsequence $s(cn + d)_{n \geq 0}$ is $k$-regular.

Proof. See [1, Theorem 2.6].

Lemma 2.3. Any periodic sequence is $k$-regular.

Proof. An ultimately periodic sequence is $k$-automatic for all $k \geq 2$, see [2, Theorem 5.4.2]. A $k$-automatic sequence is $k$-regular, see [1, Theorem 1.2].

Lemma 2.4. Let $s(n)_{n \geq 0}$ be a sequence of integers. If there exist some

$$
(1)
$$

such that the sequences $s_j(kn + i)_{n \geq 0}$, with $0 \leq i < k$ and $1 \leq j \leq r$, are $\mathbb{Z}$-linear combinations of $s_1, \ldots, s_r$, then $s(n)_{n \geq 0}$ is $k$-regular and $\langle \ker_k(s(n)_{n \geq 0}) \rangle$ is generated by $s_1, \ldots, s_r$.

Proof. It is sufficient to prove that $s(k^e n + i)_{n \geq 0} \in \langle s_1, \ldots, s_r \rangle$ for all integers $e \geq 0$ and $0 \leq i < k^e$. In fact, this claim implies that $\langle \ker_k(s(n)_{n \geq 0}) \rangle \subseteq \langle s_1, \ldots, s_r \rangle$, while by (1) we have $\langle s_1, \ldots, s_r \rangle \subseteq \langle \ker_k(s(n)_{n \geq 0}) \rangle$, hence $\langle \ker_k(s(n)_{n \geq 0}) \rangle = \langle s_1, \ldots, s_r \rangle$ and so $s(n)_{n \geq 0}$ is $k$-regular. We proceed by induction on $e$. For $e = 0$ the claim is obvious since $s = s_1$. Suppose $e \geq 1$ and that the claim holds for $e - 1$. We have $i = k^{e-1}j + i'$, for some integers $0 \leq j < k$ and $0 \leq i' < k^{e-1}$. Therefore, by the induction hypothesis,

$$
\begin{aligned}
(s(k^e n + i))_{n \geq 0} &= s(k^{e-1}(kn + j) + i')_{n \geq 0} \\
&\in \langle s_1(kn + j)_{n \geq 0}, \ldots, s_r(kn + j)_{n \geq 0} \rangle \\
&\subseteq \langle s_1, \ldots, s_r \rangle,
\end{aligned}
$$

and the claim follows.

The next lemma is well-known, we give the proof just for completeness.

Lemma 2.5. The sequence $\nu_k(n + 1)_{n \geq 0}$ is $k$-regular of rank 2. Indeed, $\langle \ker_k(\nu_k(n + 1)_{n \geq 0}) \rangle$ is generated by $\nu_k(n + 1)_{n \geq 0}$ and the constant sequence $(1)_{n \geq 0}$. 

Proof. For all nonnegative integers \( n \) and \( i < k \) we have
\[
\nu_k(kn + i + 1) = \begin{cases} 
1 + \nu_k(n + 1) & \text{if } i = k - 1, \\
0 & \text{if } i < k - 1.
\end{cases}
\]
Therefore, putting \( s_1 = \nu_k(n + 1)_{n \geq 0} \) and \( s_2 = (1 + \nu_k(n + 1))_{n \geq 0} \) in Lemma 2.4, we obtain that \( (\ker_k(\nu_k(n + 1)_{n \geq 0})) \) is generated by \( \nu_k(n + 1)_{n \geq 0} \) and \( (1 + \nu_k(n + 1))_{n \geq 0} \), hence it is also generated by \( \nu_k(n + 1)_{n \geq 0} \) and \( (1)_{n \geq 0} \), which are obviously linearly independent. Thus \( \nu_k(n + 1)_{n \geq 0} \) is \( k \)-regular of rank 2. \( \square \)

Now we state a lemma that relates the \( k \)-adic valuation of an integer with its \( p \)-adic valuations. The proof is quite straightforward and we leave it to the reader.

Lemma 2.6. We have
\[
\nu_k(m) = \min_{i=1, \ldots, h} \left\lfloor \frac{\nu_{p_i}(m)}{a_i} \right\rfloor,
\]
for any integer \( m \neq 0 \).

We conclude this section with two lemmas on the rank of apparition \( \tau(n) \).

Lemma 2.7. For each prime number \( p \) not dividing \( b \),
\[
\tau(p) \mid p - (-1)^{p-1} \left( \frac{\Delta}{p} \right),
\]
where \( \left( \frac{\cdot}{p} \right) \) denotes the Legendre symbol. In particular, if \( p \mid \Delta \) then \( \tau(p) = p \).

Proof. The case \( p = 2 \) is easy. For \( p > 2 \) see [17, Lemma 1]. \( \square \)

Lemma 2.8. If \( m \) and \( n \) are two positive integers relatively prime to \( b \), then
\[
\tau(\text{lcm}(m, n)) = \text{lcm}(\tau(m), \tau(n)).
\]

Proof. See [13, Theorem 1(a)]. \( \square \)

3. Proof of Theorem 1.6

Thanks to Lemma 2.6, we know that
\begin{equation}
\nu_k(u_n) = \min_{i=1, \ldots, h} \left\lfloor \frac{\nu_{p_i}(u_n)}{a_i} \right\rfloor.
\end{equation}
Moreover, from Lemma 2.8 it follows that
\[
\tau(p_1 \cdots p_h) = \text{lcm}\{\tau(p_1), \ldots, \tau(p_h)\}.
\]
Therefore, on the one hand, if \( \tau(p_1 \cdots p_h) \nmid n \) then \( \tau(p_i) \nmid n \) for some \( i \in \{1, \ldots, h\} \), so that by Theorem 1.5 we have \( \nu_{p_i}(u_n) = 0 \), which together with (2) implies \( \nu_k(u_n) = 0 \), as claimed.

On the other hand, if \( \tau(p_1 \cdots p_h) \mid n \) then \( \tau(p_i) \mid n \) for \( i = 1, \ldots, h \). Hence, from (2), Theorem 1.5, and Lemma 2.6, we obtain
\[
\nu_k(u_n) = \min_{i=1, \ldots, h} \left\lfloor \frac{\nu_{p_i}(n) + a_{p_i}(n)}{a_i} \right\rfloor = \min_{i=1, \ldots, h} \left\lfloor \frac{\nu_{p_i}(c_k(n)n)}{a_i} \right\rfloor = \nu_k(c_k(n)n),
\]
so that the proof is complete.
4. Proof of Theorem 1.3

Clearly, if $k$ is fixed, then $c_k(n)$ depends only of the parity of $n$. Thus it follows easily from Theorem 1.6 that

\begin{equation}
\nu_k(u_{n+1}) = \nu_k(c_k(1)(n + 1)) s(n) + \nu_k(c_k(2)(n + 1)) t(n),
\end{equation}

for each integer $n \geq 0$, where the sequences $s(n)_{n \geq 0}$ and $t(n)_{n \geq 0}$ are defined by

$$s(n) := \begin{cases} 1 & \text{if } \tau(p_1 \cdots p_2) | n + 1, 2 \nmid n + 1, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$t(n) := \begin{cases} 1 & \text{if } \tau(p_1 \cdots p_2) | n + 1, 2 | n + 1, \\ 0 & \text{otherwise}. \end{cases}$$

On the one hand, by Lemma 2.5 and Lemma 2.2, we know that both $\nu_k(c_k(1)(n + 1))_{n \geq 0}$ and $\nu_k(c_k(2)(n + 1))_{n \geq 0}$ are $k$-regular sequences. On the other hand, by Lemma 2.3, also the sequences $s(n)_{n \geq 0}$ and $t(n)_{n \geq 0}$ are $k$-regular, since obviously they are periodic.

In conclusion, thanks to (3) and Lemma 2.1, we obtain that $\nu_k(u_{n+1})_{n \geq 0}$ is a $k$-regular sequence.

5. Proof of Theorem 1.4

First, suppose that $p \mid \Delta$. By Lemma 2.7 we have $\tau(p) = p$. Moreover, it is clear that $\varrho_p(n) = \varrho_p$ does not depend on $n$. As a consequence, from Theorem 1.5 it follows easily that

\begin{equation}
\nu_p(u_{n+1}) = \nu_p(n + 1) + s(n),
\end{equation}

for any integer $n \geq 0$, where the sequence $s(n)_{n \geq 0}$ is defined by

$$s(n) := \begin{cases} \varrho_p & \text{if } n + 1 \equiv 0 \mod p, \\ 0 & \text{if } n + 1 \not\equiv 0 \mod p. \end{cases}$$

On the one hand, if $p \in \{2, 3\}$ and $\nu_p(u_p) = 1$, or if $p \geq 5$, then $\varrho_p = 0$. Thus $s(n)_{n \geq 0}$ is identically zero and it follows by (4) and Lemma 2.5 that $r = 2$. On the other hand, if $p \in \{2, 3\}$ and $\nu_p(u_p) \neq 1$, then $\varrho_p \neq 0$. Moreover, for $i = 0, \ldots, p - 1$ we have

$$s(pm + i) = \begin{cases} \varrho_p & \text{if } i = p - 1, \\ 0 & \text{if } i \neq p - 1, \end{cases}$$

hence from Lemma 2.4 it follows that $s(n)_{n \geq 0}$ is $p$-regular and that $\langle \ker_p(s(n)_{n \geq 0}) \rangle$ is generated by $s(n)_{n \geq 0}$ and $(\varrho_p)_{n \geq 0}$. Therefore, by (4), Lemma 2.5, and Lemma 2.1, we obtain that $\nu_p(u_{n+1})_{n \geq 0}$ is a $p$-regular sequence and that $\langle \ker_p(\nu_p(u_{n+1})_{n \geq 0}) \rangle$ is generated by $\nu_p(n + 1)_{n \geq 0}$, $s(n)_{n \geq 0}$, and $(1)_{n \geq 0}$, which are clearly linearly independent, hence $r = 3$.

Now suppose $p \nmid \Delta$. By Lemma 2.7, we know that $p \equiv \varepsilon \mod \tau(p)$, for some $\varepsilon \in \{-1, +1\}$. Furthermore, if $p = 2$ then it follows easily that $\tau(2) = 3$. As a consequence, from Theorem 1.5 we obtain that

\begin{equation}
\nu_p(u_{n+1}) = s(n) + t(n),
\end{equation}

for any integer $n \geq 0$, where the sequences $s(n)_{n \geq 0}$ and $t(n)_{n \geq 0}$ are defined by

$$s(n) := \begin{cases} \nu_p(n + 1) + v & \text{if } n + 1 \equiv 0 \mod \tau(p), \\ 0 & \text{if } n + 1 \not\equiv 0 \mod \tau(p), \end{cases}$$

with $v := \nu_p(u_{\tau(p)})$, and

$$t(n) := \begin{cases} \nu_2(u_6) - \nu_2(u_3) - 1 & \text{if } p = 2, n + 1 \equiv 0 \mod 6, \\ 0 & \text{otherwise}. \end{cases}$$
We shall show that \( s(n)_{n \geq 0} \) is a \( p \)-regular sequence of rank \( \tau(p) + 1 \). Let us define the sequences \( s_j(n)_{n \geq 0} \), for \( j = 0, \ldots, \tau(p) - 1 \), by

\[
s_j(n) := \begin{cases} 
  1 & \text{if } n + j + 1 \equiv 0 \pmod{\tau(p)}, \\
  0 & \text{if } n + j + 1 \not\equiv 0 \pmod{\tau(p)}.
\end{cases}
\]

On the one hand, for \( i = 0, \ldots, p - 2 \) we have

\[
s(pn + i) = \begin{cases} 
  \nu_p(pn + i + 1) + v & \text{if } pn + i + 1 \equiv 0 \pmod{\tau(p)}, \\
  0 & \text{if } pn + i + 1 \not\equiv 0 \pmod{\tau(p)},
\end{cases}
\]

where \( \nu_p(x) \) is the highest power of \( p \) that divides \( x \). Since \( p \nmid i + 1 \) and consequently \( \nu_p(pn + i + 1) = 0 \).

On the other hand,

\[
s(pn + p - 1) = \begin{cases} 
  \nu_p(pn + p) + v & \text{if } p(n + 1) \equiv 0 \pmod{\tau(p)}, \\
  0 & \text{if } p(n + 1) \not\equiv 0 \pmod{\tau(p)},
\end{cases}
\]

we have

\[
\begin{align*}
\nu_p(pn + p) &= \nu_p((p - 1)n + 1) + 1 \equiv 0 \pmod{\tau(p)}, \\
\gcd(p, \tau(p)) &= 1.
\end{align*}
\]

Furthermore, for \( i = 0, \ldots, p - 1 \) and \( j = 0, \ldots, \tau(p) - 1 \),

\[
s_j(pn + i) = \begin{cases} 
  1 & \text{if } pn + i + j + 1 \equiv 0 \pmod{\tau(p)}, \\
  0 & \text{if } pn + i + j + 1 \not\equiv 0 \pmod{\tau(p)},
\end{cases}
\]

we have

\[
\begin{align*}
s_j(pn + i) &= \begin{cases} 
  1 & \text{if } n + (\epsilon(i + 1) - 1) + 1 \equiv 0 \pmod{\tau(p)}, \\
  0 & \text{if } n + (\epsilon(i + 1) - 1) + 1 \not\equiv 0 \pmod{\tau(p)},
\end{cases}
\end{align*}
\]

Summarizing, the sequences \( s(pn + i)_{n \geq 0} \) and \( s_j(pn + i)_{n \geq 0} \), for \( i = 0, \ldots, p - 1 \) and \( j = 0, \ldots, \tau(p) - 1 \), are \( \mathbb{Z} \)-linear combinations of \( s(n)_{n \geq 0} \) and \( s_j(n)_{n \geq 0} \).

Moreover, for \( i = 0, \ldots, p^2 - 1 \) we have

\[
s_0(p^2n + i) = \begin{cases} 
  1 & \text{if } p^2n + i + 1 \equiv 0 \pmod{\tau(p)}, \\
  0 & \text{if } p^2n + i + 1 \not\equiv 0 \pmod{\tau(p)},
\end{cases}
\]

hence, by (7) and (6), it follows that

\[
s_{i \mod \tau(p)}(n)_{n \geq 0} = s_0(p^2n + i)_{n \geq 0} = s(p^2n + i_{n \geq 0} - s(p^2n + i)_{n \geq 0} \\
\in \ker_p(s(n)_{n \geq 0}).
\]

Since \( \tau(p) \mid p - \epsilon \), we have

\[
\tau(p) \leq p - \epsilon \leq p + 1 < p^2,
\]
hence by (8) we get that $s_j(n)_{n \geq 0} \in \langle \ker_p(s(n)_{n \geq 0}) \rangle$, for each $j = 0, \ldots, \tau(p) - 1$.

Therefore, in light of Lemma 2.4, we obtain that $s(n)_{n \geq 0}$ is a $p$-regular sequence and that $\langle \ker_p(s(n)_{n \geq 0}) \rangle$ is generated by $s(n)_{n \geq 0}$ and $s_j(n)_{n \geq 0}$, with $j = 0, \ldots, \tau(p) - 1$. It is straightforward to see that these last sequences are linearly independent, hence $s(n)_{n \geq 0}$ has rank $\tau(p) + 1$.

If $p > 2$, or if $p = 2$ and $\nu_2(u_6) = \nu_2(u_3) + 1$, then $t(n)_{n \geq 0}$ is identically zero, thus from (5) and the previous result on $s(n)$ we find that $r = \tau(p) + 1$.

So it remains only to consider the case $p = 2$ and $\nu_2(u_6) \neq \nu_2(u_3) + 1$. Recall that in such a case $\tau(2) = 3$, and put $d := \nu_2(u_6) - \nu_2(u_3) - 1$. Obviously, the sequence $t(2n)_{n \geq 0}$ is identically zero, while

$$t(2n + 1) = \begin{cases} d & \text{if } 2n + 2 \equiv 0 \mod 6, \\ 0 & \text{if } 2n + 2 \not\equiv 0 \mod 6, \end{cases}$$

$$= \begin{cases} d & \text{if } n + 1 \equiv 0 \mod 3, \\ 0 & \text{if } n + 1 \not\equiv 0 \mod 3, \end{cases}$$

$$= d \cdot s_0(n).$$

Thus, again from Lemma 2.4, we have that $t(n)$ is a 2-regular sequence and that $\langle \ker_p(t(n)_{n \geq 0}) \rangle$ is generated by $t(n)_{n \geq 0}$ and $d \cdot s_j(n)_{n \geq 0}$, for $j = 0, 1, 2$.

In conclusion, by (5) and Lemma 2.1, we obtain that $\nu_p(u_{n+1})_{n \geq 0}$ is a 2-regular sequence and that $\langle \ker_p(\nu_p(u_{n+1})_{n \geq 0}) \rangle$ is generated by $s(n)$, $t(n)$, and $s_j(n)$, for $j = 0, 1, 2$, which are linearly independent, hence $r = 5$. The proof is complete.

6. Concluding remarks

It might be interesting to understand if, actually, $\nu_k(u_{n+1})_{n \geq 0}$ is $k$-regular for every integer $k \geq 2$, so that Theorem 1.3 holds even by dropping the assumption that $k$ and $b$ are relatively prime. A trivial observation is that if $k$ and $b$ have a common prime factor $p$ such that $p \nmid a$, then $p \nmid u_n$ for all integers $n \geq 1$, and consequently $\nu_k(u_{n+1})_{n \geq 0}$ is $k$-regular simple because it is identically zero. Thus the nontrivial case occurs when each of the prime factors of $\gcd(b, k)$ divides $a$.

Another natural question is if it is possible to generalize Theorem 1.4 in order to say something about the rank of $\nu_k(u_{n+1})_{n \geq 0}$ when $k$ is composite. Probably, the easier cases are those when $k$ is squarefree, or when $k$ is a power of a prime number.

We leave these as open questions to the reader.

References


