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## THE QUOTIENT SET OF $k$ -GENERALIZED FIBONACCI NUMBERS IS DENSE IN $\mathbb{Q}_p$

CARLO SANNA

### Abstract

The quotient set of  $A \subseteq \mathbb{N}$  is defined as  $R(A) := \{a/b : a, b \in A, b \neq 0\}$ . Using algebraic number theory in  $\mathbb{Q}(\sqrt{5})$ , Garcia and Luca proved that the quotient set of Fibonacci numbers is dense in the  $p$ -adic numbers  $\mathbb{Q}_p$ , for all prime numbers  $p$ . For any integer  $k \geq 2$ , let  $(F_n^{(k)})_{n \geq -(k-2)}$  be the sequence of  $k$ -generalized Fibonacci numbers, defined by the initial values  $0, 0, \dots, 0, 1$  ( $k$  terms) and such that each term afterwards is the sum of the  $k$  preceding terms. We use  $p$ -adic analysis to generalize Garcia and Luca's result, by proving that the quotient set of  $k$ -generalized Fibonacci numbers is dense in  $\mathbb{Q}_p$ , for any integer  $k \geq 2$  and any prime number  $p$ .

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### 1. Introduction

Given a set of nonnegative integers  $A$ , the *quotient set* of  $A$  is defined as

$$R(A) := \{a/b : a, b \in A, b \neq 0\}.$$

The question of when  $R(A)$  is dense in  $\mathbb{R}^+$  is a classical topic and has been studied by many researchers. Strauch and Tóth [15] proved that if  $A$  has lower asymptotic density at least equal to  $1/2$  then  $R(A)$  is dense in  $\mathbb{R}^+$  (see also [1]). Bukor, Šalát, and Tóth [3] showed that if  $A \cup B$  is a partition of  $\mathbb{N}$  then at least one of  $R(A)$  or  $R(B)$  is dense in  $\mathbb{R}^+$ . Moreover, the density of  $R(\mathbb{P})$  in  $\mathbb{R}^+$ , where  $\mathbb{P}$  is the set of prime numbers, is a well-known consequence of the Prime Number Theorem [10].

On the other hand, the analog question of when  $R(A)$  is dense in the  $p$ -adic numbers  $\mathbb{Q}_p$ , for some prime number  $p$ , has been studied only recently [7, 8]. Let  $(F_n)_{n \geq 0}$  be the sequence of Fibonacci numbers, defined by  $F_0 = 0$ ,  $F_1 = 1$ , and  $F_n = F_{n-1} + F_{n-2}$ , for all integers  $n > 1$ . Using algebraic number theory in the field  $\mathbb{Q}(\sqrt{5})$ , Garcia and Luca [8] proved the following result.

**THEOREM 1.1.** *For any prime  $p$ , the quotient set of Fibonacci numbers is dense in  $\mathbb{Q}_p$ .*

One of the many generalizations of the Fibonacci numbers is the sequence of  $k$ -generalized Fibonacci numbers  $(F_n^{(k)})_{n \geq -(k-2)}$ , also called *Fibonacci  $k$ -step sequence*,

*Fibonacci  $k$ -sequence*, or  *$k$ -bonacci sequence*. For any integer  $k \geq 2$ , the sequence  $(F_n^{(k)})_{n \geq -(k-2)}$  is defined by

$$F_{-(k-2)}^{(k)} = \cdots = F_0^{(k)} = 0, F_1^{(k)} = 1,$$

and

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)},$$

for all integers  $n > 1$ .

Usually, the study of the arithmetic properties of the  $k$ -generalized Fibonacci numbers is more difficult than that of Fibonacci numbers. Indeed, for  $k \geq 3$  the sequence of  $k$ -generalized Fibonacci numbers lacks several nice properties of the sequence of Fibonacci numbers, like: being a strong divisibility sequence [13, p. 9], having a Primitive Divisor Theorem [17], and having a simple formula for its  $p$ -adic valuation [11, 14].

We give the following generalization of Theorem 1.1.

**THEOREM 1.2.** *For any integer  $k \geq 2$  and any prime number  $p$ , the quotient set of the  $k$ -generalized Fibonacci numbers is dense in  $\mathbb{Q}_p$ .*

It seems likely that Theorem 1.2 could be extended to other linear recurrences over the integers. However, in our proof we use some specific features of the  $k$ -generalized Fibonacci numbers sequence. Therefore, we leave the following open question to the interested readers:

**QUESTION 1.3.** *Let  $(S_n)_{n \geq 0}$  be a linear recurrence of order  $k \geq 2$  satisfying*

$$S_n = a_1 S_{n-1} + a_2 S_{n-2} + \cdots + a_k S_{n-k},$$

*for all integers  $n \geq k$ , where  $a_1, \dots, a_k, S_0, \dots, S_{k-1} \in \mathbb{Z}$ , with  $a_k \neq 0$ .*

*For which prime numbers  $p$  is the quotient set of  $(S_n)_{n \geq 0}$  dense in  $\mathbb{Q}_p$ ?*

Clearly, without loss of generality, one can suppose that  $\gcd(S_0, \dots, S_{k-1}) = 1$ . Also, it seems reasonable assuming that  $(S_n)_{n \geq 0}$  is nondegenerate, which in turn implies that  $(S_n)_{n \geq 0}$  is definitely nonzero [5, §2.1]. Finally, a necessary condition for  $(S_n)_{n \geq 0}$  to be dense in  $\mathbb{Q}_p$  is that  $(v_p(S_n))_{n \geq 0, S_n \neq 0}$  is unbounded. This is certainly the case if  $S_0 = 0$  and  $p \nmid a_k$  (since  $p \nmid a_k$  implies that  $(S_n)_{n \geq 0}$  is periodic modulo  $p^h$ , for any positive integer  $h$  [5, §3.1]), so this could be an useful additional hypothesis.

## 2. Proof of Theorem 1.2

From now on, fix an integer  $k \geq 2$  and a prime number  $p$ . In light of Theorem 1.1, we can suppose  $k \geq 3$ . Let

$$f_k(X) = X^k - X^{k-1} - \cdots - X - 1$$

be the characteristic polynomial of the  $k$ -generalized Fibonacci numbers sequence.

It is known [16, Corollary 3.4] that  $f_k$  is separable. Let  $K$  be the splitting field of  $f_k$  over  $\mathbb{Q}_p$  and let  $\alpha_1, \dots, \alpha_k \in K$  be the  $k$  distinct roots of  $f_k$ . We have [4, Theorem 1]

$$F_n^{(k)} = \sum_{i=1}^k c_i \alpha_i^n, \quad (2.1)$$

for all integers  $n \geq 0$ , where

$$c_i := \frac{\alpha_i - 1}{(k+1)\alpha_i^2 - 2k\alpha_i}, \quad (2.2)$$

for  $i = 1, \dots, k$ .

Now we shall interpolate a subsequence of  $(F_n^{(k)})_{n \geq 0}$  by an analytic function over  $\mathbb{Z}_p$ . This is a classical method in the study of linear recurrences, which goes back at least to the proof of the Skolem–Mahler–Lech theorem [5, Theorem 2.1].

We refer the reader to [9, Ch. 4–6] for the  $p$ -adic analysis used hereafter. Let  $\mathcal{O}_K$  be the valuation ring of  $K$ ;  $e$  and  $f$  be the ramification index and the inertial degree of  $K$  over  $\mathbb{Q}_p$ , respectively; and  $\pi$  be a uniformizer of  $K$ .

Since  $f_k(0) = -1$ , we have that each  $\alpha_i$  ( $i = 1, \dots, k$ ) is a unit of  $\mathcal{O}_K$ , so that  $|\alpha_i|_p = 1$ . Hence, in particular,  $\alpha_i \not\equiv 0 \pmod{\pi}$ . Thus, since  $\mathcal{O}_K/\pi\mathcal{O}_K$  is a finite field of  $p^f$  elements, we obtain that  $\alpha_i^{p^f-1} \equiv 1 \pmod{\pi}$ . Now pick any positive integer  $s$  such that  $p^s \geq e + 1$ . Since  $|\pi|_p = p^{-1/e}$ , we have  $\pi^{p^s} \equiv 0 \pmod{p\pi}$ , and, in turn, it follows that  $\alpha_i^t \equiv 1 \pmod{p\pi}$ , where  $t := p^s(p^f - 1)$ . At this point,

$$|\alpha_i^t - 1|_p \leq |p\pi|_p = p^{-1-1/e} < p^{-1/(p-1)}, \quad (2.3)$$

for  $i = 1, \dots, k$ .

Now let  $\log_p$  and  $\exp_p$  denote the  $p$ -adic logarithm and the  $p$ -adic exponential functions, respectively. Thanks to (2.3) we have that

$$\alpha_i^t = \exp_p(\log_p(\alpha_i^t)),$$

for  $i = 1, \dots, k$ , which together with (2.1) implies that  $F_{nt}^{(k)} = G(n)$  for all integer  $n \geq 0$ , where

$$G(z) := \sum_{i=1}^k c_i \exp_p(z \log_p(\alpha_i^t)),$$

is an analytic function over  $\mathbb{Z}_p$ .

Let  $r > 0$  be the radius of convergence of the Taylor series of  $G(z)$  at  $z = 0$ , and let  $\ell \geq 0$  be an integer. On the one hand, the radius of convergence of the Taylor series of  $G(p^\ell z)$  at  $z = 0$  is  $p^\ell r$ . On the other hand,

$$G(p^\ell z) = \sum_{i=1}^k c_i \exp_p(p^\ell z \log_p(\alpha_i^t)) = \sum_{i=1}^k c_i \exp_p(z \log_p(\alpha_i^{p^\ell t})).$$

Therefore, taking  $s$  sufficiently large, we can assume  $r > 1$ .

In particular, we have

$$G(z) = \sum_{j=0}^{\infty} \frac{G^{(j)}(0)}{j!} z^j, \quad (2.4)$$

for all  $z \in \mathbb{Z}_p$ .

Now we shall prove that  $G'(0) \neq 0$ . For the sake of contradiction, assume that

$$G'(0) = \sum_{i=1}^k c_i \log_p(\alpha_i^t) = 0.$$

Since  $f_k(0) = -1$  and  $t$  is even, we have  $\alpha_1^t \cdots \alpha_k^t = 1$ , so that

$$\log_p(\alpha_k^t) = -\log_p(\alpha_1^t) - \cdots - \log_p(\alpha_{k-1}^t),$$

and consequently

$$\sum_{i=1}^{k-1} (c_i - c_k) \log_p(\alpha_i^t) = 0. \quad (2.5)$$

We need the following lemma [6, Lemma 1], which is a special case of a general result of Mignotte [12] on Pisot numbers.

**LEMMA 2.1.** *The roots  $\alpha_1, \dots, \alpha_{k-1}$  are multiplicatively independent, that is,  $\alpha_1^{e_1} \cdots \alpha_{k-1}^{e_{k-1}} = 1$  for some integers  $e_1, \dots, e_{k-1}$  if and only if  $e_1 = \cdots = e_{k-1} = 0$ .*

Thanks to Lemma 2.1, we know that  $\alpha_1^t, \dots, \alpha_{k-1}^t$  are multiplicatively independent. Hence,  $\log_p(\alpha_1^t), \dots, \log_p(\alpha_{k-1}^t)$  are linearly independent over  $\mathbb{Z}$ . Then by [2, Theorem 1] we get that  $\log_p(\alpha_1^t), \dots, \log_p(\alpha_{k-1}^t)$  are linearly independent over the algebraic numbers, hence (2.5) implies

$$c_1 = c_2 = \cdots = c_k. \quad (2.6)$$

At this point, from (2.2) and (2.6), it follows that  $\alpha_1, \dots, \alpha_k$  are all roots of the polynomial

$$c_1(k+1)X^2 - (2c_1k+1)X + 1,$$

but that is clearly impossible, since  $k \geq 3$ . Hence, we have proved that  $G'(0) \neq 0$ .

Taking  $z = 1$  in (2.4), we find that  $v_p(G^{(j)}(0)/j!) \rightarrow +\infty$ , as  $j \rightarrow +\infty$ . In particular, there exists an integer  $\ell \geq 0$  such that  $v_p(G^{(j)}(0)/j!) \geq -\ell$ , for all integers  $j \geq 0$ . As a consequence of this, and since  $G(0) = F_0^{(k)} = 0$ , taking  $z = mp^h$  in (2.4) we get that

$$G(mp^h) = G'(0)mp^h + O(p^{2h-\ell}),$$

for all integers  $m, h \geq 0$ . Therefore, for  $h > h_0 := \ell + v_p(G'(0))$ , we have

$$\frac{G(mp^h)}{G(p^h)} - m = \frac{G'(0)mp^h + O(p^{2h-\ell})}{G'(0)p^h + O(p^{2h-\ell})} - m = \frac{O(p^{h-\ell})}{G'(0) + O(p^{h-\ell})} = O(p^{h-h_0}),$$

that is,

$$\lim_{h \rightarrow +\infty} \left| \frac{G(mp^h)}{G(p^h)} - m \right|_p = 0.$$

In conclusion, we have proved that

$$\lim_{v \rightarrow +\infty} \left| \frac{F_{mp^v(p^f-1)}^{(k)}}{F_{p^v(p^f-1)}^{(k)}} - m \right|_p = 0,$$

for all integers  $m \geq 0$ . In other words, the closure (with respect to the  $p$ -adic topology) of the quotient set of  $k$ -generalized Fibonacci numbers contains the nonnegative integers  $\mathbb{N}$ .

The next easy lemma is enough to conclude.

**LEMMA 2.2.** *Let  $A \subseteq \mathbb{N}$ . If the closure of  $R(A)$  contains  $\mathbb{N}$ , then  $R(A)$  is dense in  $\mathbb{Q}_p$ .*

**PROOF.** Let  $C$  be the closure of  $R(A)$  as a subspace of  $\mathbb{Q}_p$ . Since  $\mathbb{N}$  is dense in  $\mathbb{Z}_p$ , we have  $\mathbb{Z}_p \subseteq C$ . Moreover, the inversion  $\iota : \mathbb{Z}_p^\times \rightarrow \mathbb{Q}_p : x \rightarrow x^{-1}$  is continuous and, obviously, sends nonzero elements of  $R(A)$  to  $R(A)$ , hence  $\iota(\mathbb{Z}_p) \subseteq C$ . Finally,  $\mathbb{Q}_p = \mathbb{Z}_p \cup \iota(\mathbb{Z}_p)$ , thus  $C = \mathbb{Q}_p$  and  $R(A)$  is dense in  $\mathbb{Q}_p$ .  $\square$

The proof of Theorem 1.2 is complete.

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