ON AHARONOV-BOHM OPERATORS WITH TWO COLLIDING POLES

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Abstract. We consider Aharonov-Bohm operators with two poles and prove sharp asymptotics for simple eigenvalues as the poles collapse at an interior point out of nodal lines of the limit eigenfunction.

Dedicated to Professor Ireneo Peral on the occasion of his 70th birthday.

1. Introduction

The present paper is concerned with asymptotic estimates of the eigenvalue variation for magnetic Schrödinger operators with Aharonov–Bohm potentials. These special potentials generate localized magnetic fields, as they are produced by infinitely long thin solenoids intersecting perpendicularly the plane at fixed points (poles), as the radius of the solenoids goes to zero and the magnetic flux remains constant.

The aim of the present paper is the investigation of eigenvalues of these operators as functions of the poles on the domain. This study was initiated by the set of papers [1, 2, 4, 10, 19], where the authors consider a single point moving in the domain, providing sharp asymptotics as it goes to an interior point or to a boundary point. On the other hand, to the best of our knowledge the only paper considering different poles is [17], providing a continuity result for the eigenvalues and an improved regularity for simple eigenvalues as the poles are distinct and far from the boundary.

Additional motivations for the study of eigenvalue functions of these operators appear in the theory of spectral minimal partitions. We refer the interested reader to [7, 9, 13, 20] and references therein.

For \( a = (a_1, a_2) \in \mathbb{R}^2 \), the Aharonov-Bohm magnetic potential with pole \( a \) and circulation \( 1/2 \) is defined as

\[
A_a(x) = \frac{1}{2} \left( \frac{-x_2 - a_2}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2} \right), \quad x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{a\}.
\]

In this paper we consider potentials which are the sum of two different Aharonov–Bohm potentials whose singularities are located at two different points in the domain moving towards each other. For \( a > 0 \) small, let \( a^- = (-a, 0) \) and \( a^+ = (a, 0) \) be the poles of the following Aharonov–Bohm potential

\[
A_{a^-, a^+}(x) := -A_{a^-} + A_{a^+} = -\frac{1}{2} \left( \frac{-x_2, x_1 + a}{(x_1 + a)^2 + x_2^2} + \frac{1}{2} \left( \frac{-x_2, x_1 - a}{(x_1 - a)^2 + x_2^2} \right) \right).
\]

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The main result of the present paper is a sharp asymptotics for the eigenvalue variation

Theorem 1.1. [3, Theorem 1.13]

Let \( \lambda \) be the order of vanishing of \( \sigma_N \) at the collision point, the assumption on the symmetry of \( \sigma \). Let us assume that there exists \( N \geq 1 \). Let \( 0 < \alpha < \pi \) be such that the minimal slope of nodal lines of \( u_N \) is equal to \( \frac{\alpha}{2} \), so that

\[
\lim_{a \to 0} \lambda_k^a = \lambda_k.
\]

The main result of the present paper is a sharp asymptotics for the eigenvalue variation \( \lambda_k^a - \lambda_k \) as the two poles \( a^-, a^+ \) coalesce towards a point where the limit eigenfunction does not vanish.

A first result in this direction was given in [3], under a symmetry assumption on the domain.

Theorem 1.1. [3, Theorem 1.13]

Let \( \sigma : \mathbb{R}^2 \to \mathbb{R}^2 \), \( \sigma(x_1, x_2) = (x_1, -x_2) \). Let \( \Omega \) be an open, bounded, and connected set in \( \mathbb{R}^2 \), satisfying \( \sigma(\Omega) = \Omega \) and \( 0 \in \Omega \). Let \( \lambda_N \) be a simple eigenvalue of the Dirichlet Laplacian on \( \Omega \) and \( u_N \) be a \( L^2(\Omega) \)-normalized eigenfunction associated to \( \lambda_N \). Let \( k \in \mathbb{N} \cup \{0\} \) be the order of vanishing of \( u_N \) at \( 0 \) and \( \alpha \in [0, \pi] \) be such that the minimal slope of nodal lines of \( u_N \) is equal to \( \frac{\alpha}{2} \), so that

\[
u_N(r(\cos t, \sin t)) \sim r^k \beta \sin(\alpha - kt) \quad \text{as } r \to 0^+ \text{ for all } t,
\]

for some \( \beta \in \mathbb{R} \setminus \{0\} \) (see e.g. [12]). Let us assume that \( \alpha \neq 0 \).

For \( a > 0 \) small, let \( a^- = (-a, 0) \), \( a^+ = (a, 0) \in \Omega \), and let \( \lambda_N^a \) be the \( N \)-th eigenvalue for \( (i\nabla + A_{a^-, a^+})^2 \). Then

\[
\lambda_N^a - \lambda_N = \begin{cases} 
\frac{2\pi}{|\log a|}|u_N(0)|^2 (1 + o(1)), & \text{if } k = 0, \\
C_k \pi^2 a^{2k} \sin^2 \alpha (1 + o(1)), & \text{if } k \geq 1,
\end{cases}
\]

as \( a \to 0^+ \), \( C_k > 0 \) being a positive constant depending only on \( k \).

In the present paper, we are able to remove, in the case \( k = 0 \) (i.e. when the limit eigenfunction \( u_N \) does not vanish at the collision point), the assumption on the symmetry of the domain, proving the following result.

Theorem 1.2. [3, Theorem 1.17]

Let \( \Omega \) be an open, bounded, and connected set in \( \mathbb{R}^2 \) such that \( 0 \in \Omega \). Let us assume that there exists \( N \geq 1 \) such that the \( N \)-th eigenvalue \( \lambda_N \) of the Dirichlet Laplacian in \( \Omega \) is simple. Let \( u_N \) be a \( L^2(\Omega) \)-normalized eigenfunction associated to \( \lambda_N \). If \( u_N(0) \neq 0 \) then

\[
\lambda_N^a - \lambda_N = \frac{2\pi}{|\log a|} u_N(0)^2 (1 + o(1))
\]

as \( a \to 0^+ \).

It is worthwhile mentioning that in [17] simple magnetic eigenvalues are proved to be analytic functions of the configuration of the poles provided the limit configuration is made of interior distinct poles. A consequence of our result is that the latter assumption is even necessary and simple eigenvalues are not analytic in a neighborhood of configurations of poles collapsing outside nodal lines of the limit eigenfunction.
The proof of Theorem 1.2 relies essentially on the characterization of the magnetic eigenvalue as an eigenvalue of the Dirichlet Laplacian in $\Omega$ with a small set removed, in the flavor of \[3\] (see \[3.2\]). In \[3\] only the case of symmetric domains was considered and the magnetic problem was shown to be spectrally equivalent to the eigenvalue problem for the Dirichlet Laplacian in the domain obtained by removing the segment joining the poles; in the general non-symmetric case, we can still derive a spectral equivalence with a Dirichlet problem in the domain obtained by removing from $\Omega$ the nodal lines of magnetic eigenfunctions close to the collision point. The general shape of this removed set (which is not necessarily a segment as in the symmetric case) creates some further difficulties; in particular, precise information about the diameter of such a set is needed in order to apply the following result from \[3\].

Theorem 1.3. \[3\] Theorem 1.7] Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set containing 0. Let $\lambda_N$ be a simple eigenvalue of the Dirichlet Laplacian in $\Omega$ and $u_N$ be a $L^2(\Omega)$-normalized eigenfunction associated to $\lambda_N$ such that $u_N(0) \neq 0$. Let $(K_\varepsilon)_{\varepsilon > 0}$ be a family of compact connected sets contained in $\Omega$ such that, for every $r > 0$, there exists $\bar{\varepsilon}$ such that $K_\varepsilon \subseteq D_r$ for every $\varepsilon \in (0, \bar{\varepsilon})$ ($D_r$ denoting the disk of radius $r$ centered at 0). Then

$$\lambda_N(\Omega \setminus K_\varepsilon) - \lambda_N = \frac{2\pi}{\log (\text{diam } K_\varepsilon)} + o\left(\frac{1}{\log (\text{diam } K_\varepsilon)}\right),$$

as $\varepsilon \to 0$,

where $\lambda_N(\Omega \setminus K_\varepsilon)$ denotes the $N$-th eigenvalue of the Dirichlet Laplacian in $\Omega \setminus K_\varepsilon$.

In order to apply Theorem 1.3, a crucial intermediate step in the proof of Theorem 1.2 is the estimate of the diameter of nodal lines of magnetic eigenfunctions near the collision point. More precisely, we prove that, when $a$ is sufficiently small, locally near 0 suitable (magnetic-real) eigenfunctions have a nodal set consisting in a single regular curve connecting $a^-$ and $a^+$. If $d_a$ denotes the diameter of such a curve, we obtain that

$$\lim_{a \to 0^+} \frac{\log a}{\log d_a} = 1,$$

see \[4\].

The paper is organized as follows. In section 2 we obtain some preliminary upper bounds for the eigenvalue variation $\lambda_a^N - \lambda_N$ testing the Rayleigh quotient for eigenvalues with proper test functions constructed by suitable manipulation of limit eigenfunctions. In section 3 we prove that, as the two poles of the operator (2) move towards each other colliding at 0, then $\lambda_a^N$ is equal to the $N$-th eigenvalue of the Laplacian in $\Omega$ with a small piece of nodal line of the magnetic eigenfunction removed. Combining the upper estimates of section 2 with Theorem 1.3 in section 4 we succeed in estimating the diameter of the removed small set as in (4); we then conclude the proof of Theorem 1.2 by combining (4) and Theorem 1.3.

2. Estimates from above

We denote by $\mathcal{H}_a$ the closure of $C_c^\infty(\Omega \setminus \{a^+, a^-, a\}, \mathbb{C})$ with respect to the norm

$$\|u\|_{\mathcal{H}_a} = \left(\int_{\Omega} |(i\nabla + A_{a-a^+})u|^2 \, dx\right)^{1/2}.$$ 

We observe that, by Poincaré and diamagnetic inequalities together with the Hardy type inequality proved in \[16\], $\mathcal{H}_a \subset H_0^1(\Omega)$ with continuous inclusion. In order to estimate from
above the eigenvalue $\lambda_N$, we recall the well-known Courant-Fisher minimax characterization:

\begin{equation}
\lambda_N = \min \left\{ \max_{u \in F \setminus \{0\}} \frac{\int_{\Omega} |(i\nabla + A_{a^{-a^+}})u|^2 dx}{\int_{\Omega} |u|^2 dx} : F \text{ is a subspace of } H_a, \dim F = N \right\}.
\end{equation}

**Lemma 2.1.** Let $\tau \in (0,1)$. For every $0 < \varepsilon < 1$, there exists a continuous radial cut-off function $\rho_{\varepsilon, \tau} : \mathbb{R}^2 \rightarrow \mathbb{R}$, such that $\rho_{\varepsilon, \tau} \in H^1_{\text{loc}}(\mathbb{R}^2)$ and

(i) $0 \leq \rho_{\varepsilon, \tau}(x) \leq 1$ for all $x \in \mathbb{R}^2$;
(ii) $\rho_{\varepsilon, \tau}(x) = 0$ if $|x| \leq \varepsilon$ and $\rho_{\varepsilon, \tau}(x) = 1$ if $|x| \geq \varepsilon^\tau$;
(iii) $\int_{\mathbb{R}^2} |\nabla \rho_{\varepsilon, \tau}|^2 dx = \frac{2\pi}{(\tau - 1) \log \varepsilon}$;
(iv) $\int_{\mathbb{R}^2} (1 - \rho_{\varepsilon, \tau}^2) dx = O(\varepsilon^{2\tau})$ as $\varepsilon \rightarrow 0^+$.

**Proof.** We set

$$\rho_{\varepsilon, \tau}(x) = \begin{cases} 
0, & \text{if } |x| \leq \varepsilon, \\
\frac{\log(|x|) - \log(\varepsilon)}{\log(\varepsilon^\tau) - \log(\varepsilon)}, & \text{if } \varepsilon < |x| < \varepsilon^\tau, \\
1, & \text{if } |x| \geq \varepsilon^\tau.
\end{cases}$$

The function $\rho_{\varepsilon, \tau}$ is continuous and locally in $H^1$, with $0 \leq \rho_{\varepsilon, \tau} \leq 1$. The function $1 - \rho_{\varepsilon, \tau}^2$ is supported in the disk of radius $\varepsilon^\tau$ centered at 0. We therefore have

$$\int_{\mathbb{R}^2} (1 - \rho_{\varepsilon, \tau}^2(x)) dx \leq \pi \varepsilon^{2\tau},$$

which proves (iv). We have $\nabla \rho_{\varepsilon, \tau}(x) = 0$ if $|x| < \varepsilon$ or $|x| > \varepsilon^\tau$, and

$$\nabla \rho_{\varepsilon, \tau}(x) = \frac{x}{(\tau - 1) \log(|x|) x^2}$$

if $\varepsilon < |x| < \varepsilon^\tau$. From this we directly obtain identity (iii). $\square$

**Lemma 2.2.** For all $a > 0$, there exists a smooth function $\psi_a : \mathbb{R}^2 \setminus s_a \rightarrow \mathbb{R}$ satisfying

$$\nabla \psi_a = A_{a^{-a^+}},$$

where $s_a$ is the segment in $\mathbb{R}^2$ defined by $s_a := \{(t, 0) : -a \leq t \leq a\}$. Furthermore, for every $x \in \mathbb{R}^2 \setminus \{(0, 0)\}$, $\lim_{a \rightarrow 0^+} \psi_a(x) = 0$.

**Proof.** See [3, Lemma 3.1]. $\square$

The first step in the proof of Theorem 1.2 is the following upper bound for the eigenvalue $\lambda_N$.

**Proposition 2.3.** For every $\tau \in (0,1)$

$$\lambda_N^a \leq \lambda_N + \frac{2\pi}{(1 - \tau) \log |a|} \left( u_N^a(0) + o(1) \right)$$

as $a \rightarrow 0^+$.

The proof of Proposition 2.3 is based on estimates from above of the Rayleigh quotient for $\lambda_N^a$ computed at some proper test functions constructed by suitable manipulation of limit eigenfunctions. To this aim, let us consider, for each $j \in \{1, \ldots, N\}$, a real eigenfunction $u_j$ of
$-\Delta$ with homogeneous Dirichlet boundary conditions associated with $\lambda_j$, with $\|u_j\|_{L^2(\Omega)} = 1$. Furthermore, we choose these eigenfunctions so that

$$\int_{\Omega} u_j u_k \, dx = 0 \text{ for } j \neq k. \tag{6}$$

For $j \in \{1, \ldots, N\}$ and $a > 0$ small enough, we set

$$v_{j,\tau}^a := e^{i\psi_a} \rho_{2a,\tau} u_j. \tag{7}$$

We have that $v_{j,\tau}^a \in \mathcal{H}_a$. Lemma 2.1 and the Dominated Convergence Theorem imply that $v_{j,\tau}^a$ tends to $u_j$ in $L^2(\Omega)$ when $a \to 0^+$. This implies in particular that the functions $v_{j,\tau}^a$ are linearly independent for $a$ small enough.

Hence, for $a > 0$ small enough, $E_{N,\tau}^a = \text{span}\{v_{1,\tau}^a, \ldots, v_{N,\tau}^a\}$ is an $N$-dimensional subspace of $\mathcal{H}_a$, so that, in view of (5),

$$\lambda_N^a \leq \max_{u \in E_{N,\tau}^a \setminus \{0\}} \frac{\int_{\Omega} |(i\nabla + A_{a^-} - a^+) u|^2 \, dx}{\int_{\Omega} |u|^2 \, dx} = \frac{\int_{\Omega} |(i\nabla + A_{a^-} - a^+) v_{\tau}^a|^2 \, dx}{\int_{\Omega} |v_{\tau}^a|^2 \, dx} \tag{8}$$

with

$$v_{\tau}^a = \sum_{j=1}^N \alpha_{j,\tau}^a v_{j,\tau}^a \quad \text{for some } \alpha_{1,\tau}^a, \ldots, \alpha_{N,\tau}^a \in \mathbb{C} \text{ such that } \sum_{j=1}^N |\alpha_{j,\tau}^a|^2 = 1. \tag{9}$$

**Lemma 2.4.** For $a > 0$ small, let $v_{\tau}^a$ be as in (8) – (9) above. Then

$$\int_{\Omega} |v_{\tau}^a|^2 \, dx = 1 + O(a^2 \tau) \tag{10}$$

as $a \to 0^+$.

**Proof.** Taking into account (9), (7), and (6), we can write

$$\int_{\Omega} |v_{\tau}^a|^2 \, dx = \sum_{j,k=1}^N \alpha_{j,\tau}^a \overline{\alpha_{k,\tau}^a} \int_{\Omega} \rho_{2a,\tau}^2 u_j u_k \, dx$$

$$= 1 + \sum_{j=1}^N |\alpha_{j,\tau}^a|^2 \int_{\Omega} (\rho_{2a,\tau}^2 - 1) u_j^2 \, dx + \sum_{j \neq k} \alpha_{j,\tau}^a \overline{\alpha_{k,\tau}^a} \int_{\Omega} (\rho_{2a,\tau}^2 - 1) u_j u_k \, dx.$$

Hence the conclusion follows from Lemma 2.1 (iv). \qed

**Lemma 2.5.** For $a > 0$ small, let $v_{\tau}^a$ be as in (8) – (9) above. Then

$$\int_{\Omega} |(i\nabla + A_{a^-} - a^+) v_{\tau}^a|^2 \, dx \tag{11}$$

$$= \sum_{j,k=1}^N \alpha_{j,\tau}^a \overline{\alpha_{k,\tau}^a} \left( \frac{\lambda_j + \lambda_k}{2} \int_{\Omega \setminus D_{2a}} \rho_{2a,\tau}^2 u_j u_k \, dx + \int_{D_{(2a)^+} \setminus D_{2a}} u_j u_k |\nabla \rho_{2a,\tau}|^2 \, dx \right),$$

where, for all $r > 0$, $D_r = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < r\}$ denotes the disk of center $(0, 0)$ and radius $r$. 

Proof. Let us fix $j$ and $k$ in $\{1, \ldots, N\}$ (possibly equal). We have that, in $\Omega \setminus D_{2a}$,
\begin{align*}
(i \nabla + A_{a^{-},a^+}) v^a_{j,\tau} \cdot (i \nabla + A_{a^{-},a^+}) v^a_{k,\tau} &= \nabla (\rho_{2a,\tau} u_j) \cdot \nabla (\rho_{2a,\tau} u_k) \\
&= \rho_{2a,\tau}^2 \nabla u_j \cdot \nabla u_k + u_j u_k |\nabla \rho_{2a,\tau}|^2 + (u_j \nabla u_k + u_k \nabla u_j) \cdot \rho_{2a,\tau} \nabla \rho_{2a,\tau},
\end{align*}
and, since $\rho_{2a,\tau} \nabla \rho_{2a,\tau} = \frac{1}{2} \nabla (\rho_{2a,\tau}^2)$,
\begin{align*}
(12) \quad \int_{\Omega} (i \nabla + A_{a^{-},a^+}) v^a_{j,\tau} \cdot (i \nabla + A_{a^{-},a^+}) v^a_{k,\tau} \, dx &= \int_{\Omega \setminus D_{2a}} \rho_{2a,\tau}^2 \nabla u_j \cdot \nabla u_k \, dx \\
&+ \int_{D(2a)^r \setminus D_{2a}} u_j u_k |\nabla \rho_{2a,\tau}|^2 \, dx + \frac{1}{2} \int_{\Omega \setminus D_{2a}} (u_j \nabla u_k + u_k \nabla u_j) \cdot \nabla (\rho_{2a,\tau}^2) \, dx.
\end{align*}

An integration by part on the last term of (12) gives us
\begin{align*}
\int_{\Omega} (i \nabla + A_{a^{-},a^+}) v^a_{j,\tau} \cdot (i \nabla + A_{a^{-},a^+}) v^a_{k,\tau} \, dx &= \int_{\Omega \setminus D_{2a}} \rho_{2a,\tau}^2 \nabla u_j \cdot \nabla u_k \, dx \\
&+ \int_{D(2a)^r \setminus D_{2a}} u_j u_k |\nabla \rho_{2a,\tau}|^2 \, dx - \frac{1}{2} \int_{\Omega \setminus D_{2a}} (u_j \Delta u_k + 2 \nabla u_k \cdot \nabla u_j + \Delta u_j u_k) \rho_{2a,\tau}^2 \, dx.
\end{align*}

After cancellations, we get
\begin{align*}
(13) \quad \int_{\Omega} (i \nabla + A_{a^{-},a^+}) v^a_{j,\tau} \cdot (i \nabla + A_{a^{-},a^+}) v^a_{k,\tau} \, dx &= \frac{\lambda_k + \lambda_j}{2} \int_{\Omega \setminus D_{2a}} \rho_{2a,\tau}^2 u_j u_k \, dx + \int_{D(2a)^r \setminus D_{2a}} u_j u_k |\nabla \rho_{2a,\tau}|^2 \, dx.
\end{align*}

From (9), bilinearity, and (13) we obtain (11). □

From (8) and (11) it follows that
\begin{align*}
(14) \quad \lambda_N^2 - \lambda_N &\leq \frac{1}{\int_{\Omega} |v^a|^2 \, dx} \left[ Q_{a}(\alpha^a_{1,\tau}, \alpha^a_{2,\tau}, \ldots, \alpha^a_{N,\tau}) + \lambda_N \left( 1 - \int_{\Omega} |v^a|^2 \, dx \right) \right]
\end{align*}
where $Q_{a} : \mathbb{C}^N \to \mathbb{R}$ is the quadratic form defined as
\begin{align*}
(15) \quad Q_{a}(z_1, z_2, \ldots, z_N) = \sum_{j,k=1}^{N} M^a_{jk} z_j \overline{z_k}
\end{align*}
with
\begin{align*}
(16) \quad M^a_{jk} = \frac{\lambda_j + \lambda_k}{2} \int_{\Omega \setminus D_{2a}} \rho_{2a,\tau}^2 u_j u_k \, dx + \int_{D(2a)^r \setminus D_{2a}} u_j u_k |\nabla \rho_{2a,\tau}|^2 \, dx - \lambda_N \delta_{jk}
\end{align*}
being $\delta_{jk}$ the Kronecker delta.

To estimate the largest eigenvalue of the quadratic form $Q_{a}$, we will use the following technical lemma.

Lemma 2.6. For every $\varepsilon > 0$ let us consider a quadratic form
\begin{align*}
Q_{\varepsilon} : \mathbb{C}^N \to \mathbb{R}, \quad Q_{\varepsilon}(z_1, z_2, \ldots, z_N) = \sum_{j,k=1}^{N} m_{j,k}(\varepsilon) z_j \overline{z_k},
\end{align*}
with \( m_{j,k}(\varepsilon) \in \mathbb{C} \) such that \( m_{j,k}(\varepsilon) = m_{k,j}(\varepsilon) \). Let us assume that there exist real numbers \( C > 0 \) and \( K_1, K_2, \ldots, K_{N-1} < 0 \) such that
\[
\begin{align*}
m_{N,N}(\varepsilon) &= C\varepsilon(1 + o(1)) \text{ as } \varepsilon \to 0^+, \\
m_{j,j}(\varepsilon) &= K_j + o(1) \text{ as } \varepsilon \to 0^+ \text{ for all } j < N, \\
m_{j,k}(\varepsilon) &= m_{k,j}(\varepsilon) = O(\varepsilon) \text{ as } \varepsilon \to 0^+ \text{ for all } j \neq k.
\end{align*}
\]
Then
\[
\max \left\{ Q_{\varepsilon}(z_1, \ldots, z_N) : (z_1, \ldots, z_N) \in \mathbb{C}^N, \sum_{j=1}^N |z_j|^2 = 1 \right\} = C\varepsilon(1 + o(1)) \text{ as } \varepsilon \to 0^+.
\]

**Proof.** The result is contained in [Lemma 6.1], hence we omit the proof. \( \qed \)

**Lemma 2.7.** For \( a > 0 \) small, let \( Q_a : \mathbb{C}^N \to \mathbb{R} \) be the quadratic form defined in (15)–(16). Then
\[
\max \left\{ Q_a(z_1, \ldots, z_N) : (z_1, \ldots, z_N) \in \mathbb{C}^N, \sum_{j=1}^N |z_j|^2 = 1 \right\} = \frac{2\pi u_N^2(0)}{(1 - \tau) |\log(a)|} (1 + o(1))
\]
as \( a \to 0^+ \).

**Proof.** Since \( \int_{\Omega} u_N^2 = 1 \), we can write
\[
M_{NN}^a = \lambda_N \int_{\Omega} (\rho_{2a,\tau}^2 - 1) u_N^2 \, dx + \int_{D_{(2a)^\tau} \setminus D_{2a}} u_N^2 |\nabla \rho_{2a,\tau}|^2 \, dx.
\]
Since \( u_N \in L^\infty_{\text{loc}}(\Omega) \), from Lemma 2.1 (iv) it follows that
\[
\int_{\Omega} (\rho_{2a,\tau}^2 - 1) u_N^2 \, dx = \int_{D_{(2a)^\tau} \setminus D_{2a}} (\rho_{2a,\tau}^2 - 1) u_N^2 \, dx = O(a^2 \tau) \text{ as } a \to 0^+.
\]
Since \( u_N \in C^\infty_{\text{loc}}(\Omega) \) we have that \( u_N^2(x) - u_N^2(0) = O(|x|) \) as \( |x| \to 0^+ \), then Lemma 2.1 (iii) implies that
\[
\int_{D_{(2a)^\tau} \setminus D_{2a}} u_N^2 |\nabla \rho_{2a,\tau}|^2 \, dx
\]
\[
= u_N^2(0) \int_{D_{(2a)^\tau} \setminus D_{2a}} |\nabla \rho_{2a,\tau}|^2 \, dx + \int_{D_{(2a)^\tau} \setminus D_{2a}} (u_N^2(x) - u_N^2(0)) |\nabla \rho_{2a,\tau}(x)|^2 \, dx
\]
\[
= (u_N^2(0) + O(a^\tau)) \int_{D_{(2a)^\tau} \setminus D_{2a}} |\nabla \rho_{2a,\tau}|^2 \, dx
\]
\[
= \frac{2\pi}{(\tau - 1) \log(2a)} (u_N^2(0) + O(a^\tau)) = \frac{2\pi}{(\tau - 1) \log(a)} u_N^2(0) (1 + o(1))
\]
as \( a \to 0^+ \). Then
\[
M_{NN}^a = \frac{2\pi}{(\tau - 1) \log(a)} u_N^2(0) (1 + o(1)) \text{ as } a \to 0^+.
\]
For all \( 1 \leq j < N \) we have that
\[
M^a_{jj} = \lambda_j \int_{\Omega \setminus D_{2a}} \rho^2_{2a,\tau} u_j^2 \, dx + \int_{D_{(2a)^-} \setminus D_{2a}} u_j^2 |\nabla \rho_{2a,\tau}|^2 \, dx - \lambda_N
\]
and hence, since \( u_j \in C^\infty(\Omega) \) and in view of Lemma 2.1, we get
\[
(\lambda_j - \lambda_N) + \lambda_j \int_{\Omega} (\rho^2_{2a,\tau} - 1) u_j^2 \, dx + \int_{\Omega} u_j^2 |\nabla \rho_{2a,\tau}|^2 \, dx
\]
which is (18)
\[
M^a_{jj} = (\lambda_j - \lambda_N) + O \left( \frac{1}{|\log a|} \right) = (\lambda_j - \lambda_N) + o(1) \quad \text{as} \ a \to 0^+.
\]
Moreover, for all \( j, k = 1, \ldots, N \) with \( j \neq k \), in view of (6) and Lemma 2.1 we have that
\[
M^a_{jk} = \frac{\lambda_j + \lambda_k}{2} \int_{\Omega \setminus D_{2a}} (\rho^2_{2a,\tau} - 1) u_j u_k \, dx + \int_{D_{(2a)^-} \setminus D_{2a}} u_j u_k |\nabla \rho_{2a,\tau}|^2 \, dx = O \left( \frac{1}{|\log a|} \right)
\]
as \( a \to 0^+ \).

In view of estimates (17), (18), and (19), we have that \( Q_a \) satisfies the assumption of Lemma 2.6 (with \( \varepsilon = \frac{1}{|\log a|} \)), hence the conclusion follows from Lemma 2.6.

**Proof of Proposition 2.3.** Combining (14), Lemma 2.7 and estimate (10) we obtain that
\[
\lambda^a_N - \lambda_N \leq \frac{1}{1 + O(a^z)} \left[ \frac{2\pi u^2 N(0)}{(1 - \tau) |\log(a)|} (1 + o(1)) + O(a^{2z}) \right]
\]
and hence
\[
\lambda^a_N - \lambda_N \leq \frac{2\pi u^2 N(0)}{(1 - \tau) |\log(a)|} (1 + o(1)) \quad \text{as} \ a \to 0^+
\]
thus completing the proof.

\[\Box\]

### 3. Gauge invariance, nodal sets and reduction to the Dirichlet-Laplacian

In the following, we mean by a *path* \( \gamma \) a piecewise-\(C^1\) map \( \gamma : I \to \mathbb{R}^2 \), with \( I = [a, b] \subset \mathbb{R} \) a closed interval. It follows from the definition of \( A_{a^-, a^+} \) (see (1)) that for any closed path \( \gamma \) (i.e. \( \gamma(a) = \gamma(b) \))
\[
\frac{1}{2\pi} \oint_{\gamma} A_{a^-, a^+} \cdot ds = \frac{1}{2} \text{ind}_\gamma(a^+) - \frac{1}{2} \text{ind}_\gamma(a^-),
\]
where \( \text{ind}_\gamma(a^+) \) (resp. \( \text{ind}_\gamma(a^-) \)) is the winding number of \( \gamma \) around \( a^+ \) (resp. \( a^- \)).

#### 3.1. Gauge invariance.** Let us give some results concerning the gauge invariance of our operators. In view of applying them to several different situations, we give statements valid for a magnetic Hamiltonian in an open and connected domain \( D \), without restricting ourselves to the Aharonov-Bohm case.

In the following, the term *vector potential* (in an open connected domain \( D \)) stands for a smooth real vector field \( A : D \to \mathbb{R}^2 \). In order to define the quantum mechanical Hamiltonian for a particle in \( D \), under the action of the magnetic field derived from the vector potential \( A \), we first consider the differential operator
\[
P = (i\nabla + A)^2,
\]
acting on smooth functions compactly supported in \( D \). Using integration by parts (Green’s formula), one can easily see that \( P \) is symmetric and positive. This is formally the desired Hamiltonian, but to obtain a self-adjoint Schrödinger operator, we have to specify boundary
conditions on \( \partial \Omega \), which we choose to be Dirichlet boundary conditions everywhere. More specifically, our Hamiltonian is the Friedrichs extension of the differential operator \( P \). We denote it by \( H^D_A \), and we call it the magnetic Hamiltonian on \( D \) associated with \( A \).

We observe that the Aharonov-Bohm operator \( H^\Omega_{\mathbf{a}^-, \mathbf{a}^+} \) with poles \( \mathbf{a}^- = (-a, 0), \mathbf{a}^+ = (a, 0) \) in \( \Omega \) introduced in \([2]\) can be defined as the magnetic Hamiltonian \( H^\Omega_{\mathbf{a}^-, \mathbf{a}^+} \) on \( \hat{\Omega} \), where \( \hat{\Omega} = \Omega \setminus \{\mathbf{a}^-, \mathbf{a}^+\} \), and that the spectrum of \( H^\Omega_{\mathbf{a}^-, \mathbf{a}^+} \) consists of the eigenvalues defined by \((5)\).

The space \( \mathcal{H}_a \) is the form domain of \( H^\Omega_{\mathbf{a}^-, \mathbf{a}^+} \).

**Definition 3.1.** We call a gauge function a smooth complex valued function \( \psi : D \to \mathbb{C} \) such that \( |\psi| \equiv 1 \). To any gauge function \( \psi \), we associate a gauge transformation acting on pairs magnetic potential-function as \((A, u) \mapsto (A^\ast, u^\ast)\), with

\[
\begin{cases}
A^\ast = A - i\nabla \psi \\
u^\ast = \psi u
\end{cases}
\]

where \( \nabla \psi = \nabla(\Re \psi) + i\nabla(\Im \psi) \). We notice that, since \(|\psi| = 1\), \( i\nabla \psi \) is a real vector field. Two magnetic potentials are said to be gauge equivalent if one can be obtained from the other by a gauge transformation (this is an equivalence relation).

The following result is a consequence \([18, \text{Theorem } 1.2]\).

**Proposition 3.2.** If \( A \) and \( A^\ast \) are two gauge equivalent vector potentials, the operators \( H^D_A \) and \( H^D_{A^\ast} \) are unitarily equivalent.

The equivalence between two vector potentials (which is equivalent to the fact that their difference is gauge-equivalent to 0) can be determined using the following criterion.

**Lemma 3.3.** Let \( A \) be a vector potential in \( D \). It is gauge equivalent to 0 if and only if

\[
\frac{1}{2\pi} \oint_{\gamma} A(s) \cdot ds \in \mathbb{Z}
\]

for every closed path \( \gamma \) contained in \( D \).

**Remark 3.4.** The reverse implication in Lemma 3.3 is contained in \([14, \text{Theorem } 1.1]\), for the Neumann boundary condition.

**Proof.** Let us first prove the direct implication. We assume that \( A \) is gauge equivalent to 0, that is to say that there exists a gauge function \( \psi \) such that

\[
A \equiv i\nabla \psi \psi.
\]

Let us fix a closed path \( \gamma : I = [a, b] \to D \) and consider the mapping \( z = \psi \circ \gamma \) from \( I \) to \( U \), where \( U = \{z \in \mathbb{C} : |z| = 1\} \). By the lifting property, there exists a piecewise-\( C^1 \) function \( \theta : I \to \mathbb{R} \) such that \( z(t) = \exp(i\theta(t)) \) for all \( t \in I \). This implies that

\[
\nabla \psi(\gamma(t)) \cdot \gamma'(t) = (\psi \circ \gamma)'(t) = z'(t) = i\theta'(t) \exp(i\theta(t)),
\]

and therefore

\[
\frac{i}{\psi} \nabla \psi (\gamma(t)) \cdot \gamma'(t) = -\theta'(t).
\]
This implies that
\[ \oint_\gamma \mathbf{A}(s) \cdot d\mathbf{s} = \int_a^b \nabla_\psi \psi(t) \cdot \mathbf{A}(t) dt = -\int_a^b \theta'(t) dt = \theta(a) - \theta(b). \]

Since \( \gamma \) is a closed path, \( \exp(i\theta(a)) = \exp(i\theta(b)) \), and therefore
\[ \frac{\theta(a) - \theta(b)}{2\pi} \in \mathbb{Z}. \]

Let us now consider the reverse implication. We define a gauge function \( \psi \) in the following way. We fix an (arbitrary) point \( X_0 = (x_0, y_0) \in D \). Let us show that, for \( X = (x, y) \in D \), the quantity
\[ \exp \left( -i \int_\gamma \mathbf{A}(s) ds \right) \]
does not depend on the choice of a path \( \gamma \) from \( X_0 \) to \( X \). Indeed, let \( \gamma_1 \) and \( \gamma_2 \) be two such paths, and let \( \gamma_3 \) be the closed path obtained by going from \( X_0 \) to \( X \) along \( \gamma_1 \) and then from \( X \) to \( X_0 \) along \( \gamma_2 \). On the one hand, we have
\[ \oint_{\gamma_3} \mathbf{A}(s) ds = \int_{\gamma_1} \mathbf{A}(s) ds - \int_{\gamma_2} \mathbf{A}(s) ds. \]

On the other hand, if (21) holds, we have
\[ \oint_{\gamma_3} \mathbf{A}(s) ds \in 2\pi \mathbb{Z}. \]

This implies that
\[ \exp \left( -i \int_{\gamma_1} \mathbf{A}(s) ds \right) = \exp \left( -i \int_{\gamma_2} \mathbf{A}(s) ds \right). \]

By connectedness of \( D \), there exists a path from \( X_0 \) to \( X \) for any \( X \in \Omega \) (we can even choose it piecewise linear). We can therefore define, without ambiguity, a function \( \psi : \Omega \to \mathbb{C} \) by
\[ \psi(X) = \exp \left( -i \int_\gamma \mathbf{A}(s) ds \right). \]

It is immediate from the definition that \( |\psi| \equiv 1 \) and that \( \psi \) is smooth, with
\[ \nabla \psi(X) = -i\psi(X)\mathbf{A}(X). \]

It is therefore a gauge function sending \( \mathbf{A} \) to 0. \( \square \)

Lemma 3.3 can be used to define a set of eigenfunctions for \( \mathcal{H}_{\Omega}^{\Omega} \), \( a^-, a^+ \), having especially nice properties. It is analogous to the set of real eigenfunctions for the usual Dirichlet-Laplacian. To define it, we will construct a conjugation, that is an antilinear antiunitary operator, which commutes with \( \mathcal{H}_{\Omega}^{\Omega} \). To simplify notation, we denote \( \mathbf{A}_{a^-, a^+} \) by \( \mathbf{A} \) and \( \mathcal{H}_{\Omega}^{\Omega} \) by \( \mathcal{H} \) in the rest of this section.

According to (20), the vector potential \( 2\mathbf{A} \) satisfies condition (21) of Lemma 3.3 on \( \Omega \), and therefore is gauge equivalent to 0. Therefore there exists a gauge function \( \psi \) in \( \Omega \) such that
\[ 2\mathbf{A} = -i\frac{\nabla \psi}{\psi} \quad \text{in} \quad \Omega. \]

We now define the antilinear antiunitary operator \( K \) by
\[ Ku = \psi \tilde{u}. \]
For all \( u \in C_0^\infty(\bar{\Omega}, \mathbb{C}) \),
\[
(i\nabla + A)(\psi \bar{u}) = \psi \left( i\nabla + i\frac{\nabla \psi}{\psi} + A \right) \bar{u} = \psi (i\nabla - A) \bar{u} = -\psi (i\nabla + A) \bar{u}.
\]
The above formula, and the fact that \( K \) is antilinear and antiunitary, imply that, for all \( u \) and \( v \) in \( C_0^\infty(\bar{\Omega}, \mathbb{C}) \),
\[
\langle K^{-1} HKu, v \rangle = \langle Ku, HKu \rangle = \int_\Omega \left( (i\nabla + A) (\psi \bar{u}) \cdot (i\nabla + A) (\psi \bar{v}) \right) dx
= \int_\Omega \left( (i\nabla + A) v \cdot (i\nabla + A) u \right) dx = \langle Hu, v \rangle,
\]
where \( \langle f, g \rangle = \int_\Omega f \bar{g} \, dx \) denotes the standard scalar product on the complex Hilbert space \( L^2(\Omega, \mathbb{C}) \). By density, we conclude that
\[
K^{-1} HK = H.
\]

**Definition 3.5.** We say that a function \( u \in L^2(\Omega, \mathbb{C}) \) is **magnetic-real** when \( Ku = u \).

Let denote by \( \mathcal{R} \) the set of magnetic-real functions in \( L^2(\Omega, \mathbb{C}) \). The restriction of the scalar product to \( \mathcal{R} \) gives it the structure of a real Hilbert space. The commutation relation \( HK = KH \) implies that \( \mathcal{R} \) is stable under the action of \( H \); we denote by \( H^R \) the restriction of \( H \) to \( \mathcal{R} \). There exists an orthonormal basis of \( \mathcal{R} \) formed by eigenfunctions of \( H^R \). Such a basis can be seen as a basis of magnetic-real eigenfunctions of the operator \( H \), in the complex Hilbert space \( L^2(\Omega, \mathbb{C}) \).

Let us now fix an eigenfunction \( u \) of \( H^R \) (or, equivalently, a magnetic-real eigenfunction of \( H \)). We define its **nodal set** \( \mathcal{N}(u) \) as the closure in \( \bar{\Omega} \) of the zero-set \( u^{-1}(\{0\}) \). Let us describe the local structure of \( \mathcal{N}(u) \). In the sequel, by a **regular curve** or **regular arc** we mean a curve admitting a \( C^{1,\alpha} \) parametrization, for some \( \alpha \in (0,1) \).

**Theorem 3.6.** The set \( \mathcal{N}(u) \) has the following properties.

(i) \( \mathcal{N}(u) \) is, locally in \( \bar{\Omega} \), a regular curve, except possibly at a finite number of singular points \( \{X_j\}_{j \in \{1, \ldots, n\}} \).

(ii) For \( j \in \{1, \ldots, n\} \), in the neighborhood of \( X_j \), \( \mathcal{N}(u) \) consists in an even number of regular half-curves meeting at \( X_j \) with equal angles (so that \( X_j \) can be seen as a cross-point).

(iii) In the neighborhood of \( a^+ \) (resp. \( a^- \)), \( \mathcal{N}(u) \) consists in an odd number of regular half-curves meeting at \( a^+ \) (resp. \( a^- \)) with equal angles (in particular this means that \( a^+ \) and \( a^- \) are always contained in \( \mathcal{N}(u) \)).

**Proof.** The proof is essentially contained in [20, Theorem 1.5]; for the sake of completeness we present a sketch of it. Let the eigenfunction \( u \) be associated with the eigenvalue \( \lambda \), so that \( Hu = \lambda u \). Let \( x_0 \) be a point in \( \bar{\Omega} \). For \( \varepsilon > 0 \), we denote by \( D(x_0, \varepsilon) \) the open disk \( \{x : |x - x_0| < \varepsilon\} \). Let us show that we can find \( \varepsilon > 0 \) small enough and a local gauge transformation \( \varphi : D(x_0, \varepsilon) \to \mathbb{C} \) such that \( A^* = A - i\frac{\nabla \varphi}{\varphi} = 0 \) and \( u^* = \varphi u \) is a real-valued function in \( D(x_0, \varepsilon) \). Indeed, let us define, as before, the gauge function \( \varphi \) such that \( 2A = -i\frac{\nabla \psi}{\psi} \). For \( \varepsilon > 0 \) small enough, we can define a smooth function \( \varphi : D(x_0, \varepsilon) \to \mathbb{C} \) such that \( \psi(x) = (\varphi(x))^2 \) for all \( x \in D(x_0, \varepsilon) \): take
\[
\varphi(x) = \exp \left( -\frac{i}{2} \arg(\psi(x)) \right)
\]
with arg a determination of the argument in $\psi(D(x_0, \varepsilon))$. A direct computation shows that, for $x \in D(x_0, \varepsilon)$,

$$i \frac{\nabla \varphi(x)}{\varphi(x)} = i \frac{\nabla \psi(x)}{\psi(x)} = A(x).$$

The gauge transformation on $D(x_0, \varepsilon)$ associated with $\varphi$ therefore sends $A$ to 0. Furthermore, since $u$ is $K$-real, we have $\psi \bar{u} = \overline{\varphi u} = u$ in $D(x_0, \varepsilon)$, and therefore $\overline{\varphi u} = \varphi u$. The real-valued function $v = \varphi u$ satisfies $-\Delta v = \lambda v$, and, since $|\varphi| \equiv 1$ on $D(x_0, \varepsilon)$, we have that $\mathcal{N}(v) \cap D(x_0, \varepsilon) = \mathcal{N}(u) \cap D(x_0, \varepsilon)$. Points (i) and (ii) of Theorem 3.6 then follow from classical results on the nodal set of Laplacian eigenfunctions (see for instance [15, Theorem 2.1] and [20, Theorem 4.2]).

To prove point (iii) of Theorem 3.6, we use the regularity result of [20] for the Dirichlet problem associated with a one-pole Aharonov-Bohm operator. Indeed, let $\varepsilon > 0$ be small enough so that $D = D(a^+, \varepsilon) \subset \Omega$ and $a^- \not\in D$. By this choice of $\varepsilon$, $A_{a^-} = \nabla f$ on $D$, with $f$ a smooth function, so that the domain $D$ and the magnetic potential $A$, restricted to $D$, satisfy the hypotheses of [20, Theorem 1.5]. The function $u$ is a solution of the Dirichlet problem

$$\begin{cases}
(i \nabla + A)^2 u - \lambda u = 0, & \text{in } D, \\
u = \gamma, & \text{on } \partial D,
\end{cases}$$

with $\gamma = u_{\partial D} \in W^{1, \infty}(\partial D)$. A direct application of [20, Theorem 1.5] gives property (iii) around $a^+$. We obtain property (iii) around $a^-$ by exchanging the role of $a^+$ and $a^-$. □

### 3.2. Reduction to the Dirichlet-Laplacian

Our aim in this subsection is to show that, as the two poles of the operator (2) coalesce into a point at which $u_N$ does not vanish, then $\lambda_N^a$ is equal to the $N$-th eigenvalue of the Laplacian in $\Omega$ with a small subset concentrating at 0 removed.

**Theorem 3.7.** Let us assume that there exists $N \geq 1$ such that the $N$-th eigenvalue $\lambda_N$ of the Dirichlet Laplacian in $\Omega$ is simple. Let $u_N$ be a $L^2(\Omega)$-normalized eigenfunction associated $\lambda_N$ and assume that $u_N(0) \neq 0$. Then, for all $a > 0$ sufficiently small, there exists a compact connected set $K_a \subset \Omega$ such that

$$\lambda_N^a = \lambda_N(\Omega \setminus K_a)$$

and $K_a$ concentrates around 0 as $a \to 0^+$, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $a < \delta$ then $K_a \subset D_\varepsilon$.

We will divide the proof into two lemmas.

**Lemma 3.8.** Let $R > 0$ be such that $\overline{D_R} \subset \Omega$ and $u_N(x) \neq 0$ for all $x \in \overline{D_R}$. Let $r \in (0, R)$. We denote by $C_{r, R}$ the closed ring

$$C_{r, R} = \{x \in \mathbb{R}^2 : r \leq |x| \leq R\}.$$ 

There exists $\delta > 0$ such that, if $0 < a < \delta$ and if $u$ is a magnetic-real eigenfunction associated with $\lambda_N^a$, then $u$ does not vanish in $C_{r, R}$.

**Proof.** Let us assume, by contradiction, that there exists a sequence $a_n \to 0^+$ such that, for all $n \geq 1$, $\lambda_N^{a_n}$ admits an eigenfunction $\varphi_n$ which vanishes somewhere in $C_{r, R}$. Let us denote by $X_n$ a zero of $\varphi_n$ in $C_{r, R}$.

According to [17, Section III], we can assume, up to extraction and a suitable normalization of $\varphi_n$, that $\varphi_n \to u_N$ in $L^2(\Omega)$. Since $H$ is a uniformly regular elliptic operator in a
neighborhood of $C_{r,R}$, $\varphi_n$ converges to $u_N$ uniformly on $C_{r,R}$. Furthermore, up to one additional extraction, we can assume that $X_n \to X_\infty \in C_{r,R}$. This implies that $u_N(X_\infty) = 0$, contradicting the fact that $u_N(x) \neq 0$ for all $x \in D_R$. \hfill \Box

**Lemma 3.9.** For all $R > 0$ such that $D_R \subset \Omega$ and $u_N(x) \neq 0$ for all $x \in \overline{D_R}$, there exists $\delta > 0$ such that, if $0 < a < \delta$ and if $u_N^a$ is a magnetic-real eigenfunction associated with $\lambda_N^a$, then $\mathcal{N}(u_N^a) \cap D_R$ consists in a single regular curve connecting $a^-$ and $a^+$.

**Proof.** By continuity of $(a^-, a^+) \mapsto \lambda_N^a$ (see [17]), we have that

\begin{equation}
\Lambda = \max_{a \in [0,R]} \lambda_N^a \in (0, +\infty).
\end{equation}

Let us choose $r \in (0, R)$ such that

\begin{equation}
r < \sqrt{\frac{\lambda_1(D_1)}{\Lambda}},
\end{equation}

where $\lambda_1(D_1)$ is the 1-st eigenvalue of the Laplacian in the unit disk $D_1$. According to Lemma 3.8 there exists $\delta(r) > 0$ such that, if $a < \delta(r)$, any eigenfunction associated to $\lambda_N^a$ does not vanish in the closed ring $C_{r,R}$.

Let us assume that $0 < a < \delta(r)$ and $a < r$ and let $u_N^a$ be an eigenfunction associated with $\lambda_N^a$. The proof relies on a topological analysis of $\mathcal{N}' := \mathcal{N}(u_N^a) \cap D_R$, inspired by previous work on nodal sets and minimal partitions (see [8] Section 6 and references therein). Lemma 3.8 implies that $\mathcal{N}'$ is compactly included in $D_r$. Theorem 3.6 implies that $\mathcal{N}'$ consists of a finite number of regular arcs connecting a finite number of singular points. In other words, $\mathcal{N}'$ is a regular planar graph. Let us denote by $V$ the set of vertices of $\mathcal{N}'$, by $b_1$ the number of its connected components and by $\mu$ the number of its faces. By face, we mean a connected component of $\mathbb{R}_2 \setminus \mathcal{N}'$. There is always one unbounded face, so $\mu \geq 1$. Furthermore, for all $v \in V$, we denote by $\nu(v)$ the degree of the vertex $v$, that is to say the number of half-curves ending at $v$. Let us note that, according to Theorem 3.6 both $a^-$ and $a^+$ belong to $V$ and have an odd degree, and any other vertex can only have an even degree. These quantities are related through Euler’s formula for planar graphs:

\begin{equation}
\mu = b_1 + \sum_{v \in V} \left( \frac{\nu(v)}{2} - 1 \right) + 1.
\end{equation}

For this classical formula, see for instance [8] Theorems 1.1 and 9.5]. Note that this reference treats the case of a connected graph. The generalization used here is easily obtained by linking the $b_1$ connected components of the graph with $b_1 - 1$ edges, in order to go back to the connected case.

Let us show by contradiction that $\mu = 1$. If $\mu \geq 2$, there exists a bounded face of the graph $\mathcal{N}'$, which is a nodal domain of $u_N^a$ entirely contained in $D_r$. Let us call it $\omega$. We denote by $\lambda_k(\omega, a^-, a^+)$ the $k$-th eigenvalue of the operator $(i\nabla + A_{a^-, a^+})^2$ in $\omega$, with homogeneous Dirichlet boundary condition on $\partial \omega$. Since $\omega$ is a nodal domain, we have that, for some $k(a) \in \mathbb{N} \setminus \{0\}$ depending on $a$,

$$
\lambda_N^a = \lambda_{k(a)}(\omega, a^-, a^+) \geq \lambda_1(\omega, a^-, a^+).
$$

By the diamagnetic inequality

$$
\lambda_1(\omega, a^-, a^+) \geq \lambda_1(\omega)
$$


where $\lambda_1(\omega)$ is the 1-st eigenvalue of the Dirichlet Laplacian in $\omega$. By domain monotonicity

$$\lambda_1(\omega) \geq \lambda_1(D_r) = \frac{\lambda_1(D_1)}{r^2}.$$ 

Hence we obtain that

$$r \geq \sqrt{\frac{\lambda_1(D_1)}{\lambda_N^r}},$$

thus contradicting (23). We conclude that $\mu = 1$.

Going back to Euler’s formula (24), we obtain

$$\sum_{v \in V} \left( \frac{\nu(v)}{2} - 1 \right) = -b_1 \leq -1.$$

According to Theorem 3.6, we have $\nu(v)/2 - 1 \geq -1/2$ if $v \in \{a^-, a^+\}$ and $\nu(v)/2 - 1 \geq 1$ if $v \in V \setminus \{a^-, a^+\}$. Inequality (25) can therefore be satisfied only if $V = \{a^-, a^+\}$ and $\nu(a^-) = \nu(a^+) = 1$, that is to say if $\mathcal{N}$ is a regular arc connecting $a^-$ and $a^+$.

We are now in position to prove Theorem 3.7.

**Proof of Theorem 3.7.** From Lemma 3.9 it follows that, for $a$ sufficiently small, there exists a curve $K_a$ in $\mathcal{N}(u_N^a)$ connecting $a^-$ and $a^+$ and (in view of Lemma 3.8) concentrating at $0$, where $u_N^a$ is a magnetic-real eigenfunction associated with $\lambda_N^a$.

Let us write $\Omega'_a = \Omega \setminus K_a$. Since $K_a$ is contained in $\mathcal{N}(u_N^a)$, we have that there exists $k(a) \in \mathbb{N} \setminus \{0\}$ (depending on $a$) such that

$$\lambda_N^a = \lambda_{k(a)}(\Omega'_a, a^-, a^+),$$

where $\lambda_{k(a)}(\Omega'_a, a^-, a^+)$ denotes the $k(a)$-th eigenvalue of $H_{a^-, a^+}^{\Omega'_a}$.

Let us consider a closed path $\gamma$ in $\Omega'_a$. By definition of $\Omega'_a$, $\gamma$ does not meet $K_a$, which means that $K_a$ is contained in a connected component of $\mathbb{R}^2 \setminus \gamma$. Since the function $X \mapsto \text{Ind}_\gamma(X)$ is constant on all connected components of $\mathbb{R}^2 \setminus \gamma$, we have that $\text{Ind}_\gamma(a^-) = \text{Ind}_\gamma(a^+)$. According to (20), this implies that

$$\frac{1}{2\pi} \oint_{\gamma} A_{a^-, a^+} \cdot ds = 0.$$ 

In view Lemma 3.3, we conclude that $A_{a^-, a^+}$ is gauge equivalent to $0$ in $\Omega'_a$ and hence Proposition 3.2 ensures that

$$\lambda_{k(a)}(\Omega'_a, a^-, a^+) = \lambda_{k(a)}(\Omega'_a).$$

Combining (26) and (27) we obtain

$$\lambda_N^a = \lambda_{k(a)}(\Omega'_a).$$

We observe that $a \mapsto k(a)$ stays bounded as $a \to 0^+$; indeed if, by contradiction, $k(a_n) \to +\infty$ along some sequence $a_n \to 0^+$, by (28) we should have $\lambda_N^{a_n} = \lambda_{k(a_n)}(\Omega'_a) \geq \lambda_{k(a_n)}(\Omega) \to +\infty$ thus contradicting (22).

Then, for any sequence $a_n \to 0^+$, there exists a subsequence $a_{n_j}$ such that $k(a_{n_j}) \to k$ for some $k$; since $k(a)$ is integer-valued we have that necessarily $k(a_{n_j}) = k \in \mathbb{N} \setminus \{0\}$ for $j$ sufficiently large. Hence (28) yields $\lambda_N^{a_{n_j}} = \lambda_k(\Omega \setminus K_{a_{n_j}})$. It is well known (see e.g. Theorem 1.2) that $\lambda_k(\Omega \setminus K_{a_{n_j}}) \to \lambda_k(\Omega)$ as $j \to +\infty$; hence, taking into account (3), we conclude that $k = N$. Moreover, since the limit of $k(a_{n_j})$ does not depend on the subsequence
and \(a \to k(a)\) is integer-valued, we conclude that \(k(a) = N\) for all \(a\) sufficiently small, so that (28) becomes
\[
\lambda_N^a = \lambda_N (\Omega_a')
\]
and the proof is complete. \(\square\)

4. Proof of Theorem 1.2

We are in position to complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** For \(a > 0\) small, let \(K_a \subset \Omega\) be as in Theorem 3.7. We denote as
\[
d_a := \text{diam } K_a
\]
the diameter of \(K_a\). From Theorem 1.3 it follows that
\[
\lambda_N (\Omega \setminus K_a) - \lambda_N = u_N^2 (0) \frac{2\pi}{|\log d_a|} + o \left( \frac{1}{|\log d_a|} \right), \quad \text{as } a \to 0^+.
\]
Hence, in view of Theorem 3.7,
\[
\lambda_N^a - \lambda_N = u_N^2 (0) \frac{2\pi}{|\log d_a|} + o \left( \frac{1}{|\log d_a|} \right), \quad \text{as } a \to 0^+.
\]
From (29) and Proposition 2.3 it follows that, for every \(\tau \in (0, 1)\),
\[
\frac{1}{|\log d_a|} (1 + o(1)) \leq \frac{1}{(1 - \tau)|\log a|} (1 + o(1))
\]
and then
\[
|\log a| \leq \frac{1}{(1 - \tau)} (1 + o(1)), \quad \text{as } a \to 0^+.
\]
On the other hand, since \(a^-, a^+ \in K_a\), we have that \(d_a \geq 2a\) so that \(|\log a| \geq |\log d_a| + \log 2\)

and
\[
\frac{|\log a|}{|\log d_a|} \geq 1 + O \left( \frac{1}{|\log d_a|} \right) = 1 + o(1), \quad \text{as } a \to 0^+.
\]
Combining (30) and (31) we conclude that
\[
1 \leq \liminf_{a \to 0^+} \frac{|\log a|}{|\log d_a|} \leq \limsup_{a \to 0^+} \frac{|\log a|}{|\log d_a|} \leq \frac{1}{(1 - \tau)}
\]
for every \(\tau \in (0, 1)\), and then, letting \(\tau \to 0^+\), we obtain that
\[
\lim_{a \to 0^+} \frac{|\log a|}{|\log d_a|} = 1.
\]
The conclusion then follows from (29) and (32). \(\square\)
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